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Discretionary Policy and Multiple Equilibria in LQ RE Models

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Discretionary Policy and Multiple Equilibria in LQ RE Models[*]

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Abstract

We study discretionary equilibria in dynamic linear-quadratic rational expectations models. In contrast to the assumptions that pervade this literature we show that these models *do* have multiple equilibria in some situations. We investigate general properties of discretionary equilibria. We demonstrate that for multiple equilibria to exist, models must have some 'history dependence' that implies sluggish adjustment. This creates possibility to have different but mutually consistent beliefs of economic agents about the future speed of stabilization. Multiple equilibria are likely to occur, in particular if there are complementarities in the model.

We demonstrate the existence of multiple discretionary equilibria by example. In a simple New Keynesian model of optimal monetary policy, but with fiscal solvency constraint, monetary policy can be either 'active' or 'passive' in the sense of Leeper (1991), depending on the strength of fiscal control of debt. There is an intermediate strength of fiscal control when both active and passive policies are possible.

Key Words: Time Consistency, Discretion, Multiple Equilibria

JEL References: E31, E52, E58, E61, C61

1   Introduction

How often do policymakers change their minds? And how often are policymakers' minds changed for them? If we take the analysis of Davig and Leeper (2006) on fluctuating macro policies as a benchmark, it would seem to be that at least one of these events occurs quite often. Policymakers seem to switch from an active control of inflation with monetary policy in one period to an accommodative monetary policy in another period with no apparent change in fundamentals.

Of course, even if their preferences remain the same, policymakers typically cannot choose an optimal policy once and forever – this would require an excessive degree of credible commitment (Kydland and Prescott (1977)). Rather, they claim to operate under (constrained) discretion (King (1997)). Such policies, in theory, are optimal subject to the restriction of time consistency and are therefore credible. At every moment in time rational private agents anticipate the next realized policy move and no incentive to renege on policy announcements arises.

These very properties of discretionary policy may suggest an explanation of frequent policy changes: Multiple equilibria is a known feature of policy discretion. Indeed, discretionary policy decisions are based on a forecast of future economic conditions which themselves are determined by beliefs of the private sector about the course of the future policy. A credible discretionary policymaker will find it optimal to validate the forecast. A coordination failure then occurs – agents can coordinate on *any* of the multiple equilibria. The final choice is potentially arbitrary: A 'sunspot' might decide upon which equilibrium the agents coordinate.

As a result, the observed frequent and seemingly large changes in economic policies and corresponding fluctuations of macroeconomic variables may have nothing to do with changing preferences. These changes can be driven by the interactions between policymakers and agents in the presence of multiple equilibria. Both the policy chosen and the private sector reactions to similar events might seem to be very different in different time periods simply because the economy can move from one equilibrium to another.

This paper demonstrates how the different monetary and fiscal policy regimes documented by Davig and Leeper (2006) fit into a framework of multiple discretionary equilibria. However, we demonstrate it as a logical conclusion of our analysis rather than make it our main theme. We believe this to be the first paper to demonstrate the existence of multiple discretionary equilibria in the linear-quadratic rational expectations (LQ RE) framework and its main contribution is a thorough investigation of the nature of discretionary equilibria in this class of models. Ultimately, it is this investigation helps us to understand why policies fluctuate.

Before applying models with multiple equilibria to the analysis of observed policy a lot of work needs to be done. Although we present the first (and so far the only) example of multiple equilibria under discretion in LQ RE models, we shall argue that multiplicity should be a recognized general feature of these models. We shall present our arguments in general mathematical terms. We discuss how the multiplicity of equilibria can arise, what form it can take and whether we can find all of them. Having established these results, we present a detailed account of multiplicity in a model of monetary and fiscal interactions and discuss the empirical implications.

We shall defer any detailed discussion of the literature until we explain our main results where we contrast our findings with those from the relevant models. We simply note that despite the obvious circularity in determining discretionary policy, multiple equilibria are a documented feature of comparatively few models. Some of the early literature (Rogoff (1987), Ireland (1997))

discussed simple static models with trigger strategies in repeated play that generate a continuum of steady state levels of inflation. Recently, using a non-linear static model, Albanesi et al. (2003) demonstrated 'policy traps', in which discretionary monetary policy is forced to accommodate private sector expectations about a particular policy. Further multiple equilibria are demonstrated in a non-linear dynamic New Keynesian model by King and Wolman (2004) who find multiple policy-induced private sector equilibria. Their setting is most naturally compared with our and we discuss differences in results when we obtain them.

There are two features that make LQ RE models of discretionary policy very different from existing models with multiple discretionary equilibria. First, existing models are highly non-linear and so can have equilibria of different types, including those that are not possible in LQ RE models. Moreover, in all of the referenced papers it is shown that the non-linearity of the model was essential for the result. Second, *all* LQ RE models that can generate multiple equilibria have to be solved numerically using iterative algorithms, while the existing models admitted simplifications and analytical solutions. The algebraic complexity of the first order conditions even for the most basic LQ RE model with predetermined state variables is very high: such systems do not admit much simplification. Iterative numerical routines become the only way of solving these problems. And of course, any specific iterative procedure, if it converges, can deliver only one fixed point and multiplicity of equilibria can remain undiscovered. Moreover, the choice of numerical algorithm may affect the outcome if there is more than one equilibrium.

These features make it absolutely necessary to investigate the equilibrium properties of discretionary solutions before searching for them numerically. We need to know where iterative routines can possibly converge. The existing literature on discretionary policy in LQ models instead either focuses on placing discretionary policy within the wider class of time-consistent policies (e.g. de Zeeuw and van der Ploeg (1991)), or studies how *an* equilibrium can be obtained (e.g. Cohen and Michel (1988)). This clearly reflects a belief that discretionary equilibria are unique. Indeed, Oudiz and Sachs (1985), whose early influential paper is the source of a popular iterative approach, explicitly state that they believe that discretionary equilibrium is unique expressly because of its LQ nature. Although subsequent theoretical analysis has never ruled out multiple solutions, examples in LQ RE models seem never to have been reported.[1]

An important goal of this paper is to improve understanding of discretionary policy and of the nature of multiple discretionary equilibria. We work within the traditional framework for the literature on LQ RE models as these are the class of models most commonly used in policy analysis.[2] It is also possible to define equilibria as the solution to a dynamic game; such interpretations allow equilibrium refinements to be brought into play to help with equilibrium selection. However, this interpretation is not necessary to discuss the general mathematical properties of discretionary solutions including multiplicity, and so in what follows we do not stress this interpretation.

Our results can be summarized as follows:

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[1] As we make explicit below this is perhaps surprising given the structure of the underlying problem. Papavassilopoulos and Olsder (1984) noted that LQ Nash dynamic games, which involve the solution of similar (but crucially not the same) sets of equations as the problem we consider, can admit multiple solutions. This was exploited by Lockwood and Philippopoulos (1994), who provide an example of multiple equilibria in a bargaining model. However, this setup is the one without rational expectations, and this is important.

[2] See Currie and Levine (1993), Clarida et al. (1999), Woodford (2003a) among many others.

1. In general mathematical terms we demonstrate the following results.

   (a) Predetermined endogenous state variables are necessary for multiple equilibria to exist. The resulting 'history dependence' implies sluggish adjustment, so that stabilization can be conducted at different speeds. Economic agents can affect the speed of stabilization through their choice of beliefs. Multiple equilibria can arise, in particular if there are dynamic complementarities. Where there are multiple equilibria, the policymaker may not be able to choose the Pareto-dominant equilibrium as a coordination failure occurs.

   (b) If multiple equilibria exist then (i) there is a finite number of them, (ii) all of them are determinate, and (iii) there can be multiple private sector equilibria among them, i.e. a policy-induced multiplicity of private sector responses. Part of our discussion is the choice of appropriate algorithms to find all of them.

2. We investigate how multiple equilibria can arise in a mainstream model of monetary and fiscal policy interactions.

   (a) We use a standard New Keynesian model augmented to include fiscal policy under a solvency constraint. In it, fiscal policy operates as a simple feedback rule designed to keep government debt under control, with spending adjusted by some fixed proportion of excess debt. We show that if the fiscal control of debt is *either* strong *or* absent then, correspondingly, monetary policy is either 'active' or 'passive' in the sense of Leeper (1991), i.e. it is either predominantly devoted to the control of inflation and output, or taking part in the control of debt. However, there is an intermediate strength of fiscal control when both 'active' and 'passive' monetary policies are possible.

   (b) Multiple equilibria arise in this model because there are dynamic complementarities: a low interest rate and a high inflation rate both reduce the real value of debt on the one hand and a low interest rate leads to high inflation on the other. Private agents and the policymaker have common beliefs about whether monetary policy is predominantly concerned with debt or inflation stabilization. The prevailing equilibrium is the outcome of a 'race' between the two where agents can abruptly switch beliefs. These equilibria imply different speed of stabilization of the economy and therefore have different welfare consequences.

Our finding of multiple equilibria in a plausible monetary policy model helps us understand why policies fluctuate. As an example, we look at the chronology of monetary-fiscal policy regimes documented by Davig and Leeper (2006). We argue that the four regimes identified in this paper can be explained by switches between two discretionary equilibria.

