Lab 5 - Parte teórica - PDS

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Preg (1)

a) Sea $X \sim f_X(x)$ y $Y = aX + b$

V.A.

$Y = g(X)$ $\quad g(x) = ax+b \xrightarrow{\text{inversa}} \boxed{g^{-1}(y) = \frac{y-b}{a}}$

$$\boxed{\frac{dg^{-1}(y)}{dy} = \frac{1}{a}}$$

Por teoría de cambio de variable:

$$f_Y(y) = f_X(g^{-1}(y)) \left|\frac{dg^{-1}(y)}{dy}\right| \quad \text{(fórmula)}$$

$$\rightarrow f_Y(y) = \left|\frac{1}{a}\right| f_X\left(\frac{y-b}{a}\right) \quad \text{Lqqd}$$

b) Sea $X \sim N(m, \sigma^2)$ y $Y = aX + b$

de nuevo $\boxed{g^{-1}(y) = \frac{y-b}{a}}$ y $\boxed{\frac{dg^{-1}(y)}{dy} = \frac{1}{a}}$

$$\rightarrow f_Y(y) = f_X(g^{-1}(y)) \left|\frac{dg^{-1}(y)}{dy}\right|$$

$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$$

/ /

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-m)^2}{2\sigma^2}}$$

$$f_Y(y) = \frac{1}{\sqrt{2\pi\sigma^2 a^2}} e^{-\frac{1}{2\sigma^2}\left(\frac{y-b}{a}-m\right)^2}$$ acomodando

$$= \frac{1}{\sqrt{2\pi(\sigma a)^2}} e^{-\frac{1}{2(\sigma a)^2}(y-(ma+b))^2}$$

$(ma+b)$ → media

(σa) → desviación

→ $f_Y(y)$ = Normal con media $ma+b$ y varianza $(\sigma a)^2$

→ $\cancel{f_Y(y)} = Y \sim N(ma+b, (\sigma a)^2)$ lqqd

Otra forma: Por teoría sabemos que un cambio lineal igual sigue siendo Normal. Entonces hallamos parámetros.

→ $E\{Y\} = E\{aX+b\} = a\,E\{X\} + b = am+b$ (Media)

determinística

→ $E\{Y^2\} = E\{(aX+b)^2\} = a^2 E\{X^2\} + 2ab E\{X\} + b^2$

$E\{Y^2\} = a^2 E\{X^2\} + 2abm + b^2$

$E\{Y^2\} = a^2(\sigma^2 + m^2) + 2abm + b^2$

$\sigma_Y^2 = E\{Y^2\} - E^2\{Y\}$

$= a^2(\sigma^2+m^2) \cancel{+2abm} \cancel{+b^2} - (am)^2 \cancel{-b^2} \cancel{-2amb}$

$= a^2\sigma^2 + m^2a^2 - (am)^2 \rightarrow \boxed{\sigma_Y^2 = (a\sigma)^2}$ → (Varianza)

