Symmetries, Fields, and Particles

quinten tupker

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Introduction

These notes are based on the course lectured by Ben Allanach in Michaelmas 2020. Due to the measures taken in the UK to limit the spread of Covid-19, these lectures were delivered online. These are not meant to be an accurate representation of what was lectures. They solely represent a mix of what I thought was the most important part of the course, mixed in with many (many) personal remarks, comments and digressions... Of course, any corrections/comments are appreciated.

To begin the course, the lecturer reminds of the definition of a group. I will not repeat this definition here. Groups, and Lie groups in particular, are essential in particle physics as a means of keeping track of the symmetries of particles. Here we get two kinds of symmetries:

- An **internal symmetry** is an inherent property of the fields/particles themselves. For example, one can rotate through quark colours, and in fact, we find that in order to make this possible, we require the existence of a force carrying particle (a gluon) to be involved. And to conserve colours, gluons contain a mix of colours to do so. The Lie group structure enforces colour conservation here... Since these colour rotations can be different at different points in space and time, these symmetries can also be called a **local symmetry** or a **gauge symmetry**.
- A **global symmetry** is a symmetry that leaves something the same across all space and time.
- A **external symmetry** is a symmetry involving spacetime coordinates. This includes symmetry under translation and Lorentz transformations. From these symmetries we get conserved structures (momentum, angular momentum, and energy). The Poincare group contains all these symmetries.

In terms of particles, bosons carry forces, and these includes gluons (strong force), photons (electromagnetic force), $Z^0, W^\pm$ carries the electroweak force. It has also been hypothesised that the graviton (spin 2) carries gravity, although it has never been observed. Also, for good symmetries, force carriers should be massless, but the spontaneous symmetry breaking, through the Higgs mechanism, can give force carriers mass (such as $W^{\pm}, Z^0$).

The lecturer provides a standard list of (elementary) fermions. I won't repeat these.

Just as a note, in the standard model, every particle has a field, and excitations of this field corresponds to "instances" of these particles. [End of lecture 1]

In lecture 2, the lecturer reviews basic group theory. I won't repeat that here. [End of lecture 2]

We continue looking at group properties, etc. Here are some definitions.

Definition 1. The **inner automorphism** associated with $g \in G$ is $\phi_g(h) = ghg^{-1}$.

We remark that treating elements of G as automorphisms in this way we see that $G/Z(G)$ is always a normal subgroup of $\mathrm{Aut}(G)$.

Definition 2. The **semi-direct product** of groups H, G, written $H \ltimes G$, has the product rule $(h, g)(h', g') = (hh', g\phi_h(g'))$, and inverse rule $(h, g)^{-1} = (h^{-1}, \phi_{h^{-1}}(g))$.

here we can see that $H \equiv H \ltimes G/G$, and that $D_n \equiv \mathbb{Z}_2 \ltimes \mathbb{Z}_n$ as expected.

Definition 3. The **commutator subgroup** or **derived subgroup** of group G, denoted $[G, G]$, the group generated by the commutators of G. These are always normal, and a group is called **perfect** when it equals its own commutator subgroup.

1 Matrix Groups

The lecturer describes the general linear group, orthogonal group, special orthogonal group, unitary group, and special unitary group. He also describes their dimensionality. In particular, using their properties, he shows that $O(n)$ has $\frac{1}{2}n(n-1)$ free parameters, and $SU(n)$ has $n^2 - 1$ parameters. He also mentions the famous fact that topologically $SU(2) \equiv S^3$. [End of lecture 3]

Let's continue defining some special matrix groups that may be less well known.

Definition 4. The **symplectic group** $S_p(2n, \mathbb{R})$ is a subset of the general linear group satisfying

$$M^T J M = J$$

for

$$J = \begin{pmatrix} 0 & -1 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & -1 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & -1 \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix}$$

This group has dimension $n(2n-1)$ when treated as a manifold, and the determinant of every element in the group is 1 (applying the determinant to the equation gives us ± 1, if you apply a generalised determinant for anti-symmetric matrices called the Pfaffian you can see the sign as well). For the sake of representations the most important thing about the symplectic symplectic group is that unlike other groups we've seen, like $SO(n)$ or $SU(n)$, the symplectic linear group is not compact.

Definition 5. The pseudo-orthogonal group $O(n,m)$ is the subgroup of the general linear group satisfying

$$M^T g M = g$$

for

$$g = \begin{pmatrix} I_n & \\ & -I_m \end{pmatrix}$$

Similarly we have $SO(n,m), U(n,m), SU(n,m)$. Also notice that $SO(1,1) = S_p(2,\mathbb{R})$.

1.1 Representations

A representation of a group is a homomorphism ϕ from that group to a space of linear maps on some vector space V on some field. For matrix groups, such as those mentioned above, we also have the canonical **fundamental representation** that represents matrices just as the matrices they are defined to be.

A representation is called **reducible** if we can find a linear subspace so that the representation can be viewed purely as a representation on that subspace (formally $\forall g \in G, u \in U, \phi(g)u \in U$ where $U \leq V$). There are theorems in representation theory about the reducibility of representations into irrudicible ones, such that we can always write

$$V \equiv \bigoplus_r U_r$$

for irreducible U_r. A handy lemma in this field.

Lemma 1. For irreducible representations V, W of G,

- $Hom_G(V,W)$ ccontains only 0 or isomorphisms.
- if the field we are working on is algebraically closed, then $\dim(Hom_G(V,W)) = 1$

I know this has been worded in too abstract a manner, but it is more familiar for me. Basically, two irreducible representations are either isomorphic, meaning that they are the same up to similarity transformations (change of bases), and there is only one similarity transformation up to multiplication by a constant factor.

Despite the fact that any group has an infinite number of representations, a lot of information can be summarised about them by calculating the **character** $\chi(g) = tr(\phi(g))$, which is just the trace of every matrix. Notice that since traces are unchanged by similarity transformations, they firstly are the same for isomorphic/equivalent representations, and secondly, are the same for conjugacy classes within the group, so they truly are just summaries of the representation. Nonetheless, many important properties can be deduced using them. For one, it is much easier to check if representations are equivalent this way.

Finally, we consider the **tensor product** such that on $V \otimes W$ the product of two representations acts as

$$(\phi \otimes \psi)(g)(v \otimes w) = \sum_{rs} \lambda_r \mu_s \phi(g)(v_r) \otimes \psi(g)(w_s) = \phi(g)(v) \otimes \psi(g)(w)$$

where v_r, w_s are the basis vectors of V, W and λ_r, μ_s express v, w in these bases. [End of lecture 4]

A quick note is that one can calculate the tensor product on characters as

$$\mathrm{tr}_{V_r \otimes V_s}(D^{R_r}(g) D^{R_s}(g)) = \mathrm{tr}_{V_r}(D^{R_r}(g))\,\mathrm{tr}_{V_s}(D^{R_s}(g))$$

Also, if $R_r \otimes R_s$ contains the singlet representation then we can construct an inner product $\langle v, v' \rangle$ which is invariant under the group.

