Quantum Information Theory

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Introduction

These notes are based on the course lectured by Professor Matthew Wingate in Lent 2020. This was lectured online due to measures taken to counter the spread of Covid-19 in the UK. These are not necessarily an accurate representation of what was lectures, and represent solely my personal notes on the content of the course, combinged with probably, very very many personal notes and digressions... Of course, any corrections/comments would be appreciated.

[the lecturer outlines the course] This course is an extension of the Michaelmas Quantum Field Theory course that introduces renormalisation and the path integral formulation of quantum field theory.

The Path Integral in Quantum Mechanics

We start by reformulating the Schrödinger equation as an integral equation, which turns out to be a path integral. Anyways, starting with Schrödinger's equation for a Hamiltonian $H(x,p), [x,p] = i\hbar$ with

$$H = \frac{p^2}{2m} + V(x) \tag{1}$$

we have

$$i\hbar\partial_t \left|\psi(t)\right\rangle = H\left|\psi(t)\right\rangle \implies \left|\psi(t)\right\rangle = e^{-iHt/\hbar}\left|\psi(0)\right\rangle \tag{2}$$

where in the Schrödinger picture the states evolve, but the operators remain constant, and the wavefunction $\Psi(x,t) = \langle x|\psi(t)\rangle$. As such we can rewrite our equation as

$$\left\langle x\right| H \left|\psi(x)\right\rangle = \left(\frac{-\hbar^2}{2m}\partial_x^2 + V(x)\right)\langle x|\psi(t)\rangle \tag{3}$$

so we can write

$$
\begin{aligned}
\Psi(x,t) &= \langle x|\psi(t)\rangle \\
&= \langle x|\, e^{-iHt/\hbar}\, |\psi(0)\rangle \\
&= \int_{-\infty}^{\infty} dx_0\, \langle x|\, e^{-iHt/\hbar}\, |x_0\rangle \langle x_0|\psi(0)\rangle \\
&= \int_{-\infty}^{\infty} dx_0 K(x,x_0,t)\Psi(x_0,0)
\end{aligned}
$$

for **kernel** $K(x,x_0,t) = \langle x|\, e^{-iHt/\hbar}\, |x_0\rangle$. Now, if it is hard to calculate K for large t, it can be beneficial to split this into many intervals for many values of t, such as $0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = T$ leaving

$$
K(x,x_0,T) = \int_{-\infty}^{\infty} \prod_{r=1}^{n} dx_r\, \langle x_{r+1}|\, e^{-iH(t_{r+1}-t_r)/\hbar}\, |x_r\rangle \langle x_1|\, e^{-iH(t_1-t_0)/\hbar}\, |0\rangle \quad (4)
$$

which is in a sense an integral over all possible sequences of values of x.

In free field theory ($V = 0$) this can be explicitly evaluated using a Gaussian integral by rewriting things in the momentum basis as (use $\langle x|p\rangle = e^{ipx/\hbar}$)

$$
\begin{aligned}
K_0(x,x',t) &= \langle x|\, e^{\frac{-ip^2 t}{2m\hbar}} \int \frac{dp}{2\pi\hbar}\, |p\rangle \langle p|x'\rangle \\
&= \int_{-\infty}^{\infty} \frac{dp}{2\pi\hbar} e^{\frac{-ip^2 t}{2m\hbar}} e^{ip(x-x')/\hbar} \\
&= e^{\frac{ip(x-x')^2}{2\hbar t}} \sqrt{\frac{m}{2\pi i\hbar t}}
\end{aligned}
$$

where we note that the limit as $t \to 0$ is $\delta(x-x')$ which indeed matches $\langle x|x'\rangle = \delta(x-x')$ as expected.

Now in an interacting theory, we struggle with the Baker-Campbell-Hausdorff fact that $e^A e^B \neq e^{A+B}$ so using Suzuki-Trotter we separate into steps size $t_{r+1} - t_r = \delta t << T$ meaning that

$$
e^{-iH\delta t/\hbar} \approx e^{\frac{-ip^2\delta t}{2m\hbar}} e^{\frac{-iV(x)\delta}{\hbar}} (1 + O(\delta t^2)) \quad (5)
$$

so for $T = n\delta t$ we find that

$$
K(x,x_0,T) = \int \prod_{r=1}^{n} dx_r \left(\frac{m}{2\pi i\hbar\delta t}\right)^{\frac{n+1}{2}} e^{i\sum_{r=0}^{n}\left(\frac{m}{2\hbar}\left(\frac{x_{r+1}-x_r}{\delta t}\right)^2 - V(x_r)/\hbar\right)\delta t} \quad (6)
$$

which in the limit $n \to \infty, \delta t \to 0$ while keeping T constant leaves

$$
\frac{1}{\hbar}\int_0^T dt \left(\frac{1}{2}m\dot{x}^2 - V(x)\right) = \int_0^T dt L(x,\dot{x}) = S \quad (7)
$$

for classical Lagrangian L and action S. This is what we refer to as a path integral or function integral:

$$K(x, x_0, t) = \int \mathcal{D}x e^{iS/\hbar} \tag{8}$$

where $\mathcal{D}x$ is the limit describd above. Of course, many questions about the existence and uniqueness, etc. of such limits exists, and in fact often this limit does not exist, but in the cases we are interested it, it works well enough... [End of lecture 1]

We make the following remarks

- In the classical limit $\hbar \to 0$ the lowest frequencies dominate K. This is equivalent to Hamilton's principle (the principle of least action), as expected.
- it is common and helpful to extend analytically to imaginary time $\tau = it$ leaving $\langle x| e^{-H\tau/\hbar} |x_0\rangle = \int \mathcal{D}x e^{-S/\hbar}$ which has better convergence properties and is easier to interpret than the complex version (Hamilton's principle appears more easily as well).

