Statistical Field Theory

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Introduction

These notes are based on the course lectured by Dr Christopher E Thomas in Michaelmas 2020. Due to the measures taken in the UK to limit the spread of Covid-19, these lectures were delivered online. These are not meant to be an accurate representation of what was lectures. They solely represent a mix of what I thought was the most important part of the course, mixed in with many (many) personal remarks, comments and digressions... Of course, any corrections/comments are appreciated.

Statistical Field Theory is an extension of statistical physics. It assumes one is familiar with statistical physics, and in particular, focuses on the study on phase transitions. This course in particular follows David Tong's notes quite closely.

1 From Spin to Fields

So why do we consider fields? Here we look at a model that shows the origin of this connection.

1.1 The Ising model

The Ising model studies a lattice where each point on the lattice is assigned a spin $S_i = \pm$. Consequently, the energy of the lattice is

$$E = -B\sum_i S_i - J\sum_{<i,j>} S_i S_j$$

where B is the external magnetic field strength, J is the strength of neighbour-neighbour interactions and $< i, j >$ is any nearest neighbour pair. When $J > 0$, states tend to align (ferromagnetic behaviour) whereas if $J < 0$ states will prefer not to align (anti-ferromagnetic behaviour). We will focus on the $J > 0$ case. But of course, due to heat there some statistical randomness, and we want to include that. As such, we consider the canonical ensemble with $\mathbb{P}(S_i) = e^{-\beta E(S_i)}/Z$ where Z is the partition function, $\beta = 1/T$, and the Boltzmann constant $k_B = 1$. As ever in statistical physics, we can derive everything

from the partition function. Particularly important in our case is the Free Energy, $F = \langle E \rangle - TS = -T\ln(Z)$, and $dF = -SdT - pdV - MdB$ (so we can find S, p and M from F as well).

In statistical physics, we are particularly interested in the equilibrium state (which occurs at the minimum of the free energy when temperature is constant), and since we are looking at a magnetic system, we are particularly interested in the equilibrium magnetisation. This can be calculated as

$$m = \frac{1}{N}\sum_i \langle S_i \rangle = \frac{1}{N\beta}\partial_B \ln(Z)$$

Now all that remains is to calculate Z to find m. Unfortunately, in dimension 3 or greater this is impossible (how impossible?), and it is still hard in lower dimension. Consequently we take a different approach using the so-called "effective free energy", which is defined such that

$$\sum_m \sum_{\{S_i\}|m} e^{-\beta E[S_i]} = \sum_m e^{-\beta F(m)}$$

where $F(m)$ is the effective free energy. Since we can assume N is large (around 10^{23}), we can then write18

$$Z = N/2 \int_{-1}^{1} dm\, e^{-\beta F(m)} = N/2 \int_{-1}^{1} dm\, e^{-\beta N f(m)}$$

where $f(m) = F(m)/N$. From here, we can calculate the equilibrium field by considering that since N is large, the value contributing the most to the integral is where $\partial_m f = 0$, and this is how the equilibrium m is calculated. This approach is called the **steepest descent approximation**, and we find here that $F_{\text{thermodynamic}} \approx F(m_{\min})$.

This is all very well and nice, but the issue is that we still don't know how to calculate $F(m)$, and it turns out that this is about as hard as calculating Z. As such, we use the **mean field approximation**

$$E \approx -B\sum_i m - J \sum_{<i,j>} m^2 = -BNm - \frac{1}{2}NJqm^2$$

where $q = 2\text{dim(space)}$ for a cubic latice in a space dimension dim. Now,

$$e^{-\beta N f(m)} = \sum e^{-\beta E(S_i) \approx \Omega(m) e^{-\beta E(m)}}$$

so in order to find $f(m)$ we need to know $\Omega(m)$ which is the number of ways for a given energy state to occur, but since m depends only on the number of positive and negative spins states, $N_\uparrow, N_\downarrow$, and in particular $N = N_\uparrow + N_\downarrow$ we see the number of total states is

$$\ln(\Omega) = \ln\binom{N}{N_\uparrow} \approx N(\ln(2) - \frac{1}{2}(1+m)\ln(1+m) - \frac{1}{2}(1-m)\ln(1-m))$$

by Stirling's approximation. Consequently we find that

$$f(m) \approx -Bm - \frac{1}{2}Jqm^2 - \frac{1}{\beta}(\ln(2) - \frac{1}{2}(1+m)\ln(1+m) - \frac{1}{2}(1-m)\ln(1-m)).$$

From here by taking $\partial_m f = 0$, we find

$$\beta(B + Jqm) = \frac{1}{2}\ln\left(\frac{1+m}{1-m}\right)$$

or equivalently

$$m = \tanh(\beta(B + Jqm)).$$

From here, we can solve implicitly, but there is another approach we can take, which is the way we will go about this. Just as a side note, we can see the reason this is called the mean field approximation here as well: it is as if m just shifts to external field to $B_{\text{eff}} = B + Jqm$. [End of lecture 1.]

1.2 Landau Theory of Phase Transitions

The remarkable part about the Ising model described above is not its correctness. In fact, it is often incorrect, but rather its universality. It can be applied in a wide variety of situations. As such, Landau tried to develop a more general theory of phase transitions. Here we work through an extended example to examine the general features of these. In particular, we study the behaviour of the equilibrium $m_{\min}$ when various quantities are varied.