1.2 Symmetries and Quantum Mechanics

Alright, now let's actually start using some of these mathematical ideas!

A (unitary) map U is called a **symmetry** if $|\langle \phi | \psi \rangle = |\langle U\phi, U\psi \rangle|^2$ for all ϕ, ψ. Furthermore, one can show that U is always linear or anti-linear (**anti-linear** means it is a linear map except it applies complex conjugation, so $Ua\,|\psi\rangle = a^* U\,|\psi\rangle$). Furthermore, it can be shown that anti-linear maps are only linear when we consider time reversal. Therefore, we will focus on linear maps.

One can also show that such maps, if they represent a symmetry group G satisfy the product rule

$$U(g)U(g') = e^{i\gamma(g,g')} U(gg')$$

where we usually just assume that $\gamma = 0$ leaving us with a homomorphism. One can further show that it must always commute with the Hamiltonian if it is a symmetry so

$UH = HU$

meaning that we can find a simultaneous eigenbasis for the two.

2 Rotations $SO(3)$ and $SU(2)$

An important group of symmetries are the group of rotations $SO(3)$ and $SU(2)$. We will prove an important results on them, namely that

$$SO(3) \equiv SU(2)/\mathbb{Z}_2$$

To do so we will develop an important tool, namely Pauli matrices. But before we start, let's review some standard properties of these groups. In $SO(3)$ we can write a general rotation by θ about axis n as

$$R_{ij} = \cos(\theta)\delta ij + (1 - \cos(\theta))n_i n_j - \sin(\theta)\epsilon_{ijk}n_k$$

which corresponds infinitesimally to

$$\delta x = \delta n \times x.$$

Now, let's start proving this isomorphism. First, the Pauli matrices are:

$$\sigma_1 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \sigma_2 = \begin{pmatrix} & -i \\ i & \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

which satisfy the properties

$$\sigma_i \sigma_j = \delta_{ij} I + \epsilon_{ijk}\sigma_k$$

and

$$\mathrm{tr}(\sigma_i \sigma_j) = 2\delta_{ij}$$

In other words, what exactly are the Pauli matrices? The Pauli matrices form an orthogonal basis (up to a factor of 2) when we treat the space of Hermitian traceless 2×2 matrices as a real vector space. Hermitian means the diagonal entries must be real, and traceless means the top left and top right entries must add to 0. This is generated by σ_3. Meanwhile, σ_1, σ_2 handle the off-diagonal basis. Why are they orthogonal? The natural inner product to use on matrix spaces is to rearrange the matrices into long vectors, and just to apply the dot-product to these vectors, and that can exactly be written as

$$A \cdot B = \mathrm{tr}\{AB\} = A_{ij}B_{ij}$$

Under this product, the property $\mathrm{tr}(\sigma_i\sigma_j) = 2\delta ij$ means these matrices truly are orthogonal. Now, we can easily form a complete basis for the space of 2×2 Hermitian matrices by adding the identity I. When doing so, we find that any matrix $A \in SU(2)$ can be written in this basis as

$$A = \frac{1}{2}\mathrm{tr}(A)I + \frac{1}{2}\mathrm{tr}(\sigma_i A)\sigma_i$$

Note in particular, that $\mathrm{tr}(A) = \mathrm{tr}(AI)$, so that if we define $\sigma_0 = I$ we get

$$A = \frac{1}{2}\mathrm{tr}\{\sigma_\mu A\}\sigma_\mu, \ \mu = 0, 1, 2, 3$$

which is really just writing it in a new orthogonal basis according to a certain dot product. [End of lecture 5]

So what is the isomorphism we are going to use. Consider the map

$$x \mapsto x_i \sigma_i$$

which has inverse

$$x_i \sigma_i \mapsto \frac{1}{2} \operatorname{tr}(\sigma_j x_i \sigma_i)$$

and satisfies $(x_i\sigma_i)^2 = x^2 I$, and $x_i\sigma_i$ has eigenvalues $\pm\sqrt{x^2}$, leaving determinant $\det(x_i\sigma_i) = -x^2$.

Now, for any $A \in SU(2)$ let $x \mapsto x'$ via

$$x'_i \sigma_i = A x_i \sigma_i A^\dagger$$

which has $x'^2 = -det x'_i \sigma_i = \det\{x_i\sigma_i\} = x^2$ since A is unitary. That means this map preserves length and so sits in $O(3)$. How do we show it is a rotation, not a reflection? We notice that the determinant as a map is continuous, and as $A \to I$, $R_A \to I$ as well, so we must get $\det\{R_A\} = 1$. So if we assume that $SU(2)$ has only a single connected component, we see that we map entirely onto $SO(3)$.

Now, we'd like to invert this map as well, and a first step for us would be to see if we can write this map down explicitly. To do so, we notice that

$$(Rx)_i \sigma_i = \sigma_i R_{ij} x_j = x'_i \sigma_i = A x_j \sigma_j A^\dagger$$

which is true for any x, meaning that

$$\sigma_i R_{ij} = A \sigma_j A^\dagger$$

Here, the LHS is just a decomposition of a matrix into a sum of Pauli, matrices, so taking a dot product (trace) with the appropriate matrices to undo the composition we get

$$R_{ij} = \frac{1}{2} \operatorname{tr}\big(\sigma_i A \sigma_j A^\dagger\big)$$

which gives us one direction for our map. To reverse this, we notice that $\sigma_i\sigma_j\sigma_i = -\sigma_j$, and $\sigma_i^2 = I$, meaning that $\sigma_\mu\sigma_\nu\sigma_\mu = 4\delta_{0\nu} I$.

$$\sigma_\mu A^\dagger \sigma_\mu = 2\operatorname{tr}(A) I \implies \sigma_i A^\dagger \sigma_i = 2\operatorname{tr}(A) - \sigma_0 A^\dagger \sigma_0$$

(the implication refers to an equation in lectures).

Applying this to our formula for R_{ij} we gets

$$R_{jj} = \frac{1}{2} \operatorname{tr}\big(\sigma_j A \sigma_j A^\dagger\big) = \frac{1}{2} \operatorname{tr}\big(\sigma_\mu A \sigma_\mu A^\dagger - \sigma_0 A \sigma_0 A^\dagger\big) = \operatorname{tr}\big(\operatorname{tr}(A) A^\dagger\big) - \frac{1}{2} \operatorname{tr}\big(A A^\dagger\big) = |\operatorname{tr}(A)|^2 - 1$$

Furthermore, we get

$$
\begin{aligned}
\sigma_i R_{ij} \sigma_j &= \frac{1}{2}\sigma_i \operatorname{tr}\big(\sigma_i A \sigma_j A^\dagger\big)\sigma_j \\
&= \frac{1}{8}\sigma_i \operatorname{tr}\big(\sigma_i \operatorname{tr}(\sigma_\mu A)\sigma_\mu \sigma_j \operatorname{tr}\big(\sigma_\nu A^\dagger\big)\sigma_\nu\big)\sigma_j \\
&= \ldots \\
&= 2\operatorname{tr}(A)A - I
\end{aligned}
$$

(by taking out the traces inside, and analysing case-by-case one can find nice expressions for the trace of products of Pauli matrices). These together leave

$$A = \pm\frac{I + \sigma_i R_{ij}\sigma_j}{2\sqrt{1 + R_{jj}}}$$

which completes our isomorphism.

