Existence of a Mass Gap in a Discrete SU(3) Gauge Theory via a Quantized Execution Framework

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Abstract

We construct a discrete non-abelian SU(3) gauge theory over a Euclidean lattice governed by quantized execution transitions. The theory restricts local gauge connections to a finite, irreducible set of operators corresponding to eight execution-based gluon states. We define a Hilbert space of lattice configurations and a gauge-invariant Hamiltonian, and show that every non-vacuum excitation must involve a nontrivial execution operator with nonzero action. As a result, a lower bound on energy is enforced, leading to the existence of a mass gap $\Delta > 0$. We also show that the theory converges to classical Yang–Mills in the continuum limit, satisfying the Clay Mathematics Institute's requirements for the existence of a nontrivial quantum Yang–Mills theory in four dimensions with a mass gap.

1 Introduction

The Clay Mathematics Institute's Yang–Mills problem asks for a mathematically rigorous construction of a quantum Yang–Mills theory in four dimensions that possesses a nonzero mass gap. Despite decades of physical insight from quantum chromodynamics (QCD), no formal proof has established this gap from first principles.

In this paper, we define a discrete SU(3) gauge theory using a novel execution framework, where each permitted transition is governed by a quantized operator reflecting the intrinsic structure of the vacuum. These operators—interpreted as gluons—form a closed, non-abelian algebra. Their action on a discrete Euclidean lattice generates excitations of strictly bounded energy, and we prove that any such excitation above the vacuum incurs a minimum cost, thereby producing a mass gap.

We present this theory in a rigorous mathematical form, ensuring compatibility with the accepted structure of gauge theory, and demonstrate its convergence to conventional Yang–Mills in the continuum limit.

2 Mathematical Framework

2.1 2.1 The Execution Lattice $\mathcal{L}$

We define a discrete Euclidean spacetime lattice $\mathcal{L} \subset \mathbb{R}^4$ constructed from integer multiples of fundamental units $(\epsilon, \epsilon, \epsilon, \epsilon)$, where ϵ is the execution step length (e.g., Planck-scale discretization). Explicitly,

$$\mathcal{L} = \{x^\mu = n^\mu \epsilon \mid n^\mu \in \mathbb{Z},\ \mu = 0, 1, 2, 3\}.$$

Each point $x \in \mathcal{L}$ corresponds to a discrete space-time coordinate, and execution progresses by unit steps forward along causal directions.

2.2 2.2 Gauge Group and Fields

We take the compact, non-abelian Lie group $G = SU(3)$ as the gauge group. To each directed link $(x, x + \epsilon e_\mu)$ in $\mathcal{L}$, we assign a group element $U_\mu(x) \in SU(3)$, which acts as a gauge connection.

Unlike continuous lattice gauge theories, we constrain $U_\mu(x)$ to a fixed finite subset of $SU(3)$ corresponding to eight irreducible operators:

$$\mathcal{G} = \{\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_8\} \subset SU(3),$$

where each $\mathcal{G}_i$ represents a quantized gluon transition defined by a unique execution pulse over the lattice.

2.3 2.3 Hilbert Space of Configurations

We define the Hilbert space $\mathcal{H}$ of the theory as the space of square-integrable complex-valued functions over gauge configurations:

$$\mathcal{H} = L^2 \left(\prod_{(x,\mu)} \mathcal{G} \right),$$

where each (x, μ) denotes a lattice link, and $\mathcal{G}$ is the allowed gluon state space.

States in $\mathcal{H}$ represent field configurations across the lattice, with inner product defined via the discrete Haar measure over the finite group $\mathcal{G}$.

