3P1c **Quantum Field Theory: Example Sheet 3** Michaelmas 2020

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1. The chiral representation of the Clifford algebra is

$$\gamma^0 = \begin{pmatrix} 0 & 1_2 \\ 1_2 & 0 \end{pmatrix} \quad , \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} .$$

Show that these indeed satisfy $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbf{1}$. Find a unitary matrix U such that $(\gamma')^\mu = U\gamma^\mu U^\dagger$, where $(\gamma')^\mu$ form the Dirac representation of the Clifford algebra

$$(\gamma')^0 = \begin{pmatrix} 1_2 & 0 \\ 0 & -1_2 \end{pmatrix} \quad , \quad (\gamma')^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} .$$

2. Show that if $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$, then

$$[\gamma^\mu\gamma^\nu \, , \, \gamma^\rho\gamma^\sigma] = 2\eta^{\nu\rho}\gamma^\mu\gamma^\sigma - 2\eta^{\mu\rho}\gamma^\nu\gamma^\sigma + 2\eta^{\nu\sigma}\gamma^\rho\gamma^\mu - 2\eta^{\mu\sigma}\gamma^\rho\gamma^\nu \, .$$

Show further that the matrices $S^{\mu\nu} := \frac{1}{4}\,[\gamma^\mu \, , \, \gamma^\nu] = \frac{1}{2}(\gamma^\mu\gamma^\nu - \eta^{\mu\nu})$ form a representation of the Lie algebra of the Lorentz group.

3. Using just the Clifford algebra $\{\gamma^\mu, \ \gamma^\nu\} = 2\eta^{\mu\nu}$ (without reference to a particular representation) and defining $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, $\not{p} = p_\mu\gamma^\mu$ and $S^{\mu\nu} = \frac{1}{4}\,[\gamma^\mu, \ \gamma^\nu]$, prove the following results:

   (a) $\mathrm{Tr}\gamma^\mu = 0$
   (b) $\mathrm{Tr}(\gamma^\mu\gamma^\nu) = 4\eta^{\mu\nu}$
   (c) $\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho) = 0$
   (d) $(\gamma^5)^2 = 1$
   (e) $\mathrm{Tr}\gamma^5 = 0$
   (f) $\not{p}\not{q} = 2p \cdot q - \not{q}\not{p} = p \cdot q + 2S^{\mu\nu}p_\mu q_\nu$
   (g) $\mathrm{Tr}(\not{p}\not{q}) = 4p \cdot q$
   (h) $\mathrm{Tr}(\not{p}_1 \ldots \not{p}_n) = 0$ if n is odd
   (i) $\mathrm{Tr}(\not{p}_1\not{p}_2\not{p}_3\not{p}_4) = 4\,[(p_1 \cdot p_2)(p_3 \cdot p_4) + (p_1 \cdot p_4)(p_2 \cdot p_3) - (p_1 \cdot p_3)(p_2 \cdot p_4)]$
   (j) $\mathrm{Tr}(\gamma^5\not{p}_1\not{p}_2) = 0$
   (k) $\gamma_\mu\not{p}\gamma^\mu = -2\not{p}$
   (l) $\gamma_a\not{p}_1\not{p}_2\gamma^a = 4p_1 \cdot p_2$
   (m) $\gamma_\mu\not{p}_1\not{p}_2\not{p}_3\gamma^\mu = -2\not{p}_3\not{p}_2\not{p}_1$
   (n) $\mathrm{Tr}(\gamma^5\not{p}_1\not{p}_2\not{p}_3\not{p}_4) = 4i\,\epsilon_{\mu\nu\rho\sigma}\,p_1^\mu\,p_2^\nu\,p_3^\rho\,p_4^\sigma$

4. The plane-wave solutions to the Dirac equation are

$$u^s(\vec{p}) = \begin{pmatrix} \sqrt{p \cdot \sigma}\xi^s \\ \sqrt{p \cdot \bar{\sigma}}\xi^s \end{pmatrix} \text{ and } v^s(\vec{p}) = \begin{pmatrix} \sqrt{p \cdot \sigma}\xi^s \\ -\sqrt{p \cdot \bar{\sigma}}\xi^s \end{pmatrix},$$

where $\sigma^\mu = (1, \vec{\sigma})$ and $\bar{\sigma}^\mu = (1, -\vec{\sigma})$ and ξ^s, with $s \in \{1, 2\}$, is a basis of orthonormal two-component spinors, satisfying $(\xi^r)^\dagger \cdot \xi^s = \delta^{rs}$. Show that

$$\begin{aligned} u^r(\vec{p})^\dagger \cdot u^s(\vec{p}) &= 2p_0 \delta^{rs} \\ \bar{u}^r(\vec{p}) \cdot u^s(\vec{p}) &= 2m\delta^{rs} \end{aligned}$$

and similarly,

$$\begin{aligned} v^r(\vec{p})^\dagger \cdot v^s(\vec{p}) &= 2p_0 \delta^{rs} \\ \bar{v}^r(\vec{p}) \cdot v^s(\vec{p}) &= -2m\delta^{rs}. \end{aligned}$$

