Differential Geometry

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Introduction

These notes are based on the course lectured by dr Jack E Smith in Michaelmas 2020. Due to the measures taken in the UK to limit the spread of Covid-19, these lectures were delivered online. These are not meant to be an accurate representation of what was lectures. They solely represent a mix of what I thought was the most important part of the course, mixed in with many (many) personal remarks, comments and digressions... Of course, any corrections/comments are appreciated.

Unlike some of the other courses, there is no real introduction here, and we jump straight into the content!

1 Manifolds and Smooth Maps

Manifolds are spaces that locally look like $\mathbb{R}^n$. Formally this is:

Definition 1. X is a **topological** n **manifold** if X is a second countable Hausdorff topological space such that $\forall p \in X \exists \text{open} U \ni p$ and open $V \subseteq \mathbb{R}^n$ and homeomorphism $\phi : U \to V$.

Here,

Definition 2. Topological space X is **Hausdorff** if for every distinct $x, y \in X$ there exists open $U \ni x, V \ni y$ in X such that $U \cap V = \emptyset$.

and

Definition 3. Topological space X is **second countable** if there exists a set of open sets $\mathcal{U}$ st that every open set in X can be written as a union of sets in $\mathcal{U}$

Since these two properties transfer to subsets, any subset of a topological n manifold is also a topological n manifold. Also, to give some more intuition, the condition that X is a seond countable Hausdorff topological space is exactly equivalent to the condition that X is metrizable and has countably many components. It is just tradition that it is defined as above. Some more defintions.

Above, ϕ is called the **chart**, U is called the **coordinate patch**, although in some cases can also be called the chart. The functions $x_1 \circ \phi, \ldots, x_n \circ \phi$ (so the components of the result) are called the **local coordinates**, and ϕ^{-1} is called the **paramaterisation**, although that term is not used that frequently. Finally, for overlapping charts, we can define the **transition map** between them as $\phi_2 \circ \phi_1^{-1} : \phi_1(U_1 \cap U_2) \to \phi_2(U_1 \cap U_2)$.

Now, we want to generalise calculus to manifolds, so it makes sense to start by trying to generalise the notion of smoothness. The simplest approach would be to say that f is smooth on X if it is smooth on the local coordinates. The issue then arises that this may not be consistent with smoothness on other charts (where these overlap). As such, we need to require that the transition maps are smooth as well. Consequently we do the following:

Definition 4. The **atlas** of a manifold is a collection of charts of a topological n manifold that covers all of X.

An atlas is **smooth** if all transition maps are smooth, and a map f is **smooth** on atlas $\mathfrak{A}$ if $f \circ \phi_\alpha^{-1}$ is smooth $\forall \alpha$. As a result all local coordinate functions are smooth. Now, really, specifying the atlas precisely all the time is somewhat tedious, and somehow not the point, so we want a degree of flexibility. For this we define

Definition 5. Two atlases are **smoothly equivalent** if their union is smooth.

Note that this forms an equivalence relation (apply the chain rule on transition functions for transitivity).

Definition 6. A **smooth structure** is an equivalence class of atlases.

As we hope, we do indeed have that if a function is smooth wrt to an atlas, it is smooth wrt to any atlas in its smooth structure. Also, we can define a maximal atlas to be the union of all the atlases in a smooth structure, if we deem that to be convenient (includes trivial changes like translating or scaling the local coordinates).

Definition 7. A **smooth** n **manifold** is a topological n manifold with a smooth structure.

Note that under the product topology, the product of two manifolds naturally forms a new (smooth) manifold. Also, a remarkable fact is that for $n = 1, 2, 3$ all topological n manifolds have an essentially unique smooth structure, whereas this breaks down for $n \geq 4$. Also, a new chart is said to be **compatible** with an atlas if when added to the atlas, the atlas remains smooth.

Finally, to give a concrete example of a manifold, we may consider S^n, which forms a manifold with two charts: one being the sphere without the North pole, and the other being the sphere without the South pole, $U_\pm$ with charts

$$\phi_\pm(y_0, \ldots, y_n) = \frac{1}{1 \mp y_0}(y_1, \ldots, y_n),$$

where the local coordinates are referred to as $x^\pm$. [End of DG1]

2 Forming Manifolds from Sets (Instead of Topological Spaces)

We observe that an atlas can generate a topology. In particular if we consider the data

- set X
- subsets $U_\alpha \subseteq X$
- open sets $V_\alpha \subseteq \mathbb{R}^n$
- bijections $\phi_\alpha : U_\alpha \to V_\alpha$ that have smooth transition functions, and $\forall \alpha, \beta, \phi_\alpha(U_\alpha \cap U_\beta)$ is open in V_α (weird but useful)

then we see that if we declare U to be open iff $\phi_\alpha(U \cap U_\alpha)$ is open $\forall \alpha$, then this forms a topology (easy) and

Proposition 1. Apart from the possible failure of Hausdorff and second countable, using the above data as specified turns X into a topological n manifold, and $\{\phi_\alpha\}$ into a smooth atlas (so we have a smooth manifold).

Proof. It suffices to show that U_α are open and that ϕ_α are homeomorphisms (smoothness follows from the smoothness of the transition functions). As such it is sufficient to show that some $U \subseteq U_\alpha$ is open iff $\phi_\alpha(U)$ is open in V_α (this is to show that ϕ_α is a homeomorphism, which implies taht U_α is open by the openness of V_α). One direction is clear: if U is open, then by declaration, $\phi(U \cap U_\alpha = U)$ is open. Conversely, if $\phi(U)$ is open then we want that $\forall \beta, \phi_\beta(U \cap U_\beta)$ is open, but we observe that

$$\begin{aligned}
\phi_\beta(U \cap U_\beta) &= \phi_\beta \circ \phi_\alpha^{-1}(\phi_\alpha(U \cap U_\beta)) \\
&= (\phi_\alpha \circ \phi_\beta^{-1})^{-1}(\phi_\alpha(U) \cap \phi_\alpha(U_\alpha \cap U_\beta))
\end{aligned}$$

Here $\phi_\alpha(U_\alpha \cap U_\beta)$ is open as an intersection of $V_\alpha \cap V_\beta$, and $\phi_\alpha(W)$ is open by assumption. The transition function is continuous by assumption. So done. □

Finally, we note that we can define this set of ϕs and Us and Vs, etc. to be a "pseudo-chart", and can define a "smooth pseudo-structure," etc. from here. All we really want to show is that the toplogy is secondary once we have a good set of functions. In fact, using this approach we can skip X entirely, and start from sets U_α that we identify with different real spaces, and then stitch together with arbitrary pseudo-charts. That can definitely be done, but generally is quite complicated with many more moving parts, and so we usually, at least for the purpose of a course, start with a structure in mind, and turn that into a manifold, instead of stitching an arbitrary one together (although that may be a good source of counter examples).

Unfortunately, it does remain the case that showing Hausdorff and second-countable is still hard, although there are some tricks to do so. For second countability, using the subset of all rational balls will always work if the number of charts is countable. For Hausdorffness, as long as two points live in the same chart, we are immediately done, so combining charts and considering the exceptions can be an efficient approach.