1 Integrals and their diagrammatic expansion

The above considered quantum mechanics, which is in a sense the 0+1 dimension vrsion of QFT (since x is treated as an operator, while t is treated as a variable). To move to more general QFT, we start, strangely, with 0 dimensional QFT, for $\phi : \{\cdot\} \to \mathbb{R}$ a field on a single point. Here,

$$\mathcal{Z} = \int_{\mathbb{R}} d\phi e^{-S(\phi)/\hbar} \tag{9}$$

where we assume S is an even polynomial in ϕ for convergence reasons, and we are interested in expectation values

$$\langle f \rangle = \frac{1}{\mathcal{Z}} \int d\phi f(\phi) e^{-S(\phi)/\hbar} \tag{10}$$

1.1 Free Theory

For N fields $\phi_a, a = 1, \ldots, N$, let $S(\phi) = \frac{1}{2}\phi^T m \phi$ for a symmetric positive definite matrix $m = P\Delta P^T$ for orthogonal P. As such, we can write this essentially Gaussian integral as

$$\mathcal{Z}_0 = \int d^N \phi e^{-\frac{1}{2\hbar}\phi^T m \phi} = \sqrt{\frac{(2\pi\hbar)^N}{\det m}} \tag{11}$$

From here, we can turn this into a generating function to calculate expectation values by taking derivatives by turning $S_0(\phi) \mapsto S_0(\phi) - J^T\phi$, and writing

$\mathcal{Z}_0 = \mathcal{Z}_0(J)$ (now a generating function(al) - functional later on). We then remark

$$\mathcal{Z}_0(J) = \mathcal{Z}_0(0)e^{-\frac{1}{2\hbar}J^T m^{-1} J} \tag{12}$$

Then we can calculate the correlation functions as

$$\langle \phi_a \phi_b \rangle = \frac{1}{Z_0(0)} \hbar^2 \partial_{J_a} \partial_{J_b} \mathcal{Z}_0(J)|_{J=0} = h(m^{-1})_{ab} \tag{13}$$

Conveniently, this can be diagrammatically interpreted as a connecting two vertices on indices a, b with an undirected edge. We can generalise this to linear operator $l(\phi) = \sum l_a \phi_a$ as (for p such operators)

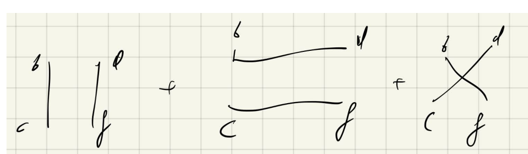
$$\langle l^{(1)}(\phi) \dots l^{(p)}(\phi) \rangle = \hbar^p \prod_{i=1}^{p} l^{(i)}(\partial_J) e^{\frac{1}{\hbar}J^T m^1 J} \tag{14}$$

If p is odd, this is always 0 by symmetry, but if p is even this corresponds to a linear combination of products $m_{ab}^{-1} m_{cd}^{-1} \dots$

Example 1. For $p = 4, l_a^{(1)} = \delta_{ab}, l_a^{(2)} = \delta_{ac}, l_a^{(3)} = \delta_{ad}, l_a^{(4)} = \delta_{af}$ then

$$\langle \phi_b \phi_c \phi_d \phi_f \rangle = \hbar^2 (m_{bc}^{-1} m_{df}^{-1} + m_{bd}^{-1} m_{cf}^{-1} + m_{bf}^{-1} m_{cd}^{-1}) \tag{15}$$

which also corresponds to the ways in which we can connect four vertices with undirected edges



[End of lecture 2]

1.2 Interacting Theory

We start investigating interacting theory by doing a series expansion of

$$\int_{\mathbb{R}^N} N\phi f(\phi) e^{-S/\hbar} \tag{16}$$

in $\hbar$. However, we find that in general, the radius of convergence of these perturbed series is 0, since if we take $\hbar < 0$ these do not converge. As such, we get asymptotic behaviour along the lines described by saying that

Definition 1. $I(\hbar)$ is **asymptotic** to $\sum_{n=0}^{\infty} c_n \hbar^n$ (denoted by $\sim$) if $\lim_{\hbar \to 0^+} \frac{1}{\hbar^N} |I(\hbar) - \sum_{n=0}^{N} c_n \hbar^n| = 0$ for fixed N.

This is much weaker than convergence since we find that adding new terms may in fact make things worse. But it does allow us to account for transcendental terms like $e^{-1/\hbar^2}$ 0 (called **nonperturbative contributions**).

Now we work out the case where

$$S(\phi) = \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4 \tag{17}$$

and expand about the minimum of S at $\phi = 0$ to get

$$\mathcal{Z} = \int d\phi e^{-S/\hbar} = \int d\phi e^{-S/\hbar} \sum_{v}^{\infty} \frac{1}{v!}\left(\frac{-\lambda}{4!\hbar}\phi^4\right)^v \tag{18}$$

but since we don't converge, we certainly can't swap $\int$ and $\sum$ so here we first truncate, and then swap. As such, if we write $x = \frac{1}{2\hbar}m^2\phi^2$,

$$\mathcal{Z} \sim \frac{\sqrt{2\hbar}}{m} \sum_{v=0}^{N} \frac{1}{v!}\left(\frac{-\hbar\lambda}{4!m^4}\right)^v 2^{2v} \int_0^\infty dx e^x x^{2v+1/2-1} \tag{19}$$

where the integral is just the gamma function $\Gamma(2v+1/2) = \frac{(4v)!\sqrt{\pi}}{4^{2v}(2v)!}$ so

$$\mathcal{Z} \sim \frac{\sqrt{2\hbar}}{m} \sum_{v=0}^{N} \frac{1}{v!}\left(\frac{-\hbar\lambda}{4!m^4}\right)^v \frac{1}{(4!)v} \frac{(4v)!\sqrt{\pi}}{2^{2v}(2v)!} \tag{20}$$