From before, we can approximate f for small m as

$$f(m) \approx \cancel{-\frac{1}{\beta}\ln(2)} - Bm + \frac{1}{2}(\frac{1}{\beta} - Jq)m^2 + \frac{1}{12\beta}m^4 + \dots$$

where we ignore the constant term since it does not affect anything. We start with the $B = 0$ case where we find that

$$f(m) = \frac{1}{2}(T - T_c)m^2 + \frac{1}{12}T^4m^4$$

where we define the **critical temperature** to be $Tc = Jq$. Then we get the following scenarios.

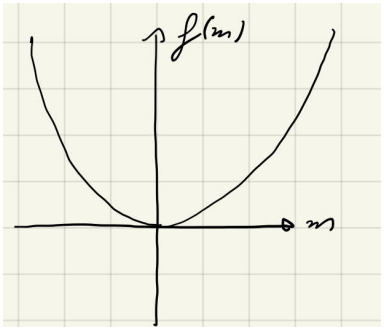

Figure 1: $T > T_c$

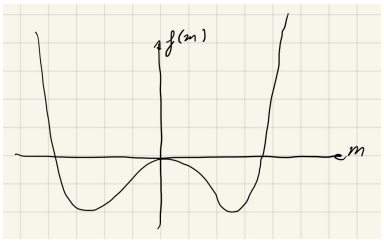

Figure 2: $T < T_c$

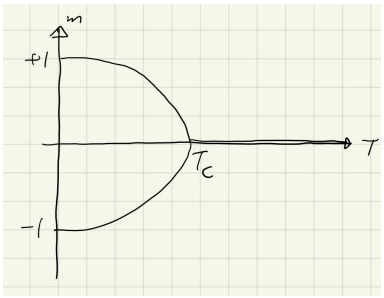

Figure 3: $m_{\min}(T)$ for $B = 0$

The middle graph is meant to be symmetric, and we see here that $m_{\min}$ can take two different values

$$m_{\min} = \pm m_0 = \pm\sqrt{\frac{3(T_c - T}{T}}$$

Now some definitions apply here. This is a **second order** or **continuous** phase transition since m is continuous in the quantity being varied. $m = 0$ is called the **disordered phase**, and $m \neq 0$ is called an **ordered phase**.

Spontaneous symmetry breaking is what we call the loss of symmetry in m when T decreases below T_0 (since m is forced to choose a positive or negative value). Finally, these terms all extend to arbitrary **order parameter** m. We finally compute

$$f(m_{\min}) = \begin{cases} 0 & T > T_c \\ -\frac{3}{4}\frac{(T_c - T)^2}{T} & T < T_c \end{cases}$$

We can also look at the heat capacity here, and how that various with temperature. Heat capacity may be defined as

$$C = \partial_T \langle E \rangle = \beta^2 \partial_\beta^2 \ln(Z)$$

since recall that $\langle E \rangle = -\partial_\beta \ln(Z)$. Using $Z \approx -\beta N f(m_{\min})$ we find $c = C/N$ has

$$c = \begin{cases} 0 & T \to T_c^+ \\ 3/2 & T \to T_c^- \end{cases}$$

so c is a first order phase transition - ie. discontinuous.

The above scenario considered $B \neq 0$, but when $B = 0$ we get a different situation. In particular, we do not get spontaneous symmetry breaking since the system is no longer symmetric. In this case, the above graphs are skewed to the right or the left unevenly leaving global minimum (the "true" minimum) and a so-called **metastable** state.

Finally, we can observe another discontinuous phase transition when we study $m_{\min}$ as a function of B at low T ($T < T_c$). Since we're below the critical temperature, the global minimum does not smoothly slide through 0, but instead abruptly shifts from negative to postive (or the other way around) giving a 1st order phase transition. Incidentally, I find this a bit odd, since this really only makes sense in the 1D case, but as we will see later Mean Field Theory (MFT) does not work in 1D. But in higher dimensions, m is a vector and instead of getting two local minima as above, we get a tilted (for $B \neq 0$) circular valley where $f(m)$ is low. In this case, no phase transition occurs, since the minimum just smoothly moves around this circular valley as B is varied...

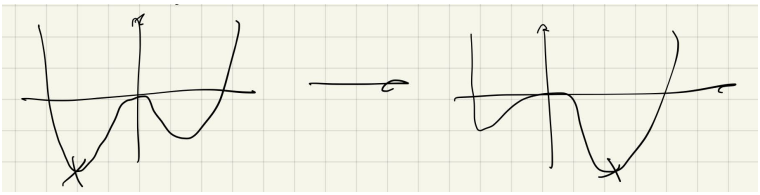

Figure 4: $m_{\min}$ vs B

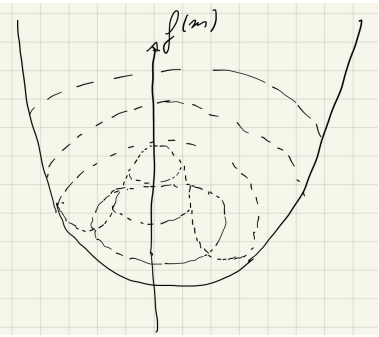

Figure 5: $f(m)$ vs m in 2D

Anyways, some other asymptotic behaviour includes that when $T \approx T_c$, we get

$$f(m) \approx -Bm + \frac{1}{12} T_c m^4$$

leaving

$$m \sim B^{1/3}$$

If we define magnetic susceptibility (the **response function** in this context) to be $\chi = \partial_B m|_T$ we see that for $T > T_c$ we have