We further argue that finding multiple equilibria in LQ RE models has implications for the academic approach to monetary policymaking. For example, the literature on optimal delegation suggests that delegating policy to a discretionary policymaker with appropriately modified objectives can improve social welfare.[3] However, this rests on the assumption that the discretionary

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[3]To mention only few, Walsh (2003) suggests speed-limit policies, Woodford (2003b) suggests interest rate smoothing and Svensson (1999) and Vestin (2006) suggest price level targeing.

equilibrium associated with each candidate policymaker is unique. If the multiplicity of discretionary equilibria is not eliminated by a delegation scheme then an announcement of a new policy target can trigger a switch to a new equilibrium that might be worse in welfare terms than the anticipated equilibrium. Further, many policy recommendations are based on models where it is assumed that the many of the simplifications used are innocuous. It may not be so; for example the explicit modeling of saving decisions and the introduction of capital can generate multiple equilibria.[4]

Given what we have just discussed, the structure of this paper is perhaps a little unconventional. We begin with a general discussion of discretionary equilibria and their mathematical properties. This part precedes the economics, to facilitate an unambiguous interpretation of the numerical results. We formally set up a standard linear-quadratic rational expectations optimization problem, again in general mathematical terms, and obtain the first order conditions for the discretionary optimization problem in Section 2. We highlight the fixed-point aspect of common algorithms to find the solution. In Section 3 we report our results on multiplicity. We study the equilibrium responses of the private sector and of the policymaker separately and describe their properties. We prove that all discretionary equilibria, if they exist, are determinate and we describe all possible types of them. We then discuss the problem of finding all equilibria numerically. In Section 6 we turn to where it all matters, and describe a New Keynesian model that has multiple discretionary equilibria. We discuss the model and its implications in detail, and discuss not only the mere fact of obtaining multiple solutions, but the economics behind these equilibria. We discuss how our findings are related to those in the existing literature. We also compare some implications of our model with empirical findings in Davig and Leeper (2006). Section 7 concludes.

2   The Framework

Our framework is a linear discrete time rational expectations economy. The private sector consists of large number of small agents who each minimize their own infinite-horizon intertemporal loss function. Key linearities are preserved so to allow aggregation of dynamic optimality equations of the private agents. The government's own criterion is an infinite-horizon intertemporal loss function which only depends upon aggregate variables. We assume that the aggregate dynamic equations can be written in a linear form, while the flow policy objective function is quadratic in goal variables, where goal variables are a linear combination of aggregate variables.[5] We next set up a model of this kind in general mathematical terms.

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[4]In fact, we know that this is true. In the subsequent paper Blake and Kirsanova (2008b) we demonstrate that a New Keynesian model with capital accumulation (Sveen and Weinke (2005), Woodford (2005)) has several discretionary equilibria.

[5]A standard approach in the modern literature on monetary policy is to view the quadratic objectives and the linear system as a linear-quadratic approximation of the forward-looking economic model, see Clarida et al. (1999) for example.

2.1 A class of models

We assume a non-singular linear deterministic rational expectations model of the type described by Blanchard and Kahn (1980), augmented by a vector of control instruments. Specifically, the evolution of the economy is explained by the following system:

$$\begin{bmatrix} y_{t+1} \\ x_{t+1} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} y_t \\ x_t \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} [u_t], \tag{1}$$

where y_t is an n_1-vector of predetermined variables with initial conditions y_0 given, x_t is n_2-vector of non-predetermined (or jump) variables, and u_t is a $k-$vector of policy instruments of the policymaker. For notational convenience we define the n-vector $z_t = (y_t', x_t')'$ where $n = n_1 + n_2$.

Typically, the second block of equations in this system represents the solved out optimization problem for the private sector. The private sector has choice variables, represented by x_t (one can think of consumption or inflation as examples). Additionally, there is an equation explaining the evolution of the predetermined state variables y_t (capital or the stock of debt are examples). These two equations together describe the 'evolution of the economy' as observed by the policymaker. One can draw an example with the behavior of the private sector in macroeconomic models, presented by the Euler consumption equation and the Phillips curve; see Clarida et al. (1999) among many others.

At time t the policymaker has the following optimization problem ($\mathcal{E}_t$ denotes expectations operator, conditional on information available at time t):

$$\min_{u_t} \mathcal{E}_t W_t \tag{2}$$

with the loss function

$$W_t = \frac{1}{2} \sum_{s=t}^{\infty} \beta^{s-t} g_s' \mathcal{Q} g_s = \frac{1}{2} \sum_{s=t}^{\infty} \beta^{s-t} \left(z_s' Q z_s + 2 z_s' P u_s + u_s' R u_s \right), \tag{3}$$

subject to system (1). In addition, any solution to this optimization problem should satisfy the time-consistency constraint: for any $s > t$ the policymaker will choose

$$u_s = \mathcal{E}_t u_s. \tag{4}$$

The elements of vector g_s are the goal variables of the policymaker, $g_s = \mathcal{C}(z_s', u_s')'$. The matrix $\mathcal{Q}$ is assumed to be symmetric and positive semi-definite. In our formulation the quadratic loss function includes instrument costs, but no assumptions about the invertibility of R need be made.[6] We are looking for solutions that ensure that the loss is finite, i.e. $W_t < \infty$.

We have outlined a deterministic setup with the consequent assumption of perfect foresight. None of the results depend on this, as we can always add an appropriate vector of shocks and appeal to the certainty equivalence property of LQ models. This, however, would complicate the analysis unnecessarily in order to demonstrate the main point.[7]

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[6] For example, our analysis is valid for $R = 0$.

[7] Shocks can be included into vector y_t, see e.g. Anderson et al. (1996).

2.2 Timing of actions

The sequence of actions within a period in this paper is as follows. In the first stage of every period t the policymaker chooses the instrument u_t, knowing the state y_t and taking the process by which private agents behave as given. In the second stage the private sector adjusts its choice variable x_t. The optimal x_t, u_t and given y_t result in the level of y_{t+1} by the beginning of the next period $t+1$.

2.3 Discretionary equilibrium as a 'triplet' of matrices

Since the problem is linear-quadratic, we seek a time-consistent solution in a class of matrices with time-invariant coefficients. Given this structure, the solution in any time t gives a value function which is quadratic in the state variables,

$$W_t = \frac{1}{2} y_t' S y_t$$

a linear relation between the forward-looking variables

$$x_t = -N y_t$$

and a linear policy reaction function

$$u_t = -F y_t.$$

Given y_0 and system matrices A and B, matrices N and F define trajectories $\{y_s, x_s, u_s\}_{s=t}^{\infty}$ in a unique way and vise versa: if we know that $\{y_s, x_s, u_s\}_{s=t}^{\infty}$ solve discretionary optimization problem then, by construction, there are unique time-invariant linear relationships between them which we label by N and F. Matrix S defines the cost-to-go along a trajectory. Given the one-to-one mapping between equilibrium trajectories and $\{y_s, x_s, u_s\}_{s=t}^{\infty}$ and the triplet of matrices $\mathcal{T} = \{N, S, F\}$, it is convenient to continue with definition of policy equilibrium in terms of $\mathcal{T}$, not trajectories. This approach has become standard since Oudiz and Sachs (1985) and Backus and Driffill (1986).

The following proposition defines conditions on matrices F, S and N. We provide an outline proof,[8] primarily with the purpose to introduce all notation and create the basis for further analysis.

Proposition 1 (First order conditions) *The first-order conditions to the optimization problem (1) – (4) can be written in the following form:*

$$S = Q^* + \beta A^{*\prime} S A^* - \left(P^{*\prime} + \beta B^{*\prime} S A^*\right)' \left(R^* + \beta B^{*\prime} S B^*\right)^{-1} \left(P^{*\prime} + \beta B^{*\prime} S A^*\right), \quad \text{(DARE)}$$
$$F = \left(R^* + \beta B^{*\prime} S B^*\right)^{-1} \left(P^{*\prime} + \beta B^{*\prime} S A^*\right), \quad \text{(POLICY)}$$
$$0 = N C_{11} + C_{21} - N C_{12} N - C_{22} N, \quad \text{(NCARE)}$$

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[8]It is similar to the one given in e.g. Söderlind (1999), although there is an important difference: we derive general conditions for a solution rather than suggest a particular iterative algorithm to solve the problem. We discuss the difference further in the text.

where matrix C describes the evolution of the system under control F:

$$\begin{bmatrix} y_{t+1} \\ x_{t+1} \end{bmatrix} = \begin{bmatrix} A_{11} - B_1 F & A_{12} \\ A_{21} - B_2 F & A_{22} \end{bmatrix} \begin{bmatrix} y_t \\ x_t \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \begin{bmatrix} y_t \\ x_t \end{bmatrix}, \tag{5}$$

and matrices Q^*, P^*, R^*, A^*, and B^* are defined as follows:

$$\begin{align}
Q^* &= Q_{11} - Q_{12}J - J'Q_{21} + J'Q_{22}J, \tag{6} \\
P^* &= J'Q_{22}K - Q_{12}K + P_1 - J'P_2, \tag{7} \\
R^* &= K'Q_{22}K + R - K'P_2 - P_2'K, \tag{8} \\
A^* &= A_{11} - A_{12}J, \tag{9} \\
B^* &= B_1 - A_{12}K, \tag{10} \\
J &= (A_{22} + NA_{12})^{-1}(A_{21} + NA_{11}), \tag{11} \\
K &= (A_{22} + NA_{12})^{-1}(B_2 + NB_1). \tag{12}
\end{align}$$

Proof. The value function of the policymaker in t satisfies the Bellman equation

$$y_t'Sy_t = \min_{u_t} \left(z_t'Qz_t + 2z_t'Pu_t + u_t'Ru_t \right) + \beta \left(y_{t+1}'Sy_{t+1} \right), \tag{13}$$

subject to $x_t = -Ny_t$, the second equation in (1) and y_t given.

