Non-Equilibrium Statistical Field Theory

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Introduction

These notes are based on the course lectured by Johannes Pausch in Michaelmas 2020. Due to the measures taken in the UK to limit the spread of Covid-19, these lectures were delivered online. These are not meant to be an accurate representation of what was lectures. They solely represent a mix of what I thought was the most important part of the course, mixed in with many (many) personal remarks, comments and digressions... Of course, any corrections/comments are appreciated.

Non-Equilibrium Statistical Field Theory is the study of statistical properties of field theory that is changing in time (which is what non-equilibrium refers to here).

1 Master Equations

Firstly, we try to derive some master equations to use in our model. How do we model our system? We first discretise time into steps t_n. For some reason, in this field by convention, time travels from right to left. Anyways, having discretised time, we model systems as Markov chains with $N(t_n)$ being the number of particles "present" at t_n.

We can then write the **Chapman-Kolmogorov equation** as

$$\mathbb{P}(N(t_3)|N(t_1)) = \sum_{N(t_2)} \mathbb{P}(N(t_3)|N(t_2))\mathbb{P}(N(t_2)|N(t_1))$$

Now we want to take the continuum limit of this equation by defining

$$W_t(N'|N) = \partial_{t'}\mathbb{P}(N'(t')|N(t))|_{t'=t}$$

then we can write

$$\mathbb{P}(N'(t+\Delta t)|N(t)) \approx \mathbb{P}(N'(t)|N(t)) + \Delta t W_t(N'|N) = \delta_{N'(t),N(t)} + \Delta W_t(N'|N)$$

Notice that since probabilities add to 1, $\sum_{N'} W_t(N'|N) = 0$, so in particular $W_t N|N = -\sum_{N' \neq N} W_t(N'|N)$. Using that we can rewrite the Chapman-Kolmogorov equation as

$$\begin{aligned}
\mathbb{P}(N(t_3)|N(t_1)) - \mathbb{P}(N(t_2)|N(t_1)) &= \sum_{N(t_2)} \mathbb{P}(N(t_3)|N(t_2))\mathbb{P}(N(t_2)|N(t_1)) \\
&= \Delta t \sum_{N_2 \neq N_3} W_{t_2}(N_3|N_2)\mathbb{P}(N_2|N_1) - W_{t_3}(N_2|N_3)\mathbb{P}(N_3|N_1)
\end{aligned}$$

Taking limits and removing the $|N_1$ part in the conditional probabilities for notational simplicity we get the final form of our **Master Equation** (which has no other name?)

$$\partial_t \mathbb{P}(N) = \sum_{N' \neq N} W_t(N|N')\mathbb{P}(N') - W_t(N'|N)\mathbb{N}$$

where the first term in the sum can be called the **gain**, and the second subtracted term can be called the **loss** (we can interpret them as such). To build intuition we provide a simple example.

Example 1. We consider a simple extinction process where a group of particles are gradually disappearing. Here $W(N-1|N) = N\epsilon, W(N|N) = -N\epsilon$ for some $\epsilon > 0$ while all other values of the matrix W are zero. As such we get a simple master equation

$$\partial_t \mathbb{P}(N) = \epsilon(N+1)\mathbb{P}(N+1) - \epsilon N \mathbb{P}(N)$$

[End of lecture 1]