$\sigma^2 = E\{x^2\} - E^2\{x\}$

→ media $= am+b$ y Varianza $= (a\sigma)^2$

→ $Y \sim N\left(am+b, (\sigma a)^2\right)$ Lqqd

Preg ②

$f_{xy}(x,y) = 1/2$ en la Región R

⇒ Hallamos $f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$

Para $0<x<1$

$f_X(x) = \int_{1-x}^{x+1} \frac{1}{2}\,dy$

$f_X(x) = \frac{1}{2}\left(x+1-1+x\right)$

$\boxed{f_X(x) = x}$

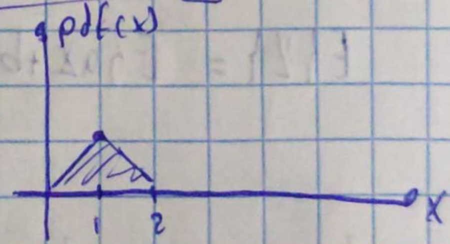
Para $1<x<2$

$f_X(x) = \int_{x-1}^{-x+3} \frac{1}{2}\,dy$

$f_X(x) = \frac{1}{2}\left(-x+3-x+1\right)$

$\boxed{f_X(x) = 2-x}$

→ $f_X(x) = \begin{cases} x, & 0<x<1 \\ 2-x, & 1<x<2 \\ 0, & c.c. \end{cases}$

→ 

⇒ Hallamos $f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$

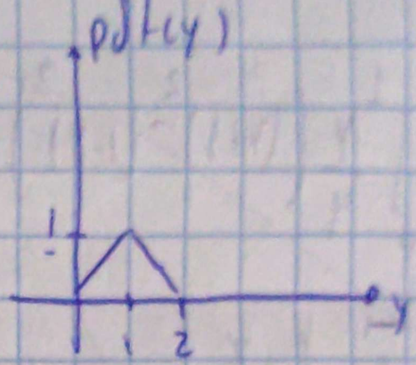
Para $0<y<1$

$f_Y(y) = \int_{1-y}^{y+1} \frac{1}{2}\,dy = y$

Para $1<y<2$

$f_Y(y) = \int_{y-1}^{-y+3} \frac{1}{2}\,dy = 2-y$

$$f_Y(y) = \begin{cases} y & , 0 < y < 1 \\ 2-y & , 1 < y < 2 \\ 0 & , c.c. \end{cases}$$



b) Independencia: Como $f_X(x) \cdot f_Y(y) \neq f(x,y)$
$\Sigma \cdot 1/2$

→ X y Y no son Independientes

Para que sean independientes el producto de pdf's marginales debe ser igual a la pdf conjunta.

Correlacionadas:

→ interpretamos como centro masa

$$E\{X\} = \int_{-\infty}^{\infty} x \, pdf(x) \, dx = 1$$

y $E\{Y\} = 1$

similar

$$E\{XY\} = \int_0^1 \frac{y}{4}\left(\int_{1-y}^{y+1} \frac{x}{2}\,dx\right)dy + \int_1^2 y\left(\int_{y-1}^{3-y} \frac{x}{2}\,dx\right)dy$$

$$= \int_0^1 \frac{y}{4}\left(x^2\right)\Big|_{1-y}^{y+1} dy + \int_1^2 \frac{y}{4}\left(x^2\right)\Big|_{y-1}^{3-y} dy$$

$$= \int_0^1 \frac{y}{4}\left(y^2+1+2y-y^2-1+2y\right)dy + \int_1^2 \frac{y}{4}\left((3-y)^2-(y-1)^2\right)dy$$

$$= \int_0^1 \frac{y}{4}(4y)\,dy + \int_1^2 \frac{y}{4}\left(9+y^2-6y-y^2-1+2y\right)dy$$

$$= \int_0^1 y^2\,dy + \int_1^2 \frac{y}{4}(-4y+8)\,dy$$

$$= \int_0^1 y^2\,dy + \int_1^2 y(2-y)\,dy$$

$$= \frac{y^3}{3}\Big|_0^1 + y^2 - \frac{y^3}{3}\Big|_1^2 \rightarrow E\{XY\} = \frac{1}{3} + \left(4-\frac{8}{3}\right) - 1 + \frac{1}{3}$$

$$= 3 + \frac{2-8}{3}$$

$$= 3 - 2$$

$$E\{XY\} = 1$$

$$\underbrace{1}\cdot\underbrace{1} = \underbrace{1}$$

$$\rightarrow E\{X\}E\{Y\} = E\{XY\}$$

$\rightarrow$ X y Y no son correlacionados pq

Preg ③ AR[2] $\rightarrow$ ARMA (2,0)

$$y[n] = \frac{5}{6} y[n-1] - \frac{1}{6} y[n-2] + x[n]$$

$x[n]$ es ruido AWGN $\mu = 0$ y $\sigma^2 = 1$

Hallar $r_y[l]$, $l \geq 0$ $\qquad r_x[l] = E\{x[n]\, x[n-k]\}$

Solución:

$$Y(z) = \frac{5}{6} Y(z) z^{-1} - \frac{1}{6} Y(z) z^{-2} + X(z)$$

$$\angle \qquad H(z) = \frac{1}{1 - \frac{5}{6} z^{-1} + \frac{1}{6} z^{-2}}$$

$$\rightarrow H(z) = \frac{A}{1 - \frac{z^{-1}}{3}} + \frac{B}{1 - \frac{z^{-1}}{2}} \quad \ldots \text{x FP} \begin{cases} A = -2 \\ B = 3 \end{cases}$$

$$H(z) = \frac{-2}{1 - z^{-1}/3} + \frac{3}{1 - z^{-1}/2} \Bigg\} \text{ todo Causal}$$

$$h[n] = (-2)\left(\frac{1}{3}\right)^n u[n] + 3\left(\frac{1}{2}\right)^n u[n]$$