We rewrite constraints excluding jump variables. Relationship $x_t = -Ny_t$ can be taken with one lead forward and y_{t+1} is substituted from the first equation (1). We obtain:

$$\begin{align}
x_{t+1} = -Ny_{t+1} &= -N(A_{11}y_t + A_{12}x_t + B_1 u_t) \tag{14} \\
&= A_{21}y_t + A_{22}x_t + B_2 u_t,
\end{align}$$

from where it follows:

$$x_t = -(A_{22} + NA_{12})^{-1}[(A_{21} + NA_{11})y_t + (B_2 + NB_1)u_t] = -Jy_t - Ku_t. \tag{15}$$

where J and K are defined as in (11)-(12). To derive this formula we have used the timing assumption that the private sector moves second and so responds to the policy, i.e. $\partial x_t/\partial u_t = -K \neq 0$ and the private sector can be manipulated by the policymaker. Matrix J is used to denote $\partial x_t/\partial y_t = -J$.

Substitute $u_t = -Fy_t$ into (15) to obtain $x_t = -\left(J - KF \right) y_t = -Ny_t$, so

$$N = J - KF. \tag{16}$$

A straightforward substitution of (11)-(12) into (16) leads to

$$(A_{22} + NA_{12})^{-1}((A_{21} - B_2 F) + N(A_{11} - B_1 F)) - N = 0, \tag{17}$$

and multiplication of both sides by $A_{22} + NA_{12}$ produces equation (NCARE), a Non-symmetric Continuous Algebraic Riccati Equation for N, conventionally abbreviated as 'NCARE'.

We substitute $W_t = \frac{1}{2}y_t'Sy_t$ into formula (13) and, using (15) and $y_{t+1} = (A_{11} - A_{12}J)y_t + (B_1 - A_{12}K)u_t$, obtain that along the optimal path:

$$
\begin{aligned}
y_t'Sy_t = &\; y_t'(Q^* + \beta A^{*\prime}SA^*)y_t + u_t'(P^{*\prime} + \beta B^{*\prime}SA^*)y_t \\
&+ y_t'(P^* + \beta A^{*\prime}SB^*)u_t + u_t'(R^* + \beta B^{*\prime}SB^*)u_t,
\end{aligned}
\tag{18}
$$

where Q^*, P^*, R^*, A^*, and B^* are defined as in (6)-(10).

The optimal feedback rule can be determined from (18) by differentiating the loss function with respect to u_t:

$$
u_t = -(R^* + \beta B^{*\prime}SB^*)^{-1}(P^{*\prime} + \beta B^{*\prime}SA^*)y_t = -Fy_t,
$$

from where the policymaker's reaction function is defined by (POLICY). Now, we substitute the reaction rule $u_t = -Fy_t$ into (18) and obtain equations for S:

$$
S = T_0 + \beta T_1'ST_1,
\tag{19}
$$

where $T_0 = Q^* + F'R^*F - F'P^{*\prime} - P^*F$, $\quad T_1 = A^* - B^*F$.

Equation (19) is equivalent to (DARE), provided we can use (POLICY), with (DARE) a Discrete Algebraic Riccati Equation, again conventionally 'DARE'. In order to see this, we can do the following manipulations, adding and subtracting additional terms:

$$
\begin{aligned}
S &= Q^* + \beta A^{*\prime}SA^* - (\beta B^{*\prime}SA^* + P^{*\prime})'(\beta B^{*\prime}SB^* + R^*)^{-1}(\beta B^{*\prime}SA^* + P^{*\prime}) \\
&= Q^* + \beta A^{*\prime}SA^* - (\beta B^{*\prime}SA^* + P^{*\prime})'F \\
&= Q^* + F'R^*F - F'P^{*\prime} - P^*F + \beta(A^* - B^*F)'S(A^* - B^*F) \\
&\quad + F'P^{*\prime} - F'R^*F + \beta F'B^{*\prime}S(A^* - B^*F) \\
&= T_0 + \beta T'ST + F'\left[\beta B^{*\prime}S(A^* - B^*F) + P^{*\prime} - R^*F\right] \\
&= T_0 + \beta T'ST.
\end{aligned}
$$

as the term in square brackets in the penultimate line is zero because of (POLICY). ∎

Definition 1 *The triplet $\mathcal{T} = \{N, S, F\}$ is a discretionary equilibrium if it satisfies the system of FOCs (DARE)-(NCARE).*

Definition 1 implicitly assumes that the first order conditions are necessary and sufficient conditions of optimality. We proceed with this assumption and demonstrate later in Section 3.1 that, under the assumption of symmetric positive semi-definite $\mathcal{Q}$, the second order conditions for the minimum are always satisfied.

2.4 Discretionary equilibrium as a fixed point

Proposition 1 implies that in order to find a discretionary solution to an optimization problem, we need to solve a simultaneous system of the first order conditions (DARE), (POLICY) and (NCARE). It is evident that although the decision rules are linear for both the policymaker and the private sector, the first order conditions are non-linear in the parameters: they constitute a

non-linear *polynomial* system of $q = (n + k) \times n_1$ equations for q unknown coefficients of matrices S, F and N. Such systems generally can have many solution sets of different dimensions: that is, a system could have isolated solution points, curves, surfaces, etc., all simultaneously. Because of the system's complexity, all but the most trivial of models and policy problems need to be solved numerically. All numerical algorithms we are aware of are iterative. We must search for an appropriate *fixed point*.

It should be clear why we have to adopt an iterative approach. In engineering, there is a voluminous literature on solving DARE- and NCARE- type matrix equations. For example, either DARE or NCARE can be separately solved using an appropriate eigenvalue decomposition which can pick up *all* solutions for a particular equation. But we have a system of equations of different types where the coefficients of DARE depend on the solution of NCARE and vice versa. Each matrix equation now has order higher than two and no generalized decomposition method is possible except in very special cases, one of which we discuss below. This implies that we must iterate to a solution.

Of course, the convergence of an iterative algorithm neither guarantees the uniqueness of the solution, nor does it imply any particular properties of the resulting equilibrium. In what follows we address this issue: we first examine the set of possible equilibria and then discuss the choice of the algorithm. This is a necessary step before we present any examples, because the choice of the algorithm might decide which of potentially many solutions we find.

3 Multiplicity of discretionary equilibria

We start the section on *multiple* equilibria with demonstration of a particular case where the discretionary equilibrium is *unique*. It will be convenient to contrast our coming general results with this particular case, which is proved in Appendix A.[9]

Proposition 2 (A Special Case) *Suppose A_{22} is non-singular, $A_{12} = 0$ and $B_1 = 0$. Then if the discretionary equilibrium exists it is unique.*

In this example, $B_1 = 0$ suggests that predetermined state variables cannot be affected by policy and $A_{12} = 0$ suggests that they cannot be affected by private sector's expectations. A typical example of models in this class is a system where the only predetermined variables are potentially (auto-)correlated shocks, which are *exogenous* predetermined state variables. This is by no means uncommon in models that omit potentially important *endogenous* predetermined state variables such as capital or debt for the sake of simplicity.

Formally, our assumptions result in two of three non-linear first order conditions, (DARE)-(NCARE), becoming linear and disconnected from the third equation in this special case. It is clear that this is unlikely to happen under more general conditions. In what follows, we shall study the first-order conditions in their most general form. Proposition 2 suggests that the model has to have predetermined endogenous state variables in order to be able to generate multiple equilibria under discretion.

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[9]Proposition 2 proves what appears to be a well known fact, but we were unable to find a published proof. Typically, when dealing with a particular problem in this class of models, researchers easily find the particular solution, and it is clear that it is unique by construction, see e.g. Clarida et al. (1999).

As we shall see, the complete characterization of all equilibria is linked to the solution and stability properties of appropriate matrix Riccati equations. These are very like those found in standard control problems but have a number of special properties that allow us to derive useful conditions on the existence of different types of equilibria.

This characterization of the reaction function of agents as a Riccati equation is implicit in much of the literature, including Blanchard and Kahn (1980).[10] It is explicit in much of the literature on dynamic LQ Nash games, but the results are not transferable. A dynamic LQ Nash game would consist of a system of (POLICY)-type decision rules for each player dependent on appropriately defined an interrelated DARE-type equations. For the rational expectations problems the stability and existence conditions are necessarily different as we instead have a mixture of DARE- and NCARE-type equations as part of simultaneous system.[11]

The rest of the section describes the necessary properties of a discretionary equilibrium, if it exists. It is convenient to split system (DARE)–(NCARE) into two blocks, the first block consisting of the two matrix equations (DARE) and (POLICY), and the second block consisting of the matrix equation (NCARE). We shall discuss solutions to these blocks separately and then summarize the results. In the final sub-section we demonstrate that multiple discretionary equilibria, if exist, are all determinate and there can be policy-induced private sector equilibria among them.

3.1 The policy function and the value function

Equation (DARE) is for a square symmetric *value function* matrix S. It is a quadratic matrix equation which coefficients Q^*, P^*, R^*, A^*, and B^* are determined by structural system matrices, given that the household reaction function is given in the form of linear rule $x_t = -\left(J - KF\right) y_t = -N y_t$. Having solved (DARE) we can construct F in a unique way from (POLICY). Of course, both S and F are functions of N.

The following results are standard to the literature on optimal control (all proofs and references are given in Appendix B).

Proposition 3 *Suppose N is given. The following two results hold:*

1. *There is a unique symmetric positive semi-definite solution S to (DARE) if the matrix pair (A^*, B^*) is controllable, i.e. if the controllability matrix $[B^*, A^*B^*, A^{*2}B^*, ..., A^{*n_1-1}B^*]$ has full row rank.[12]*

2. *The policy function F, which is uniquely determined from (POLICY) for given S, is such that all eigenvalues of transition matrix Ω (that defines the evolution of the system under control):*

$$
\begin{aligned}
y_{t+1} &= A_{11}y_t + A_{12}x_t + B_1 u_t = (A_{11} - A_{12}J) y_t + (B_1 - A_{12}K) u_t \\
&= (A^* - B^*F) y_t = \Omega \left(F(S) \right) y_t,
\end{aligned}
\tag{20}
$$

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[10]One of few exceptions is Blake (2004).

