Quantum Field Theory

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Introduction

These notes are based on the course lectured by Professor Nicholas Dorey in Michaelmas 2020. This was lectured online due to measures taken to counter the spread of Covid-19 in the UK. These are not necessarily an accurate representation of what was lectures, and represent solely my personal notes on the content of the course, combinged with probably, very very many personal notes and digressions... Of course, any corrections/comments would be appreciated.

But, let's actually introduce the content of this course. What is quantum field theory? Quantum field theory (QFT) essentially succeeds in merging special relativity and quantum mechanics. Why is this so difficult? The first relativistic theory was electromagnetism, and the biggest idea that was introduced, and what truly set it apart from older theories was the idea of fields. In Newtonian theory, and much of what followed, the protagonist of the theory was a particle, or if not a particle, at least a body of some kind. Field theory changed. Now, there was a new protoganist: the field. What difference does that make, architectually? Firstly, the field theory is often simpler. But more importantly, the biggest structural difference is that field theory excels in describing "delayed interactions." When the particle is the progagonist, it is very difficult describe theories where forces are not instantaneous. Field theory avoids this. All interactions are made through the field, through which they propogate through space. Now, delayed responses become natural. In electromagnetism, the simplest expression thereof is electromagnetic waves: light.

What does this have to do with relativity? Well, as soon as high speeds become relevant, forces can no longer be considered instantaneous. As such, it is difficult to keep using particles as the protoganist of these theories. Consequently, the natural step is to make, instead of particles, fields the protoganist of this new quantum theory we are developing.

That is the goal of QFT. There is one important consequence though, once particles are no longer the protagonists of the theory. That is that particle number no longer has to be conserved. In the most elegant fashion, by removing the supremacy of the "particle" in our theory, and replacing it with the more powerful notion of the field, particles merely become phenomenon to be observed, and tools of analysis. In this context, it is only natural that particle number is no longer conserved. Whereas before, the wavefunction was often associated with a wave-particle like object, now the wavefunction (which is a field) describes a multiparticle state. Well, really it describes the field, and the multiparticle state is something that can be deduced from it. Somehow, although this is just the beginning of the course, I feel that that's not that important anymore. It is deeply intriguing though, how the imposition of boundary conditions somehow forces a degree of discreteness onto this theory...

Well then, the overall architecture is more or less the same as standard quantum theory. It is probabilistic, and we assume a degree of symmetry under boosts, and rotations (isotropy and translation invariance). The fundamental approach to making predictions still boils down to the same calculation: evaluating

$$A_{i\to f} = \langle f| \, e^{iHT} \, |i\rangle$$

for probability amplitude A, initial state i, final state f, Hamiltonian (time translation generator) H, and time interval T.

There are two caveates with most of these field theories, though. Firstly, they have not been mathematically formalised, so often there are areas that are somewhat ambiguous. Secondly, contributing to this ambiguity, many of the sums are divergent, so the meaning of some calculations can really be somewhat ambiguous... How curious! I'd like to think about this a bit more...

1 Preliminaries

Anyways, getting down to business. We'll be mostly using natural units during this course. That means that $c = \hbar = 1$, and these can be added back into the calculation using dimensional analysis. The effect of this, is that the only unit used throughout all calculations is really a unit of mass-energy. As such, all quantities scale by a power of the unit of energy.

Definition 1 (dimension of X)**.** Denoted $[X]$, this is δ, such that for unit of mass-energy M, X scales as M^{δ}. δ may also be called the scaling or the engineering dimension of X.

Also, for special relativity we use the convention that we are working on Minkowsky space-time $\mathbb{R}^{3,1}$, with metric tensor

$$\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1).$$

2 Classical Field Theory

Fortunately for me, we are starting with a description of classical field theory, and we are starting, very simply, with scalar fields.

Definition 2 (Scalar Field)**.** $\phi(t, \underline{x}) = \phi(x) : \mathbb{R}^{3,1} \to \mathbf{R}$ is a sclar field if it is Lorentz invariant, meaning that it follows the transformation rule

$$\phi(x) \to \phi(\Lambda^{-1}(x))$$

Here the domain is called the spacetime, and the codomain is called the **field space**. These may, and will be replaced with other spaces.

Changing the spacetime here corresponds to implementing gravity in some way or another, by changing the manifold we are working on. The field space corresponds to the complexity of what is being descibed. Since we will be describing multi-particle states with our wavefunction, this will become significantly more complex. And, I intentionally left the description of what it means to be Lorentz invariant a bit vague, since, well, in our case, it just means being invariant under the Lorentz transformations, which is the group of linear transformations that preserves the Minkowsky metric (ie. the set of matrices such that $\Lambda\eta\Lambda^T = \eta$). But really, while linearity makes a lot of sense in the context of linear Minkowsky space, I doubt (though I have no familiarity with this area) this remains the case when we are on an arbitrary manifold, which happens when we consider general relativity. As such, I prefer to think of Λ as any arbitrary invertible map on the manifold, corresponding to the symmetries we impose.

The difficulties that arise in comibing quantum theory with general relativity are also quite clear. It does seem tremendously difficult. If we do not simply assuming that general relativity simply bends space, which I will assume is not entirely the case, or else I feel a theory reconciling the two would have already been developed long ago, in spite of the tremendous difficulty of the calculations involved, and it also does not seem to make much sense of Hawking radiation, since in general relativity, the space at the centre of a black hole truly is cut off... Nevertheless, from what I've heard, if you want to turn gravity into a quantised force with mediator particle, then somehow the protagonist of that theory would be not only be a field, but somehow span the space of possible manifolds as well. Purely intuitively, I would imagine that we would be getting fields of the form $\phi : \mathcal{D} \to V$ where $\mathcal{D}$ is an object that stitches many manifolds together. Brrr... I have not thought too deeply about this, but that does seem like a truly terrifying object indeed! Or perhaps not quite. Hm, it might be worth thinking a bit more about this...

Anyways, I also wanted to remark that having fields transform as $\phi(\Lambda^{-1}x)$ is more or less an arbitrary definition that is called the "active" definition of field transformations. Oh well, you can assign some intuition to it, but it is more or less convention to use the inverse of the matrix instead of the matrix itself.

Extending our notion of fields to vector fields, first note that notation wise, we use $\partial_\mu \phi = \frac{\partial \phi}{\partial x^\mu}$, and we define

Definition 3 (ϕ^μ transforms as a vector field)**.** if

$$\phi \to \Lambda^\mu_\nu \phi^\nu(\lambda^{-1}x)$$

This is just the transformation rule for rank 1 tensors, so is nothing particularly remarkable. The only remarkable part is that, as a result $\partial^\mu \phi$ transformas as vector, and so the following becomes a rank 0 tensor (ie, a scalar field) $\partial^\mu \phi \partial_\mu \phi$.

Well, that ends [lecture 1], for those of you interested.

2.1 The Lagrangian

We will review some of the tools from classical mechanics that we are using. The first is the Lagrangian. There are three advantages to using the lagrangian here:

- they are independent of coordinates
- the symmetries of the system can be easily expressed
- the path integral formulation follows immediately

although the Lagrangian is used much more heavily in the Advanced Quantum Field Theory course than the current course. Anyways, we review the Lagrangian, except that now we implement it on fields, so $L = L(\phi, \partial_\mu \phi)$ for (scalar) fields ϕ, and we make three requirements of our action/Lagrangian

- The action S is Lorentz invariant
- we require locality (see below)
- we have at most a second order time derivative

Locality essentially means that $L = \int dx^3\, \mathcal{L}(\phi(x), \partial_\mu \phi(x))$, so the Lagrangian has an associated local Lagrangian density $\mathcal{L}$. From hereon, this $\mathcal{L}$ will usually be referred to as the Lagrangian instead. We note that the action (over an infinite time scale) may now be expressed as

$$S = \int dx^4 \mathcal{L}(\phi, \partial_\mu \phi)$$

Now, Lorentz invariance essentially means that the Lagrangian density is a scalar field (a tensor really), meaning that it transforms as $\mathcal{L}(x) \mapsto \mathcal{L}(\Lambda^{-1} x)$. This corresponds to the action being Lorentz invariant since the Jacobian of a Lorentz transformation (the determinant) is 1, so the relevant change of variables, the action does not change.

Finally, using no more than a second order time derivative in the context of relativity means using no more than a second order derivative of any kind, and since we also are rank-0 tensors, we in fact, cannot have first order derivatives either. Consequently, we can write the general form of the Lagrangian as

$$\mathcal{L} = \frac{1}{2} F(\phi) \partial_\mu \phi \partial^\mu \phi - V(\phi)$$

(note that $\phi\partial_\mu\partial^\mu\phi$ is related to the above by integration by parts so can be safely ignored). Also, in practice in quantum theory we may neglect $F(\phi)$ leaving us with quite a simple general form.

To make all this work, we apply the principle of least action, which gives us the Euler Lagrange equation

$$\partial_\mu \partial_{\partial_m u \phi} \mathcal{L} = \partial_\phi \mathcal{L}$$

An important special case to consider is the case when the potential is quadratic, so $V(\phi) = \frac{1}{2}M^2\phi^2$, which means the Euler-Lagrange case is linear, leaving us with the so-called **Klein-Gordon Equation**

$$\partial_\mu \partial^\mu \phi + M^2 \phi = 0$$

As one might expect, since in Minkowski spacetime $\partial_\mu\partial^\mu = \partial_t^2 - \nabla^2$ we get wavelike solutions

$$\phi \sim e^{ix\cdot p}$$

where $x \cdot p = \omega t - \vec{k} \cdot \vec{x}$, and the dispersion relation requires $\omega_k = \sqrt{k^2 + M^2}$.

On a philosophical note, I was wondering what difference using the principle of least action makes compared to just Newton's equations (on a philosophical level - it is obviously more practical), and as such I was wondering to what extent the Lagrangian formulation in a sense is just a tensor formulation of Newton's equations?

I also was wondering what locality means. When discussing life, I have remarked that really as long as one is not starving, etc., reality is little more than a medium for communication between people. This is certainly the case for particles, and objects, which do not have to worry about starvation, etc. But then I wonder, do particles really not need to worry about starvation? Many particles decay after all, although I doubt that has to do with a kind of starvation of any kind... [End of lecture 2]

2.2 Maxwell's Theory

Maxwell's theory uses a 4-vector potential $A^\mu = (\phi, \vec{A})$. This being a rank 1 tensor means it transforms as

$$A^\mu(x) \mapsto \Lambda^\mu_\nu A^\nu(\Lambda^{-1}x)$$

In electromagnetism, the **field strenght tensor** is given by

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

and as a tensor, it transforms as appropriate. But it satisfies a further condition that it is invariant under Gauge transformations $A^\mu \mapsto A^\mu + \partial^\mu \lambda$. This leads to the **Bianchi identity**

$$\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0$$

which captures 2 out of Maxwell's 4 equations. The other two arise from the principle of least action applied to the **Maxwell Lagrangian**

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

We should note that there really are not that many options of Lagrangians here once one introduces the standard assumptions in addition to assuming it must be expressed in terms of physical quantities (the only candidate is the field strength vector used above)... Writing

$$\mathcal{L} = -\frac{1}{2}\partial_\mu A_\nu \partial^\mu A^\nu + \frac{1}{2}(\partial_\mu A^\mu)^2 \eta^{\mu\nu}$$

gives us (after Euler-Lagrange)

$$\partial_\mu F^{\mu\nu} = 0$$

which provides us with the rest of the Maxwell equations.

2.3 Symmetries in QFT

Symmetries (see the Symmetries, Fields, and Particles (SFP) course) are variations of fields that leave the action invariant. Symmetries have several effects in physics. Firstly, by Noether's theorem, they lead to conservation laws. Secondly, they restrict the form of the Lagrangian. In particular, Gauge symmetry is very powerful due to strong restrictions it puts on the form of the Lagrangian.

Common symmetries include (See SFP) translation invariance and Lorentrz transformation invariance. Another symmetry occurs when either $m = 0, V = 0$ or the action is propoertional to the field. In this case we get an additional "scale invariance" where $x^\mu \mapsto \lambda x^\mu$ and $\phi \mapsto \lambda^{-\Delta}\phi(\lambda^{-1}x)$ for engineering constant Δ.

Internal symmetries like charge, flavour or colour conservation also occur, further restricting the theory. These are not related to the space, and are somehow better expressed by the quantity they conserve. Another small curious symmetry is the following:

Example 1. For complex scalar fields with $\mathcal{L} = \partial_\mu \psi^* \partial^\mu \psi - V(|\psi|^2)$ the following is also a symmetry: $\psi \mapsto e^{i\alpha}\psi$ for real α. The lecturer does not go into any more detail about this.

Finally, we note that for continuous symmetries, which form Lie groups, we can take the set of elements of the group "near" the identity, $g = e^{\alpha X}$ where X sits in the Lie algebra of G. Working to 1st order we now may write

$$g \cdot \psi \mapsto \psi + \delta\psi = \psi + \alpha X \psi$$

to first order. [End of lecture 3]

Just as a remark on terminology, a Lorentz transformation is a linear transformation preserving the Minkowski metric. On the other hand, a **proper Lorentz transformation** is a Lorentz transformation with determinant 1 (which ensures causality). Our goal now is to work towards Noether's theorem. Noether's theorem states that

Theorem 1 (Noether's Theorem)**.** For every continuous symmetry, there exists a conserved current, given by

$$j^\mu = \partial_{\partial_\mu \phi}\mathcal{L}\delta\phi - F^\mu(\phi(x))$$

where $\delta\phi$ is the variation in ϕ and $\delta\lambda = F^\mu$. Conservation here means that $\partial_\mu j^\mu = 0$.

The conserved quantity associated with the conserved current then can be shown to be $Q = \int dx^3 j^0$ since

$$\begin{aligned}
\frac{d}{dt}Q &= \int dx^3 \partial_t j^0 \\
&= -\int dx^3 \nabla \cdot J \\
&= \int_S dS \cdot J \\
&= 0
\end{aligned}$$

A good example is $j^\mu = (\rho, J)$ for charge density and current in electromagnetism. Anyways, here is our "proof" of Noether's theorem:

Proof. Using the Euler-Lagrange equation applied to the field ϕ we see that

$$\begin{aligned}
\delta\mathcal{L} &= \partial_\phi \mathcal{L}\delta\phi + \partial_{\partial_\mu \phi}\mathcal{L}\delta\partial_\mu\phi \\
&= (\partial_\phi\mathcal{L} - \partial_\mu(\partial_{\partial_\mu\phi}\mathcal{L}))\delta\phi + \partial_\mu(\partial_{\partial_\mu\phi}\mathcal{L}\delta\phi) \\
&= \partial_\mu(\partial_{\partial_\mu\phi}\mathcal{L}\delta\phi)
\end{aligned}$$

Then using our definition of j^μ as above we see that $\partial_\mu j^\mu = 0$. □

Here it is assumed that $\delta\mathcal{L} = \partial_\mu F^\mu$ which means we can apply Stokes' theorem and integrate on the surface only. When we do so, F^μ becomes that surface term we integrate with. But Euler-Lagrange also means that our expression above, for the first term in the conserved current, is also a surface term (often determined by the boundary conditions) for $\delta\mathcal{L}$. So what exactly is the difference? Is it that the first term depends on the variation whereas the second does not?

[End of lecture 4 - this bit needs rewriting]

[Start of Noether's theorem rewrite]

Alright, so now to the main topic of this lecture: Noether's theorem. Noether's theorem states that for every continuous symmetry we get a conserved current (and quantity), which she gives an equation for. Now how do we get that?

Firstly, we observe that for a Lorentz transformation

$$\Lambda \approx I + s\Omega$$

where Ω is antisymmetric (to ensure we preserve the Minkowski metric). If that the case, we see that if we tree $\mathcal{L}$ as a scalar field $\mathcal{L}(x)$ instead of a function of the field, then

$$\delta\mathcal{L} = -s\Omega^\mu_\nu x^\nu \partial_\mu(\mathcal{L}(x)) = -s\Omega^\mu_\nu \partial_\mu(x^\mu \mathcal{L}(x)).$$

This last equality only holds because Ω is asymmetric (so its diagonal is all zeros). But importantly, we see that $\delta\mathcal{L}$ can be written as a total derivative when x is varied.

