Part III, Lent 2021
Quantum Information Theory

Exercises prefaced by $\star$ are eligible for marking.

Example Sheet 2

Exercise 1

1. Prove that if A is positive semi-definite, then A is Hermitian.

2. The action of a flip operator $\mathbb{F} \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$, where $\mathcal{H}_A, \mathcal{H}_B \simeq \mathbf{C}^d$, is defined through the following equation:

$$\mathbb{F}|i\rangle \otimes |j\rangle = |j\rangle \otimes |i\rangle,$$

where $\{|i\rangle\}_{i=1}^d$ denotes an orthonormal basis of $\mathbf{C}^d$. Write an explicit expression for $\mathbb{F}$ in terms of the basis vectors $|i\rangle$. What are the eigenvalues of $\mathbb{F}$ and what are their multiplicities? Express $\mathbb{F}$ in terms of the MES

$$|\Omega\rangle := \frac{1}{\sqrt{d}}|i\rangle \otimes |i\rangle.$$

Hint: Think of what mathematical operation you would use to relate one to the other.

Exercise 2 Show that any density operator of a qubit can be written as

$$\varrho = \tfrac{1}{2}\left(\mathbb{1} + \vec{s} \cdot \vec{\sigma}\right) = \tfrac{1}{2}\sum_{k=0}^{3} s_k \sigma_k, \tag{1}$$

where $\vec{s} = (s_1, s_2, s_3)$ is a real three–dimensional vector such that $||\vec{s}|| \leq 1$. In the above $\vec{\sigma} := (\sigma_1, \sigma_2, \sigma_3)$, with $\sigma_1, \sigma_2, \sigma_3$ being the three Pauli matrices $\sigma_x, \sigma_y, \sigma_z$. Moreover, σ_0 denotes the 2×2 identity matrix.

The vector $\vec{s}$ is usually referred to as the Bloch vector. The set of all Bloch vectors defines a sphere which is called the *Bloch sphere*. It provides a useful geometrical representation of the states of a qubit.

Show that $s_k = \mathrm{Tr}\, \sigma_k \varrho$. Show that $||\vec{s}|| = 1$ for a pure state ρ. What is the Bloch representation of the completely mixed state $\rho = \mathbf{I}/2$? What do the North Pole and the South Pole of the Bloch sphere correspond to?

$\star$ **Exercise 3** Given the density matrix

$$\rho = \begin{pmatrix} \frac{3}{5} & \frac{1}{4} - i\frac{1}{6} \\ \frac{1}{4} + i\frac{1}{6} & \frac{2}{5} \end{pmatrix}$$

what is the Bloch vector $\vec{s} = (s_x, s_y, s_z)$ for ρ? Is it a pure state or a mixed state? What is the probability that a measurement of the spin of the qubit along the Z-axis will yield a value $+1$?

Exercise 4 Show that the set of density operators acting on a Hilbert space $\mathcal{H}$, where dim $\mathcal{H} = d$ is a *convex subset* of the real vector space of $d \times d$ Hermitian matrices. Show that pure states are extremal points of this set.

Exercise 5 Find the Schmidt ranks for each of the following states

$$|\,\phi_1\rangle = \tfrac{1}{2}(|\,00\rangle + |\,11\rangle + |\,22\rangle + |\,44\rangle) \qquad |\,\phi_2\rangle = \tfrac{1}{2}(|\,00\rangle - |\,01\rangle + |\,10\rangle - |\,11\rangle)$$

$$|\,\phi_3\rangle = \tfrac{1}{2}(|\,00\rangle + |\,01\rangle + |\,10\rangle - |\,11\rangle) \qquad |\,\phi_4\rangle = \frac{1}{\sqrt{3}}(|\,00\rangle + |\,01\rangle + |\,11\rangle)$$