Ahora planteamos ecuaciones de Yule-Walker, ✓ regrw

$$r_y[k] + \sum_{l=1}^{p} a_p[l]\, r_y[k-l] = \begin{cases} \sigma_v^2\, C_q[k] , & 0 \leq k \leq q \\ 0 , & k > q \end{cases}$$

Para $\boxed{k=0}$

$$r_y[0] + \sum_{l=1}^{2} a_l r_y[0-l] = \sigma^2 \sum_{l=0}^{0} b_l h[l]$$

$$r_y[0] + a_1 r_y[-1] + a_2 r_y[-2] = (2)\, h[0]$$

Autocorrelación Par

$$r_y[0] + a_1 r_y[1] + a_2 r_y[2] = 2 \quad (1)$$

$$\rightarrow r_y[0] = 2 - a_1 r_y[1] - a_2 r_y[2] \quad (I)$$

Para $k=1$, $k>q$

$$\rightarrow r_y[1] + \sum_{l=1}^{2} a_l r_y[1-l] = 0$$

$$r_y[1] = -a_1 r_y[0] - a_2 r_y[-1]$$

$$r_y[1] = -a_1 r_y[0] - a_2 r_y[1] \quad (II)$$

Para $k=2$

$$r_y[2] + \sum_{l=1}^{2} a_l r_x[2-l] = 0$$

$$r_y[2] + a_1 r_y[1] + a_2 r_y[0] = 0$$

$$r_y[2] = -a_1 r_y[1] - a_2 r_y[0] \quad (III)$$

Tenemos

$$\rightarrow \begin{cases} 2 = r_y[0] + a_1 r_y[1] + a_2 r_y[2] \\ -r_y[1] = a_1 r_y[0] + a_2 r_y[1] \\ -r_y[2] = a_1 r_y[1] + a_2 r_y[0] \end{cases}$$

$$y[n] - \frac{5}{6} y[n-1] + \frac{1}{6} y[n-2] = x[n]$$

$a_1 \quad a_2$

$$\begin{cases} 2 = r_y[0] + (-5/6)\, r_y[1] + (1/6)\, r_y[2] \\ -r_y[1] = (-5/6)\, r_y[0] + (1/6)\, r_y[1] \\ -r_y[2] = (-5/6)\, r_y[1] + (1/6)\, r_y[0] \end{cases}$$

Sea $r_y[0] = x$
$r_y[1] = y$
$r_y[2] = z$

$$\begin{cases} 12 = 6x - 5y + z \\ -6y = -5x + y \\ -6z = -5y + x \end{cases} \rightarrow \begin{cases} 12 = 6x - 5y + z \\ -7y = -5x \\ -6z = -5y + x \end{cases}$$

$$\begin{cases} 12 = 6\left(\frac{7}{5}y\right) - 5y + z \\ -6z = -5y + \frac{7}{5}y \end{cases} \rightarrow \begin{cases} 60 = 42y - 25y + 5z \\ -30z = -25y + 7y \end{cases}$$

$$\begin{cases} 60 = 17y + 5z \\ -30z = -18y \end{cases} \rightarrow \quad 60 = 17\left(\frac{30z}{18}\right) + 5z$$

$$\frac{60}{\frac{17 \cdot 30}{18} + 5} = z \rightarrow z = \frac{12}{\frac{17}{3} + 1}$$

$$z = \frac{36}{17 + 3}$$

$$\boxed{z = \frac{36}{20} = \frac{9}{5}}$$

$$Y = \frac{30}{18} Z \rightarrow Y = \frac{\cancel{30}^{\cancel{15}\,3}}{\cancel{18}}\left(\frac{\cancel{9}}{\cancel{5}}\right)$$

$$\boxed{Y = 8}$$

$$\rightarrow \quad x = \frac{7}{5} y \rightarrow \boxed{x = \frac{21}{5}}$$

$\rightarrow r_x[0] = 21/5$

$r_x[1] = 3$

$r_x[2] = 9/5$

y para $l > 2$

$$\sum r_x[l] = 0$$

$$\rightarrow r_x[l] = \frac{21}{5}\delta[l] + 3\delta[l-1] + \frac{9}{5}\delta[l-2]$$