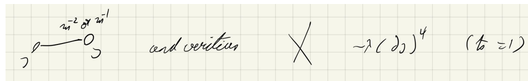
(the lecture notes seem to omit the $\sqrt{\pi}$). Here the $\frac{1}{(4!)v}$ comes from expanding the interaction term $e^{-S_1/\hbar}$ and the second fraction comes from the number of ways to pair $4v$ fields with v copies of ϕ^4. Applying stirling's formula, this series grows as $v!$ so this series is definitely not convergent, but still asymptotic. Now, as before, we want to insert our J somehow to get a generating function. Here we get

$$\begin{aligned}
\mathcal{Z}(J) &= \int d\phi e^{-\frac{1}{\hbar}(S_0(\phi)+S_1(\phi)-J\phi)} \\
&= e^{-\frac{1}{\hbar}S_1(\hbar\partial_J)} \int d\phi e^{-\frac{1}{\hbar}(S_0(\phi)-J\phi)} \\
&\sim e^{-\frac{\lambda}{4!\hbar}(\hbar\partial_J)^4} e^{\frac{1}{2\hbar}J^T m^{-1} J}
\end{aligned}$$

which works out to

$$\mathcal{Z}(J) \sim \sum_{v=0}^{N} \frac{1}{v!}\left(\frac{-\lambda}{4!\hbar}(\hbar\partial_J)^4\right)^v \sum_{p=0} \frac{1}{p!}\left(\frac{1}{2\hbar}J^T m^{-1} J\right)^p \tag{21}$$

where diagramatically v corresponds to the number of vertices, and p to the number of propagators.



where in order to get a nonzero term we need the number of derivatives ($4v$ vertex line ends - 4 per vertex here) to match the number of sources ($2p$ for each propagator) to match. However, we can have a predetermined number of external sources $E = 2p - 4v$. For example, the first two nontrivial terms in the $Z(0)$ expansion for $E = 0$ are $(v, p) = (1, 2), (2, 4)$ which corresponds diagrammatically to

Note that each diagram may have a factor in front of it determined by how often it repeats itself (affect by the product rule in taking derivatives). [end of lecture 3]

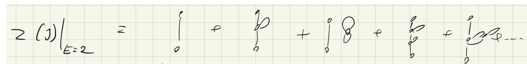
To work out these prefactors, consider the first non-constant term above, the figure eight. We can split this into its components: a vertex and two propagators - it's so-called "pre-diagram." We can work out the prefactor by considering the number of ways to connect these while still forming a figure eight, $A = 4!$, and dividing by a denominator given by the coefficients in the series

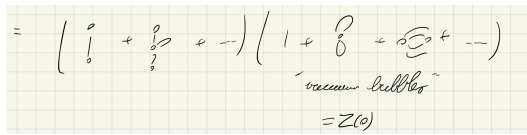
$$F = v!(4!)^v(p!)2^p = 1 \cdot 4! \cdot 2 \cdot 2^2 = 4!2^3 \tag{22}$$

leaving prefactor $A/F = 1/8$. Note here that F accounts for

- $v!$ ways to permute the vertices
- $4!$ ways to permute the vertex legs
- $p!$ ways to permute the propagator legs
- 2^p ways to swap propagator direction

Now, another interpretation is that $A/F = 1/S$ where S is the **symmetry factor** counting the number of ways of redrawing the unlabelled graph to leave its overall structure the same (the number of graph isomorphisms). So, for example, for the figure eight, we get $S = 2 \times 2 \times 2 = 8$ for swapping the direction of loop 1, swapping the direction of loop 2, and swapping loops 1 and 2. Similarly for the basketball, we get $S = 4! \cdot 2 = 48$ for $4!$ ways to rearrange the lines and 2 ways to swap vertices (what happened to swapping the orientations of the lines?). Working by brute force, to verify we get $A = 8 \cdot 6 \cdot 4 \cdot 2 \cdot 4! = 3^2 2^{10}$ for the number of ways to connect propagators and the number of their permutations. Similarly, $F = 2(4!)^2 4! 2^4 = 3^3 2^{14}$, leaving $A/F = 1/48 = 1/S$. Overall, in this case we get





$$\mathcal{Z}(0)/\mathcal{Z}_0(0) = 1 - \frac{\hbar\lambda}{8m^4} + \frac{\hbar^2\lambda^2}{m^8}\left(\frac{1}{48} + \frac{1}{16} + \frac{1}{128}\right) + O(\hbar^3) = 1 - \frac{\hbar\lambda}{8m^4} + \frac{35}{384}\frac{\hbar^2\lambda^2}{m^8} + \ldots \tag{23}$$

Now to work out the $E = 2$ case we get that

where we get disconnected graphs of a different kind, and also get loose outputs really. Importantly, we can factor out the "vacuum bubbles" (or the $E = 0$ case described earlier)

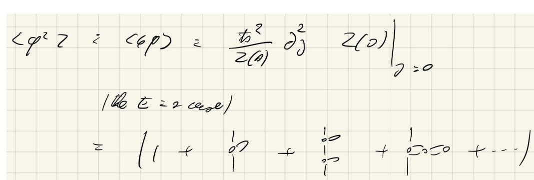
Note that this corresponds to the expectation value of $\langle\phi\phi\rangle$

1.3 Effective Actions

Our next step is to simplify these calculations by showing that we only have to work hard on connected graphs. In particular, we define **effective action** $W(J) = -\hbar \ln Z(J)$ and a diagram D. Any such D can be written as a product of connect diagrams as

$$D = \frac{1}{S_D}\prod_i (C_i)^{n_i} \tag{24}$$

where each C_i is a distinct diagram, and we assume each C_i contains its own symmetry factor, meaning that $S_D = \prod_i n_i!$ is only the number of ways to rearrange the various connected diagrams. Consequently,



$$
\begin{aligned}
\mathcal{Z}/\mathcal{Z}_0 &= \sum_{\{n_i\}} \prod_i \frac{1}{n_i!}(C_i)^{n_i} \\
&= \prod_{i=1}^{\infty} \sum_{n_i} \frac{1}{n_i!}(C_i)^{n_i} \\
&= e^{\sum_i C_i} \\
&= e^{-(W-W_0)/\hbar}
\end{aligned}
$$

leaving

$$
\mathcal{Z}/\mathcal{Z}_0 = e^{\sum_i C_i} = e^{-(W-W_0)/\hbar} \tag{25}
$$

which is quite a remarkable decomposition into only connected graphs. [End of lecture 4] Let's work out how to use the effective action, W in general.