$$f(m) \approx -Bm + \frac{1}{2}(T - T_c)m^2 + \dots$$

so

$$m \approx \frac{B}{T - T_c}$$

so

$$\chi \sim \frac{1}{T - T_c}$$

and for $T < T_c$

$$\chi \sim \frac{1}{|T - T_c|}$$

1.3 The Validity of MFT

The purpose of all these calculations at the end of the last section is to find some important set of numbers that we can use to check our theory with in the real world. In particular, what we looked for were the **critical exponents**, which are

$$
\begin{aligned}
m &\sim (T_c - T)^{\beta} && \beta = 1/2 && T < T_c \\
c &\sim c_{\pm}|T - T_c|^{-\alpha} && \alpha = 0 \\
\chi &\sim |T - T_c|^{-\gamma} && \gamma = 1 \\
m &\sim B^{1/\delta} && \delta = 3
\end{aligned}
$$

in our worked example. Now for results, we find that in general, MFT fails entirely below a certain **lower critical dimension**, d_i and works correctly above a certain **upper critical dimension**, d_u. In between it is structurally accurate (can detect the right phase transitions) but inaccurate (finds the wrong critical exponents). The Ising fails for dimension 1, gets the right structure for dimensions 2 and 3, and works for dimension 4 and above. It is, as expected, closer to the correct value for dimension 3 than for 2, but still can be quite off.

Nevertheless, as mentioned earlier, the power of the MFT is not in its correctness, but in its universality. Also, often structure is more important. After all, if you can predict a phenomenon will happen, you can then carry out an experiment to measure it more accurately. On the other hand, it is far harder to use experiment to search for interesting phenomenon without knowing where to look. So it is still quite powerful. Also, the fact that critical exponents exist, and that critical points exist is a powerful bit of universality that MFT gives.

To illustrate its universality, we can consider liquid-gas transitions. Here, van der Waals analysis is completely off in its critical exponents, although it does predict the correct type of phenomenon. However in three dimensions, the Ising model agrees well. How do we implement the Ising model for a gas. Although it's not at all obvious it will work, the implementation is actually quite natural. Assume that a space consists of a lattice, and that every lattice is assigned a value of 1 or 0 based on whether or not it is occupied by a particle. Also assume that any point on the lattice can be occupied by no more than 1 particle at a time. Then we get

$$E = -4J \sum_{<ij>} n_i n_j - \mu \sum n_i$$

We can proceed from there as usual. [End of lecture 2]

1.4 Landau-Ginzburg Theory

We aim to improve on Landau theory, and somehow the obvious generalisation is to allow m to vary through space as $m(x)$, but doing so in a way that it is analytic, so that the calculations involved still stay relatively simple...

Concretely, that means assuming we have a large number of sites N, but that these sites are all grouped into clusters of N' sites, which is still a large number, but significantly smaller than N. Then we define for each cluster, a centre x, and we assign $m(x)$ the value of the average magnetisation of this

cluster. However, since $1 << N' << N$, everything is still analytic, etc. Or so we assume.

Consequently we can calculate the partition function in much the same way with the same effective free energy, except now the effective free energy is a functional instead of a function:

$$Z = \sum_{m(x)} e^{-\beta F[m(x)]}$$

This can be written as a functional integral (otherwise known as a path integral)

$$Z = \int Dm[x] e^{-\beta F[m(x)]}$$

and now instead of probabilities we get probabilty density

$$p(m(x)) = \frac{e^{-\beta F[m(x)]}}{Z}.$$

Now, we have a number of conditions that we require this effective free energy to satisfy:

- locality: basically, it takes the form

  $$F[m(x)] = \int d^d x f[m(x)]$$

  where f depends on $m, \nabla m, \nabla^2 m, \ldots$
- $\mathbb{Z}_2$ symmetry when $B = 0$ under $m \mapsto -m$
- Analyticity - we assume it has a Taylor expansion
- we are only interested in m that are slowly varying in space

Now, focusing on the $B = 0$ case we see that the effective free energy must take the form

$$F[m(x)] = \int dx (\frac{1}{2}\alpha_2(T)m^2 + \frac{1}{4}\alpha_4(T) + \frac{1}{2}\gamma(T)(\nabla m)^2)$$

where the odd terms are eliminated due to symmetry (of course, when $B \neq 0$ we can add in these odd terms again).

Unfortunately it is hard to find an general way to calculate $\alpha_2, \alpha_4, \gamma$. but we can say that $\alpha_4, \gamma > 0$ and α_2 changes sign at T_c/ In particular, in MFT,

$$\alpha_2 \sim (T - T_c), \alpha_4 \sim \frac{1}{3}T.$$

Just as in plain Landau theory, we use the saddle point approximation that $Z = \int dm(x) e^{-\beta F(m(x))}$ is focused on the saddle point of F, but this time to

find it we need to use the Euler-Lagrange approach, yielding the equation for the equilibrium:

$$\gamma \nabla^2 m = \alpha_2 m + \alpha_4 m^3$$

How well does this theory work? We can check that if we assume m is constant, then we get back to Landau theory, so we know that at the very least, this theory cannot be worse than Landau theory. As our first development we then look at the notion of domain walls, and see whether or not those can be of any help. In particular, we want to see what a solution to the following looks like in the one dimensional case:

$$\partial_x^2 m = \alpha_2 m + \alpha_4 m$$

where $m = \pm m_0$ as $x \to \pm\infty$. The solution works out to $m = m_0 \tanh\left(\frac{x-x_0}{W}\right)$ where the width, $W = \sqrt{-\gamma/\alpha_2}$.

