# Statistical Field Theory

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October 9 2020 - October 17, 2020

## Introduction

These notes are based on the course lectured by Dr Christopher E Thomas in Michaelmas 2020. Due to the measures taken in the UK to limit the spread of Covid-19, these lectures were delivered online. These are not meant to be an accurate representation of what was lectures. They solely represent a mix of what I thought was the most important part of the course, mixed in with many (many) personal remarks, comments and digressions... Of course, any corrections/comments are appreciated.

Statistical Field Theory is an extension of statistical physics. It assumes one is familiar with statistical physics, and in particular, focuses on the study on phase transitions. This course in particular follows David Tong's notes quite closely.

## 1 From Spin to Fields

So why do we consider fields? Here we look at a model that shows the origin of this connection.

### 1.1 The Ising model

The Ising model studies a lattice where each point on the lattice is assigned a spin $S_i = \pm$. Consequently, the energy of the lattice is

$$E = -B\sum_i S_i - J\sum_{<i,j>} S_i S_j$$

where $B$ is the external magnetic field strength, $J$ is the strength of neighbour-neighbour interactions and $< i, j >$ is any nearest neighbour pair. When $J > 0$, states tend to align (ferromagnetic behaviour) whereas if $J < 0$ states will prefer not to align (anti-ferromagnetic behaviour). We will focus on the $J > 0$ case. But of course, due to heat there some statistical randomness, and we want to include that. As such, we consider the canonical ensemble with $\mathbb{P}(S_i) = e^{-\beta E(S_i)}/Z$ where $Z$ is the partition function, $\beta = 1/T$, and the Boltzmann constant $k_B = 1$. As ever in statistical physics, we can derive everything

from the partition function. Particularly important in our case is the Free Energy, $F = \langle E \rangle - TS = -T\ln(Z)$, and $dF = -SdT - pdV - MdB$ (so we can find S, p and M from F as well).

In statistical physics, we are particularly interested in the equilibrium state (which occurs at the minimum of the free energy when temperature is constant), and since we are looking at a magnetic system, we are particularly interested in the equilibrium magnetisation. This can be calculated as

$$m = \frac{1}{N}\sum_i \langle S_i \rangle = \frac{1}{N\beta}\partial_B \ln(Z)$$

Now all that remains is to calculate $Z$ to find $m$. Unfortunately, in dimension 3 or greater this is impossible (how impossible?), and it is still hard in lower dimension. Consequently we take a different approach using the so-called "effective free energy", which is defined such that

$$\sum_m \sum_{\{S_i\}|m} e^{-\beta E[S_i]} = \sum_m e^{-\beta F(m)}$$

where $F(m)$ is the effective free energy. Since we can assume $N$ is large (around $10^{23}$), we can then write18

$$Z = N/2 \int_{-1}^{1} dm\, e^{-\beta F(m)} = N/2 \int_{-1}^{1} dm\, e^{-\beta N f(m)}$$

where $f(m) = F(m)/N$. From here, we can calculate the equilibrium field by considering that since $N$ is large, the value contributing the most to the integral is where $\partial_m f = 0$, and this is how the equilibrium $m$ is calculated. This approach is called the **steepest descent approximation**, and we find here that $F_{\text{thermodynamic}} \approx F(m_{\min})$.

This is all very well and nice, but the issue is that we still don't know how to calculate $F(m)$, and it turns out that this is about as hard as calculating $Z$. As such, we use the **mean field approximation**

$$E \approx -B\sum_i m - J \sum_{<i,j>} m^2 = -BNm - \frac{1}{2}NJqm^2$$

where $q = 2\text{dim(space)}$ for a cubic latice in a space dimension dim. Now,

$$e^{-\beta N f(m)} = \sum e^{-\beta E(S_i) \approx \Omega(m) e^{-\beta E(m)}}$$

so in order to find $f(m)$ we need to know $\Omega(m)$ which is the number of ways for a given energy state to occur, but since $m$ depends only on the number of positive and negative spins states, $N_\uparrow, N_\downarrow$, and in particular $N = N_\uparrow + N_\downarrow$ we see the number of total states is

$$\ln(\Omega) = \ln\binom{N}{N_\uparrow} \approx N(\ln(2) - \frac{1}{2}(1+m)\ln(1+m) - \frac{1}{2}(1-m)\ln(1-m))$$

by Stirling's approximation. Consequently we find that

$$f(m) \approx -Bm - \frac{1}{2}Jqm^2 - \frac{1}{\beta}(\ln(2) - \frac{1}{2}(1+m)\ln(1+m) - \frac{1}{2}(1-m)\ln(1-m)).$$

From here by taking $\partial_m f = 0$, we find

$$\beta(B + Jqm) = \frac{1}{2}\ln\left(\frac{1+m}{1-m}\right)$$

or equivalently

$$m = \tanh(\beta(B + Jqm)).$$

From here, we can solve implicitly, but there is another approach we can take, which is the way we will go about this. Just as a side note, we can see the reason this is called the mean field approximation here as well: it is as if $m$ just shifts to external field to $B_{\text{eff}} = B + Jqm$. [End of lecture 1.]

