Statistical Field Theory

quinten tupker

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Introduction

These notes are based on the course lectured by Dr Christopher E Thomas in Michaelmas 2020. Due to the measures taken in the UK to limit the spread of Covid-19, these lectures were delivered online. These are not meant to be an accurate representation of what was lectures. They solely represent a mix of what I thought was the most important part of the course, mixed in with many (many) personal remarks, comments and digressions... Of course, any corrections/comments are appreciated.

Statistical Field Theory is an extension of statistical physics. It assumes one is familiar with statistical physics, and in particular, focuses on the study on phase transitions. This course in particular follows David Tong's notes quite closely.

1 From Spin to Fields

So why do we consider fields? Here we look at a model that shows the origin of this connection.

1.1 The Ising model

The Ising model studies a lattice where each point on the lattice is assigned a spin $S_i = \pm$. Consequently, the energy of the lattice is

$$E = -B \sum_i S_i - J \sum_{<i,j>} S_i S_j$$

where B is the external magnetic field strength, J is the strength of neighbour-neighbour interactions and $< i, j >$ is any nearest neighbour pair. When $J > 0$, states tend to align (ferromagnetic behaviour) whereas if $J < 0$ states will prefer not to align (anti-ferromagnetic behaviour). We will focus on the $J > 0$ case. But of course, due to heat there some statistical randomness, and we want to include that. As such, we consider the canonical ensemble with $\mathbb{P}(S_i) = e^{-\beta E(S_i)}/Z$ where Z is the partition function, $\beta = 1/T$, and the Boltzmann constant $k_B = 1$. As ever in statistical physics, we can derive everything

from the partition function. Particularly important in our case is the Free Energy, $F = \langle E \rangle - TS = -T \ln(Z)$, and $dF = -SdT - pdV - MdB$ (so we can find S, p and M from F as well).

In statistical physics, we are particularly interested in the equilibrium state (which occurs at the minimum of the free energy when temperature is constant), and since we are looking at a magnetic system, we are particularly interested in the equilibrium magnetisation. This can be calculated as

$$m = \frac{1}{N} \sum_i \langle S_i \rangle = \frac{1}{N\beta} \partial_B \ln(Z)$$

Now all that remains is to calculate Z to find m. Unfortunately, in dimension 3 or greater this is impossible (how impossible?), and it is still hard in lower dimension. Consequently we take a different approach using the so-called "effective free energy", which is defined such that

$$\sum_m \sum_{\{S_i\}|m} e^{-\beta E[S_i]} = \sum_m e^{-\beta F(m)}$$

where $F(m)$ is the effective free energy. Since we can assume N is large (around 10^{23}), we can then write18

$$Z = N/2 \int_{-1}^{1} dm \, e^{-\beta F(m)} = N/2 \int_{-1}^{1} dm \, e^{-\beta N f(m)}$$

where $f(m) = F(m)/N$. From here, we can calculate the equilibrium field by considering that since N is large, the value contributing the most to the integral is where $\partial_m f = 0$, and this is how the equilibrium m is calculated. This approach is called the **steepest descent approximation**, and we find here that $F_{\text{thermodynamic}} \approx F(m_{\min})$.

This is all very well and nice, but the issue is that we still don't know how to calculate $F(m)$, and it turns out that this is about as hard as calculating Z. As such, we use the **mean field approximation**

$$E \approx -B \sum_i m - J \sum_{<i,j>} m^2 = -BNm - \frac{1}{2} NJqm^2$$

where $q = 2\dim(\text{space})$ for a cubic latice in a space dimension dim. Now,

$$e^{-\beta N f(m)} = \sum e^{-\beta E(S_i) \approx \Omega(m) e^{-\beta E(m)}}$$

so in order to find $f(m)$ we need to know $\Omega(m)$ which is the number of ways for a given energy state to occur, but since m depends only on the number of positive and negative spins states, $N_\uparrow, N_\downarrow$, and in particular $N = N_\uparrow + N_\downarrow$ we see the number of total states is

$$\ln(\Omega) = \ln \binom{N}{N_\uparrow} \approx N(\ln(2) - \frac{1}{2}(1+m)\ln(1+m) - \frac{1}{2}(1-m)\ln(1-m))$$

by Stirling's approximation. Consequently we find that

$$f(m) \approx -Bm - \frac{1}{2}Jqm^2 - \frac{1}{\beta}(\ln(2) - \frac{1}{2}(1+m)\ln(1+m) - \frac{1}{2}(1-m)\ln(1-m)).$$

From here by taking $\partial_m f = 0$, we find

$$\beta(B + Jqm) = \frac{1}{2}\ln\left(\frac{1+m}{1-m}\right)$$

or equivalently

$$m = \tanh(\beta(B + Jqm)).$$

From here, we can solve implicitly, but there is another approach we can take, which is the way we will go about this. Just as a side note, we can see the reason this is called the mean field approximation here as well: it is as if m just shifts to external field to $B_{\text{eff}} = B + Jqm$. [End of lecture 1.]