Edexcel Advanced Level GCE Mathematics FP2

William Bevington   Callum O'Brien   Alex Pace

Contents

1 Complex Numbers

1.1 De Moivre's Theorem

De Moivre's theorem states

$$z = r\left(\cos\theta + i\sin\theta\right) \Leftrightarrow z^n = r^n\left(\cos\left(n\theta\right) + i\sin\left(n\theta\right)\right)$$

Using the exponential form, this is equivalent to

$$z = re^{i\theta} \Leftrightarrow z^n = \left(re^{i\theta}\right)^n = r^n e^{in\theta}$$

1.1.1 Proof of DMT

Let P_n be a predicate:

$$P_n \Leftrightarrow \left(r\left(\cos\theta + i\sin\theta\right)\right)^n = r^n\left(\cos\left(n\theta\right) + i\sin\left(n\theta\right)\right)$$

Let $n \to 1$. Evidently,

$$P_1 \Leftrightarrow r\left(\cos\theta + i\sin\theta\right) = r\left(\cos\theta + i\sin\theta\right))$$

thus

$$P_1 \text{ is True}$$

Assume P_k:

$$\left(r\left(\cos\theta + i\sin\theta\right)\right)^k = r^k\left(\cos\left(k\theta\right) + i\sin\left(k\theta\right)\right)$$

Now, consider P_{k+1}:

$$\left(r\left(\cos\theta + i\sin\theta\right)\right)^{k+1} = r\left(\cos\theta\right) \times r^k\left(\cos\left(k\theta\right) + i\sin\left(k\theta\right)\right)$$

$$= r^{k+1}\left(\cos\left(\theta\left(k+1\right)\right) + i\sin\left(\theta\left(k+1\right)\right)\right) \Leftrightarrow P_{k+1}$$

therefore

$$P_k \Rightarrow P_{k+1}$$

Thus, by induction,

$$P_n \quad \forall n \in \mathbb{N} n \geq 0$$

1.1.2 Uses

DMT is great, because it can be used to simplify expressions like this:

$$\frac{\left(\cos\left(\frac{9\pi}{17}\right) + i\sin\left(\frac{9\pi}{17}\right)\right)^5}{\left(\cos\left(\frac{2\pi}{17}\right) - i\sin\left(\frac{2\pi}{17}\right)\right)^3}$$

First we must put the denominator into the correct polar form (with a + inbetween cos and sin), and then we can apply de Moivre's Theorem.

$$\frac{\left(\cos\left(\frac{9\pi}{17}\right) + i\sin\left(\frac{9\pi}{17}\right)\right)^5}{\left(\cos\left(\frac{-2\pi}{17}\right) + i\sin\left(\frac{-2\pi}{17}\right)\right)^3}$$

And now appying de Moivre's Theorem:

$$\frac{\cos\left(\frac{45\pi}{17}\right)+i\sin\left(\frac{45\pi}{17}\right)}{\cos\left(\frac{-6\pi}{17}\right)+i\sin\left(\frac{-6\pi}{17}\right)}$$

From this point we can easily simplify the complex number down.

$$\cos\left(\frac{51\pi}{17}\right)+i\sin\left(\frac{51\pi}{17}\right)$$

$$\cos(3\pi)+i\sin(3\pi)$$

$$\cos(\pi)+i\sin(\pi)=-1$$

1.1.3 De Moivre's Theorem and Trigonometric Identities

De Moivre's Theorem can also be applied to problems consisting of trigonometric functions by using identites and binomial expansion. First though, it is worth noting the following, where Z is a complex number $r(\cos(\theta)+i\sin(\theta))$

$$Z+\tfrac{1}{Z} \;=\; 2\cos(\theta) \qquad Z^n+\tfrac{1}{Z^n} \;=\; 2\cos(n\theta)$$

$$Z-\tfrac{1}{Z} \;=\; 2i\sin(\theta) \qquad Z^n-\tfrac{1}{Z^n} \;=\; 2i\sin(\theta)$$

Trigonometric functions can be replaced with these arrangements of complex numbers and then manipulated far more easily. The following is an example of such a situation.

