Differential Geometry

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Introduction

These notes are based on the course lectured by Dr Jack E Smith in Michaelmas 2020. Due to the measures taken in the UK to limit the spread of Covid-19, these lectures were delivered online. These are not meant to be an accurate representation of what was lectures. They solely represent a mix of what I thought was the most important part of the course, mixed in with many (many) personal remarks, comments and digressions... Of course, any corrections/comments are appreciated.

Unlike some of the other courses, there is no real introduction here, and we jump straight into the content!

1 Manifolds and Smooth Maps

Manifolds are spaces that locally look like $\mathbb{R}^n$. Formally this is:

Definition 1. X is a **topological** n **manifold** if X is a second countable Hausdorff topological space such that $\forall p \in X \exists \text{open} U \ni p$ and open $V \subseteq \mathbb{R}^n$ and homeomorphism $\phi : U \to V$.

Here,

Definition 2. Topological space X is **Hausdorff** if for every distinct $x, y \in X$ there exists open $U \ni x, V \ni y$ in X such that $U \cap V = \emptyset$.

and

Definition 3. Topological space X is **second countable** if there exists a set of open sets $\mathcal{U}$ st that every open set in X can be written as a union of sets in $\mathcal{U}$

Since these two properties transfer to subsets, any subset of a topological n manifold is also a topological n manifold. Also, to give some more intuition, the condition that X is a seond countable Hausdorff topological space is exactly equivalent to the condition that X is metrizable and has countably many components. It is just tradition that it is defined as above. Some more defintions.

Above, ϕ is called the **chart**, U is called the **coordinate patch**, although in some cases can also be called the chart. The functions $x_1 \circ \phi, \dots, x_n \circ \phi$ (so the components of the result) are called the **local coordinates**, and ϕ^{-1} is called the **paramaterisation**, although that term is not used that frequently. Finally, for overlapping charts, we can define the **transition map** between them as $\phi_2 \circ \phi_1^{-1} : \phi_1(U_1 \cap U_2) \to \phi_2(U_1 \cap U_2)$.

Now, we want to generalise calculus to manifolds, so it makes sense to start by trying to generalise the notion of smoothness. The simplest approach would be to say that f is smooth on X if it is smooth on the local coordinates. The issue then arises that this may not be consistent with smoothness on other charts (where these overlap). As such, we need to require that the transition maps are smooth as well. Consequently we do the following:

Definition 4. The **atlas** of a manifold is a collection of charts of a topological n manifold that covers all of X.

An atlas is **smooth** if all transition maps are smooth, and a map f is **smooth** on atlas $\mathfrak{A}$ if $f \circ \phi_\alpha^{-1}$ is smooth $\forall \alpha$. As a result all local coordinate functions are smooth. Now, really, specifying the atlas precisely all the time is somewhat tedious, and somehow not the point, so we want a degree of flexibility. For this we define

Definition 5. Two atlases are **smoothly equivalent** if their union is smooth.

Note that this forms an equivalence relation (apply the chain rule on transition functions for transitivity).

Definition 6. A **smooth structure** is an equivalence class of atlases.

As we hope, we do indeed have that if a function is smooth wrt to an atlas, it is smooth wrt to any atlas in its smooth structure. Also, we can define a maximal atlas to be the union of all the atlases in a smooth structure, if we deem that to be convenient (includes trivial changes like translating or scaling the local coordinates).

Definition 7. A **smooth** n **manifold** is a topological n manifold with a smooth structure.

Note that under the product topology, the product of two manifolds naturally forms a new (smooth) manifold. Also, a remarkable fact is that for $n = 1, 2, 3$ all topological n manifolds have an essentially unique smooth structure, whereas this breaks down for $n \geq 4$. Also, a new chart is said to be **compatible** with an atlas if when added to the atlas, the atlas remains smooth.

Finally, to give a concrete example of a manifold, we may consider S^n, which forms a manifold with two charts: one being the sphere without the North pole, and the other being the sphere without the South pole, $U_\pm$ with charts

$$\phi_\pm(y_0, \dots, y_n) = \frac{1}{1 \mp y_0}(y_1, \dots, y_n),$$

where the local coordinates are referred to as $x^\pm$. [End of DG1]

1.1 Forming Manifolds from Sets (Instead of Topological Spaces)

We observe that an atlas can generate a topology. In particular if we consider the data

- set X
- subsets $U_\alpha \subseteq X$
- open sets $V_\alpha \subseteq \mathbb{R}^n$
- bijections $\phi_\alpha : U_\alpha \to V_\alpha$ that have smooth transition functions, and $\forall \alpha, \beta, \phi_\alpha(U_\alpha \cap U_\beta)$ is open in V_α (weird but useful)

then we see that if we declare U to be open iff $\phi_\alpha(U \cap U_\alpha)$ is open $\forall \alpha$, then this forms a topology (easy) and

Proposition 1. Apart from the possible failure of Hausdorff and second countable, using the above data as specified turns X into a topological n manifold, and $\{\phi_\alpha\}$ into a smooth atlas (so we have a smooth manifold).

