3P2a **Symmetries: Examples Sheet 1** Michaelmas 2020

Corrections and suggestions should be emailed to `B.C.Allanach@damtp.cam.ac.uk`. *Starred questions may be handed in prior to the class to your examples class supervisor, for feedback.*

1. Consider the set $\Re^2$ consisting of pairs of real numbers. For $(x, y) \in \Re^2$, find which of the following group operations make a group on $\Re^2$ (and if not, find why not):

   (a) $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$,

   (b) $(x_1, y_1) \cdot (x_2, y_2) = (x_1 x_2, y_1 y_2)$,

   (c) $(x_1, y_1) \circ (x_2, y_2) = (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1)$.

   Identify a well known mathematical object that is isomorphic to $\{\Re^2 \backslash (0, 0)\})$ under $\circ$ as a group.

2. Write the group multiplication table for the dihedral group D_4. Enumerate the subgroups, the normal subgroups and the conjugacy classes. Can D_4 be written as the *non-trivial* product of some of its subgroups?

3. Consider the possibility that a set G of $n \times n$ matrices forms a group with respect to matrix multiplication.

   (a) Prove that if G is a group and if one of the elements of G is a non-singular matrix then all of the elements of G must be non-singular matrices. Conclude that all the elements of G are either non-singular matrices or singular matrices.

   (b) Consider the set of 2×2 singular matrices G of the form
   $$\begin{pmatrix} x & x \\ x & x \end{pmatrix}, \tag{1}$$
   where $x \in \Re$ and $x \neq 0$. Prove that G is a group with respect to matrix multiplication. Determine the matrix corresponding to the identity element of G. Determine the inverse element of (1).

   (c) The group defined in question 3b is isomorphic to a well known group. Identify this group.

4. The *centre* of a group G, denoted by $\mathcal{Z}(G)$, is defined as the set of elements $z \in G$ that commute with all elements of the group. That is,
   $$\mathcal{Z}(G) = \{z \in G \,|\, zg = gz \,, \forall\, g \in G\} \,.$$

   (a) Show that $\mathcal{Z}(G)$ is an abelian subgroup of G.

   (b) Show that $\mathcal{Z}(G)$ is a normal subgroup of G.

   (c) Find the centre of D_4 and construct the group $D_4/\mathcal{Z}(D_4)$. Determine whether the isomorphism $D_4 \cong [D_4/\mathcal{Z}(D_4)] \times \mathcal{Z}(D_4)$ is valid.

5. An automorphism is defined as an isomorphism of a group G onto itself.

   (a) Show that for any $g \in G$, the mapping $T_g(x) = gxg^{-1}$ is an automorphism (called an *inner automorphism*), where $x \in G$.

(b) Show that the set of all inner automorphisms of G, denoted by $\mathcal{I}(G)$, is a group.

(c) Show that $\mathcal{I}(G) \simeq G/\mathcal{Z}(G)$, where $\mathcal{Z}(G)$ is the centre of G.

(d) Show that the set of all automorphisms of G, denoted by $\mathcal{A}(G)$, is a group and that $\mathcal{I}(G)$ is an invariant subgroup. (The factor group $\mathcal{A}(G)/\mathcal{I}(G)$ is called the group of *outer automorphisms* of G.)

6.* For the tensor product space $\mathcal{V}_{j_1} \otimes \mathcal{V}_{j_2}$ then the total angular momentum is $\mathbf{J} = \mathbf{J}_1 + \mathbf{J}_2$, which can be written in terms of components $J_\pm,\ J_3$. Let $\mathcal{U}_M$ be the subspace for which J_3 has the eigenvalue M.

(a) Determine the dimension of $\mathcal{U}_M$. Show that if $M \geq |j_1 - j_2|$ there is a one-dimensional subspace of $\mathcal{U}_M$ which is orthogonal to $J_- \mathcal{U}_{M+1}$.

(b) Hence show that there is a single normalised state, unique up to an overall phase factor, $|\phi\rangle \in \mathcal{U}_M$ such that $J_+|\phi\rangle = 0$ if $M \geq |j_1 - j_2|$ and that we may identify $|JJ\rangle = |\phi\rangle$ for $J = M$. What happens if $M < |j_1 - j_2|$?

7.* If $\mathbf{u}$ and $\mathbf{v}$ are vectors in three dimensional Euclidean space, show that

$$T_{ij} = u_i\, v_j = \tilde{T}_{ij} + \tfrac{1}{2}\epsilon_{ijk} V_k + \tfrac{1}{3}\delta_{ij} S$$

separates the components of T_{ij} into subsets of length $5, 3, 1$, respectively, that transform amongst themselves under $SO(3)$ rotations, where

$$\tilde{T}_{ij} = \tfrac{1}{2}(u_i v_j + u_j v_i) - \tfrac{1}{3}\delta_{ij} u_k v_k \quad , \quad V_k = (\mathbf{u} \times \mathbf{v})_k \quad , \quad S = \mathbf{u}\cdot\mathbf{v}\,.$$

Explain the relation to the result that, if $\mathcal{V}_j$ is the vector space for angular momentum j, then $\mathcal{V}_1 \otimes \mathcal{V}_1 \simeq \mathcal{V}_2 \oplus \mathcal{V}_1 \oplus \mathcal{V}_0$.

8. (a) Show that $SO(n)$ is a normal subgroup of $O(n)$.

(b) If n is odd, show that $\mathbb{Z}_2 \cong \{I_n\,,\, -I_n\}$ is a normal subgroup of $O(n)$, where I_n is the $n \times n$ identity matrix. Prove that $O(n)$ can be written as an internal direct product, $O(n) \cong SO(n) \times \mathbb{Z}_2$.

(c) Explain why the results of part (b) do not apply to the case of even n.

(d) The group $SO(2)$ consists of all 2×2 orthogonal matrices with unit determinant. Prove that $SO(2)$ is an abelian group.

(e) The group $O(2)$ consists of all 2×2 orthogonal matrices, with no restriction on the sign of its determinant. Is $O(2)$ abelian or non-abelian? (If the latter, find two $O(2)$ matrices that do not commute.)