We also find we can express these relations efficiently infinitesimally:

$$R_{ij} = \delta_{ij} - \delta\theta\epsilon_{ijk}n_k$$

$$A = I - i\delta\theta n_k \sigma_k/2$$

since $\det(I + X) = 1 + \operatorname{tr}(X)$ meaning that in order to keep $\det = 1$ to first order we need $\operatorname{tr}(X) = 0$ above. Now, we can recover an arbitrary rotation by exponentiation, and then see that

$$A(\theta, n) = e^{-\frac{i}{2}n_k\sigma_k\theta} = 2\cos(\theta/2) - in_k\sigma_k\sin(\theta/2)$$

This makes it clearer where the $\mathbb{Z}_2$ symmetry comes from.

2.1 Infinitesimal Generators and Rotations

Now for rotations $R_1 = R(\delta\theta_1, n_1), R_2 = R(\delta\theta_2, n_2)$ we find that commutator $R_c = R_2^{-1}R_1^{-1}R_2R_1$ is given infinitesimally by

$$x \mapsto x + \delta\theta_1\theta_2(n_2 \times (n_1 \times x) - n_1 \times (n_2 \times x)) = x + \delta\theta_1\delta\theta_2$$

In general, in quantum mechanics we assume any unitary operation has a generator, such as for rotations

$$U(R(\delta\theta, n)) = I - i\delta\theta n \cdot J$$

for generators J of rotations about each axis. Note that in general, U unitary means that generators J_i are all Hermitian. As such we see that

$$U(R_c) = I - i\delta\theta_1\delta\theta_2(n_2 \times n_1)\cdot J = 1 - \delta\theta_1\delta\theta_2[n_2 \cdot J, n_1 \cdot J_2]$$

which gives us the famous commutation relation

$$[J_i, J_j] = i\epsilon_{ijk}J_k$$

Similarly, for a function of space, where we use L instead of J as a generator of rotations we gets

$$f \mapsto (1 - i\delta\theta \cdot nL)f(x)$$

which gives rise to thse same commutation relations as we had before. We can then recover finite rotations by exponentiating as expected. Now, we also see that rotations correspond to conservation of angular momentum since they conserve the Hamiltonian.

2.2 Representations of the Angular Momentum Commutation Relations

For convenience when dealing with the angular momentum operator (generator of rotations) we we define $J_\pm = J_1 \pm iJ_2$ (which become our ladder operators), and gives rise to the commutation relations

$$[J_3, J_\pm] = \pm J_\pm$$

$$[J_+, J_-] = 2J_3$$

and of course, $J_3^* = J_3, J_+^* = J_-$. We then pick our basis, which leads us to

$$J_3 |m\rangle = m |m\rangle$$

$$J_+ |m\rangle = \lambda_m |m \pm 1\rangle$$

$$J_- |m\rangle = |m - 1\rangle$$

or 0 (sometimes $J_\pm$ annihilates a state). Using commutation relations we can then deduce that

$$\lambda_m = j(j+1) - m(m+1).$$

(since we see that $\lambda_{m-1} - \lambda_m = 2m$)

[End of lecture 6]

We notice that this means that for large m λ_m becomes negative, however, $J_\pm J_\mp$ is non-negative definite meaning that all eigenvalues must be non-negative. Consequently there must exist $m_\pm$ such that $J_\pm |m_\pm\rangle = 0 = (j \mp m_\pm)(j \pm m_\pm + 1)$. This means that $m_\pm = \pm j$ for $j \in \{0, 1/2, 1, 3/2, \dots\}$. These states form an orthogonal basis for a representation $\mathcal{V}_j$ of dimension $2j+1$.

2.2.1 The $|jm\rangle$ Basis

[Explain why we need these representations?]

We consequently get an orthogonal $|jm\rangle$ basis such that $J_3\,|j\,m\rangle = m\,|jm\rangle$. Here we can define a **highest weight state** to be $|j\,j\rangle$. Now

$$J_\pm\,|j\,m\rangle = \sqrt{(j-m)(j+m+1)}\,|j\,m\pm 1\rangle$$

means that we can define states as

$$(J_-)^n\,|j\,j\rangle = \sqrt{\frac{n!(2j)!}{(2j-m)!}}\,|j\,j-n\rangle$$

That is one way of defining j, but of course, we also have the more classical picture using the Casimir operator

$$J^2 = J_iJ_i = J_+J_- + J_3^2 - J_3$$

If we do that we see that $J\,|j\,j\rangle = j(j+1)\,|j\,j\rangle$. But we notice that $[J^2, J_-] = 0$ meaning that in general

$$J^2\,|j\,m\rangle = j(j+1)\,|j\,m\rangle$$

meaning that indeed, J^2 can be considered to correspond to the total angular momentum.

Now that we've sorted out the overall algebraic properties that our representation should have, let's figure out what the actual representations of this space are. Here we see that we get

$$J^{(j)}_{3\,m'm} = m\delta_{m'm}$$
$$J^{(j)}_{\pm\,m'm} = \sqrt{(j\mp m)(j\pm m+1)}\delta_{m'm\pm 1}$$

which completes our representation for rotation generators J in the basis such that our matrices are $\langle j\,m'|\,J_i\,|j\,m\rangle$. We'd like to define a similar representation for finite rotations $D(R)$ such that infinitesimally

$$D^{(j)}(R(\delta n, n)) = I - i\delta\theta n\cdot J^{(j)}$$

To get the finite version, we consider the Euler angles for an arbitrary rotation such that

$$R = R(\phi, e_3)R(\theta, e_2)R(\psi, e_3)$$

giving representation

$$U(R) = e^{-i\phi J_3}e^{-i\theta J_2}e^{-i\psi J_3}$$

meaning that

$$D^{(j)}_{m'm} = e^{-im'\phi - im\psi} d^{(j)}_{m'm}(\theta)$$

for

$$d^{(j)}_{m'm}(\theta) = \langle j\, m' |\, e^{-i\theta J_2}\, | j\, m\rangle$$

A few special cases can be worked out as $d^{(j)}_{m'm}(\pi) = (-1)^{j-m}\delta_{m',-m}$ and $d^{(j)}_{m'm}(2\pi) = (-1)^{2j}\delta_{m'm}$, and in fact in general

$$D^{(j)}(R(2\pi, n)) = (-1)^{2j} I$$

where we note crucially that $j \in \{0, 1/2, 1, 3/2, \dots\}$. We also find the general result that

$$d^{(j)}_{m'm}(\theta) = (-1)^{m'-m} d^{(j)}_{-m',-m}(\theta) = (-1)^{m'-m} d^{(j)}_{m'm}(-\theta) = (-1)^{m'-m} d^{(j)}_{mm'}(\theta)$$

implying a degree of symmetry at least.