Example 2. Consider 0-dimension action with two fields

$$
S(\phi,\chi) = \frac{m^2}{2}\phi^2 + \frac{M^2}{2}\chi^2 + \frac{\lambda}{4}\phi^2\chi^2 \tag{26}
$$

which has Feynman rules

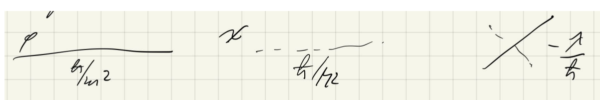


Figure 1: Feynman Rules

and so we have a sum of connected diagrams given by

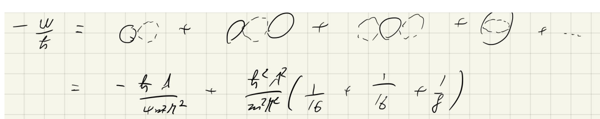


Figure 2: Connected Diagrams Expansion

in the so-called "full theory". Note here that the free theory involves

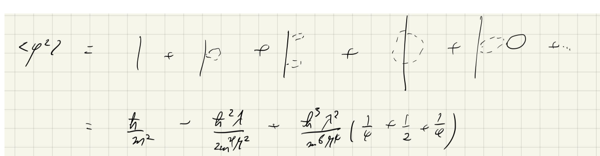


Figure 3: Free Theory Expected Value

We can reduce the complexity of these calculations by removing the explicit χ depending by "integrating it out" (which can make sense if χ is very massive, for example, so doesn't contribute strongly), then we define our effective action to be $W(\phi)$ such that

$$e^{-W(\phi)/\hbar} = \int d\chi e^{-S(\phi,\chi)/\hbar} \tag{27}$$

where we in effect treat $\phi^2\chi^2$ as a local source for χ^2 (with $J = -\phi^2\lambda/4$). Consequently, correlatino functions are given by

$$\langle f(\phi)\rangle = \frac{1}{\mathcal{Z}}\int d\phi d\chi f(\phi)e^{-S(\phi,\chi)/\hbar} = \frac{1}{\mathcal{Z}}\int d\phi f(\phi)e^{-W(\phi)/\hbar} \tag{28}$$

In this special case, we can evaluate the integral explicitly as

$$\int d\chi e^{-S(\phi,\chi)/\hbar} = e^{-m^2\phi^2/2\hbar}\sqrt{\frac{2\pi\hbar}{M^2+\lambda\phi^2/2}} \tag{29}$$

meaning

$$W(\phi) = \frac{1}{2}m\phi^2 + \frac{\hbar}{2}\ln\left(1+\frac{\lambda}{2M^2}\phi^2\right) + \frac{\hbar^2}{2}\ln\left(\frac{M^2}{2\pi\hbar}\right) \tag{30}$$

Here the last constant term does not effect QFT, and is ignored. It does, however, have interpretations relating to the energy density of the universe, and thus, the cosmological constant. Expanding in ϕ (since $\phi = 0$ is the local minimum here) gives

$$W(\phi) = \frac{m_{eff}^2\phi^2}{2} + \frac{\lambda_4}{4!}\phi^4 + \cdots + \frac{\lambda_{2k}(2k)!}{\phi}^{2k} \tag{31}$$

where $m_{eff}^2 = m^2 + \frac{\hbar\lambda}{4M^2}, \lambda_{2k} = (-1)^{k+1}\hbar\frac{(2k)!}{2^{k+1}k}\frac{\lambda^k}{M^{2k}}$. (notice how we get many more terms in the effective theory than the full theory. This is a standard effect.). However, integration, as done here, is usually not possible, so we resort to pertubations, treating $\frac{\lambda}{4}\phi^2\chi^2$ as a source with rules

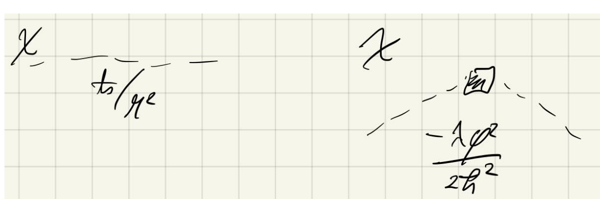


Figure 4: $\frac{\lambda}{4}\phi^2\chi^2$ as a source

leaving

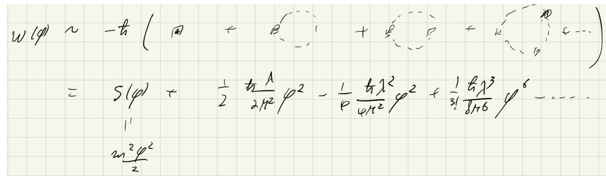


Figure 5: Effective Action Expansion

which as usual can be used to calculate correlation functions

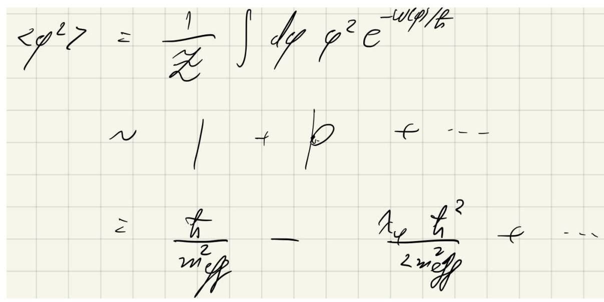


Figure 6: Effective Action Correlation

which is the same as the full theory.

