

Differential Geometry

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Introduction

These notes are based on the course lectured by Dr Jack E Smith in Michaelmas 2020. Due to the measures taken in the UK to limit the spread of Covid-19, these lectures were delivered online. These are not meant to be an accurate representation of what was lectures. They solely represent a mix of what I thought was the most important part of the course, mixed in with many (many) personal remarks, comments and digressions... Of course, any corrections/comments are appreciated.

Unlike some of the other courses, there is no real introduction here, and we jump straight into the content!

1 Manifolds and Smooth Maps

Manifolds are spaces that locally look like \mathbb{R}^n . Formally this is:

Definition 1. X is a **topological n manifold** if X is a second countable Hausdorff topological space such that $\forall p \in X \exists \text{open } U \ni p$ and open $V \subseteq \mathbb{R}^n$ and homeomorphism $\phi : U \rightarrow V$.

Here,

Definition 2. Topological space X is **Hausdorff** if for every distinct $x, y \in X$ there exists open $U \ni x, V \ni y$ in X such that $U \cap V = \emptyset$.

and

Definition 3. Topological space X is **second countable** if there exists a set of open sets \mathcal{U} st that every open set in X can be written as a union of sets in \mathcal{U}

Since these two properties transfer to subsets, any subset of a topological n manifold is also a topological n manifold. Also, to give some more intuition, the condition that X is a second countable Hausdorff topological space is exactly equivalent to the condition that X is metrizable and has countably many components. It is just tradition that it is defined as above. Some more definitions.

Above, ϕ is called the **chart**, U is called the **coordinate patch**, although in some cases can also be called the chart. The functions $x_1 \circ \phi, \dots, x_n \circ \phi$ (so the components of the result) are called the **local coordinates**, and ϕ^{-1} is called the **paramaterisation**, although that term is not used that frequently. Finally, for overlapping charts, we can define the **transition map** between them as $\phi_2 \circ \phi_1^{-1} : \phi_1(U_1 \cap U_2) \rightarrow \phi_2(U_1 \cap U_2)$.

Now, we want to generalise calculus to manifolds, so it makes sense to start by trying to generalise the notion of smoothness. The simplest approach would be to say that f is smooth on X if it is smooth on the local coordinates. The issue then arises that this may not be consistent with smoothness on other charts (where these overlap). As such, we need to require that the transition maps are smooth as well. Consequently we do the following:

Definition 4. The **atlas** of a manifold is a collection of charts of a topological n manifold that covers all of X .

An atlas is **smooth** if all transition maps are smooth, and a map f is **smooth** on atlas \mathfrak{A} if $f \circ \phi_\alpha^{-1}$ is smooth $\forall \alpha$. As a result all local coordinate functions are smooth. Now, really, specifying the atlas precisely all the time is somewhat tedious, and somehow not the point, so we want a degree of flexibility. For this we define

Definition 5. Two atlases are **smoothly equivalent** if their union is smooth.

Note that this forms an equivalence relation (apply the chain rule on transition functions for transitivity).

Definition 6. A **smooth structure** is an equivalence class of atlases.

As we hope, we do indeed have that if a function is smooth wrt to an atlas, it is smooth wrt to any atlas in its smooth structure. Also, we can define a maximal atlas to be the union of all the atlases in a smooth structure, if we deem that to be convenient (includes trivial changes like translating or scaling the local coordinates).

Definition 7. A **smooth n manifold** is a topological n manifold with a smooth structure.

Note that under the product topology, the product of two manifolds naturally forms a new (smooth) manifold. Also, a remarkable fact is that for $n = 1, 2, 3$ all topological n manifolds have an essentially unique smooth structure, whereas this breaks down for $n \geq 4$. Also, a new chart is said to be **compatible** with an atlas if when added to the atlas, the atlas remains smooth.

Finally, to give a concrete example of a manifold, we may consider S^n , which forms a manifold with two charts: one being the sphere without the North pole, and the other being the sphere without the South pole, U_\pm with charts

$$\phi_\pm(y_0, \dots, y_n) = \frac{1}{1 \mp y_0}(y_1, \dots, y_n),$$

where the local coordinates are referred to as x^\pm . [End of DG1]

1.1 Forming Manifolds from Sets (Instead of Topological Spaces)

We observe that an atlas can generate a topology. In particular if we consider the data

- set X
- subsets $U_\alpha \subseteq X$
- open sets $V_\alpha \subseteq \mathbb{R}^n$
- bijections $\phi_\alpha : U_\alpha \rightarrow V_\alpha$ that have smooth transition functions, and $\forall \alpha, \beta, \phi_\alpha(U_\alpha \cap U_\beta)$ is open in V_α (weird but useful)

then we see that if we declare U to be open iff $\phi_\alpha(U \cap U_\alpha)$ is open $\forall \alpha$, then this forms a topology (easy) and

Proposition 1. Apart from the possible failure of Hausdorff and second countable, using the above data as specified turns X into a topological n manifold, and $\{\phi_\alpha\}$ into a smooth atlas (so we have a smooth manifold).

