

Quantum Information Theory

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Introduction

These notes are based on the course lectured by Professor Matthew Wingate in Lent 2020. This was lectured online due to measures taken to counter the spread of Covid-19 in the UK. These are not necessarily an accurate representation of what was lectures, and represent solely my personal notes on the content of the course, combined with probably, very very many personal notes and digressions... Of course, any corrections/comments would be appreciated.

[the lecturer outlines the course] This course is an extension of the Michaelmas Quantum Field Theory course that introduces renormalisation and the path integral formulation of quantum field theory.

The Path Integral in Quantum Mechanics

We start by reformulating the Schrödinger equation as an integral equation, which turns out to be a path integral. Anyways, starting with Schrödinger's equation for a Hamiltonian $H(x, p), [x, p] = i\hbar$ with

$$H = \frac{p^2}{2m} + V(x) \quad (1)$$

we have

$$i\hbar\partial_t |\psi(t)\rangle = H |\psi(t)\rangle \implies |\psi(t)\rangle = e^{-iHt/\hbar} |\psi(0)\rangle \quad (2)$$

where in the Schrödinger picture the states evolve, but the operators remain constant, and the wavefunction $\Psi(x, t) = \langle x|\psi(t)\rangle$. As such we can rewrite our equation as

$$\langle x| H |\psi(x)\rangle = \left(\frac{-\hbar^2}{2m} \partial_x^2 + V(x) \right) \langle x|\psi(t)\rangle \quad (3)$$

so we can write

$$\begin{aligned}
\Psi(x, t) &= \langle x | \psi(t) \rangle \\
&= \langle x | e^{-iHt/\hbar} | \psi(0) \rangle \\
&= \int_{-\infty}^{\infty} dx_0 \langle x | e^{-iHt/\hbar} | x_0 \rangle \langle x_0 | \psi(0) \rangle \\
&= \int_{-\infty}^{\infty} dx_0 K(x, x_0, t) \Psi(x_0, 0)
\end{aligned}$$

for **kernel** $K(x, x_0, t) = \langle x | e^{-iHt/\hbar} | x_0 \rangle$. Now, if it is hard to calculate K for large t , it can be beneficial to split this into many intervals for many values of t , such as $0 = t_0 < t_1 < \dots < t_n < t_{n+1} = T$ leaving

$$K(x, x_0, T) = \int_{-\infty}^{\infty} \prod_{r=1}^n dx_r \langle x_{r+1} | e^{-iH(t_{r+1}-t_r)/\hbar} | x_r \rangle \langle x_1 | e^{-iH(t_1-t_0)/\hbar} | 0 \rangle \quad (4)$$

which is in a sense an integral over all possible sequences of values of x .

In free field theory ($V = 0$) this can be explicitly evaluated using a Gaussian integral by rewriting things in the momentum basis as (use $\langle x | p \rangle = e^{ipx/\hbar}$)

$$\begin{aligned}
K_0(x, x', t) &= \langle x | e^{-\frac{ip^2 t}{2m\hbar}} \int \frac{dp}{2\pi\hbar} | p \rangle \langle p | x' \rangle \\
&= \int_{-\infty}^{\infty} \frac{dp}{2\pi\hbar} e^{-\frac{ip^2 t}{2m\hbar}} e^{ip(x-x')/\hbar} \\
&= e^{\frac{ip(x-x')^2}{2\hbar t}} \sqrt{\frac{m}{2\pi i \hbar t}}
\end{aligned}$$

where we note that the limit as $t \rightarrow 0$ is $\delta(x - x')$ which indeed matches $\langle x | x' \rangle = \delta(x - x')$ as expected.

Now in an interacting theory, we struggle with the Baker-Campbell-Hausdorff fact that $e^A e^B \neq e^{A+B}$ so using Suzuki-Trotter we separate into steps size $t_{r+1} - t_r = \delta t \ll T$ meaning that

$$e^{-iH\delta t/\hbar} \approx e^{-\frac{ip^2 \delta t}{2m\hbar}} e^{-\frac{iV(x)\delta}{\hbar}} (1 + O(\delta t^2)) \quad (5)$$

so for $T = n\delta t$ we find that

$$K(x, x_0, T) = \int \prod_{r=1}^n dx_r \left(\frac{m}{2\pi i \hbar \delta t} \right)^{\frac{n+1}{2}} e^{i \sum_{r=0}^n \left(\frac{m}{2\hbar} \left(\frac{x_{r+1} - x_r}{\delta t} \right)^2 - V(x_r)/\hbar \right) \delta t} \quad (6)$$

which in the limit $n \rightarrow \infty, \delta t \rightarrow 0$ while keeping T constant leaves

$$\frac{1}{\hbar} \int_0^T dt \left(\frac{1}{2} m \dot{x}^2 - V(x) \right) = \int_0^T dt L(x, \dot{x}) = S \quad (7)$$

for classical Lagrangian L and action S . This is what we refer to as a path integral or function integral:

$$K(x, x_0, t) = \int \mathcal{D}x e^{iS/\hbar} \quad (8)$$

where $\mathcal{D}x$ is the limit described above. Of course, many questions about the existence and uniqueness, etc. of such limits exist, and in fact often this limit does not exist, but in the cases we are interested in, it works well enough... [End of lecture 1]

