FP3

William Bevington Callum O'Brien Alex Pace

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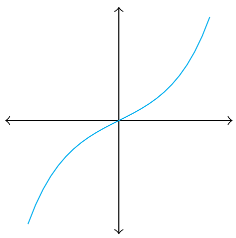
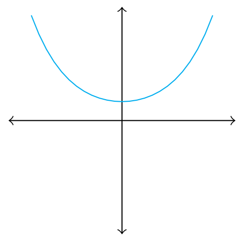
Contents

1 Hyperbolic Functions

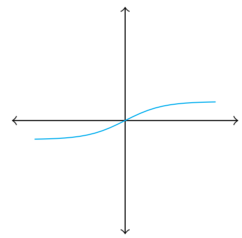
The hyperbolic functions are analogs of the ordinary trigonometric, or circular functions. The basic hyperbolic functions are, as one might expect, analygous to sine and cosine; they are hyperbolic sine and hyperbolic cosine.
The following graphs plot the various hyperbolic trigonometric functions, allong the left hand side is the hyperbolic function of sin and its counterparts, down the middle hyperbolic cos, and the right hyperbolic tan. The domain and range of each function is also described below their respective graphs.



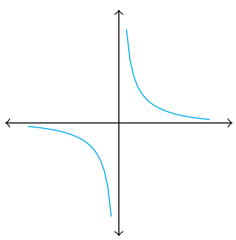
$\sinh(x)$
Domain $\{x \in \mathbb{R}\}$
Range $\{y \in \mathbb{R}\}$



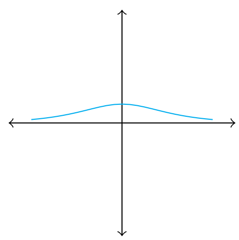
$\cosh(x)$
Domain $\{x \in \mathbb{R}\}$
Range $\{y \in \mathbb{R} \,|\, y \geq 1\}$



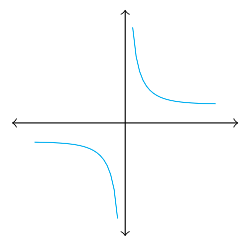
$\tanh(x)$
Domain $\{x \in \mathbb{R}\}$
Range $\{y \in \mathbb{R} \,|\, -1 < y < 1\}$



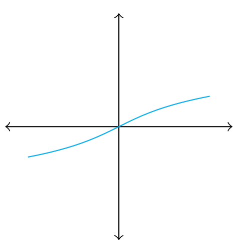
csch(x)
Domain $\{x \in \mathbb{R} \,|\, x \neq 0\}$
Range $\{y \in \mathbb{R} \,|\, y \neq 0\}$



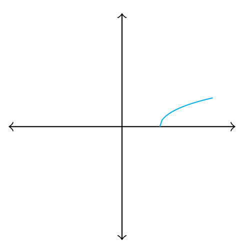
sech(x)
Domain $\{x \in \mathbb{R}\}$
Range $\{y \in \mathbb{R} \,|\, 0 < y < 1\}$



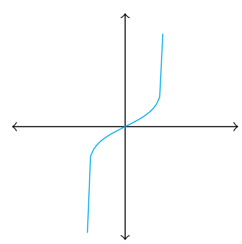
coth(x)
Domain $\{x \in \mathbb{R} \,|\, x \neq 0\}$
Range $\{y \in \mathbb{R} \,|\, y < -1 \text{ or } y > 1\}$



$\sinh^{-1}(x)$
Domain $\{x \in \mathbb{R}\}$
Range $\{y \in \mathbb{R}\}$



$\cosh^{-1}(x)$
Domain $\{x \in \mathbb{R} \,|\, x \geq 1\}$
Range $\{y \in \mathbb{R}\}$



$\tanh^{-1}(x)$
Domain $\{x \in \mathbb{R} \,|\, -1 < x < 1\}$
Range $\{y \in \mathbb{R}\}$

1.1 Inverse Hyperbolic Trigonometric Functions

The inverse of a hyperbolic function reverses the operation of the corresponding trigonometric function. Therefore, these functions can be used to find a certain input of a hyperbolic trigonometric function haven being given its output.

$$\operatorname{arsinh}(z) = \ln\left(z + \sqrt{1+z^2}\right)$$

this identity can be found through the following process:

$$z = \sinh(x)$$

$$z = \frac{e^x - e^{-x}}{2} \rightarrow 2z = e^x - e^{-x}$$

$$2ze^x = e^{2x} - 1 \rightarrow (e^x)^2 - 2z(e^x) - 1 = 0$$

This should be recognizable as a quadratic equation with e^x instead of the regular x, the variable z will be considered a constant.
Using the quadratic formula, you should obtain:

$$e^x = z \pm \sqrt{z^2+1}$$

At this point you should realise that $e^x > 0$ for all x, and considering $\sqrt{z^2+1} > y$, we can discard the solution with the minus sign and then take the natural logarithm of both sides to arrive at our solution for x:

$$e^x = z + \sqrt{z^2+1} \rightarrow x = \ln\left(z + \sqrt{z^2+1}\right)$$

similarly, the inverse of $\cosh(x)$ is defined thus:

$$\operatorname{arcosh}(z) = \ln\left(z + \sqrt{z+1}\sqrt{z-1}\right)$$

It is also derived in a similar fashion:

$$z = \cosh(x)$$

$$z = \frac{e^x + e^{-x}}{2} \rightarrow 2z = e^x + e^{-x}$$

$$2ze^x = e^{2x} + 1 \rightarrow (e^x)^2 - 2z(e^x) + 1 = 0$$

again this is a quadratic equation which can be solved to find the following:

$$e^x = z \pm \sqrt{z^2-1}$$

since $x \geq 0$ we know that $e^x \geq 1$ for all x. And since $z - \sqrt{z^2-1}$ does not exceed 1 for some y, we must discard the solution with the minus sign. This leaves us with a difference of two squares which can be sepearted to find all solutions of x.