Prog (4)

a)

$$\iint f_{X,Y}(x,y)\,dx\,dy = 1$$

$$k_1 \int_0^{\infty}\int_0^{\infty} (x+y)e^{-x-y}\,dx\,dy = 1$$

$$k_1 \int_0^{\infty} e^{-y} \int_0^{\infty} (x+y)e^{-x}\,dx\,dy = 1$$

	D	I
+	x+y	e^{-x}
−	1	$-e^{-x}$
	0	e^{-x}

$$k_1 \int_0^{\infty} e^{-y}\left(-(x+y)e^{-x} - e^{-x}\right)\Big|_{x=0}^{x\to\infty} dy = 1$$

$$k_1 \int_0^{\infty} e^{-y}\left((0-0) - (-y-1)\right) dy = 1$$

$$\lim_{x\to\infty} \frac{(x+y)}{e^x} = \lim_{x\to\infty} \frac{1}{e^x} = 0$$

$$(\mathrm{I}) \ldots k_1 \int_0^{\infty} e^{-y}(y+1)\,dy = 1$$

	D	I
+	y+1	e^{-y}
−	1	$-e^{-y}$
	0	e^{-y}

$$k_1\left(-(y+1)e^{-y} - e^{-y}\right)\Big|_{y=0}^{y\to\infty} = 1$$

$k_1\left((0)-(-1-1)\right)=1 \rightarrow 2k_1=1$

$\rightarrow k_1=1/2$

b) Qe en (I) Ahí integramos sobre x

$\rightarrow f_Y(y)=k_1 e^{-y}(1+y)$

$\rightarrow f_Y(y)=\frac{1}{2}e^{-y}(1+y)\,u(y)$

y como x y y son similares al operar

$f_X(x)=\frac{1}{2}e^{-x}(1+x)\,u(x)$

Ahora hallamos

$$f_{\underline{Y}|\underline{X}}(y/x)=\frac{f_{XY}(x,y)}{f_X(x)}=\frac{\cancel{k_1}(x+y)\,e^{-(\cancel{x}+y)}\,u(x)u(y)}{\cancel{k_1}\cancel{e^{-x}}(1+x)\,u(x)}$$

$$\rightarrow f_{Y|\underline{X}}(y/x)=\left(\frac{x+y}{x+1}\right)e^{-y}\,u(y)$$

Podemos notar que

$f_Y(x) \neq f_{Y|X}(y/x) \rightarrow \underline{X}$ y $\underline{Y}$ no son independientes

Preg (5)

$$H(z) = \frac{1 - 0{,}1 z^{-1} - 0{,}72 z^{-2}}{1 - 0{,}9 z^{-1} + 0{,}81 z^{-2}}$$

$x[n] \sim N(0,1)$

a) Dado que $x[n]$ es ruido blanco (es un proceso WSS) y $H(z)$ representa un sistema LTI

Entonces la salida del sistema tmb será WSS.
($y[n]$)

Aclaración $x[n]$ es WSS (estacionario en el sentido amplio) porque sus métricos son invariantes en el tiempo es decir su media y varianza son cte.

Conclusión: $Y[n]$ es estacionario en el sentido Amplio // dada la entrada $x[n] \sim N(0,1)$

b) Media:

$$m_y = E\{y[n]\} = \underset{0}{m_x} \sum_{k=-\infty}^{\infty} h[k] = 0$$

$\rightarrow \boxed{m_y = 0}$ Media

Varianza: $\sigma_y^2 = E\{Y^2[n]\} - \underbrace{(E\{Y[n]\})^2}_{0} = E\{Y[n]\, Y[n+0]\}$

$$\sigma_y^2 = r_y[0]$$

y Autocorrelación:

$$r_y[k] = E\{Y[n+k]\, Y[n]\} = r_x[k] * h[k] * h[-k]$$

Como $X \sim N(0,1) \rightarrow r_x[k] = \underbrace{\sigma_w^2}_{1} \delta[k]$

$$\rightarrow r_y[k] = h[k] * h[-k]$$

→ Debemos hallar Autocorrelación y Varianza es lo mismo evaluado en 0

Calculamos $h[n]$

Sea $z^{-1} = x$

$$\frac{1-0{,}1x-0{,}72x^2}{1-0{,}9x+0{,}81x^2} = \left(\frac{-0{,}72}{0{,}81}\right)\left(1 + \frac{0{,}9x-1}{0{,}81x^2-0{,}9x+1}\right) + \frac{1-0{,}1x}{0{,}81x^2-0{,}9x+1}$$

$$= \frac{-8}{9} + \frac{8/9-0{,}8x}{0{,}81x^2-0{,}9x+1} + \frac{1-0{,}1x}{0{,}81x^2-0{,}9x+1}$$

$$= \frac{-8}{9} + \frac{1}{0{,}81}\left(\frac{A}{x-\frac{5}{9}(1+\sqrt{3}j)} + \frac{B}{x-\frac{5}{9}(1-\sqrt{3}j)}\right)$$

x F P $\rightarrow$ $\boxed{B = \frac{17/9-(1-\sqrt{3}j)/2}{\left(-10\sqrt{3}/9\right)}}$, $\boxed{A = -0{,}9 - B}$

$$H(z) = -\frac{8}{9} + \frac{1}{0{,}81}\left(\frac{A}{z^{-1}-\frac{5}{9}(1+\sqrt{3}j)} + \frac{B}{z^{-1}-\frac{5}{9}(1-\sqrt{3}j)}\right)$$

$$\frac{1}{z^{-1}-a} = \frac{-a^{-1}}{1-a^{-1}z^{-1}} = -(a)^{-1}(a^{-1})^{n}u[n]$$

$$\rightarrow h[n] = \frac{-8}{9}\delta[n] + \frac{1}{0{,}81}\left(-A\left(\frac{5}{9}(1+\sqrt{3}j)\right)^{-1}\left(\frac{5}{9}(1+\sqrt{3}j)\right)^{-n}u[n]\right.$$

$$\left. + -B\left(\frac{5}{9}(1-\sqrt{3}j)\right)^{-1}\left(\frac{5}{9}(1-\sqrt{3}j)\right)^{-n}u[n]\right)$$

$$\rightarrow C = \frac{-A\left(\left(\frac{5}{9}\right)(1+\sqrt{3}j)\right)^{-1}}{0{,}81} \qquad D = \frac{-B\left(\left(\frac{5}{9}\right)(1-\sqrt{3}j)\right)^{-1}}{0{,}81}$$

$\rightarrow h[n] = -\frac{8}{9}\delta[n] + C\left(\frac{5}{9}(1+\sqrt{3}j)\right)^{-n} u[n] + D\left(\frac{5}{9}(1-\sqrt{3}j)\right)^{-n} u[n]$

$\rightarrow h[-n] = -\frac{8}{9}\delta[n] + C\left(\frac{5}{9}(1+\sqrt{3}j)\right)^{n} u[-n] + D\left(\frac{5}{9}(1-\sqrt{3}j)\right)^{n} u[-n]$

~~Se sabe que $(\gamma_1)^n u[n] * (\gamma_2)^n u[-n]$~~

Se sabe que $(\gamma_1)^n u[-n-1] * (\gamma_2)^n u[n] = \dfrac{\gamma_2}{\gamma_1 + \gamma_2} (\gamma_2)^n u[n] + \dfrac{\gamma_1}{\gamma_1 - \gamma_2} (\gamma_1)^n u[-n-1]$

$$\to h[n] * h[-n] = \left(-\frac{8}{9}\right)^2 \delta[n] + \frac{-8C}{9}\left(\frac{5}{9}(1+\sqrt{3}j)\right)^n u[-n] + \frac{-8D}{9}\left(\frac{5}{9}(1-\sqrt{3}j)\right)^n u[-n]$$