Proof. It suffices to show that U_α are open and that ϕ_α are homeomorphisms (smoothness follows from the smoothness of the transition functions). As such it is sufficient to show that some $U \subseteq U_\alpha$ is open iff $\phi_\alpha(U)$ is open in V_α (this is to show that ϕ_α is a homeomorphism, which implies taht U_α is open by the openness of V_α). One direction is clear: if U is open, then by declaration, $\phi(U \cap U_\alpha = U)$ is open. Conversely, if $\phi(U)$ is open then we want that $\forall \beta, \phi_\beta(U \cap U_\beta)$ is open, but we observe that

$$\begin{aligned}\phi_\beta(U \cap U_\beta) &= \phi_\beta \circ \phi_\alpha^{-1}(\phi_\alpha(U \cap U_\beta)) \\ &= (\phi_\alpha \circ \phi_\beta^{-1})^{-1}(\phi_\alpha(U) \cap \phi_\alpha(U_\alpha \cap U_\beta))\end{aligned}$$

Here $\phi_\alpha(U_\alpha \cap U_\beta)$ is open as an intersection of $V_\alpha \cap V_\beta$, and $\phi_\alpha(W)$ is open by assumption. The transition function is continuous by assumption. So done. $\square$

Finally, we note that we can define this set of ϕs and Us and Vs, etc. to be a "pseudo-chart", and can define a "smooth pseudo-structure," etc. from here. All we really want to show is that the toplogy is secondary once we have a good set of functions. In fact, using this approach we can skip X entirely, and start from sets U_α that we identify with different real spaces, and then stitch together with arbitrary pseudo-charts. That can definitely be done, but generally is quite complicated with many more moving parts, and so we usually, at least for the purpose of a course, start with a structure in mind, and turn that into a manifold, instead of stitching an arbitrary one together (although that may be a good source of counter examples).

Unfortunately, it does remain the case that showing Hausdorff and second-countable is still hard, although there are some tricks to do so. For second countability, using the subset of all rational balls will always work if the number of charts is countable. For Hausdorffness, as long as two points live in the same chart, we are immediately done, so combining charts and considering the exceptions can be an efficient approach.

1.2 Projective Spaces and Grassmannians

Today we do an extended example to develop some more interesting manifolds. First, we look at the real projective linear space $\mathbb{RP}^n$, which is the space $\mathbb{R}^{n+1} \setminus \{0\}$ under the equivalence relation $x \sim y$ iff $\exists \lambda \in \mathbb{R} \setminus \{0\}, x = \lambda y$. As such, any point can be labelled by the *ratio* of coordinates $[x_0 : x_1 : \cdots : x_n]$, and for the purpose of comparison, we are free to set any nonzero coordinate x_i to 1.

To turn this into a smooth n-manifold, let's consider the following pseudo-chart:

$$U_i = \{[x_0 : x_1 : \cdots : x_n] | x_i \neq 0\}$$
$$\phi_i([x_0 : x_1 : \cdots : x_n]) = \frac{1}{x_i}(x_0, \ldots, \hat{x_i}, \ldots, x_n)$$

where $\hat{x_i}$ means omit x_i from the list.

Lemma 1. These form a pseudo-atlas.

Proof. Need to show $\phi_i(U_i \cap U_j)$ open and $\phi_j \circ \phi_i^{-1}$ smooth. wlog, $i = 0, j = 1$ and s, t are our local coordinates of interest. With a little thought we see $\phi_0(U_0 \cap U_1) = \{s_1 \neq 0\}$ which is open. To see $\phi_1 \circ \phi_0^{-1}$ is smooth, notice that

$$\phi_0^{-1}(s) = [1 : s_1 : \cdots : s_n]$$
$$\phi_1^{-1}(t) = [t_1 : 1 : \cdots : t_n]$$

and consequently we see that certainly

$$\phi_1 \circ \phi_0^{-1}(s) = \frac{1}{s_1}[1 : s_1 : \cdots : s_n]$$

is smooth, so indeed we have pseudo-atlas. $\square$

All that is left now is to show that $\mathbb{RP}^n$ is second countable and Hausdorff. Second countability follows from the finite number of covers that we are using. Hausdorffness is harder though. We cannot use our previous tactic of relying on the fact that any two points will lie in at least some chart. So instead we expand our atlas until that is the case. Above, we essentially formed charts using the standard unit vector basis, and omitted that each time. This can be generalised in two ways. Firstly, we can expand to any basis, which copies the above approach and is entirely straight forward. Secondly, we can consider a space W to be our line in $\mathbb{R}^n$, and any complement W' of it. This abstracts the notion, but it has its uses as well.

Taking this approach, we see that for a given line input T the projection $\pi_W|_T$ restricted to T is a bijection (as these have forcibly the same dimension) so taking

$$\psi_T = \pi'_W \circ (\pi_W|_T)^{-1} : W \to W'$$

allows us to biject from our manifold to W' which is isomorphic to $\mathbb{R}^n$, giving us a chart. It is an exercise on the example sheet to show this is compatible. Consequently, filling these all, we can show that the space is Hausdorff as well, meaning that $\mathbb{RP}^n$ forms a smooth n-manifold. Now, just as a note, really, we are not identifying the line T with a point in W'. Rather, we are identifying T with the map ψ_T. This will be relevant.