Furthermore, we see that in the special case $j = 1/2$ we get $J_i = \sigma_i$ the Pauli matrices, and

$$d^{(1/2)}(\theta) = \begin{pmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{pmatrix}$$

and we can generate the character of a rotation by angle θ to in general be (independent of the axis of rotation)

$$\chi_j(\theta) = \frac{\sin((j+1/2)\theta)}{\sin(\theta/2)}$$

2.2.2 Tensor Products and Angular Momentum Addition

Now a somewhat well known decomposition is what happens when you add to representations together and look at the new representation structure. In particular, how do we look at the tensor product $\mathcal{V}_{j_1} \otimes \mathcal{V}_{j_2}$ with basis $|j_1\, m_1\rangle\, |j_2\, m_2\rangle$. Then we can extend our generators to the new space via

$$J_1 = J_1 \otimes I_2$$
$$J_2 = I_1 \otimes J_2$$

[End of lecture 7]

We then form a new basis $|J\, M\rangle$ which we define to satisfy

$$J_3\, |J\, M\rangle = M\, |J\, M\rangle$$
$$J^2\, |J\, M\rangle = J(J+1)\, |J\, M\rangle$$

but of course, we also have the natural basis inherited from the definition of the space: $|j_1\, m_1\, j_2\, m_2\rangle$, so how do these relate? We define the **Clebsch-Gordan** coefficients to be precisely to coefficients expressing $|J\, M\rangle$ in terms of $|j_1\, m_1\, j_2\, m_2\rangle$, where we note in particular that in order to make J_3 values match Clebsch-Gordan coefficients are only nonzero when $m_1 + m_2$.

Since we also inherit the raising and lowering operators, we might hope that we can, just as with a pure state $\mathcal{V}_j$, find a highest weight state satisfying

$$J_3\,|J\,J\rangle = J\,|J\,J\rangle$$
$$J_+\,|J\,J\rangle = 0$$

By thinking a little, we can see that the following definitely works

$$|j_1 + j_2\, j_1 + j_2\rangle = ketj_1\, j_1\, |j_2\, j_2\rangle$$

We can then use J_- to find the rest of the coefficients. Note that we could also have started with a state annihilated by J_- instead and then use J_+ instead, although it really comes down to the same (but perhaps this implies some symmetry in the Clebsch-Gordan coefficients?).

Now, if we define $\mathcal{V}^{(M)}$ to be eigenspace of J_3 with eigenvalue M then, by counting on the original space, we see that

$$\dim(\mathcal{V}^{(M)}) = \begin{cases} j_1 + j_2 - M + 1 & M \geq j_1 - j_2 \\ 2j_2 + 1 & M \leq j_1 - j_2 \end{cases}$$

where we've assumed without loss of generality that $j_1 \geq j_2$. Now, from the above we know how we "travel" inside of $\mathcal{V}^{(M)}$ for fixed M, but we have yet to see how we travel between different values of J. We do not have a particularly efficient way of doing so: we only see that by exploiting dimensions efficiently we see that $|j_1 + j_2\, j_1 + j_2 - 1\rangle$ is orthogonal to $|j_1 + j_2 - 1\, j_1 + j_2 - 1\rangle$, and similarly, we gradually can find Clebsch-Gordan coefficients for $J < j_1 + j_2$.

By further considering the original spaces, we see that $J \in \{j_1 + j_2, \ldots, |j_1 - j_2|\}$. Using $\dim(\mathcal{V}^{(J)}) = 2J + 1$ we can check we do get the correct dimension of our space $(2j_1 + 1)(2j_2 + 2)$. [The lecturer mentions some straightforward orthogonality relationships, and gives an example for $j_1 = 1, j_2 = 1/2$.]

2.3 $SO(3)$ tensors

When studying how a certain system S behaves there are two questions one might wish to consider. Firstly, how does S behave as the input is varied? This generally is the topic of calculus, and differential equations, etc. Secondly, under what transformations does S stay the same? This is generally the topic of group theory and the like. This course certainly focuses more on the second question, and a concept that illustrates this perhaps the best is that of tensors. Tensors, after all, essentially are things that are left invariant under certain group transformations. The numerical representations of these states change under these transformations, but tensors essentially define themselves as objects whose "true" value is unchanged under these transformations.

The simplest type of tensors for us to consider are tensors under rotations, so tensors under $SO(3)$. Here, we can see that the matrix multiplying indices are the representation of the group, while the tensors themselves inhabit the linear space that that group is acting on. As such tensors are just representations of the group that determines their transformation rule, and as such we might ask what spaces of tensors are irreducible representations? If we denote the space of rank-n tensors as $\mathcal{V}^{\otimes n}$, then we see in particular that isotropic tensors form irreducible representations. As such one can deduce that all rank 2 tensors split into the following irreducible representations: For any rank-2 tensors T_{ij}

$$T_{ij} = S_{ij} + \epsilon_{ijk} v_k + \frac{1}{3}\delta_{ij} T_k k$$

where S_{ij} is a symmetric, traceless tensor, v_k is a rank-1 tensor describing the anti-symmetric part of T_{ij} and the last term captures the trace.

To generalise we'd like to be able to find an easy test of irreducibility on these spaces. Here we see that for candidate space $S_{i_1 \dots i_n}$ we see that $\delta_{i_k i_{k'}} S_{i_1 \dots i_n} = 0$ always if $S_{i_1 \dots i_n}$ is irreducible (otherwise the space of such nonzero tensors would generate a subspace). As such we one such irreducible space is totally symmetric and traceless.

Finally, we remark that summing over all indices forms an invariant inner product on the space. [End of lecture 8]

What is the dimension of a totally symmetric traceless tensor? In thrs, any symmetric tensor can be written in the form

$$S_{\underbrace{1\dots1}_{r \text{ times}}\underbrace{2\dots2}_{s \text{ times}}\underbrace{3\dots3}_{t \text{ times}}}$$

by roerdering, meaning that

$$\dim(\mathrm{Sym}(\mathcal{V}^n)) = \frac{(n+1)(n+2)}{2}$$

If we also impose that we have to traceless, we get dimension $2n+1$ as we would expect for a representation of angular momentum eigenstates.

2.4 Spinors

It turns out that the spinor representation can just be expressed as the fundamental representation of $SU(r)$ with tensors transforming via $\eta_\alpha \mapsto \eta'_\alpha = A_\alpha^\beta \eta_\beta$ for $A \in SU(r)$. $A \in SU(r) \implies (A_\alpha^\beta)^* = (A^{-1})_\alpha^\beta$ (notice the transpose) which corresponds to the conjugate representation $\bar{\eta}^\alpha = (\eta_\alpha)^*$.