Now, we vary the same $\mathcal{L}$ but instead of varying with respect to x, we vary with respect to the field ϕ. Doing so, and assuming Euler-Lagrange we find that:

$$\begin{aligned}
\delta\mathcal{L} &= \partial_\phi \mathcal{L}\delta\phi + \partial_{\partial_\mu \phi}\mathcal{L}\delta\partial_\mu\phi \\
&= (\partial_\phi\mathcal{L} - \partial_\mu(\partial_{\partial_\mu\phi}\mathcal{L}))\delta\phi + \partial_\mu(\partial_{\partial_\mu\phi}\mathcal{L}\delta\phi) \\
&= \partial_\mu(\partial_{\partial_\mu\phi}\mathcal{L}\delta\phi)
\end{aligned}$$

Remarkably, we find once again that we may write $\delta\mathcal{L}$ as a total derivative. Let us now denote these variations as $\delta_x\mathcal{L}$ for the first, and $\delta_\phi\mathcal{L}$ for the second. Now, since we have a symmetry, we know that the action is unchanged when we apply a Lorentz transformation. In particular, we know that the "partial variation" of $\mathcal{L}$ with respect to x alone, but ignoring ϕ should be "constant." What do we mean by constant? The partial derivative with respect to space is a 4-vector, so really we mean that the Minkowski metric is unchanged. In other words, to first order the quantity

$$v_M \cdot \partial_x \delta\mathcal{L} = 0$$

where we use $\partial_x\delta$ to denote this "pure" partial derivative and

$$v_M = \begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

is the "Minkowski vector" (the first order derivative of the Minkowski metric in a sense). This would mean that the Minkowski metric is unchanged under a pure variation of spacetime. So how do we write this pure variation of space time? We notice that $\delta\partial_x\mathcal{L} = \delta_x\mathcal{L} - \delta_\phi\mathcal{L}$ since in a sense $\delta_x\mathcal{L}$ is the total derivative of $\mathcal{L}$ with respect to x, and $\delta_\phi\mathcal{L}$ is the total derivative of $\mathcal{L}$ with respect to ϕ, and so, by the chain rule, what is left over is the partial variation with respect to x.

This is exactly what Noether's theorem refers to. Noether's theorem defines the **conserved current** to be

$$j^\mu = \partial_{\partial_\mu \phi}\mathcal{L}\delta\phi - F^\mu$$

where $\delta_x\mathcal{L} = \partial_\mu F^\mu$, and $\delta_\phi\mathcal{L} = \partial_\mu(\partial_{\partial_\mu\phi}\mathcal{L}\delta\phi)$, and so we see that our partial variation of $\mathcal{L}$ with respect to only x is

$$\delta\partial_x\mathcal{L} = -\frac{\partial}{\partial x_\mu}j^\mu$$

(no summation convention - this is a vector here), and importantly

$$\partial_\mu j^\mu = v_M \cdot \delta\partial_x\mathcal{L} = 0$$

as required since $\partial_\mu j^\mu = \delta\mathcal{L} - \delta\mathcal{L} = 0$.

That is Noether's theorem. I do not fully understand it, and to me it seems weird that the "generators" of these variations (the terms like F^μ inside the derivative) are not the same. But anyhow, Noether's theorem states that j^μ is conserved, so $\partial_\mu j^\mu = 0$ (which means the Minkowski metric stays the same in the way we described). Aside from that we will remark that we assume in general that for any symmetry $\partial_x\mathcal{L} = \partial_\mu F^\mu$ for some F^μ,

Now where does a conserved quantity arise from. Here we note that if $Q = \int dx^3 j^0$ then it satisfies

$$\begin{aligned}\frac{d}{dt}Q &= \int dx^3 \partial_t j^0 \\ &= -\int dx^3 \nabla \cdot J \\ &= \int_S dS \cdot J \\ &= 0\end{aligned}$$

and os is conserved. One can check that in the case of electromagnetism, Q corresponds precisely to electromagnetic charge.

[End of Noether's theorem rewrite]

Today we will look at certain examples of Noether's theorem. In particular, we will define the energy-momentum constant as the Noether current arising from translation invariance. Under the symmetry $x^\mu \mapsto x^\mu + \epsilon^\mu$, we get $\phi \mapsto \phi(x-\epsilon) \approx \phi(x) - \epsilon^\mu\partial_\mu\phi + \ldots$. We also get $\delta\mathcal{L} = -\epsilon^\mu\partial_\mu\mathcal{L}$. Consequently we get energy-momentum tensor

$$T^\mu_\nu = j^\mu_\nu = \partial_{\partial_\mu\phi}\mathcal{L}\partial_\nu\phi - \delta^\mu_\nu\mathcal{L}$$

As Noether current, this is conserved, meaning $\partial_\mu T^\mu_\nu = 0$. We consequently get conserved quantities:

$$E = \int d^3x T^{00}$$

$$p^i = \int d^3x T^{0i}$$

or equivalently, the 4-momentum is conserved:

$$p^\nu = \int d^3x T^{0\nu}.$$

[Lecturer computes example in Klein-Gordon case.] In many cases, such as the Klein-Gordon case we find the energy-momentum tensor to be symmetric, but this need not be the case in general. However, just as the Lagrangian gives rise to the same physics if we add a total derivative, one can show that the energy-momentum tensor has a similar symmetry, and if a particular choice of total derivative is made, we find that we can always make $T^{\mu\nu}$ symmetric.

More specifically, we find that if we use the formula from general relativity for the energy-momentum tensor:

$$T_{\mu\nu}(x) = \frac{-2}{\sqrt{-g}} \partial_{g^{\mu\nu}} (\sqrt{-g}\tilde{\mathcal{L}})$$

for $g = \det(g_{\mu\nu})$ then we always get a symmetric result. By using the Minkowski metric we can recover our original result.

3 Canonical Quantisation

In the Advanced Quantum Field Theory course in Lent, we will use the Lagrangian directly to formulate the path integral version of quantum mechanics. Here, however, we will stick to the Hamiltonian approach. For that we do a small review of Hamiltonian mechanics by starting with the Lagrangian

$$L(q, \dot{q}) = \frac{1}{2}\dot{q}^2 - V(q)$$

We then define the **momentum conjugate to** q to be $p = \partial_{\dot{q}} L$. Performing the Legendre transform we then can define the **Hamiltonian** to be

$$H = p\dot{q} - \mathcal{L} = \frac{1}{2}p^2 + V(q) = H(p, q)$$

since H is really seen as a functino of p and q. It is also clear that Hamiltonian can be seen as the total energy of a point in state space (p, q).

Now, we generalise our system to N particles as

$$H = \sum_{i=1}^{N} p_i q_i - \mathcal{L}$$

and introduce the **Poisson bracket** (which is related to the commutator) as

$$\{F, G\} = \sum_{i=1}^{N} \partial_{q_i} F \partial_{p_i} G - \partial_{p_i} F \partial_{q_i} G$$

for functions F, G of p, q. We can then get the following important result: the Hamiltonian is the generator of time evolution, meaning that $\forall F(p, q)$ we find

$$\dot{F} = \{H, F\}.$$

An important special case of this is that we can write the principle of least action by applying the time evolution property to p and q asking

$$\begin{aligned} \dot{q}_i &= \{H, q_i\} = \partial_{p_i} H \\ \dot{p}_i &= -\partial_{q_i} H \end{aligned}$$

and for any conserved Q, we get $\{H, Q\} = 0$. Finally, we note that these also satisfy the property that

$$\{q_i, p_j\} = \delta_{ij}$$

[End of lecture 5]

How do translate this Hamiltonian to field theory? Firstly we define the **momentum conjugate to the field** as

$$\pi = \partial_{\partial_{\dot{\phi}}} \mathcal{L}$$

and we apply the Legendre transform to get the **Hamiltonian density**

$$\mathcal{H} = \pi \dot{\phi} - \mathcal{L} = \mathcal{H}(\phi, \pi)$$

and, as expected the normal Hamiltonian can then be calculated as

$$H = \int d^3x \mathcal{H}.$$

As a check, and example, if we use $\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi)$ (a "suitably" general Lagrangian) we get

$$\mathcal{H} = \frac{1}{2} \dot{\phi}^2 + \frac{1}{2} |\nabla \phi|^2 + V(\phi) = T^{00}$$

the energy density, as one might hope.

3.1 The Canonical Quantisation

Now to actually quantise things we convert the Poisson bracket {} to the commutator $\frac{1}{i\hbar}[]$, with overall quantities staying the same. That means that for π, ϕ we get that

$$[\pi_i, \pi_j] = [\phi_i, \phi_j] = 0$$

but that

$$[\pi_i, \phi_j] = i\hbar\delta(x - y),$$

as one might expect. The time evolution property of H becomes

$$H |\psi\rangle = -i\hbar\partial_t |\psi\rangle .$$

In particular that means that for any conserved quantity (commutes with H), we can simultaneously diagonalise these to get a simultaneous basis H and the conserved quantity. Also, note here that we later set $\hbar = 1$.

Anyways, using the standard Hamiltonian approach we then find that

$$H = \int d^3x \frac{1}{2}\pi^2 + \frac{1}{2}|\nabla\phi|^2 + V(\phi)$$

In practice, this is very hard to calculate, so not very helpful (one would have to diagonalise this function to find the energy eigenstates...). Instead we find some other ways.

Finally, on interpretation, we note that the eigenvalues have become functions $f : \mathbb{R}^3 \to \mathbb{R}$, which we assume span the Hilbert space as usual. However, this has never been formalised mathematically, so it is a bit of a grey area... [End of lecture 6]

In general, solving this Hamiltonian works out to be a functional differential equation, which is predictably hard. Instead, we tend to work with special cases. That is how we will proceed.

3.2 Free Field Theory

The simplest case is to assume everything is non-interacting, and that the Lagrangian obeys the Klein-Gordon equation. That means we are left with

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 \implies \partial_\mu\partial^\mu\phi + m^2\phi = 0$$

We can solve this using a Fourier transform to get

$$(\partial_t^2 + |p|^2 + m^2)\tilde{(\phi)} = 0$$

which has solutions for $\omega_p = \sqrt{|p|^2 + m^2}$ of the form

$$\tilde{\phi} = A_p e^{-i\omega_p t} + B_p e^{-i\omega_p t}$$

which form the modes of a Harmonic oscillator. In fact, calculating the action we find

$$S = \frac{1}{2}\int dt \int \frac{d^3p}{(2\pi)^2}\tilde{\phi}^*(-\partial_t^2 - |p|^2 - m^2)\tilde{\phi}$$

which means really do get a infinite number of decoupled complex simple harmonic oscillators. This is essential, since it is this simplicity locally that gives us a chance at solving the problem at all!

3.2.1 The Quantum Simple Harmonic Oscillator (review)

Firstly, though, let's review the quantum simple harmonic oscillator (SHO) so that we can use it for comparison. In this case

$$L = \frac{1}{2}\dot{q}^2 - \frac{\omega^2}{2}q^2 \implies H = \frac{p^2}{2} + \frac{\omega^2}{2}q^2$$

We can quantise this as usual for p, q, and then solve it using general methods. The unique aspect of the SHO though is that we can use purely its algebra to gain a solution by considering the raising and lowering/ladder operators

$$a = \sqrt{\frac{\omega}{2}}q + \frac{i}{\sqrt{2\omega}}p, a^\dagger = \sqrt{\frac{\omega}{2}}q - \frac{i}{\sqrt{2\omega}}p$$

where we find $[a, a^\dagger] = \hbar$, which crucially means that

$$H = \frac{1}{2}\omega(aa^\dagger + a^\dagger a) = \omega(a^\dagger a + \frac{1}{2}\hbar)$$

where one can identify $N = a^\dagger a$ as the number operator. Even more importantly we see that

$$[H, a] = -\omega\hbar a, [H, a^\dagger] = \omega\hbar a^\hbar$$

The fact that the commutator is still proportional to $a, a^\dagger$ means that when applied to an energy eigenstate, $a, a^\dagger$ either annihilate the state, keep the same state, or create a new state. As such, we create the ladder operators as usual, and we find a state $|0\rangle$ such that $a|0\rangle = 0$, but we can define an inifite ladder of states $|n\rangle = (a^\dagger)^n|0\rangle$. The energies of these states are given by $E_n = \hbar\omega(n + \frac{1}{2})$.

3.2.2 Back to Field Theory

To generalise to field theory we might define our ladder operators as:

$$\phi = \int \frac{d^3p}{(2\pi)^2}\frac{1}{\sqrt{2\omega_p}}\left(a_p e^{ip\cdot x} + a_p^\dagger e^{-ip\cdot x}\right)$$

$$\pi = \int \frac{d^3p}{(2\pi)^2}(-i)\sqrt{\frac{\omega_p}{2}}\left(a_p e^{ip\cdot x} - a_p^\dagger e^{-ip\cdot x}\right)$$

If we gain the same commutation relations, from these definitions then we can call ourselves satisfied that these definitions have the properties what we want. As such, we wish to show that the following commutation relations

$$[\phi(x), \phi(y)] = [\pi(x), \pi(y)] = 0, [\phi(x), \pi(y)] = i\delta(x - y)$$

if and only ifoddpage

$$[a_p, a_q] = [a_p^\dagger, a_q^d agger] = 0, [a_p.a_q^\dagger] = i\delta(x - y)$$

Some algebra shows that indeed this is the case.

Having shown that these definitions can make sense, we want to see if we can calculate the energy eigenstates from here, as we do in the non-field theory case. As such we calculate

$$H = \frac{1}{2} \int d^3x(\pi^2 + |\nabla p|^2 + m^2\phi^2)$$

by writing every term in terms of $a, a^\dagger$ and find that as expected

$$H = \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \omega_p(a_p a_p^\dagger + a_p^\dagger a_p)$$

[End of lecture 7] This calculation can be simplified by creating and eliminating delta functions as soon as possible. So our first step here would be to calculate the ground state energy, which in the context of field theory is called the "vacuum" (or the energy of the vacuum). Doing so for $|0\rangle$ such that $a\,|0\rangle = 0$ we find that

$$H\,|0\rangle = \frac{1}{2} \int d^3p\omega_p\delta(0)\,|0\rangle = \infty\,|0\rangle$$

which points us to a general aspect of field theory: many things diverge. How do we get around this? There are two types of divergence present here, which are called **infrared divergence** (IR divergence) and **ultraviolet divergence** (UV divergence). IR divergence refers to divergence that occurs due to the large distances (but possibly low energies) involved. Ultraviolet divergence on the other hand occurs due to the high energies (but often short distances) involved. The infrared divergence is relatively simple to solve: we consider the energy density $\mathcal{E}$ instead of the total energy, which makes sense since almost everything else we're working with is a density. Formally this can be expressed as restricting the energy to a box of limited size.

That solves the IR divergence, leaving us with

$$\mathcal{E}_0 = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2}\omega_p \sim \int |p|^3 d|p| = \infty$$

(where exactly does this come from?), but we still have UV divergence. To resolve this, we use a rather crude UV cutoff, which is an energy cutoff, limiting our integrals to only energies lower than $\Lambda >> 1/m$ where $1/m$ is the only natural length scale available to us in this context. Calculating $\mathcal{E}_0$ in this limit for Λ gives us

$$\mathcal{E}_0^\Lambda = \frac{1}{16\pi^2}\Lambda^4(1 + O(m^2/\Lambda^2)).$$

An alternative, and more intuitive approach would be to replace continuous spacetime with a lattice. This does work, and gives a similar result in fact, but is much harder to do. The equivalence between this relates to a somewhat profound relationship between small distances and high energies, and in fact the lattice spacing a is inversely proportional to our energy cutoff in a certain sense.

Some interpretations can be offered for the situations we get here. One involves the fact that we don't really know what happens at high energies, and so we focus on where we know our field theory works (referred to as **effective field theory**). Another approach is that the method that we quantise our theory is more or less arbitrary and so we could instead use a different quantisation called the "natural ordering" which forces all terms involving $a, a^\dagger$ to put as in front, leaving $\mathcal{E}_0 = 0$. I don't know how well this works in general.

Finally, another approach to this issue is simply not to care, since why should inifinite energies really make a difference (closest to my attitude to the situation). After all, we can only really measure energy differences in experiment. That's not entirely true though, since once one starts to incorporate gravity, one sees that through the energy-momentum tensor relativity does introduce a certain absolute measure of energy... This relation remains a relatively mysterious aspect of physics. [End of lecture 8]

To give some more detail on this relationship, we notice that the Einstein equations $R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu}$ mean that if we have constant vacuum energy $\langle 0| T_{\mu\nu} |0\rangle = \mathcal{E}_0 g_{\mu\nu}$ essentially is equivalent to introducing a new cosmological constant λ. This was significant, especially since until around 20 years ago (2000) the cosmological constant was assumed to be 0. In truth though it remains one of the greatest mysteries in physics since $\lambda \sim \mathcal{E}_0 G \sim (10^{-3}\text{eV})^2$, while if instead we assume that there is an energy cutoff around the Planck length $\Lambda \sim M_L = \sqrt{\frac{\hbar c}{G}}$ then we find $\lambda \sim 10^{28}\text{eV}$. So we really don't know what we're doing when it comes to the cosmological constant.

Another approach is that some argue that the vacuum energy is something that is an input to the theory, and that it should be fitted instead of fixed by some prediction, but of course, that unsatisfactory to some people. One can also wonder whether it really has much meaning, since - excluding relativity - there are no obvious implications of it. There is in fact, however, a place where even ignoring general relativity, quantum field theory has very physical effects, although this only depends on local changes in the vacuum energy - unlike the relativity.