Now, one generalisation we will make, is to that of the **Grassmannian**. The Grassmannian, $Gr(k,n)$ is thespace of k-subspaces within $\mathbb{R}^n$. Note in particular that $\mathbb{RP}^n = Gr(1, n+1)$. And since we identify subspaces T with ψ_T we also see that $\dim Gr(k,n) = \dim \mathcal{L}(W, W') = k(n-k)$. Finally, we want to remark that we can use $\mathbb{C}$ instead of $\mathbb{R}$, to give use $\mathbb{CP}^n$, and $Gr_{\mathbb{C}}(k,n)$. Incidentally, the transition maps here are not only smooth, but also holomorphic, meaning that these form complex manifolds. [End of DG 3]

1.3 Smoothness of Maps Between Manifolds

So how do we define the smoothness of a map between manifolds (instead of to a Euclidean space)? As expected, we just work with local coordinates. Consequently for smooth manifolds X, Y of respective dimension n, m with respective atlases $\phi_\alpha : U_\alpha \to V_\alpha, \psi_\beta : S_\beta \to T_\beta$, then

Definition 8. $F : X \to Y$ is **smooth** wrt to these atleses iff $\forall \alpha, \beta$

$$\psi_\beta \circ F \circ \phi_\alpha^{-1} : \phi_\alpha(F^{-1}(S_\beta)) \to T_\beta$$

is smooth as a map between open subsets in $\mathbb{R}^n$ and $\mathbb{R}^m$. Importantly, this definition only makes sense if F is **continuous**, and in particular we require $F^{-1}(S_\beta)$ to be open for this definition to make sense.

From here, we remark some trivial lemmas [which I won't write up properly] that due to the smoothness of transition functions, it suffices to check that F is smooth in the neighbourhood of some point of a single chart. If that holds for all points, then F is smooth as well. We also see that smoothness depends only on the smooth structure, and not the precise atlas, as one may expect.

Finally, again some more sanity check lemmas. It is easy to see that this definition coincides with the usual definition of smoothness between finite dimesinonal real spaces, and also corresponds to the definition of smoothness from a manifold to $\mathbb{R}$ used earlier in this course. More importantly, we also see that a composition of smooth maps as smooth, and finally, this is one result I will write down as a lemma, we see that smoothness is a local property of the map:

Lemma 2. Smoothness is **local in source**, meaning $\forall F : X \to Y$, F is smooth iff $\exists$open coverW_γ st $\forall \gamma, F|_{W_\gamma}$ is smooth.

This is indeed a neat way of saying that a property is truly local in nature.

In line with our previous approach, we'd like to be able to specify smoothness without relying on the topology of the space. It is a bit pedantical, but nevertheless here we go:

Proposition 2. $F : X \to Y$ is smooth iff $\exists$ cover (not necessarily smooth - we can't rely on the topology) $W_{\gamma \in \mathcal{C}}$ of X st $\forall \gamma \exists \alpha(\gamma) \in \mathcal{A}, \beta(\gamma) \in \mathcal{B}$ such that:

1. $W_\gamma \subseteq U_{\alpha(\gamma)}$ and $F(W_\gamma) \subseteq S_{\beta(\gamma)}$
2. $\phi_{\alpha(\gamma)}(W_\gamma)$ open in $V_{\alpha(\gamma)}$ (equivalent to saying W_γ open in X once the topology has been enforced)
3. $\psi_{\beta(\gamma)} \circ F \circ \phi_{\alpha(\gamma)}|_{W_\gamma}$ is smooth.

Proof.
- only if: pick $\mathcal{C} = \mathcal{A} \times \mathcal{B}$, and α, β as projection maps, and $W_\gamma = U_\alpha \cap F^{-1}(S_\beta)$. Then the result follows.
- if: we only need to check that F is cont, so that $F^{-1}(S)$ is open for open S, or equivalently, since we know the W_γ will turn out to be open, that $F^{-1}(S) \cap W_\gamma$ is open each time. To do so we note that ϕ_α is a homeomorphism, we we're done if the following is open:

$$\begin{aligned} \phi_\alpha(F^{-1}(S) \cap W_\gamma) &= \phi_\alpha(F^{-1}(S \cap S_\beta) \cap W_\gamma) \\ &= \phi_\alpha(F^{-1}(S \cap S_\beta)) \cap \phi_\alpha(W_\gamma) \\ &= (\psi_\beta \circ F \circ \phi_\alpha^{-1})^{-1}(\psi_\beta(S)) \cap \phi_\alpha(W_\gamma) \end{aligned}$$

  which is certainly open.

□

Example 1. $H : S^{2n+1} \to \mathbb{CP}^n$, the Hopf map, is an example of a smooth map.

Finally, we'd like to define some notion of equivalence on manifolds. Here we define

Definition 9. A **diffeomorphism** is a smooth invertible map between manifolds with a smooth inverse.

Definition 10. Two manifolds are diffeomorphic when there exists a diffeomorphism between them.

You might think that since we're doing geometry, and not topology, we care about the values involved, and these equivalence classes are not of interest, but really, these equivalence classes have less to do with the space, and more to do with the smooth structure on them. If you change charts, then you also may have to change the functions involved. As such, diffeomorphism establish an equivalence between smooth structures more than between manifolds. This is particularly relevant for the time, as before, when we said that for $n \leq 3$ all manifolds have an essentially unique smooth structure. Here "essentially unique" meant unique up to diffeomorphism.

Example 2. A particularly famous manifold is the Riemann sphere, which is in this context often defined to be $\mathbb{CP}^1$, but is diffeomorphic to S^2, and $\mathbb{C} \cup \{\infty\}$ via either the stereographic projection or identifying via normalising (normalise $[z_0 : z_1]$ to $[1 : z]$ or "∞" for $\mathbb{C} \cup \{\infty\}$, etc.).