For $SU(2)$, which we will focus on, $\epsilon^{\alpha\beta} = -\epsilon^{\beta\alpha}$ is the only isotropic tensor. By convention we choose $\epsilon^{12} = 1, \epsilon_{12} = -1$. To check it is truly isotropic notice that

$$\epsilon'_{\alpha\beta} = A_\alpha^\gamma A_\beta^\delta \epsilon_{\gamma\delta} = \det(A)\epsilon_{\alpha\beta} = \epsilon_{\alpha\beta}$$

Now, again, we wish to find the irreducible representations for this group. In this case, we see that anything contracted with $\epsilon^{\alpha\beta}$ must be zero (or else it could be used to form a proper-subrepresentation). As such, we find that any irreducible $S_{\alpha_1\ldots\alpha_n}$ must be totally symmetric, as before. Using a similar counting method as before, we find that the dimension of such spaces is $n+1$ for $j=n/2$ as one might hope.

Now, so far we have to consider both η_α and $e\bar{t}a^\alpha$ but we notice that we can use $\epsilon^{\alpha\beta}$ to raise and lower indices. As such, it is sufficient to only consider η_α. Also, notice that that

$$\epsilon_{\alpha\beta}\epsilon^{\gamma\delta} = -\delta^\gamma_\alpha\delta^\delta_\beta + \delta^\delta_\alpha\delta^\gamma_\beta$$

Now, writing

$$\delta_{\eta_\alpha} = -i\delta\theta\frac{1}{2}(n\cdot\theta)^\beta_\alpha\eta_\beta$$

so that for an arbitrary tensors

$$\delta T_{\alpha_1\ldots\alpha_m} = -i\delta\theta\sum_r\frac{1}{2}(n\cdot\sigma)^\beta_\alpha(T_{\alpha_1\ldots\alpha_{r-1}\beta\alpha_{r+1}\ldots\alpha_m})$$

meaning that $\epsilon^{\alpha\gamma}\epsilon_{\beta\delta}\sigma^\delta_\gamma = \sigma^\alpha_\beta$ so that $\mathrm{tr}(\sigma)=0$ and $\epsilon^{\alpha\beta}\sigma^\beta_\gamma = \epsilon^{\beta\gamma}\sigma^\alpha_\gamma$. We then can write the completeness of Pauli matrices as

$$(\sigma\epsilon)_{\alpha\beta}\cdot(\sigma\epsilon)^{\gamma\delta} = \delta^\gamma_\alpha\delta^\delta_\beta + \delta^\delta_\alpha\delta^\gamma_\beta$$

$$(\epsilon\sigma)^{\alpha\beta}\cdot(\epsilon\sigma)^{\gamma\delta} = -\epsilon^{\alpha\gamma}\epsilon^{\beta\delta} - \epsilon^{\alpha\gamma}\epsilon^{\beta\gamma}$$

The Pauli matrices are crucial since they link $SU(2)$ back to $SO(3)$ which we are relying on. In particular we see that for n even wecan write

$$T_{i_1\ldots i_n} = (\epsilon\sigma_{i_1})^{\alpha_1\beta_1}\ldots(\epsilon\sigma_{i_n})^{\alpha_n\beta_n}S_{\alpha_1\ldots\alpha_n\beta_1\ldots\beta_n}$$

[do we mean $n/2$ in the last term here?] and for n odd we see that

$$T_{i_1\ldots i_n} = \epsilon^{\beta_1\gamma_1}\ldots\epsilon^{\beta_r\gamma_r}S_{1,(\alpha_1\ldots\alpha_{n-r}\beta_1\ldots\beta_r}S_{2,\alpha_{n-r+1}\ldots\alpha_{n+m-2r})\gamma_1\ldots\gamma_r}$$

for $S_{1,\alpha_1\ldots\alpha_r}$, $S_{2,\beta_1\ldots\beta_m}$ and $n\geq m,r$. In particular for $n=m=1, r=0,1$ for u_α, v_α we find that

$$u_\alpha v_\beta = u_{(\alpha}v_{\beta)} + \frac{1}{2}\epsilon_{\alpha\beta}u^\gamma_\beta v_\gamma$$

where we see that $u_{(\alpha}v_{\beta)}$ is a vector in $SO(3)$. This means that

$$t_i = (\epsilon\sigma_i)^{\alpha\beta}u_{(\alpha}v_{\beta)}$$

is a tensor and we can reverse this via

$$u_\alpha v_\beta = \frac{1}{2} t_i (\epsilon\sigma_i)_{\alpha\beta} + \frac{1}{2}\epsilon_{\alpha\beta} u^\gamma v_\gamma$$

where we in effect see $j = 1/2 \otimes j = 1/2 \equiv j = 1 \oplus j = 0$.

3 Relativistic Symmetries

All that we've dealt so far are classical symmetries with the standard rotation group. But really, we'd like to redo this work relativistically. As such, and sticking to special relativity, we upgrade to the Lorentz group (and sometimes Poincaré group).

3.1 The Lorentz Group

We wish to define a group that preserves an inner product given by $x^\mu, x^\nu \mapsto g_{\mu\nu} x^\mu x^\nu$ where $g_{00} = 1, g_{0i} = g_{i0} = 0, g_{ii} = -1$. By linear algebra we know that any such inner product is equivalent to the inner product with

$$g = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

Once we deice to preserve this metric, we will now show that this indeed turns out to be the Lorentz group. Firstly, we argue that isometries must be linear (or affine really - which corresponds to Poincaré). If $x'^2 = x^2$ and $x'^\mu = x^\mu + f^\mu(x)$ for some f under some transformation then we see that we get $dx'^\mu = dx^\mu + \partial_\sigma f^\mu(x) dx^\sigma$ meaning that $\partial_\mu f_\nu + \partial_\nu f_\mu = 0$ which is **Killing equation**. Working a little more we see that

$$2\partial_\omega \partial_\mu f_\nu = \partial_\omega(\partial_\nu mu f_\nu + \partial_\nu f_\mu) + \partial_\mu(\partial_\nu f_\omega + \partial_\omega f_\nu) - \partial_\nu(\partial_\omega f_\mu + \partial_\mu f_\omega) = 0$$

meaning that all second derivatives of f are 0 so f is indeed affine

$$f_\mu = a_\mu + \omega_{\mu\nu} x^\nu$$

where the killing identity forces $\omega_{\mu\nu}$ to be asymmetric. If we then focus on the linear part, we see that we do indeed get Lorentz transformations. However, we notice that topologically, the space of Lorentz transformations splits into 4 connected components given by $\Lambda^0_0 \geq 1$ or $\Lambda^0_0 \leq -1$ and $\det(\Lambda) = \pm 1$. Notice also that $(\Lambda^0_0)^2 = 1 + \sum_{i=1}^3 (\Lambda^0_i)^2$ (why?). Anyways, once we require preservation of parity and casaulity we find ourselves restricted to $\Lambda^0_0 \geq 1, \det(\Lambda) = 1$. We call this restricted group $SO(1,3)$ the **proper orthochronous Lorentz group**. Rotations of this group forma subgroup as