We make the following remarks

- In the classical limit $\hbar \rightarrow 0$ the lowest frequencies dominate K . This is equivalent to Hamilton's principle (the principle of least action), as expected.
- it is common and helpful to extend analytically to imaginary time $\tau = it$ leaving $\langle x | e^{-H\tau/\hbar} | x_0 \rangle = \int \mathcal{D}x e^{-S/\hbar}$ which has better convergence properties and is easier to interpret than the complex version (Hamilton's principle appears more easily as well).

1 Integrals and their diagrammatic expansion

The above considered quantum mechanics, which is in a sense the 0+1 dimension version of QFT (since x is treated as an operator, while t is treated as a variable). To move to more general QFT, we start, strangely, with 0 dimensional QFT, for $\phi : \{\cdot\} \rightarrow \mathbb{R}$ a field on a single point. Here,

$$\mathcal{Z} = \int_{\mathbb{R}} d\phi e^{-S(\phi)/\hbar} \quad (9)$$

where we assume S is an even polynomial in ϕ for convergence reasons, and we are interested in expectation values

$$\langle f \rangle = \frac{1}{\mathcal{Z}} \int d\phi f(\phi) e^{-S(\phi)/\hbar} \quad (10)$$

1.1 Free Theory

For N fields $\phi_a, a = 1, \dots, N$, let $S(\phi) = \frac{1}{2} \phi^T m \phi$ for a symmetric positive definite matrix $m = P \Delta P^T$ for orthogonal P . As such, we can write this essentially Gaussian integral as

$$\mathcal{Z}_0 = \int d^N \phi e^{-\frac{1}{2\hbar} \phi^T m \phi} = \sqrt{\frac{(2\pi\hbar)^N}{\det m}} \quad (11)$$

From here, we can turn this into a generating function to calculate expectation values by taking derivatives by turning $S_0(\phi) \mapsto S_0(\phi) - J^T \phi$, and writing

$\mathcal{Z}_0 = \mathcal{Z}_0(J)$ (now a generating function(al) - functional later on). We then remark

$$\mathcal{Z}_0(J) = \mathcal{Z}_0(0)e^{-\frac{1}{2\hbar}J^T m^{-1}J} \quad (12)$$

Then we can calculate the correlation functions as

$$\langle \phi_a \phi_b \rangle = \frac{1}{\mathcal{Z}_0(0)} \hbar^2 \partial_{J_a} \partial_{J_b} \mathcal{Z}_0(J)|_{J=0} = \hbar(m^{-1})_{ab} \quad (13)$$

Conveniently, this can be diagrammatically interpreted as a connecting two vertices on indices a, b with an undirected edge. We can generalise this to linear operator $l(\phi) = \sum l_a \phi_a$ as (for p such operators)

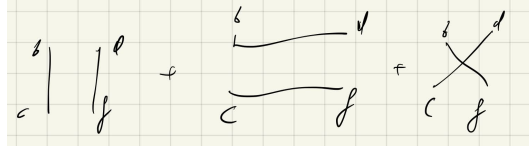
$$\langle l^{(1)}(\phi) \dots l^{(p)}(\phi) \rangle = \hbar^p \prod_{i=1}^p l^{(i)}(\partial_J) e^{\frac{1}{\hbar}J^T m^{-1}J} \quad (14)$$

If p is odd, this is always 0 by symmetry, but if p is even this corresponds to a linear combination of products $m_{ab}^{-1} m_{cd}^{-1} \dots$

Example 1. For $p = 4$, $l_a^{(1)} = \delta_{ab}$, $l_a^{(2)} = \delta_{ac}$, $l_a^{(3)} = \delta_{ad}$, $l_a^{(4)} = \delta_{af}$ then

$$\langle \phi_b \phi_c \phi_d \phi_f \rangle = \hbar^2 (m_{bc}^{-1} m_{df}^{-1} + m_{bd}^{-1} m_{cf}^{-1} + m_{bf}^{-1} m_{cd}^{-1}) \quad (15)$$

which also corresponds to the ways in which we can connect four vertices with undirected edges