3.2.3 Casimir Effect

Here we consider two parallel conducting plates each of area A separated by distance d. We assume that boundary requirements mean that a scalar field (sufficient for electromagnetism) on conducting plates mean that the field $\phi = 0$ on these plates. In particular, this restricts the momentum to $p = (n\pi/d, p_y, p_z)$ so one of the components has been discretised. If we then calculate the change in energy compared to the vacuum between the plates $\mathcal{E}_0 - \tilde{\mathcal{E}}_0$ where

$$\mathcal{E}_0 = \int \frac{d^3p}{(2\pi)^2}\frac{1}{2}\omega_p$$

$$\tilde{\mathcal{E}}_0 = \frac{1}{d}\sum_{n=1}^{\infty}\int \frac{dp_y dp_z}{(2\pi)^2}\frac{1}{2}\sqrt{(n\pi/d)^2 + p_y^2 + p_z^2}$$

then we find that the energy shift $\Delta E(d) = (\tilde{\mathcal{E}}_0 - \mathcal{E}_0)Ad$ is nontrivial (see David Tong's notes for more detail), and in particular, an energy change means a force, which means a pressure, meaning that $p = -\partial_d \Delta E(d)/A = \frac{-\pi^2}{480d^4}$ can be calculated (note that d^4 is the right scaling constant for pressures).

This can be measured in experiment, and turns out to agree well with this calculation, giving some extra weight to the notion of at least some vacuum energy, although admittedly, this only detects changes in it.

3.2.4 Excited States

Let's denote a state with "momentum" p by $|p\rangle = a_p^\dagger |0\rangle$. We want to calculate its energy $H - E_0$. Since this would be 0 for the vacuum, this is equivalent to calculating the energy under normal ordering, meaning that we can write this as

$$H - E_0 = \int \frac{d^3q}{(2\pi)^3}\omega_q a_q^\dagger a_q$$

Using commutation relations $[a_q^\dagger a_q, a_p^\dagger] = a_q^\dagger[a_q, a_p^\dagger] = (2\pi)^3\delta(p-q)a_q^\dagger$, meaning

$$[H - E_0, a_p^\dagger] = \omega_p a_p^\dagger$$

which is exactly what we require of a ladder operator. Using the standard procedure to calculate energies from ladder operators we then find that

$$(H - E_0)ketp = \omega_p a_p^\dagger |0\rangle$$

so the energy of a state of this kind is $E_p = \omega_p = \sqrt{|p|^2 + m^2}$. This corresponds exactly to the energy of a particle with momentum p in standard quantum theory. To confirm it really does resemble that though, we should calculate the "momentum" of the state to verify that it truly is p. This is indeed the case, as is seen in Example Sheet 2, Question 1. Combining the number "density"-like operator with the momentum integral, we can then write the momentum operator as

$$\hat{p} = \int \frac{d^3p}{(2\pi)^3} p a_p^\dagger a_p$$

which certainly has the property that $\hat{p}\,|p\rangle = p\,|p\rangle$. Finally, to really check it is a particle, one might want to verify that it has 0 intrinsic angular momentum, since this theory so far deals with scalar particles with no spin. This is indeed the case, as one can find that $\langle 0|\, J\, |0\rangle = 0$.

3.2.5 Multiparticle States

To expand our notation we may write $|p_1 \dots p_n\rangle = a_{p_1}^\dagger \dots a_{p_n}^\dagger\, |0\rangle$. We can consequently check that the energy and momentum are additive as one might hope (note that energies as such are only additive in free field theory which is non-interactive).

In this particular scenario we may also formulise the notion of the number operator as

$$N = \int \frac{d^3p}{(2\pi)^3} a_p^\dagger a_p$$

which indeed gives us the total number of particles in the system. It also is a true quantum number in free field theory meaning that it commutes with H or is conserved.

Finally, we might wonder if we truly get identical particles. It is easy to see that

$$|p_1 p_2\rangle = |p_2 p_1\rangle$$

meaning that we indeed get the bosonic behaviour expected for scalar particles (in scalar field theory).

3.2.6 Normalisation

We can simply define $\langle 0|0\rangle = 1$, but getting all states to normalise properly after arbitrary ladder operators is a bit trickier. As such, we might calculate

$$\langle p|q\rangle = \langle 0|\, a_p a_q^\dagger\, |0\rangle = \langle 0|\, [a_p, a_q^\dagger]\, |0\rangle = (2\pi)^3 \delta(p - q)$$

but the issue with this RHS is that it is not Lorentz invariant, since it uses a 3-vector instead of a 4-vector. We can fix this with a completeness operators

$$\mathbb{1} = \int \frac{d^3p}{(2\pi)^3}\, |p\rangle \langle p|$$

where the LHS is manifestly invariant, but the RHS is a bit tricker [End of lecture 9]. (What is a completeness relation?)

To establish equivalence, we consider the Lorentz invariant measure

$$\int d\mu_p = \int d^4p\delta(p_\mu p^\mu - m^2)\Theta(p_0)$$

where Θ is the Heaviside step function. This implicitly imposes the constraint that $p_\mu p^\mu - m^2$ and $p_0 > 0$ which equivalent to the condition that $p_0 = \sqrt{|p|^2 + m^2}$ as we require. In particular, we may write this

$$\int dp_\mu = \int d^3p \int_0^\infty \frac{d(p_0)^2}{2p_0}\delta(p_0^2 - |p|^2 - m^2) = \int \frac{d^3p}{2E_p}$$

So if we can somehow squeeze in a $2E_p$ factor, we'd be fine. Hence we redefine our states as

$$|p\rangle = \sqrt{2E_p} a_p^\dagger |p\rangle$$

meaning from before we get

$$\mathbb{1} = \int \frac{d^3p}{(2\pi)^3} \frac{\langle p|p\rangle}{2E_p}$$

is indeed Lorentz invariant. We can similarly generalise our definition of $|p_1 \dots p_n\rangle$ by multiplying by $\prod \sqrt{2E_{p_i}}$.

3.2.7 Complex Scalar Fields

(see David Tong page 33) here we consider field $\psi : \mathbb{R}^{3,1} \to mathbbC$ with Lagrangiange

$$\mathcal{L} = \partial_\mu\psi^*\partial^\mu\psi - M^2\psi^*\psi$$

Now, if one writes $\psi = (\phi_1 + i\phi_2)/\sqrt{2}$ one can convert this to two scalar fields, but there is some additional insight we can gain by treating it purely as a complex field. Importantly, just as seen on example sheet 1, we have global symmetry $\psi \mapsto e^{-i\alpha}\psi$ which has conserved current

$$j^\mu = i(\partial^\mu\psi^*)\psi - i\psi^*(\partial^\mu\psi)$$

and the conserved quantity $Q = \int d^3xj^0$. Here this corresponds to the probability current, and the total probability in standard theory.

Developing further to find the Hamiltonian we see that the conjugate momentum is

$$\pi = \partial_{\dot\psi}\mathcal{L} = \psi^*$$

and quantising we get the standard commutation relations with in particular that

$$[\psi(x), \pi(y)] = i\delta(x - y)$$

$$[\psi^\dagger(x), \pi^\dagger(y)] = i\delta(x-y)$$

(see the latter by taking the Hermitian conjugate of the first). By taking a mode expansion, and repeating the approach we employed earlier we find that

$$\psi = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} (b_p e^{ip\cdot x} + c_p e^{-ip\cdot x})$$

for two different creation operators b_p, c_p. Writing the corresponding states as $|p,+\rangle, |p,-\rangle$ for the states created by b_p, c_p respectively we find that

$$Q = i \int d^3x (\pi\psi - \psi^\dagger \pi^\dagger) = \cdots = \int \frac{d^3p}{(2\pi)^3} (c_p^\dagger c_p - b_p^\dagger b_p)$$

which corresponds to the number operators of each ladder operators. This quantity being conserved, we find this corresponds precisely to the notion of particles and anti-particles, and these always being created and annihilated in pairs.

3.2.8 Time Dependence

To complete our discussion of free field theory we'd like to examine time dependence. In particular, we take this opportunity to review the notions of the Schrödinger and Heisenberg pictures. The Schrödinger picture assumes that operators are constant in time, while states are not, and the Heisenberg picture does it the other way around. To translate between them we may use that

$$|\psi\rangle_H = e^{iHt} |\psi\rangle_S$$

for states, and so operators translate by the conjugate of that

$$O_H = e^{iHt} O_S e^{-iHt}.$$

The Schrödinger picture can be helpful since there the time evolution reduces to a simple differential equation

$$i\partial_t |\psi\rangle = H |\psi\rangle$$

whereas the Heisenberg picture can be helpful in expressing Lorentz invariance (the time dependence is no longer special) as we can write

$$\phi_H(x) = e^{iHT} \phi_S(\vec{x}) e^{-iHt}$$

where the Heisenberg version of the field operator uses the 4-vector x and the Schrödinger version uses the 3-vector $\vec{x}$ (although I often just write x). In particular, by expanding in modes as

$$\phi_H = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} (a_{p,H} e^{ip_\mu x^\mu} + a_{p,H}^\dagger e^{-ip_\mu x^\mu})$$

we find that

$$a_{p,H}(t) = e^{-iE_n t} a_{p,H}(0) = e^{-iE_n t} a_{p,H}$$

and the same for $a_{p,H}^{\dagger}$.

4 Interacting QFT

We now consider Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu \tilde{\phi} \partial^\mu p\tilde{h}i - V(\tilde{\phi})$$

where we assume, wlog, $\partial_{\tilde{\phi}} V|_{\tilde{\phi}_0} = 0$, and $\partial^2_{\tilde{\phi}} V|_{\tilde{\phi}_0} \geq 0$ and $V(\tilde{\phi}_0) = 0$ so that if $\phi = \tilde{\phi} - \tilde{\phi}_0$ and writing

$$V(\tilde{\phi}) = \sum_0^\infty \frac{1}{n!} \partial^n_{\tilde{\phi}} V|_{\tilde{\phi}_0} \phi^n = \frac{1}{2}\lambda_2 \phi^2 + \frac{1}{6}\lambda_3 \phi^3 + \dots$$

then we can write our Lagrangian as $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{int}$ where $\mathcal{L}_0$ is free where as $\mathcal{L}_I$ is interacting. Here $\mathcal{L}_0$ takes the Klein-Gordon form:

$$\mathcal{L}_0 = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2/2$$

where $\lambda_2 \geq 0$ means that m is real. The interacting bit, on the other hand is

$$\mathcal{L}_{int} = -\sum_3^\infty \lambda_n \phi^n / n!$$

Now, in this pertubation, it would be helpful to have a sense for when the λ_n are small. A simple, and effective approach is dimensional analysis. It is also wrong. Well, it's wrong after tree level (first order pertubations), but we only are interested in tree level, so this does not affect us here. Anyways, starting with $[S] = 0$ for the action (since $[\hbar] = [S] = 0$ after we set $\hbar = 1$), we can deduce that $[\phi] = 1$ and consequently $[\lambda_n] = 4 - n$. To normalise to a dimensionless pertubation parameter (which is certainly desirable) we use the energy E and define dimensionless pertunbation paraemeters

$$\tilde{\lambda_n} = \lambda_n E^{n-4}$$

however this depends on the energy scale. We consequently get three cases

- when $n < 4$ we get the **relevant coupling** which is weakly coupled at high energies, but strongly coupled at low energies. Fortunately, $E \geq m$ provides a lower bound on the energy, and so we say that if $\lambda_n << m^{4-n}$ our pertubation is good for all energies.

- we get the **marginal** coupling for $n = 4$ which is good as long as $\lambda_4 << 1$ for any energy.
- we get **irrelevant couplings** for $n > 4$ since these only work for small energies $\lambda_n << E^{4-n}$. This also means we cannot use the continuum limit here, although under certain interpretations we can partially circumvent this.

Despite its simplicity, this analysis is quite possible, and gives a simple explanation of why some solutions are not renormalisable, but more on that later.

Example 2. A key example will be ϕ^4-theory, or the marginal case. We start this as a running example here, and will expand below, but for reference, here we get

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - m^2\phi^2/2 - \lambda/4!\phi^4$$

Example 3. Scalar-Yukawa theory refers to a combination of two fields $\phi : \mathbb{R}^{3,1} \to \mathbb{R}, \psi : \mathbb{R}^{3,1} \to \mathbb{C}$ for masses m, M where in additional to the momentum, and mass terms, we also get a

$$g\psi^*\psi\phi$$

term in the expansion (a cubic term?). We are weakly coupled for $g << m, M$, and notice we use the complex norm squared to maintain our global phase shift symmetry for complex values.

4.1 Scattering

An obvious question of interest, since it can be verified by experiment, is that of how particles scatter. Now, unlike classical mechanics or the quantum mechanics that has been covered before, the number of particles entering the collision, and exiting is not constant here, which leads to extra complexity. Nonetheless, we seek to find

$$A_{i\to f} = \lim_{T\to\infty} \langle f, t = T/2| e^{-iHT} |i, t = T/2\rangle$$

where $|i\rangle, |f\rangle$ are the initial and final states of interest. Also, we assume that we can take $t = \pm\infty$ for $|i\rangle$ and $|f\rangle$ and that the free theory approximation holds in these cases. Then we write that

$$H = H_0 + H_{int}$$

Example 4. In ϕ^4 theory we have

$$H_0 = \frac{1}{2}\int d^3x(\pi^2 + |\nabla\phi|^2 + m^2\phi^2)$$

$$H_{int} = \frac{\lambda}{4!}\int d^3x\phi^4$$

for $\lambda << 1$. We can see that the particle number might change, as unlike the free field theory case we now get terms of the form $a^\dagger_{p_1} a^\dagger_{p_2} a^\dagger_{p_3} a_{p_4}$, which changes the overall particle number (here $p_1 + p_2 + p_3 = p_4$). Also, notice importantly now that many of our operators no longer commute, such as H_0, H_{int}, N.

Now, the key advantage of taking $t = \pm\infty$ is that we can assume that $|i\rangle, |f\rangle$ can be eigenstates of H_0. Now, our key to reasoning about scattering will be the **interaction picture**. The interaction picture is a weird mix of the Schrödinger and Heisenbergs picture where H_0 is kept as Heisenberg, but H_{int} is kept as Schrödinger. As such

$$|\psi\rangle_I = e^{iH_0 t} |\psi\rangle_S, O_I = e^{iH_0 t} O_S e^{-iH_0 t}$$

In particular, we use H_I the interaction picture version of H_{int} in this manner. Notice importantly that $[H_I(t), H_I(t')] \neq 0$. [End of lecture 11] As such we see that

$$i\partial_t |\psi\rangle_I = H_I |\psi\rangle_I$$

which is quite nice since we get rid of the free field noise, and can focus on what's "actually" happening. Assuming that $H_I = \lambda\mathbb{H}$ then we see that for$|\psi\rangle_I = U(t)|\psi(0)\rangle_I$

$$U = \sum_0^\infty \lambda^n \mathbb{K}_n(t)$$

where substituting the Schrödinger equation leaves

$$i\partial_t \mathbb{K}_n(t) = \mathbb{H}(t)\mathbb{K}_n(t)$$

so we find $\mathbb{K}_0 = \mathbb{1}$, $\mathbb{K}_t = \frac{1}{i}\int_0^t dt'\mathbb{H}(t')$, $\mathbb{K}_2(t) = \frac{1}{i^2}\int_0^t dt'\mathbb{H}(t')\int_0^{t'} dt''\mathbb{H}(t'')$, etc. As such we define the time ordering symbol T which replaces a list of operators $A_1(t_1)A_2(t_2)\dots$ with $A_{n_1}(t_{n_1})A_{n_2}(t_{n_2})\dots$ where $t_{n_1} \geq t_{n_2} \geq \dots$. By forcing this ordering symbol onto our expression we can write our solution in a nice form

$$|\psi\rangle_I = T(e^{\frac{1}{i}\int_0^t dt' H_I(t')})\,|\psi(0)\rangle_I$$

where the T is applied to each term in the power series. Using the time ordering symbol, verifying the validity of this formula is straigthforward.

4.1.1 Dyson Formula

The **Dyson formula** is the name of our solution to the scattering problem, and we explain it here. In particular we write

$$
\begin{aligned}
A_{i\to f} &= \lim_{T\to\infty} \langle f, t = T/2|_S \, e^{-iHt} \, |i, t = -T/2\rangle \\
&= \lim_{T\to\infty} \langle f, t = T/2|_S i, t = -T/2\rangle_S \\
&= \lim_{T\to\infty} \langle f, t = T/2|_S i, t = -T/2\rangle \\
&= \langle f| \, S \, |i\rangle
\end{aligned}
$$

where the **scattering operator** (S-matrix) S is

$$S = T(e^{\frac{1}{i}\int_{-\infty}^{\infty} H_I(t)dt})$$

This expression for $A_{i\to f}$ is called Dyson's formula.