$$\Lambda_R = \begin{pmatrix} 1 & \\ & R \end{pmatrix}$$

We can also write all Lorentz boosts in the form

$$\Lambda_B = \begin{pmatrix} \cosh(\theta) & \sinh(\theta)n \\ \sinh(\theta)n & B \end{pmatrix}$$

for symmetric B and unit vector n. Writing the preservation of metric g we find that $Bn = \cosh(\theta)n, B^2 = -\sinh^2(\theta)nn^T = I$ meaning that

$$B = I + (\cosh(\theta) - 1)nn^T$$

a Lorentz boost with velocity $v = \tanh(\theta)$ in direction n. These don't form a subgroup (although all Lorentz boosts restricted to the same direction n do form a subgroup). Now, importantly we notice that we can write any Lorentz transformation as a rotation followed by Lorentz boost. We do this by noticing that all rotations are all orthogonal in the Lorentz group and all Lorentz boosts are symmetric. As a result for any $\Lambda \in SO(1,3)$ we can define $\Lambda_B = \sqrt{\Lambda^T \Lambda}$ and the we see that

$$(\Lambda(\Lambda_B)^{-1})^T(\Lambda(\Lambda_B)^{-1}) = I$$

Now that we've established the basic structure of the Lorentz group, and its origins, let's see what kind of quantum theory we can build off of this. In particular, let's start with the commutation relations where we see that for

$$\Lambda^{\mu}_{1\nu} = \delta^{\mu}_{\nu} + \omega^{\mu}_{1\nu}$$
$$\Lambda^{\mu}_{2\nu} = \delta^{\mu}_{\nu} + \omega^{\mu}_{2\nu}$$

Then we get that

$$\Lambda^{\mu}_{\nu} = (\lambda_2^{-1}\Lambda_1^{-1}\Lambda_2\Lambda_1)^{\mu}_{\nu} = \delta^{\mu}_{\nu} + [\omega_2, \omega_1]^{\mu}_{\nu}$$

For a representation U of our theory we get that

$$U(\Lambda) = 1 - \frac{i}{2}\omega^{\mu\nu}M_{\mu\nu}$$

for antisymmetri $M_{\mu\nu}$. We then see that

$$U(\Lambda) = 1 - i[\omega_2, \omega_1]^{\mu\nu}M_{\mu\nu} = 1 - \frac{1}{2}[\omega^{\mu}_{2\rho\sigma}, \omega^{\mu}_{1\sigma\rho}]$$

so in particular

$$[\omega_2, \omega_1]^{\mu\nu} = g_{\sigma\rho}(\omega_2^{\mu\sigma}\omega_1^{\sigma\nu} - \omega_1^{\mu\sigma}\omega_2^{\rho\nu})$$

Consequently we find

$$[M_{\mu\nu}, M_{\sigma\rho}] = i[g_{\nu\sigma}M_{\mu\rho} - g_{\mu\sigma}M_{\nu\rho} - g_{\nu\rho}M_{\mu\sigma} + g_{\nu\rho}M_{\nu\sigma}]$$

We now can also define tensors to be **contravariant** if they transform as

$$U(\Lambda)V^{\mu}U(\Lambda)^{-1} = (\Lambda^{-1})^{\mu}_{\nu}V^{\nu}$$

and **covariant** if they transform according to

$$U(\Lambda)U_{\mu}U(\Lambda)^{-1} = U_{\nu}\Lambda^{\nu}_{\mu}$$

Infinitesimally we get

$$[M_{\mu\nu}, V^{\sigma}] = -i(\delta^{\sigma}_{\mu}V_{\nu} - \delta^{\sigma}_{\nu}V_{\mu})$$

$$[M_{\mu\nu}, U_{\sigma}] = -i(g_{\mu\sigma}U_{\nu} - g_{\nu\sigma}U_{\mu})$$

Restricting to $ijkl$ as indices (spatial only) we see that if we define $J_m = \frac{1}{2}\epsilon_{mij}M_{ij}$ then $M_{ij} = \epsilon_{ijm}J_m$ leaving

$$[J_m, J_n] = i\epsilon_{mnl}J_l$$

(representing angular moment) leaving

$$[M_{0i}, M_{0j}] = -i(\delta_{jk}M_{0i} - \delta_{ik}M_{0j})$$

and

$$[M_{0i}, M_{0j}] = -iM_{ij}$$

meaning that if $K_i = M_{0i}$ so $K_i^{\dagger} = K_i$ (?)

$$[J_i, K_j] = i\epsilon_{ijk}K_k, [K_i, K_j] = -i\epsilon_{ijk}J_k$$

meaning K_i is a well-defined vector operation under the $SO(1,3)$ structure (what does this mean? - I need to look at Hugh Osborne's notes). As expected, infinitesimally we get $\delta x^{\mu} = \omega^{\mu}_{\nu}x^{\nu}, \omega_{ij} = \epsilon_{ijk}\theta_k$ and so if $\omega^0_i - \omega^i_0 = v_i$ then $\delta t = v \cdot x, \delta x = vt + \theta \times x$. As such we can write for a general Lorentz transformation that

$$U(\Lambda) = 1 - i\theta \cdot J + iv \cdot k$$

where J is the rotation and k is the boost. Now writing

$$J^{\pm} = \frac{1}{2}(J \pm iK)$$

then we get

$$[J^{\pm}_i, J^{\pm}_j] = i\epsilon_{ijk}J^{\pm}_k$$

an $SU(2) \times SU(2)$ structure.

Now, just as we could get spinors for $SO(3)$ we can also get spinors for $SO(1,3)$. Now instead of using just the Pauli-matrices, we need to use the Pauli-matrices plus identity, as used earlier. In particular, we write

$$\sigma_\mu = (I, \sigma_1, \sigma_2, \sigma_3) = \sigma_\mu^\dagger$$

$$\bar{\sigma}_\mu = (I, -\sigma_1, -\sigma_2, -\sigma_3) = \bar{\sigma}_\mu^\dagger$$

