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The paper proposes an alternative general equilibrium formulation of financial asset economies with transactions costs. Transaction costs emerge endogenously at equilibrium and reflect agents decisions of intermediating financial activities at the expense of providing labor services. An equilibrium is shown to exist in the case of real asset structures.

Keywords: Competitive equilibrium, Incomplete markets, Endogenous transaction costs.

1. INTRODUCTION

Transactions costs such as, brokerage commissions, market impact costs[1] and transaction taxes are an inherent feature of modern financial markets. Researchers have questioned the impact of transaction costs on financial markets focusing particularly on the way these costs affect the trading behavior of market participants as well as on the role they have in determining equilibrium prices and trading volume in asset markets. Proponents of transactions costs argue that their presence discourage destabilizing speculation that can threaten high employment and price stability. For instance, imposing higher transacting rates to investors with short-term horizon may reduce the trading and influence of those investors on the market allowing investors with a long horizon to engage in investment strategies that promote research and development (see Stiglitz (1989)). In addition, the presence of transaction costs may diminish the activities of noise traders. Because the actions of those traders are not based on information about underlying values, their trades may drive prices away from their intrinsic value, reducing price informativeness while increasing volatility. Opponents of transaction costs argue for a careful design of transacting rates. High rates may affect price discovery, in the sense that can make prices less informative. They may also have negative effects on market liquidity[2] by causing competing, informed traders to scale back their trades (see Subrahmanyam (1998b) and Subrahmanyam (1998a)). Therefore, transactions costs may reduce overall market efficiency and cause a misallocation of resources. The magnitude of the aforementioned effects is still subject to considerable controversy and discussion. Solid empirical evidence is required to establish which effects dominate in the real world (see Jones and Seguin (1997), Habermeier and Kirilenko (2003), Hau (2006)).

Roughly speaking, theoretical models dealing with the implications of transaction costs in financial markets have their roots in two broad branches of economic literature: dynamic asset pricing theory and general equilibrium theory.

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[1]The market impact cost occurs because the transaction itself may change the market price of the asset. The difference between the transaction price and what the market price would have been in the absence of the transaction is termed the market impact of the transaction. The market impact is a price-per-share amount. Multiplying the market impact by the number of shares traded gives the market impact cost of the transaction.

[2]Market liquidity corresponds to the ease with which market participants can transact, or the ability of markets to absorb large purchases or sales without much effect on prices.

Contributions on asset pricing under transaction costs include partial equilibrium models of optimal portfolio design (see Magill and Constantinides (1976), Constantinides (1986), Duffie and Sun (1990)) as well as general equilibrium models of endogenous price formation (see Vayanos (1998), Lo, Mamysky, and Wang (2004)). Transaction costs have also been proposed (see Heaton and Lucas (1986), Aiyagari and Gertler (1991), Aiyagari (1993)) as an explanation to various asset pricing puzzles that emerge in empirical literature (e.g. the equity premium puzzle, Mehra and Prescott (1985)).

Transaction costs were originally introduced in the standard general equilibrium model of Arrow and Debreu (1954) in order to explain the sequential opening of commodity markets.[3] The traditional approach to transaction costs assumes that real resources are used in the process of transaction. Early studies visualize market as a profit maximizing intermediary who uses real resources to transform sold goods into bought goods according to an exogenously specified technology. Hahn (1971) and Hahn (1973) refer to such a technology as the transaction technology for the economy. Kurz (1974a) and Kurz (1974b) push the theory further by allowing traders to have their own exchange technology. Shefrin (1981) introduced uncertainty to account for the possibility of inactive future markets.

Further developments encompass economies with incomplete financial markets and transaction costs. Repullo (1987) shows that the set of equilibria in a Radner (1972) economy with incomplete markets coincides with the set of equilibria of a Kurz (1974a) economy where agents have a special kind of transaction technology.[4] Laitenberger (1996) considers a model where transaction costs take the form of real resources that are used to enforce financial contracts. From the angle of a buyer transaction costs induce a reduce real claim on assets while from the angle of a seller they induce an increased real obligation. In this setting transactions costs are incurred at the second period and their level is exogenously specified. In a recent paper Arrow and Hahn (1999) have shown that in sequential economies with transactions costs equilibria are always constrained Pareto-inefficient even in cases where security-payoffs span the set of all states of nature.[5] Jin and Milne (1999) (see also Milne and Neave (2003)) propose a model where financial activities are intermediated by some brokers/intermediaries that trade assets between buyers and sellers using a costly technology. The intermediaries act competitively taking buying and selling prices as given. They also are allowed to hold portfolios and trade on their own account. At equilibrium their revenues are eventually redistributed back to consumers. [6]

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[3]In the classical multi-period Arrow-Debreu economy, agents' trading opportunities are not affected by opening markets at futures dates.

[4]Each agent has a transaction technology that assigns zero cost if at some date-event the markets are open, and infinite otherwise.

[5]Herings and Schmedders (2006) provide an algorithm for the computation of equilibria in an particular economy with one good (wealth).

[6]There are two issues that arise in the framework proposed by Jin and Milne (1999). First, in an environment where the outcome of production activity is uncertain and financial markets are incomplete, treating intermediaries as privately-owned firms is difficult to justify since there may be disagreement

An alternative to real costs approach argues for commission fees derived from the monopolistic power of a privately owned brokerage house. In this setting the transacting cost is interpreted as an intermediation cost since it only implies a transfer of income across individuals (no real resources are used in the process of transaction). Préchac (1996) considers a model where a privately owned brokerage house acts as an intermediary charging an exogenous commission fee on all selling and buying orders. The commissions are subsequently redistributed to consumers according to their equity shares. In Préchac (1996) the commission fee is proportional to the value of trade. This may be problematic when assets have non-positive returns.[7] Markeprand (2008) allows for arbitrary asset returns and for a general intermediation cost function that takes into account the volume of trade. Both Préchac (1996) and Markeprand (2008) deal with real asset markets. In both models intermediation costs serve to bound endogenously portfolios, ruling out situations where equilibria fail to exist (see Hart (1975)). Duffie and Shafer (1985) and Duffie and Shafer (1986) showed that Hart's example is an exception and that the existence of Radner equilibrium is assured generically. However, subsequent contributions showed that the generic existence argument is no more valid in cases where asset dividends are not linear with respect to prices (see Polemarchakis and Ku (1990)) or preferences are not strictly convex (see Busch and Govindan (2004)).

Préchac (1996) and Markeprand (2008) don't provide any rationale for the level of intermediation costs. The bid-ask spread in their models is specified outside the functioning of the economy. Moreover, there is no market for equity shares and their initial distribution is exogenous. In an interesting paper Bisin (1998) addresses these shortcoming. He considers a model where intermediaries have a monopoly power in a fix number of security markets. Intermediaries act as a monopolist choosing optimally the securities' payoffs[8] and the bid-ask spread they charge. Intermediation involves fixed costs as well as costs that are proportional to trading volume. Equilibrium prices and asset demands are rationally anticipated by intermediaries when evaluating their profits. In this setting, strategic interaction (Bertrand competition) among firms leads to endogenous bid-ask spreads.

The aim of this paper is to propose an alternative general equilibrium formulation of financial asset economies with transactions costs. As in Bisin (1998) we believe that a model with transaction costs should provide an economic rationale for the emergence of these costs. Bisin (1998) proposed an explanation based on Bertrand competition among monopolists. We follow another route and explain how transaction costs can emerge endogenously at equilibrium in a competitive market of

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among the share holders on the objective assigned to the firm. Second, assets pay in unspecified units of account (i.e. the asset structure is nominal). This in turn raises a number of serious questions. What is the role of these units of account? Are they related to a system of monetary equations that specify the amount of money that is available for transactions? If this is the case, then money is not neutral and can affect price levels at equilibrium. In sequential markets the assumption of perfect foresight amounts to assuming that agents correctly anticipate current and future monetary policy. One can refer to Magill and Quinzii (1996) for a thorough discussion on both issues.

[7]In Préchac (1996) returns are non-negative.

[8]Payoffs are assumed to be nominal.

brokers.[9]

Consistent with the traditional approach of modeling transaction costs, we argue for real transaction costs. We feel, however, that in its early formulation the real cost approach is not suited to the study of financial markets. On the other side it is unreasonable to neglect any form of real transaction costs as Préchac (1996) and Markeprand (2008) do. Although some physical inputs are burnt in the transaction process,[10] empirical evidence reports that the financial sector is more labor intensive. Indeed, when comparing the labor cost figures for financial intermediation with those for the whole economy it is clear that the former are in many cases higher than the latter (see Carley (2002)). In addition, the financial sector in developed economies exhibits higher wage differentials with respect to other industry-specific sectors (see Genre, Momferatou, and Mourre (2005)). It is therefore reasonable to think that an important part of the costs generated in modern financial markets are associated with the provision of labor services required to intermediate the financial activities. From this perspective real transaction costs can be viewed as the individual opportunity cost of labor or effort.

We formulate a two-period financial asset economy with endogenous transaction costs that reflect agents decisions of intermediating financial activities at the expense of providing labor services. In that respect our formulation is close to Kurz (1974a) and Kurz (1974b) in the sense that transactions are directly employed by consumers.

There is a fixed number of real assets available for trade. Each agent is characterized by a consumption and investment set as well as by an abstract set representing his available labor or effort. Agents' actions involve three activities. The exchange of commodity goods, the purchase or sale of assets and the intermediation of financial orders. The demand for commodities is determined by the prices of goods. Investment decisions are driven by security prices and the commission fee per unit of transactions. The choice of supplying labor to intermediate transactions is determined by the price of labor (salaries). The provision of labor services implies a revenue that agents can dispose in the consumption of goods but simultaneously it incurs a loss in utility due to the reduction in leisure. Consumers can intermediate in any security market and can act simultaneously as investors and intermediaries.

We assume that there are no barriers to entry in intermediating financial activities. The entry costs are low enough such that any agent can be a potential broker.[11]

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[9]The empirical assessments on the competitiveness of financial sector in specific economies is mixed. Shaffer (1989) and Shaffer (1993) present results that strongly reject collusive conduct and support perfect competition in U.S. and Canada while Shaffer (2001) reports that European banks appear to suffer from a measurable but limited degree of banking market power. Evidence from Battelino (2008) suggests that the Australian financial sector is competitive. Bandt and Davis (2000) find that the U.S. banking sector is highly competitive whereas in some European countries like Germany and France the banking system is characterized by monopolistic competition.

[10]For instance, the use of capital goods like computers, the rent of infrastructures, etc.