1.4 Quantum Effective Action Γ

The previous effective action accounts for some quantum effects, we can also introduce the quantum effective action, which accounts for all quantum effects (?). As such we first define

$$\Phi = -\partial_J W = \langle \phi \rangle \tag{32}$$

for $J \neq 0$. Then we can do a Legendre transform (we assume convexity, and in practice, this is justified)

$$\Gamma(\Phi) = W(J) + \Phi J \tag{33}$$

which has the property that $\partial_\Phi \Gamma = J$ (so at $J = 0$ we get 0, meaning we have a local extremum). In higher dimensions, we can use this to define the **effective/quantum** potential $V(\Phi)$ which can be more useful than the action

$$\Gamma(\Phi) = \int d^4x \left(-V(\Phi) - \frac{1}{2}\partial^\mu \Phi \partial_\mu \Phi + \dots \right) \tag{34}$$

We can draw analogies to statistical mechanics with this Legendre transform, and the Gibbs free energy, for example. [End of lecture 5]

We can interpret the effective action $\Gamma(\Phi)$ nicely in Feynman diagrams, but first

Definition 2. we define a **bridge** to be an internal line of a connected graph such that if it is cut, the graph becomes disconnected. (in the context of Feynman diagrams, graphs can contain both internal and external lines.)

Definition 3. Furthermore, we call a connected graph a **one particle irreducible** or 1PI if it contains no bridges.

Claim 1. From here we claim that when we expand $\Gamma(\Phi)$ we sum over the 1PI graphs of the $S(\phi)$ theory with vertices

$$\gamma(\Phi) = \Gamma^{(0)} + \Gamma^{(1)}\Phi + \frac{1}{2!}\Gamma^{(2)}\Phi^2 + \dots \tag{35}$$

which can be thought of interactions.

Example 3. For example, in a theory with ϕ^3, ϕ^4 terms we may have

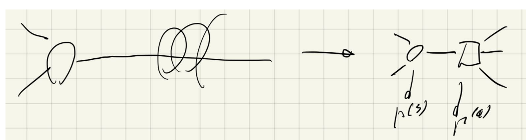


Figure 7: Expanding the Effective Action

where the $3, 4$ powers come from the number of vertices connecting to each 1PI graph.

But we should provide some justification for this statement. Let's imagine we treat $\Gamma(\Phi)$ just as we treat the normal action, and so we get

$$e^{W_\Gamma(J)/g} = \int d\Phi e^{-(\Gamma(\Phi) - J\Phi)/g} \tag{36}$$

for fictitious Planck's constant g and sum of connected diagrams W_Γ. We can expand this as $W_\Gamma(J) = \sum_{l=0}^{\infty} g^l W_\Gamma^{(l)}(J)$ and then, as with $S(\phi), W(J)$ only the tree graphs (so the graphs without loops) contribute to $W_\Gamma(J)$ as $g \to 0$.

$$\lim_{g \to 0} W_\Gamma(J) = W_\Gamma^{(0)}(J) \tag{37}$$

Note also that as $g \to 0$ the minimum dominates, so we have $\partial_\Phi \Gamma = J$ (Legendre transform) and so

$$W_\Gamma(J) = W_J^{(0)}(J) = \Gamma(\Phi) - J\Phi = W(J) \tag{38}$$

where $W(J)$ is given by

$$e^{-W(J)/\hbar} = \int d\phi e^{-(S(\phi) - J\phi)/\hbar} \tag{39}$$

meaning that the sum of connected diagrams from $W(J)$ can be represented as a sum of tree diagrams in $W_\Gamma(J)$ for $g \to 0$ (is this only true for $\hbar \to 0$?).

Note also, that for a theory with N fields ϕ_a we have

$$\begin{aligned}
\langle\phi_a\phi_b\rangle_J^{conn} &= \langle\phi_a\phi_b\rangle_J - \langle\phi_a\rangle_J\langle\phi_b\rangle_J \\
&= -\hbar\partial_{J_a}\partial_{J_b}W \\
&= \hbar\partial_{J_a}\Phi_b \\
&= \hbar\left(\partial_{\Phi_b)J_a}\right) \\
&= \hbar(\partial_{\Phi_a}\partial_{\Phi_b}\Gamma)^{-1}
\end{aligned}$$

so the full propagator including loops is $\hbar$ times the inverse quadratic term in Γ.

1.5 Fermions

The current theory works well for bosons, but to allow for the antisymmetric nature of fermions, we need a little extra work. Here we introduce the **Grassmannian numbers** θ_a obeying

$$\theta_a\theta_b = -\theta_b\theta_a \tag{40}$$

meaning for example that $\theta_a^2 = 0$ always. This last property is convenient, since it means in particular, that all series expansions are finite. Also, for $\phi \in \mathbb{C}$ we simply define $\phi\theta = \theta\phi$. Then we have that series here take the form

$$F(\theta) = f + \phi_a\theta_a + \frac{1}{2!}g_{ab}\phi_a\phi_b + \cdots + \frac{1}{N!}k_{a_1\ldots a_N}\theta_{a_1}\ldots\theta_{a_N} \tag{41}$$

which is indeed a finite sum. Notice also that we require here that the tensors of coefficients are totally antisymmetric in the swapping of indices $g_{ab} = -g_{ba}$. Also note that even products of Grassmannians are not Grassmannian, so $\theta_a\theta_b$ is not Grassmannian since

$$(\theta_1\theta_2)(\theta_3\theta_4) = (\theta_3\theta_4)(\theta_1\theta_2) \tag{42}$$