$$e^x = z + \sqrt{z^2-1} \rightarrow e^x = z + \sqrt{z+1}\sqrt{z-1}$$

$$x = \ln\left(e^x = z + \sqrt{z+1}\sqrt{z-1}\right)$$

finally, inverse $\tanh(x)$ is defined:

$$\operatorname{artanh}(z) = \frac{1}{2}\left(\ln(1+z) - \ln(1-z)\right)$$

and is derived:

$$z = \tanh(x)$$
$$z = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^{2x}-1}{e^{2x}+1}$$
$$\left(e^{2x}+1\right)z = e^{2x} - 1 \;\rightarrow\; (z-1)\,e^{2x} + z + 1 = 0$$

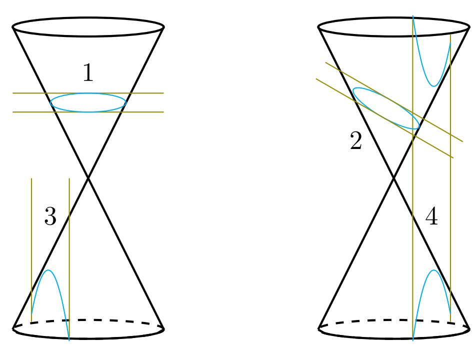
solving this quadratic equation yields:

$$e^x = \pm\sqrt{\frac{z+1}{1-z}}$$

because $e^x > 0$ we must disregard the negative solution. Then from here we can take the natural logarithm of both sides and simplify to find our solution:

$$e^x = \sqrt{\frac{z+1}{1-z}} \;\rightarrow\; x = \ln\left(\sqrt{\frac{z+1}{1-z}}\right)$$
$$x = \frac{1}{2}\ln\left(\frac{z+1}{1-z}\right) \;\rightarrow\; x = \frac{1}{2}\left(\ln(1+z) - \ln(1-z)\right)$$

2 Conic Sections



1. Circle 2. Ellipse 3. Parabola 4. Hyperbola

All conic sections can be described in terms of loci; for any point P on a conic section,

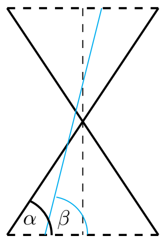
$$\frac{|PM|}{|PS|} = e$$

where e is the eccentricity, S is the focus and M is the closest point to P that lies on the directrix.

$$0 \leq e < 1 \Rightarrow P \text{ describes an ellipse}$$

$$e = 1 \Rightarrow P \text{ describes a parabola}$$

$$e > 1 \Rightarrow P \text{ describes a hyperbola}$$



The eccentricity can also be defined graphically, in terms of the cone and the intersection. Where e is the ratio between the angle of the cone and the intersection

$$e = \frac{\sin(\beta)}{\sin(\alpha)}$$

2.1 Ellipses

Ellipses are conic sections with eccentricty $e \in [0, 1)$. Considering their geometry, it is clear that, when expressed as a parametric equation, ellipses take the form;

$$x = a\cos\theta,\ y = b\sin\theta$$

Hence;

$$\left(\frac{x}{a}\right)^2 = \cos^2\theta,\ \left(\frac{y}{b}\right)^2 = \sin^2\theta$$

Making apparent an ellipse expressed in cartesian form;

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

Theorem For $e \in [0, 1)$, an ellipse of focus $(ae, 0)$ and directrix $x = \frac{a}{e}$ is described by the equation;

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

Proof

$$\frac{|PS|^2}{|PM|^2} = e \Rightarrow |PS|^2 = e^2|PM|^2$$

By pythagoras,

$$|PS|^2 = (x - ae)^2 + y^2$$

$$|PM|^2 = \left(\frac{a}{e} - x\right)^2 + y^2$$

Thus

$$(x - ae)^2 + y^2 = \left(\frac{a}{e} - x\right)^2 e^2$$

The derivative of an ellipse can be found by implicitly differentiating its cartesian form or using the chain rule on its parametric form. By the former method;

$$\frac{\mathrm{d}}{\mathrm{d}\,x}\left(\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2\right) = \frac{\mathrm{d}}{\mathrm{d}\,x}1$$

$$\frac{2x}{a^2} + \frac{2y}{b^2} \times \frac{\mathrm{d}\,y}{\mathrm{d}\,x} = 0$$

$$\frac{\mathrm{d}\,y}{\mathrm{d}\,x} = -\frac{b^2 x}{a^2 y}$$

And by the latter;

$$\frac{\mathrm{d}\,x}{\mathrm{d}\,\theta} = -a\sin\theta$$

$$\frac{\mathrm{d}\,y}{\mathrm{d}\,\theta} = b\cos\theta$$

$$\frac{\mathrm{d}\,y}{\mathrm{d}\,x} = \frac{\mathrm{d}\,y}{\mathrm{d}\,\theta} \times \left(\frac{\mathrm{d}\,x}{\mathrm{d}\,\theta}\right)^{-1} = -\frac{b\cos\theta}{a\sin\theta}$$

From this we can derive the general equation of a tangent to an ellipse at a point $(i,\, j)$:

$$y - j = -\frac{b^2 i}{a^2 j}(x - i)$$

$$a^2 jy + b^2 ix = (aj)^2 + (bi)^2$$

Similarly, we can derive the equation for a normal:

$$y - j = \frac{a^2 j}{b^2 i}(x - i)$$

$$b^2 iy + a^2 ij = a^2 jx + b^2 ij$$

2.2 Hyperbolae

A hyperbola is a conic section with $e > 1$.