[11]Modeling explicitly entry costs leads to non-convex action sets. This is because the consumers' maximization problem involves a choice variable that takes discrete values: one value represents the decision to pay the entry cost and be a broker while a second value represents the decision not to be

Therefore, instead of considering that there is a finite set of brokerage houses with monopoly power, we assume that brokers act competitively in the sense that there is a competitive market for their labor services. In this setting, the price of labor coincides with the commission fee at equilibrium. This in turn implies that the commission fee is endogenously determined, reflecting agents' productivity of intermediating transactions and their intratemporal preferences for labor and leisure.

The paper is structure as follows. In Section 2 we present the model, introduce notation and assumptions and define agents' objectives. In section 3 we present the concept of the competitive equilibrium, highlight its properties and prove its existence. Section 4 is technical in nature and is devoted to prove an intermediate result (i.e. existence of equilibria with bounded action sets) that is used in the proof of our main existence theorem. Section 5 relates our work with the literature on exogenous intermediation costs and in particular with the models of Préchac (1996) and Markeprand (2008). We argue that both models are incomplete since they do not provide any rational for the level of commission fees. An attempt to make the choice of commission fees endogenous in Préchac (1996) raises serious difficulties that stem from the fact that profits depend on the equilibrium outcome which in turn depends on the level of the chosen commission fees. The problem becomes more serious in the setting proposed by Markeprand (2008). The specification of his cost function induces a budget restriction at $t = 0$ that is no longer homogeneous of degree zero with respect to prices. This may raise questions about the normalization of prices that already appear in the literature of financial markets with nominal assets while simultaneously may jeopardize the existence of equilibrium.

2. THE MODEL

We consider a pure exchange economy that extends over two periods $t \in \{0, 1\}$. There is exogenous uncertainty about consumers' characteristics at $t = 1$ represented by a finite set S of events. Markets open sequentially. The economy consists of a finite set I of agents, indexed by $i \in I$. Each consumer i has unlimited abilities to form expectations and thus can perfectly forecast endogenous macroeconomic variables. At every period t there is a finite set L_t of commodities available for trade. Let $X_t^i \subset X_t \equiv \mathbb{R}_+^{L_t}$ denote agent i's set of commodity bundles available for consumption and $P_t \equiv \mathbb{R}_+^{L_t}$ denote the set of commodity prices at period t.

At the first period $t = 0$ consumers can trade commodities and a finite set J of financial assets. The financial structure is assumed to be exogenous. The payoff of asset j in state s, denominated in units of account, is given by $V_j(p_1(s), s)$ where $p_1(s)$ is the commodity price prevailing in that state. In order to avoid issues related to monetary policies, we restrict our attention to *real assets*.

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a broker. In such a setting the salary should be such that the entry costs of those who decided to be brokers have to be compensated by the utility of the goods they can consume using the revenue from the commission fees. One could deal with non-convexities by introducing a setting with a continuum of agents. While such a formulation is interesting, tackling the existence problem becomes technically more complex. We postpone such an attempt in a future paper.

ASSUMPTION 2.1   The payoffs of each asset j are real in the sense that for each state s, the function $V_j(\cdot,s) : P_1 \to \mathbb{R}$ is homogenous of degree 1, i.e.,

$$\forall \lambda \in \mathbb{R}_+, \quad \forall p_1(s) \in P_1, \quad V_j(\lambda p_1(s),s) = \lambda V_j(p_1(s),s).$$

Moreover, we assume that the function $V_j(\cdot,s) : P_1 \to \mathbb{R}$ is continuous.

REMARK 2.1   In contrast with Préchac (1996) we don't impose that asset payoffs are non-negative. As in Markeprand (2008) we **relax** the assumption of linear dependent payoff-functions to allow for general assets like (real) options[12] and futures. However, we rule out nominal assets.

At $t = 0$, each agent chooses to purchase an amount $\theta_j \geqslant 0$ and to sale an amount $\varphi_j \geqslant 0$ of asset j. Let $\Theta^i \subset \Theta \equiv \mathbb{R}_+^J$ denote the space of available purchases and $\Phi^i \subset \Phi \equiv \mathbb{R}_+^J$ the space of available sales. We let $Q = \mathbb{R}_+^J$ be the space of asset prices at $t = 0$.

ASSUMPTION 2.2   For each i, the sets Θ^i and Φ^i are closed and convex. Moreover, every agent i is allowed to sell or purchase a small amount of each asset, i.e.,

$$\exists v \in \mathbb{R}_{++}^J, \quad [0,v] \subset \Theta^i \cap \Phi^i.$$

We assume that markets for consumption are frictionless: there are no transaction costs. However, purchasing or selling assets requires the intermediation of financial brokers. Every agent i is a potential broker. He is characterized by an abstract set Y^i representing potential labor or effort and a function $f^i : Y^i \to \mathbb{R}_+^J$ representing his *production function*. If agent i chooses a level of effort y, then he can intermediate a volume $f_j^i(y)$ of financial transactions for each asset j. We assume that agent i's ability to intermediate transactions is asset-dependent but independent of the type of transactions: buying or selling.

Each agent i has a utility/disutility function $U_0^i : X_0 \times Y^i \to \mathbb{R}$ for consumption and labor at $t = 0$ and an initial endowment of goods $e_0^i \in X_0$. For each possible realization of exogenous uncertainty s at $t = 1$, agent i has a utility function $U_1^i(\cdot,s) : X_1 \to \mathbb{R}$ for consumption at $t = 1$ and an initial endowment $e_1^i(s) \in X_1$. At $t = 0$ agent i discounts future consumption with a discount factor $\beta^i > 0$ and has beliefs about exogenous uncertainty represented by a probability $\nu^i \in \mathrm{Prob}(S)$.

ASSUMPTION 2.3   Every agent i considers that every state of nature s is probable, i.e.,

$$\forall s \in S, \quad \nu^i(s) > 0.$$

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[12]Consider that there is an asset j which dividend in state s is the market value of a bundle $A_j(s) \in \mathbb{R}_+^{L_1}$. Our framework allows for a call option on this asset with strike the market value of a fixed bundle $\xi \in \mathbb{R}_+^{L_1}$. The dividend of this call option in state s is given by $[p_1(s) \cdot A_j(s) - p_1(s) \cdot \xi]^+$ where $p_1(s)$ is the vector of commodity market prices in state s.

Purchasing or selling assets induces a commission fee that is paid to those agents who act as intermediaries (brokers). The decision of participating in the intermediation process involves the devotion of working time. For a fixed volume of financial transactions, different agents may need a different level of labor to intermediate this volume. We assume that there are no barriers to entry in intermediating financial activities. The entry costs are low enough such that any agent can be a potential broker. At equilibrium the wage (the price of labor) received by brokers for the provision of their labor services will coincide with the commission fee paid by investors. Therefore, in our framework the commission fee is endogenously determined, reflecting agents' productivity of intermediating transactions and their intratemporal preferences for labor and leisure. We denote by $\kappa_j \in \mathbb{R}$ the salary paid to each broker for intermediating a unit of asset j. Given the previous discussion, κ_j is also the transaction cost paid by investors per unit of asset j's purchase or sale.

It is assumed that the transacting costs are proportional to the volume of transactions a consumer/broker can intermediate. We allow for the possibility of an agent to be simultaneously an investor (purchasing or selling an asset j) and a broker (intermediating exchanges of asset j).

When agent i chooses at $t = 0$ to consume x_0^i and supplies the labor/effort y^i, he gets the payoff $U_0^i(x_0^i, y^i)$ and intermediates the vector $z^i = f^i(y^j)$ of asset transactions. We propose to represent agent i's action by the couple (x_0^i, z^i) of consumption/intermediation instead of the couple (x_0^i, y^i) of consumption/effort. We denote by Z^i the set of transactions that agent i can intermediate, i.e.,

$$Z^i \equiv f^i(Y^i) \subset Z \equiv \mathbb{R}_+^J.$$

ASSUMPTION 2.4   The production set Z^i is a closed convex subset of $\mathbb{R}_+^J$. Moreover, inaction is possible, i.e., $0 \in Z^i$ and every agent is able to make an effort sufficient to produce a non-negative amount of transactions for each asset, i.e.,

$$\exists v \in \mathbb{R}_{++}^J, \quad [0, v] \subset Z^i.$$

We subsequently define the payoff that agent i obtains when he chooses an action (x_0^i, z^i) in the feasible set $X_0^i \times Z^i$. The natural definition of the payoff $\Pi_0^i(x_0^i, z^i)$ is the following:

$$\Pi_0^i(x_0^i, z^i) \equiv \sup\{U_0^i(x_0^i, y) : \ y \in Y^i \quad \text{and} \quad f^i(y) = z^i\}.$$

REMARK 2.2   If there is a topological structure on the effort space Y^i such that Y^i is compact and the production function f^i is continuous, then the continuity of the (indirect) payoff Π_0^i follows from the continuity of U_0^i.

In what follows we abstract from specific assumptions on the space Y^i and consider that agent i is characterized by a function $\Pi_0^i : X_0 \times Z \to [-\infty, +\infty)$ where $Z \equiv \mathbb{R}_+^J$. We make the following assumptions on preferences and endowments.

ASSUMPTION 2.5   For each agent i,
  (a) initial endowments in commodities are strictly positive, i.e.,

$$e_0^i \in \mathrm{int} X_0^i \subset \mathbb{R}_{++}^{L_0} \quad \text{and} \quad \forall s \in S, \quad e_1^i(s) \in \mathrm{int} X_1^i \subset \mathbb{R}_{++}^{L_1};$$

  (b) the payoff function $\Pi_0^i : X_0 \times Z \to [-\infty, \infty)$ is continuous, concave, strictly increasing in consumption x_0 and strictly decreasing in production z;
  (c) the utility function $U_1^i(\cdot, s)$ is continuous, concave and strictly increasing.

Consider a consumption plan $x^i = (x_0^i, x_1^i) \in X^i$ and a production vector $z^i \in Z^i$. The expected discounted payoff of the action $a^i = (x_0^i, z^i, x_1^i)$ is defined by

$$\Pi^i(a^i) = \Pi_0^i(x_0^i, z^i) + \beta^i \sum_{s \in S} v^i(s) U_1^i(s, x^i(s)).$$

Consider now another production plan $\widetilde{z}^i$ such that

$$\widetilde{z}^i > z^i.$$

We assume that higher levels of production give rise to lower expected payoffs, that is we assume that

$$\Pi^i(x_0^i, \widetilde{z}^i, x_1^i) < \Pi^i(x_0^i, z^i, x_1^i).$$

One may consider the following representation of the payoff Π_0^i:

$$\forall (x_0, z) \in X_0 \times Z, \quad \Pi_0^i(x_0, z) = U_0^i(x_0) - E^i(z)$$

where $U_0^i : X_0 \to [-\infty, \infty)$ is continuous, concave and strictly increasing and the function $E^i : Z \to [0, \infty)$ is continuous, convex, strictly increasing and satisfies $E^i(0) = 0$. While the set Z^i represents the production capacities of agent i, the amount $E^i(z)$ represents the negative impact in terms of utility of the effort to intermediate the vector z of transactions.