Now, we extend to some extra definitions for other operations. Here we define **Grassmannian differentiation** such that

$$\partial_{\theta_a}(\theta_b F(\theta)) = -\theta_b\partial_{\theta_a}F + \delta_{ab}F \tag{43}$$

Integration is a bit different, but we require linearity of differentiation, and so if η is a Grassmannian constant we have that $\int d\theta\eta = 0$, so we see that $\int d\theta = 0, \int d\theta\theta = 1$ (normalising). As such we define **Grassmannian integration** such that

$$\int d\theta(f + \phi\theta) = \tag{44}$$

These rules are called the **Berezin rules**. We also notice that $\int d\theta\partial_\theta F(\theta) = 0$. [End of lecture 6]

From here we note that if we take an integral $\int d^N\theta F$ the only term that does not vanish is the one that has each θ_a appearing exactly once. In particular, we find that

$$\int d\theta^N \theta_{a_1} \dots \theta_{a_N} = \epsilon^{a_1 \dots a_N} \tag{45}$$

Now suppose we want to change variables $\theta'_a = X_{ab}\theta_b, X_{ab} \in \mathbb{C}$ then

$$\begin{aligned}
\int d^N\theta \theta'_{a_1} \dots \theta'_{a_N} &= X_{a_1 b_1} \dots X_{a_N b_N} \int d^N\theta \theta_{b_1} \dots \theta_{b_N} \\
&= X_{a_1 b_1} \dots X_{a_N b_N} \epsilon^{b_1 \dots b_N} \\
&= \det(X) \epsilon^{a_1 \dots a_N} \\
&= \det(X) \int d^N\theta' \theta'_{a_1} \dots \theta'_{a_N}
\end{aligned}$$

which leads to an interesting contrast with commuting scalars:

$$\theta' = X\theta \implies d^N\theta' = \frac{1}{\det(X)} d^N\theta \tag{46}$$

$$\phi' = Y\phi \implies d^N\phi' = \det(Y) d^N\phi \tag{47}$$

1.5.1 Free fermion field theory (0 dimensions)

For $N = 2m$ fermionic fields $m \in \mathbb{N}$ we see that the action must take the form $S = \frac{1}{2}A_{ab}\theta_a\theta_b$ since S must be bosonic. Then we see that

$$\begin{aligned}
\mathcal{Z}_0 &= \int d^{2m}\theta e^{-S/\hbar} \\
&= \frac{(-1)^m}{(2\hbar)^m m!} \int d^{2m}\theta A_{a_1 a_2} \dots A_{a_{2m-1} a_{2m}} \theta_{a_1} \dots \theta_{a_{2m}} \\
&= \frac{(-1)^m}{(2\hbar)^m m!} \epsilon^{a_1 \dots a_{2m}} A_{a_1 a_2} \dots A_{a_{2m-1} a_{2m}} \\
&= \frac{(-1)^m}{\hbar^m} \mathrm{Pf}(A) = \pm\sqrt{\frac{\det(A)}{\hbar^N}}
\end{aligned}$$

where we remove the exponential by noticing that in its power series the only term that does not vanish is the term that contains all θs exactly once. We also note that the Pfaffian of an antisymmetric matrix here can be defined as

$$\mathrm{Pf}(A) = \frac{1}{2^m m!} \epsilon^{a_1 \dots a_m} A_{a_1 a_2} \dots A_{a_{2m-1} a_{2m}} \tag{48}$$

(the Pfaffian is necessary content in this course. It apparently also has uses in supersymmetry.) Consequently, we can summarise our results as

$$\mathcal{Z}_{0,bosons} = \sqrt{\frac{(2\pi\hbar)^N}{\det(M)}} \tag{49}$$

$$\mathcal{Z}_{0,fermions} = \pm\sqrt{\frac{\det(A)}{\hbar^N}} \tag{50}$$

As such in computer simulations, which struggle with Grassmanns, one often treats fermions as bosons with $M = A^{-1}$...

1.5.2 Adding external sources

To add external source we introduce Grassmann η and say that

$$S(\theta) - \eta\theta = \frac{1}{2}A_{ab}\theta_a\theta_b - \eta_a\theta_a \tag{51}$$

and so completing the square we find

$$S(\theta) - \eta\theta = \frac{1}{2}((\theta_a - \eta_c(A^{-1})_{ca})A_{ab}(\theta_b - \eta_d(A^{-1})_{db})) + \frac{1}{2}\eta_a(A^{-1})_{ab}\eta_b \tag{52}$$

and so we use translational invariance to get

$$\mathcal{Z}_0(\eta) = e^{-\frac{1}{2\hbar}\eta^T A^{-1}\eta}\mathcal{Z}_0(0) \tag{53}$$

and propagator

$$\langle\theta_a\theta_b\rangle = \frac{\hbar^2}{\mathcal{Z}_0}\partial_{\eta_a}\partial_{\eta_b}\mathcal{Z}_0(\eta)|_{\eta=0} = \hbar(A^{-1})_{ab} \tag{54}$$

2 LSZ Reduction Formula

(LSZ means Lehmann-Symanzik-Zimmerman) The LSZ reduction formula is a useful tool, but for us it relates the concepts of scattering amplitudes, correlation functions, and vacuum expectation values (VEVs - really the same as correlation functions). It also introduces the normalisation of fields. Here, the lecturer tries to keep the same conventions as David Tong notes, though he warns he may not do so perfectly. In particular, that means that for $\hbar = 1$ we have