This models how the system changes state half-way, and since these don't occur in ordinary Landau theory. As such, to compare the two it would be helpful to see what the free energy cost of having a domain wall is. We may calculate this as

$$\begin{aligned} F_W &\sim L^{d-1} \int dx \gamma (\partial_x)^2 \\ &\sim L^{d-1} \gamma m_0^2 / W \\ &= L^{d-1} \sqrt{-\gamma \alpha_2^3 / \alpha_4^2} \end{aligned}$$

where L is the "length" of the system. Now curiously, this allows us to see why Landau fails in 1 dimension, but not (as much) in 2 dimensions.

Consider a 1 dimensional space length L, starting in state m_0 and ending in some other state, for $T < T_c$, meaning that $\alpha_2 < 0$. We then see that

$$\mathbb{P}(\text{wall at } x = x_0) = \frac{e^{-\beta F_W}}{Z}$$

$$\mathbb{P}(\text{wall somewhere}) \sim \frac{L}{W} \frac{e^{-\beta F_W}}{Z}$$

meaning that

$$\mathbb{P}(n \text{ walls}) = e^{-n\beta F_W}/Z \int_0^L dx_1 \int_{x_1}^L dx_2 \cdots \int_{x_{n-1}}^L dx_n = \frac{1}{Zn!} \left(\frac{L e^{-\beta F_W}}{W} \right)^n.$$

Consequently we can calculate the probabilities of how the final state of the rod ends up:

$$\mathbb{P}(m \to m_0) = \frac{1}{Z} \cosh \left(\frac{L e^{-\beta F_W}}{W} \right)$$

$$\mathbb{P}(m \to -m_0) = \frac{1}{Z} \sinh\left(\frac{Le^{-\beta F_W}}{W}\right)$$

Now as $L \to \infty$, we see that these probabilities stay about the same, meaning that the state of the rad stays more or less random. Why does this argument fail in two dimensions? Well, the calculation does change a bit since we're in two dimensions, but we also have the fact that F_W may depend on L in this scenario, meaning that in effect, the probabilities above instead converge to 0 and 1, meaning that a phase transition does indeed occur. [End of lecture 4]

1.4.1 Path Integrals

So how do we evaluate the path integrals we deal with

$$Z = \int \mathcal{D}m(x) e^{-\beta F[m(x)]}$$

For $B =$ we get the general form

$$F[m(x)] = \int d^d x (\frac{1}{2}\gamma(\nabla\phi)^2 + \frac{1}{2}\alpha_2\phi^2 + \alpha_4\phi^4 + \dots)$$

the path integral can be done when F is at most quadratic in ϕ. Small higher order terms are hard, but still feasible, and that's more or less it. Importantly, for $\alpha_2 = \mu^2$ when μ is small, this becomes hard to evaluate since higher order terms do matter. This case will be treated separately.

Now, to perturb we write $\tilde{\phi} = \phi - \langle\phi\rangle$ and then

$$F(\tilde{\phi}) = F(m_0) + \frac{1}{2}\int d^d x \gamma(\nabla\tilde{\phi})^2 + \alpha_2\tilde{\phi}^2$$

where $\langle\phi\rangle = \pm m_0$ for $T < T_C$. To start evaluating things, we use the Fourier space, so $\phi_k = \int d^d x e^{-ik\cdot x}\phi(x)$ for mommentum k. Since we're on a lattice, we take $\phi_k = 0$ for $k > \lambda = \pi/a$ the UV cut-off. Writing $\phi = \int \frac{d^d k}{(2\pi)^d} e^{ik\cdot x}\phi_k$, we see that

$$\begin{aligned}
F(\phi_k) &= \frac{1}{2}\int \frac{d^d k_1}{(2\pi)^d}\frac{d^d k_2}{(2\pi)^d}\int d^d x(-\gamma k_1\cdot k_2 + \mu^2)\phi_{k_1}\phi_{k_2}e^{i(k_1+k_2)\cdot x} \\
&= \frac{1}{2}\int \frac{d^d k}{(2\pi)^d}(-\gamma(-k)\cdot k + \mu^2)\phi_k\phi_{-k} \\
&= \frac{1}{2}\int \frac{d^d k}{(2\pi)^d}(\gamma k^2 + \mu^2)|\phi_k|^2 \\
&= \frac{1}{2V}\sum_k (\gamma k^2 + \mu^2)|\phi_k|^2
\end{aligned}$$

Now, we can calculate that the measure in our path integral is

$$\int \mathcal{D}\phi(x) = \prod_k \left(N \int d\phi_k d\phi_k^* \right)$$

where N is some normalisation constant which we will ignore from hereon since it does not matter for the calculations we do. Consequently we can claulate

$$\begin{aligned} Z &= \prod_k \int d\phi_k d\phi_k^* e^{-\frac{\beta}{2V}\sum_k (\gamma k^2+\mu^2)|\phi_k|^2} \\ &= \prod_k \int d\phi_k d\phi_k^* e^{-\frac{\beta}{2V}(\gamma k^2+\mu^2)|\phi_k|^2} \end{aligned}$$

where the last result is just a Gaussian leaving us with

$$Z = \prod_k \sqrt{\frac{2\pi VT}{\gamma k^2+\mu^2}}$$

Relating this to F_{therm} via $Z = e^{-\beta F_{therm}}$ we find that

$$F_t herm/V = -\frac{T}{2V}\sum_k \ln\left(\frac{2\pi VT}{\gamma k^2+\mu^2}\right) = \frac{-T}{2}\int \frac{d^d k}{(2\pi)^d} \ln\left(\frac{2\pi VT}{\gamma k^2+\mu^2}\right)$$

which might look like it diverges. This is the one place where the normalisation constant N does matter, and does fix the issue. Regardless, we continue to ignore it.

