Symmetries, Fields, and Particles

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Introduction

These notes are based on the course lectured by Ben Allanach in Michaelmas 2020. Due to the measures taken in the UK to limit the spread of Covid-19, these lectures were delivered online. These are not meant to be an accurate representation of what was lectures. They solely represent a mix of what I thought was the most important part of the course, mixed in with many (many) personal remarks, comments and digressions... Of course, any corrections/comments are appreciated.

To begin the course, the lecturer reminds of the definition of a group. I will not repeat this definition here. Groups, and Lie groups in particular, are essential in particle physics as a means of keeping track of the symmetries of particles. Here we get two kinds of symmetries:

- An **internal symmetry** is an inherent property of the fields/particles themselves. For example, one can rotate through quark colours, and in fact, we find that in order to make this possible, we require the existence of a force carrying particle (a gluon) to be involved. And to conserve colours, gluons contain a mix of colours to do so. The Lie group structure enforces colour conservation here... Since these colour rotations can be different at different points in space and time, these symmetries can also be called a **local symmetry** or a **gauge symmetry**.
- A **global symmetry** is a symmetry that leaves something the same across all space and time.
- A **external symmetry** is a symmetry involving spacetime coordinates. This includes symmetry under translation and Lorentz transformations. From these symmetries we get conserved structures (momentum, angular momentum, and energy). The Poincare group contains all these symmetries.

In terms of particles, bosons carry forces, and these includes gluons (strong force), photons (electromagnetic force), $Z^0, W^\pm$ carries the electroweak force. It has also been hypothesised that the graviton (spin 2) carries gravity, although it has never been observed. Also, for good symmetries, force carriers should be massless, but the spontaneous symmetry breaking, through the Higgs mechanism, can give force carriers mass (such as $W^{\pm}, Z^0$).

The lecturer provides a standard list of (elementary) fermions. I won't repeat these.

Just as a note, in the standard model, every particle has a field, and excitations of this field corresponds to "instances" of these particles. [End of lecture 1]

In lecture 2, the lecturer reviews basic group theory. I won't repeat that here. [End of lecture 2]

We continue looking at group properties, etc. Here are some definitions.

Definition 1. The **inner automorphism** associated with $g \in G$ is $\phi_g(h) = ghg^{-1}$.

We remark that treating elements of G as automorphisms in this way we see that $G/Z(G)$ is always a normal subgroup of $\mathrm{Aut}(G)$.

Definition 2. The **semi-direct product** of groups H, G, written $H \ltimes G$, has the product rule $(h, g)(h', g') = (hh', g\phi_h(g'))$, and inverse rule $(h, g)^{-1} = (h^{-1}, \phi_{h^{-1}}(g))$.

here we can see that $H \equiv H \ltimes G/G$, and that $D_n \equiv \mathbb{Z}_2 \ltimes \mathbb{Z}_n$ as expected.

Definition 3. The **commutator subgroup** or **derived subgroup** of group G, denoted $[G, G]$, the group generated by the commutators of G. These are always normal, and a group is called **perfect** when it equals its own commutator subgroup.

1 Matrix Groups

The lecturer describes the general linear group, orthogonal group, special orthogonal group, unitary group, and special unitary group. He also describes their dimensionality. In particular, using their properties, he shows that $O(n)$ has $\frac{1}{2}n(n-1)$ free parameters, and $SU(n)$ has $n^2 - 1$ parameters. He also mentions the famous fact that topologically $SU(2) \equiv S^3$. [End of lecture 3]

Let's continue defining some special matrix groups that may be less well known.

Definition 4. The **symplectic group** $S_p(2n, \mathbb{R})$ is a subset of the general linear group satisfying

$$M^T J M = J$$

for

$$J = \begin{pmatrix} 0 & -1 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & -1 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & -1 \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix}$$

This group has dimension $n(2n-1)$ when treated as a manifold, and the determinant of every element in the group is 1 (applying the determinant to the equation gives us ± 1, if you apply a generalised determinant for anti-symmetric matrices called the Pfaffian you can see the sign as well). For the sake of representations the most important thing about the symplectic symplectic group is that unlike other groups we've seen, like $SO(n)$ or $SU(n)$, the symplectic linear group is not compact.

Definition 5. The pseudo-orthogonal group $O(n,m)$ is the subgroup of the general linear group satisfying

$$M^T g M = g$$

for

$$g = \begin{pmatrix} I_n & \\ & -I_m \end{pmatrix}$$

Similarly we have $SO(n,m), U(n,m), SU(n,m)$. Also notice that $SO(1,1) = S_p(2,\mathbb{R})$.

1.1 Representations

A representation of a group is a homomorphism ϕ from that group to a space of linear maps on some vector space V on some field. For matrix groups, such as those mentioned above, we also have the canonical **fundamental representation** that represents matrices just as the matrices they are defined to be.

A representation is called **reducible** if we can find a linear subspace so that the representation can be viewed purely as a representation on that subspace (formally $\forall g \in G, u \in U, \phi(g)u \in U$ where $U \leq V$). There are theorems in representation theory about the reducibility of representations into irrudicible ones, such that we can always write

$$V \equiv \bigoplus_r U_r$$

for irreducible U_r. A handy lemma in this field.