[11]NCARE is a *non-symmetric* equation that substantially widens the set of possible solutions as we shall see.

[12]The requirement of controllability of (A^*, B^*) is standard for the linear-quadratic optimal control. We use this condition as a sufficient condition. We do not discuss whether this is necessary condition.

are strictly less than $1/\sqrt{\beta}$ in modulus.

It follows that $\lim_{t \to \infty} \beta^{-t/2} y_t = 0$. Thus, the policy reaction function ensures finite loss. It also follows that necessary conditions for optimality (DARE)-(NCARE) are sufficient as with symmetric and positive semi-definite matrix S the second-order conditions for minimum are always satisfied.

3.2   The private sector's response

Equation (NCARE) is for a non-square matrix N that describes the *response of the private sector*.

For a given policy response written in the form of linear rule $u_t = -F y_t$ the coefficients C_{jk} of (NCARE) depend only on the structural system matrices A and B. The following results are proved in Appendix C.

Proposition 4 *Under the following conditions:*

 i) *matrix C can be diagonalized,*

 ii) *for some $\rho \in \mathbb{R}$, $\rho > 0$, m is the number of eigenvalues of C that are strictly less than ρ in modulus,*

 one of three situations is almost always possible:

 1. *If $m = n_2$ then there is a unique solution N to (NCARE) such that all eigenvalues of transition matrix $\Omega(N)$:*

$$y_{t+1} = C_{11} y_t + C_{12} x_t = (C_{11} - C_{12} N) y_t = \Omega(N) y_t, \tag{21}$$

 are strictly less than ρ in modulus.

 2. *If $m > n_2$ then there are no solutions to (NCARE) such that all eigenvalues of matrix $\Omega(N)$ are strictly less than ρ in modulus.*

 3. *If $m < n_2$ then there are at most $\binom{n-m}{n_2-m}$ of different solutions N to (NCARE) such that all eigenvalues of matrix $\Omega(N)$ are strictly less than ρ in modulus.*

This proposition generalizes the well known Blanchard and Kahn (1980) conditions for rational expectations equilibrium, and their uniqueness condition is reflected in the first point.

Conditions i)-ii) that we impose have a number of implications. First, if matrix C cannot be diagonalized then there is a continuum of solutions to (NCARE), as shown by Freiling (2002). We do not consider this case further. Second, it follows that ρ defines the growth rate of y_t at infinity, and its choice is open to the researcher. Because in LQ infinite-horizon optimization problems transversality conditions are *necessary* conditions, i.e. they follow from optimization, we cannot set ρ to be too small as this will restrict the set of solutions. In what follows we set $\rho = 1/\sqrt{\beta}$ that will allow all solutions that deliver finite loss; this choice is most nonrestrictive and will not rule out any relevant solutions.[13]

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[13]It is sometimes suggested to set $\rho = 1$ (which is the case in Blanchard and Kahn (1980)). However, in some

3.3 Equilibrium outcome

So far, none of our discussion rules out the existence of multiple discretionary equilibria. Nothing prevents the economy that, in the face of some shock, would evolve along one of several paths, each of which satisfies conditions for optimality and time consistency. The cost-to-go along each path would be finite, because the eigenvalues of the corresponding transition matrix Ω were shown to be less than $1/\sqrt{\beta}$. All solutions would be local minima of the loss function, because S was proved to be positive semi-definite so the second-order conditions would always be satisfied. In this section we summarize further properties of multiple equilibria under the assumption that they exist. We shall demonstrate that there is a finite number of them, all of them are determinate and there can be multiple policy-induced private sector equilibria among them.

3.3.1 Determinacy of discretionary equilibria

The following proposition concerns determinacy. Suppose we can find one discretionary solution and can compute all the system matrices. Define the following scalar:

$$
\begin{aligned}
\mathcal{J} \;=\;& \det \left(\Omega' \otimes I_{n_2} - I_{n_1} \otimes \Lambda + \left(\Phi \Lambda^{-1} \otimes \Lambda K \Theta^{-1} B^{*\prime} \right) K_{n_2 n_1} + \Omega' \otimes \Lambda K \Theta^{-1} \Psi \Lambda^{-1} \right. \qquad (22) \\
& \left. + 2 \left(\Omega' \otimes \Lambda K \Theta^{-1} \beta B^{*\prime} \right) \left(I_{n_1^2} - \beta \Omega' \otimes \Omega' \right)^{-1} N_{n_1} \left(\Omega' \otimes Y \Lambda^{-1} \right) \right),
\end{aligned}
$$

where:

$$
\Omega = A^* - B^* F = A_{11} - B_1 F - A_{12} N = C_{11} - C_{12} N,
$$

is the transition matrix, and matrices

$$
\begin{aligned}
\Lambda &= A_{22} + N A_{12}, \\
\Theta &= R^* + \beta B^{*\prime} S B^*,
\end{aligned}
$$

were used before and are both invertible. Matrices

$$
\begin{aligned}
\Phi &= J' Q_{22} - Q_{12} - F' \left(Q_{22} K - P_2 \right)' - \Omega \beta S A_{12}, \\
\Psi &= \left(Q_{22} K - P_2 \right)' - B^{*\prime} \beta S A_{12}
\end{aligned}
$$

are introduced to simplify the expression. Here N_{n_1} is a symmetrizer matrix and $K_{n_2 n_1}$ is a commutation matrix.

The following statement is proved in Appendix D.

Proposition 5 *Suppose we can find a solution to problem (1) – (4) and compute $\mathcal{J}$. If $\mathcal{J} \neq 0$ then:*

1. There can be at most a finite number of other discretionary equilibria.

---

applications of LQ problems it is *optimal* to have either unit root or slowly explosive solutions. This can only happen in models with endogenous predetermined state variables like debt, see Benigno and Woodford (2004), but these *are* some of the models of interest to us. The over-restrictiveness of $\rho = 1$ is discussed, for example, in Sims (2001) for a similar class of LQ RE problems.

2. *All discretionary equilibria are locally isolated i.e. if $\mathcal{T}^i = \{N^i, S^i, F^i\}$, $i = 1, 2, .., p$ is a discretionary equilibrium then there exist no other equilibrium $\mathcal{T}^j = \{N^j, S^j, F^j\}$, $i \neq j$ such that $\|\mathcal{T}^i - \mathcal{T}^j\| < \varepsilon$, where $\varepsilon > 0$ is any arbitrary small real number.*

Proposition 5 essentially proves that if $\mathcal{J} \neq 0$ then the Jacobian of the system of FOCs (consisting of (DARE)-(NCARE)) is not equal to zero identically, and uses the fact that the system of FOCs, as a polynomial system, can then *only* have a finite number of locally isolated solutions. We can say nothing about the case of $\mathcal{J} = 0$ as it can either result in the Jacobian being equal to zero *identically* and so a continuum of solutions is possible, or it might be that the Jacobian has an isolated zero in this point and, again, we end up with the finite number of locally isolated discretionary equilibria. It seems likely, however, that condition $\mathcal{J} \neq 0$ is satisfied in most economic applications.

Corollary 1 *The local isolation of discretionary equilibria implies that all discretionary equilibria are determinate, in the terminology advocated by Woodford (1984), i.e. if $(z_t, \ t = 0, 1, 2, ..)$ is a path under optimal discretionary policy then there exist no other path $(\tilde{z}_t, \ t = 0, 1, 2, ..)$ such that $\|z_t - \tilde{z}_t\| < \varepsilon$ in each period, where $\varepsilon > 0$ is any arbitrary small real number. Thus determinacy is viewed as a property of trajectories and not of their limit points.*

3.3.2 Types of Discretionary Equilibria

Propositions 3 and 4 do not rule out multiple equilibria of two conceptually different types. First, there can be multiple *policy-induced private sector equilibria*, which arise if *one* policy action makes it possible for the private sector to reply in *several* ways. Second, if there is a unique response of the private sector to the every action of the policymaker, we can still have multiple pairs of policy actions and corresponding private sector responses. We call such situation as multiple *policy equilibria*. Both types of equilibria are *discretionary equilibria*.

These two different types of equilibria are easily characterized mathematically. Consider the system (NCARE)–(POLICY). We can substitute F into equation (NCARE) and end up with a system of $(n_1 + n_2) n_1$ equations for the n_1^2 coefficients of S and the $n_1 n_2$ coefficients of N. Solutions to (DARE) (if $N \in \mathbb{R}^{n_1 n_2}$) define a unique $n_1 n_2$-dimensional manifold in $\mathbb{R}^{(n_1+n_2)n_1}$ and solutions to (NCARE) (if $S \in \mathbb{R}^{n_1^2}$) define $L \geq 1$ sets of $n_1 n_1$-dimensional manifold(s) in $\mathbb{R}^{(n_1+n_2)n_1}$. All points of intersection of these manifolds are discretionary solutions. We show how they can be plotted in Section 6.5.

How does this correspond to our above classification? We say that we have multiple *policy equilibria* if there are several points of intersection of a particular N^l with S. We say that we have multiple policy-induced *private sector* equilibria if $L > 1$ and S intersects several N^l, $l = 1, .., \bar{l}$, $1 < \bar{l} \leq L$.

We note the distinction between these types of equilibria but warn that they cannot be distinguished when we use numerical methods to compute the fixed points. This is because we always end up with distinct matrix triplets, as even in the case of multiple policy-induced private sector equilibria the multiple private sector responses do not have the same policy action as the best reply. Policy reacts optimally in a different way to the different household action which are

induced.[14] As we discuss in the next section, the different types of equilibria are likely to arise in different economic situations and so might result in different policy implications. Although we leave this for future research, we also conjecture that equilibrium refinements might affect the different types of equilibria in different ways, ruling out some types of equilibria more easily.