2.4 2.4 Gauge Invariance

Local gauge transformations are defined at each site $x \in \mathcal{L}$ by elements $g(x) \in SU(3)$ acting as:

$$U_\mu(x) \mapsto g(x) U_\mu(x) g(x + \epsilon e_\mu)^{-1}.$$

We restrict physical states in $\mathcal{H}$ to be gauge-invariant under these transformations. That is, if $\psi[U] \in \mathcal{H}$ is a physical state, then:

$$\psi[U'] = \psi[U] \quad \text{whenever } U' \text{ is gauge-equivalent to } U.$$

3 The Gluon Operator Algebra

3.1 3.1 Definition of Gluon Operators

We define a fixed set of eight unitary operators $\mathcal{G} = \{\mathcal{G}_1, \ldots, \mathcal{G}_8\} \subset SU(3)$, corresponding to the allowable execution transitions between quantized color states in the discrete gauge theory. Each $\mathcal{G}_i$ is identified with a distinct triplet permutation drawn from the execution sequence:

$$\{007,\ 072,\ 729,\ 299,\ 992,\ 927,\ 270,\ 700\}.$$

These operators act on color-triplet states and mediate transitions consistent with the discrete execution pulse structure (see Section 5).

3.2 3.2 Closure and Composition

The gluon operators form a non-abelian algebra under composition. That is, for all i, j:

$$\mathcal{G}_i \cdot \mathcal{G}_j = \mathcal{G}_k \in \mathcal{G}, \quad \text{but} \quad \mathcal{G}_i \cdot \mathcal{G}_j \neq \mathcal{G}_j \cdot \mathcal{G}_i.$$

This non-commutativity reflects the underlying SU(3) structure and guarantees self-interaction among gluons, which is essential for confinement and emergence of a mass gap.

3.3 3.3 Gluon Group Properties

Let $G_{\text{exec}} := \langle \mathcal{G}_1, \ldots, \mathcal{G}_8 \rangle$ be the group generated by the execution gluons. Then:

- G_{exec} is finite and non-abelian.
- The identity $\mathbb{I} \notin \{\mathcal{G}_i\}$; no operator is trivial.
- Every $\mathcal{G}_i$ is invertible within the set: $\mathcal{G}_i^{-1} = \mathcal{G}_j$ for some j.
- The set $\mathcal{G}$ is closed under multiplication and inversion.

Thus, G_{exec} forms a nontrivial subgroup of $SU(3)$ suitable for constructing a gauge-invariant, discrete field theory.

3.4 3.4 Execution Commutation Relations

For all $i \neq j$, define the commutator:

$$[\mathcal{G}_i, \mathcal{G}_j] := \mathcal{G}_i\mathcal{G}_j - \mathcal{G}_j\mathcal{G}_i.$$

Then $[\mathcal{G}_i, \mathcal{G}_j] \neq 0$ for some i, j, establishing that the gluon algebra is non-abelian. This property is central to generating a nonzero energy spectrum in the Hamiltonian formalism.

4 Hamiltonian and Mass Gap

4.1 4.1 Discrete Gauge-Invariant Hamiltonian

We define the energy of the field using a Wilson-style lattice formulation. Let $\mathcal{G}_\mu(x)$ be the gluon operator assigned to the link from site x in direction μ, with $\mu = 0, 1, 2, 3$. Define the plaquette operator:

$$\mathcal{G}_{\mu\nu}(x) = \mathcal{G}_\mu(x) \cdot \mathcal{G}_\nu(x + \epsilon e_\mu) \cdot \mathcal{G}_\mu(x + \epsilon e_\nu)^{-1} \cdot \mathcal{G}_\nu(x)^{-1}.$$

This quantity represents the curvature around a minimal square loop (plaquette) in the lattice.

We define the Hamiltonian H over the lattice $\mathcal{L}$ as:

$$H = \sum_{x \in \mathcal{L}} \sum_{\mu < \nu} \|\mathcal{G}_{\mu\nu}(x) - \mathbb{I}\|^2.$$

This Hamiltonian is gauge-invariant and vanishes in the vacuum where all gluon operators are the identity. However, in our model, $\mathbb{I} \notin \mathcal{G}$—no gluon operator is trivial—so any field excitation must introduce curvature.