Example 5. In ϕ^4-theory we find $H_I = \lambda/4! \int d^3x\phi_H^4$. And if we write $\phi = \phi_+ + \phi_-$ where

$$\phi_+ = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}} a_p e^{-p_\mu x^\mu}, \phi_- = \int \frac{d^3p}{(2\pi)s}\frac{1}{\sqrt{2E_p}} a_p^\dagger e^{ip_\mu x^\mu}$$

And so we get Dyson's formula

$$A_{i\to f} = \langle f| \, Te^{\frac{i\lambda}{4!}} \int d^4x\phi x$$

but we see we still have no choice but expanding in powers of λ here. [End of lecture 12]

As such we define

$$A_{i\to f}^{(n)} = \frac{1}{l!}\left(\frac{1}{i4!}\right)^l = \int d^4x_1 \cdots \int d^4x_l \, \langle f| \, T(\phi^4(x_1)\dots\phi^4(x_l)) \, |i\rangle$$

but how do we compute these. Computing these leads us eventually to certain diagrams expressing these calculations, which of course are Feynman diagrams. We start in the simplest case with $T_2 = T(\phi(x)\phi(y))$. Our goal to calculate these is to reduce them to normal form, where calculations simplify significantly. In other words, we have to convert T_2 from time ordered to normal form. Expanding T_2 into $\phi^\pm$ we find that in both the $x^0 \geq y^0$ and $x^0 < y^0$ case we have a single commutator

$$D(x-y) = [\phi^+(x), \phi^-(y)] = \int \frac{d^3pd^3q}{(2\pi)^6}\frac{1}{2\sqrt{E_pE_q}} e^{i(q\cdot y - p\cdot x)}[a_p, a_q^\dagger] = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p} e^{ip\cdot(x-y)}$$

where

$$T(\phi(x)\phi(y)) =: \phi(x)\phi(y) : + D(x-y)$$

for $x^0 > y^0$ and the order of x, y in D is swapped when $x^0 < y^0$. We generalise this to the **Feynman propogator** Δ_F. We claim we can calculate it as

Claim 1.

$$\Delta_F(x-y) = \int_{C_F} \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2} e^{-p_\mu (x-y)^\mu}$$

where C_F is the Feynman countour.

This is just the Fourier transform of our expression before, but it turns out to be more useful. Notice the above can be written instead as

$$\Delta_F(x-y) = \int \frac{d^3p}{(2\pi)^3} i e^{ip\cdot(x-y)} \int_{C_F} dp^0 \mathcal{I}(p^0)$$

where

$$\mathcal{I}(p^0) = \frac{e^{-p^0(x^0-y^0)}}{(p^0)^2 - E_p^2}$$

for $E_p = \sqrt{|p|^2 + m^2}$. Using partial fractions we get poles $p^0 = \pm E_p$ with residues $\frac{\pm 1}{2E_p} e^{\mp i E_p (x^0 - y^0)}$. Diagramatically:

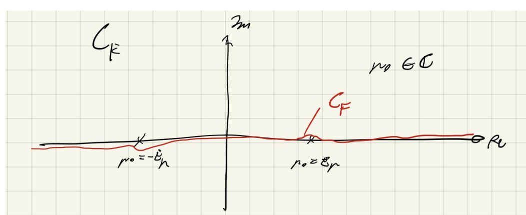

Figure 1: Feynman Contour

Now to prove our claim.

Proof. Notice that $|\mathcal{I}|$ depends only on $\mathrm{Im}(p^0)$ meaning that if $x^0 > y^0$ then we can complete the contour with a semicircle in the lower half plane of the complex plane and apply Jordan's theorem since we are decaying exponentially fast. In this case we only include the rightmost residue. On the contrary if $x^0 < y^0$ we decay in the upper half plane and can complete with a semicircle contour there to get the same result but now with the leftmost residue. $\square$

Alternatively, one can justify this instead calculating

$$\Delta_F(x-y) = \lim \epsilon \to 0^+ \int \frac{d^4p}{(2\pi)^3} \frac{i e^{-ip_\mu (x-y)^\mu}}{p^2 - m^2 + i\epsilon}$$

4.1.2 Wick's Theorem

Now, this is fine for T_2 but what about arbitrary l? Here Wick's Theorem comes into play. First let's define the **contraction** of ϕ_i, ϕ_j to be (for example in the case $i = 1, j = 3$) as $\phi_1\phi_2\phi_3\phi_4 \mapsto \Delta_F(x_1 - x_3)\phi_2\phi_4$ where $\phi_i = \phi(x_i)$. Then we define the symbol $\mathcal{C}(\phi_1 \ldots \phi_n)$ to denote the sum over all possible contractions (including all numbers of contractions - so no contractions at all is a possibility, and so is 5 different contractions), then

Theorem 2. Wick's Theorem states that

$$T(\phi_1 \ldots \phi_n) =: \phi_1 \ldots \phi_n : + \mathcal{C}(\phi_1 \ldots \phi_n)$$

[End of lecture 13]

Proof. We proceed by induction. If $n = 2$ then $T[\phi_1\phi_2] =: \phi_1\phi_2 : + \Delta_F(x_1 - x_2) =: \phi_1\phi_2 : + : (\phi_1\phi_2) :=: \mathcal{C}(\phi_1\phi_2) :$ (we use parentheses here in LateX to denote contraction). For the inductive step we write that if wlog $\forall i, x_1 \geq x_i$, then

$$\begin{aligned}
T(\phi_1 \ldots \phi_n) = \phi_1 T(\phi_2 \ldots \phi_n)& \\
&= (\phi_1^+ + \phi_1^-) : \mathcal{C}(\phi_2 \ldots \phi_n) \\
&=: \phi_1^- \mathcal{C}(\phi_2 \ldots \phi_n) : + : \phi_1^+ \mathcal{C}(\phi_2 \ldots \phi_n) : + : \sum_{i=2}^{n} [\phi_1^+, \phi_i] \phi_\in \ldots \phi_{\rangle - \infty} \phi_{\rangle + \infty} \ldots \phi_\backslash \\
&=: \phi_1 \mathcal{C}(\phi_2 \ldots \phi_n) : + \sum_{i=2}^{n} : \mathcal{C}((\phi_1)\phi_i)\phi_2 \ldots \phi_{i-1}\phi_{i+1} \ldots \phi_n : \\
&=: \mathcal{C}(\phi_1 \ldots \phi_n)
\end{aligned}$$

as required. Note here the we split the terms into a term where ϕ_1 is never contracted and a term where ϕ_1 is always contracted (with every ϕ_i). ☐

Bizarrily, it turns out that the following quantity, the transition rate from the vacuum to the vacuum, is nonzero, and this can be interpreted physically. Anyways, here we find that

$$\begin{aligned}
\langle 0 | S | 0 \rangle &= \langle 0 | T e^{\frac{1}{i} \frac{\lambda}{4!} \int d^4 x \phi(x)} | 0 \rangle \\
&= \sum \frac{1}{l!} \left(\frac{1}{i} \frac{\lambda}{4!} \right)^l \int dx_1^4 \cdots \int dx_l^4 M(x_1, \ldots, x_l)
\end{aligned}$$

For

$$M(x_1, \ldots, x_l) = \langle 0 | T(\phi_1^4 \ldots \phi_l^4) | 0 \rangle = \langle 0 | : \mathcal{C}(\phi_1^4 \ldots \phi_l^4) | 0 \rangle$$

The only surviving terms are the terms where everything is contracted (or else one of the vacuum states will be annihilated). The terms we then get can be represented diagramatically (Feynman diagrams), such as

$$\langle 0| (\phi_1\phi_1)(\phi_1(\phi_1\phi_2)(\phi_2\phi_2)\phi_2) |0\rangle$$

We can represent this by the following diagram:

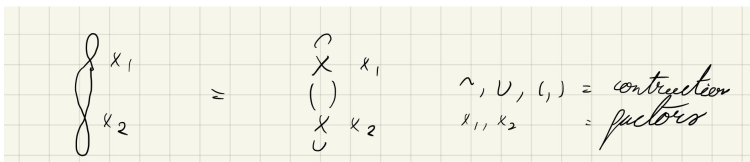

Figure 2: Simple Feynman Diagram

Here the number of vertices is l, and the number of connecting points at each vertex is the number of times ϕ_i occurs. The contractions are then given by the lines connecting the vertices.

4.1.3 Scattering in Scalar-Yukawa

We will work out one extended example of scattering in Scalar-Yukawa. Here we have Lagrangian

$$\mathcal{L} = \partial_\mu \psi^* \partial^\mu - M^2 \psi^* \psi + \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{M^2}{2}\phi^2 - g\phi\psi^*\psi$$

We get ladder operators

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}(a_p e^{-ip\cdot x} + a_p^\dagger e^{ip\cdot x})$$
$$\psi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}(b_p e^{-ip\cdot x} + c_p^\dagger e^{ip\cdot x})$$
$$\psi^\dagger(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}(b^\dagger e^{ip\cdot x} + c_p e^{-ip\cdot x})$$

We want to calculate the scattering amplitude between states

$$|i\rangle = \sqrt{2E_{p_1}}\sqrt{2E_{p_2}} b_{p_1}^+ b_{p_2}^+ |0\rangle$$
$$|f\rangle = \sqrt{2E_{p_1'}}\sqrt{2E_{p_2'}} b_{p_1'}^+ b_{p_2'}^+ |0\rangle$$

Now to get a non-zero transition we need at least the terms $b^2, b^{\dagger 2}$, which means we need to consider at least the second order term of the Dyson equation, or in order words we use

$$A_{i\to f} = \langle f| S |i\rangle = Te^{\frac{g}{i}\int d^4x\phi\psi^\dagger\psi} \approx A^{(2)}_{i\to f} = \frac{1}{2}\frac{g^2}{i^2}\int d^4x_1 \int d^4x_2 M(x_1,x_2)$$

where

$$\begin{aligned}
M(x_1,x_2) &= \langle f| T(\phi_1\psi_1^\dagger\psi_1\phi_2\psi_2^\dagger\psi_2) |i\rangle \\
&= \Delta_F^{(\phi)}(x_1-x_2) \langle f| : \psi_1^\dagger\psi_1\psi_2^\dagger\psi_2 |i\rangle \\
&= \Delta_F^{(\phi)} \langle f| \psi_1^\dagger\psi_2^\dagger\psi_1\psi_2 |i\rangle \\
&= \Delta_F^{(\phi)} N
\end{aligned}$$

since we care only for b terms, so we can ignore c, and we've contracted ϕ since nothing happens to those terms. [End of lecture 14] [The lecturer mentions that his notes extend Wick's theorem to $\mathbb{C}$].

Writing

$$N = \langle f| \psi_1^\dagger\psi_2^\dagger \mathbb{1} \psi_1\psi_2 |i\rangle$$

where $\mathbb{1} = \sum |\psi\rangle\langle\psi|$ means we can write

$$N = N_i N_f$$

where $N_i = \langle 0| \psi_1\psi_2 |f\rangle$, and similar for N_f. Then

$$N_i = \sqrt{2E_{p_1}}\sqrt{2E_{p_2}}\int\frac{d^3p}{(2\pi)^3}\int\frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\frac{1}{\sqrt{2E_q}}e^{-ip\cdot x_1 - iq\cdot x_2}\langle 0| b_p b_q b^\dagger_{p_1} b^\dagger_{p_2} |0\rangle$$

where we set $A = \langle 0| b_p b_q b^\dagger_{p_1} b^\dagger_{p_2} |0\rangle$ and $[b_p, b^\dagger_{p'}] = (2\pi)^3\delta(p-p')$ so

$$A = (2\pi)^6(\delta(p_1-p)\delta(p_2-q) + \delta(p_2-p)\delta(p_1-q))$$

(notice the apparent Bose symmetry). Consequently,

$$N_i = e^{-ip_1'\cdot x_1 + ip_2'\cdot x_2} + e^{ip_2'\cdot x_1 + ip_1\cdot x_2}$$

As such,

$$\begin{aligned}
A^{(2)}_{i\to f} &= \frac{1}{2}\frac{g^2}{i^2}\int d^4x_1\int d^4x_2 N_f N_i \int\frac{d^4k}{(2\pi)^4}\frac{ie^{ik(x_1-x_2)}}{k^2+m^2+i\epsilon} \\
&= \frac{g^2}{2i^2}\int\frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2+i\epsilon}\int d^4x_1\int d^4x_2\left(e^{ix_1\cdot(k-p_1'-p_1)}e^{ix_2\cdot(k-p_2'+p_2)} + e^{ix_1\cdot(k+p_2'-p_1)}e^{ix_2\cdot(k-p_1'+p_2)}\right)
\end{aligned}$$

where the two terms are related by swapping x_1, x_2. Doing the x_1, x_2 integral using $\int d^4x e^{ipx} = (2\pi)^4\delta(p)$ we get

$$A_{i\to f}^{(2)} = i(-ig)^2(2\pi)^4\delta(p_1+p_2-p_1'-p_2')\left(\frac{1}{(p_1-p_1')^2-m^2+i\epsilon}+\frac{1}{(p_1-p_1')^2-m^2+i\epsilon}\right)$$

Now, $(p_1-p_1')^2, (p_2-p_2')^2 < 0$ means that we can just set $\epsilon = 0$.

4.1.4 Feynman Diagrams

We can systemise the previous calculations through the use of Feynman diagrams. In particular, we see that terms of $\langle f|\, S-1\,|i\rangle$ correspond to Feynman diagrams.

In Feynman diagrams, we represent terms via lines (or **propagators**) and connect these according to the Hamiltonian. We also always specify a direction for time. So if time flows from left to right we get

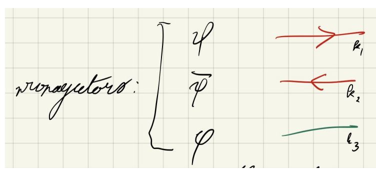

Figure 3: Feynman Diagram Propagators

meaning that in our previous example we get leading terms as

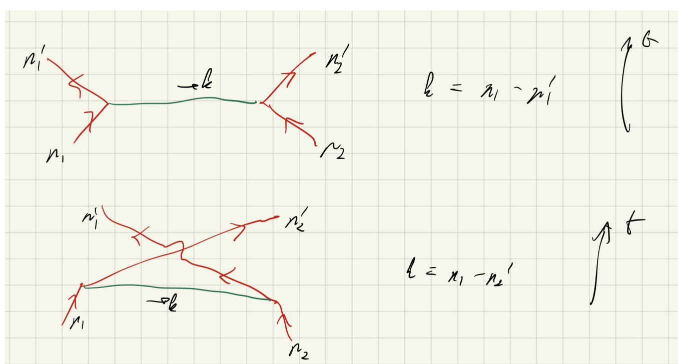

Figure 4: Feynman Diagram Leading Terms

Note that here every propagator has a momentum associated with it, and that particles/anti-particles have "directions" associated with them. Now to convert these to calculations we have the following correspondance:

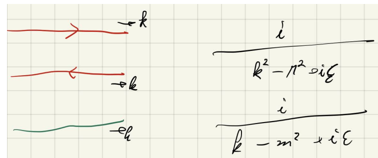

Figure 5: Feynman Propagator Meaning

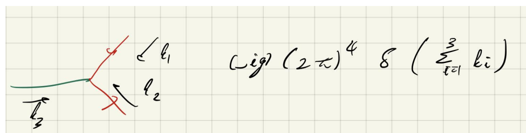

Figure 6: Feynman Vertex Meaning

Consequently, the top diagram we had correspondance to (to appropritae order)

$$A_{i\to f}|_{O(g^2)} = i(-ig)^2 \left(\frac{1}{(p_1 - p_1')^2 - m^2} + \frac{1}{(p_2 - p_2') - m^2} \right) (2\pi)^4 \delta(p_1 + p_2 - p_1' - p_2')$$

That completes the Feynman rules for this theory!

5 Fermions

We've now completed the scalar field theory of the course, and now want to move to scalar fields of spins $1/2, 1$. To do so, we now need to start considering representation theory. Now, what is "spin" here, just so we see how we might generalise. We can roughly define spin to be the angular momentum of a particular in its rest frame, $\langle 0| J |0\rangle$. Non-relativistically, we get $2s+1$ such spin states.

5.1 Relativistic Spin

To represent relativistic spins, it turns out we have to use representations of the Lorentz group. [End of lecture 15] For that, let's review how representation theory fits into quantum field theory. In particular, we move from a scalar field to a vector field, and we specify how it transforms via a representation D of the Lorentz group such that

$$\psi_A(x) \mapsto \psi_A'(x') = D(\Lambda)_{AB} \psi_B(\Lambda^{-1} x)$$

How can we get more representations? In general, we have a Lie group G, which is quite hard to describe, being on a manifold. Fortunately, we can greatly simplify our analysis by focusing on the Lie algebra $\mathbb{L}(G)$, which is linear approximation of the Lie group near the identity. We then can use the exponential map $\text{Exp} : \mathbb{L}(G) \to G$ to map back into the Lie group. This, however, is just a local bijection, but for most purposes this is enough. A representation of a Lie algebra is a map $R : \mathbb{L}(G) \to \text{Mat}_N(\mathbb{C})$ that is linear and preserves the Lie bracket

$$R([X,Y]) = [R(X), R(Y)].$$

where we can then recover the Lie group by using $D(\Lambda) = Exp(R(X))$. However, this is not a surjective map, and only gives us one representation of a large set of possible representations. More importantly, we dont actually get a representation itself, but rather of the cover of G, denoted $\tilde{G}$, just like us getting $SL(2, \mathcal{C})$ as a double-cover of $SO(3,1)$. This is important to know since we will actually be using representations of the cover, not the group itself.