Proof. It suffices to show that U_α are open and that ϕ_α are homeomorphisms (smoothness follows from the smoothness of the transition functions). As such it is sufficient to show that some $U \subseteq U_\alpha$ is open iff $\phi_\alpha(U)$ is open in V_α (this is to show that ϕ_α is a homeomorphism, which implies that U_α is open by the openness of V_α). One direction is clear: if U is open, then by declaration, $\phi(U \cap U_\alpha) = \phi(U)$ is open. Conversely, if $\phi(U)$ is open then we want that $\forall \beta, \phi_\beta(U \cap U_\beta)$ is open, but we observe that

$$\begin{aligned}\phi_\beta(U \cap U_\beta) &= \phi_\beta \circ \phi_\alpha^{-1}(\phi_\alpha(U \cap U_\beta)) \\ &= (\phi_\alpha \circ \phi_\beta^{-1})^{-1}(\phi_\alpha(U) \cap \phi_\alpha(U_\alpha \cap U_\beta))\end{aligned}$$

Here $\phi_\alpha(U_\alpha \cap U_\beta)$ is open as an intersection of $V_\alpha \cap V_\beta$, and $\phi_\alpha(W)$ is open by assumption. The transition function is continuous by assumption. So done. \square

Finally, we note that we can define this set of ϕ s and U s and V s, etc. to be a “pseudo-chart”, and can define a “smooth pseudo-structure,” etc. from here. All we really want to show is that the topology is secondary once we have a good set of functions. In fact, using this approach we can skip X entirely, and start from sets U_α that we identify with different real spaces, and then stitch together with arbitrary pseudo-charts. That can definitely be done, but generally is quite complicated with many more moving parts, and so we usually, at least for the purpose of a course, start with a structure in mind, and turn that into a manifold, instead of stitching an arbitrary one together (although that may be a good source of counter examples).

Unfortunately, it does remain the case that showing Hausdorff and second-countable is still hard, although there are some tricks to do so. For second countability, using the subset of all rational balls will always work if the number

of charts is countable. For Hausdorffness, as long as two points live in the same chart, we are immediately done, so combining charts and considering the exceptions can be an efficient approach.

1.2 Projective Spaces and Grassmannians

Today we do an extended example to develop some more interesting manifolds. First, we look at the real projective linear space \mathbb{RP}^n , which is the space $\mathbb{R}^{n+1} \setminus \{0\}$ under the equivalence relation $x \sim y$ iff $\exists \lambda \in \mathbb{R} \setminus \{0\}, x = \lambda y$. As such, any point can be labelled by the *ratio* of coordinates $[x_0 : x_1 : \dots : x_n]$, and for the purpose of comparison, we are free to set any nonzero coordinate x_i to 1.

To turn this into a smooth n -manifold, let's consider the following pseudo-chart:

$$U_i = \{[x_0 : x_1 : \dots : x_n] | x_i \neq 0\}$$

$$\phi_i([x_0 : x_1 : \dots : x_n]) = \frac{1}{x_i}(x_0, \dots, \hat{x}_i, \dots, x_n)$$

where \hat{x}_i means omit x_i from the list.

Lemma 1. These form a pseudo-atlas.

Proof. Need to show $\phi_i(U_i \cap U_j)$ open and $\phi_j \circ \phi_i^{-1}$ smooth. wlog, $i = 0, j = 1$ and s, t are our local coordinates of interest. With a little thought we see $\phi_0(U_0 \cap U_1) = \{s_1 \neq 0\}$ which is open. To see $\phi_1 \circ \phi_0^{-1}$ is smooth, notice that

$$\phi_0^{-1}(s) = [1 : s_1 : \dots : s_n]$$

$$\phi_1^{-1}(t) = [t_1 : 1 : \dots : t_n]$$

and consequently we see that certainly

$$\phi_1 \circ \phi_0^{-1}(s) = \frac{1}{s_1}[1 : s_1 : \dots : s_n]$$

is smooth, so indeed we have pseudo-atlas. \square

All that is left now is to show that \mathbb{RP}^n is second countable and Hausdorff. Second countability follows from the finite number of covers that we are using. Hausdorffness is harder though. We cannot use our previous tactic of relying on the fact that any two points will lie in at least some chart. So instead we expand our atlas until that is the case. Above, we essentially formed charts using the standard unit vector basis, and omitted that each time. This can be generalised in two ways. Firstly, we can expand to any basis, which copies the above approach and is entirely straight forward. Secondly, we can consider a space W to be our line in \mathbb{R}^n , and any complement W' of it. This abstracts the notion, but it has its uses as well.