[End of lecture 2]

1.2 Interacting Theory

We start investigating interacting theory by doing a series expansion of

$$\int_{\mathbb{R}^N} N \phi f(\phi) e^{-S/\hbar} \quad (16)$$

in \hbar . However, we find that in general, the radius of convergence of these perturbed series is 0, since if we take $\hbar < 0$ these do not converge. As such, we get asymptotic behaviour along the lines described by saying that

Definition 1. $I(\hbar)$ is **asymptotic** to $\sum_{n=0}^{\infty} c_n \hbar^n$ (denoted by \sim) if $\lim_{\hbar \rightarrow 0^+} \frac{1}{\hbar^N} |I(\hbar) - \sum_{n=0}^N c_n \hbar^n| = 0$ for fixed N .

This is much weaker than convergence since we find that adding new terms may in fact make things worse. But it does allow us to account for transcendental terms like $e^{-1/\hbar^2} 0$ (called **nonperturbative contributions**).

Now we work out the case where

$$S(\phi) = \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4 \quad (17)$$

and expand about the minimum of S at $\phi = 0$ to get

$$\mathcal{Z} = \int d\phi e^{-S/\hbar} = \int d\phi e^{-S/\hbar} \sum_v \frac{1}{v!} \left(\frac{-\lambda}{4!\hbar} \phi^4 \right)^v \quad (18)$$

but since we don't converge, we certainly can't swap \int and \sum so here we first truncate, and then swap. As such, if we write $x = \frac{1}{2\hbar}m^2\phi^2$,

$$\mathcal{Z} \sim \frac{\sqrt{2\hbar}}{m} \sum_{v=0}^N \frac{1}{v!} \left(\frac{-\hbar\lambda}{4!m^4} \right)^v 2^{2v} \int_0^\infty dx e^x x^{2v+1/2-1} \quad (19)$$

where the integral is just the gamma function $\Gamma(2v + 1/2) = \frac{(4v)!\sqrt{\pi}}{4^{2v}(2v)!}$ so

$$\mathcal{Z} \sim \frac{\sqrt{2\hbar}}{m} \sum_{v=0}^N \frac{1}{v!} \left(\frac{-\hbar\lambda}{4!m^4} \right)^v \frac{1}{(4!)^v} \frac{(4v)!\sqrt{\pi}}{2^{2v}(2v)!} \quad (20)$$

(the lecture notes seem to omit the $\sqrt{\pi}$). Here the $\frac{1}{(4!)^v}$ comes from expanding the interaction term $e^{-S_1/\hbar}$ and the second fraction comes from the number of ways to pair $4v$ fields with v copies of ϕ^4 . Applying stirling's formula, this series grows as $v!$ so this series is definitely not convergent, but still asymptotic. Now, as before, we want to insert our J somehow to get a generating function. Here we get

$$\begin{aligned} \mathcal{Z}(J) &= \int d\phi e^{-\frac{1}{\hbar}(S_0(\phi) + S_1(\phi) - J\phi)} \\ &= e^{-\frac{1}{\hbar}S_1(\hbar\partial_J)} \int d\phi e^{-\frac{1}{\hbar}(S_0(\phi) - J\phi)} \\ &\sim e^{-\frac{\lambda}{4!\hbar}(\hbar\partial_J)^4} e^{\frac{1}{2\hbar}J^T m^{-1}J} \end{aligned}$$

which works out to

$$\mathcal{Z}(J) \sim \sum_{v=0}^N \frac{1}{v!} \left(\frac{-\lambda}{4!\hbar}(\hbar\partial_J)^4 \right)^v \sum_{p=0} \frac{1}{p!} \left(\frac{1}{2\hbar}J^T m^{-1}J \right)^p \quad (21)$$

where diagrammatically v corresponds to the number of vertices, and p to the number of propagators.

$$g \xrightarrow{\partial^2 g \partial^2} 0 \quad \text{and written} \quad \diagup \rightarrow (\partial)^4 \quad (b=1)$$

where in order to get a nonzero term we need the number of derivatives ($4v$ vertex line ends - 4 per vertex here) to match the number of sources ($2p$ for each propagator) to match. However, we can have a predetermined number of external sources $E = 2p - 4v$. For example, the first two nontrivial terms in the $Z(0)$ expansion for $E = 0$ are $(v, p) = (1, 2), (2, 4)$ which corresponds diagrammatically to

$$Z(0) \sim 1 + \text{loop} + \text{bubble} + \text{two loops} + \text{two loops with external lines}$$

Note that each diagram may have a factor in front of it determined by how often it repeats itself (affect by the product rule in taking derivatives).