Express $\sin^3(\theta)$ in the form $d\cos(4\theta)+e\cos(2\theta)+f$

$$Z-\frac{1}{Z} \;=\; 2i\sin(\theta) \;\therefore\; \left(Z^4-\frac{1}{Z^4}\right)^4 \;=\; 16\sin(4\theta)$$

$$\left(Z^4-\frac{1}{Z^4}\right)^4 \;=\; Z^4+4Z^3\left(\frac{-1}{Z}\right)+6Z^2\left(\frac{-1}{Z}\right)^2+4Z\left(\frac{-1}{Z}\right)^3+\left(\frac{-1}{Z}\right)^4$$

$$\rightarrow\; Z^4+4Z^2+6-4Z^{-2}+Z^{-4}$$

$$\rightarrow\; \left(Z^4+\frac{1}{Z^4}\right)-4\left(Z^2+\frac{1}{Z^2}\right)+6$$

$$\therefore\; 16\sin^4(\theta) \;=\; 2\cos(4\theta)-8\cos(2\theta)+6$$

$$\therefore\; \sin^4(\theta) \;=\; \frac{1}{8}\cos(4\theta)-\frac{1}{2}\cos(2\theta)+\frac{3}{8}$$

1.1.4 Finding the $n^{\mathbf{th}}$ Root of a Polynomial Using De Moivre's Theorem

Solve $Z^3 = 1$. Represent Solutions on an argand diagram.

$$|Z^3| = 1 \;\; arg(Z^3) = \pi$$

$$Z^3 = 1\left(\cos\left(\pi\right) + i\sin\left(\pi\right)\right)$$

$$Z^3 = \left(\cos\left(\pi + 2k\pi\right) + i\sin\left(\pi + 2k\pi\right)\right)$$

$$Z = \left(\cos\left(\pi + 2k\pi\right) + i\sin\left(\pi + 2k\pi\right)\right)^{\frac{1}{3}}$$

$$Z = \cos\left(\frac{\pi + 2k\pi}{3}\right) + i\sin\left(\frac{\pi + 2k\pi}{3}\right)$$

We only need consider all values of k for which $\pi < \theta < \pi$

$$k = 0 \;\rightarrow\; Z = \cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)$$

$$\therefore\; Z \;=\; \frac{1}{2} + \frac{\sqrt{3}}{2}i$$

$$k = 1 \;\rightarrow\; Z = \cos\left(\pi\right) + i\sin\left(\pi\right)$$

$$\therefore\; Z \;=\; -1$$

$$k = -1 \;\rightarrow\; Z = \cos\left(\frac{-\pi}{3}\right) + i\sin\left(\frac{-\pi}{3}\right)$$

$$\therefore\; Z \;=\; \frac{1}{2} - \frac{\sqrt{3}}{2}i$$

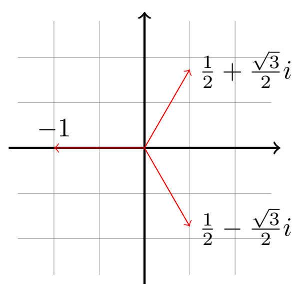
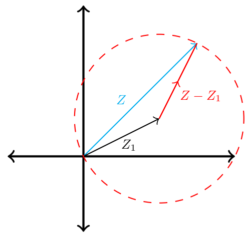


Figure 1: Third roots of -1 shown on an argand diagram

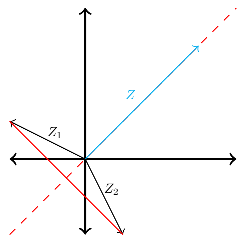
1.2 Loci with Complex Numbers

Finding the Loci of a complex number is stating the possible value(s) of Z (Where Z is a variable point given by $Z = x + iy$. There are a few special cases which are required to be memorised and these are described below.



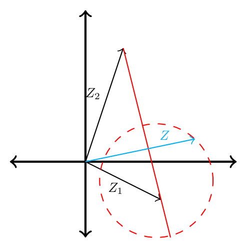
$$|Z - Z_1| = r$$

The solution to this general loci is a circle with center (x, y), and with a radius r.