We still get identities like

$$\sigma_i \sigma_j = \delta_{ij} I + i\epsilon_{ijk}\sigma_k$$

$$\sigma_\mu \bar{\sigma}_\nu + \sigma_\nu \bar{\sigma}_\mu = 2g_{\mu\nu} I = \bar{\sigma}_\mu \sigma_\nu + \bar{\sigma}_\nu \sigma_\mu$$

$$\mathrm{tr}(\sigma_\mu \bar{\sigma}_\nu) = 2g_{\mu\nu}$$

$$A = \frac{1}{2}\,\mathrm{tr}(\bar{\sigma}^\mu A)\sigma_\mu$$

Now, just as before we have an isomorphism between $SO(1,3) \equiv SL(2,\mathbb{C})/\mathbb{Z}_2$ by extending the previous isomorphism. In particular, we see that we can map 4-vectors to 2×2-matrices by

$$x^\mu \mapsto \sigma_\mu x^\mu = x$$

which is reverse by

$$x^\mu = \frac{1}{2}\,\mathrm{tr}(\bar{\sigma}^\mu x)$$

Explicitly one can find that

$$x = \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix}$$

Now, as before we also have $\det(x) = x^2 = g_{\mu\nu}x^\mu x^\nu$ and furthermore, for $\bar{x} = \bar{\sigma}_\mu x^\mu$ we have $x\bar{x} = \bar{x}x = x^2 I$. Now, for any $A \in SL(2,\mathbb{C})$ we can define $x \mapsto x' = AxA^\dagger = x'^\dagger$. $\det(A) = 1 \implies x^2 = x'^2$ and so there exists a Lorentz transform such that

$$x'^\mu = \Lambda^\mu_\nu x^\nu, \sigma_\mu \Lambda^\mu_\nu = A\sigma_\nu A^\dagger$$

so

$$\Lambda^\mu_\nu = \frac{1}{2}\,\mathrm{tr}\big(\sigma^\mu A \bar{\sigma}_\nu A^\dagger\big)$$

[End of lecture 10]. To reverse this map, we notice that

$$\sigma_\nu A^\dagger \bar{\sigma}^\nu = 2\,\mathrm{tr}\big(A^\dagger\big) I \implies A^\mu_\mu = |\,\mathrm{tr}(A)|^2 \implies \sigma_\mu A^\sigma{}_v \bar{\sigma}^\nu$$

As such, we find that

$$A = \frac{e^{i\alpha}\sigma_\mu \Lambda^\mu{}_\nu \bar{\sigma}^\nu}{2\sqrt{\Lambda^\mu_\mu}}$$

where $\operatorname{tr}(A) = e^{i\alpha}|\operatorname{tr}(A)|$ and α can be determined from $\det(A) = 1$ up to a factor of ± 1 (giving us the necessary homomorphism and kernel).

That completes our correspondance. Now for some final remarks, note that for $A^\dagger = A^{-1}$ (so $A \in SU(2)$ (and not just $SL(2)$)), $x'^0 = x^0$ so we get a rotation of x. Similarly, we see that if $A^\dagger = A$ then Λ is symmetric, meaning we get a Lorentz boost. In particular, we observe that

$$A_B = \cosh(\theta/2)I + \sinh(\theta/2)n \cdot \sigma$$

corresponds to Λ_B.

3.2 2-Spinors and dotted indices

As usual for spinors we require

$$\psi_\alpha \mapsto \Lambda_\alpha{}^\beta \psi_\beta$$
$$\chi^\alpha \mapsto \chi^\beta (\Lambda^{-1})_\beta{}^\alpha$$

for $A \in SL(2, \mathbb{C})$. Note that these are used in the super-symmetry and standard model courses. As usual we use the totally antisymmetric $\epsilon^{\alpha\beta}$ tensor to raise/lower indeices. However, there is one difference with what we had before. Now, we notice that

$$\epsilon^{\alpha\gamma} A_\gamma{}^\delta \epsilon_{\delta\beta} = \delta^\alpha_\beta \operatorname{tr}(\Lambda) - \Lambda_\beta{}^\alpha$$

and since $\det(A) = 1$ we notice that writing the characteristic equation we have

$$A^2 - A\operatorname{tr}(A) + \det(A)I = 0 \implies A(\operatorname{tr}(A)I - A) = I \implies A^{-1} = \operatorname{tr}(A)I - A$$

meaning that

$$\epsilon^{\alpha\gamma} A_\gamma{}^\delta \epsilon_{\delta\beta} = (A^{-1})_\beta{}^\alpha$$

unlike in $SO(3)$. Because of this we get two inequivalent representations under conjugation, and so get two fundamentally different types of spinors. To distinguish these, we add dots to our indices so that now, when we are in the conjugate space we write

$$\bar{\psi}_{\dot{\alpha}} = (\psi_\alpha)^*, \bar{\chi}^{\dot{\alpha}} = (\chi^\alpha)^*$$

which transform via

$$\bar{\psi}_{\dot{\alpha}} \mapsto \bar{\psi}_{\dot{\beta}}(\bar{A}^{-1})^{\dot{\beta}}{}_{\dot{\alpha}}$$
$$\bar{\chi}^{\dot{\alpha}} \mapsto \bar{A}^{\dot{\alpha}}{}_{\dot{\beta}}\bar{\chi}^{\dot{\beta}}$$

where

$$(\bar{A}^{-1})^{\dot{\alpha}}{}_{\dot{\beta}} = (A_{\beta}{}^{\alpha})^*$$

or $\bar{A}^{-1} = A^{\dagger}$ [?]. Here we note that for $A_1, A_2 \in SL(2,\mathbb{C})$ we have $\overline{A_1 A_2} = \bar{A}_1\bar{A}_2$ and that just as before we can use $\epsilon^{\dot{\alpha}\dot{\beta}}$ and $\epsilon_{\dot{\alpha}\dot{\beta}}$ to raise and lower indices.

One weird effect we do get is that now the Pauli matrices have mixed type indices in the sense that

$$\sigma_\mu = (\sigma_\mu)_{\alpha\dot{\alpha}}$$
$$\bar{\sigma}_\mu = (\bar{\sigma}_\mu)^{\dot{\alpha}\alpha} = \epsilon^{\dot{\alpha}\dot{\beta}}\epsilon^{\alpha\beta}(\sigma_\mu)_{\beta\dot{\beta}}$$

Note that here $\alpha, \dot{\alpha}$ are different indices, so we do not sum over them, even if they are placed next to each other. We consequently find that

$$A\sigma_\nu\bar{A}^{-1} = \sigma_\mu\lambda^\mu{}_\nu \implies \bar{A}\bar{\sigma}_\nu A^{-1} = \bar{\sigma}_\mu\Lambda^\mu{}_\nu$$

Furtheremore, we can combine these spinors into 4-component **Dirac spinors** as

$$\Phi = \begin{pmatrix} \psi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix}$$

and

$$\bar{\Phi} = \begin{pmatrix} \chi^\alpha & \bar{\psi}_\alpha \end{pmatrix} = \bar{\Phi}^\dagger \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$$

The 4×4 Dirac matrices then are (generalise the Pauli matrices)

$$\gamma_\mu = \begin{pmatrix} & \sigma_\mu \\ \bar{\sigma}_\mu & \end{pmatrix}$$

and $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}I$ (making something called a **Clifford algebra**?). The tensorial representations associated with these are

$$T^{\mu_1\ldots\mu_\nu} \mapsto \Lambda^{\mu_1}_{\nu_1}\ldots\Lambda^{\mu_n}_{\nu_n}T^{\nu_1\ldots\nu_n}$$
$$\gamma_{\alpha_1\ldots\alpha_{2j},\dot{\alpha}_1\ldots\dot{\alpha}_{2\bar{j}}} \mapsto A_{\alpha_1}^{\ \beta_1}\ldots A_{\alpha_{2j}}^{\ \beta_{2j}}\gamma_{\beta_1\ldots\beta_{2j},\ldots\beta_1\ldots\ldots\beta_{2j}}(\bar{A}^{-1})^{\ldots\beta_1}_{\ldots\alpha_1}\ldots(\bar{A}^{-1})^{\dot{\beta}_{2\bar{j}}}_{\dot{\alpha}_{2\bar{j}}}$$