REMARK 2.3   Observe that the payoff function Π_0^i is defined on the whole space

$$X_0 \times Z \equiv \mathbb{R}_+^{L_0} \times \mathbb{R}_+^{J}$$

and the contingent utility function $U_1^i(s, \cdot)$ is also defined on the whole space

$$X_1 = \mathbb{R}_+^{L_1}$$

while agent i's actions may be restricted to strict subsets

$$X_0^i \subset X_0, \quad Z^i \subset Z \quad \text{and} \quad X_1^i \subset X_1.$$

These restrictions may be problematic if agents are satiated on some feasible consumption plans. To avoid this problem, we make the following non-satiation assumption.

ASSUMPTION 2.6 The restrictions on each agent's consumption sets do not prevent them to purchase the aggregate endowment, i.e.,

$$\forall i \in I, \quad e_0 \equiv \sum_{k\in I} e_0^k \in \operatorname{int} X_0^i \quad \text{and} \quad e_1(s) \equiv \sum_{k\in I} e_1^k(s) \in \operatorname{int} X_1^i, \quad \forall s \in S.$$

We denote by A_0^i the set of actions at $t=0$,

$$A_0^i \equiv X_0^i \times \Theta^i \times \Phi^i \times Z^i$$

and we denote by A_1^i the set of perfectly anticipated actions for $t=1$, i.e.,

$$A_1^i = [X_1^i]^S.$$

The set $A^i = A_0^i \times A_1^i$ represents the set of intertemporal actions available for agent i.

3. COMPETITIVE EQUILIBRIUM

We assume that markets are competitive and agents are price-takers. At $t=0$, given

- a family

$$(p_0, q, \kappa)$$

  of commodity prices $p_0 \in P_0$, asset prices $q \in Q$ and transaction costs/salaries $\kappa \in \mathbb{R}_+^J$;
- a family of perfectly anticipated future prices

$$\boldsymbol{p}_1 = (p_1(s))_{s\in S}$$

each agent i chooses an action

$$a^i = (a_0^i, \boldsymbol{x}_1^i)$$

where

$$a_0^i = (x_0^i, \theta^i, \varphi^i, z^i) \in A_0^i$$

is the vector of actions at $t=0$ and

$$\boldsymbol{x}_1^i = (x_s^i)_{s\in S} \in A_1^i$$

is the vector of perfectly anticipated consumption bundles, such that

(3.1) $\quad p_0 \cdot x_0^i + (q+\kappa)\cdot\theta^i \leqslant p_0\cdot e_0^i + (q-\kappa)\cdot\varphi^i + \kappa\cdot z^i$

and for each $s\in S$,

(3.2) $\quad p_1(s)\cdot x_1^i(s) + V(p_1(s),s)\cdot\varphi^i \leqslant p_1(s)\cdot e_1^i(s) + V(p_1(s),s)\cdot\theta^i.$

The set of all actions satisfying the budget restrictions (3.1) and (3.2) is called the budget set and denoted by

$$B^i(p, q, \kappa).$$

We recall that the (discounted and expected) payoff of an action $a^i = (a_0^i, x_1^i)$ is defined by

$$\Pi^i(a^i) = \Pi_0^i(x_0^i, z^i) + \beta^i \sum_{s \in S} \nu^i(s) U_1^i(s, x^i(s)).$$

Definition 3.1 A competitive equilibrium of an economy

$$\mathscr{E} \equiv \left\{ X^i, \Theta^i, \Phi^i, Z^i \right\}_{i \in I}$$

is a family

$$\left\{ (p, q, \kappa), (a^i)_{i \in I} \right\}$$

composed of prices (p, q, κ) and an allocation $(a^i)_{i \in I}$ of intertemporal actions

$$a^i = (a_0^i, a_1^i) \quad \text{with} \quad a_0^i = (x_0^i, \theta^i, \varphi^i, z^i) \quad \text{and} \quad a_1^i = (x_1^i(s))_{s \in S}$$

such that
  (a) actions are optimal, i.e.,

$$(3.3) \qquad \forall i \in I, \quad a^i \in \operatorname{argmax}\{\Pi^i(a) : a \in B^i(p, q, \kappa)\}$$

  (b) commodity markets clear, i.e.,

$$(3.4) \qquad \sum_{i \in I} x_0^i = \sum_{i \in I} e_0^i \quad \text{and} \quad \sum_{i \in I} x_1^i(s) = \sum_{i \in I} e_1^i(s)$$

  (c) asset markets clear, i.e.,

$$(3.5) \qquad \forall j \in J, \quad \sum_{i \in I} \theta_j^i = \sum_{i \in I} \varphi_j^i$$

  (d) transactions are feasible, i.e.,

$$(3.6) \qquad \sum_{i \in I} \varphi^i + \theta^i = \sum_{i \in I} z^i.$$

3.1. *Equilibrium properties*

Before presenting general conditions that ensure existence of a competitive equilibrium, we propose to underly some of its properties.

DEFINITION 3.2 The asset structure is said *non-degenerate and positive* if for every possible vector of expected strictly positive prices $(p_1(s))_{s \in S}$[13] and for every asset j, payoffs are non-negative

$$\forall s \in S, \quad V_j(p_1(s), s) \geqslant 0$$

and non-degenerate

$$\exists s_j \in S, \quad V_j(p_1(s_j), s_j) > 0.$$

REMARK 3.1 If each asset j promises to deliver the units of account corresponding to the market value of a bundle $A_j(s)$ contingent to state s, then the asset structure is non-degenerate and positive if $A_j(s)$ belongs to $\mathbb{R}_+^{L_1}$ and for at least one state s_j the promise $A_j(s_j)$ is not zero.

Consider a competitive equilibrium

$$\left\{ (p, q, \kappa), (a^i)_{i \in I} \right\}.$$

A direct consequence of Assumptions (2.5) and (2.6) is that commodity prices are strictly positive, i.e.,

$$p_0 \in \mathbb{R}_{++}^{L_0} \quad \text{and} \quad p_1(s) \in \mathbb{R}_{++}^{L_1}, \quad \forall s \in S.$$

Assume that the asset structure is non-degenerate and positive. If there are no restrictions on portfolios, i.e.,

$$\forall i \in I, \quad \Theta^i = \Theta \quad \text{and} \quad \Phi^i = \Phi$$

then the vector of asset prices must be strictly positive, i.e.,

$$q \in \mathbb{R}_{++}^J.$$

In that case we can reinterpret our model by considering that transaction costs and salaries are proportional to transaction *values* instead of volumes. Indeed, let $c = (c_j)_{j \in J}$ be defined by

$$\forall j \in J, \quad c_j = \frac{\kappa_j}{q_j}.$$

The number c_j represents the transaction cost paid by investors per unit of account invested or borrowed on asset j. The budget set at $t = 0$ can be rewritten in the following way:

$$(3.7) \qquad p_0 \cdot x_0 + \sum_{j \in J} (1 + c_j) q_j \theta_j \leqslant p_0 \cdot e_0^i + \sum_{j \in J} (1 - c_j) q_j \varphi_j + \sum_{j \in J} c_j q_j z_j^i.$$

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[13]In the sense that $p_1(s) \in \mathbb{R}_{++}^{L_1}$.

This budget set is similar to the one proposed by Préchac (1996). Two main differences are in order. In Préchac (1996) the cost c_j is an exogenous parameter and the "reward" of each agent i is a fixed share on the profits of a monopolistic brokerage house. We will highlight these differences in Section 5.

Observe that if $c_j > 1$ in the budget restriction (3.7) then there is a problem.[14] An interesting property of our equilibrium is that, provided that there is trade in asset j, the cost c_j is endogenously determined at a level strictly lower than 1. Indeed, assume that there is trade in the market of asset j, i.e.,

$$\sum_{i \in I} \theta_j^i + \varphi_j^i > 0.$$

The transaction cost κ_j must necessarily be strictly lower than the asset price q_j. By way of contradiction, assume that it is not the case, i.e., $q_j \leqslant \kappa_j$. There exists at least one agent k that is selling an amount $\varphi_j^k > 0$ of that asset. By doing so, he is paying at $t = 0$ the amount $\kappa_j - q_j$ in exchange of the obligation to deliver non-negative amounts in every state and a strictly positive amount in state s_j.[15] Agent k would be better-off canceling his sales and replacing them by strictly positive consumption at least in state s_j. This contradicts the optimality of action a^k. The above reasoning illustrates the importance of allowing the transaction costs to be determined endogenously and not to be fixed exogenously as in Préchac (1996) and Markeprand (2008). We come back on this issue in Section 5. In particular, we argue that the lack of a rationale for specifying the level of transaction costs in the models of Préchac (1996) and Markeprand (2008) raises a number of serious questions.

DEFINITION 3.3   An economy

$$\mathscr{E} = \left\{ X^i, \Theta^i, \Phi^i, Z^i \right\}_{i \in I}$$

is said *particular* if
- there is only one good at $t = 0$ and in every state s at $t = 1$;
- for each agent i,
    - there are no restrictions, i.e.,

    $$X_0^i = X_0, \quad X_1^i = X_1, \quad \Theta^i = \Theta, \quad \Phi^i = \Phi \quad \text{and} \quad Z^i = Z;$$

    - the payoff function Π_0^i is decomposed as follows:

    $$\forall (x_0, z) \in X_0 \times Z, \quad \Pi_0^i(x_0, z) = U_0^i(x_0) - E^i(z);$$

---

[14]In Préchac (1996) it is not explicitly assumed that the exogenous cost parameter should satisfy $c_j \leqslant 1$.

[15]Remember that in state s_j we have $V_j(p_1(s_j), s_j) > 0$.