Example 1. Compute $c = C/V$.

$$\begin{aligned} C &= \frac{\beta^2}{V}\partial_\beta^2 \ln(Z) \\ &= \frac{1}{2}(T^2\partial_T^2 + 2T\partial_T)\int \frac{d^d k}{(2\pi)^d} \ln\left(\frac{2\pi VT}{\gamma k^2+\mu^2}\right) \\ &= \frac{1}{2}\int \frac{d^d k}{(2\pi)^d}(1 - \frac{2T}{\gamma k^2+\mu^2} + \frac{T^2}{(\gamma k^2+\mu^2)^2}) \end{aligned}$$

Here we see that this depends on d quite strongly and the convergence as $\Lambda \to \infty$ and we also only care about k^2 not $\hat{k}$. For the two non-constant terms we see that

$$\int_0^\Lambda dk \frac{k^{d-1}}{\gamma k^2+\mu^2} \sim \begin{cases} \Lambda^{d-2} + \dots & d > 2 \\ \ln(\Lambda) + \dots & d = 2 \\ \mu^{-1} + \dots & d = 1 \end{cases}$$

$$\int_0^\Lambda dk \frac{d^d k}{\gamma k^2+\mu^2} \sim \begin{cases} \Lambda^{d-4} + \dots & d > 4 \\ \ln(\Lambda) + \dots & d = 4 \\ \mu^{d-4} + \dots & d < 4 \end{cases}$$

Consequently, we see that for $d \geq 4$, c is independent of T whereas for $d < 4$, $c \sim \mu^{d-4} \sim |T - T_c|^{-\alpha} \implies \alpha = 2 - d/2$. Do remember that we cannot use this calculation near the critical point, since the ϕ^4 term matters there. [End of lecture 2.1]

In general, this approach can help us refine our critical exponent calcuations. Firstly, note that

$$\langle \phi(x) \rangle = \begin{cases} 0 & T > T_c \\ \pm m & T < T_c \end{cases}$$

Here we define the **correlation** to be

$$\langle \phi(x)\phi(y) \rangle >$$

and the **connected correlation** to be

$$\langle \phi(x)\phi(y) \rangle_c = \langle \phi(x)\phi(y) \rangle - \langle \phi \rangle^2$$

We will see that we are often less interested in the correlation itself, but more in the length scales on which it is significant. Now to see how we might examine correlatoins, we consider free energy

$$F = \int d^d x \langle \frac{1}{2}\gamma(\nabla\phi)^2 + \frac{1}{2}\mu^2\phi^2 - B(x)\phi(x) \rangle$$

with $Z(B(x)) = \int \mathcal{D}\phi e^{-\beta F}$ as usual. As usual, the partition function contains a lot of information

Example 2.

$$\frac{1}{\beta}\frac{\delta \ln(Z)}{\delta B(x)} = \frac{1}{\beta^2}\frac{\delta Z}{\delta B(x)} = -\langle \phi \rangle_B$$

which is minus the expected value of ϕ for $B = B(x)$.

Example 3. Similarly,

$$\begin{aligned} \frac{1}{\beta^2}\frac{\delta^2 \ln(Z)}{\delta B(x)\delta B(y)} &= \frac{1}{\beta^2 Z}\frac{\delta^2 Z}{\delta B(x)\delta B(y)} - \frac{1}{\beta^2 Z^2}\frac{\delta Z}{\delta B(x)}\frac{\delta Z}{\delta B(y)} \\ &= \langle \phi(x)\phi(y) \rangle_B - \langle \phi(x) \rangle_B \langle \phi(y) \rangle_B \end{aligned}$$

leaving

$$\langle \phi(x)\phi(y) \rangle_c = \frac{1}{\beta^2}\frac{\delta^2 \ln(Z)}{\delta B(x)\delta B(y)}|_{B=0}$$

All we need to do to make use of these nice properties is actually calculating $Z(B(x))$. We use the same approach as we used earlier for $B = 0$, which is to use the Fourier space, but now also with Fourier modes B_k of B.

$$F(\phi_k) = \int \frac{d^d k}{(2\pi)^d} \left(\frac{1}{2}(\gamma k^2 + \mu^2)\phi_k \phi_{-k} - B_{-k}\phi_k \right)$$

using Gaussian integrals with $\hat{\phi}_k = \phi_k - \frac{Bk}{\gamma k^2+\mu^2}$ we get

$$F(\hat{phi}_k) = \int \frac{d^d k}{(2\pi)^d} \left(\frac{1}{2}(\gamma k^2 + \mu^2)|\hat{\phi}_k|^2 - \frac{1}{2}\frac{|B_k|^2}{\gamma k^2 + \mu^2} \right)$$

meaning that

$$\begin{aligned} Z(B(x)) &= \prod_k \int d\hat{\phi}_k d\hat{\phi}_k^* e^{-\beta F(\phi)} \\ &= e^{-\beta F_{thermo}} e^{\frac{\beta}{2} \int \frac{d^d k}{(2\pi)^d} \frac{|B_k|^2}{\gamma k^2+\mu^2}} \\ &= e^{-\beta F_{thermo}} e^{\frac{\beta}{2} \int d^d x d^d y B(x) G(x-y) B(y)} \end{aligned}$$