$$+ \frac{-8}{9} C \left(\frac{5}{9}(1+\sqrt{3}j)\right)^{-n} u[n] + (C)^2 \frac{\left(\frac{5}{9}(1+\sqrt{3}j)\right)^{-|n|}}{1 - \left(\frac{5}{9}(1+\sqrt{3}j)\right)^2}$$

$$+ \frac{-8}{9} D \left(\frac{5}{9}(1-\sqrt{3}j)\right)^{-n} u[n] + (D)^2 \frac{\left(\frac{5}{9}(1-\sqrt{3}j)\right)^{-|n|}}{1 - \left(\frac{5}{9}(1-\sqrt{3}j)\right)^2}$$

$$+ (C)(D)\left(\frac{5}{9}(1-\sqrt{3}j)\right)^{-n} u[n] * \left(\frac{5}{9}(1+\sqrt{3}j)\right)^n u[-n] \to \ldots (\mathrm{I})$$

$$+ C.D\left(\frac{5}{9}(1+\sqrt{3}j)\right)^{-n} u[n] * \left(\frac{5}{9}(1-\sqrt{3}j)\right)^n u[-n] \to \ldots (\mathrm{II})$$

$$(I)\ldots \left(\left(\frac{5}{9}(1-\sqrt{3})\right)^{-1}\right)^{n} u[n] * \left(\left(\frac{5}{9}(1+\sqrt{3})\right)\right)^{n} u[-n-1] + \delta[n]$$

$$= \left(\frac{5}{9}(1-\sqrt{3})\right)^{-n} u[n] + \frac{\left(\frac{5}{9}(1-\sqrt{3})\right)^{-1}}{\frac{5}{9}(1+\sqrt{3}) - \left(\frac{5}{9}(1-\sqrt{3})\right)^{-1}} \left(\frac{5}{9}(1-\sqrt{3})\right)^{-n} u[n]$$

$$+ \frac{\frac{5}{9}(1+\sqrt{3})}{\frac{5}{9}(1+\sqrt{3}) - \left(\frac{5}{9}(1-\sqrt{3})\right)^{-1}} \frac{5}{9}(1+\sqrt{3})^{-n} u[-n-1]$$

$$(II)\ldots \frac{5}{9}(1+\sqrt{3})^{-n} u[n] + \frac{\left(\frac{5}{9}(1+\sqrt{3})\right)^{-1}}{\frac{5}{9}(1-\sqrt{3}) - \left(\frac{5}{9}(1+\sqrt{3})\right)^{-1}} \left(\frac{5}{9}(1+\sqrt{3})\right)^{-n} u[n]$$

$$+ \frac{\frac{5}{9}(1-\sqrt{3})}{\frac{5}{9}(1+\sqrt{3}) - \left(\frac{5}{9}(1+\sqrt{3})\right)^{-1}} \frac{5}{9}(1+\sqrt{3})^{n} u[-n-1]$$

$$r_y[n] = \left(\frac{-8}{9}\right)^2 u[n] + \frac{-8C}{9}\left(\frac{5}{9}(1+\sqrt{3})\right)^{n} u[-n] - \frac{8D}{9}\left(\frac{5}{9}(1-\sqrt{3})\right)^{n} u[-n]$$

$$- \frac{8}{9}C\left(\frac{5}{9}(1+\sqrt{3})\right)^{-n} u[n] + C^2 \frac{\left(\frac{5}{9}(1+\sqrt{3})\right)^{-|n|}}{1-\left(\frac{5}{9}(1+\sqrt{3})\right)^2}$$

$$- \frac{8}{9}D\left(\frac{5}{9}(1-\sqrt{3})\right)^{-n} u[n] + D^2 \frac{\left(\frac{5}{9}(1-\sqrt{3})\right)^{-|n|}}{1-\left(\frac{5}{9}(1-\sqrt{3})\right)^2}$$