Exercise 6 Let $|\Psi_{AR}\rangle$ and $|\Phi_{AR}\rangle$ be two purifications of a state ρ of a system A to a composite system AR. Prove that there exists a unitary transformation U_R acting on system R alone such that

$$|\Phi_{AR}\rangle = (\mathbf{I}_A \otimes U_R)|\Psi_{AR}\rangle.$$

Exercise 7 (Surjectivity of $\Lambda \mapsto J$)

The Choi-Jamilkowski matrix $J \in \mathcal{B}(\mathbf{C}^{d'} \otimes \mathbf{C}^d)$ and a linear map $\Lambda : \mathcal{M}_d \to \mathcal{M}_{d'}$ are related by

$$\begin{aligned} J &= (\Lambda \otimes \mathrm{id}_d)(|\,\Omega\rangle\,\langle\Omega\,|) \\ \mathrm{Tr}\,[A\,\Lambda(B)] &= d\,\mathrm{Tr}\,[J(A \otimes B^T)] \end{aligned}$$

for any $A \in \mathcal{M}_{d'}$ and $B \in \mathcal{M}_d$, where $|\,\Omega\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |\,i\rangle\,|\,i\rangle$. Prove that the map $\Lambda \mapsto J$ is surjective.

Hint: Decompose J into a linear combination of rank-1 operators $|\,\psi_i\rangle\,\langle\psi_j\,|$ and use $|\,\psi_i\rangle = (R_i \otimes I)\,|\,\Omega\rangle$.

Exercise 8 (Unitary freedom in the Kraus decomposition)

Suppose $\{A_i\}_{i=1}^n$ and $\{B_i\}_{i=1}^n$ are Kraus operators giving rise to quantum operations Λ and Λ', respectively. Prove that $\Lambda = \Lambda'$ if and only if there exist complex numbers u_{ij} such that $A_i = \sum_{j=1}^n u_{ij} B_j$, and u_{ij} are the elements of an $n \times n$ unitary matrix.

Exercise 9 (Number of Kraus operators)

1. Consider a quantum operation $\Lambda : \mathcal{B}(\mathcal{H}_\mathcal{A}) \to \mathcal{B}(\mathcal{H}_\mathcal{B})$. What is the maximal rank of its Choi state $J(\Lambda)$?

2. Prove that any quantum operation Λ on the state ρ of a system with a d–dimensional Hilbert space, there exists a Kraus decomposition with at most d^2 elements, i.e.,

$$\Lambda(\rho) = \sum_{i=1}^n A_i \rho A_i^\dagger,$$

   where $n \leq d^2$.

Exercise 10 Consider two quantum operations Λ_1 and Λ_2 acting on a single qubit, having Kraus representations

$$\Lambda_1(\rho) = \sum_{k=1}^2 A_k \rho A_k^\dagger; \quad \Lambda_2(\rho) = \sum_{k=1}^2 V_k \rho V_k^\dagger,$$

with

$$A_1 = \frac{1}{\sqrt{2}}\sigma_0 \; ; A_2 = \frac{1}{\sqrt{2}}\sigma_z$$

and

$$V_1 = |0\rangle\langle 0| \; ; V_2 = |1\rangle\langle 1|.$$

How do the actions of Λ_1 and Λ_2 differ from each other?

⋆ **Exercise 11** Show that the following three operators form a POVM.

$$\begin{aligned}
E_1 &= \frac{\sqrt{2}}{1+\sqrt{2}} \,|\,1\rangle\,\langle 1\,| \\
E_2 &= \frac{\sqrt{2}}{2+2\sqrt{2}} (|\,0\rangle - |\,1\rangle)(\langle 0\,| - \langle 1\,|) \\
E_3 &= \mathbb{1} - E_1 - E_2
\end{aligned}$$

Suppose Alice gives Bob a state prepared in one of the two states $|\,\psi_1\rangle = |\,0\rangle$ or $|\,\psi_2\rangle = (|\,0\rangle + |\,1\rangle)/\sqrt{2}$. Show that if Bob does a measurement characterized by these POVM elements on the state he receives, he never makes an error of misidentification. Discuss the possible outcomes.