Finally, note that although we described each discretionary equilibrium as a triplet of matrices, propositions 3 and 4 suggest that distinct discretionary equilibria have different matrices N and vice versa. Therefore, we could label a discretionary equilibrium not by a triplet of matrices, $\mathcal{T} = \{N, S, F\}$ but by a single matrix 'identifier' N. We note this but continue with triplets as it is convenient to discuss applications.

As we cannot rule out multiple equilibria and Section 6 contains an example of a model with multiple equilibria with the described properties proving their existence, we now proceed in this presumption.

4   Discussion

Having derived all mathematical results, we can now discuss the intuition behind the multiplicity, necessary conditions for its existence and the ability of the policymaker to choose a particular equilibrium.

We have argued that finding multiple equilibria under discretionary policy in infinite-horizon models should not be surprising. Under discretion the policymaker's choices are conditional on household's beliefs about future policy. Indeed, policy is chosen to minimize future losses, that in turn depend on the expected private sector response. Private sector decisions depend on the forecast of future policy and, so, the current policy depends on beliefs about future policy. Therefore, in principle, we can have multiple equilibria that would correspond to different beliefs about future policy: in each of them the future policy responds to states that were determined based on a forecast of future policy.

The existence of different beliefs about the future course of stabilization seems to be an intrinsic feature of LQ RE models with endogenous predetermined state variables.[15] Indeed, without such variables and under optimal discretionary policy the economy, once disturbed, converges back to the equilibrium within a *single* period. All optimal policy can do is to reduce the *amplitude* of the *immediate* reactions of economic variables to shocks. In contrast, the presence of persistent endogenous variables, like capital or debt, necessarily creates 'history dependence' and implies sluggish adjustment. In this case the policy can also reduce persistence (the *half-life*) of the effects of shocks already in the system.[16] It means that stabilization can be conducted at different speeds. As both the private sector and the policymaker affect the speed of stabilization, there

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[14]In order to distinguish between the two types we would need to compute points along all manifolds and investigate the general topological properties of their union and not only the intersection to identify the difference. We do this below for our example where the equilibria are of the policy and therefore there is only one 'pair', but in general this would be very burdensome.

[15]The presence of such variables is necessary for multiplicity, see Proposition 2.

[16]These two tasks are completely orthogonal to each other in LQ problems, i.e. two optimal policies F_1 and F_2 which only differ in the feedback coefficients on shocks will ensure the same half-life, and two rules which only differ by feedback coefficients on predetermined (dynamic) states will identically reduce the amplitude of concurrent shocks.

can be different but mutually consistent beliefs of economic agents about *speed of stabilization* that will prevail.

Of course, the presence of persistent endogenous state variable does not guarantee the multiplicity of discretionary equilibria. The existing examples of multiple equilibria under discretion often emphasize the importance of dynamic complementarities for the existence of multiplicity, see King and Wolman (2004). More generally, complementarity of strategies can provide us with important information about properties of equilibrium sets in games, see e.g. Vives (2005). It may, therefore, be instructive to think of dynamic interactions between the policymaker and the private sector as a type of a dynamic coordination game. In this game we have two players, the policymaker and the private sector. Of course, the private sector being atomistic is not assumed to set their beliefs strategically, but in a resulting equilibrium the collective action of private agents has a strategic effect on the policymaker's choice of policy.

The analogy is not exact: there is one difference between the model we work with and a mainstream model used in game theory literature. The mainstream model typically characterizes each agent by an objective function, its set of instruments and by constraints on optimization. Our class of problems treats the policymaker and the aggregate private sector in a non-symmetric way, as the optimization problem of the private sector is solved out. As a result, the private sector's behavior is only described by constraints on its strategies, which are the appropriate first order conditions (Euler equations) and the time-consistency requirement.[17] The policymaker's behavior is conventionally described by the objective function, instruments and constraints. Namely this non-symmetric treatment is responsible for the possibility of different types of multiple equilibria, which we discussed above. This non-symmetry also limits the role of complementarities for multiple equilibria, as we now discuss.

Following Cooper and John (1988), we say that we have a complementarity if the optimal stabilization 'strategy' of one agent is increasing in 'strategies' of the other, see also Vives (2005).[18] Such a situation is, of course, possible in LQ RE models as both the policymaker and the private sector affect the speed of adjustment to equilibrium. If the optimal action of one agent reinforces the optimal action of other agent then their actions are complementary in achieving stabilization targets and multiplicity may arise. (Our example in Section 6 demonstrates such a case.)[19] But this is not the only situation which that can lead to multiplicity in LQ RE models.

As we discussed above, in LQ RE models a different kind of multiplicity can also arise.[20] If the RE private sector has several instruments then, because of the linearity of the model, we can think about the private sector's optimal response as of an outcome in a game of several players, each of which has a single instrument and a strategy that satisfies appropriate constraints. Again, strategic complementarities can be helpful to generate the multiplicity of policy-induced private sector equilibria, if, following a policy action, an optimal response using one instrument can

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[17]One way to redefine 'new' objectives of the private sector can be the following: if the strategy satisfies constraints then the payoff is positive but constant, otherwise the payoff is minus infinity.

[18]Vives (2005) distinguishes contemporaneous and intertemporal complementarities. This distinction is unimportant for our informal discussion.

[19]We can also argue that complementarities do not have much predictive power in this case, because if the best responses of the agents are unique, continuous and monotonic, *and* we have multiple equilibria, then we can always find an order on strategies – not necessarily a 'natural' one – that makes the game of one of strategic comlementarities (see discussion in Vives (2005)).

[20]See Propositions 3 and 4.

reinforce the optimal response using another instrument.[21]

In both cases, complementarity can be useful to explain why multiplicity happens. It is hard, however, to come out with more definite results, for example, about the number of equilibria. Because the first order conditions in LQ RE models have quadratic structure and so best reply functions are likely to have several disconnected branches of manifolds, then complementarity is likely to be a local property. Further investigation of these issues is beyond the scope of this paper and we leave it for future research that would utilize the game-theoretical framework.

Finally, as is common in coordination games, and despite the clear Pareto-ranking of different discretionary equilibria, the discretionary policymaker is unable to choose the best one unilaterally. There exists a multiplicity of beliefs, shared by the private sector and the policymaker, about the future course of adjustment. The discretionary policymaker is unable to manipulate the private sector's beliefs in order to choose the best equilibrium *globally*. It is instructive to compare commitment and discretionary equilibria. Under commitment the policymaker is able to manipulate the private sector's expectations along *the whole future path* including its terminal point, and thus, by implication, is able to choose the best trajectories for all variables including beliefs. The discretionary policymaker is only able to manipulate beliefs of the private sector *within* a single period as it chooses the best policy given the intra-period response of the private sector.[22] The intra-period response of the rational private sector is based on the forecast of the future state of economic variables that is in turn based on the future policy. Recall that because we work with Euler equations as a sole description of the private sector behavior, the private sector can appear to act like separate players, because the information structure of the underlying optimization problem has been lost in LQ RE formulation of the problem. Therefore, the private sector can coordinate on any of private sector equilibria, each of which is conditional on their forecast. As the discretionary policymaker takes this forecast as given in every period it can find it optimal to validate the forecast. A coordination failure occurs – all agents can coordinate on any of the multiple equilibria. Anything, even a sunspot, might dictate which equilibrium agents coordinate on.

5 Finding All Equilibria

We have algebraic equations for the components of $\mathcal{T}$, (NCARE)–(POLICY), but the next difficulty we face is how to find *all* of the solutions. It may be that some of the equilibria can be safely discarded as they have no meaningful economic interpretation, and so the complete set need not be found.

In practice, researchers usually use some numerical procedure that iterates in its search for the fixed point of the system. But is it possible to establish to which of many possible solutions a given numerical routine converges? Note that all solutions are likely to be *stable in a conventional sense,* i.e. all eigenvalues of corresponding transition matrices Ω are likely to be within the unit circle (we have established that they will be smaller than $1/\sqrt{\beta}$ in modulus). Each of these solutions may still have particular properties that differentiate between them. We shall discuss

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[21]Here 'optimal strategy' simply means any time-consistent strategy that satisfies first order conditions.

[22]It means that the discretionary policymaker acts as an *intra-period* Stackelberg leader, by optimizing on the 'reaction function' of the household, see Cohen and Michel (1988), Sections 4 and 5.

these issues using a particular approach, to which we now turn.

Oudiz and Sachs (1985) suggest searching for discretionary equilibria for problem (1)–(4) using the solution to the following 'corresponding' finite-horizon problem. They search for a finite sequence $\left\{ \tilde{N}_s, \tilde{S}_s, \tilde{F}_s \right\}_{s=t}^T$, where $x_s = -\tilde{N}_s y_s$, $u_s = -\tilde{F}_s y_s$ and $W_s^T = \frac{1}{2} y_s' \tilde{S}_s y_s$, so $\{u_s, x_s, y_s\}_{s=t}^T$ solve the finite-horizon problem: $\min_{u_t} \frac{1}{2} \mathcal{E}_t \sum_{s=t}^T \beta^{s-t} g_s' \mathcal{Q} g_s$ subject to (1), the time-consistency constraint *and* terminal conditions.