Finally, we do have a somewhat important result that comes out from here:

Lemma 3. If smooth manifolds X, Y are diffeomorphic, then they are of the same dimension.

Proof. Fix diffeomorphism $F : X \to Y, p \in X, U \ni p, V \ni F(p), \phi : U \to V, \psi : S \to T$ such that U, V, S, T open and shrink them such that $F(U) = S$, then let $G = \psi \circ F \circ \phi^{-1}$, $H = \phi \circ F^{-1} \circ \phi^{-1}$, then both of these are differentiable, and mutually inverse maps on subsets of $\mathbb{R}^n, \mathbb{R}^m$. Taking derivatives we get two linear maps that are mutually inverse, and so must be of the same dimension, meaning $m = n$. □

[End of DG 4]

1.4 Tanget Spaces

We want to define tangent spaces on abstract manifolds.

Definition 11. A **curve based at** p is a curve, $\gamma : I \to X$ where I is an open subset of $\mathbb{R}$ such that $\gamma(0) = p$. (they also have to be somewhat smooth)

Definition 12. Two curves γ_1, γ_2 based at p **agree to first order** iff $\exists$ a chart such that
$$(\phi \circ \gamma_1)'(0) = (\phi \circ \gamma_2)'(0)$$

This forms an equivalence relation. Reflexivity, and symmetry are clear, however, transitivity is not entirely obvious. Hence we have the lemma

Lemma 4. If two curves agree to first order according to one chart, they agree to first order according to any chart.

Proof. Just apply the chain rule to the transition functions. □

As such we then say

Definition 13. The **tangent space** of X at p, T_pX is the set of curves at p modulo agreement to first order. Elements of this set are denoted $[\gamma]$ (for the equivalence class).

Now, if we define the map

$$\begin{aligned} \pi_p^\phi : \{\text{curves based at } p\} &\to \mathbb{R}^n \\ \gamma &\mapsto (\phi \circ \gamma)'(0) \end{aligned}$$

then we see that modulo agreement to first order this is injective. Furthermore, if we show it surjective, we so T_pX is isomorphic to $\mathbb{R}^n$:

Proposition 3. T_pX is isomorphic to $\mathbb{R}^n$ and π_p^ϕ is an isomorphism.

Proof. We just need to show π_p^ϕ is surjective. $\forall v \in \mathbb{R}^n$ consider $\gamma(t) = \phi^{-1}(\phi(p)+tv)$. This works. $\square$

Now, as a matter of notation we use ∂_i or ∂_{x_i} to denote $(\pi_p^\phi)^{-1}(e_i)$. Clearly, this notation has connotations, but interestingly these turn out to be well-founded, for the following two reasons:

Proposition 4. Vectors transform as

$$\partial_{y_i} = \sum_j \partial_{y_i} x_j \partial_{x_j}$$

Proof.

$$\begin{aligned}
\partial_{y_i} &= (\pi_p^{\phi_2})^{-1}(e_j) \\
&= (\pi_p^{\phi_1})^{-1}(\sum_j \partial_{y_i} x_j e_j) \\
&= \sum_j \partial_{y_i} x_j (\pi_p^{\phi_1})^{-1}(e_j) \\
&= \sum_j \partial_{y_i} x_j \partial_{x_j}
\end{aligned}$$

$\square$

Furthermore, we can see that when written in the standard unit basis ∂_i, any $[\gamma]$ has its coefficients precisely the values it gets as its derivatives, so one can interpret these vectors as derivatives to a certain extent. [End of DG 5]

1.5 Vectors as Differential Operators

In the last section, we discussed a geometric way to derive and think about tangent vectors. However, we can also do so in an algebraic manner. That is what we do here.

If we are to describe tangent vectors algebraically as differential operators, then we want to describe them as "hitting" something. What will they hit, well, quite naturally, they will hit functions from the manifold. Consequently, we consider a function $f : U \to \mathbb{R}$ for open $U \ni p$ for some point p, and then the result of the hitting along some path would be $(f \circ \gamma)'(0)$ (so we are interested in what this "hitter" may be).

But we also want to relate this to our original picture of vectors, so we have:

Lemma 5. $(f \circ \gamma)'(0)$ depends only on $[\gamma]$, and for $[\gamma] = \sum a_i \partial_i$, we have

$$(f \circ \gamma)'(0) = \sum a_i \partial_i f|_p$$

Proof.

$$
\begin{aligned}
(f \circ \gamma)'(0) &= \frac{d}{dt}|_{t=0}((f \circ \phi^{-1}) \circ (\phi \circ \gamma))(t) \\
&= \sum_i \partial_{x_i} f|_p (x_i \circ \gamma)'(0)
\end{aligned}
$$

since $f \circ \phi^{-1}$ is just f written in terms of the local coordinates. □

So we see that combined with a hitter, we can turn any γ into a differential operator on f, but we can make a further simplification. We notice in particular that we don't really care that much about the open neighbourhood U containing p. As such we define

Definition 14. The **germ** of smooth f at p is the equivalence class $[(U, f)]$ for smooth $f : U \to \mathbb{R}$, open $U \ni p$ modulo the equivalence relation $(U_1, f_1) \sim (U_2, f_2)$ iff there exists open $V \ni p$ such that $V \subseteq U_1, U_2$, and $f_1|_V = f_2|_V$.