## 1.2 Landau Theory of Phase Transitions

The remarkable part about the Ising model described above is not its correctness. In fact, it is often incorrect, but rather its universality. It can be applied in a wide variety of situations. As such, Landau tried to develop a more general theory of phase transitions. Here we work through an extended example to examine the general features of these. In particular, we study the behaviour of the equilibrium $m_{\min}$ when various quantities are varied.

From before, we can approximate $f$ for small $m$ as

$$f(m) \approx \cancel{-\frac{1}{\beta}\ln(2)} - Bm + \frac{1}{2}(\frac{1}{\beta} - Jq)m^2 + \frac{1}{12\beta}m^4 + \dots$$

where we ignore the constant term since it does not affect anything. We start with the $B = 0$ case where we find that

$$f(m) = \frac{1}{2}(T - T_c)m^2 + \frac{1}{12}T^4m^4$$

where we define the **critical temperature** to be $Tc = Jq$. Then we get the following scenarios.

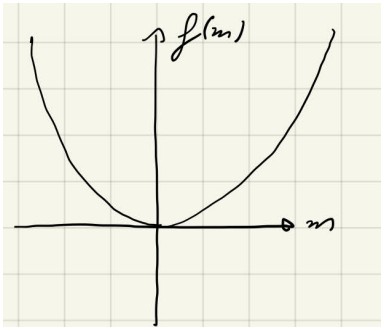


Figure 1: $T > T_c$

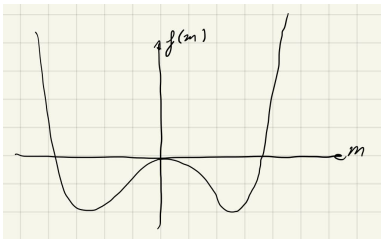


Figure 2: $T < T_c$

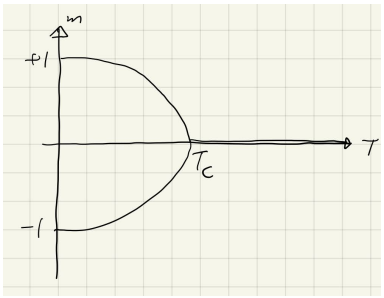


Figure 3: $m_{\min}(T)$ for $B = 0$

The middle graph is meant to be symmetric, and we see here that $m_{\min}$ can take two different values

$$m_{\min} = \pm m_0 = \pm\sqrt{\frac{3(T_c - T}{T}}$$

Now some definitions apply here. This is a **second order** or **continuous** phase transition since $m$ is continuous in the quantity being varied. $m = 0$ is called the **disordered phase**, and $m \neq 0$ is called an **ordered phase**.

**Spontaneous symmetry breaking** is what we call the loss of symmetry in $m$ when $T$ decreases below $T_0$ (since $m$ is forced to choose a positive or negative value). Finally, these terms all extend to arbitrary **order parameter** $m$. We finally compute

$$f(m_{\min}) = \begin{cases} 0 & T > T_c \\ -\frac{3}{4}\frac{(T_c - T)^2}{T} & T < T_c \end{cases}$$

We can also look at the heat capacity here, and how that various with temperature. Heat capacity may be defined as

$$C = \partial_T \langle E \rangle = \beta^2 \partial_\beta^2 \ln(Z)$$

since recall that $\langle E \rangle = -\partial_\beta \ln(Z)$. Using $Z \approx -\beta N f(m_{\min})$ we find $c = C/N$ has

$$c = \begin{cases} 0 & T \to T_c^+ \\ 3/2 & T \to T_c^- \end{cases}$$

so $c$ is a first order phase transition - ie. discontinuous.

The above scenario considered $B \neq 0$, but when $B = 0$ we get a different situation. In particular, we do not get spontaneous symmetry breaking since the system is no longer symmetric. In this case, the above graphs are skewed to the right or the left unevenly leaving global minimum (the "true" minimum) and a so-called **metastable** state.

Finally, we can observe another discontinuous phase transition when we study $m_{\min}$ as a function of $B$ at low $T$ ($T < T_c$). Since we're below the critical temperature, the global minimum does not smoothly slide through 0, but instead abruptly shifts from negative to postive (or the other way around) giving a 1st order phase transition. Incidentally, I find this a bit odd, since this really only makes sense in the 1D case, but as we will see later Mean Field Theory (MFT) does not work in 1D. But in higher dimensions, $m$ is a vector and instead of getting two local minima as above, we get a tilted (for $B \neq 0$) circular valley where $f(m)$ is low. In this case, no phase transition occurs, since the minimum just smoothly moves around this circular valley as $B$ is varied...

