Quantum Information Theory

quinten tupker

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Introduction

These notes are based on the course lectured by Professor Matthew Wingate in Lent 2020. This was lectured online due to measures taken to counter the spread of Covid-19 in the UK. These are not necessarily an accurate representation of what was lectures, and represent solely my personal notes on the content of the course, combinged with probably, very very many personal notes and digressions... Of course, any corrections/comments would be appreciated.

[the lecturer outlines the course] This course is an extension of the Michaelmas Quantum Field Theory course that introduces renormalisation and the path integral formulation of quantum field theory.

The Path Integral in Quantum Mechanics

We start by reformulating the Schrödinger equation as an integral equation, which turns out to be a path integral. Anyways, starting with Schrödinger's equation for a Hamiltonian $H(x, p), [x, p] = i\hbar$ with

$$H = \frac{p^2}{2m} + V(x) \tag{1}$$

we have

$$i\hbar\partial_t |\psi(t)\rangle = H |\psi(t)\rangle \implies |\psi(t)\rangle = e^{-iHt/\hbar} |\psi(0)\rangle \tag{2}$$

where in the Schrödinger picture the states evolve, but the operators remain constant, and the wavefunction $\Psi(x, t) = \langle x|\psi(t)\rangle$. As such we can rewrite our equation as

$$\langle x| H |\psi(x)\rangle = \left(\frac{-\hbar^2}{2m}\partial_x^2 + V(x)\right) \langle x|\psi(t)\rangle \tag{3}$$

so we can write

$$\begin{aligned}\Psi(x,t) &= \langle x|\psi(t)\rangle \\ &= \langle x|\, e^{-iHt/\hbar}\,|\psi(0)\rangle \\ &= \int_{-\infty}^{\infty} dx_0\, \langle x|\, e^{-iHt/\hbar}\,|x_0\rangle\,\langle x_0|\psi(0)\rangle \\ &= \int_{-\infty}^{\infty} dx_0 K(x,x_0,t)\Psi(x_0,0)\end{aligned}$$

for **kernel** $K(x,x_0,t) = \langle x|\, e^{-iHt/\hbar}\,|x_0\rangle$. Now, if it is hard to calculate K for large t, it can be beneficial to split this into many intervals for many values of t, such as $0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = T$ leaving

$$K(x,x_0,T) = \int_{-\infty}^{\infty} \prod_{r=1}^{n} dx_r\, \langle x_{r+1}|\, e^{-iH(t_{r+1}-t_r)/\hbar}\,|x_r\rangle\,\langle x_1|\, e^{-iH(t_1-t_0)/\hbar}\,|0\rangle \quad (4)$$

which is in a sense an integral over all possible sequences of values of x.

In free field theory ($V = 0$) this can be explicitly evaluated using a Gaussian integral by rewriting things in the momentum basis as (use $\langle x|p\rangle = e^{ipx/\hbar}$)

$$\begin{aligned}K_0(x,x',t) &= \langle x|\, e^{\frac{-ip^2t}{2m\hbar}} \int \frac{dp}{2\pi\hbar}\,|p\rangle\,\langle p|x'\rangle \\ &= \int_{-\infty}^{\infty} \frac{dp}{2\pi\hbar} e^{\frac{-ip^2t}{2m\hbar}} e^{ip(x-x')/\hbar} \\ &= e^{\frac{ip(x-x')^2}{2\hbar t}} \sqrt{\frac{m}{2\pi i\hbar t}}\end{aligned}$$

where we note that the limit as $t \to 0$ is $\delta(x - x')$ which indeed matches $\langle x|x'\rangle = \delta(x - x')$ as expected.

Now in an interacting theory, we struggle with the Baker-Campbell-Hausdorff fact that $e^A e^B \neq e^{A+B}$ so using Suzuki-Trotter we separate into steps size $t_{r+1} - t_r = \delta t << T$ meaning that

$$e^{-iH\delta t/\hbar} \approx e^{\frac{-ip^2\delta t}{2m\hbar}} e^{\frac{-iV(x)\delta}{\hbar}} (1 + O(\delta t^2)) \quad (5)$$

so for $T = n\delta t$ we find that

$$K(x,x_0,T) = \int \prod_{r=1}^{n} dx_r \left(\frac{m}{2\pi i\hbar\delta t}\right)^{\frac{n+1}{2}} e^{i\sum_{r=0}^{n}\left(\frac{m}{2\hbar}\left(\frac{x_{r+1}-x_r}{\delta t}\right)^2 - V(x_r)/\hbar\right)\delta t} \quad (6)$$