We call the space of germs at p, $\mathcal{O}_{X,p}$. Now we add some algebraic definitions that, as far as I can tell, serve little purpose in this context: note that by using constant functinos we can easily construct a homomorphism $\mathbb{R} \to \mathcal{O}_{X,p}$. turning it into an $\mathbb{R}$-algebra. Also,

Lemma 6. As a ring (which it forms), $\mathcal{O}_{X,p}$ has a unique maximal ideal $\mathfrak{m}$ which is precisely the set of functions that vanish at p.

Proof. $\mathfrak{m}$ forms a maximal ideal since it is the kernel of a homomorphism to a field: $[(U, f)] \mapsto f(p)$. It is unique, since for any element outside of this ideal, choosing U to be small, we can see f is always nonzero, meaning that the ring contains a multiplicative inverse of our element $[(U, 1/f)]$, but if an ideal contains an invertible element, that ideal is the whole ring, and so $\mathfrak{m}$ is the unique maximal ideal. □

Precisely because of the situation that occurs here, rings with a unique maximal ideal are called **local** rings. Even in other context, one still imagines that maximal ideal as being the place where some function vanishes.

Having worked so far (although the algebra bit still seems unnecessary to me), we see that we can hit functions with vectors to get real numbers via

$$T_p X \times \mathcal{O}_{X,p} \to \mathbb{R}$$

or equivalently there exists

$$D : T_P X \to \mathcal{O}_{X,p}^{\vee}$$

where $\mathcal{O}_{X,p}^{\vee}$ is the dual of $\mathcal{O}_{X,p}$, ie. D converts vectors into hitters of functions (so as originally described, combining a vector v with something D allows us to hit functions). Now, really, we hope that somehow D forms an isomorphism. Certainly D is well defined (will see), and injective (see how it transforms the unit basis vectors ∂_i), but we see it is not surjective. However, on closer observation, we see that D maps only to a very small subset of $\mathcal{O}_{X,p}^\vee$. In particular, it maps to a subset satisfying the property

Lemma 7.

$$D(v)(fg) = D(v)(f)g(p) + f(p)D(v)(g)$$

Proof. Apply the chain rule to $(fg) \circ \gamma$. $\square$

As such, we make the following (far too general for this context) definition:

Definition 15. For ring R, R-algebra S, and S-module M, an R-linear **derivation** from S to M is an R-linear map $d : S \to M$ satisfying

$$d(fg) = d(f)g + fd(g)$$

The set of derivations is denoted by $\mathrm{Der}_R(S, M)$, which is a submodule of $\mathrm{Hom}_R(S, M)$.

I mean, this is clearly just an algebraic definition of something like a first order differential operator. We would also like to note that $d(r) = 0 \forall r \in R$, since $d(r) = rd(1)$ and $d(1) = d(1) + d(1) = 2d(1)$ so $d(1) = 0$.

Now certainly, $D(v) \in \mathrm{Der}_{\mathbb{R}}(\mathcal{O}_{X,p}, \mathbb{R})$ where the relevant bits form $\mathbb{R}$ algebras once we consider constant functions. Of course, the whole point of this otherwise random introduction of derivations is that we get an isomorphism:

Proposition 5. $D : T_pX \to \mathrm{Der}_{\mathbb{R}}(\mathcal{O}_{X,p}, \mathbb{R})$ is an isomorphism.

Proof. Certainly D is linear and injective. Then it remains for us to show that D is surjective. In order to do so we pick any derivation δ and want to show $\exists v, D(v) = \delta$. To do so we consider something like the Taylor series of f for any $[U, f] \in \mathcal{O}_{X,p}$. If we then pick local coordinates such that $x(p) = 0$, then we see that all constant terms in the series are eliminated ($d(\mathrm{const}) = 0$), and quadratic terms and higher are eliminated as well, leaving us with

$$\delta(f) = \delta(\sum_i x_i \partial_{x_i} f|_p) = \sum_i \delta(x_i) D(\partial_{x_i})(f)$$

From here we see clearly that

$$v = \sum_i \delta(x_i)\partial_{x_i}$$

does the trick. That's all fine and well, but the above formula depends on the existence of a kind of Taylor series. To find something like that we consider the following. For any germ $[(U, f)]$, consider $g : U \to \mathbb{R}$ given by

$$g = \begin{cases} \frac{f(x_1,\ldots,x_n) - f(x_1,\ldots,x_{n-1},0)}{x_n} & x_n \neq 0 \\ \partial_{x_n} f & x_n = 0 \end{cases}$$

then by l'Hopital's rule, this is continuous. Now define $f_n(x_1, \dots, x_n) = f(x_1, \dots, x_{n-1}, 0)$, then we get first order pseudo-Taylor series

$$f = f_n + x_n g$$

with property

$$\delta(f) = \delta(f_n) + \delta(x_n)g(p) + x_n(p)\delta(g) = \delta(f_n) + \delta(x_n)\partial_{x_n} f|_p$$

which once we apply it to each coordinate completes our proof. □

[End of DG 6]

1.6 Derivatives of Smooth Maps

The natural next step is to find a way to express derivatives on maps between smooth manifolds. That is, to find linear approximations. Since the spaces themselves aren't linear, the only plausible way to implement any kind of linear map is to consider a linear map between the tangent spaces $T_pX \to T_{F(p)}Y$ for $F : X \to Y$. As such we define

Definition 16. The **derivative** of F at p is the map $D_pF : T_pX \to T_{F(p)}Y$ given by $[\gamma] \mapsto [F \circ \gamma]$. This map may also be denoted by F_* the pushforward by F on tangent vector spaces.