Lemma 1. For irreducible representations V, W of G,

- $Hom_G(V, W)$ ccontains only 0 or isomorphisms.
- if the field we are working on is algebraically closed, then $\dim(Hom_G(V, W)) = 1$

I know this has been worded in too abstract a manner, but it is more familiar for me. Basically, two irreducible representations are either isomorphic, meaning that they are the same up to similarity transformations (change of bases), and there is only one similarity transformation up to multiplication by a constant factor.

Despite the fact that any group has an infinite number of representations, a lot of information can be summarised about them by calculating the **character** $\chi(g) = tr(\phi(g))$, which is just the trace of every matrix. Notice that since traces are unchanged by similarity transformations, they firstly are the same for isomorphic/equivalent representations, and secondly, are the same for conjugacy classes within the group, so they truly are just summaries of the representation. Nonetheless, many important properties can be deduced using them. For one, it is much easier to check if representations are equivalent this way.

Finally, we consider the **tensor product** such that on $V \otimes W$ the product of two representations acts as

$$(\phi \otimes \psi)(g)(v \otimes w) = \sum_{rs} \lambda_r \mu_s \phi(g)(v_r) \otimes \psi(g)(w_s) = \phi(g)(v) \otimes \psi(g)(w)$$

where v_r, w_s are the basis vectors of V, W and λ_r, μ_s express v, w in these bases. [End of lecture 4]

A quick note is that one can calculate the tensor product on characters as

$$\mathrm{tr}_{V_r \otimes V_s}(D^{R_r}(g) D^{R_s}(g)) = \mathrm{tr}_{V_r}(D^{R_r}(g))\, \mathrm{tr}_{V_s}(D^{R_s}(g))$$

Also, if $R_r \otimes R_s$ contains the singlet representation then we can construct an inner product $\langle v, v' \rangle$ which is invariant under the group.

1.2 Symmetries and Quantum Mechanics

Alright, now let's actually start using some of these mathematical ideas!

A (unitary) map U is called a **symmetry** if $|\langle \phi | \psi \rangle = |\langle U\phi, U\psi \rangle|^2$ for all ϕ, ψ. Furthermore, one can show that U is always linear or anti-linear (**anti-linear** means it is a linear map except it applies complex conjugation, so $Ua\,|\psi\rangle = a^* U\,|\psi\rangle$). Furthermore, it can be shown that anti-linear maps are only linear when we consider time reversal. Therefore, we will focus on linear maps.

One can also show that such maps, if they represent a symmetry group G satisfy the product rule

$$U(g)U(g') = e^{i\gamma(g,g')} U(gg')$$

where we usually just assume that $\gamma = 0$ leaving us with a homomorphism. One can further show that it must always commute with the Hamiltonian if it is a symmetry so

$UH = HU$

meaning that we can find a simultaneous eigenbasis for the two.

2 Rotations $SO(3)$ and $SU(2)$

An important group of symmetries are the group of rotations $SO(3)$ and $SU(2)$. We will prove an important results on them, namely that

$$SO(3) \equiv SU(2)/\mathbb{Z}_2$$

To do so we will develop an important tool, namely Pauli matrices. But before we start, let's review some standard properties of these groups. In $SO(3)$ we can write a general rotation by θ about axis n as

$$R_{ij} = \cos(\theta)\delta ij + (1 - \cos(\theta))n_i n_j - \sin(\theta)\epsilon_{ijk}n_k$$

which corresponds infinitesimally to

$$\delta x = \delta n \times x.$$

Now, let's start proving this isomorphism. First, the Pauli matrices are:

$$\sigma_1 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \sigma_2 = \begin{pmatrix} & -i \\ i & \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

which satisfy the properties

$$\sigma_i\sigma_j = \delta_{ij}I + \epsilon_{ijk}\sigma_k$$

and

$$\mathrm{tr}(\sigma_i\sigma_j) = 2\delta_{ij}$$

In other words, what exactly are the Pauli matrices? The Pauli matrices form an orthogonal basis (up to a factor of 2) when we treat the space of Hermitian traceless 2×2 matrices as a real vector space. Hermitian means the diagonal entries must be real, and traceless means the top left and top right entries must add to 0. This is generated by σ_3. Meanwhile, σ_1, σ_2 handle the off-diagonal basis. Why are they orthogonal? The natural inner product to use on matrix spaces is to rearrange the matrices into long vectors, and just to apply the dot-product to these vectors, and that can exactly be written as

$$A \cdot B = \mathrm{tr}\{AB\} = A_{ij}B_{ij}$$

Under this product, the property $\mathrm{tr}(\sigma_i\sigma_j) = 2\delta ij$ means these matrices truly are orthogonal. Now, we can easily form a complete basis for the space of 2×2 Hermitian matrices by adding the identity I. When doing so, we find that any matrix $A \in SU(2)$ can be written in this basis as

$$A = \frac{1}{2}\mathrm{tr}(A)I + \frac{1}{2}\mathrm{tr}(\sigma_i A)\sigma_i$$

Note in particular, that $\mathrm{tr}(A) = \mathrm{tr}(AI)$, so that if we define $\sigma_0 = I$ we get

$$A = \frac{1}{2}\mathrm{tr}\{\sigma_\mu A\}\sigma_\mu, \;\; \mu = 0, 1, 2, 3$$

which is really just writing it in a new orthogonal basis according to a certain dot product. [lecture 5]