Show also that the orthogonality condition between u and v is

$$\bar{u}^s(\vec{p}) \cdot v^r(\vec{p}) = 0,$$

while taking the inner product using † requires an extra minus sign

$$u^s(\vec{p})^\dagger \cdot v^r(-\vec{p}) = 0.$$

5. Using the same notation as Question 4, show that

$$\begin{aligned} \sum_{s=1}^{2} u^s(\vec{p})\bar{u}^s(\vec{p}) &= \not{p} + m, \\ \sum_{s=1}^{2} v^s(\vec{p})\bar{v}^s(\vec{p}) &= \not{p} - m, \end{aligned}$$

where, rather than being contracted, the two spinors on the left-hand side are placed back to back to form a 4×4 matrix.

6. The Fourier decomposition of the Dirac field operator $\psi(x)$ and the hermitian conjugate field $\psi^\dagger(\vec{x})$ is given by

$$\begin{aligned} \psi(\vec{x}) &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_{s=1}^{2} \left[b^s_{\vec{p}} u^s(\vec{p}) e^{i\vec{p}\cdot\vec{x}} + c^{s\dagger}_{\vec{p}} v^s(\vec{p}) e^{-i\vec{p}\cdot\vec{x}} \right], \\ \psi^\dagger(\vec{x}) &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_{s=1}^{2} \left[b^{s\dagger}_{\vec{p}} u^s(\vec{p})^\dagger e^{-i\vec{p}\cdot\vec{x}} + c^s_{\vec{p}} v^s(\vec{p})^\dagger e^{i\vec{p}\cdot\vec{x}} \right]. \end{aligned}$$

The creation and annihilation operators are taken to satisfy

$$\begin{aligned} \{b^r_{\vec{p}}, b^{s\dagger}_{\vec{q}}\} &= (2\pi)^3 \delta^{rs}\, \delta^{(3)}(\vec{p} - \vec{q}), \\ \{c^r_{\vec{p}}, c^{s\dagger}_{\vec{q}}\} &= (2\pi)^3 \delta^{rs}\, \delta^{(3)}(\vec{p} - \vec{q}), \end{aligned}$$

with all other anticommutators vanishing. Show that these imply that the field and its conjugate field satisfy the anti-commutation relations

$$\begin{aligned} \{\psi_\alpha(\vec{x}), \psi_\beta(\vec{y})\} &= \{\psi^\dagger_\alpha(\vec{x}), \psi^\dagger_\beta(\vec{y})\} = 0, \\ \{\psi_\alpha(\vec{x}), \psi^\dagger_\beta(\vec{y})\} &= \delta_{\alpha\beta}\delta^{(3)}(\vec{x} - \vec{y}). \end{aligned}$$

7. Using the results of Question 6, show that the quantum Hamiltonian

$$H = \int d^3x \; \bar{\psi}(-i\gamma^i\partial_i + m)\psi$$

can be written, after normal ordering, as

$$H = \int \frac{d^3p}{(2\pi)^3}\, E_{\vec{p}} \sum_{r=1}^{2} \left[b_{\vec{p}}^{r\dagger} b_{\vec{p}}^{r} + c_{\vec{p}}^{r\dagger} c_{\vec{p}}^{r} \right].$$

8. The Lagrangian density for a fermionic Yukawa theory is given by

$$\mathcal{L} = \tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{2}\mu^2\phi^2 + \bar{\psi}(i\,\partial\!\!\!/ - m)\psi - \lambda\phi\bar{\psi}\psi.$$

Consider $\psi\bar{\psi} \to \psi\bar{\psi}$ scattering, with initial and final states given by,

$$\begin{aligned} |i\rangle &= \sqrt{4E_{\vec{p}}E_{\vec{q}}}\, b_{\vec{p}}^{s\dagger} c_{\vec{q}}^{r\dagger} \,|0\rangle \\ |f\rangle &= \sqrt{4E_{\vec{p}}E_{\vec{q}}}\, b_{\vec{p}'}^{s'\dagger} c_{\vec{q}'}^{r'\dagger} \,|0\rangle\,. \end{aligned}$$

Show that the amplitude is given by

$$\mathcal{A} = -(-i\lambda)^2 \left(\frac{[\bar{u}^{s'}(\vec{p'}) \cdot u^s(\vec{p})][\bar{v}^r(\vec{q}) \cdot v^{r'}(\vec{q'})]}{t - \mu^2} - \frac{[\bar{v}^r(\vec{q}) \cdot u^s(\vec{p})][\bar{u}^{s'}(\vec{p'}) \cdot v^{r'}(\vec{q'})]}{s - \mu^2} \right).$$

where $t = (p-p')^2$ and $s = (p+q)^2$. Draw the two Feynman diagrams that correspond to these two terms.