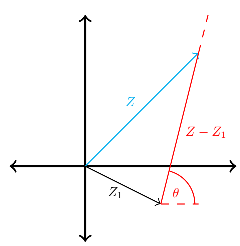
$$|Z - Z_1| = |Z - Z_2|$$

The solution to this general loci is a perpendicular bisector of the segment that passes through (x_1, y_1) and (x_2, y_2).



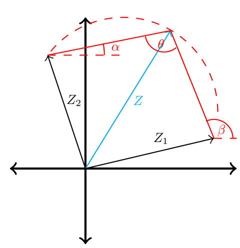
$$|Z - Z_1| = \lambda |Z - Z_2|$$

The solution to this general loci is a circle whose form can be found through algebraic methods.



$$arg(Z - Z_1) = \theta$$

The solution to this general loci is a "half line" from (x, y) at an angle θ to the x-axis.

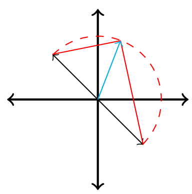


$$\frac{arg\,(Z - Z_1)}{arg\,(Z - Z_2)} = \theta$$

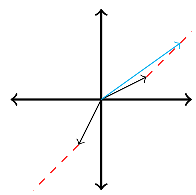
The solution to this general loci is the arc of a circle through (a, b) and (c, d) whose form can be found through geometric approach.

1.2.1 Special Cases of Complex Loci

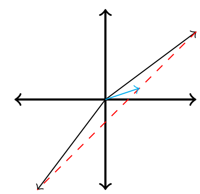
$$arg\left(\frac{Z-a}{Z-b}\right)=\frac{\pi}{2}$$



$$arg\left(\frac{Z-a}{Z-b}\right)=0$$



$$arg\left(\frac{Z-a}{Z-b}\right)=\pi$$



2 Polynomial Expansions

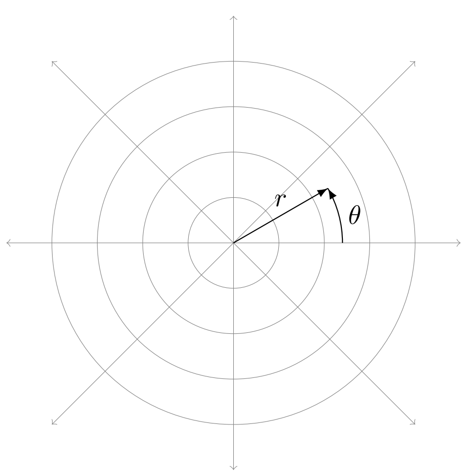
Maclaurin's expansion is a way to fit a polynomial to an arbitrary function. It states

$$f(x)\approx f(0)+xf'(0)+\frac{x^2f''(0)}{2}+\frac{x^3f'''(0)}{6}+\cdots+\frac{x^nf^n(0)}{n!}=\sum_{r=0}^{n}\frac{x^rf^r(0)}{r!}$$

Mapping this out to infinite terms gives an exact equivalence, called a Maclaurin series;

$$f(x)=\sum_{r=0}^{\infty}\frac{x^rf^r(0)}{r!}$$

3 Polar Coordinates



Points in a space described by polar coordinates are defined in terms of the magnitude and direction of their displacement from the origin. In a two dimensional space, (r,θ) is an arbitrary point, where either $\theta\in(-\pi,\pi\,]$ or $\theta\in[\,0,2\pi)$. This is equivalent to (x,y), the cartesian form, where

$$x=r\cos\theta,\quad y=r\sin\theta$$

$$r=\left(x^2+y^2\right)^{\frac{1}{2}},\quad \theta=\arctan\left(\frac{y}{x}\right)$$

Cartesian equations can hence be converted into equivalent polar equations and vice versa.

4 Differential Equations

4.1 First Order Linear Inhomogeneous Ordinary Differential Equations

First order linear inhomogeneous ordinary differential equations can be expressed in the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)$$

To solve an equation in this form, one multiplies by an integrating factor λ where

$$\lambda = e^{\int P(x)\,\mathrm{d}x}$$

This puts the equation in the form

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(y\lambda\right) = Q(x)$$

hence

$$y\lambda = \int Q(x)\,\mathrm{d}x + \text{const.}$$

4.2 Second Order Linear Homogeneous ODEs