As such we see that for proper Lorentz transformations $\Lambda = e^{\omega}$ we have ω antisymmetric (by considering on a linear level $\Lambda\eta\Lambda^T = \eta$). Consequently the Lie alegbra of the Lorentz transformations is just the set of antisymmetric 4×4 real matrices.

Antisymmetry means that treated as a linear space, this space has dimension $\frac{1}{2}4 \cdot 3 = 6$ (3 rotations and 3 boosts) which can express with the basis $M^{\rho\sigma} = (M^{\rho\omega})^{\mu\nu} = \eta^{\rho\mu}\eta^{\sigma\nu} - \eta^{\sigma\mu}\eta^{\rho\nu}$ where the generators are labelled such that $M^{\rho\sigma} = -M^{\sigma\rho}$. We can then write a general element of the Lie algebra $\mathbb{L}(G_L)$ is

$$\omega^{\mu}{}_{\nu} = \frac{1}{2}\Omega_{\rho\sigma}(M^{\rho\sigma})^{\mu}{}_{\nu}$$

where the parameters $\Omega_{\rho\sigma} = -\Omega_{\rho\sigma}$ (the symmetric part does not contribute). To finish showing that this truly is a representation of the Lie Algebra we can observe that

$$[M^{\rho\sigma}, M^{\tau\nu}] = \eta^{\sigma\tau}M^{\rho\nu} - \eta^{\rho\tau}M^{\sigma\nu} + \eta^{\rho\nu}M^{\sigma\tau} - \eta^{\sigma\nu}M^{\rho\tau}$$

[why?] A trick due to Dirac in organising representations of the Lorentz group is to start with the Clifford alegbra γ^{μ} for $\mu = 0, 1, 23$ with the property

$$\{\gamma^{\mu}, \gamma^{\nu}\} = 2\eta^{\mu\nu}I$$

This is satisfied by the Pauli matrices, however, there are only 3 of those, and it turns out the lowest-dimensional set of 4 matrices satisfying these properties are

$$\gamma^0 = \begin{pmatrix} & I \\ I & \end{pmatrix}$$
$$\gamma^i = \begin{pmatrix} & \sigma^i \\ -\sigma^i & \end{pmatrix}$$

The Lorentz Lie Algebra generators then arise (for reasons not discussed here) as

$$S^{\rho\sigma} = \frac{1}{4}[\gamma^\rho, \gamma^\sigma] = \frac{1}{2}\gamma^\rho\gamma^\sigma - \frac{1}{2}\eta^{\rho\sigma}$$

On example sheet 3, one shows that as required

$$[S^{\mu\nu}, S^{\rho\sigma}] = \eta^{\nu\rho}S^{\mu\sigma} - \eta^{\mu\rho}S^{\nu\sigma} + \eta^{\mu\sigma}S^{\nu\rho} - \eta^{\nu\sigma}S^{\mu\rho}$$

We can then exponentiate to get a representation of the double cover $SL(2, \mathbb{C})$.

$$\Lambda = \mathrm{Exp}(\frac{1}{2}\Omega_{\rho\sigma}M^{\rho\sigma}), S(\Lambda) = \mathrm{Exp}(\frac{1}{2}\Omega_{\rho\sigma}S^{\rho\sigma})$$

(we establish a representation of the Lorentz group by establish a simple correspondance between generators of the Lorentz group as a vector space and similar generators for the representation). (Ms generate the Lie algebra, Ss generate the Lie group) [End of lecture 16]

Now, for rotations we can write

$$\Omega_{ik} = -\epsilon_{ijk}a^k$$

and so the Spinor representation is given by (for $i \neq j$)

$$S^{ij} = \frac{1}{4}[\gamma^i, \gamma^j] = \frac{1}{2}\gamma^i\gamma^j = -i\frac{1}{2}\epsilon^{ijk}\begin{pmatrix} \sigma^k & \\ & \sigma^k \end{pmatrix}$$

in the Chiral representation. This corresponds to Lorentz transformation

$$S(\Lambda) = \mathrm{Exp}(\frac{1}{2}\Omega_{ij}S^{ij}) = \begin{pmatrix} e^{ia\cdot\sigma/2} & \\ & e^{ia\cdot\sigma/2} \end{pmatrix}$$

Consequently for a rotation around the z-axis where $a^1 = a^2 = 0$ we get

$$S(a^3) = S(\Lambda) = \begin{pmatrix} e^{ia^3/2} & & & \\ & e^{-ia^3/2} & & \\ & & e^{ia^3/2} & \\ & & & e^{-ia^3/2} \end{pmatrix}$$

for $\Lambda = \mathrm{Exp}(\frac{1}{2}\Omega_{ij}M^{ij})$. In this case we see that

$$\Lambda(a^3) = \mathrm{Exp}(-a^3M^{12})$$

and

$$M^{12} = \begin{pmatrix} 0 & & & \\ & 0 & -1 & \\ & 1 & 0 & \\ & & & 0 \end{pmatrix}$$

meaning that by pairing sines and cosines

$$\Lambda(a^3) = \begin{pmatrix} 0 & & \\ & M(a^3) & \\ & & 0 \end{pmatrix}$$

for standard rotation matrix

$$M(a^3) = \begin{pmatrix} \cos\left(a^3\right) & \sin\left(a^3\right) \\ -\sin\left(a^3\right) & \cos\left(a^3\right) \end{pmatrix}$$

Note here that for $a^3 = 2\pi$ we get $\Lambda(a^3) = I$. However, the spinors in this case correspond to

$$S(a^3 = 2\pi) = -I$$

confirming indeed that we have a representation of the double-cover and not the space itself. This in particular, is a characteristic of fermion representations.

5.2 Spinor Representations

Our Clifford algebra forms a "invariant tensor" in the sense that [what exactly is meant here]

$$S^{-1}(\Lambda)\gamma^\mu S(\Lambda) = \Lambda^\mu{}_\nu \gamma^\nu$$

We only verify this infinitesimally where we see that to first order

$$\begin{aligned}
\Lambda &= \text{Exp}\left(\frac{1}{2}\Omega_{\rho\sigma}M^{\rho\sigma}\right) \\
&\approx \left(S^\mu_{\ \nu} + \frac{1}{2}\Omega_{\rho\sigma}(M^{\rho\sigma})^\mu{}_\nu)\right)\gamma^\nu + O(\Omega^2) \\
&= \gamma^\nu + \frac{1}{2}\Omega_{\rho\sigma}(M^{\rho\sigma})^\mu{}_\nu\gamma^\nu + O(\Sigma) \\
&= \gamma^\mu + \frac{1}{2}\Omega_{\rho\sigma}(\eta^{\rho\mu}\gamma^\sigma - \eta^{\sigma\mu}\gamma^\rho)
\end{aligned}$$

And so on the LHS we find

$$\begin{aligned}
S(\Lambda) &= \text{Exp}\left(\frac{1}{2}\Omega_{\rho\sigma}S^{\rho\sigma}\right) \\
&\approx I - \frac{1}{2}\Sigma_{\rho\sigma}S^{\rho\sigma}\cdot\gamma^\mu\cdot\left(I + \frac{1}{2}\Omega_{\rho\sigma}S^{\rho\sigma}\right) + O(\Sigma^2) \\
&\approx \gamma^\mu + \frac{1}{2}\Omega_{\rho\sigma}[\gamma^\mu, S^{\rho\sigma}] + O(\Omega^2)
\end{aligned}$$

As such we get equality to linear order, or in other words, we get

$$[\gamma^\mu, S^{\rho\sigma}] = \eta^{\rho\sigma}\gamma^\sigma - \eta^{\sigma\mu}\gamma^\rho$$

5.3 Spinor Fields

So how do we construct fields (and eventually equations of motion) in this case. For field $\psi : \mathbb{R}^{3,1} \to \mathbb{C}^4$ we have transformation property

$$\psi^\alpha(x) \to \psi'^\alpha(x') = S(\Lambda)^\alpha_\beta \psi^\beta(\Lambda^{-1} \cdot x) = S(\Lambda) \cdot \psi(\Lambda^{-1} x)$$

which fortunately satisfies the key property that under a 2π rotation we have $\psi \mapsto -\psi$. This is all nice and well, but we need a Lorentz transformation that is Lorentz invariant, and it is not immediately obvious how we construct that from these terms that transform in a more complex manner. First we observe that

$$\partial_\mu \psi(x) \mapsto (\Lambda^{-1})^\nu{}_\mu S(\Lambda) \cdot \partial_\nu(\Lambda^{-1} \cdot x)$$

and

$$\psi^\dagger_\alpha(x) \mapsto \psi^\dagger \cdot S(\Lambda)^\dagger$$

Unlike before $S(\Lambda)$ is not always unitary, which means $\psi^\dagger \cdot \psi$ is no longer a scalar quanttiy. Furthermore, we have the theorem that

Theorem 3. A simple non-compact Lie group has no non-trivial finite dimensional unitary representation.

which severely restricts our options here. As such, notice first that for our Clifford algebra using the Chiral representation we get

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} I, (\gamma^0)^2 = I, (\gamma^i)^2 = -I, (\gamma^i)^\dagger = -\gamma^i, (\gamma^0)^\dagger = \gamma^0$$

Consequently we can see that

$$(\gamma^\mu)^\dagger = \gamma^0 \gamma^\mu \gamma^0$$

[End of lecture 17]
meaning that

$$\begin{aligned}
(S^{\mu\nu})^\dagger &= \frac{1}{4}([\gamma^\mu, \gamma^\nu])^\dagger \\
&= \frac{1}{4}[\gamma^0\gamma^\nu\gamma^0, \gamma^0\gamma^\mu\gamma^0] \\
&= -\frac{1}{4}\gamma^0[\gamma^\mu, \gamma^\nu]\gamma^0 \\
&= -\gamma^0 S^\mu \gamma^0
\end{aligned}$$

and

$$S(\Lambda)^\dagger = (\mathrm{Exp}(\frac{1}{2}\Omega_{\rho\sigma} S^{\rho\sigma}))^\dagger = \gamma^0 S(\Lambda)^{-1} \gamma^0$$

so while we don't get orthogonality, we get something like it.

That means that our spinor field $\psi^\alpha(x)$ transforms as

$$\psi^\dagger \mapsto \psi^\dagger(\Lambda^{-1})\gamma^0 S^{-1}(\Lambda)\gamma_0$$

As such $\psi^\dagger\psi$ is not scalar, however, if we define the **Dirac adjoint** to be $\bar{\psi}(x) = \psi^\dagger(x) \cdot \gamma^0$ we see that

$$\bar{\psi}(x) \mapsto \bar{\psi}'(x) = \bar{\psi}(\Lambda^{-1} \cdot x) \cdot S^{-1}(\Lambda)$$

which is a nicer rule and means that

$$\Sigma(x) = \bar{\psi}(x) \cdot \psi(x)$$

is indeed a scalar. Furthermore, one can also form a vector field as

$$V^\mu(x) = \bar{\psi}(x)\gamma^\mu\psi(x)$$

5.4 Dirac Action

From here we can finally state the **Dirac action** associated with non-scalar fields. We require the following conditions:

- it is Lorentz invariant
- it is real
- it is local, so action $S = \int d^4x\mathcal{L}(x)$ for scalar real field $\mathcal{L}$

We consequently get

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 = \bar{\psi}(x) \cdot i\gamma^\mu\partial_\mu \cdot \psi(x) - m\bar{\psi} \cdot \psi = \bar{\psi}(x) \cdot (i\gamma^\mu\partial_\mu - m)\psi(x)$$

where $\mathcal{L}_2$ is the mass term. Notice here that $\mathcal{L}_1$ is real (take the Hermitian conjugate) and scalar (can prove - done in handwritten notes - this uses $S(\Lambda)^{-1} \cdot \gamma^\mu \cdot S(\Lambda)$).

So what do the equations of motion — the so-called **Dirac equation** — look like here? Setting $\delta S = 0$ as usual we find that

$$(i\gamma^\mu\partial_\mu - m) \cdot \psi(x) = 0$$

with conjugate equation

$$i(\partial_\mu\bar{\psi})\gamma^\mu + m\bar{\gamma} = 0$$

We often use the notation here that $\not{V} = \gamma^\mu V_\mu$. Doing so we can write the Dirac equation as

$$(i\not{\partial} - m) \cdot \psi = 0$$

For our spinors

$$\Lambda = \text{Exp}(\frac{1}{2}\Omega_{\rho\sigma}M^{\rho\sigma})$$
$$S(\Lambda) = \text{Exp}(\frac{1}{2}\Omega_{\rho\sigma}M^{\rho\sigma})$$

with rotations given by $\Omega_{ij} = -\epsilon ijka^k, \Omega_{i0} = 0$ (anti-symmetric) and boost given by $\Omega_{i0} = -\Omega_{0i} = \chi_i, \Omega_{ij} = 0$. We covered rotations before, so focusing on boosts now we find that

$$S^{0i} = \frac{1}{4}[\gamma^0, \gamma^i] = \frac{1}{2}\begin{pmatrix} -\sigma^i & \\ & \sigma^i \end{pmatrix}$$

so

$$S(\chi) = S(\Lambda) = \begin{pmatrix} e^{\chi\cdot\sigma/2} & \\ & e^{-\chi\cdot\sigma/2} \end{pmatrix}$$

meaning both rotations and boosts result in block diagonal matrices, meaning that these represntations are reducible. To do so, it is convenient to define

$$\gamma^5 = -i\gamma^0\gamma^1\gamma^2\gamma^3$$

which in the Chiral representation corresponds to

$$\gamma^5 = \begin{pmatrix} I_2 & \\ & -I_2 \end{pmatrix}$$

which satisfies $\{\gamma^\mu, \gamma^5\} = 0, (\gamma^5)^2 = I$. To reduce this representation we can use the projection operators

$$P_\pm = \frac{1}{2}(I \pm \gamma^5), P_+P_- = 0$$

leaving

$$\psi_\pm = P_\pm\psi = \begin{pmatrix} U_+ \\ \end{pmatrix}, \begin{pmatrix} \\ U_- \end{pmatrix}$$

for **Weyl spinors** $U_\pm$. [End of lecture 18]

5.5 Lorentz Group Representations

For the Lorentz group $G_L = SO(3,1)_+$ where the + denotes the proper orthochronousness of matrices, and for the cover $\overline{G}_L = SL(2,\mathbb{C}) \equiv Spin(3,1)_+$. As discussed we use the Lie alegbra to study these Lie groups, but since the Lie algebra depends only on the behaviour of the group in the neighbourhood of the identity, and these groups differ only by the number of connected components we see that

$$\mathbb{L}(SL(2,\mathbb{C})) = \mathbb{L}(SO(3,1)) = \mathbb{L}(SO(3,1)_+)$$

Also, note that

$$\mathbb{L}_{\mathbb{C}}(SO(2,1)) \equiv \mathbb{L}_{\mathbb{C}}(SO(4))$$

and from a theorem classifying Lie algebras we know that

$$\mathbb{L}(SO(4)) = \mathbb{L}(SU(2)) \oplus \mathbb{L}(SU(2))$$

Now, all finite dimensional (irreducible?) representations of $\mathbb{L}(SU(2))$ are given by $j = 0, 1/2, \ldots$ and some $-j \leq j_z \leq j$ such that the representation R_j has dimension $2j+1$. Consequently, by the correspondance given above we know that finite dimensional irreducible representations of $\mathbb{L}(SO(3,1))$ are given by pairs (j_+, j_-) for $j_+, j_- = 0, 1/2, \ldots$ with dimension $(2j_+ + 1)(2j_- + 1)$. We can then classify the smallest irreducible representations to be

- when $j_+ = j_- = 0$ we get the scalar (trivial) representation of dimension 1.
- $j_+ = 1/2, j_- = 0$ we get the left Weyl spinor.
- $j_+ = 0, j_- = 1/2$ we get the right Weyl spinor,
- and for $j_+ = j_- = 1/2$ we get the 4-vector represntation from above (?)

Note that the Dirac spinor from before is reducible (into the right and left spinors as $S \equiv (1/2, 0) \oplus (0, 1/2)$).

