Quantum Field Theory

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Introduction

These notes are based on the course lectured by Professor Nicholas Dorey in Michaelmas 2020. This was lectured online due to measures taken to counter the spread of Covid-19 in the UK. These are not necessarily an accurate representation of what was lectures, and represent solely my personal notes on the content of the course, combinged with probably, very very many personal notes and digressions... Of course, any corrections/comments would be appreciated.

But, let's actually introduce the content of this course. What is quantum field theory? Quantum field theory (QFT) essentially succeeds in merging special relativity and quantum mechanics. Why is this so difficult? The first relativistic theory was electromagnetism, and the biggest idea that was introduced, and what truly set it apart from older theories was the idea of fields. In Newtonian theory, and much of what followed, the protagonist of the theory was a particle, or if not a particle, at least a body of some kind. Field theory changed. Now, there was a new protoganist: the field. What difference does that make, architectually? Firstly, the field theory is often simpler. But more importantly, the biggest structural difference is that field theory excels in describing "delayed interactions." When the particle is the progagonist, it is very difficult describe theories where forces are not instantaneous. Field theory avoids this. All interactions are made through the field, through which they propogate through space. Now, delayed responses become natural. In electromagnetism, the simplest expression thereof is electromagnetic waves: light.

What does this have to do with relativity? Well, as soon as high speeds become relevant, forces can no longer be considered instantaneous. As such, it is difficult to keep using particles as the protoganist of these theories. Consequently, the natural step is to make, instead of particles, fields the protoganist of this new quantum theory we are developing.

That is the goal of QFT. There is one important consequence though, once particles are no longer the protagonists of the theory. That is that particle number no longer has to be conserved. In the most elegant fashion, by removing the supremacy of the "particle" in our theory, and replacing it with the more powerful notion of the field, particles merely become phenomenon to be observed, and tools of analysis. In this context, it is only natural that particle number is no longer conserved. Whereas before, the wavefunction was often associated with a wave-particle like object, now the wavefunction (which is a field) describes a multiparticle state. Well, really it describes the field, and the multiparticle state is something that can be deduced from it. Somehow, although this is just the beginning of the course, I feel that that's not that important anymore. It is deeply intriguing though, how the imposition of boundary conditions somehow forces a degree of discreteness onto this theory...

Well then, the overall architecture is more or less the same as standard quantum theory. It is probabilistic, and we assume a degree of symmetry under boosts, and rotations (isotropy and translation invariance). The fundamental approach to making predictions still boils down to the same calculation: evaluating

$$A_{i\to f} = \langle f|\, e^{iHT}\, |i\rangle$$

for probability amplitude A, initial state i, final state f, Hamiltonian (time translation generator) H, and time interval T.

There are two caveates with most of these field theories, though. Firstly, they have not been mathematically formalised, so often there are areas that are somewhat ambiguous. Secondly, contributing to this ambiguity, many of the sums are divergent, so the meaning of some calculations can really be somewhat ambiguous... How curious! I'd like to think about this a bit more...

0.1 Preliminaries

Anyways, getting down to business. We'll be mostly using natural units during this course. That means that $c = \hbar = 1$, and these can be added back into the calculation using dimensional analysis. The effect of this, is that the only unit used throughout all calculations is really a unit of mass-energy. As such, all quantities scale by a power of the unit of energy.

Definition 1 (dimension of X)**.** Denoted $[X]$, this is δ, such that for unit of mass-energy M, X scales as M^δ. δ may also be called the scaling or the engineering dimension of X.

Also, for special relativity we use the convention that we are working on Minkowsky space-time $\mathbb{R}^{3,1}$, with metric tensor

$$\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1).$$

0.2 Classical Field Theory

Fortunately for me, we are starting with a description of classical field theory, and we are starting, very simply, with scalar fields.

Definition 2 (Scalar Field)**.** $\phi(t, \underline{x}) = \phi(x) : \mathbb{R}^{3,1} \to \mathbf{R}$ is a sclar field if it is Lorentz invariant, meaning that it follows the transformation rule

$$\phi(x) \to \phi(\Lambda^{-1}(x))$$

Here the domain is called the spacetime, and the codomain is called the **field space**. These may, and will be replaced with other spaces.

Changing the spacetime here corresponds to implementing gravity in some way or another, by changing the manifold we are working on. The field space corresponds to the complexity of what is being descibed. Since we will be describing multi-particle states with our wavefunction, this will become significantly more complex. And, I intentionally left the description of what it means to be Lorentz invariant a bit vague, since, well, in our case, it just means being invariant under the Lorentz transformations, which is the group of linear transformations that preserves the Minkowsky metric (ie. the set of matrices such that $\Lambda\eta\Lambda^T = \eta$). But really, while linearity makes a lot of sense in the context of linear Minkowsky space, I doubt (though I have no familiarity with this area) this remains the case when we are on an arbitrary manifold, which happens when we consider general relativity. As such, I prefer to think of Λ as any arbitrary invertible map on the manifold, corresponding to the symmetries we impose.

The difficulties that arise in comibing quantum theory with general relativity are also quite clear. It does seem tremendously difficult. If we do not simply assuming that general relativity simply bends space, which I will assume is not entirely the case, or else I feel a theory reconciling the two would have already been developed long ago, in spite of the tremendous difficulty of the calculations involved, and it also does not seem to make much sense of Hawking radiation, since in general relativity, the space at the centre of a black hole truly is cut off... Nevertheless, from what I've heard, if you want to turn gravity into a quantised force with mediator particle, then somehow the protagonist of that theory would be not only be a field, but somehow span the space of possible manifolds as well. Purely intuitively, I would imagine that we would be getting fields of the form $\phi : \mathcal{D} \to V$ where $\mathcal{D}$ is an object that stitches many manifolds together. Brrr... I have not thought too deeply about this, but that does seem like a truly terrifying object indeed! Or perhaps not quite. Hm, it might be worth thinking a bit more about this...

Anyways, I also wanted to remark that having fields transform as $\phi(\Lambda^{-1}x)$ is more or less an arbitrary definition that is called the "active" definition of field transformations. Oh well, you can assign some intuition to it, but it is more or less convention to use the inverse of the matrix instead of the matrix itself.

Extending our notion of fields to vector fields, first note that notation wise, we use $\partial_\mu\phi = \frac{\partial\phi}{\partial x^\mu}$, and we define

Definition 3 (ϕ^μ transforms as a vector field)**.** if

$$\phi \to \Lambda^\mu_\nu \phi^\nu(\lambda^{-1}x)$$

This is just the transformation rule for rank 1 tensors, so is nothing particularly remarkable. The only remarkable part is that, as a result $\partial^\mu\phi$ transformas as vector, and so the following becomes a rank 0 tensor (ie, a scalar field) $\partial^\mu\phi\partial_\mu\phi$.

Well, that ends lecture 1, for those of you interested.