Lemma 8. This is well defined and linear, and

$$D_pF(\partial_{x_i}) = \sum_j \partial_{x_i} y_j|_p \partial_{y_j}$$

Proof. Linearity and well-definedness can be checked easily. What remains is for us to check the formula. As such,

$$\frac{d}{dt}|_p (y_j \circ F \circ \phi^{-1})(\phi(p) + te_i) = \partial_{x_i} y_j$$

as required. □

We note that this coincides with the standard multivariate calculus definition of the derivative. Also, we notice that any vector $[\gamma]$ can be written as $D_0\gamma(\partial_t) = [\gamma]$. Furthermore, the chain rule follows straight from this definition as

Proposition 6. The **chain rule** states that

$$D_p(G \circ F) = D_{F(p)}G \circ D_pF$$

Proof.

$$D_p(G \circ F)([\gamma]) = [G \circ F \circ \gamma] = D_{F(p)}G \circ D_pF([\gamma])$$

□

That finishes the main topic for this section, the definition of the derivative, but for completeness, we will include how one may define the derivative from the algebraic point of view of derivations. As a side note, since it may not be clear what the purpose of the algebraic perspective is. I hear this is the primary focus in Algebraic Geometry. Anyhow, moving on

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Definition 17. The **pullback**, F^* by F from $\mathcal{O}_{Y,F(p)}$ to $\mathcal{O}_{X,p}$ mapping $[(U, f)]$ to $[(F^{-1}(U), f \circ F)]$.

This is well defined, and since it is $\mathbb{R}$-linear we have dual map

$$(F^*)^\vee : \mathcal{O}_{X,p}^\vee \to \mathcal{O}_{Y,F(p)}^\vee$$

.

but really, one may observe that these maps preserves the spaces we want it to preserve. Namely, it restricts nicely to a map

$$(F^*)^\vee : \mathrm{Der}_{\mathbb{R}}(\mathcal{O}_{X,p}, \mathbb{R}) \to \mathrm{Der}_{\mathbb{R}}(\mathcal{O}_{Y,F(p)}, \mathbb{R})$$

but of course, since these are equivalent to tangent spaces, this is just another way of implementing the derivative.

Lemma 9. The appropriate diagram commutes, meaning that $D \circ D_pF = (F^*)^\vee \circ D$.

Proof. $\forall [\gamma] \in T_pX, f \in \mathcal{O}_{Y,F(p)}$ it holds that

$$\begin{aligned} D(D_pF([\gamma]))(f) &= D([F \circ \gamma])(f) \\ &= (f \circ F \circ \gamma)'(0) \\ &= D([\gamma])(F^* f) \\ &= (F^*)^\vee (D([\gamma]))(f) \end{aligned}$$

$\square$

So we now have two parallel ways to define the derivative. In different situations, a different approach will be more useful than the other. Overall, a lot of notation and terms are used, but fundamentally, the concepts we've seen so far are nothing unsurprising. [End of DG 7]

1.7 Immersions, Submersions and Local Diffeomorphisms

Recall the inverse function theorem:

Theorem 1. Inverse Function Theorem (IFT) Given continuously differentiable $G : V \to T$, a map between open subsets of $\mathbb{R}^n$ and if the derivative at $p \in V$ is a linear isomorphism, then $\exists$open $V' \ni p, T' \ni G(p)$ such that $G|_{V'}$ bijects $V` \to T'$ and the inverse is also continuously differentiable.

Corollary 1. If the same conditions hold and G is smooth then G^{-1} is smooth as well.

Proof. $D(G^{-1}) = (DG)^{-1}$ and we can differentiate as many times as we want to get our result. My only question here [?] why we can assume that the function stays invertible. ☐

Now some definitions

Definition 18. $F : X \to Y$ is an **immersion at** p iff D_pF is injective (this means it is like an inclusion map).

Definition 19. $F : X \to Y$ is an **submersion at** p iff D_pF is surjective (this means it is like a projection map).

Definition 20. $F : X \to Y$ is an **local diffeomorphism at** p iff D_pF is bijective.

Having these, we now state and prove some rather intuitive results using these concepts.

Proposition 7. If D_pF then $\exists$ open $U \ni p, S \ni F(p)$ such that $F|_U : U \to S$ is a diffeomorphism.

Proof. Pick $\phi : U \to V, \psi : S \to T$ about p and $F(p)$ and shrink until $F(U) \subseteq S$ then applying the IFT to

$$G = \psi \circ F \circ \phi^{-1} : V \to T$$

we can find $V' \subseteq V, T' \subseteq T$ such that $G|_V'$ is a diffeomorphism $V' \to T'$. Replacing with $\phi^{-1}(V')$ and $\psi(T')$ we get what we wanted. ☐

Now notice how we are now using functions between manifolds to construct charts. If used effectively, we may be able to find a way to translate charts from one manifold onto another, which could be very helpful. We may for example use this to get a chart on polar coordinates by taking the natural map converting from polar to Cartesian coordinates $\mathbb{R}_{>0} \times \mathbb{R} \to \mathbb{R}^2$. Working to generalise this we get the following

Lemma 10. If F is a local diffeomorphism at p with local coordinates x around p, then there exists local coordinates y such that $y \circ F = x$ (so F locally acts as the identity map). The reverse can also be done.