- the function U_0^i is continuous, strictly increasing and strictly concave on $[0, \infty)$, differentiable on $(0, \infty)$, **and** satisfies an Inada's condition at $x_0 = 0$, i.e.,

$$\lim_{h \to 0^+} \frac{U_0^i(h) - U_0^i(0)}{h} = \infty;$$

- for each asset j, there exists a function $z_j \mapsto E_j^i(z_j)$ differentiable, strictly increasing and strictly convex on $[0, \infty)$ such that

$$\forall z \in Z, \quad E^i(z) = \sum_{j \in J} E_j^i(z_j).$$

The term $E_j^i(z_j)$ represents the loss in terms of utility due to the effort required to intermediate a volume z_j of asset j.

Assume that the economy is particular. Let (π, a) be a competitive equilibrium with

$$\pi = (p, q, \kappa) \quad \text{and} \quad a^i = (a_0^i, a_1^i) \quad \text{where} \quad a_0^i = (x_0^i, \theta^i, \varphi^i, z^i).$$

Since the function $x_0 \mapsto \Pi_0^i(x_0, z^i)$ is strictly increasing, we can assume without any loss of generality that $p_0 = 1$. Fix an asset j for which there exists trade. Then, there exists at least one agent i such that $z_j^i > 0$. From the Inada's condition we have $x_0^i > 0$. It follows from the first order conditions that[16]

$$\kappa_j = \frac{\nabla E_j^i(z_j^i)}{\nabla U_0^i(x_0^i)}.$$

At equilibrium, the salary for intermediation of one unit of asset j equals the marginal rate of substitution between the disutility of effort and the utility of consumption.

When there are no transaction costs, existence of equilibrium is not ensured. The non-existence arises form the discontinuity of demand for assets when commodity prices converge to prices for which the payoff matrix drops in rank (see Hart (1975)). We provide a framework where transaction costs may imply the existence of an ex-ante bound on individually rational and physically feasible portfolios. In what follows we discuss two cases that imply such a bound. First, we consider the case where the bound is a consequence of limited intermediation possibilities. This case is close to the model proposed by Laitenberger (1996) where the bound comes from the scarcity of commodities. However, the level of transaction costs in Laitenberger (1996) is specified exogenously. The second case has no counterpart in the literature and replace limited intermediation by making the bound to be a consequence of a strong disutility for effort.

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[16]If $f : [0, \infty) \to \mathbb{R}$ is a differentiable function, the differential of f is denoted by ∇f.

3.2. *Limited intermediation*

An allocation $\boldsymbol{a} = (a^i)_{i \in I}$ where

$$a^i = (a_0^i, a_1^i) \quad \text{with} \quad a_0^i = (x_0^i, \theta^i, \varphi^i, z^i) \quad \text{and} \quad a_1^i = (x_1^i(s))_{s \in S}$$

is said physically feasible if

- commodity markets clear, i.e.,

$$(3.8) \qquad \sum_{i \in I} x_0^i = \sum_{i \in I} e_0^i \quad \text{and} \quad \sum_{i \in I} x_1^i(s) = \sum_{i \in I} e_1^i(s)$$

- asset markets clear, i.e.,

$$(3.9) \qquad \forall j \in J, \quad \sum_{i \in I} \theta_j^i = \sum_{i \in I} \varphi_j^i$$

- transactions are feasible, i.e.,

$$(3.10) \qquad \sum_{i \in I} \varphi^i + \theta^i = \sum_{i \in I} z^i.$$

The set of physically feasible actions is denoted by

$$F((X^i, \Theta^i, \Phi^i, Z^i)_{i \in I}).$$

It is trivial to exhibit an exogenous upper-bound on the consumption allocations that are physically feasible since

$$F((X^i, \Theta^i, \Phi^i, Z^i)_{i \in I}) \subset \prod_{i \in I} \left\{ [0, e_0] \times \Theta^i \times \Phi^i \times Z^i \times \prod_{s \in S} [0, e_1(s)] \right\}.$$

In particular, Assumption 2.6 implies that each agent satisfies a non-satiation assumption at each feasible consumption plan.

DEFINITION 3.4   An economy

$$\mathscr{E} \equiv \left\{ X^i, \Theta^i, \Phi^i, Z^i \right\}_{i \in I}$$

is said to have *limited intermediation* if the production set Z^i of each agent i is bounded.

If the production possibilities of each agent is bounded then it is possible to exhibit an exogenous bound on portfolio allocations that are physically feasible.

PROPOSITION 3.1   If an economy has limited intermediation then the set of physically feasible allocations is bounded.

PROOF OF PROPOSITION 3.1: Consider an economy

$$\mathscr{E} \equiv \left\{ X^i, \Theta^i, \Phi^i, Z^i \right\}_{i \in I}$$

with limited intermediation. It follows that

$$(3.11) \qquad \forall i \in I, \quad \exists \bar{z}^i \in \mathbb{R}_+^J, \quad Z^i \subset [0, \bar{z}^i].$$

As a consequence we obtain that

$$\mathrm{F}((X^i, \Theta^i, \Phi^i, Z^i)_{i \in I})$$

is a subset of the following bounded set

$$\prod_{i \in I} \left\{ [0, e_0] \times [0, \bar{z}] \times [0, \bar{z}] \times [0, \bar{z}^i] \times \prod_{s \in S} [0, e_1(s)] \right\}.$$

where

$$\bar{z} \equiv \sum_{i \in I} \bar{z}^i$$

is the maximum level of transactions that can be implemented by the labor market. *Q.E.D.*

There is another natural situation where an exogenous bound on portfolios can be exhibited.

3.3. *Effort versus consumption*

We recall that an allocation $a = (a^i)_{i \in I}$ is individually rational if

$$\forall i \in I, \quad \Pi^i(a^i) \geqslant \Pi^i(e_0^i, 0, e_1^i).$$

If agent i does not participate to any market, neither consumption nor labor market, he gets the payoff $\Pi^i(e_0^i, 0, e_1^i)$. The set of individually rational and physically feasible allocations is denoted by

$$\mathrm{Ir\text{-}F}((X^i, \Theta^i, \Phi^i, Z^i)_{i \in I}).$$

We propose to replace the assumption that the volume of intermediation of each agent is limited by the assumption that agents need a huge effort to intermediate large amounts of financial activities.

DEFINITION 3.5 Consider an economy

$$\mathscr{E} \equiv \left\{ X^i, \Theta^i, \Phi^i, Z^i \right\}_{i \in I}.$$

An agent i is said to exhibit *strong disutility for effort* if there is no physically feasible consumption that can compensate the effort for intermediating an arbitrarily large volume of transactions, i.e.,[17]

$$(3.12) \qquad \liminf_{\|z\| \to \infty} \Pi^i(e_0, z, e_1) < \Pi^i(e_0^i, 0, e_1^i).$$

PROPOSITION 3.2   If every agent in an economy exhibits strong disutility for effort then the set of individually rational and physically feasible allocations is bounded.

PROOF OF PROPOSITION 3.2:   Assume by contradiction that the set of individually rational and physically feasible allocations is not bounded. Then there exists an unbounded sequence of allocation $(a_n)_n$ satisfying[18]

$$\forall n \in \mathbb{N}, \quad a_n \in \text{Ir-F}((X^i, \Theta^i, \Phi^i, Z^i)_{i \in I}).$$

Each consumption allocation $x_n = (x_n^i)_{i \in I}$ is physically feasible and in particular we have

$$(3.13) \qquad \forall i \in I, \quad 0 \leqslant x_n^i \leqslant e$$

where $e = (e_0, e_1)$ is the aggregate intertemporal endowment. By market clearing of the labor markets, we have

$$\forall n \in \mathbb{N}, \quad \sum_{i \in I} z_n^i = \sum_{i \in I} \theta_n^i + \varphi_n^i.$$

Since the sequence $(a_n)_n$ is unbounded, we must have that the sequence of production allocations $(z_n)_{n \in \mathbb{N}}$ is unbounded. Passing to a subsequence if necessary, we can assume that there exists an agent $i \in I$ such that

$$\lim_{n \to \infty} \|z_n^i\| = \infty.$$

Since for each n, the allocation a_n is individually rational, we must have

$$\forall n \in \mathbb{N}, \quad \Pi^i(x_{0,n}^i, z_n^i, x_{1,n}^i) \geqslant \Pi^i(e_0^i, 0, e_1^i).$$

Since the function $(x_0, x_1) \mapsto \Pi^i(x_0, z, x_1)$ is strictly increasing for any z, it follows from (3.13) that

$$\forall n \in \mathbb{N}, \quad \Pi^i(e_0, z_n^i, e_1) \geqslant \Pi^i(e_0^i, 0, e_1^i).$$

This leads to the following contradiction

$$\liminf_{n \to \infty} \Pi^i(e_0, z_n^i, e_1) \geqslant \Pi^i(e_0^i, 0, e_1^i).$$

Q.E.D.

---

[17]If K is a finite set then $\|\cdot\|$ represents the norm defined by $\|y\| = \sum_{k \in K} |y_k|$ for each vector $y = (y_k)_{k \in K}$ in $\mathbb{R}^K$.

[18]Recall that $a_n = (a_n^i)_{i \in I}$ and $a_n^i = (a_{0,n}^i, a_{1,n}^i)$ where $a_{0,n}^i = (x_{0,n}^i, \theta_n^i, \varphi_n^i, z_n^i)$ and $a_{1,n}^i = (x_{1,n}^i(s))_{s \in S}$. The intertemporal consumption plan $(x_{0,n}^i, (x_{1,n}^i(s))_{s \in S})$ is denoted by x_n^i. The production allocation $(z_n^i)_{i \in I}$ is denoted by z_n.

3.4. *Existence result*

If it is possible to exhibit an exogenous bound on individually rational and physically feasible allocations, then existence of a competitive equilibrium follows from standard fixed-point arguments. The only novelty is to show that it is possible to clear at $t = 0$ via competitive prices, the three markets: commodities, assets and labor. Actually we propose to show that under condition (b) of Assumption 2.5 (i.e., the payoff function is concave and strictly decreasing in production) the economy has either limited intermediation or agents exhibit strong disutility for effort.

PROPOSITION 3.3   Consider an economy satisfying the aforementioned assumptions. Every agent i has either limited intermediation in the sense that Z^i is bounded, or exhibits strong disutility for effort. In particular, the set of individually rational and physically feasible allocations is bounded.