Taking this approach, we see that for a given line input T the projection $\pi_W|_T$ restricted to T is a bijection (as these have forcibly the same dimension) so taking

$$\psi_T = \pi'_W \circ (\pi_W|_T)^{-1} : W \rightarrow W'$$

allows us to biject from our manifold to W' which is isomorphic to \mathbb{R}^n , giving us a chart. It is an exercise on the example sheet to show this is compatible. Consequently, filling these all, we can show that the space is Hausdorff as well, meaning that \mathbb{RP}^n forms a smooth n -manifold. Now, just as a note, really, we are not identifying the line T with a point in W' . Rather, we are identifying T with the map ψ_T . This will be relevant.

Now, one generalisation we will make, is to that of the **Grassmannian**. The Grassmannian, $Gr(k, n)$ is the space of k -subspaces within \mathbb{R}^n . Note in particular that $\mathbb{RP}^n = Gr(1, n+1)$. And since we identify subspaces T with ψ_T we also see that $\dim Gr(k, n) = \dim \mathcal{L}(W, W') = k(n-k)$. Finally, we want to remark that we can use \mathbb{C} instead of \mathbb{R} , to give use \mathbb{CP}^n , and $Gr_{\mathbb{C}}(k, n)$. Incidentally, the transition maps here are not only smooth, but also holomorphic, meaning that these form complex manifolds. [End of DG 3]

1.3 Smoothness of Maps Between Manifolds

So how do we define the smoothness of a map between manifolds (instead of to a Euclidean space)? As expected, we just work with local coordinates. Consequently for smooth manifolds X, Y of respective dimension n, m with respective atlases $\phi_\alpha : U_\alpha \rightarrow V_\alpha, \psi_\beta : S_\beta \rightarrow T_\beta$, then

Definition 8. $F : X \rightarrow Y$ is **smooth** wrt to these atlases iff $\forall \alpha, \beta$

$$\psi_\beta \circ F \circ \phi_\alpha^{-1} : \phi_\alpha(F^{-1}(S_\beta)) \rightarrow T_\beta$$

is smooth as a map between open subsets in \mathbb{R}^n and \mathbb{R}^m . Importantly, this definition only makes sense if F is **continuous**, and in particular we require $F^{-1}(S_\beta)$ to be open for this definition to make sense.

From here, we remark some trivial lemmas [which I won't write up properly] that due to the smoothness of transition functions, it suffices to check that F is smooth in the neighbourhood of some point of a single chart. If that holds for all points, then F is smooth as well. We also see that smoothness depends only on the smooth structure, and not the precise atlas, as one may expect.

Finally, again some more sanity check lemmas. It is easy to see that this definition coincides with the usual definition of smoothness between finite dimensional real spaces, and also corresponds to the definition of smoothness from a manifold to \mathbb{R} used earlier in this course. More importantly, we also see that a composition of smooth maps is smooth, and finally, this is one result I will write down as a lemma, we see that smoothness is a local property of the map:

Lemma 2. Smoothness is **local in source**, meaning $\forall F : X \rightarrow Y$, F is smooth iff \exists open cover W_γ st $\forall \gamma, F|_{W_\gamma}$ is smooth.

This is indeed a neat way of saying that a property is truly local in nature.

In line with our previous approach, we'd like to be able to specify smoothness without relying on the topology of the space. It is a bit pedantic, but nevertheless here we go:

Proposition 2. $F : X \rightarrow Y$ is smooth iff \exists cover (not necessarily smooth - we can't rely on the topology) $W_{\gamma \in \mathcal{C}}$ of X st $\forall \gamma \exists \alpha(\gamma) \in \mathcal{A}, \beta(\gamma) \in \mathcal{B}$ such that:

1. $W_{\gamma} \subseteq U_{\alpha(\gamma)}$ and $F(W_{\gamma}) \subseteq S_{\beta(\gamma)}$
2. $\phi_{\alpha(\gamma)}(W_{\gamma})$ open in $V_{\alpha(\gamma)}$ (equivalent to saying W_{γ} open in X once the topology has been enforced)
3. $\psi_{\beta(\gamma)} \circ F \circ \phi_{\alpha(\gamma)}|_{W_{\gamma}}$ is smooth.