which in the limit $n \to \infty, \delta t \to 0$ while keeping T constant leaves

$$\frac{1}{\hbar}\int_0^T dt\left(\frac{1}{2}m\dot{x}^2 - V(x)\right) = \int_0^T dt L(x,\dot{x}) = S \quad (7)$$

for classical Lagrangian L and action S. This is what we refer to as a path integral or function integral:

$$K(x, x_0, t) = \int \mathcal{D}x e^{iS/\hbar} \tag{8}$$

where $\mathcal{D}x$ is the limit describd above. Of course, many questions about the existence and uniqueness, etc. of such limits exists, and in fact often this limit does not exist, but in the cases we are interested it, it works well enough... [End of lecture 1]

We make the following remarks

- In the classical limit $\hbar \to 0$ the lowest frequencies dominate K. This is equivalent to Hamilton's principle (the principle of least action), as expected.
- it is common and helpful to extend analytically to imaginary time $\tau = it$ leaving $\langle x| e^{-H\tau/\hbar} |x_0\rangle = \int \mathcal{D}x e^{-S/\hbar}$ which has better convergence properties and is easier to interpret than the complex version (Hamilton's principle appears more easily as well).

1 Integrals and their diagrammatic expansion

The above considered quantum mechanics, which is in a sense the 0+1 dimension vrsion of QFT (since x is treated as an operator, while t is treated as a variable). To move to more general QFT, we start, strangely, with 0 dimensional QFT, for $\phi : \{\cdot\} \to \mathbb{R}$ a field on a single point. Here,

$$\mathcal{Z} = \int_{\mathbb{R}} d\phi e^{-S(\phi)/\hbar} \tag{9}$$

where we assume S is an even polynomial in ϕ for convergence reasons, and we are interested in expectation values

$$\langle f \rangle = \frac{1}{\mathcal{Z}} \int d\phi f(\phi) e^{-S(\phi)/\hbar} \tag{10}$$

1.1 Free Theory

For N fields $\phi_a, a = 1, \ldots, N$, let $S(\phi) = \frac{1}{2}\phi^T m \phi$ for a symmetric positive definite matrix $m = P \Delta P^T$ for orthogonal P. As such, we can write this essentially Gaussian integral as

$$\mathcal{Z}_0 = \int d^N \phi e^{-\frac{1}{2\hbar}\phi^T m \phi} = \sqrt{\frac{(2\pi\hbar)^N}{\det m}} \tag{11}$$

From here, we can turn this into a generating function to calculate expectation values by taking derivatives by turning $S_0(\phi) \mapsto S_0(\phi) - J^T\phi$, and writing

$\mathcal{Z}_0 = \mathcal{Z}_0(J)$ (now a generating function(al) - functional later on). We then remark

$$\mathcal{Z}_0(J) = \mathcal{Z}_0(0)e^{-\frac{1}{2\hbar}J^T m^{-1} J} \tag{12}$$

Then we can calculate the correlation functions as

$$\langle \phi_a \phi_b \rangle = \frac{1}{Z_0(0)} \hbar^2 \partial_{J_a} \partial_{J_b} \mathcal{Z}_0(J)|_{J=0} = h(m^{-1})_{ab} \tag{13}$$

Conveniently, this can be diagrammatically interpreted as a connecting two vertices on indices a, b with an undirected edge. We can generalise this to linear operator $l(\phi) = \sum l_a \phi_a$ as (for p such operators)

$$\langle l^{(1)}(\phi) \dots l^{(p)}(\phi) \rangle = \hbar^p \prod_{i=1}^{p} l^{(i)}(\partial_J) e^{\frac{1}{\hbar}J^T m^1 J} \tag{14}$$

If p is odd, this is always 0 by symmetry, but if p is even this corresponds to a linear combination of products $m^{-1}_{ab} m^{-1}_{cd} \dots$

Example 1. For $p = 4, l^{(1)}_a = \delta_{ab}, l^{(2)}_a = \delta_{ac}, l^{(3)}_a = \delta_{ad}, l^{(4)}_a = \delta_{af}$ then

$$\langle \phi_b \phi_c \phi_d \phi_f \rangle = \hbar^2 (m^{-1}_{bc} m^{-1}_{df} + m^{-1}_{bd} m^{-1}_{cf} + m^{-1}_{bf} m^{-1}_{cd}) \tag{15}$$

which also corresponds to the ways in which we can connect four vertices with undirected edges