The last of these is particularly important. *Given appropriate terminal conditions*, the finite horizon problem has a *unique* solution. At each moment in time t all agents can rationally predict $\tilde{F}_{t+1}, \tilde{S}_{t+1}, \tilde{N}_{t+1}$, which must be the same for all of them. They apply backwards induction – which is initialized in some unique way – to arrive to $\tilde{F}_t, \tilde{S}_t, \tilde{N}_t$, again common to each of them. The solution can be written as the following recursion (where for $k > 0 : s + k = T - k$) :

$$\tilde{S}_{s+1} = \tilde{Q}_s^* + \beta \tilde{A}_s^{*\prime} \tilde{S}_s \tilde{A}_s^* - \left(\tilde{P}_s^{*\prime} + \beta \tilde{B}_s^{*\prime} \tilde{S}_s \tilde{A}_s^* \right)' \tilde{F}_s, \tag{23}$$

$$\tilde{F}_s = (\tilde{R}_s^* + \beta_s' \tilde{B}_s^* \tilde{S}_s \tilde{B}_s^*)^{-1} \left(\tilde{P}_s^{*\prime} + \beta \tilde{B}_s^{*\prime} \tilde{S}_s \tilde{A}_s^* \right), \tag{24}$$

$$\tilde{N}_{s+1} = (A_{22} + \tilde{N}_s A_{12})^{-1}((A_{21} - B_2 \tilde{F}_s) + \tilde{N}_s (A_{11} - B_1 \tilde{F}_s)), \tag{25}$$

$$\tilde{Q}_s^* = Q_{11} - Q_{12} \tilde{J}_s - \tilde{J}_s' Q_{21} + \tilde{J}_s' Q_{22} \tilde{J}_s, \quad \tilde{P}_s^* = \tilde{J}_s' Q_{22} \tilde{K}_s - Q_{12} \tilde{K}_s + P_1 - \tilde{J}_s' P_2, \tag{26}$$

$$\tilde{R}_s^* = \tilde{K}_s' Q_{22} \tilde{K}_s + R - \tilde{K}_s' P_2 - P_2' \tilde{K}_s, \quad \tilde{A}_s^* = A_{11} - A_{12} \tilde{J}_s, \quad \tilde{B}_s^* = B_1 - A_{12} \tilde{K}_s, \tag{27}$$

$$\tilde{J}_s = (A_{22} + \tilde{N}_s A_{12})^{-1}(A_{21} + \tilde{N}_s A_{11}), \quad \tilde{K}_s = (A_{22} + \tilde{N}_s A_{12})^{-1}(B_2 + \tilde{N}_s B_1). \tag{28}$$

Indeed, we can apply this recursion to find solutions to the infinite-horizon problem (1)–(4). Clearly, if as s tends to infinity and if this recursion converges, it finds a solution to (DARE)–(NCARE) as these equations describe the steady state of dynamic system (23)–(28). So this recursion picks up *an* equilibrium of the infinite-horizon problem.

But this algorithm converges to a *particular* infinite-horizon equilibrium, that is determined by a particular initialization $\left\{ \tilde{S}_0, \tilde{F}_0, \tilde{N}_0 \right\}$. All other equilibria – either those that are either *not* asymptotically stable steady states of system (23)-(28) or those which *are* asymptotically stable fixed points but are in the basin of attraction of *different terminal conditions* of the corresponding finite-horizon problem – will remain undetected.

But we are not confined to use the recursion (23)-(28) to find discretionary equilibria for infinite-horizon problems. Finite- and infinite-horizon problems are different. When solving a finite-horizon problem using the Bellman principle, we have a recursion *as a part of solution*. When solving an infinite-horizon problem, we have algebraic equations that define the equilibrium that have no recursive element defined by the problem. Different recursive algorithms can converge to different discretionary equilibria. But not all discretionary equilibria can be picked up by a particular algorithm, even if we try all possible initializations for that algorithm: Not all discretionary equilibria are necessarily asymptotically stable steady states of a chosen algorithm.

It is not difficult to come up with alternatives. The algebraic equations define a fix point that can be solved iteratively but *not* recursively if desired. If we restrict ourselves to a recursive approach[23] perhaps rather than use an arbitrary algorithm we should base our choice on some

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[23]See Section 2.4 above.

additional assumptions about economic behavior. For example, we can assume that bounded-rational agents solve the problem with a finite horizon and then tend the time index of the last period to infinity. In this case recursion (23)-(28) demonstrate which of infinite-horizon equilibria can be obtained.[24] Alternatively, but not exclusively, agents might use some learning algorithms.

We are almost ready to proceed to the example. In our model we find *all* discretionary equilibria, but only because the model is extremely simple. At this point a definition will prove useful. In order to demonstrate which equilibria are selected by a particular recursion we introduce the concept of R-stability, where R stands for 'recursive'. This concept of R-stability works as an equilibrium refinement, and we base it on the recursion (23)-(28), but it is clear how it can be generalized for wider class of recursions that can also follow from particular assumptions about behavior of economic agents.

Definition 2 *A triplet $\mathcal{T} = \{N, S, F\}$ is a R-stable discretionary equilibrium if it is an asymptotically stable steady state of the difference system (23)-(28). If the equilibrium is not R-stable, we call it R-unstable.*

In what follows we shall search for all R-stable solutions using different initializations of algorithm (23)-(28).

6 Fiscal (Mis-)Conduct and Multiple Equilibria in a New Keynesian Model

In the foregoing we have established a number of conditions that hold in the analysis of discretionary equilibria. Nothing so far implies that multiple equilibria will generally be a feature of interesting models. We now turn to a case where we can demonstrate important implications of a fiscal solvency constraint on discretionary monetary policy making in a very standard model, where we do find multiple equilibria.

6.1 The Model

We employ a conventional model with monopolistic competition and sticky prices in the goods markets (as in Woodford (2003a), Ch. 6), extended to include government spending and nominal government debt.[25] The model leads to a three-equation dynamic system that describes dynamics of the out-of-steady-state economy, this consists of an IS curve, a Phillips curve and an intertemporal budget constraint. We assume a simple fiscal rule for government spending, that is the fiscal authorities stabilize domestic debt. Specifically, the evolution of the economy is determined by the following equations, written in log-linearized form around the steady state with inflation $\pi = 0$, real output Y, real private consumption $C = \theta Y$, real government spending $G = (1 - \theta)Y$, real debt $B = \chi Y$, and the interest rate $1 + R = 1/\beta$. The deterministic part of

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[24]Of course, we should allow the bounded-rational agents to start their backward induction with all possible guesses about the final period, i.e. they should include all rational expectations equilibria.

[25]Detailed exposition is in the working paper version of this paper, Blake and Kirsanova (2008a), and is similar to the one in Schmitt-Grohe and Uribe (2007)

the model can be written as:[26]

$$c_t = \mathcal{E}_t c_{t+1} - \sigma(i_t - \mathcal{E}_t \pi_{t+1}), \tag{29}$$

$$\pi_t = \beta \mathcal{E}_t \pi_{t+1} + \kappa_c c_t + \kappa_y y_t, \tag{30}$$

$$y_t = (1 - \theta) g_t + \theta c_t, \tag{31}$$

$$b_{t+1} = \chi i_t + \frac{1}{\beta} \left(b_t - \chi \pi_t + (1 - \theta) g_t - \tau y_t \right), \tag{32}$$

$$g_t = -\lambda b_t, \tag{33}$$

where endogenous variables are aggregate real output $y_t = \ln(Y_t/Y)$, real private consumption $c_t = \ln(C_t/C)$, real government spending $g_t = \ln(G_t/G)$, real debt $b_t = \chi \ln(B_t/B)$, nominal interest rate i_t and inflation π_t.[27] All structural parameters are defined in Appendix E. This system describes the dynamic behavior of the economy as observed by a policymaker. Equations (29) and (30) describe the decision rules of the private sector. The private sector chooses consumption and inflation at each period in time, such that their future utility and profits are maximized, given the evolution of state variables and policy. The fiscal authority is not an optimizing agent in this set-up, as it mechanically reacts to the level of domestic debt. The private sector explicitly treats monetary policy as given when making decisions.[28]

The central bank uses the nominal interest rate and acts under discretion. We assume that the central bank explicitly maximizes the aggregate utility function, and this implies the following quadratic loss function, where terms independent of policy and all higher order terms are ignored

$$\frac{1}{2} \sum_{s=t}^{\infty} \beta^{s-t} \left(\pi_s^2 + a_c c_s^2 + a_g g_s^2 + a_y y_s^2 \right). \tag{34}$$

where $a_c, a_g, a_y > 0$ and given in Appendix E. This quadratic approximation to the social loss is obtained assuming that there is a production subsidy that eliminates the distortion caused by both monopolistic competition and income taxes. As a result, welfare can be written in terms of deviations from the natural rate levels for output, consumption and government spending, and inflation.[29]

6.2 A canonical representation

In this section we start analytical characterization of the solution. In order to apply results that we obtained in Section 3 we rewrite the problem in a matrix form.

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[26] We omit a stochastic component, that could contain cost-push and other shocks, as it would only increase the number of equations without any additional insights.

[27] Real debt B_t is defined as deflated nominal debt: $\mathcal{B}_t/P_{t-1}$.

[28] Note that this setup implicitly requires $\lambda \ll \infty$. This is a consequence of using a linear rule for G, which is a 'local' fiscal rule.

[29] This derivation follows Woodford (2003a). The alternative way of deriving social welfare of Sutherland (2002) and Benigno and Woodford (2004) is inappropriate, as it assumes some sort of precommitment to a policy which is not the case under discretion. Additionally, elimination of monopolistic and tax distortions is a virtue for us as it allows us to concentrate on the effect of the solvency constraint on monetary policy.