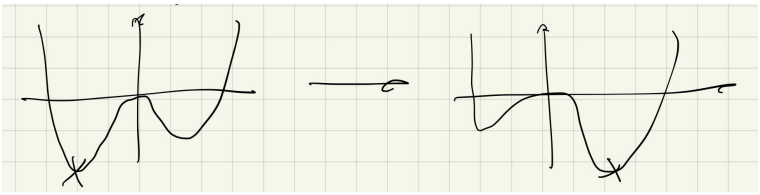

Figure 4: $m_{\min}$ vs $B$

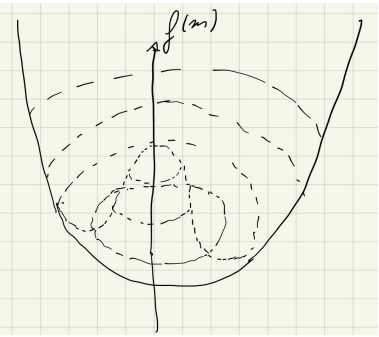


Figure 5: $f(m)$ vs $m$ in 2D

Anyways, some other asymptotic behaviour includes that when $T \approx T_c$, we get

$$f(m) \approx -Bm + \frac{1}{12} T_c m^4$$

leaving

$$m \sim B^{1/3}$$

If we define magnetic susceptibility (the **response function** in this context) to be $\chi = \partial_B m|_T$ we see that for $T > T_c$ we have

$$f(m) \approx -Bm + \frac{1}{2}(T - T_c)m^2 + \dots$$

so

$$m \approx \frac{B}{T - T_c}$$

so

$$\chi \sim \frac{1}{T - T_c}$$

and for $T < T_c$

$$\chi \sim \frac{1}{|T - T_c|}$$

## 1.3 The Validity of MFT

The purpose of all these calculations at the end of the last section is to find some important set of numbers that we can use to check our theory with in the real world. In particular, what we looked for were the **critical exponents**, which are

$$
\begin{aligned}
m &\sim (T_c - T)^{\beta} && \beta = 1/2 && T < T_c \\
c &\sim c_{\pm}|T - T_c|^{-\alpha} && \alpha = 0 \\
\chi &\sim |T - T_c|^{-\gamma} && \gamma = 1 \\
m &\sim B^{1/\delta} && \delta = 3
\end{aligned}
$$

in our worked example. Now for results, we find that in general, MFT fails entirely below a certain **lower critical dimension**, $d_i$ and works correctly above a certain **upper critical dimension**, $d_u$. In between it is structurally accurate (can detect the right phase transitions) but inaccurate (finds the wrong critical exponents). The Ising fails for dimension 1, gets the right structure for dimensions 2 and 3, and works for dimension 4 and above. It is, as expected, closer to the correct value for dimension 3 than for 2, but still can be quite off.

Nevertheless, as mentioned earlier, the power of the MFT is not in its correctness, but in its universality. Also, often structure is more important. After all, if you can predict a phenomenon will happen, you can then carry out an experiment to measure it more accurately. On the other hand, it is far harder to use experiment to search for interesting phenomenon without knowing where to look. So it is still quite powerful. Also, the fact that critical exponents exist, and that critical points exist is a powerful bit of universality that MFT gives.

To illustrate its universality, we can consider liquid-gas transitions. Here, van der Waals analysis is completely off in its critical exponents, although it does predict the correct type of phenomenon. However in three dimensions, the Ising model agrees well. How do we implement the Ising model for a gas. Although it's not at all obvious it will work, the implementation is actually quite natural. Assume that a space consists of a lattice, and that every lattice is assigned a value of 1 or 0 based on whether or not it is occupied by a particle. Also assume that any point on the lattice can be occupied by no more than 1 particle at a time. Then we get

$$E = -4J \sum_{<ij>} n_i n_j - \mu \sum n_i$$

We can proceed from there as usual. [End of lecture 2]

## 1.4 Landau-Ginzburg Theory

We aim to improve on Landau theory, and somehow the obvious generalisation is to allow $m$ to vary through space as $m(x)$, but doing so in a way that it is analytic, so that the calculations involved still stay relatively simple...

Concretely, that means assuming we have a large number of sites $N$, but that these sites are all grouped into clusters of $N'$ sites, which is still a large number, but significantly smaller than $N$. Then we define for each cluster, a centre $x$, and we assign $m(x)$ the value of the average magnetisation of this

cluster. However, since $1 << N' << N$, everything is still analytic, etc. Or so we assume.