(we split off the parts depending on B) where

$$G(x) = \int \frac{d^d k}{(2\pi)^d} \frac{e^{-ik\cdot(x-y)}}{\gamma k^2 + \mu^2}$$

We then see, from our earlier example, that

$$\langle \phi(x)\phi(y) \rangle_c = \frac{1}{\beta} G(x-y)$$

We then define the **correlation length** to be $\xi^2 = \gamma/\mu^2$, which from our expression for the connected correlation we can see that this is strongly related the length scale the meaningful correlation occurs on. In particular, we see that

$$G(x) = G(|x|) = \frac{1}{\gamma} \int \frac{d^d x}{(2\pi)^d} \frac{e^{-ik\cdot x}}{k^2 + 1/\xi^2}$$

A such (see example sheet 2) we see that

$$G(r) \sim \begin{cases} \frac{1}{r^{d-2}} & r << \xi \\ \frac{e^{-r/\xi}}{r^{(d-1)/2}} & r >> \xi \end{cases}$$

A such, we see that meaningful fluctation sizes occur for lengths $r < \xi$ as we'd hoped. In particular, we see that near the critical point ($\mu^2 \sim |T - T_c \implies \xi \to \infty$ as $T \to T_c$) meaningful fluctuations occur on all length scales. Let's try to understand what happens here. In particular, let's define two **new critical exponents** (yay!)

$$\xi \sim |T - T_c|^{-\nu}, \nu = 1/2$$

$$\langle \phi(x)\phi(y) \rangle \sim \frac{1}{r^{d-2+\eta}}, \eta = 0$$

These values turn out to be correct for $d \geq 4$, but wrong for $d < 4$, leaving another mystery. Nevertheless, the upper critical dimension seems to fit with

our pattern. To think a bit more about the upper critical dimension notice that MFT is not obviously wrong if $\langle\phi^2\rangle < \langle\phi\rangle^2$. As such we find it useful to define

$$R = \frac{\int_0^\xi d^d x \langle\phi(x)\phi(0)\rangle}{\int_0^\xi d^d x m_0^2}$$

where $\langle\phi\rangle = \pm m_0$ for $T < T_c$. Here R is roughly indicative of the size of the fluctuations we see. In particular, we notice that

$$R \sim \frac{1}{m_0^2 \xi^d} \int_0^\xi dr \frac{r^{d-1}}{r^{d-2}} \sim \frac{\xi^{2-d}}{m_0^2}$$

so from our critical exponents

$$R \sim |T - T_c|^{(d-4)/2}$$

In particular, we see that $d < 4 \implies R \to \infty$ as $T \to T_c$ so MFT is certainly wrong.

So we can see information about the upper critical dimension using this approach, but let's see if we can get a bit of a better picture on the fluctuations as well. [End of lecture 2.2]

2 Renormalisation

Here we use the more famous technique of renormalisation. First note that we treat renormalisation in general, meaning that now

$$F(\phi) = \int d^d x \left(\frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}\mu^2\phi^2 + g\phi^4 + \dots \right)$$

where all we require is that

- F is analytic about $\phi = 0$
- F is even, so the same for $\phi \mapsto -\phi$
- We conventially set the coefficient in front of $(\nabla\phi)^2$ to be $\frac{1}{2}$ which is really the same as a rescaling up to units so has no effect.

We thus get an infinite list of constants g_n to describe our theory, and in particular, when we renormalise we study how we "move between theories". We in a sense, establish a way to travel between theories and a way to check if two theories are close enough. This is particularly insightful, as it allows us to see why universality exists, which naturally arises from many theories being close to each other in some sense, and we can also clearly see when it breaks down. As such, we refer to the coefficients g_n as corresponding to a point in **theory space**.

So how do we establish the "closeness" of theories? Probably you could build some kind of metric and that certainly would be interesting on its own.

However, now instead of using a metric, we look at the convergence of theories. In particular, we notice that since we always approximate theories when we study them, we really only care about theories "up to approximation" or we only care about them up to the point that they approximate the same thing. In particular, rescaling shouldn't change the nature of the theory, but it may change the constants, and seeing which theories converge to the same type when we rescale is how we determine which theories are close to one another. That is what renormalisation is. The rescaling of theories, and analysis of their convergence under this rescaling.

When we rescale, we need to remember that we are almost always using a UV cutoff $\Lambda \sim 1/a$ for spacing a allowing us to ignore ϕ_k for $k >> 1/L$ when we are working on length scale L. As such, when we rescale by $\zeta > 1$ we also need to rescale the cutoff to $\Lambda' = \Lambda/\zeta$. We then also split

$$\phi_k = \phi_k^- + \phi_k^+$$

where

$$\phi_k^- = \begin{cases} \phi_k & |k| < \Lambda' \\ 0 & |k| > \Lambda' \end{cases}$$

and similar for ϕ_k^+ (so we want to ignore ϕ_k^+). We then write

$$F(\phi_k) = F_0(\phi_k^-) + F_0(\phi_k^+) + F_I(\phi_k^-, \phi_k^+)$$

where F_I is the interaction term. We then write

$$\begin{aligned} Z &= \int \prod_{k<\Lambda} d\phi_k e^{-F(\phi_k)} \\ &= \int \prod_{k<\Lambda'} d\phi_k^- e^{-F_0(\phi_k^-)} \int \prod_{\Lambda'<k<\Lambda} d\phi_k^+ e^{-F_0(\phi_k^+)} e^{-F_I(\phi_k^-, \phi_k^+)} \\ &= \int \prod_{k<\Lambda'} d\phi_k^- e^{-F'(\phi_k^-)} \end{aligned}$$

for **Wilsonian effective free energy** F' (we need to reformat the theory into one with a UV cutoff after all). The Wilsonian effective free energy takes the form, again,

$$F'(\phi) = \int d^d x \left(\frac{1}{2}\gamma'(\nabla\phi)^2 + \frac{1}{2}\mu'^2\phi^2 + g'\phi^4 + \dots \right)$$