Proof. • only if: pick $\mathcal{C} = \mathcal{A} \times \mathcal{B}$, and α, β as projection maps, and $W_{\gamma} = U_{\alpha} \cap F^{-1}(S_{\beta})$. Then the result follows.

- if: we only need to check that F is cont, so that $F^{-1}(S)$ is open for open S , or equivalently, since we know the W_{γ} will turn out to be open, that $F^{-1}(S) \cap W_{\gamma}$ is open each time. To do so we note that ϕ_{α} is a homeomorphism, we we're done if the following is open:

$$\begin{aligned} \phi_{\alpha}(F^{-1}(S) \cap W_{\gamma}) &= \phi_{\alpha}(F^{-1}(S \cap S_{\beta}) \cap W_{\gamma}) \\ &= \phi_{\alpha}(F^{-1}(S \cap S_{\beta})) \cap \phi_{\alpha}(W_{\gamma}) \\ &= (\psi_{\beta} \circ F \circ \phi_{\alpha}^{-1})^{-1}(\psi_{\beta}(S)) \cap \phi_{\alpha}(W_{\gamma}) \end{aligned}$$

which is certainly open. □

Example 1. $H : S^{2n+1} \rightarrow \mathbb{CP}^n$, the Hopf map, is an example of a smooth map.

Finally, we'd like to define some notion of equivalence on manifolds. Here we define

Definition 9. A **diffeomorphism** is a smooth invertible map between manifolds with a smooth inverse.

Definition 10. Two manifolds are diffeomorphic when there exists a diffeomorphism between them.

You might think that since we're doing geometry, and not topology, we care about the values involved, and these equivalence classes are not of interest, but really, these equivalence classes have less to do with the space, and more to do with the smooth structure on them. If you change charts, then you also may have to change the functions involved. As such, diffeomorphism establish an equivalence between smooth structures more than between manifolds. This is particularly relevant for the time, as before, when we said that for $n \leq 3$ all manifolds have an essentially unique smooth structure. Here "essentially unique" meant unique up to diffeomorphism.

Example 2. A particularly famous manifold is the Riemann sphere, which is in this context often defined to be \mathbb{CP}^1 , but is diffeomorphic to S^2 , and $\mathbb{C} \cup \{\infty\}$ via either the stereographic projection or identifying via normalising (normalise $[z_0 : z_1]$ to $[1 : z]$ or “ ∞ ” for $\mathbb{C} \cup \{\infty\}$, etc.).

Finally, we do have a somewhat important result that comes out from here:

Lemma 3. If smooth manifolds X, Y are diffeomorphic, then they are of the same dimension.

Proof. Fix diffeomorphism $F : X \rightarrow Y, p \in X, U \ni p, V \ni F(p), \phi : U \rightarrow V, \psi : S \rightarrow T$ such that U, V, S, T open and shrink them such that $F(U) = S$, then let $G = \psi \circ F \circ \phi^{-1}, H = \phi \circ F^{-1} \circ \psi^{-1}$, then both of these are differentiable, and mutually inverse maps on subsets of $\mathbb{R}^n, \mathbb{R}^m$. Taking derivatives we get two linear maps that are mutually inverse, and so must be of the same dimension, meaning $m = n$. \square

[End of DG 4]

1.4 Tanget Spaces

We want to define tangent spaces on abstract manifolds.

Definition 11. A **curve based at** p is a curve, $\gamma : I \rightarrow X$ where I is an open subset of \mathbb{R} such that $\gamma(0) = p$. (they also have to be somewhat smooth)

Definition 12. Two curves γ_1, γ_2 based at p **agree to first order** iff \exists a chart such that

$$(\phi \circ \gamma_1)'(0) = (\phi \circ \gamma_2)'(0)$$

This forms an equivalence relation. Reflexivity, and symmetry are clear, however, transitivity is not entirely obvious. Hence we have the lemma

Lemma 4. If two curves agree to first order according to one chart, they agree to first order according to any chart.

Proof. Just apply the chain rule to the transition functions. \square

As such we then say

Definition 13. The **tangent space** of X at p , $T_p X$ is the set of curves at p modulo agreement to first order. Elements of this set are denoted $[\gamma]$ (for the equivalence class).

Now, if we define the map

$$\begin{aligned} \pi_p^\phi : \{\text{curves based at } p\} &\rightarrow \mathbb{R}^n \\ \gamma &\mapsto (\phi \circ \gamma)'(0) \end{aligned}$$

then we see that modulo agreement to first order this is injective. Furthermore, if we show it surjective, we so $T_p X$ is isomorphic to \mathbb{R}^n :

Proposition 3. $T_p X$ is isomorphic to \mathbb{R}^n and π_p^ϕ is an isomorphism.