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3 2E_k}\left(a(k)e^{-ik\cdot x} + a^\dagger(k)e^{ik\cdot x}\right) \tag{55}$$

and use the mostly minus sign convention in Minkowski spacetime $(+---)$ so

$$k \cdot x = E_k t - \mathbf{k} \cdot \mathbf{x} \tag{56}$$

and the relativistic normalisation for $a(k)$ mentioned below. We want to express a in terms of ϕ and as such we note that

$$\int d^3x e^{ik\cdot x}\phi(x) = \frac{1}{2E}a(k) + \frac{1}{2E}e^{2iEt}a^\dagger(-k) \tag{57}$$

$$\int d^3x e^{ik\cdot x}\partial_a\phi(x) = \frac{-i}{2}a(k) + \frac{i}{2}e^{2iEt}a^\dagger(-k) \tag{58}$$

2.0.1 Initial States

To describe initial states then, for example in free theory with 1 particle, we take

$$|k\rangle = a^\dagger(k)\,|\Omega\rangle \tag{59}$$

for the full vacuum state $|\Omega\rangle$ (Tong includes a discussion of full vacuum states and another type) such that $a(k)\,|\Omega\rangle = 0\forall k$ and $\langle\Omega|\Omega\rangle = 1$) and we use the relativistic normalisation

$$\langle k|k'\rangle = (2\pi)^3(2E)\delta(k-k'), E = \sqrt{k^2+m^2} \tag{60}$$

[End of lecture 7]

We wish to compute an (approximation) to scattering amplitudes, and as such it makes sense to consider Gaussian wavepackets $a_1^\dagger = \int d^3k f_1(k) a^\dagger(k)$ for $f(k) \propto e^{\frac{-(k-k_1)^2}{2\sigma^2}}$ and similarly for $a_2^\dagger$ for a different k_2. These represent waves of different momenta, of course, and we imagine these collide giving a certain outcome. Here we assume that interactions are weak, and that for time $t=\pm\infty$ we can approximate our theory with free theory, and so we define

$$|i\rangle = \lim_{t\to\infty} a_1^\dagger(t)a_2^\dagger(t)\,|\Omega\rangle \tag{61}$$

$$|f\rangle = \lim_{t\to\infty} a_{2'}^\dagger(t)a_{2'}^\dagger(t)\,|\Omega\rangle \tag{62}$$

for $k_1 \neq k_2, k_{1'} \neq k_{2'}$. Then we see that

$$\begin{aligned}
a_1^\dagger(\infty) - a_1^\dagger(-\infty) &= \int_{-\infty}^{\infty} dt\partial_0 a_1^\dagger(t) \\
&= \int d^3k f_1(k)\int d^4x\partial_0\left(e^{-ik\cdot x}(-i\partial_0\phi + E\phi)\right) \text{ (why?)} \\
&= -i\int d^3k f_1(k)\int d^4x e^{-ik\cdot x}(\partial_0^2 + E^2\phi) \\
&= -i\int d^3k f_1(k)\int d^4x e^{-ik\cdot x}(\partial_0^2 + |k|^2 + m^2)\phi
\end{aligned}$$

where we can turn the $|k|^2$ into $-\overleftarrow{\nabla}^2$ (derivative applied to the left) which by integration by parts becomes $\vec{\nabla}^2$ which then combines with ∂_0^2 to becomes ∂^2 leaving

$$a_1^\dagger(\infty) - a_1^\dagger(-\infty) = -i\int d^3k f_1(k)\int d^4x e^{-ik\cdot x}(\partial^2 + m^2)\phi \tag{63}$$

where we notice the last $\partial^2 + m^2$ is the Klein-Gordon operator which should be 0 in free theory, so in a sense this measures deviation from free theory. We then write

$$\langle f|i\rangle = \langle\Omega| T a_{1'}(\infty)a_{2'}(\infty)a_1^\dagger(-\infty)a_2^\dagger(-\infty)|\Omega\rangle \tag{64}$$

so we can use the previous relation to reorder and notice that the only non-zero term is

$$\begin{aligned}\langle f|i\rangle = i^4\int d^4x_1 d^4x_2 d^4x_{1'} d^4x_{2'} e^{-ik_1\cdot x_1 - ik_2\cdot x_2 + ik_{1'}\cdot x_{1'} + ik_{2'}\cdot x_{2'}}\\ (\partial_1^2 + m^2)(\partial_2^2 + m^2)(\partial_{1'}^2 + m^2)(\partial_{2'}^2 + m^2)\\ \langle\Omega| T\phi(x_1)\phi(x_2)\phi(x_{1'})\phi(x_{2'})|\Omega\rangle\end{aligned} \tag{65}$$

This is the **LSZ reduction formula**. It is fortunately easily generalised to more than two wavepackets by considering $T\phi(x_1)\ldots\phi(x_n)\phi(x_{1'})\ldots\phi(x_{2'})|\Omega\rangle$. It works under relatively standard assumptions in QFT, but it does occasionally encounter a need for renormalisation, which will be a big topic later on. [End of lecture 8]

3 Scalar field theory

3.1 Wick rotation

We now finally start looking at scalar field theory in more than 0 dimensions. Here we first introduce the Wick rotation, which is to say we introduce imaginary time as a means of turning the Minkowski metric into a standard Euclidean one. In many applications this is useful.