[so I guess renormalisation is more of a rescaling of the cutoff, and examining how the theory varies under different cutoffs - makes sense]. Continuing to reformat things, we set $k' = \zeta k, x' = x/\zeta$ (want to preserve $k \cdot x$) and rescale the field as $\phi_k' = \sqrt{\gamma'}\phi_j^-$ to get

$$F'(\phi') = \int d^d x' \left(\frac{1}{2} (\nabla' \phi')^2 + \frac{1}{2} \mu^2(\zeta) \phi^2 + g(\zeta) \phi^4 \right)$$

In short then, renormalisation corresponds today

1. Integrate out modes $\Lambda/\zeta < k < \Lambda$ for $\zeta > 1$
2. rescale $k' = \zeta k$
3. rescale fields to get $1/2$ gradient coefficient [End of lecture 3.0]

So what kind of behaviour can we see when we renormalise a theory? Basically, we can diverge to ∞, approach a fixed point, we can end up in a fixed cycle or we can wander around aimlessly. Of these, the last two are "uncommon," so we will focus on the first two. In particular, fixed points of ξ are of note, since if $\xi' = \xi/\zeta$ then either $\xi = 0$ or $\xi = \infty$. In the first case, then for T small we get an ordered phase, and T big we get a disordered phase. This is not so interesting. The interesting case here is $\xi = \infty$ so we get global behaviour, which is interesting (see examples at here).

For flows near a fixed point we use the following terms:

- a flow is **irrelevant** if it flows back to the fixed point.
- a flow is **relevant** if it flows away from the fixed point.
- a flow is **marginal** if it moves somewhere else within an overall "fixed hyperplane"

Usually you get an infinite number of irrelevant directions and a finite number of relevant directions around a fixed point. This is what leads to the universality that we generally observe. This universality makes the low energy regime easier to understand, but makes it harder to extrapolate to high energy physics, since many high energy theories lead to the same low energy physics. [End of lecture 3.1]

2.1 Understanding Critical Exponents

We can use renormalisation to improve our understanding of critical exponents. To start with we see that at a fixed point there is no sense of scale (?) so we can write

$$\langle \phi(x) \phi(0) \rangle = \frac{1}{r^{d-2+\eta}}$$

(any other functional would need a sense of scale on dimensional grounds). Dimensional analysis tells us that $[x] = 1 \implies [\partial_x] = 1$ so in particular, $[F] = 0, [d^d x] = -d, [\nabla] = 1$, so $[\phi] = \frac{d-2}{2}$, so $\eta = 0$ as before. This disagrees with experiment despite our analysis being as simple as dimensional analysis.

This is because there is a "hidden scale" that comes from the energy cut-off a and so instead we get

$$\langle \phi(x)\phi(y)\rangle \sim \frac{a^\eta}{r^{d-2+\eta}}$$

which gives us a slightly modified dimensional analysis. In particular, if we rescale $x \mapsto x/\zeta$, so $\phi(x) \mapsto \phi'(x') = \zeta^{\Delta_\phi}\phi(x)$ with scaling dimension Δ_ϕ leaving

$$\langle \phi(x)\phi(y)\rangle \sim \frac{1}{r^{d-2+\eta}}$$

giving

$$\Delta_\phi = \frac{d-2+\eta}{2}$$

with so-called **anomolous dimension** $\eta/2$. Consequently, if $t = |T - T_c|/T_c$ then

$$\xi \sim t^{-\nu}$$

has $\xi \sim 1/\mu \sim 1/t^{1/2}$ which does not agree with experiment, but if $[\xi] = -1$ then $t \mapsto \zeta t^{\Delta_t} t$ where $\Delta_t = 1/\nu$.

Now, if ξ is the only length scale in our theory, then all other critical exponents follow from η, ν. We work this out now.

1. Note that $F_{thermo} = \int d^d x f(t) \implies f \sim t^{d\nu}$. Also, $c = \partial_t^2 f \sim t^{d\nu-2} \sim t^{-\alpha} \implies \alpha = 2 - d\nu$. This is known as the **hyperscaling relation**.
2. $m \sim \phi \sim t^\beta$ and $\Delta_\phi = \beta\Delta_t \implies \beta = \nu\Delta_\phi = \nu(d-2+\eta)/2$
3. $F \sim \int d^d x B\phi \implies \Delta_\phi + \Delta_B = d \implies \Delta_B = (d+2-\eta)/2 \implies \gamma = \nu(2+\eta)$. This is known as **Fisher's identity**.
4. We also see that $\delta = \Delta_B/\Delta_\phi = (d+2-\eta)/(d-2+\eta)$.

This works well for the $d = 2, 3$ Ising model and for MFT with $d = 4$. However, it fails for MFT $d > 4$ for β, δ because the $g\phi^4$ term means we get $m \sim \sqrt{t/g}$.