Continúa

$+ (2)$

$+ (C)(D)(I) \quad + CD(II)$

fusion hallados Antes

$$y \quad C = \frac{(-A)\left(\left(\frac{5}{9}\right)(1+\sqrt{3})\right)^{-1}}{0,81}$$

$$D = -B\,\frac{\left(\left(\frac{5}{9}\right)(1-\sqrt{3})\right)^{-1}}{0,81}$$

$$\text{Con} \quad A = -0,9 - B$$

$$B = \frac{17/9 - ((1-\sqrt{3}))/2}{\left(-10\sqrt{3}/9\right)}$$

y la Varianza es solo evaluar en 0

$$\rightarrow \sigma^2_y = \left(\frac{-8}{9}\right)^2 - \frac{8C}{9} - \frac{8D}{9} - \frac{8C}{9} + \frac{C^2}{1-\left(\frac{5}{9}(1+\sqrt{3})\right)^2}$$

$$-\frac{8}{9}D + \frac{D^2}{1-\left(\frac{5}{9}(1-\sqrt{3})\right)^2}$$

$$+ CD\left(1 + \frac{\left(\frac{5}{9}(1-\sqrt{3})\right)^{-1}}{\frac{5}{9}(1+\sqrt{3}) - \left(\frac{5}{9}(1-\sqrt{3})\right)^{-1}} + 0\right)$$

$$+ CD\left(1 + \frac{\left(\frac{5}{9}(1+\sqrt{3})\right)^{-1}}{\frac{5}{9}(1+\sqrt{3}) - \left(\frac{5}{9}(1+\sqrt{3})\right)^{-1}} + 0\right)$$

Preg (6) AR(1)

$w[n] \sim N(0, \sigma_w^2)$

$x[n] = \rho\, x[n-1] + w[n]$

a) Hallar $r_x[k]$:

x def fmb $\longrightarrow r_x[k] = r_w[k] * h[k] * h[-k] \quad \ldots (I)$

$X(z) = \rho X(z) z^{-1} + W(z)$

$\rightarrow \dfrac{X(z)}{W(z)} = \dfrac{1}{1-\rho z^{-1}} \rightarrow H(z) = \dfrac{1}{1-\rho z^{-1}}$

output (X(z)), input (W(z))

Asumiendo causalidad $\rightarrow h[n] = (\rho)^n u[n]$

$\rightarrow$ x Propiedad de convolución $h[n] * h[-n] = \dfrac{\rho^{|n|}}{1-\rho^2}$

Reemplazando en (I)

$\rightarrow r_x[k] = (r_w[k]) * \left(\dfrac{\rho^{|k|}}{1-\rho^2}\right)$

Como $w[n]$ es ruido blanco $\rightarrow r_w[k] = \sigma^2 \delta[k]$

$\rightarrow r_x[k] = \dfrac{\sigma^2}{1-\rho^2} \left(\delta[k] * \rho^{|k|}\right)$

$r_x[k] = \frac{\sigma^2}{1-\rho^2} \rho^{|k|}$ → Autocorrelación de salida $x[n]$

b) Hallar $S_x(e^{jw}) = ?$ $DTFT\{r_x[k]\}$

→ $DTFT\{r_x[k]\} = Z\{r_x[k]\}\Big|_{z=e^{jw}}$

→ $Z\{r_x[n]\} = \frac{\sigma^2}{1-\rho^2}\left(\frac{1}{1-\rho z^{-1}} + \frac{1}{1-\rho z^{-1}} - 1\right)$

→ $S_x(e^{jw}) = \frac{\sigma^2}{1-\rho^2}\left(\frac{2-(1-\rho e^{-jw})}{1-\rho e^{-jw}}\right)$

$$= \frac{\sigma^2}{1-\rho^2}\left(\frac{(1+\rho e^{-jw})(1-\rho e^{jw})}{(1-\rho e^{-jw})(1-\rho e^{+jw})}\right)$$

$$= \frac{\sigma^2}{1-\rho^2}\left(\frac{1-\cancel{\rho e^{jw}}+\cancel{\rho e^{-jw}}-\rho^2(1)}{1-\rho e^{jw}-\rho e^{-jw}+\rho^2}\right)$$

$$= \frac{\sigma^2}{\cancel{1-\rho^2}}\left(\frac{\cancel{1-\rho^2}}{1-\rho\, 2\cos(w)+\rho^2}\right)$$

$$S_x(e^{jw}) = \frac{\sigma^2}{1-2\rho\cos(w)+\rho^2}$$ Densidad Espectral de potencia