5.6 Soultions of the Dirac Equation

We want to quantise our theory, but as before, in order to do that, we first need to find the form of the solutions to our equations of motion (after all, our quantisation really just corresponds to the superposition of solutions to our equation of motion). For the equation

$$(i\gamma^\mu \partial_\mu - mI) \cdot \psi = 0$$

we see that if we apply $-i\gamma^\nu \partial_\nu - mI$ to the LHS we get the wave equation

$$(\gamma^\mu \gamma^\nu \partial_\nu \partial_\mu + m^2 I) \cdot \psi = (\partial_m u \partial^\mu + m^2) \cdot I \cdot \psi = 0$$

since $\gamma^\mu \gamma^\nu \partial_\nu \partial_\mu = \frac{1}{2}\{\gamma^\mu, \gamma^\nu\} \partial_\nu \partial_\mu = \eta^{\mu\nu} I$. As such each component of the vector field solves the Klein Gordon equation. So

$$\psi \sim e^{\pm ip \cdot x}$$

where $p^\mu p_\mu = m^2, p^0 = E_p = \sqrt{p^2 + m^2}, p^\mu = (E_p, \vec{p})$. Now it turns out we get slightly different forms for positive and negative frequencies, as such we start with positive frequency solutions

$$\psi_\alpha(x) = u_\alpha(p)e^{-ip\cdot x}$$

Also, the Dirac equation is actually stronger than the Klein Gordon equation, so applying that we see that

$$(p^\mu\gamma_\mu - mI)\cdot(p) = 0$$

Now, applying a Lorentz boost to translate the solution, we start wlog with $p^\mu = (m, 0)$ leaving us with

$$m(\gamma^0 - I)\cdot u = 0$$

which in the Chiral representation means that

$$m\begin{pmatrix} -I_2 & I_2 \\ I_2 & -I_2 \end{pmatrix} u(p_0) = 0$$

meaning

$$u(p_0) = \sqrt{m}\begin{pmatrix} \xi \\ \xi \end{pmatrix}, \xi \in \mathbb{C}^2$$

So, if we take a boost of rapidity η in direction n to leave the rest frame, which corresponds to the Lorentz transformation

$$\Lambda(\chi) \in G_L, \chi = \eta n$$

such that

$$\Lambda(\chi)(m, 0) = (E_p, \vec{p}) = (m\cosh(\eta), m\sinh(\eta)n)$$

so

$$u(p)S(\chi)u(p_0) = \sqrt{m}\begin{pmatrix} e^{\chi\cdot\sigma/2} & \\ & e^{-\chi\cdot\sigma/2} \end{pmatrix}\begin{pmatrix} \xi \\ \xi \end{pmatrix} = \sqrt{m}\begin{pmatrix} e^{\chi\cdot\sigma/2}\xi \\ e^{-\chi\cdot\sigma/2}\xi \end{pmatrix}$$

by expanding in even powers and noticing that

$$(\chi\cdot\sigma)^2 = \chi^i\chi^j\sigma^i\sigma^j = \frac{1}{2}\chi^i\chi^j\{\sigma^i, \sigma^j\} = \chi^2 I = \eta^2 I$$

so

$$e^{\pm\chi\pm\sigma} = \cosh(\eta)I_2 \pm \sinh(\eta)(n\cdot\sigma)$$

Since these matrices are symmetric so diagonalisable, and defining $e^{\chi\cdot\sigma/2}$ to take the square root of the eigenvalues of the previous matrix, but preserving the eigenvectors and setting $\sigma^\mu = (I, \sigma), \bar{\sigma}^\mu = (I, -\sigma)$ we then get that

$$me^{\chi\cdot\sigma} = \sigma^\mu p_\mu, me^{-\chi\cdot\sigma} = \bar{\sigma}^\mu p_\mu$$

and so we can use the notation

$$\sqrt{p\cdot\sigma} = \sqrt{m}e^{\chi\cdot\sigma/2}, \sqrt{p\cdot\bar{\sigma}} = \sqrt{m}e^{-\chi\cdot\sigma/2}$$

($M = \sum_\mu p^\mu \sigma^\mu$ is symmetric). As such for positive (nonnegative frequency) we can write our solution as

$$\psi(x) = u(p)e^{-p\cdot x}, u(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\xi \\ \sqrt{p\cdot\bar{\sigma}}\xi \end{pmatrix}$$

where ξ is spanned by the orthonormal basis vectors ξ^s for $s = 1, 2$ form a basis for $\mathbb{C}^2$, and so we only get two linearly independent parts

$$u^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\xi^s \\ \sqrt{p\cdot\bar{\sigma}}\xi \end{pmatrix}$$

Similarly for negative frequencies, $\psi = v(p)e^{ip\cdot x}$ we can repeat the same process, except now we find that

$$m\begin{pmatrix} I_2 & I_2 \\ I_2 & I_2 \end{pmatrix} v(p_0) = 0$$

so instead

$$v(p_0) = \sqrt{m}\begin{pmatrix} \eta \\ -\eta \end{pmatrix}, \eta \in \mathbb{C}^2$$

giving us the general solution for negative frquencies to be

$$\psi(x) = v(p)e^{ip\cdot x}, v^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\eta^s \\ -\sqrt{p\cdot\bar{\sigma}}\eta^s \end{pmatrix}$$

for orthonormal basis η^s, $\eta^{r\dagger}\eta^s = \delta^{rs}$. Many identities can be calculated on these solutions, as we do on ES 3 [do we need to memorise these?]. [End of lecture 19]

5.7 Quantising Spinor Fields

We want to quantise our field with

$$S = \int d^4x\mathcal{L}, \mathcal{L} = i\bar{\psi}\partial_\mu^\mu\psi - m\bar{\psi}\psi$$

This action has the following symmetries

- Lorentz symmetry
- so-called **vector symmetry** corresponding to

  $$\psi \mapsto e^{-i\alpha}\psi, \bar{\psi} \mapsto e^{i\alpha}\bar{\psi}$$

  which leads to the conserved current $j_V^\mu = \bar{\psi}\gamma^\mu\psi, \partial_\mu j_V^\mu = 0$ and conserved charge $Q = \int d^3x\psi^\dagger\psi$.

- **chiral/axial symmerty** when mass $m = 0$ corresponding to

$$\psi \mapsto e^{i\alpha\gamma^5}\psi, \bar{\psi \mapsto e}^{-i\alpha\gamma^5}\bar{\psi}$$

- Finally there are the discrete charge, parity and time symmetries, which are discussed in more detail in Peskin and Schröder.

Again, as before to perform the quantisation, we first move to the Hamiltonian formulation of the theory. First we note that the conjugate momentum here is just

$$\pi = \partial_{\dot{\psi}}\mathcal{L} = i\bar{\psi}\gamma^0 = i\psi^\dagger$$

which is quite different before (it also results in half the number of degrees of freedom one might otherwise expect). The Hamiltonian then is given by

$$H = \int d^3x\mathcal{H}(x), \mathcal{H}(x) = \pi(x)\dot{\psi}(x) - \mathcal{L}(x) = \bar{\psi}(-i\gamma^i\partial_i + m)\psi$$

Just as in free tield theory, we split into many harmonic oscillators. As before, we want to expand in a complete basis of solutions of the Dirac equation, which we found in the previous section. These can be parameterised by $p^\mu p_\mu = m^2, p_0 > 0$ and $s = 1, 2$ for $\psi = u^s(p)e^{-ip\cdot x}, v^s(p)e^{ip\cdot x}$ so that we get 4 independent solutions summed as

$$\psi_\alpha(x) = \sum_{s=1}^{2}\int \frac{d^3p}{(2\pi)^3}\left(b_p^s u_\alpha^s(p)e^{-ip\cdot x} + c_p^{s*}v_\alpha^s(p)e^{ip\cdot x}\right)$$

and similar for $\psi_\alpha^\dagger(x)$ where the b_p, c_p are the ladder operators for the particle and its anti-particle. In the bosonic case we had the quantisation rules

- $[a, a^\dagger] = 1, [a, a] = [a^\dagger, a^\dagger] = 0$
- $H_B = \frac{1}{2}w(a^\dagger a + aa^\dagger) = w(N + \frac{1}{2}), N_B = a^\dagger a$
- ground state $|0\rangle$ such that $a|0\rangle = 0$
- excited states $|n\rangle = (a^\dagger)^n|0\rangle$ such that $N_F|n\rangle = n|n\rangle\,\forall n \in \mathbb{N}_0$

whil in the fermionic case we get the same as before, except we replace commutators with anti-commutators:

- $\{b, b^\dagger\} = 1, \{b, b\} = \{b^\dagger, b^\dagger\} = 0$ (which means that $b^2 = b^{\dagger 2} = 0$)
- $H_F = \frac{1}{2}w(b^\dagger b - bb^\dagger) = w(N_F - 1/2)$ for Fermionic number operator N_F.
- ground state $|0\rangle$ such that $b|0\rangle = 0$
- excited states $|n\rangle = b^{\dagger n}|0\rangle$ such that $N_F|n\rangle = n|n\rangle$

Importantly, the fermionic case shares the property that

$$[H_F, b] = -wb, [H_F, b^\dagger] = wb^\dagger$$

meaning that as before we still get the standard harmonic oscillator ladder operators!

The next big result, which will not be proven, or discussed in much detail at all, is the **spin statistics theorem**, which says that half-integer spin particles are always fermions, and whole integer spin particles are always bosons. Basically, what goes wrong is that if these are matched up the wrong way around, the energy is unbounded below, which is an issue.

Now that we have our basic quantisation, let's look at some basic properties. In the Schrödinger picture at a fixed point in time, so $\psi(x) \mapsto \psi(\vec{x})$ then we get canonical anti-commutator relations:

$$\{\psi_\alpha(x), \psi_\beta^\dagger(y)\} = \delta_{\alpha\beta}\delta(x-y), \{\psi_\alpha(x), \psi_\beta(y)\} = 0, \{\psi_\alpha^\dagger(x), \psi_\beta^\dagger(y)\} = 0$$

We then get a Schrödinger [Heisenberg?] picture mode expansion

$$\psi_\alpha(x) = \sum_{s=1}^{2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(b_p^s u_\alpha^s(p) e^{ip\cdot x} + c_p^\dagger v_\alpha^s(p) e^{-ip\cdot x} \right)$$

where we have

$$\{b_p^s, b_q^{\dagger r}\} = \{c_p^s, c_q^{\dagger r}\} = (2\pi)^3 \delta^{sr}\delta(x-y)$$

We then find

$$H = \int \frac{d^3p}{(2\pi)^3} E_p \left(\sum_{s=1}^{2} (b_p^{\dagger s} b_p^s + c_p^{\dagger s} c_p^\dagger) - (2\pi)^3 \delta(0) \right)$$

where just like the bosonic case we get a stray infinity, however, unlike before we git minus infinity not plus inifity. This changing of sign is a common occurance between bosonic and fermionic calculations. We also get

$$Q = \int \frac{d^3p}{(2\pi)^3} \left(\sum_{s=1}^{2} b_p^{\dagger s} b_p^s - c_p^{\dagger s} c_p^s \right)$$

Finally, to discuss particles, we really get an identicle situation as in the bosonic case. The vacuum is the state that solves $b_p^r |0\rangle = c_p^r |0\rangle = 0$, and we construct particle and anti-particle states by using creations operators (up to a factor) such that $b_p^{\dagger s} |0\rangle = |p, s\rangle$ with charge $Q = 1$ for $E_p = \sqrt{p^2 + m^2}$ (the anti-particle has charge $Q = -1$). [End of lecture 20]

5.8 Interacting Fermions

To generalise to interacting fermions, we write

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{int}, \mathcal{L}_0 = \bar{\psi} \cdot (i\gamma^\mu \partial_\mu - m) \cdot \psi$$

Dimensional analysis means that $[\mathcal{L} = 4], [\partial] = 1, [\psi] = [\bar{\psi}] = 3/2$ meaning that the simplest fermion self-coupling

$$\mathcal{L}_{int} = -g_F (\psi\bar{\psi})^2$$

is irrelevant since $[g_F] = -2$. As we can only develop an interesting interacting theory by introducing another field, which in its simplest form introduces a new scalar field. This is **Yukawa theory** where we add scalar field $\phi(x)$ with mass μ to get

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m) \cdot \psi + \frac{1}{2}(\partial_\mu \psi)(\partial^\mu \psi) - \frac{\mu^2}{2}\psi + \lambda\phi\bar{\psi}\psi$$

[This Lagrangian disagrees with the Lagrangian given in ES 3 Q8? The example sheet gives the following:]

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m) \cdot \psi + \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{\mu^2}{2}\phi^2 + \lambda\phi\bar{\psi}\psi$$

Here we can show that $[\lambda] = 0$ so at least this is marginal. In terms of applications, this was used as an early theory of the interactions between mesons and nucleons (protons and neutrons). We use this as a stepping stone to describe quantum electrodynamics (QED) by using the standard process

1. try to use Dyson formula
2. calculate time ordered products
3. find a version of Wick's theorem
4. develop Feynman rules

First we need to work out a propagator as in the scalar case so that we can perform contrations. As such we write

$$S_F(x-y)_{\alpha\beta} = \langle 0| T(\psi_\alpha(x)\bar{\psi}_\beta(y)) |0\rangle = \begin{cases} 0 & x^0 = y^0 \\ \langle 0| \psi_\alpha(x)\bar{\psi}_\beta(y) |0\rangle & x^0 > y^0 \\ -\langle 0| \bar{\psi}_\beta(y)\psi_\alpha(x) |0\rangle & x^0 < y^0 \end{cases}$$

where $\psi_\alpha(x), \bar{\psi}_\beta(y)$ are Heisenberg picture operators. Here the big difference is that we get the minus sign in order for things to agree. In particular, when $(x-y)^2 < 0$ we should agree where we have a spacelike overlap, giving rise to the sign (swapping operators inverts the sign).

As such we see that similar to the scalar case we may write

$$\psi_\alpha(x) = \sum_{s=1}^{2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} (b_p^s u_\alpha^s(p) e^{-ip\cdot x} + c_p^{s\dagger} v_\alpha^s e^{ip\cdot x}), \bar{\psi}_\alpha(x) = (\psi^\dagger(x) \cdot \gamma^0)_\alpha$$

As such splitting the components from the propagator we get (we only need to focus on the annihilation operators in $\psi_\alpha(x)$ and the creation operators in $\bar{\psi}_\beta(y)$)

$$\begin{aligned} M_+ &= \langle 0| \psi_\alpha(x)\bar{\psi}_\beta(y) |0\rangle \\ &= \sum_{r,s=1}^{2} \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3q}{(2\pi)^3} \frac{u_\alpha^s(p)\bar{u}_\beta^r(q)}{\sqrt{4E_pE_q}} e^{-i(p\cdot x - q\cdot y)} \langle 0| b_p^r b_q^{s\dagger} |0\rangle \end{aligned}$$

where using anti-commutation rules we see that $\langle 0| b_p^r b_q^{s\dagger} |0\rangle = (2\pi)^3\delta(p - q)\delta^{rs}$ and the d^3q integral yields

$$M_+ = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{-ip\cdot(x-y)} \sum_{s=1}^{2} u_\alpha^s(p)\bar{u}_\beta^s(p)$$

where using example sheet 3 for $\sum_s u_\alpha^s(p)\bar{u}_\beta^s(p) = (\not{p} + mI)_{\alpha\beta}$

$$M_+ = \int \frac{d^3p}{(2\pi)^3} \frac{1}{E_p} (\not{p} + m)_{\alpha\beta} e^{-ip\cdot(x-y)}$$

and similarly

$$M_- = \langle 0| \bar{\psi}_\beta(y)\psi_\alpha(x) |0\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} (\not{p} - mI)_{\alpha\beta} e^{ip\cdot(x-y)}$$

so

$$S_F(x-y)_{\alpha\beta} = \begin{cases} M_+ & x^0 > y^0 \\ -M_- & x^0 < y^0 \end{cases} = \int \frac{d^4p}{(2\pi)^4} \frac{i(\not{p} + mI)_{\alpha\beta}}{p^2 - m^2 + i\epsilon} e^{-ip\cdot(x-y)},\ \epsilon \to 0^+$$

where just as before we summarise this function with a complex integral limit.

Working to find an analogue of Wick's theorem we write

$$T[\psi_\alpha(x)\bar{\psi}_\beta(y)] =: \psi_\alpha(x)\bar{\psi}_\beta(y) : + \psi_\alpha(x)[\bar{\psi}_\beta(y)]$$

where the last square brackets denote a contraction between the two. We then compute the vacuum expectation value (VEV)

$$[\psi_\alpha(x)\bar{\psi}_] beta(y)] = \langle 0| T(\psi_\alpha(x)\bar{\psi}_\beta(y)) |0\rangle = S_F(x-y)_{\alpha\beta}$$

As hoped (here this is the only contraction, since the $\psi\psi$ and $\bar{\psi}\bar{\psi}$ terms disappear). Consequently, once we countthe number of transpositions [?] we just get Wick's theorem in its usual form:

Theorem 4. For a product π of $\psi, \bar{\psi}$s with $\psi(x_i) = \psi_i$ and similar for $\bar{\psi}$ we find

$$T(\pi) =: \mathcal{C}(\pi) :$$

where $\mathcal{C}(\pi)$ is the sum over all contractions.

To calculate our scatering amplitude then we can apply Dyson's formula to get

$$A_{i\to f} = \langle f|\, T\left(e^{\frac{1}{i}\int_{-\infty}^{\infty} dt H_I(t)}\right)|i\rangle = \langle f|\, T\left(e^{\frac{\lambda}{i}\int d^4x \phi(x)\bar{\psi}(x)\psi(x)}\right)|i\rangle$$

in Yukawa theory, and we can apply as a similar approach as we did in Scalar-Yukawa theory as we did before.