Finally, we note that we can characterise interactions by how they scale with $F(\phi) \sim \int d^d x g_O O(x)$ for some interaction order $O(x)$ where $O(x) \mapsto \zeta^{\Delta_O} O(x)$ and $\Delta_{g_O} = d - \Delta_O \implies g_O \mapsto \zeta^{d-\Delta_O} g_O$. Note however that not all interactions take this form. Just as with renormalisation flows, we can classify these interactions as **relevant** for $\Delta_O < d$, **irrelevant** for $\Delta_O > d$ and marginal for $\Delta_O = d$. [End of lecture 3.2]

2.2 Gaussian Case

There is one case where it is relatively easy to work out renormalisation explicitly, which is the so-called **Gaussian case** when

$$F_0(\phi) = \int d^d x \left(\frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}\mu_0^2\phi^2 \right) = \int^\Delta \frac{d^d x}{(2\pi)^d} \frac{1}{2}(k^2 + \mu_0^2)\phi_k\phi_{-k}$$

This case is special because here there is no interaction term:

$$F_0(\phi) == F_0(\phi^-) + F_0(\phi^+)$$

so

$$e^{-F_0'(\phi')} = \left(\int \mathcal{D}\phi^+ e^{-F_0(\phi^+)} \right) e^{-F_0(\phi^-)} = N e^{-F_0(\phi^-)}$$

So setting $k' = \zeta k, \phi_k = \zeta^{-w}\phi_k$ where w is undetermined leaves us with

$$F_0(\phi^-) = \int^{\Delta/\zeta} \frac{d^d k}{(2\pi)^k} \frac{1}{2}(k^2+\mu_0^2)\phi_k^-\phi_{-k}^- = \int^\Delta \frac{d^d k'}{(2\pi)^d} \frac{1}{2\zeta^d} \left(\frac{k'^2}{\zeta^2} + \mu_0^2 \right) \zeta^{2w}\phi_{k'}'\phi_{-k'}'$$

so picking $w = (d+2)/2$ gives us the right $1/2$ coefficient. Relabelling k' as k we get our desired result:

$$F_0'(\phi') = \int^\Delta \frac{d^d k}{(2\pi)^d} \frac{1}{2}(k^2 + \mu^2(\zeta))\phi_k'\phi_{-k}'$$

where $\mu^2(\zeta) = \zeta^2\mu_0^2$ so as mentioned before, for $\xi = 1/\mu$, $\partial_\xi\mu^2 = 0 \implies \mu^2 = \infty$ at $T = \infty$ (so ignore this case) or $\mu^2 = 0$ (which is the Gaussian fixed point).

As $\mu^2\phi^2$ gets bigger this becomes the relevant operator. On other hand, $\alpha_2(\nabla^2\phi)^2$ scales as $\alpha_2(\zeta) = \zeta^{-2}\alpha_0$ so this term is irrelevant. In the more general terms

$$F(\phi) = \int d^d x \left(\frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}\mu_0^2\phi^2 + \sum_4^\infty g_{0n}\phi^n \right)$$

where the ns are all even, we get non-trivial interactions terms leading today

$$g_{0n} \mapsto g_n' + g_{0n} + \delta g_n$$

where δg_n arises from the interactions. $\phi'(x') = \zeta^{\Delta_\phi}\phi(x)$ then means that

$$F(\phi') = \int d^d x' \zeta^d \left(\frac{1}{2}\zeta^{-2-2\Delta_\phi}(\nabla'\phi')^2 + \frac{1}{2}\mu_0^2\zeta^{-2\Delta_\phi}\phi'^2 + \sum_4^\infty g_{0n}\zeta^{-n\Delta_\phi}\phi'^n \right)$$

wher $\Delta_\phi = (d-2)/2$ again. The same approach with dimensional results allows us to return to the original form of the free energy with new coefficients

$$g_n(\zeta) = \zeta^{d-n\Delta_\phi} g_{0n} = \zeta^{n+d(1-n/2)} g_{0n}$$

Example 4. For the quartic term we find $g_4(\zeta) = \zeta^{4-d} g_{0n}$ which becomes irrelevant for $d > 4$, but relevant for $d < 4$ and marginal for $d = 4$. This agrees nicely with what we've found before. For $d > 4$, the quartic term "auto-tunes" whereas for lower dimension, an experimentalist would have to forcibly control it, to prevent it from "veering off" whereas for lower dimension, an experimentalist would have to forcibly control it, to prevent it from "veering off." The problem we find is that for $n > 4$ and n even, all g_n are always relevant (except perhaps $d = 1$ for g_6?).

Now, as a matter of language, we now will usually use the term MFT or **mean field theory** to refer to the Gaussian case specifically, even though things have changed quite a bit.

Just as a final warning, we'd like to point out the concept of *dangerous irrelevant terms*. A specific example is the quartic term in the critical exponents for MFT with $d > 4$. In this case, for β, δ our more basic analysis fails, but once we include g_4 we find that $m_0 \sim (t/g_4)^\beta$ for $\beta = 1/2$, Here, $\Delta_{g_4} = 4 - d$ gives us $\Delta_\phi = \beta(\Delta_t - \Delta_{g_4}) = \beta(d-2)$ and now $\beta = 1/2$ is fully compatible with scaling (so MFT and the scaling approach agree).

All this is just to point out that sometimes, irrelevant terms can change critical exponeents even though they are "irrelevant" in another sense. [End of lecture 3.3] [It might be good to remember the reltaions between the 4 thermodynamic energies, etc. (enthalpy, free energy, etc.)]