Proof. Pick a chart ϕ defining the local coordinates x, then leting y be the local coordinates of $\phi \circ (F|_U)^{-1}$ where U is the open set containing x that is needed to make F is a local diffeomorphism succeeds in giving us our desired coordinates.

Similarly, if y are the coordinates associated with ψ, then take x to be the coordinates to be associated with $\psi \circ F$. [Is it possible something is reversed here?] $\square$

We can use this to characterise submersions and immersions as projections and inclusions.

Lemma 11. Let F be an immersion at p, then we can find local coords such that in a neighbourhood of p, F is given by the inclusion

$$\mathbb{R}^n \to \mathbb{R}^m$$

for $m > n$. Similarly, if F is a submersion, we can find a coordinates so that F appears as the projection

$$\mathbb{R}^n \to \mathbb{R}^m$$

for $m < n$.

Proof. We only do the submersion case, since the immersion case is quite similar. Our strategy is to increase the dimension appropriately, and then to apply the IFT.

For local coordinates y on Y about $F(p)$ with chart $\psi : S \to T$, and chart $\phi : U \to V$ on X about p. Replace F with $\psi \circ F \circ \phi^{-1}$, so it becomes a map of open subsets of $\mathbb{R}^n \to \mathbb{R}^m$.

We want a change of coordinates χ on $\mathbb{R}^n$ to form the projection we want. Now, define $K = \mathbf{Ker}(D_p F)$ then we see that the projection map $\pi : \mathbb{R}^n \to \diagdown^{n-m}$ induces an isomorphism $K \to \mathbb{R}^{n-m}$. Now consider

$$\chi : X \to Y \times \mathbb{R}^{n-m} \text{ given by } (F, \pi)$$

then this is smooth, and its derivative at p, $(D_p F, \pi)$, is an isomorphism, which means χ gives a change of coordinates about p, and by construction $F \circ \chi^{-1}$ is a projection onto the first m components of the vectors involved. [End of DG 8] $\square$

1.8 Submanifolds

A natural concept to consider is to consider submanifolds. A first guess would be to assume that these are just subsets of the manifolds, and naturally translate the structure to them. The struggle with this is that these structures may not have a consistent dimension, varying between one dimensional lines, and far higher dimensional spaces at various points. Instead we define submanifolds to be subsets that can be written as if this miss a number of coordinates.

Definition 21. $Z \subseteq X$ is a **submanifold of codimension** k iff $\forall p \in Z, \exists$open $U \ni p$ such that $Z \cap U$ is given by $x_1 = x_2 = \cdots = x_k = 0$.

An example includes writing S^1 in $\mathbb{R}^2$ in polar coordinates, where a shifted radius coordinate is 0.

Proposition 8. *A submanifold of $Z \subseteq X$ of codimension k is naturally a smooth $n-k$ manifold.*

Proof. Hausdorffness and second-countability are inherited as subsets. The local coordinates in the definition of a submanifold naturally form charts. □

But this way of defining a submanifold, while better since we do force true inheritance of the submanifold structure, is of course much harder to verify. So how can we check this more conveniently? We use the following concepts to do so:

Definition 22. y is a **regular value** of $F : X \to Y$ if $\forall p \in F^{-1}(y)$ are regular points (meaning that D_pF is surjectve). A value that is a not a regular value is called a **critical value** (the same holds of points).

This corresponds to the notion of critical points/values in calculus, so critical points are saddle points/local minima/maxima.

Proposition 9. *For any regular value q, $F^{-1}(q)$ is a codimension m (the dimension of Y) submanifold of X.*

Proof. For each $p \in F^{-1}(q)$ we know that F is a submersion at p, and so we can find local coordinates x about p, and y about q such that $y \circ F = x$, so translating such that $y(q) = 0$, we find that on the domain of x we have that $F^{-1}(q)$ is given by $x_1 = \cdots = x_m = 0$ as required. In a sense Z is defined by the vanishing of the pullback here. □

Examples include $F(x, y) = xy$ which is regular everywhere except at 0, leaving a smooth 1-manifold there. At 0, the inverse is not a manifold (it is a cross, and acts weirdly at the origin). A big example is also using $F(x) = |x|^2$ to define the unit sphere as $F^{-1}(1)$.

More useful examples though, include using this to look at submanifolds of matrix spaces. If we take the n^2 dimensional space $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ a natural map to consider is the determinant map. $det^{-1}(\mathbb{R}^*)$ is an open set and so can be taken as a manifold as well. Furthermore, we can show that

Claim 1. *1 is a regular value of the determinant.*

Proof. Take $A \in \mathrm{SL}(n, \mathbb{R})$ then to show that $\partial_A \det$ is surjective it suffices to show it is nonzero at some point. Consider $\gamma : t \mapsto e^t A$ then

$$D_A \det([\gamma]) = [\det \circ \gamma] = [t \mapsto e^{nt}] = n\partial_x$$

which is indeed nonzero in the tangent space of the reals. □

Consequently $\mathrm{SL}(n, \mathbb{R})$ is indeed a manifold. Another example would be to consider the map $F : \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n) \to S$ where S the space of symmetric matrices given by $A \mapsto A^T A$. This has I as a regular value, and $O(n) = F^{-1}(I)$ so we see that the space of orthogonal matrices is indeed a smooth $n(n-1)/2$ manifold.