As our final result, we may work out the Feynman rules in this theory as for incident particles, the term $u^r_\alpha(p)$ term for 4-momentum p, spin r corresponds to

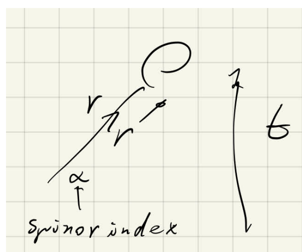

Figure 7: Incident Particle

and for an incident anti-fermion, $v^r_\alpha(p)$ corresponds to

Figure 8: Incident Anti-particle

Similarly for final states (outgoing (anti-)particles)

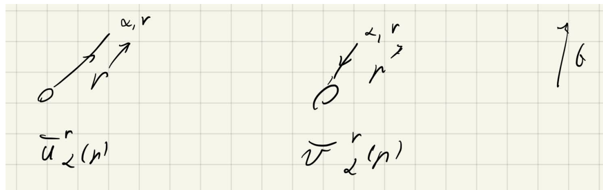

Figure 9: Final (Anti-)Particles

[Note one mistake here is that in Feynman diagrams, anti-particles should be drawn with negative momentum. Or not? I've heard something about this - see ES 3 solutions, Q8...]

Similarly for internal propagators we have for spin-1/2 fermions

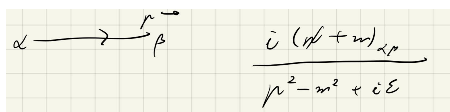

Figure 10: Spin-1/2 Propagator

and for scalar propagators

Figure 11: Scalar Propagator

and for a vertex

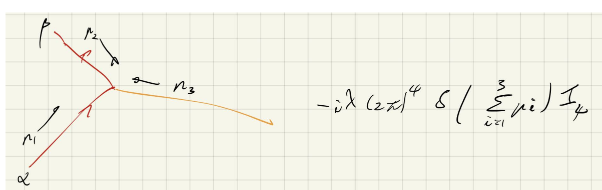

Figure 12: A Vertex

and then for all these together adding the integral

$$\int \frac{d^4p}{(2\pi)^4}$$

completes our job. [End of lecture 21]

5.9 Gauge Fields

We want to somehow combine a 4-vector field where

$$A^\mu(x) \mapsto \Lambda^\mu{}_\nu(\Lambda^{-1} \cdot x)$$

and the field strength tensor

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

which is invariant under gauge-transform

$$A^\mu \mapsto A^\mu + \partial^\mu \lambda$$

For Maxwell's theory, we usually use the action

$$S = \int d^4x\mathcal{L}, \mathcal{L} = -1/4F_{\mu\nu}F^{\mu\nu}$$

giving the equations of motion $\partial_\mu F^{\mu\nu} = 0$ and Bianchi identity $\partial_\mu * F^{\mu\nu} = 0$ where $*F^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$.

Now why is this different than the cases we covered before? The most obvious issue is that when we reformulate in a Hamiltonian manner and compute the canonical moentum we get

$$\begin{aligned}\pi^0 &= \partial_{\dot{A}^0}\mathcal{L} = 0\\ \pi^i &= -F^{0i} = E^i\end{aligned}$$

The zero here is particularly fatal, and is related to the redudancies that appear in a Gauge theory - multiple states in the configuration space lead to the same behaviour. Anyways, we need to "fix a gauge" somehow, which involves quite a horrible process really.

A convenient choice of gauge is the Lorentz gauge

$$\partial_\mu A^\mu = 0$$

but we cannot just assume this off the bat. Instead we solve the theory as if this isn't true, and then force the condition later on...

We start with a Lagrangian with a "gauge fixing term"

$$\mathcal{L}_\xi = -1/4F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2$$

Note here that the $\xi = 1$ case is called the **Feynman gauge**. Anyways, suing this approach we find that the equations of motion become

$$\begin{aligned}\partial_\mu\partial_{\partial_\mu A_\nu}\mathcal{L} &= -\partial_\mu F^{\mu\nu} - \partial_\mu(\eta^{\mu\nu}(\partial_\rho A^\rho))\\ &= -\partial_\mu\partial^\mu A^\nu + \partial_\mu\partial^\nu A^\nu - \partial^\nu\partial_\mu A^\mu\\ &= -\partial_\mu\partial^\mu A^\nu\end{aligned}$$

which means that the equations of motion correspond to the wave equation!

$$\Box A^\mu = (\partial_\rho \partial^\rho) A^\mu =$$

Then we get

$$\pi^0 = \partial_{\dot{A}_0} \mathcal{L} = -\partial_\mu A^\mu$$
$$\pi^i = \partial_{\dot{A}_i} \mathcal{L} = \partial^i A^0 - \dot{A}^i$$

As such we get the Gupta-Bleuler quantisation which involves

1. quantise using the gauge fixing term
2. impose the gauge constraint afterwards

It's quite crude. As such in the Schrödinger picture at fixed time where $A^\mu(x) = A^\mu(\vec{x})$ and the same for π^μ we are left with commutator

$$[A^\mu(x), \pi^\nu(y)] = i\eta^{\mu\nu}\delta(x-y)$$

[how is this deduced again?] To expand in modes, we notice that since we are dealing with the wave equation, our solutions are of the form.

$$\epsilon^\mu e^{\pm ip\cdot x}, p^\mu p_\mu = 0 \forall \epsilon^\mu \in \mathbb{R}^{1,3}$$

This makes sense since the theory we are developing corresponds to electromagnetism, so the particles being considered here are photons, which are light-like ($p^\mu p_\mu = 0$). As such we find that $E_p = |p|, p^\mu = (E_p, p)$. Now, normally we get a constraint on ϵ^μ restricting it to two dimensions, but here in fact, ϵ^μ is completely free until we impose the gauge constraint. As such we pick a basis $\epsilon^{(\lambda)}_\mu(p), \lambda = 0, 1, 2, 3$ such that

$$\epsilon^{(\lambda)}_\mu \eta^{\mu\nu} \epsilon^{(\rho)}_\nu(p) = \eta^{\lambda\rho}$$

and $\epsilon^{(0)}$ is timelike and $\epsilon^{(1)}$ is spacelike. By writing this as a matrix equation and noting that η is orthogonal, it is easy to see that this means that

$$\eta_{\lambda\rho} \epsilon^{(\lambda)}_\mu(p) \epsilon^{(\mu)}_\nu(p) = \eta_{\mu\nu}$$

Let's also choose $\epsilon^{(1)}, \epsilon^{(2)}$ to be transverse so $p^\mu \epsilon^{(i)}_\mu = 0$ for $i = 1, 2$, and so necessarily, $\epsilon^(3)$ is parallel to p_μ. We then get mode expansions

$$A_\mu(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2|p|}} \sum_{\lambda=0}^{3} \epsilon^{(\lambda)}_\mu(p) (a^\lambda_p e^{ip\cdot x} + a^{\lambda\dagger}_p e^{-ip\cdot x})$$

$$\pi_\mu(x) = \int \frac{d^3p}{(2\pi)^3} \sqrt{\frac{|p|}{2}} i \left(a^\lambda e^{ip\cdot x} - a^{\lambda\dagger}_p e^{-ip\cdot x} \right)$$

where we notice that since both A, π must be real, we enforce realness in the last part of these terms (so we get no anti-particles or these particles are half-anti-particle half-particle).

From here we can derive the commutation relations

$$[a_p^\lambda, a_q^{\rho\dagger}] = -\eta^{\lambda\rho}(2\pi)^3\delta(p-q)$$

As such we find Hamiltonian

$$H = \int \frac{d^3p}{(2\pi)^3}|p|\left(\sum_{i=1}^{3} a_p^i a_p^i - a_p^{0\dagger}a_p^0\right)$$

and vacuum state $|0\rangle$ such that

$$a_p^\lambda |0\rangle = 0$$

Now, if we work in the Hilbert space spanned by creation operators applied to $|0\rangle$ we encounter issues. In particular, we can get negative norms for timelike states, and energy is unbounded below! As such we now impose the gauge condition as suggested by Gupta-Bleuler so that we only allow states $|\psi\rangle$ if

$$\langle\psi| \partial_\mu A^\mu(x) |\psi\rangle = 0$$

(use the Heisenberg field operator A here). This perfectly remoevs unphysical states and does the job. It also restricts ϵ_μ to be purely transeverse instead of being any arbitrary vector, so now we get photons

$|p, l\rangle = (2|p|)^{1/2} a_p^{l\dagger} |0\rangle, l = 1, 2$

which are massless particles with energy $E = |p|$. [End of lecture 22]

Using similar approaches as before for the photon propagator we find that

$$\Delta_{\mu\nu}(x-y) = \langle 0| T(A_\mu(x)A_\nu(y)) |0\rangle = \begin{cases} M_+ & x^0 > y^0 \\ M_- & x^0 < y^0 \end{cases}$$

where

$$\begin{aligned} M_+ &= \langle 0| A_\mu(x)A_\nu(y) |0\rangle \\ &= \int \frac{d^3p}{(2\pi)^3}\int \frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2|p|}}\frac{1}{\sqrt{2|q|}}\sum_{\lambda,\eta=0}^{4}\epsilon_\mu^{(\lambda)}(p)\epsilon_\nu^{(\eta)}(q)\langle 0| (a_p^\lambda e^{-ip\cdot x} + \cancel{a_p^{\lambda\dagger}e^{ip\cdot x}})(\cancel{a_q^\mu e^{-iq\cdot y}} + a_q^{\mu\dagger}e^{iq\cdot y}) |0\rangle \end{aligned}$$

where

$$\langle 0| a_p^\lambda e^{-ip\cdot x} a_q^{\mu\dagger} e^{iq\cdot y} |0\rangle = e^{-i(p\cdot x + q\cdot y)} \langle 0| a_p^\mu a_q^{\nu\dagger} |0\rangle = e^{-i(p\cdot x+q\cdot y)}(-\eta^{\mu\nu})(2\pi)^3\delta(p-q)$$

which leaves us with

$$M_+ = -\eta_{\mu\nu} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2|p|} e^{-ip\cdot(x-y)}$$

and similar for M_- leaving us with

$$\Delta_{\mu\nu}(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{-i\eta_{\mu\nu}}{p^2 + i\epsilon} e^{-p\cdot(x-y)}, \ \epsilon > 0$$

5.9.1 Coupling to Matter (QED)

The Maxwell equations with source $J^\mu(x) = (\rho(x), \vec{J}(x))$ as

$$\partial_\mu J^\mu = 0, \partial_\mu F^{\mu\nu} = J^\nu$$

then the action is

$$S = \int d^4 \mathcal{L}(x), \mathcal{L}(x) = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - A_\mu J^\mu$$

For spinor fields on the hand we had

$$\mathcal{L}_F = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi$$

with the $U(1)$ global symmetry

$$\psi(x) \mapsto e^{-i\alpha}\psi(x), \bar{\psi}(x) \mapsto e^{i\alpha}\bar{\psi}(x)$$

where we get a Noether conserved current

$$J^\mu(x) = \bar{\psi}\gamma^\mu\psi(x), \partial_\mu J^\mu = 0$$

(proving this is conserved is rather non-trivial using other methods). In QED we then get that

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi - e\bar{\psi}\gamma^\mu \psi A_\mu$$

Note that the only new part to show that this is Lorentz invariant is to show that the last term $\bar{\psi}\gamma^\mu\psi A_\mu$ is Lorentz invariant. This follows immediately from the fact that $\bar{\psi}\gamma^\mu\psi$ and A_μ are vector fields (use that the former is a conserved current!).

This is also gauge invariant, but the proof is more involved. First of all, notice that the a gauge transform here takes the form that

$$\begin{aligned} \psi(x) &\mapsto e^{-ie\alpha(x)}\psi(x) \\ \bar{\psi}(x) &\mapsto e^{ie\alpha(x)}\bar{\psi}(x) \\ A^\mu(x) &\mapsto A^\mu(x) + \partial^\mu \alpha(x) \end{aligned}$$

As such, we find it is helpful to define **covariant derivative** $D_\mu\psi = \partial_\mu\psi + ieA_\mu\psi$. It follows then that $D_\mu\psi \mapsto e^{-ie\alpha(x)}D_\mu\psi$ and we can write

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\gamma^\mu D_\mu - m)\psi$$

which is manifestly gauge invariant. Using dimensional analysis, it follows that $[\mathcal{L}] = 4 \implies [\psi] = [\langle\psi|] = 3/2, [A] = 1 \implies [\bar{\psi}\gamma^\mu\psi A_\mu] = 4 \implies [e] = 0$, so our interaction is term is marginal, so interesting. It also worth noting that if we add in $\hbar, c$ again, we find that the dimensionaless parameter we are really expanding in is

$$\frac{e^2}{4\pi\hbar c} = \alpha \approx 1/136$$

the fine structure constant. This is certainly "small" and so using pertubation theory makes sense.

5.9.2 $\mu^+ - \mu^-$ pair creation

As an extended example, let's think about creating a muon - anti-muon pair. In particular, we are interested in the process

$$e^+ + e^- \mapsto \mu^+ + \mu^-$$

In particular that means we now get two fields ϕ for the electron and χ for the muon. As such we can write (for $D_\mu = \partial_\mu + ieA_\mu$) that

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}\cdot(i\gamma^\mu D_\mu - m_e)\cdot\psi + \bar{\chi}(i\gamma^\mu D_\mu - m_\mu)\cdot\chi$$

Doing a mode expansion for ψ, χ we get that for $E_p = \sqrt{|p|^2 + m_e^2}, \mathcal{E}_p = \sqrt{|p|^2 + m_\mu^2}$ for two spin-1/2 particles. Here we label ladder operators as b, c, d, e and spinors $u_\alpha^r, v_\alpha^r, \tilde{u}_\alpha^r, \tilde{v}_\alpha^r$ (different since these depend on the mass parameter). Here all anti-commutators vanish except for

$$\{b_p^r, b_q^{s\dagger}\} = \{c_p^r, c_q^{s\dagger}\} = \{d_p^r, d_q^{s\dagger}\} = \{e_p^r, e_q^{s\dagger}\} = \delta^{rs}(2\pi)^3\delta(p-q)$$

We then want to calculate scaterring amplitude between

$$\langle i| = \sqrt{2E_p}\sqrt{2E_q}b_q^{s\dagger}c_p^{r\dagger}|0\rangle$$
$$\langle f| = \sqrt{2\mathcal{E}_p}\sqrt{2\mathcal{E}_q}d_q^{s\dagger}e_p^{r\dagger}|0\rangle$$

Here we want to use the scattering operator $S = T\left(e^{\frac{1}{i}\int d^4x M_I}\right)$ where we find that

$$M_I = e\bar{\psi}\gamma^\mu\psi A_\mu + e\bar{\chi}\gamma^\mu\chi A_\mu$$

with $A_{i\to f} = \langle f|\, S\, |i\rangle$ as usual. Since we need to destroy and create two particles, we must work to at least second order. As such, the leading order term is O_2 where

$$O_2 = (ie)^2 \int d^4x \int d^4y : \bar{\psi}(x)\gamma^\mu\psi(x)A_\mu(x)\bar{\chi}(y)\gamma^\mu\chi(y)A_\nu(y) :$$

where only the photon (A_μ) is contracted since we have no photons. Here only the following terms turn out to be important

$$\begin{aligned}
\bar{\psi} &\sim b_p^{s\dagger} + c_p^s \mapsto c_p^s \\
\psi &\sim b_p^s + c_p^{s\dagger} \mapsto b_p^s \\
\bar{\chi} &\sim d_p^{s\dagger} + e_p^s \mapsto d_p^{s\dagger} \\
\chi &\sim d_p^s + e_p^{s\dagger} \mapsto c_p^{s\dagger}
\end{aligned}$$

[End of lecture 23]. Using standard procedures, one can turn this into an explicit expression for $A_{i\mapsto f}$. However, the resulting expression is several lines long as

$$\begin{aligned}
A_{i\to f} = \sqrt{4E_pE_q4\mathcal{E}_{p'}\mathcal{E}_{q'}} &\sum_{s_1,s_2,s_3,s_4=1}^{2} \int \frac{d^3p_1d^3p_2d^3p_3d^3p_4}{(2\pi)^4} \\
&\frac{1}{\sqrt{4E_{p_1}E_{p_2}4\mathcal{E}_{p_3}\mathcal{E}_{p_4}}}(-ie)^2 \int d^4x \int d^4y \int \frac{d^4k}{(2\pi)^4}\frac{-i\eta_{\mu\nu}}{k^2+i\epsilon}e^{-ik\cdot(x+y)} \\
&(\bar{v}^{s_1}(p_1)\gamma^\mu u^{s_2}(p_2))(\bar{\tilde{u}}^{s_3}(p_3)\gamma^\nu\tilde{v}^{s_4}(p_4))e^{-ix\cdot(p_1+p_2)+iy\cdot(p_3+p_4)} \\
&\langle 0|\, e_{q'}^{r'}d_{p'}^{s'}d_{p_3}^{s_3\dagger}e_{p_4}^{s_4\dagger}c_{p_1}^{s_1}b_{p_2}^{s_2}b_p^{s\dagger}c_q^{r\dagger}\,|0\rangle
\end{aligned}$$

where we note

$$\langle 0|\, e_{q'}^{r'}d_{p'}^{s'}d_{p_3}^{s_3\dagger}e_{p_4}^{s_4\dagger}c_{p_1}^{s_1}b_{p_2}^{s_2}b_p^{s\dagger}c_q^{r\dagger}\,|0\rangle = (2\pi)^{12}s^{s_1r}\delta^{s_2s}\delta^{s_3s'}\delta^{s_4r'}\delta(p_1-q)\delta(p_2-p)\delta(p_3-p')\delta(p_4-q')$$

Using standard procedures (not that wlog we can set $\epsilon = 0$ at some point) we find that

$$A_{i\to f} = (-ie)^2\frac{1}{(p+q)^2}(2\pi)^4\delta(p'+q'-p-q)(\bar{v}^r(q)\gamma^\mu u^s(p))(\bar{\tilde{u}}^{s'}(p')\gamma_\mu\tilde{v}^{r'}(q'))$$

This completes our extended example.