PROOF OF PROPOSITION 3.3:   Assume now that there exists an agent i whose production set Z^i is not bounded. Let $(z_n)_{n \in \mathbb{N}}$ be a sequence in Z^i such that

$$\lim_{n \to \infty} \|z_n\| = \infty.$$

Passing to a subsequence if necessary, one can assume that there exists at least one asset j such that the sequence $(z_n(j))_{n \in \mathbb{N}}$ is strictly increasing and unbounded (in particular converges to ∞). We let ξ_n be the vector in $\mathbb{R}^J_+$ defined by $\xi_n = z_n(j)\mathbf{1}_{\{j\}}$.[19] Since the function $z \mapsto \Pi^i_0(e_0, z, e_1)$ is decreasing, we have

$$\forall n \in \mathbb{N}, \quad \Pi^i_0(e_0, z_n, e_1) \leqslant \Pi^i_0(e_0, \xi_n, e_1).$$

Since the function $h \mapsto \Pi^i_0(e_0, h\mathbf{1}_{\{j\}}, e_1)$ is concave and strictly decreasing, we must have[20]

$$\lim_{n \to \infty} \Pi^i_0(e_0, \xi_n, e_1) = -\infty$$

implying that agent i exhibits strong disutility for effort.   *Q.E.D.*

As a consequence, we can state our main result.

THEOREM 3.1   *There exists a competitive equilibrium $((p, q, \kappa), a)$ with non-negative transaction costs, i.e., $\kappa \in \mathbb{R}^J_+$.*

---

[19]If K is a finite set and H is a subset of K then $\mathbf{1}_H$ denotes the vector $y = (y_k)_{k \in K}$ in $\mathbb{R}^K$ defined by $y_k = 1$ if $k \in H$ and $y_k = 0$ elsewhere.

[20]Let $f : [0, \infty) \to \mathbb{R}$ be a concave and strictly decreasing function. For every increasing sequence $(x_n)_{n \in \mathbb{N}}$ converging to ∞ one must have $\lim_n f(x_n) = -\infty$. Indeed, since the sequence converges to ∞, for n large enough we have $x_n \geqslant 1$. It implies by concavity that $f(x_n) \leqslant f(0) + x_n[f(1) - f(0)]$. Since f is strictly decreasing we have $f(1) - f(0) < 0$ and we get the desired result.

Our proof of Theorem 3.1 is based on a limiting argument. We first state that existence is ensured when the sets of actions are bounded.[21]

PROPOSITION 3.4   Assume that each action set A^i is bounded. Then, there exists a competitive equilibrium $((p,q,\kappa),\boldsymbol{a})$ with non-negative transaction costs, i.e., $\kappa \in \mathbb{R}^J_+$.

We subsequently consider a sequence of suitably truncated economies and apply Proposition 3.4 to obtain a sequence of competitive equilibria for the corresponding truncated economies. The final argument amounts to show that there exists a truncated economy for which every competitive equilibrium is actually an equilibrium of the initial economy.

PROOF OF THEOREM 3.1:   Consider an economy

$$\mathscr{E} \equiv \left\{ X^i, \Theta^i, \Phi^i, Z^i \right\}_{i \in I}$$

satisfying the assumptions of this paper and such that the set

$$\text{Ir-F}((X^i, \Theta^i, \Phi^i, Z^i)_{i \in I})$$

of individually rational and physically feasible allocations is bounded. We fix an integer $n \in \mathbb{N}$ and we propose to truncate the economy $\mathscr{E}$ in a suitable manner, such that the truncated economy $\mathscr{E}_n$ still satisfies the assumptions of the paper. Consider the economy

$$\mathscr{E}_n \equiv \left\{ X^i_n, \Theta^i_n, \Phi^i_n, Z^i_n \right\}_{i \in I}$$

where
- consumption sets are defined by

$$X^i_{0,n} = X^i_0 \cap \left[0, e_0 + n\mathbf{1}_{L_0} \right] \quad \text{and} \quad X^i_{1,n} = X^i_1 \cap \left[0, \overline{e}_1 + n\mathbf{1}_{L_1} \right]$$

  where $\overline{e}_1(\ell) = \max\{e_1(s,\ell) : s \in S\}$ for each $\ell \in L_1$;
- portfolio sets are defined by

$$\Theta^i_n = \left[0, n\mathbf{1}_J \right] \cap \Theta^i \quad \text{and} \quad \Phi^i_n = \left[0, n\mathbf{1}_J \right] \cap \Phi^i;$$

- production sets are defined by

$$Z^i_n = \left[0, n\mathbf{1}_J \right] \cap \Theta^i.$$

We can apply Proposition 3.4 to each economy $\mathscr{E}_n$ to get a sequence

$$(\pi_n, \boldsymbol{a}_n)_{n \in \mathbb{N}}$$

[21]The proof of Proposition 3.4 is postponed to Section 4.

of competitive equilibrium where $\pi_n = (p_n, q_n, \kappa_n)$ with $\kappa_n \in \mathbb{R}^J_+$.

Since the

$$\text{Ir-F}((X^i, \Theta^i, \Phi^i, Z^i)_{i \in I})$$

of individually rational and physically feasible allocations is bounded, there exists $n_0 \in \mathbb{N}$ such that for all $n \geqslant n_0$,

$$\text{Ir-F}((X^i, \Theta^i, \Phi^i, Z^i)_{i \in I}) \subset \prod_{i \in I} X^i_{0,n} \times \Theta^i_n \times \Phi^i_n \times Z^i_n \times [X^i_{1,n}]^S.$$

We let $\nu \equiv n_0 + 1$. It follows from standard arguments based on the concavity of each expected payoff function Π^i that (π_ν, a_ν) is actually a competitive equilibrium of the initial economy $\mathcal{E}$. *Q.E.D.*

4. PROOF OF PROPOSITION 3.4

Consider an economy

$$\mathcal{E} \equiv \left\{ X^i, \Theta^i, \Phi^i, Z^i \right\}_{i \in I}$$

satisfying the list of assumptions of this paper and such that the sets X^i, Θ^i, Φ^i and Z^i are compact. We let Price_0 be the auctioneer's action set at $t = 0$ where[22]

$$\text{Price}_0 = \{ \pi_0 = (p_0, q, \kappa) \in \mathbb{R}^{L_0} \times \mathbb{R}^J \times \mathbb{R}^J_+ \,:\, \|\pi_0\| \equiv \|p_0\| + \|q\| + \|\kappa\| \leqslant 1 \}.$$

Since the vector $(0, 0, 0)$ belongs to the price set Price_0, we follow Bergstrom (1976) by considering the following *slack* function

$$\forall \pi_0 = (p_0, q, \kappa) \in \text{Price}_0, \quad \gamma(\pi_0) = 1 - \|\pi_0\|.$$

We let Price_1 be the auctioneer's action set at $t = 1$ where

$$\text{Price}_1 = \{ p \in \mathbb{R}^{L_1}_+ \,:\, \|p\| = 1 \}.$$

We let Price be the set of auctioneer's intertemporal actions defined by

$$\text{Price} = \text{Price}_0 \times [\text{Price}_1]^S.$$

We slightly modify each agent's budget set as follows: for each agent i and price family $\pi = (\pi_0, \pi_1)$ with $\pi_0 = (p_0, q, \kappa)$ and $\pi_1 = (p_1(s))_{s \in S}$, we let $B^i_\gamma(\pi)$ be the set of all actions $a = (a_0, a_1)$ with

$$a_0 = (x_0, \theta, \varphi, z) \in A^i_0 = X^i \times \Theta^i \times \Phi^i \times Z^i$$

---

[22]We recall that if K is a finite set then $\|\cdot\|$ represents the norm defined by $\|y\| = \sum_{k \in K} |y(k)|$ for each $y = (y(k))_{k \in K}$ in $\mathbb{R}^K$.

and $a_1 = (x_1(s))_{s \in S}$ where $x_1(s) \in X_1^i$ such that the following budget constraints are satisfied:

$$(4.1) \qquad p_0 \cdot x_0 + (q + \kappa) \cdot \theta \leqslant p_0 \cdot e_0^i + (q - \kappa) \cdot \varphi + \kappa \cdot z + \gamma(\pi_0)$$

and for each state s,

$$(4.2) \qquad p_1(s) \cdot x_1(s) + V(p_1(s), s) \cdot \varphi \leqslant p_1(s) \cdot e_1^i(s) + V(p_1(s), s) \cdot \theta.$$

The function γ has suitably been chosen in order to obtain the following continuity result.

LEMMA 4.1 *The correspondence B_γ^i is continuous on the set* Price *for each agent i.*

PROOF: Upper semi-continuity is not an issue. To establish lower semi-continuity the only difficulty amounts to show that the strict modified budget set $\beta_\gamma^i(\pi)$ is non-empty for every price $\pi \in$ Price where $\beta_\gamma^i(\pi)$ is the subset of all actions a in $B_\gamma^i(\pi)$ such that the inequalities in (4.1) and (4.2) are strict. Fix a price $\pi \in$ Price. Since $e_1^i(s)$ belongs to the interior of X_1^i and $p_1(s) > 0$, there exists $\underline{x}_1^i(s) \in X_1^i(s)$ such that

$$0 < \varepsilon_1^i(s) \equiv p_1(s) \cdot [e_1^i(s) - \underline{x}_1^i(s)].$$

If $\gamma(\pi_0) > 0$ then the action $\underline{a}^i = (0, \underline{x}_1^i)$ belongs to the strict budget set $\beta_\gamma^i(\pi)$. Assume now that $\gamma(\pi_0) = 1$. Three cases are possible:

Case 1. The price p_0 is not zero. Since e_0^i belongs to the interior of X_0^i, there exists $\underline{x}_0^i \in X_0^i$ such that

$$0 < p_0 \cdot [e_0^i - \underline{x}_0^i].$$

The action $\underline{a}^i = (\underline{a}_0^i, \underline{x}_1^i)$ belongs to the strict budget set $\beta_\gamma^i(\pi)$ where $\underline{a}_0^i = (\underline{x}_0^i, 0, 0, 0)$.

Case 2. The price p_0 is zero but the price κ is not zero. This implies that there exists at least one asset $j \in J$ such that $\kappa_j > 0$. Since agent i has the productive capacity to intermediate at least some units of asset j, there exists $\alpha > 0$ small enough such that[23] $\underline{z}^i = \alpha \mathbf{1}_{\{j\}}$ belongs to Z^i. The action $\underline{a}^i = (\underline{a}_0^i, \underline{x}_1^i)$ belongs to the strict budget set $\beta_\gamma^i(\pi)$ where $\underline{a}_0^i = (0, 0, 0, \underline{z}^i)$.