Now it turns out that from a theoretical level, regular values are "common" in the sense that

Theorem 2. Sard's theorem: the set of critical values has measure 0 in Y.

This is certainly not proven here, but it provides some background. Nevertheless, this is not necessarily a meaningful statement since one can construct maps such that every point in X is a critical point, but then values in Y are treated as regular even when $F^{-1}(y)$ are empty sets, so it's not always that meaningful. More useful in many cases in the following corollary:

Corollary 2. The set of regular values in dense. (and so there exists at least one such value)

This can be more helpful [End of DG 9].

1.9 Embeddings

The above was a geometric definition of a submanifolds, but we can define them using a more "functional" approach. Here we define

Definition 23. An **embedding** is a smooth immersion that is homeomorphic to its image.

Here are some results that apply to immersions.

Lemma 12. Any inclusion map $i : Z \to X$ from a submanifold to its supermanifold is an immersion.

Proof. About any point $p \in Z$ we have local coordinates x such that in X, Z can be expressed as $x_1 = \cdots = x_k = 0$, and $x_{k+1}, \ldots, x_n$ represent the coordinates inside Z. Here $i : (x_{k+1}, \ldots, x_n) \mapsto (0, \ldots, 0, x_{k+1}, \ldots, x_n)$. This is clearly a homeomorphism under subspace topology. □

Since we're aiming to look at submanifolds from a different angle, one might expect that we're trying to show that the converse holds as well (the image of an embedding is a submanifold), which is exactly what we will do.

Lemma 13. Post composition with the inclusion map gives a bijection

$$\{\text{smooth maps to } Z\} \to \{\text{smooth maps to } X \text{ with image in } Z\}$$

Proof. If F mapping to Z is smooth, certainly $i \circ F$ is smooth. Conversely, if $i \circ F$ is smooth, so is F since if x are local coordinates about some p in the image of F, then we can pick $x_1 = \cdots = x_k = 0$ so that we get nice coordinates on Z. Then, if $x' = (x_{k+1}, \ldots, x_n)$, $x \circ F$ being smooth must mean that $x' \circ F$ is smooth. □

Now the converse of our first lemma for this section is

Proposition 10. If $F : Y \to X$ is an embedding, then $\text{Im}(F)$ is a submanifold of X of codomain $k = n - m$, and $\bar{F} : Y \to \text{Im}(F)$ is a diffeomorphism by the above lemma.

Proof. Let's call $\text{Im}(F), Z$ and consider the inclusion i such that $i \circ \bar{F} = F$. Now by the chain rule, $\bar{F}$ is an immersion, and by considering dimension it is a local diffeomorphism. Thus locally $\bar{F}$ is smooth and a bijection, so by knitting together local inverses we can form a global inverse (recall F is a homeomorphism), so overall as well, $\bar{F}$ is a diffeomorphism.

All that's left is to verify that the image has the right dimension $(n - m)$. Consider $q \in Y$ and let $p = F(q)$, then we see that F being an immersion at q means that there exists local coordinates y on a neighbourhood S of q in Y and x on a neighbourhood U of p in X such that $x \circ F = (y, 0, \dots, 0)$. Now, fi we can find open $U' \ni p$ such that $F(y) \cap U' = F(S) \cap U'$, then x defines coordinates on U' such that

$$F(Y) \cap U' = \{x_{m+1} = \cdots = x_n = 0\} \cap U'.$$

This is what we want since then $F(Y)$ is locally given by the vanishing of the right number of coordinates. But notice that since F is a homeomorphism onto its image, the set $F(S)$ is open in $F(Y)$ so we can find an open neighbourhood W of p such that $F(S) = F(Y) \cap W$. Taking $U' = W \cap U$ we get $F(Y) \cap U' = F(S) \cap U'$ as required. $\square$

Finally we can now show a result that we mentioned some time ago.

Proposition 11. The definition of S^n as a submanifold of $\mathbb{R}^{n+1}$ and the definition using the stereographic projection are diffeomorphic.

Proof. Start with the stereographic projection definition of S^n, and now let $F : S^n \to \mathbb{R}^{n+1}$ be the inclusion map. If we can show that F is an embedding then we're done.

Certainly F is a homeomorphism onto the unit sphere, so we just need to show it's a smooth immersion. To do so, STP $F \circ \phi_{\pm}^{-1}$ is a smooth immersion $\mathbb{R}^n \to \mathbb{R}^{n+1}$, where

$$\phi_{\pm} : (x_0, \dots, x_n) \in S^n \setminus \{(\pm 1, 0, \dots, 0)\} = \frac{1}{1 \mp x_0}(x_1, \dots, x_n) \in \mathbb{R}^n$$

are the charts. Evaluating explicitly, we see that indeed this map is smooth, asking

$$F \circ \phi_{\pm}^{-1} = \frac{1}{|y|^2 + 1}(\pm(|y|^2 - 1), 2y)$$

$\square$

Finally, for context, we mention, but do not prove, the following remarkable result:

Theorem 3. Any (smooth?) n-manifold is a submanifold of $\mathbb{R}^{2n}$

The proof of this theorem is certainly beyond the scope of this course.