5.9.3 Feynman rules for QED

Initial states can be described by

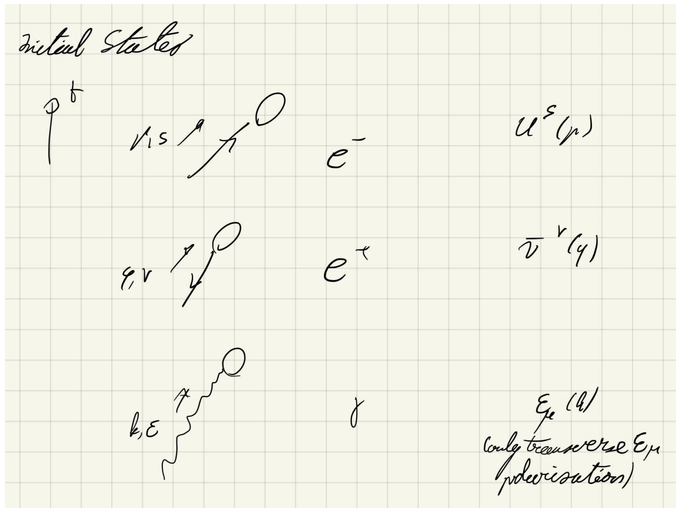

Figure 13: QED Initial States

For final states we have

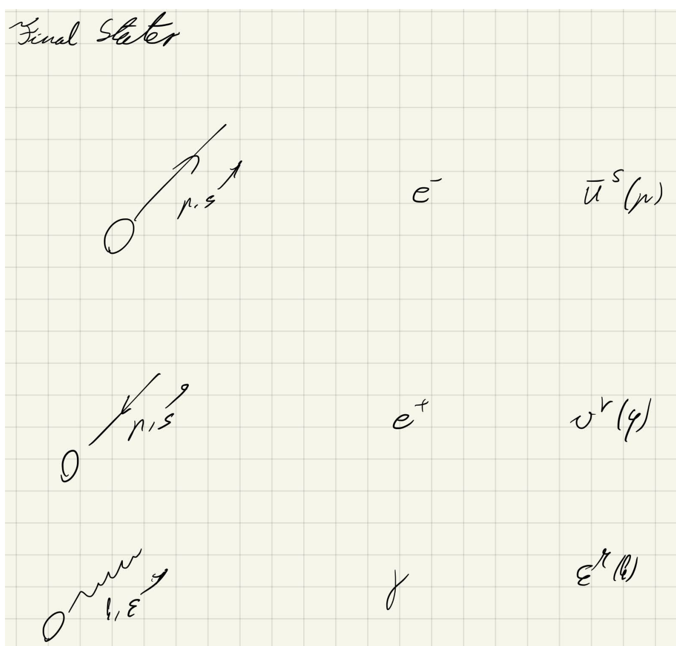

Figure 14: QED Final States

For internal lines we have

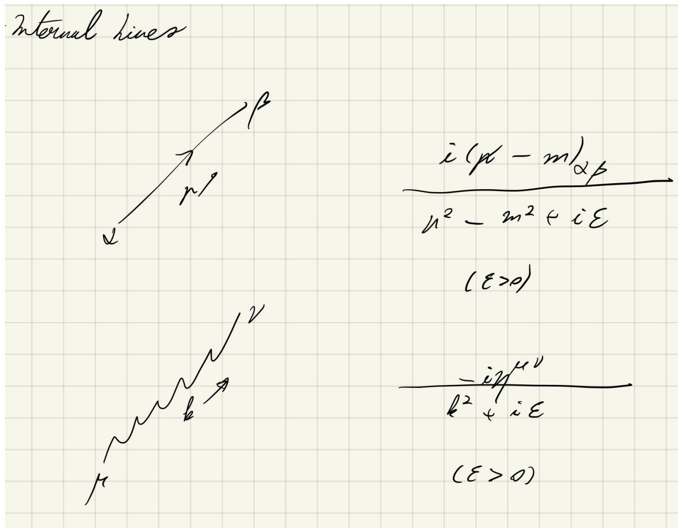

Figure 15: QED Internal Lines

and a vertex translates to

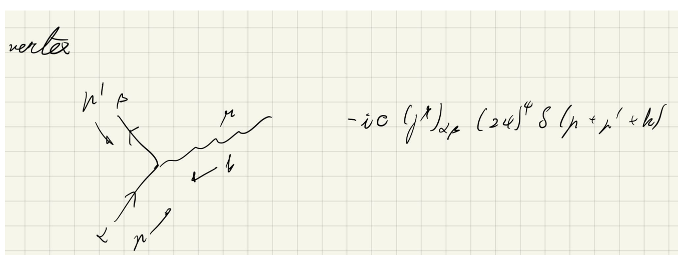

Figure 16: QED Vertex

Note that Feynman rules are in general gauge dependent, although the final results should always be the same. This specifically corresponds to the specific Feynman gauge.

For example, for electron scattering, $e^+ + e^- \mapsto e^+ + e^-$ we get

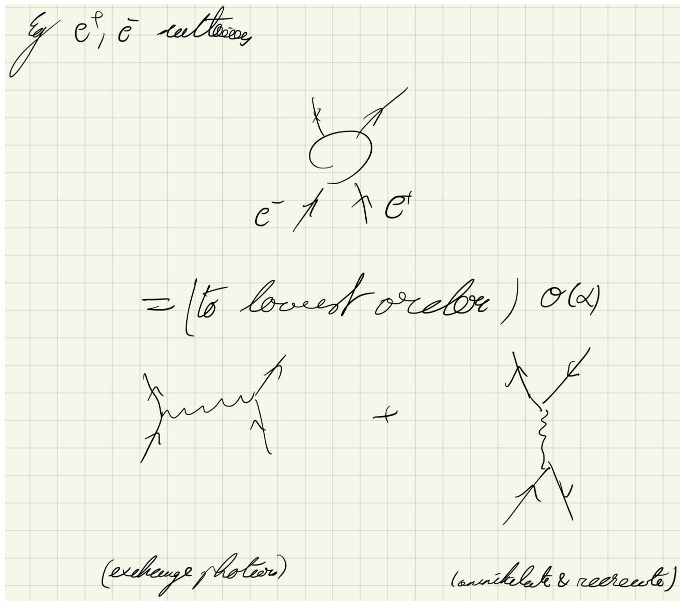

Figure 17: electron scattering to lowest order

then using the Feynman rules we find that

$$A_{i\to f} = -i(-ie)^2(2\pi)^4\delta(p' + q' - p - q)\left(\frac{-X}{(p-p')^2} + \frac{Y}{(p+q)^2}\right)$$

where

$$X = (\bar{u}^{s'}(p')\gamma^\mu u^s(p))(\bar{v}^r(q)\gamma_\mu v^{r'}(q'))$$
$$Y = (\bar{v}^r(q)\gamma^\mu u^s(p))(\bar{u}^{s'}(p')\gamma_\mu v^{r'}(q'))$$

Another example is $e^- + e^- \mapsto e^- + e^-$ is noted by

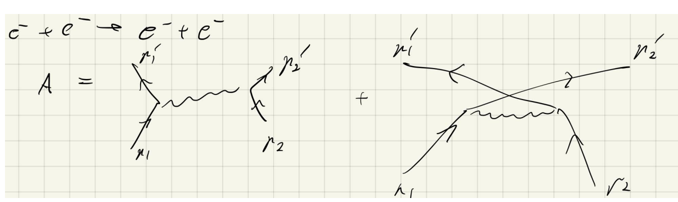

Figure 18: Another Electron Scattering?

A third is $e^+ + e^- \mapsto \gamma + \gamma$ (note, one can show one cannot have the process $e^+ + e^- \mapsto \gamma$ for some reaosn...) leaving

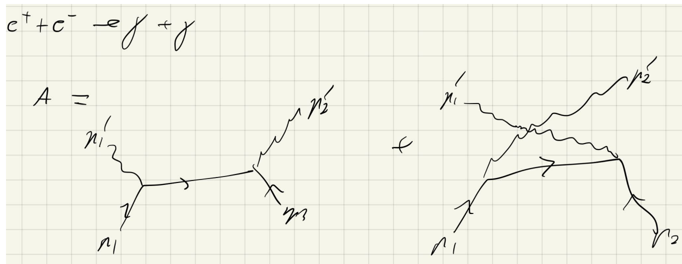

Figure 19: Electron-Position Annihilation

A fourth example is Compton scattering $\gamma + e^- \mapsto \gamma + e^-$ with

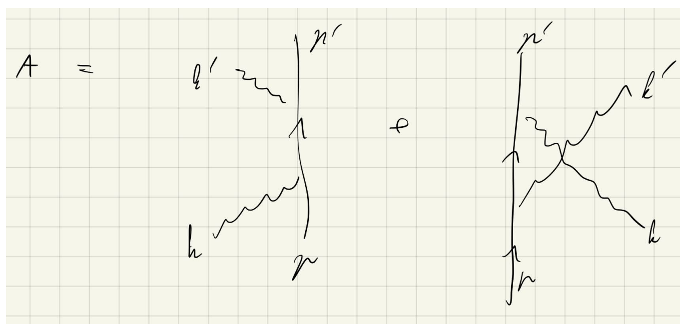

Figure 20: Compton Scattering

That completes our coverage of QED.

5.9.4 Comparison with Experiment

Basically, to compare our theoretical predictions to experiments we in practice compare them to collider experiments. In particular, we want to measure $N = F\sigma$ for flux F and cross-section σ. In particular, we are then interested in the probability of transitioning form state i to f

$$P_{i\to f} = \frac{|\langle f| S |i\rangle|^2}{\langle f|f\rangle \langle i|i\rangle}$$

(note most QFT calculations are not normalised). This is how we relate to the scattering amplitude $A_{i\to f}$. Recall relativistic normalisation for a single particle state is

$$\langle p|q\rangle = 2E_p(2\pi)^3\delta(p-q)$$

where we have the $(2\pi)^3\delta(0) = \int d^4x 1 = V$ where we can cancel the numerator and denominator as necessary.

In particular, let's look at 2×2 scattering. Let's define differential cross section $d\sigma = \frac{1}{F} dP_{i \to f} = \frac{1}{4E_{p_1}E_{p_2}} \frac{1}{FV} |M_{i \to f}|^2$ for the reduced scattering amplitude

$$A_{i \to f} = M_{i \to f} (2\pi)^4 \delta(p_1' + p_2' - p_1 - p_2)$$

and angular component

$$d\pi = \prod_{i=1}^{2} \frac{d^3 p_i'}{2E_{p_i'}} (2\pi)^4 \delta(p_1' + p_2' - p_1 - p_2)$$

and $F = |v_1 - v_2|/V, V_i = p_i/(p^0)_i$. We then can define the total cross section to be $\sigma = \int d\sigma$.

We now work out the case of Fermion scatteringm and in particular, the case we looked at before $e^+ + e^- \mapsto \mu^+ + \mu^-$ with

$$M_{s,r,s',r'} = M(p, s, q, r \to p', s', q', r') = \frac{e^2}{(p+q)^2} (\bar{v}^r(q) \gamma^\mu u^s(p)) (\bar{\tilde{u}}^{s'}(p') \gamma_\mu \tilde{v}^{r'}(q'))$$

We can then define the unpolarised differential cross section

$$P = \frac{1}{4} \sum_{r,s,r',s'=1}^{2} |M_{r,s,r',s'}|^2 = \frac{e^4}{(p+q)^4} A^{\mu\nu} B_{\mu\nu}$$

where A, B provide a specific factorisation. Here we have

$$A^{\mu\nu} = \frac{1}{4} \sum_{r,s=1}^{2} (\bar{v}^r(q) \gamma^\mu u^s(p))^* (\bar{v}^r(q) \gamma^\nu u^s(p))$$
$$B_{\mu\nu} = \frac{1}{4} \sum_{r',s'=1}^{2} (\bar{\tilde{u}}^{s'}(p') \gamma_\mu \tilde{v}^{r'}(q'))^* (\bar{\tilde{u}}^{s'}(p') \gamma_\nu \tilde{v}^{r'}(q'))$$

To evaluate these we remark that $(\gamma^\mu)^\dagger = \gamma^0 \gamma^\mu \gamma^0, (\gamma^0)^2 = I$ we get

$$\begin{aligned}
(\bar{v}^r(q) \gamma^\mu u^s(p))^* &= (v^{r\dagger}(q) \gamma^0 \gamma^\mu u^s(p))^* \\
&= u^{s\dagger}(p) (\gamma^\mu)^\dagger (\gamma^0)^\dagger v^r(q) \\
&= u^{s\dagger}(p) \gamma^0 \gamma^\mu \gamma^0 \gamma^0 \gamma^0 v^r(q) \\
&= \bar{u}^s(p) \gamma^\mu v^r(q) \\
&= (\bar{v}^s(q) \gamma^\mu u^s(p))^*
\end{aligned}$$

So we have

$$\bar{u}^s(p) \gamma^\mu v^r(q) = (\bar{v}^s(q) \gamma^\mu u^s(p))^*$$

which is helpful to known in general. As such by using some formulas from example sheet 3, we see that

$$A^{\mu\nu} = \frac{1}{4}\,\mathrm{tr}((\not{p} + m_e)\gamma^{\mu}(\not{q} - m_e)\gamma^{\nu})$$

which via more trace identities from question 3 on example sheet 3 gives

$$A^{\mu\nu} = p^{\mu}q^{\nu} + p^{\nu}q^{\mu} - \eta^{\mu\nu}(p\cdot q + m_e^2)$$

One can apply a similar approach for $B_{\mu\nu}$.

Now, working from the centre of momentum frame we find that $m_e << m_\mu$ (since $m_e \approx \frac{1}{200}m_\mu$ for actual muons) and so also that $E \sim m_\mu$. So if we then also use the approximation $m_e \approx 0$ then we get

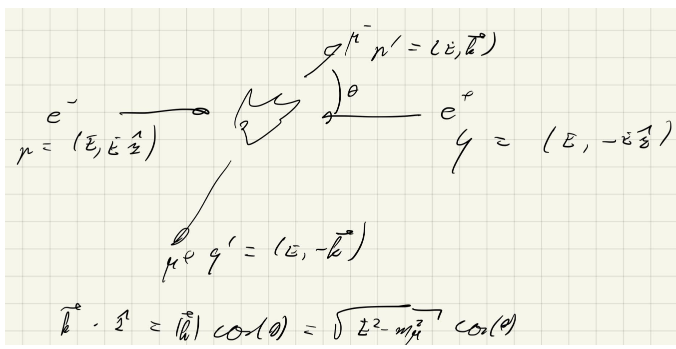

Figure 21: Centre of Momentum Frame

This calculation is finished on example sheet 4, but we conseuqnelty find that

$$P = \frac{1}{4}\sum_{r,s,r',s'}|M_{r,s,r',s'}|^2 = e^4(1 + m_\mu^2/E^2) + (1 - m_\mu^2/E^2)\cos^2(\theta)$$

Consequently one can find that

$$\frac{d\sigma}{d\Omega} = \frac{1}{16E^3}\frac{|k|}{16}P = \frac{\alpha^2}{16E^2}\sqrt{1 - m_\mu^2/E^2}\,\left((1 + m_\mu^2/E^2) + (1 - m_\mu^2/E^2)\cos^2(\theta)\right)$$

so [factor?]

$$\sigma = \int_0^{2\pi} d\phi \int_{-1}^{1} d(\cos(\theta))\frac{d\sigma}{d\Omega} = \frac{\pi^2\alpha^2}{3E^2}\sqrt{1 - m_\mu^2/E^2}(1 + m_\mu^2/E^2)$$

This finally yields a quantity that can be compared with experiment! It also works out to correct, along with many of the other results of QED. Together with the annihilation and creation processes that have been described, this overall is a big vindication that this is an effective union of special relativity and quantum mechanics.