Case 3. The vector (p_0, κ) is zero but the price q is not zero. This implies that there exists at least one asset $j \in J$ such that $q_j \neq 0$. Assume that $q_j > 0$.[24] Since agent i can short-sell at least some units of asset j, we can choose $\underline{\varphi}^i = \alpha \mathbf{1}_{\{j\}}$ for $\alpha > 0$ small enough such that

$$\forall s \in S, \quad V(p_1(s), s) \cdot \underline{\varphi}^i < \varepsilon_1^i(s).$$

Such a choice of α implies that the action $\underline{a}^i = (\underline{a}_0^i, \underline{x}_1^i)$ belongs to the strict budget set $\beta_\gamma^i(\pi)$ where $\underline{a}_0^i = (0, 0, \underline{\varphi}^i, 0)$.

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[23] We recall that if K is a finite set and H is a subset of K then $\mathbf{1}_H$ denotes the vector $y = (y(k))_{k \in K}$ in $\mathbb{R}^K$ defined by $y(k) = 1$ if $k \in H$ and $y(k) = 0$ elsewhere.
[24] We omit the similar argument for the case $q_j < 0$.

<div align="right">Q.E.D.</div>

Using the continuity and convexity assumptions made on payoff functions together with the continuity of the modified budget correspondence, it is straightforward to apply Berge's Maximum Theorem (see (Berge, 1963, pp. 115-116) or (Aliprantis and Border, 1999, Theorem 16.31)) and obtain the upper semi-continuity of the modified demand correspondence as defined hereafter.

LEMMA 4.2 *For each agent i, the correspondence d_γ^i is continuous on the set* Price, *where*

$$\forall \pi \in \text{Price}, \quad d_\gamma^i(\pi) \equiv \text{argmax}\{\Pi^i(a) : a \in B_\gamma^i(\pi)\}.$$

Moreover, for every $\pi \in$ Price the set $d_\gamma^i(\pi)$ is non-empty, convex and compact.

We let σ_0 be the correspondence from $\prod_{i\in I} A_0^i$ to Price_0 representing the auctioneer's demand at $t = 0$ and defined by

$$\sigma_0(\boldsymbol{a}_0) \equiv \text{argmax}\left\{\sum_{i\in I} p_0 \cdot [x_0^i - e_0^i] + q \cdot [\theta^i - \varphi^i] + \kappa \cdot [\theta^i + \varphi^i - z^i] : \pi_0 \in \text{Price}_0\right\}$$

for all $\boldsymbol{a}_0 = (a_0^i)_{i\in I}$. For each state s, we let σ_s be the correspondence from $\prod_{i\in I} X_1^i$ to Price_1 representing the auctioneer's demand at $t = 1$ contingent to state s and defined by

$$\sigma_s(\boldsymbol{x}_1(s)) \equiv \text{argmax}\left\{\sum_{i\in I} p_1(s) \cdot [x_1^i(s) - e_1^i(s)] : \pi_1(s) \in \text{Price}_1\right\}$$

for all $\boldsymbol{x}_1(s) = (x_1^i(s))_{i\in I}$.

We omit the standard arguments to prove that these correspondences are upper semi-continuous.

LEMMA 4.3 *The correspondence σ_0 is upper semi-continuous on $\prod_{i\in I} A_0^i$ with non-empty compact convex values, and for each state s the correspondence σ_s is upper semi-continuous on $\prod_{i\in I} X_1^i$ with non-empty, compact and convex values.*

Let K be the compact convex and non-empty set defined by

$$K \equiv \text{Price} \times \prod_{i\in I} A^i.$$

We let χ be the correspondence from K to K defined by

$$\forall (\pi, \boldsymbol{a}), \quad \chi(\pi, \boldsymbol{a}) = \left[\sigma_0(\boldsymbol{a}_0) \times \prod_{s\in S} \sigma_s(\boldsymbol{x}_1(s))\right] \times \prod_{i\in I} d_\gamma^i(\pi).$$

It follows from Lemma (4.2) and (4.3) that the correspondence χ is upper semi-continuous with compact convex and non-empty values. Applying Kakutani's Fixed-Point Theorem (see Kakutani (1941) or (Aliprantis and Border, 1999, Corollary 16.51)), we obtain the existence of a fixed-point $(\pi, \boldsymbol{a})$ of the correspondence χ, i.e.,

$$(4.3) \qquad (\pi, \boldsymbol{a}) \in \chi(\pi, \boldsymbol{a}).$$

We split the rest of the proof in several step.

LEMMA 4.4   *We have $\gamma(\pi_0) = 0$, i.e., the modified demand and the demand coincide.*

PROOF OF LEMMA 4.4:   Assume by way of contradiction that $\gamma(\pi_0) > 0$, i.e.,

$$0 < \varepsilon \equiv 1 - \left(\|p_0\| + \|q\| + \|\kappa\| \right).$$

We first prove that commodity markets at $t = 0$ must clear. It follows from (4.3) that

$$(4.4) \qquad \pi_0 = (p_0, q, \kappa) \in \sigma_0(\boldsymbol{a}_0).$$

In particular, we have

$$\widetilde{p}_0 \cdot \sum_{i \in I} [x_0^i - e_0^i] \leqslant p_0 \cdot \sum_{i \in I} [x_0^i - e_0^i]$$

for every commodity price $\widetilde{p}_0$ in $\mathbb{R}^{L_0}$ satisfying

$$\|\widetilde{p}_0\| + \|q\| + \|\kappa\| \leqslant 1$$

or equivalently

$$\|\widetilde{p}_0\| \leqslant \|p_0\| + \varepsilon.$$

This implies that

$$v \cdot \sum_{i \in I} [x_0^i - e_0^i] \leqslant 0$$

for every vector $v \in \mathbb{R}^{L_0}$ satisfying $\|v\| \leqslant \varepsilon$. As a consequence we obtain that commodity markets clear at $t = 0$, i.e.,

$$\sum_{i \in I} [x_0^i - e_0^i] = 0.$$

Similarly we can prove that asset markets clear, i.e.,

$$\sum_{i \in I} \theta^i = \sum_{i \in I} \varphi^i.$$

Now it follows from (4.3) that

$$\widetilde{\kappa} \cdot \sum_{i \in I} \theta^i + \varphi^i - z^i \leqslant \kappa \cdot \sum_{i \in I} \theta^i + \varphi^i - z^i$$

for every vector $\widetilde{\kappa} \in \mathbb{R}_+^J$ satisfying $\|\widetilde{\kappa}\| \leqslant 1$. Since $\gamma(\pi_0) > 0$, we must have

$$0 < \eta \equiv 1 - \|\kappa\|.$$

It follows that we can choose $\widetilde{\kappa} = \kappa + v$ for every vector $v \in \mathbb{R}_+^J$ with $\|v\| \leqslant \eta$. In particular, we get

$$v \cdot \sum_{i \in I} \theta^i + \varphi^i - z^i \leqslant 0$$

for every vector $v \in \mathbb{R}_+^J$ with $\|v\| \leqslant \eta$. This implies that the labor market clears with free-disposal, i.e.,

$$\sum_{i \in I} \theta^i + \varphi^i - z^i \leqslant 0.$$

Since commodity markets clear at $t = 0$, we must have

$$\forall i \in I, \quad x_0^i \leqslant e_0.$$

Following Assumption 2.6, each agent i is non-satiated at (x_0^i, z^i) in terms of the payoff Π_0^i. This implies that the budget restriction of the first period $t = 0$ must be binding, i.e.,

$$p_0 \cdot [x_0^i - e_0^i] + q \cdot [\theta^i - \varphi^i] + \kappa \cdot [\theta^i + \varphi^i - z^i] = \gamma(\pi_0).$$

Summing over i and using the fact that commodity markets clear, asset markets clear and labor markets clear with free disposal, we must have

$$0 < \#I\gamma(\pi_0) = \kappa \cdot \sum_{i \in I} [\theta^i + \varphi^i - z^i] \leqslant 0$$

which leads to a contradiction. *Q.E.D.*

Using the fact that the slack $\gamma(\pi_0)$ is zero, we can prove that commodity, asset and labor markets clear at $t = 0$.

LEMMA 4.5 *Commodity, asset and labor markets clear at $t = 0$, i.e.,*

$$\sum_{i \in I} x_0^i = \sum_{i \in I} e_0^i, \quad \sum_{i \in I} \theta^i = \sum_{i \in I} \varphi^i \quad \text{and} \quad \sum_{i \in I} \theta^i + \varphi^i = \sum_{i \in I} z^i.$$

PROOF OF LEMMA 4.5:   It follows from (4.4) that

$$\widetilde{p}_0 \cdot A + \widetilde{q} \cdot B + \widetilde{\kappa} \cdot C \leqslant p_0 \cdot A + q \cdot B + \kappa \cdot C$$

for every $(\widetilde{p}_0, \widetilde{q}, \widetilde{\kappa})$ in Price_0 and where

$$A = \sum_{i \in I} [x_0^i - e_0^i], \quad B = \sum_{i \in I} [\theta^i - \varphi^i] \quad \text{and} \quad C = \sum_{i \in I} [\theta^i + \varphi^i - z^i].$$

Since the slack $\gamma(\pi_0)$ is zero, it follows from the first period budget constraint that

$$p_0 \cdot A + q \cdot B + \kappa \cdot C \leqslant 0$$

implying that

$$\forall (\widetilde{p}_0, \widetilde{q}, \widetilde{\kappa}) \in \text{Price}_0, \quad \widetilde{p}_0 \cdot A + \widetilde{q} \cdot B + \widetilde{\kappa} \cdot C \leqslant 0.$$

Therefore, we must have that (A, B, C) belongs to the negative polar of Price_0, i.e.,

$$A = 0, \quad B = 0 \quad \text{and} \quad C \leqslant 0.$$

Since commodity markets clear at $t = 0$, the budget set restriction for the first period must be binding.[25] Therefore, we have

$$\kappa \cdot C = 0.$$

Remind that κ belongs to $\mathbb{R}_+^J$. Fix an asset j. If $\kappa_j > 0$ then we must have

$$\sum_{i \in I} \theta_j^i + \varphi_j^i - z_j^i = 0.$$

Assume next that $\kappa_j = 0$. Since the function $z \mapsto \Pi_0^i(x_0^i, z)$ is strictly decreasing, we must have $z_j^i = 0$ for every agent i. Since

$$0 \leqslant \sum_{i \in I} \theta_j^i + \varphi_j^i \leqslant \sum_{i \in I} z_j^i = 0$$

we get that the labor market for intermediation of asset j clears.   *Q.E.D.*

Once we have proved that asset markets clear at the first period, it is straightforward and standard to prove that commodity and labor markets clear at the second period.