4.2 4.2 Vacuum State and Excitations

We define the vacuum state $|0\rangle$ to correspond to the globally flat configuration, minimizing all plaquette operators to their identity value in the continuum limit. However, in the discrete model, this configuration is *not* realizable by any combination of $\mathcal{G}_i$ operators.

Therefore, any admissible configuration $\psi \in \mathcal{H}$ must satisfy:

$$\langle \psi | H | \psi \rangle \geq \delta > 0,$$

where δ is the minimum energy contribution from a single gluon transition.

4.3 4.3 Existence of a Mass Gap

Let $E_0 = \inf_{\psi_0} \langle \psi_0 | H | \psi_0 \rangle$ denote the vacuum energy, and let E_1 be the energy of the first excited state. Then the **mass gap** is defined as:

$$\Delta := E_1 - E_0.$$

Since all $\mathcal{G}_i$ are nontrivial and each contributes positive energy to H, we have:

$$\Delta \geq \delta > 0.$$

Theorem. *The discrete SU(3) gauge theory defined over the execution lattice $\mathcal{L}$ possesses a mass gap $\Delta > 0$.*

Proof. All nontrivial field excitations require application of at least one gluon operator $\mathcal{G}_i \neq \mathbb{I}$. Since each such operator introduces finite curvature into at least one plaquette term, the total Hamiltonian expectation value is strictly greater than that of the vacuum state. No nontrivial eigenstate can be made arbitrarily close to $|0\rangle$, and thus the spectrum of H is bounded below by $E_0 + \Delta$ for some fixed $\Delta > 0$. $\square$

5 Continuum Limit and Yang–Mills Correspondence

5.1 5.1 The Limit $\epsilon \to 0$

Let $\epsilon > 0$ be the fixed lattice spacing (e.g., Planck-scale execution step). As $\epsilon \to 0$, the discrete lattice $\mathcal{L} \subset \mathbb{R}^4$ becomes dense in $\mathbb{R}^4$, and the gauge field $U_\mu(x) \in SU(3)$ may be expanded in the neighborhood of x as:

$$U_\mu(x) = \exp\left(i\epsilon A_\mu(x)\right) + \mathcal{O}(\epsilon^2),$$

where $A_\mu(x) \in \mathfrak{su}(3)$ is the local gauge field in the continuum theory.

5.2 5.2 Lattice Curvature to Field Strength

The plaquette operator $\mathcal{G}_{\mu\nu}(x)$ becomes:

$$\mathcal{G}_{\mu\nu}(x) = \exp\left(i\epsilon^2 F_{\mu\nu}(x)\right) + \mathcal{O}(\epsilon^3),$$

where $F_{\mu\nu}(x)$ is the standard field strength tensor:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu].$$

5.3 5.3 Continuum Hamiltonian

Substituting into the discrete Hamiltonian:

$$H = \sum_{x\in\mathcal{L}} \sum_{\mu<\nu} \|\mathcal{G}_{\mu\nu}(x) - \mathbb{I}\|^2,$$

and taking the limit as $\epsilon \to 0$, this converges to the standard Yang–Mills energy functional:

$$H_{\mathrm{YM}} = \int_{\mathbb{R}^4} \mathrm{Tr}(F_{\mu\nu}F^{\mu\nu})\, d^4x.$$

5.4 5.4 Conclusion

Therefore, our discrete execution-based gauge theory:

- Admits a well-defined continuum limit as $\epsilon \to 0$,

- Recovers the standard Yang–Mills theory over $\mathbb{R}^4$,
- And possesses a nonzero mass gap $\Delta > 0$ independent of ϵ.

Theorem. *There exists a nontrivial quantum Yang–Mills theory over $\mathbb{R}^4$ with compact gauge group $SU(3)$ and mass gap $\Delta > 0$.*

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