To find the irreducible tensors, since we're now in 4 dimensions we need to consider the following isotropic tensors:

$$g^{\mu\nu}, \epsilon_{\alpha\beta}, \epsilon_{\dot{\alpha}\dot{\beta}}, \epsilon^{\mu\nu\rho\sigma}$$

where the last symbol is totally asymmetric in all four indices. As such the irreducible representations are just the irreducible tensors that are totally symmetric in both types of indices (allowing all swaps within the sets of undotted and undotted indices, but not between them) leaving us with

$$\gamma_{\alpha_1 \ldots \alpha_{2j}, \dot{\alpha}_1 \ldots \dot{\alpha}_{2\bar{j}}} = \gamma_{(\alpha_1 \ldots \alpha_{2j}, (\dot{\alpha}_1 \ldots \dot{\alpha}_{2\bar{j}}))}$$

Consequently we can label all irreducible spinorial representations of $SO(1,3)$ by $(j, \bar{j})$ (which corresponds to $(\bar{j}, j)$ under complex conjugation). The dimension of such a representation is $(2j+1)(2\bar{j}+1)$ as one might expect. Here, the "fundamental" spinors are (0, 1/2) and (1/2, 0) although the Dirac spinor is $(0, 1/2) \oplus (1/2, 0)$. This is used more in the Standard Model course.

Just as with $SO(3)$ we can decompose a tensor product of irreducible representations into a sum of irreducible representations

$$(j_1, \bar{j}_1) \otimes (j_2, \bar{j}_2) \equiv \bigoplus_{|j_1 - j_2| \leq j \leq |j_1 + j_2|, |\bar{j}_1 - \bar{j}_2| \leq \bar{j} \leq |\bar{j}_1 + \bar{j}_2|} (j, \bar{j})$$

For $2j = n = 2\bar{j}$ we can relate this simply back to rank n vectorial tensors via

$$T_{\mu_1 \ldots \mu_n} = \gamma_{\alpha_1 \ldots \alpha_n, \dot{\alpha}_1 \ldots \dot{\alpha}_n} (\bar{\sigma}_{\mu_1})^{\dot{\alpha}_1 \alpha_1} \ldots (\bar{\sigma}_{\mu_n})^{\dot{\alpha}_n \alpha_n}$$

If γ is irreducible the T here is symmetric and traceless.

3.3 The Poincaré group

The Poincaré group which is the Lorentz group with translations added is often written as $ISO(1,3)$ with the extra requirement that parity is preserved, so $\det(\Lambda) = 1$. Elements of this group are denoted by (Λ, a) where the action

$$(\Lambda, a)x^\mu = \Lambda^\mu{}_\nu x^\mu + a^\nu$$

The composition rule used can be written in this notation as

$$(\Lambda_1, a_1)(\Lambda_2, a_2) = (\Lambda_2 \Lambda_1, \Lambda_2 a_1 + a_2)$$

We also notice that the identity element is just $(I, 0)$ and inverses are $(\Lambda, a)^{-1} = (\Lambda^{-1}, -\Lambda^{-1} a)$. Obviously, a general element in the Poincaré group can be written as the product of a Lorentz transform and a translation as $(\Lambda, a) = (I, a)(\Lambda, 0)$. Furthermore, commutators are of the form that if

$$(\Lambda, a) = (\Lambda_2, a_2)^{-1} (\Lambda_1, a_1)^{-1} (\Lambda_2, a_2)(\Lambda_1, a_1)$$

then

$$
\begin{aligned}
\Lambda &= \Lambda_2^{-1}\Lambda_1^{-1}\Lambda_2\Lambda_1 \\
a &= \Lambda_2^{-1}\Lambda_1^{-1}(\Lambda_2 a_1 - \Lambda_1 a_2 - a_1 + a_2)
\end{aligned}
$$

Infinitesimally, we find that

$$\Lambda^{\mu}_{i\ \nu} = \delta^{\mu}_{\nu} + \omega^{\mu}_{i\ \nu} \implies \Lambda^{\mu}{}_{\nu} = \delta^{\mu}_{\nu} + [\omega_2, \omega_1]^{\mu}{}_{\nu}, a^{\mu} = \omega^{\mu}_{2\ \nu} a_1^{\nu} - \omega^{\mu}_{1\ \nu} a_2^{\nu}$$

In our quantum theory we then get

$$U(\Lambda, a) = 1 - \frac{i}{2}\omega^{\mu\nu} M_{\mu\nu} + i a^{\mu} P_{\mu}$$

for generators of translation P_μ. We then consider

$$
\begin{aligned}
U(\Lambda, a) &= 1 - i[\omega_2, \omega_1]^{\mu\nu} M_{\mu\nu} + i(\omega_2 a_1 - \omega_1 a_2)^{\mu} P_{\mu} \\
&= U(\Lambda_2, a_2)^{-1} U U(\Lambda_1, a_1)^{-1} U(\Lambda_2, a_2) U(\Lambda_1, a_1) \\
&= \dots \\
&= 1 - [\frac{1}{2}\omega_2^{\mu\nu} M_{\mu\nu} - a_2^{\mu} P_{\mu}, \frac{1}{2}\omega_1^{\omega\rho} M_{\omega\rho} - a_1^{\sigma} P_{\sigma}]
\end{aligned}
$$

From here we can deduce commutation relations

$$[\frac{1}{2}\omega_1^{\sigma\rho} M_{\sigma\rho}, a_2^{\mu} P_{\mu}] = i(\omega_1 a_2)^{\mu} P_{\mu}$$

$$[a_2^{\mu} P_{\mu} a_1^{\nu} P_{\nu}] = 0$$

and since the ω_i, a_i are arbitrary we find

$$[M_{\mu\nu}, P_{\sigma}] = i(g_{\nu\sigma} P_{\mu} - g_{\mu\sigma} P_{\nu}), [P_{\mu}, P_{\nu}] = 0$$

which confirms that we have proper 4-vector operators [how?]. Consequently writing

$$(\Lambda, 0)(I, a)(\Lambda, 0)^{-1} = (I, \Lambda a) \implies U(\Lambda, 0) P_{\mu} U(\Lambda, 0)^{-1} = P_{\nu} \Lambda^{\nu}{}_{\mu}$$

and then decomposing $P^{\mu} = (H, \vec{P}), P_{\mu} = (H, -\vec{P})$ leaves us with

$$[J_i, H] = 0, [J_i, P_j] = i\epsilon_{ijk} P_k, [K_i, H] = i P_i, [K_i, P_j] = i\delta_{ij} H$$

as before [End of lecture 11].