We can substitute out the static variables (output and government spending) and obtain the following reduced form linear-quadratic optimization problem, written in the matrix form:

$$\min_{i_t} \frac{1}{2} \mathcal{E}_t \sum_{s=t}^{\infty} \beta^{s-t} \left(\begin{bmatrix} b_s \\ \pi_s \\ c_s \end{bmatrix}' \begin{bmatrix} \left(a_g + a_y(1-\theta)^2\right)\lambda^2 & 0 & -a_y(1-\theta)\lambda\theta \\ 0 & 1 & 0 \\ -a_y(1-\theta)\lambda\theta & 0 & \left(a_c + a_y\theta^2\right) \end{bmatrix} \begin{bmatrix} b_s \\ \pi_s \\ c_s \end{bmatrix} \right), \quad (35)$$

subject to the dynamic system:

$$\begin{bmatrix} b_{s+1} \\ \pi_{s+1} \\ c_{s+1} \end{bmatrix} = \begin{bmatrix} \frac{1}{\beta}\left(1-(1-\theta)(1-\tau)\lambda\right) & -\frac{\chi}{\beta} & -\frac{\theta\tau}{\beta} \\ \frac{\lambda\kappa_y(1-\theta)}{\beta} & \frac{1}{\beta} & -\frac{(\kappa_c+\theta\kappa_y)}{\beta} \\ -\frac{\sigma(1-\theta)\lambda\kappa_y}{\beta} & -\frac{\sigma}{\beta} & 1+\frac{\sigma(\kappa_c+\theta\kappa_y)}{\beta} \end{bmatrix} \begin{bmatrix} b_s \\ \pi_s \\ c_s \end{bmatrix} + \begin{bmatrix} \chi \\ 0 \\ \sigma \end{bmatrix} [i_s]. \quad (36)$$

Variables π_s and c_s are non-predetermined, b_s is a predetermined variable and i_s is the monetary policy instrument. All system matrices and their components can be easily defined from (35)-(36).

6.3 The Policy and the Value Function

We now derive equations that characterize *the policy*. We introduce the following notation:

$$\begin{bmatrix} \pi_s \\ c_s \end{bmatrix} = -\begin{bmatrix} J_\pi \\ J_c \end{bmatrix} [b_s] - \begin{bmatrix} K_\pi \\ K_c \end{bmatrix} [i_s], \quad (37)$$

and compute the following scalars (see definitions (6)-(8)):

$$Q^* = J_\pi^2 + 2\theta\lambda a_y(1-\theta)J_c + J_c^2\left(a_c + \theta^2 a_y\right) + \lambda^2\left(a_g + a_y(1-\theta)^2\right),$$
$$P^* = J_\pi K_\pi + J_c K_c\left(a_c + \theta^2 a_y\right) + \theta\lambda K_c a_y(1-\theta), \quad R^* = K_\pi^2 + K_c^2\left(a_c + \theta^2 a_y\right),$$
$$A^* = \frac{\chi}{\beta}J_\pi + \frac{\theta\tau}{\beta}J_c + \frac{1}{\beta}\left(1-(1-\theta)(1-\tau)\lambda\right), \quad B^* = \chi + \frac{\chi}{\beta}K_\pi + \frac{\theta\tau}{\beta}K_c.$$

Then, (DARE) is a quadratic equation for a scalar variable S and can be written:

$$\beta B^{*2}S^2 + \left(R^* - \beta\left(Q^*B^{*2} - 2P^*B^*A^* + R^*A^{*2}\right)\right)S + \left(P^{*2} - Q^*R^*\right) = 0. \quad (38)$$

The product of the two eigenvalues of the quadratic equation is negative:

$$P^{*2} - Q^*R^* = -\left(\left(a_c a_g + a_y a_c(1-\theta)^2 + \theta^2 a_g a_y\right)\lambda^2 K_c^2 + a_c\left(K_\pi J_c - J_\pi K_c\right)^2\right.$$
$$\left. + a_g\lambda^2 K_\pi^2 + a_y\left(\lambda(1-\theta)K_\pi + \theta\left(K_\pi J_c - J_\pi K_c\right)\right)^2\right) < 0.$$

and, therefore, the determinant of this equation is positive:

$$D = \left(R^* - \beta\left(Q^*B^{*2} - 2P^*B^*A^* + R^*A^{*2}\right)\right)^2 - 4\beta B^{*2}\left(P^{*2} - Q^*R^*\right) > 0,$$

It follows that the two eigenvalues are always real and of different signs. Since we are looking for a positive value function S, the solution is unique. We can easily find it with conventional

methods for solving quadratic equations. Having found S we can uniquely determine optimal discretionary policy:

$$F = \frac{P^* + \beta B^* A^* S}{R^* + \beta B^{*2} S}.$$

(39)

Note that in equations (38) and (39) all coefficients depend on J_π, J_c, K_π, and K_c, which are, in their turn, functions of N_π and N_c. Therefore $F = F(S(N_\pi, N_c)) = F(N_\pi, N_c)$ depends on the two coefficients of the private sector's responses.

6.4 The Private Sector's Response

This section completes the analytical characterization of the solution to the problem. Here we derive the private sector's response to policy.

Suppose that the policymaker operates with a linear rule:

$$[i_t] = -[F][b_t],$$

(40)

then the private sector's response will necessarily have a linear form, given by:

$$\begin{bmatrix} \pi_t \\ c_t \end{bmatrix} = -\begin{bmatrix} N_\pi \\ N_c \end{bmatrix}[b_t].$$

(41)

The private sector's response solves (NCARE), i.e. it is a solution to the following system of two quadratic equations in N_π and N_c, given F:

$$\chi N_\pi^2 + \theta\tau N_\pi N_c - N_\pi \left((1-\theta)(1-\tau)\lambda + \chi\beta F\right) + (\kappa_c + \theta\kappa_y) N_c$$
$$+ \lambda\kappa_y (1-\theta) = 0,$$

(42)

$$N_c^2 + \frac{\chi}{\theta\tau} N_c N_\pi + \frac{1}{\theta\tau}\left(1 - (1-\theta)(1-\tau)\lambda - \beta\left(1 + \frac{\sigma(\kappa_c + \theta\kappa_y)}{\beta} + \chi F\right)\right) N_c$$
$$+ \frac{\sigma}{\theta\tau} N_\pi - \frac{\sigma}{\theta\tau}\left((1-\theta)\lambda\kappa_y + \beta F\right) = 0.$$

(43)

All solutions to this system can be obtained as points where the solution of (42), depicted by a dashed line, intersects solution of (43), depicted by a solid line.[30] There is a local complementarity of the private sector's choices of how much to consume and of how to price goods it produces: it is optimal to increase consumption and inflation in a response to a higher debt and higher consumption leads to higher inflation. Hence, the dashed line can intersect the solid line in two points denoted by N^1 and N^2. These two solutions ensure that all eigenvalues of transition matrix Ω are less than $1/\sqrt{\beta}$. Because of the particular polynomial structure of the system, we also have a third solution N^3 that belongs to a different branch of the solution to (42). This third solution leads to instability of the system, and so should be disregarded.

Note that if we vary F but condition on a given fiscal feedback λ we can obtain three curves $\{N^k(F) = \{N_\pi^k(F), N_c^k(F)\}, k = 1, 2, 3\}$ of the private sector's reactions to policy F. We shall use this later.

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[30] The lines are shown for some given values of λ and F. The base line calibration is discussed below and in Appendix E. Here $\lambda = 1.08$, $F = 0.1$.

6.5 Multiplicity of discretionary solutions

6.5.1 Multiple solutions for a given λ

We have derived the system of four equations: (38)-(39) and (42)-(43), that determine components of matrices N and F. We can now demonstrate multiplicity of discretionary solutions, see Figure 1, Panel II.

For every reaction of the private sector $\{N_\pi, N_c\}$ we plot the policy response, $-F(N_c, N_\pi)$, which is given by (39). This defines a two-dimensional *surface* in the three-dimensional space, $\{-F(N_c, N_\pi), \{N_c, N_\pi\} \in \mathbb{R}^2\}$, which is the optimal response of monetary policy. Similarly, for every policy F there is an optimal private sector response. The three possible responses $\{N^k(F), k = 1, 2, 3\}$ are identified in Section 6.4. Each of these responses can be presented by a *one-dimensional curve* in the three-dimensional space $\{F, N_c(F), N_\pi(F), F \in \mathbb{R}\}$. The points where these curves intersect the surface are the points of discretionary equilibria. We have only plotted one curve $\{N_c^1(F), N_\pi^1(F)\}$ in Panel II in Figure 1 (where we show *minus* F in order to be consistent with policy reactions presented further in the text) as this is the only line which intersects the surface. Specifically, line $\{N_c^1(F), N_\pi^1(F)\}$ intersects the surface in three points labelled A, B and C.[31] We used a solid line to show the reaction curve of the private sector where it goes above the surface, and used a dotted line to show the reaction curve where it goes below the surface.
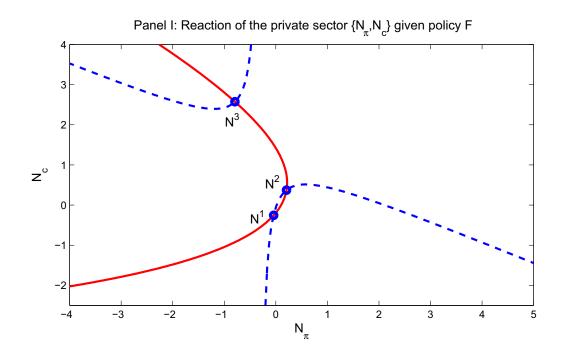
Using the iterative algorithm (23)-(28) we are only able to obtain points A and C – they are R-stable while the intermediate equilibrium B is R-unstable in the sense of Definition 2: we can only obtain equilibrium B by moving along the reaction curve with small increments and checking after each increment whether the equilibrium conditions are satisfied.[32] For a higher dimensional case this method would not work, and an iterative numerical procedure can only pick up solutions that are asymptotically stable fixed points of dynamic system that corresponds to the numerical procedure. Unstable equilibria will remain undetected. In what follows we only discuss equilibria A and C as they can be discovered by agents who make an initial guess about their location and verify it using the backward induction. As we shall argue these two equilibria are economically meaningful, and each of them has been found in a number of similar models but typically under the assumption of that policy is formulated in terms of simple rules. Equilibrium B cannot be obtained with backward induction and we do not discuss it further.