Proof. We just need to show π_p^ϕ is surjective. $\forall v \in \mathbb{R}^n$ consider $\gamma(t) = \phi^{-1}(\phi(p) + tv)$. This works. \square

Now, as a matter of notation we use ∂_i or ∂_{x_i} to denote $(\pi_p^\phi)^{-1}(e_i)$. Clearly, this notation has connotations, but interestingly these turn out to be well-founded, for the following two reasons:

Proposition 4. Vectors transform as

$$\partial_{y_i} = \sum_j \partial_{y_i} x_j \partial_{x_j}$$

Proof.

$$\begin{aligned} \partial_{y_i} &= (\pi_p^{\phi_2})^{-1}(e_j) \\ &= (\pi_p^{\phi_1})^{-1}\left(\sum_j \partial_{y_i} x_j e_j\right) \\ &= \sum_j \partial_{y_i} x_j (\pi_p^{\phi_1})^{-1}(e_j) \\ &= \sum_j \partial_{y_i} x_j \partial_{x_j} \end{aligned}$$

\square

Furthermore, we can see that when written in the standard unit basis ∂_i , any $[\gamma]$ has its coefficients precisely the values it gets as its derivatives, so one can interpret these vectors as derivatives to a certain extent. [End of DG 5]

1.5 Vectors as Differential Operators

In the last section, we discussed a geometric way to derive and think about tangent vectors. However, we can also do so in an algebraic manner. That is what we do here.

If we are to describe tangent vectors algebraically as differential operators, then we want to describe them as “hitting” something. What will they hit, well, quite naturally, they will hit functions from the manifold. Consequently, we consider a function $f : U \rightarrow \mathbb{R}$ for open $U \ni p$ for some point p , and then the result of the hitting along some path would be $(f \circ \gamma)'(0)$ (so we are interested in what this “hitter” may be).

But we also want to relate this to our original picture of vectors, so we have:

Lemma 5. $(f \circ \gamma)'(0)$ depends only on $[\gamma]$, and for $[\gamma] = \sum a_i \partial_i$, we have

$$(f \circ \gamma)'(0) = \sum a_i \partial_i f|_p$$

Proof.

$$\begin{aligned}(f \circ \gamma)'(0) &= \frac{d}{dt} \Big|_{t=0} ((f \circ \phi^{-1}) \circ (\phi \circ \gamma))(t) \\ &= \sum_i \partial_{x_i} f|_p (x_i \circ \gamma)'(0)\end{aligned}$$

since $f \circ \phi^{-1}$ is just f written in terms of the local coordinates. \square

So we see that combined with a hitter, we can turn any γ into a differential operator on f , but we can make a further simplification. We notice in particular that we don't really care that much about the open neighbourhood U containing p . As such we define

Definition 14. The **germ** of smooth f at p is the equivalence class $[(U, f)]$ for smooth $f : U \rightarrow \mathbb{R}$, open $U \ni p$ modulo the equivalence relation $(U_1, f_1) \sim (U_2, f_2)$ iff there exists open $V \ni p$ such that $V \subseteq U_1, U_2$, and $f_1|_V = f_2|_V$.

We call the space of germs at p , $\mathcal{O}_{X,p}$. Now we add some algebraic definitions that, as far as I can tell, serve little purpose in this context: note that by using constant functions we can easily construct a homomorphism $\mathbb{R} \rightarrow \mathcal{O}_{X,p}$ turning it into an \mathbb{R} -algebra. Also,

Lemma 6. As a ring (which it forms), $\mathcal{O}_{X,p}$ has a unique maximal ideal \mathfrak{m} which is precisely the set of functions that vanish at p .

Proof. \mathfrak{m} forms a maximal ideal since it is the kernel of a homomorphism to a field: $[(U, f)] \mapsto f(p)$. It is unique, since for any element outside of this ideal, choosing U to be small, we can see f is always nonzero, meaning that the ring contains a multiplicative inverse of our element $[(U, 1/f)]$, but if an ideal contains an invertible element, that ideal is the whole ring, and so \mathfrak{m} is the unique maximal ideal. \square

Precisely because of the situation that occurs here, rings with a unique maximal ideal are called **local rings**. Even in other context, one still imagines that maximal ideal as being the place where some function vanishes.

Having worked so far (although the algebra bit still seems unnecessary to me), we see that we can hit functions with vectors to get real numbers via

$$T_p X \times \mathcal{O}_{X,p} \rightarrow \mathbb{R}$$

or equivalently there exists

$$D : T_p X \rightarrow \mathcal{O}_{X,p}^\vee$$