LEMMA 4.6   *For every possible state s at the second period $t = 1$, commodity markets clear, i.e.,*

$$\sum_{i \in I} x_1^i(s) = \sum_{i \in I} e_1^i(s).$$

---

[25] See the argument used in the proof of Lemma 4.4.

PROOF OF LEMMA 4.6: Fix a state s and observe that (4.3) implies

$$(4.5) \qquad \pi_1(s) \in \sigma_s(\boldsymbol{x}_1(s)).$$

Therefore

$$\widetilde{p} \cdot \sum_{i \in I} [x_1^i(s) - e_1^i(s)] \leqslant p_1(s) \cdot \sum_{i \in I} [x_1^i(s) - e_1^i(s)]$$

for every vector $\widetilde{p} \in \mathrm{Price}_1$. Since asset markets clear, it follows from the budget constraint for state s that

$$p_1(s) \cdot \sum_{i \in I} [x_1^i(s) - e_1^i(s)] \leqslant 0$$

implying that

$$\forall \widetilde{p} \in \mathrm{Price}_1, \quad \widetilde{p} \cdot \sum_{i \in I} [x_1^i(s) - e_1^i(s)] \leqslant 0.$$

The excess demand in state s must belong to the negative polar of the simplex Price_1, i.e.,

$$\sum_{i \in I} [x_1^i(s) - e_1^i(s)] \leqslant 0.$$

This free-disposal market clearing implies that $x^i(s) \leqslant e_1(s)$ for each i. Following Assumption 2.6 each agent i is non-satiated at $x_1^i(s)$ in terms of the utility $U_1^i(s, \cdot)$. This implies that the budget restriction for state s must be binding, i.e.,

$$p_1(s) \cdot [x_1^i(s) - e_1^i(s)] \leqslant V(p_1(s), s) \cdot [\theta^i - \varphi^i].$$

Summing over i we get that

$$p_1(s) \cdot \sum_{i \in I} [x_1^i(s) - e_1^i(s)] = 0.$$

Since commodity markets clear with free-disposal, the claim is true provided that the vector $p_1(s)$ is strictly positive. This follows from the fact that the function $U_1^i(s, \cdot)$ is strictly increasing. _Q.E.D._

In order to prove that $(\pi, \boldsymbol{a})$ is a competitive equilibrium we still have to prove that actions are optimal, i.e., $a^i \in d^i(\pi)$ for each agent i. Observe that (4.3) implies

$$(4.6) \qquad \forall i \in I, \quad a^i \in d_\gamma^i(\pi).$$

The desired conclusion follows from Lemma 4.4.

5. RELATED LITERATURE

In this section we argue that specifying exogenously the level of transaction costs, as it is the case in the models proposed by Préchac (1996) and Markeprand (2008), it raises a number of serious questions. Préchac (1996) and Markeprand (2008) propose a model where no real resources are burnt in the process of transaction and labor costs are negligible. They assume that there is one firm in the market that they refer to as the brokerage house. Trade can only be implemented through this firm, i.e., this firm has the monopoly of intermediating financial activities. The brokerage house is assumed to be privately owned by consumers/investors. Each agent i is endowed with an equity share $\sigma^i \in (0,1)$ that determines at equilibrium his share of profits. Using its monopoly power, the brokerage house fixes a commission fee on transactions that can be proportional to volume and/or value of assets traded. More precisely, if an agent chooses a financial strategy (θ, φ) and if the asset price is q, then the agent should pay to the brokerage house the amount $c(q, \theta, \varphi)$ of units of account given by

$$(5.1) \qquad c(q, \theta, \varphi) = \sum_{j \in J} c_j q_j (\theta_j + \varphi_j) + \kappa_j (\theta_j + \varphi_j)$$

where $c \in \mathbb{R}_+^J$ and $\kappa \in \mathbb{R}_+^J$. Préchac (1996) assumes that assets are non-degenerate and positive, and consider the special case

$$c \in \mathbb{R}_{++}^j \quad \text{and} \quad \kappa = 0$$

while Markeprand (2008) allows for a more general asset structure but he needs to assume that[26]

$$\kappa \in \mathbb{R}_{++}^j.$$

Agents operate in a perfectly competitive environment taking not only prices as given but also the profit π of the brokerage house. We denote by $B_M^i(q, p, \pi)$ the set of agent i's actions $a^i = (a_0^i, x_1^i)$ with $a_0^i = (x_0^i, \theta^i, \varphi^i)$ satisfying the following budget constraint at $t = 0$

$$(5.2) \qquad p_0 \cdot x_0^i + q \cdot \theta^i + c(q, \theta^i, \varphi^i) \leqslant p_0 \cdot e_0^i + q \cdot \varphi^i + \sigma^i \pi$$

and the same budget constraint at $t = 1$ and each state s that we consider in our model, i.e.,

$$(5.3) \qquad p_1(s) \cdot x_1^i(s) + V(p_1(s), s) \cdot \varphi^i \leqslant p_1(s) \cdot e_1^i(s) + V(p_1(s), s) \cdot \theta^i.$$

Since labor costs are negligible in Préchac (1996) and Markeprand (2008), we denote by V^i the expected utility function of agent i defined by

$$V^i(a^i) = U_0^i(x_0^i) + \beta^i \sum_{s \in S} \nu^i(s) U_1^i(s, x_1^i(s)).$$

---

[26]Actually Markeprand (2008) allows for a more general form of cost function. When the cost function takes the form defined by (5.1), then it is needed to assume that $\kappa_j > 0$ for each asset j since the price q_j may be zero at equilibrium.

We now consider the definition of equilibrium adapted to this model and introduced by Préchac (1996).

DEFINITION 5.1   A competitive equilibrium of the economy with a monopolistic brokerage house is a family

$$\{(p, q, \pi), (a^i)_{i \in I}\}$$

composed of prices (p, q), profit π and an allocation $(a^i)_{i \in I}$ of intertemporal actions such that
  (a) actions are optimal, i.e.,

$$(5.4) \qquad \forall i \in I, \quad a^i \in \operatorname{argmax}\{V^i(a) : a \in B_M^i(p, q, \pi)\}$$

  (b) commodity markets clear, i.e.,

$$(5.5) \qquad \sum_{i \in I} x_0^i = \sum_{i \in I} e_0^i \quad \text{and} \quad \sum_{i \in I} x_1^i(s) = \sum_{i \in I} e_1^i(s)$$

  (c) asset markets clear, i.e.,

$$(5.6) \qquad \forall j \in J, \quad \sum_{i \in I} \theta_j^i = \sum_{i \in I} \varphi_j^i$$

  (d) profits are correctly anticipated, i.e.,

$$(5.7) \qquad \sum_{i \in I} c(q, \theta^i, \varphi^i) = \pi.$$

5.1. *Endogenous intermediation costs*

When assets are non-degenerate and positive, Préchac (1996) proved that an equilibrium with a monopolistic brokerage house always exists. A serious drawback of the model proposed by Préchac (1996) is the lack of a rationale for the determination of intermediation costs. Since the brokerage house has the monopoly power to choose the vector $c = (c_j)_{j \in J}$, an issue that naturally arises concerns with the way the brokerage house chooses the vector c. An obvious answer is to say that c is determined such that the brokerage house maximize its profit. However, such a modification introduces serious difficulties in tackling the existence problem. This is because, in such a setting, profits depend on the equilibrium outcome which in turn depends on the level of the chosen intermediation costs. Additional complications arise form the existence of multiple equilibria. To simplify things, consider that the primitives of the economy are such that for each vector c, there is a unique equilibrium and therefore the profit function

$$c \mapsto \pi(c) = \sum_{i \in I} c_j q_j(c)(\theta_j^i(c) + \varphi_j^i(c))$$

is well-defined. Following the idea proposed (although in a different framework) by Bisin (1998), a simple and natural way to make c endogenous is to solve the following maximization problem

$$\text{argmax}\{\pi(c) : c \in (0,1)^J\}.$$

It is not clear whether this maximization problem has always a solution. Moreover, for economies exhibiting the non-existence phenomena à la Hart (1975), it follows from Proposition 2 in Markeprand (2008) that one must have

$$\lim_{c \to 0} \sum_{i \in I} \|\theta^i(c)\| + \|\varphi^i(c)\| = \infty.$$

This observation implies that it is far from clear that the profit function $c \mapsto \pi(c)$ is bounded from above in a neighborhood of 0. Therefore, one cannot conclude whether non-existence phenomena à la Hart (1975) are ruled out when the intermediation cost becomes endogenous.

5.2. *Exogenous nominal costs*

In Préchac (1996) intermediation costs are proportional to the value of the transactions. In other words, costs are denominated in units of assets: in order to trade one unit of asset j, each agent should give to the brokerage house c_j units of the same asset as a fee. This kind of intermediation costs ensures existence when assets are non-degenerate and positive. Markeprand (2008) showed that in order to deal with a more general asset structure including options, the cost function should satisfy extra-properties since asset prices may be 0 at equilibrium. Markeprand (2008) showed that if the cost function

$$c(q, \theta, \varphi) = \sum_{j \in J} c_j q_j(\theta_j + \varphi_j) + \kappa_j(\theta_j + \varphi_j)$$

is such that $\kappa_j > 0$ for each asset j, then existence is guaranteed even if the assets are not non-degenerate and positive. The only property that the payoff function of each asset should satisfy is continuity with respect to commodity prices. This kind of cost function proposed by Markeprand (2008) introduced an important difference with respect to the model proposed by Préchac (1996). It implies that the budget restriction at $t = 0$ is no more homogeneous of degree zero with respect to prices. If we multiply the prices (p_0, q) by 2, intermediation becomes less costly, while if we divide the prices by 2 intermediation becomes more costly. This may lead to a serious problem of interpretation and raises questions that already appear in the literature of financial markets with nominal assets. Since the *level of prices* matters, who determines this level? Is it endogenously determined through market prices? If not, is there any institution or agent that is fixing the equilibrium level of prices? When does the brokerage house choose the commission fee κ? Before or after observing the price level? Markeprand (2008) does not discuss the implications of his

cost function on the homogeneity of period $t = 0$ budget restriction. More problematic is the proof of the existence result found in his paper. The inherited nominal feature due to the specification of his cost function seems to play no role. Indeed, it is claimed in Markeprand (2008) (see Section 3 p. 152) that, independently of the *nominal* level of the vector of commission fees κ, there exists a competitive equilibrium

$$\{(p, q, \pi), (a^i)_{i \in I}\}$$

where prices at $t = 0$ satisfy the following conditions[27]

$$(5.8) \qquad p_0 \in \mathbb{R}_+^{L_0}, \quad \|p_0\| = 1 \quad \text{and} \quad \|q\| \leqslant 1.$$

We propose to show that this existence result is not correct. In order to prove that a competitive equilibrium exists, the level of prices at $t = 0$ matters and should depend on the nominal commission fee κ.