Consequently we can calculate the partition function in much the same way with the same effective free energy, except now the effective free energy is a functional instead of a function:

$$Z = \sum_{m(x)} e^{-\beta F[m(x)]}$$

This can be written as a functional integral (otherwise known as a path integral)

$$Z = \int Dm[x] e^{-\beta F[m(x)]}$$

and now instead of probabilities we get probabilty density

$$p(m(x)) = \frac{e^{-\beta F[m(x)]}}{Z}.$$

Now, we have a number of conditions that we require this effective free energy to satisfy:

- locality: basically, it takes the form

  $$F[m(x)] = \int d^d x f[m(x)]$$

  where $f$ depends on $m, \nabla m, \nabla^2 m, \ldots$
- $\mathbb{Z}_2$ symmetry when $B = 0$ under $m \mapsto -m$
- Analyticity - we assume it has a Taylor expansion
- we are only interested in $m$ that are slowly varying in space

Now, focusing on the $B = 0$ case we see that the effective free energy must take the form

$$F[m(x)] = \int dx (\frac{1}{2}\alpha_2(T)m^2 + \frac{1}{4}\alpha_4(T) + \frac{1}{2}\gamma(T)(\nabla m)^2)$$

where the odd terms are eliminated due to symmetry (of course, when $B \neq 0$ we can add in these odd terms again).

Unfortunately it is hard to find an general way to calculate $\alpha_2, \alpha_4, \gamma$. but we can say that $\alpha_4, \gamma > 0$ and $\alpha_2$ changes sign at $T_c$/ In particular, in MFT,

$$\alpha_2 \sim (T - T_c), \alpha_4 \sim \frac{1}{3}T.$$

Just as in plain Landau theory, we use the saddle point approximation that $Z = \int dm(x) e^{-\beta F(m(x))}$ is focused on the saddle point of $F$, but this time to

find it we need to use the Euler-Lagrange approach, yielding the equation for the equilibrium:

$$\gamma \nabla^2 m = \alpha_2 m + \alpha_4 m^3$$

How well does this theory work? We can check that if we assume $m$ is constant, then we get back to Landau theory, so we know that at the very least, this theory cannot be worse than Landau theory. As our first development we then look at the notion of domain walls, and see whether or not those can be of any help. In particular, we want to see what a solution to the following looks like in the one dimensional case:

$$\partial_x^2 m = \alpha_2 m + \alpha_4 m$$

where $m = \pm m_0$ as $x \to \pm\infty$. The solution works out to $m = m_0 \tanh\left(\frac{x-x_0}{W}\right)$ where the width, $W = \sqrt{-\gamma/\alpha_2}$.

This models how the system changes state half-way, and since these don't occur in ordinary Landau theory. As such, to compare the two it would be helpful to see what the free energy cost of having a domain wall is. We may calculate this as

$$\begin{aligned} F_W &\sim L^{d-1} \int dx \gamma (\partial_x)^2 \\ &\sim L^{d-1} \gamma m_0^2 / W \\ &= L^{d-1} \sqrt{-\gamma \alpha_2^3 / \alpha_4^2} \end{aligned}$$

where $L$ is the "length" of the system. Now curiously, this allows us to see why Landau fails in 1 dimension, but not (as much) in 2 dimensions.

Consider a 1 dimensional space length $L$, starting in state $m_0$ and ending in some other state, for $T < T_c$, meaning that $\alpha_2 < 0$. We then see that

$$\mathbb{P}(\text{wall at } x = x_0) = \frac{e^{-\beta F_W}}{Z}$$

$$\mathbb{P}(\text{wall somewhere}) \sim \frac{L}{W} \frac{e^{-\beta F_W}}{Z}$$

meaning that

$$\mathbb{P}(n \text{ walls}) = e^{-n\beta F_W}/Z \int_0^L dx_1 \int_{x_1}^L dx_2 \cdots \int_{x_{n-1}}^L dx_n = \frac{1}{Zn!} \left( \frac{L e^{-\beta F_W}}{W} \right)^n.$$

Consequently we can calculate the probabilities of how the final state of the rod ends up:

$$\mathbb{P}(m \to m_0) = \frac{1}{Z} \cosh \left( \frac{L e^{-\beta F_W}}{W} \right)$$

$$\mathbb{P}(m \to -m_0) = \frac{1}{Z} \sinh\left(\frac{Le^{-\beta F_W}}{W}\right)$$

Now as $L \to \infty$, we see that these probabilities stay about the same, meaning that the state of the rad stays more or less random. Why does this argument fail in two dimensions? Well, the calculation does change a bit since we're in two dimensions, but we also have the fact that $F_W$ may depend on $L$ in this scenario, meaning that in effect, the probabilities above instead converge to 0 and 1, meaning that a phase transition does indeed occur. [End of lecture 4]