# 3P1b **Quantum Field Theory: Example Sheet 2** Michaelmas 2020

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1. Consider a real scalar field with Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2. \tag{1}$$

Show that, after normal ordering, the conserved four-momentum $P^\mu = \int d^3x\ T^{0\mu}$ takes the operator form

$$P^\mu = \int \frac{d^3p}{(2\pi)^3} p^\mu a^\dagger_{\vec{p}} a_{\vec{p}} \tag{2}$$

where $p^0 = E_{\vec{p}}$ in this expression. From Eq. (2), verify that if $\phi(x)$ is now in the Heisenberg picture, then

$$[P^\mu, \phi(x)] = -i\partial^\mu\phi(x).$$

2*. Show that in the Heisenberg picture,

$$\dot{\phi}(x) = i[H, \phi(x)] = \pi(x) \quad \text{and} \quad \dot{\pi}(x) = i[H, \pi(x)] = \nabla^2\phi(x) - m^2\phi(x)\,.$$

Hence show that the operator $\phi(x)$ satisfies the Klein-Gordon equation.

3. Let $\phi(x)$ be a real scalar field in the Heisenberg picture. Show that the relativistically normalised states $|p\rangle = \sqrt{2E_{\vec{p}}}a^\dagger_{\vec{p}}|0\rangle$ satisfy

$$\langle 0|\,\phi(x)\,|p\rangle = e^{-ip\cdot x}.$$

4*. In Example Sheet 1, you showed that the classical angular momentum of a field is given by

$$Q_i = \frac{1}{2}\epsilon_{ijk}\int d^3x\ \left(x^j T^{0k} - x^k T^{0j}\right).$$

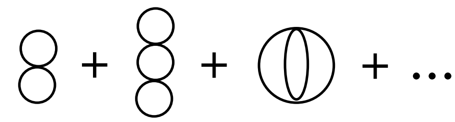
Write down the explicit form of the angular momentum for a free real scalar field with Lagrangian as in Eq.(1). Show that, after normal ordering, the quantum operator $Q_i$ can be written as

$$Q_i = \frac{i}{2}\epsilon_{ijk}\int \frac{d^3p}{(2\pi)^3} a^\dagger_{\vec{p}} \left(p^j\frac{\partial}{\partial p_k} - p^k\frac{\partial}{\partial p_j}\right) a_{\vec{p}}.$$

Hence confirm that the quanta of the scalar field have spin zero (i.e. a one-particle state $|\vec{p}\rangle$ has zero angular momentum in its rest frame).

5. Show that the time ordered product $T(\phi(x_1)\phi(x_2))$ and the normal ordered product $:\phi(x_1)\phi(x_2):$ are both symmetric under the interchange of $x_1$ and $x_2$. Deduce that the Feynman propagator $\Delta_F(x_1 - x_2)$ has the same symmetry property.

6. Examine $\langle 0| \hat{S} |0\rangle$ to order $\lambda^2$ in $\phi^4$ theory. Identify the different contributions arising from an application of Wick's theorem and derive Feynman rules representing these contributions as diagrams. Confirm that to order $\lambda^2$, the combinatoric factors work out so that the vacuum to vacuum amplitude is given by the following diagramatic expression,

$$\langle 0| \hat{S} |0\rangle = \exp\Bigg($$

$$\Bigg)$$

corresponding to the exponential of the sum of distinct vacuum bubble diagrams.

7. Consider the scalar Yukawa theory given by the Lagrangian

$$\mathcal{L} = \partial_\mu \psi^* \partial^\mu \psi + \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - M^2 \psi^* \psi - \frac{1}{2} m^2 \phi^2 - g \psi^* \psi \phi.$$

Calculate the amplitude for meson decay $\phi \to \psi\bar{\psi}$ to leading order in $g$. Show that the amplitude is only non-zero for $m > 2M$ and explain the physical interpretation of this condition using conservation laws.