We consider the simplest case: one good per date, no uncertainty at $t = 1$ and one asset delivering one unit of the unique good at $t = 1$. The cost function is defined as follows:

$$c(q, \theta, \varphi) = \kappa(\theta + \varphi).$$

To make the analysis closer to the notations in Markeprand (2008), we let $z = \theta - \varphi$ denote the net trade in the asset market.[28] Given a vector of commodity prices $p = (p_0, p_1)$, an asset price q and the (correctly anticipated) profit π, the budget set of agent i, denoted by $B_M^i(p, q, \pi)$ is the set of all actions

$$(x_0, z, x_1) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+$$

such that at $t = 0$

$$(5.9) \qquad p_0 x_0 + qz + \kappa|z| \leqslant p_0 e_0^i + \sigma^i \pi$$

and at $t = 1$

$$(5.10) \qquad p_1 x_1 \leqslant p_1(e_1^i + z).$$

Each agent i is assumed to have the same utility function given by

$$U^i(x_0, x_1) = \sqrt{x_0} + \sqrt{x_1}.$$

What differentiates agents are initial endowments. We will assume that there are two agents $I = \{i_0, i_1\}$. Agent i_0 has a larger endowment at $t = 0$ while agent i_1

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[27]We recall that if K is a finite set, we let $\|z\|$ be the norm of a vector $z = (z_k)_{k \in K}$ in $\mathbb{R}^K$ be defined by $\|z\| = \sum_{k \in K} |z_k|$.

[28]We already saw that an agent will optimally choose (θ, φ) such that $\theta \varphi = 0$. In other words, either we have $\theta = 0$ or $\varphi = 0$. This implies that the action (θ, φ) of an agent on the asset market can be replaced by $z = \theta - \varphi$.

has a larger endowment at $t = 1$. More precisely, we will assume that there exists $M > 1$ such that

$$e_0^{i_0} = M e_1^{i_0} \quad \text{and} \quad e_1^{i_1} = M e_0^{i_1}.$$

This economy satisfies all the assumptions of Theorem 1 in Markeprand (2008). Therefore, independently of κ, there exists a competitive equilibrium

$$\{(p, q, \pi), (x_0^i, z^i, x_1^i)_{i \in I}\}$$

satisfying

$$p_0 = 1, \quad p_1 = 1 \quad \text{and} \quad |q| \leqslant 1.$$

There is an obvious problem if $\kappa \geqslant 1$.

PROPOSITION 5.1   If $\kappa \geqslant 1$ then we can choose initial endowments such that there does not exist a competitive equilibrium with prices satisfying (5.8).

PROOF OF PROPOSITION 5.1:   Assume that $\kappa \geqslant 1$ and choose M large enough such that $M > (\kappa + 1)^2$. Since $|q| \leqslant 1$ we must have $q - \kappa \leqslant 0$. This implies that if agent i chooses to short-sell one unit of the asset, he has to deliver $q - \kappa$ units of the good at $t = 0$ and 1 unit at $t = 1$. He is clearly better off not short-selling. Since the asset market clears, there is no transaction in the market, i.e.,

$$z^{i_0} = z^{i_1} = 0 \quad \text{and} \quad \pi = 0.$$

However, agent i_0 prefers to transfer wealth from period $t = 0$ to period $t = 1$. Indeed, let $\widetilde{a}^{i_0}(\varepsilon)$ be the alternative action

$$\widetilde{a}^{i_0}(\varepsilon) = (\widetilde{x}_0^{i_0}(\varepsilon), \widetilde{z}^{i_0}(\varepsilon), \widetilde{x}_1^{i_0}(\varepsilon))$$

defined by

$$\widetilde{z}^{i_0}(\varepsilon) = \varepsilon, \quad \widetilde{x}_0^{i_0}(\varepsilon) = e_0^{i_0} - (q + \kappa)\varepsilon \quad \text{and} \quad \widetilde{x}_1^{i_0}(\varepsilon) = e_1^{i_0} + \varepsilon.$$

The action $\widetilde{a}^{i_0}(\varepsilon)$ belongs to the budget set $B_M^i(p, q, \pi)$ and satisfies

$$\lim_{\varepsilon \to 0^+} \frac{U^i(\widetilde{x}^{i_0}(\varepsilon)) - U^i(x^{i_0})}{\varepsilon} = -\frac{q + \kappa}{2\sqrt{e_0^{i_0}}} + \frac{1}{2\sqrt{e_1^{i_0}}}.$$

Observe that

$$q + \kappa \leqslant 1 + \kappa < \sqrt{M} = \sqrt{e_0^{i_0}/e_1^{i_0}}$$

implying that for ε small enough, we have the following contradiction

$$U^i(\widetilde{x}^{i_0}(\varepsilon)) > U^i(x^{i_0}).$$

$$Q.E.D.$$

Actually, the normalization (5.8) proposed by Markeprand (2008) is problematic even if there are no transaction costs.[29]

PROPOSITION 5.2   Assume that $\kappa = 0$ and $e_0 > e_1$ then there does not exist a competitive equilibrium with prices satisfying (5.8).

PROOF OF PROPOSITION 5.2:   Assume that $\kappa = 0$. We have complete markets and actually the optimal action $a^i = (x_0^i, z^i, x_1^i)$ of agent i is also a solution to the maximization of $U^i(x_0, x_1)$ under the constraint

$$x_0 + qx_1 \leqslant e_0^i + qe_1^i.$$

First order condition implies

$$q = \frac{\sqrt{x_0^i}}{\sqrt{x_1^i}}.$$

Since markets clear, one must have

$$e_0 = q^2 e_1.$$

If $e^0 > e^1$ we get the contradiction: $q > 1$.   *Q.E.D.*

Let us replace the normalization (5.8) by the classical one[30]

(5.11) $(p_0, q, p_1) \in \mathbb{R}_+^3, \quad p_0 + q = \chi_0 \quad \text{and} \quad p_1 = \chi_1$

where $\chi_0 > 0$ and $\chi_1 > 0$. One may wonder if the arguments in Markeprand (2008) can be corrected when considering this new normalization. The answer is yes, but if κ is larger than χ_0, the only possible equilibrium is no-trade. We don't provide the general proof of our claim. For the simplicity of the presentation, we prefer to illustrate this result using the specific economy we have been considering.

PROPOSITION 5.3   Assume that $\kappa \geqslant \chi_0$ then no-trade is the only possible competitive equilibrium with prices satisfying (5.11).

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[29]In Markeprand (2008), the existence is guaranteed only if $\kappa > 0$. However, in our specific example, assets are numéraire. This implies that we can find an exogenous bound on actions that is not binding at equilibrium. If the arguments of Lemma 1 and Lemma 3 in Markeprand (2008) were correct, existence of an equilibrium satisfying the normalization (5.8) should be ensured even if there are no transaction costs.

[30]In our specific example, the asset structure is non-degenerate and positive. If assets may have negative payoff as in Markeprand (2008), the normalization has to be adapted to the following one:

$$(p_0, q, p_1) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+, \quad p_0 + |q| = \chi_0 \quad \text{and} \quad p_1 = \chi_1.$$

PROOF OF PROPOSITION 5.3:   Assume that there exists a competitive equilibrium

$$\{(p,q,\pi),(x_0^i,z^i,x_1^i)_{i\in I}\}$$

with

$$(p_0,q,p_1)\in\mathbb{R}_+^3,\quad p_0+q=\chi_0\quad\text{and}\quad p_1=\chi_1.$$

Since $q-\kappa\leqslant\chi_0-\kappa\leqslant 0$, no agent will short-sell the asset. By market clearing, we must have no trade. Therefore no-trade is the only possible equilibrium. Actually, it is an equilibrium. We only have to choose q close enough to χ_0 such that

$$\frac{q+\kappa}{\chi_0-q}>\sqrt{M}.$$

Indeed we claim that

$$\{(p,q,\pi),(e_0^i,0,e_1^i)_{i\in I}\}\quad\text{with}\quad p_0=\chi_0-q\quad\text{and}\quad p_1=\chi_1$$

is a competitive equilibrium. We only have to prove that for each agent i, no-trade is the optimal action. Fix an agent i and assume by way of contradiction that there exists a budget feasible action $a^i=(x_0^i,z^i,x_1^i)$ such that $U^i(x^i)>U^i(e^i)$. Since $q\leqslant\chi_0\leqslant\kappa$ we have $q-\kappa\leqslant 0$ implying that agent i will not short-sell the asset. As a consequence, we must have $z^i>0$. Since a^i is budget feasible, we have

$$x_0^i\leqslant e_0^i-\frac{q+\kappa}{\chi_0-q}z^i\quad\text{and}\quad x_1^i\leqslant e_1^i+z^i.$$

We let $\widetilde{a}^i$ be the action $(\widetilde{x}_0^i,z^i,\widetilde{x}_1^i)$ defined by

$$\widetilde{x}_0^i=e_0^i-\frac{q+\kappa}{\chi_0-q}z^i\quad\text{and}\quad\widetilde{x}_1^i=e_1^i+z^i.$$

This action is budget feasible and satisfies

$$U^i(\widetilde{x}^i)-U^i(e^i)>0.$$

By concavity, this implies

$$\lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon}\left[\left(\sqrt{e_0^i-\frac{q+\kappa}{\chi_0-q}\varepsilon}-\sqrt{e_0^i}\right)+\left(\sqrt{e_1^i+\varepsilon}-\sqrt{e_1^i}\right)\right]>0.$$

Therefore, we must have the following contradiction

$$\frac{q+\kappa}{\chi_0-q}<\sqrt{\frac{e_0^i}{e_1^i}}\leqslant\max\{1/\sqrt{M},\sqrt{M}\}\leqslant\sqrt{M}.$$

                                                                                    Q.E.D.

Proposition 5.3 illustrates that the level of prices χ_0 is a relevant parameter. If it is not large enough (i.e., larger that the commission fee κ) only *no-trade* equilibrium exists. Obviously, neither the brokerage house nor the agents have an incentive to preclude trade. The model proposed by Markeprand (2008) shares with the model developed by Préchac (1996) the same drawback: the commission fee is exogenous and the objective of the brokerage house is not explicitly modeled. But in Markeprand (2008), a rationale for an endogenous level of prices is also missing. Observe that our model suffers from none of the two drawbacks.

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