In particular, in the (mostly minus) Minkowski metric we may have $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu phi - V(\phi)$ for $V(\phi) = \frac{1}{2}m^2\phi^2 + \sum_{n>2}\frac{1}{n!}V^{(n)}\phi^n$ and

$$\mathcal{Z} = \int\mathcal{D}\phi e^{i\int dx^0 L}, L = \int d^3x\mathcal{L}, S = \int d^4x\mathcal{L} \tag{66}$$

where we've adopted the convention that $\hbar = 1$. Then here we have propagator

$$\frac{i}{k^2 - m^2 + i\epsilon} = \frac{i}{(k^0)^2 - |k|^2 - m^2 + i\epsilon} \tag{67}$$

To introduce the **Wick rotation** or imaginary time then we set $x_4 = ix^0$ and use a standard $(++++)$ Euclidean metric we now have $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + V(\phi)$ with

$$\mathcal{Z} = \int\mathcal{D}\phi e^{-\int dx_4 L} \tag{68}$$

and momentum satisfies $k_0 x^0 - k \cdot x = -ik_0 x_4 - k \cdot x = -k_4 x_4 - k \cdot x$ leaving $k_4 = ik_0$. Notice, however that this implies that

$$\pm ik \cdot x \mapsto \mp ik \cdot x \tag{69}$$

Some use this to justify the use of the mostly plus Minkowski metric. As such our propagator becomes

$$\tilde{\Lambda}_0(k) = \frac{1}{k^2 + m^2} = \frac{1}{k_4^2 + |k|^2 + m^2} \tag{70}$$

Note that we are able to get rid of the $i\epsilon$ in the denominator since now the the denominator never hits 0 (essentially by integrating over k_4 we have swapped the real and imaginary axes moving the singularities onto what is now the imaginary axis of k_4 instead).

3.2 Feynman rules

Let's develop Feynman rules. For a free propagator we see that if, as usual,

$$S_0(\phi, J) = \int d^4x \left(\frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{1}{2} m^2 \phi^2 - J(x)\phi(x) \right) \tag{71}$$

where we remove the derivatives by using a Fourier transform $\phi = \int \frac{d^4k}{(2\pi)^4} e^{ik\cdot x} \tilde{\phi}(k)$. Also, we split the $J(x)\phi(x) = \frac{1}{2}(J(x)\phi(x) + J(x)\phi(x))$ and integrate each term differently, which you see in the Fourier transform.

$$\begin{aligned} S_0(\tilde{\phi}, \tilde{J}) &= \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \left(\tilde{\phi}(-k)(k^2 + m^2)\tilde{\phi}(k) - \tilde{J}(-k)\tilde{\phi}(k) - \tilde{J}(k)\tilde{\phi}(-k) \right) \\ &= \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \left(\tilde{\chi}(-k)(k^2 + m^2)\tilde{\chi}(k) - \frac{\tilde{J}(-k)\tilde{J}(k)}{k^2 + m^2} \right) \end{aligned}$$

where we completed the square with $\tilde{\chi} = \tilde{\phi} - \tilde{J}/(k^2 + m^2)$. This leaves

$$\mathcal{Z}_0(J) = e^{\frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \frac{\tilde{J}(-k)\tilde{J}(k)}{k^2 + m^2}} \tag{72}$$

(where we assume $\mathcal{Z}_0(0) = 1$). This leaves us with propagator

$$\tilde{\Lambda}_0(J) = \frac{\delta^2 \mathcal{Z}_0(J)}{\delta \tilde{J}(-q) \delta \tilde{J}(q)} = \frac{1}{q^2 + m^2} \tag{73}$$

for functional derivatives

$$\frac{\delta}{\delta f(x_1)} f(x_2) = \delta(x_1 - x_2), \frac{\delta}{\delta \tilde{g}(k_1)} \tilde{g}(k_2) = (2\pi)^4 \delta(k_1 - k_2) \tag{74}$$

Transforming the propagator we then finally get

$$\Lambda_0(x - x') = \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik\cdot(x - x')}}{k^2 + m^2} \tag{75}$$

which is the same as we get via

$$\mathcal{Z}_0(J) = e^{\frac{1}{2}\int d^4x d^4x' J(x)\Lambda_0(x-x')J(x')} \tag{76}$$

Moving from free theory to interacting theory we write

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1, \mathcal{L}_0 = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}m^2\phi^2 \tag{77}$$

and as in the 0-dimensional case we get the asymptotic series given by

$$\begin{aligned}
\mathcal{Z}_0(J) &= \int \mathcal{D}\phi e^{-\int d^4(\mathcal{L}_0+\mathcal{L}_1-J\phi)} \\
&= e^{-\int d^4y\mathcal{L}_1\left(\frac{\delta}{\delta J(y)}\right)} e^{\frac{1}{2}\int d^4xd^4x'J(x)\Lambda_0(x-x')J(x')} \\
&= \sum_{v=0}^{N}\frac{1}{v!}\left(-\int d^4y\mathcal{L}_1\left(\frac{\delta}{\delta J(y)}\right)\right)\sum_{p=0}\frac{1}{p!}\left(\frac{1}{2}\int d^4xd^4x'J(x)\Lambda_0(x-x')J(x')\right)
\end{aligned}$$

where similarly each term corresponds to a graph where

- propagators connect points (here x, x' act like indices on the points)
- vertices with n legs correspond to powers of $\mathcal{L}_1\left(\frac{\partial}{\partial J(y)}\right)$
- we integrate over the positions of internal lines
- we include symmetry factors as before

which give us the position space Feynman rules (momentum space Feynman rules are also common and described later).

Example 4. We have correlation function

$$\langle\phi(x_2)\phi(x_1)\rangle = -\frac{\delta}{\delta J(x_2)}\frac{\delta}{\delta J(x_1)}W(J) \tag{78}$$

and we take $\mathcal{L}_1 = \frac{\lambda}{3!}\phi^3$. Then we have

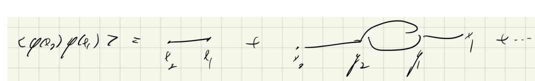


Figure 8: $\frac{\lambda}{3!}\phi^3$ expansion

so that if the more complex second term is D we have

$$D = \frac{\lambda^2}{2}\int d^4y_1d^4y_2\Lambda_0(x_2-y_2)\Lambda_0(y_1-x_1)(\Lambda_0(y_0-y_1))^2 \tag{79}$$