6.5.2 'Active' and 'Passive' Monetary Policy

What is the economic intuition behind the two stable discretionary solutions of the model outlined above? As our model is entirely deterministic, the initial deviation of debt from its steady state value is the only reason for dynamic behavior of the economy. Suppose initial debt is higher

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[31] Sufficient conditions in Garcia and Li (1980) suggest that we have no more than $2^2 3^2 = 36$ solutions. Of course, we have different restrictions that reduce the number of them: S must be symmetric positive semi-definite, N and F should deliver certain eigenvalues of Ω, all solutions must be real, etc. We nevertheless were not able to prove analytically that there are no other points of intersection. We have checked the much bigger area numerically and did not find any more solutions: the surface and the curves were diverging asymptotically.

[32] Equilibrium B does exist as it is a point of intersection of smooth one-dimensional curve with smooth two-dimensional boundless manifold in three-dimensional space. We have shown numerically that the curve can be either below or above the manifold, by the continuity argument there exist a point of intersection.

Panel I: Reaction of the private sector $\{N_\pi, N_c\}$ given policy F
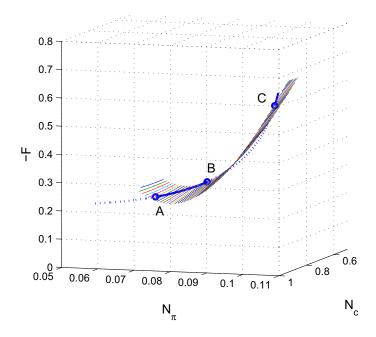
Panel II: Multiple discretionary equilibria

Figure 1: Discretionary equilibria for a given weak control λ.

than its steady state level, $b_t > 0$. Further suppose that debt is tightly controlled by fiscal policy, i.e. λ is relatively large. Higher b_t results in lower spending, lower output, negative inflation and higher debt, unless monetary policy intervenes. Monetary policy intervenes and the interest rate is reduced to stimulate economic activity. Only a small reduction in interest rates is needed to deal with the inflation problem, because the RE private sector sees no problem with debt sustainability and does not set inflation expectations high. As a result, inflation rises only a little, interest rate falls by correspondingly little and debt *slowly* converges back to the steady state.

By contrast, if fiscal policy does not react to debt at all, i.e. $\lambda = 0$, then monetary policy acts to reduce interest rates *by more* in response to the positive initial debt displacement, as it must help stabilize debt and *also* validate private sector expectations of high inflation. Rational private agents know that debt has to be stabilized without fiscal adjustment, that the interest rate is going to be low, and so they set expectations of future inflation high. As a result, inflation rises by more, interest rates fall by more and debt *quickly* converges back to the steady state.
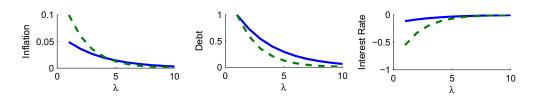
However, for the intermediate fiscal regime where debt is only weakly controlled, *both* these policies can be consistent with a long run where all variables converge to their steady states. We plot responses of the economy for two discretionary equilibria with a positive but small fiscal feedback λ in Panel I in Figure 2. The existence of multiple discretionary solutions in our case depends on the possibility of forming varying but consistent beliefs by both agents about the future course of stabilization in the presence of dynamic complementarities.

There is a complementarity between the sets of choice variables of the two agents in their effect on debt. First, the policymaker chooses the interest rate, which has a first-order effect on debt if the level of steady state debt, χ, is not zero. A lower interest rate reduces the stock of debt. Second, inflation and consumption, which are chosen by the private sector, also affect debt. Inflation has a direct effect on debt if $\chi \neq 0$: higher inflation reduces the stock of debt. Consumption affects debt via output and tax collection in an unambiguous way: higher consumption reduces the debt. Additionally, there is an indirect effect which remains if $\chi = 0$: lower interest rate and higher inflation increase consumption, and therefore reduce debt. Crucially, for any χ, the private sector's actions reinforce the effect of the monetary policy on debt: lower interest rate results in higher inflation. (Table 1 reports the *equilibrium* responses of the economic agents. It is apparent that $\partial c_t / \partial (-i_t) > 0$, $\partial \pi_t / \partial (-i_t) > 0$ in both equilibrium points A and B; we cannot compute the signs of components of $K(F) = -\partial N(F)/\partial F$ for any value of F but using the argument of continuity we suggest that there is an area where the derivative preserves its sign.) As a result, there are several combinations of time-consistent responses of interest rate and inflation that lead to the debt stabilization; they result in stabilization with different speed, see Figure 1.
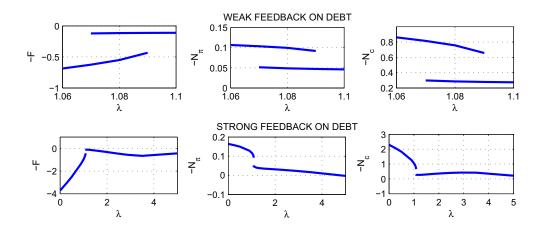
The difference in response to a debt displacement in the two equilibria reflects the distinction between 'active' and 'passive' monetary regimes identified in Leeper (1991) and Leith and Wren-Lewis (2000). One can see certain similarities with our results: Panel II in Figure 2 demonstrates that only Plan C survives for small λ, in particular for $\lambda = 0$; if λ is sufficiently large, then only Plan A survives.[33] If λ is sufficiently large then monetary policy reacts to debt only weakly and

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[33]When λ is large enough and increases then F, N_π and N_c will grow in line with λ, as monetary policy has to offset the excessive tightness of fiscal control.

Panel I: Optimal response to a unit debt displacement if the fiscal feedback is weak

Panel II: Feedback parameters as a function of fiscal feedback, λ

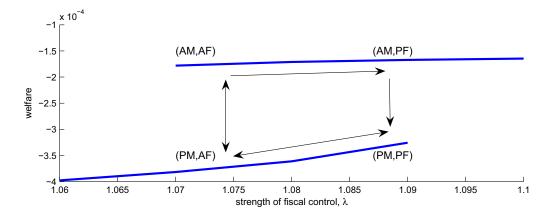Panel III: Welfare, Different monetary/fiscal regimes in post−war US data and Multiple Equilibria

Figure 2: The two discretionary equilibria

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Table 1: Equilibrium Responses of the Monetary Policymaker and the Private Sector

	Policymaker $u_t = -Fy_t;$	Private Sector $x_t = -Jy_t - Ku_t = -Ny_t$
Plan A :	$[i_t] = -[0.115][b_t];$	$\begin{bmatrix} \pi_t \\ c_t \end{bmatrix} = - \begin{bmatrix} -0.047 \\ -0.234 \end{bmatrix}[b_t] - \begin{bmatrix} 0.017 \\ 0.442 \end{bmatrix}[i_t] = - \begin{bmatrix} -0.05 \\ -0.29 \end{bmatrix}[b_t]$
Plan C:	$[i_t] = -[0.551][b_t];$	$\begin{bmatrix} \pi_t \\ c_t \end{bmatrix} = - \begin{bmatrix} -0.097 \\ -0.557 \end{bmatrix}[b_t] - \begin{bmatrix} 0.004 \\ 0.363 \end{bmatrix}[i_t] = - \begin{bmatrix} -0.10 \\ -0.76 \end{bmatrix}[b_t]$

stabilizes inflation and output in a conventional way, so can be classified as 'active'. Fiscal policy can be classified as 'passive' as it is totally devoted to control of domestic debt. If λ is small then monetary policy controls debt tightly in order to ensure that it will converge back to equilibrium, so it can be classified as 'passive'. Fiscal policy can be classified as 'active' as it pursues some other targets but not the control of domestic debt. In this regime inflation is accommodated as it actually helps to reduce real debt. Our classification resembles the one in Leeper (1991), but it differs in that we consider discretionary policy, not a policy formulated in terms of simple rules. Leeper (1991) bases his classification on the determinacy properties of a rules-based equilibrium, while, as as we have argued above, a discretionary equilibrium is always determinate. We make the link between interpretations tighter in Section 6.7.

6.6   Related Literature

At this point we should position our results within the existing literature on multiple equilibria in related models. King and Wolman (2004) (henceforth, KW) provide an analytical survey of the recent literature on multiple equilibria under discretion. It is convenient to distinguish between static and dynamic models. The class of dynamic models includes the KW model and Siu (2008). The class of static models includes Albanesi et al. (2003) and Armenter (2007). Non-linearity is a necessary property of all of these models in order to generate a multiplicity. Our findings can be most naturally compared with those in KW, even though our (dynamic) model is linear.

A key non-linearity in the KW model originates in the model of price setting behavior. There are many identical price-setters who choose prices optimally, based on the forecast of future monetary policy and the corresponding aggregate price level. The authors study a symmetric equilibrium and demonstrate that the pricing strategies of different price-setters are complementary to each other. Because of these complementarities multiple *private sector* equilibria arise: monetary policy determines the aggregate price level on which the private sector coordinate, the private sector can be either optimistic or pessimistic about the future price level and monetary policy has to respect these expectations.[34] In contrast, our model of price-setting behavior is linearized. This eliminates the possibility to have complementarity of the price-setting actions

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[34]Siu (2008) has taken the analysis in KW further and investigated how endogenous pricing affects multiplicity. The type of equilibria remain the same.

in a symmetric equilibrium. There is still a possibility, however, to have multiple *policy-induced private sector equilibria* because of complementarity between the two choice variables of the private sector, inflation and consumption, in their effect on debt, as higher inflation leads to higher consumption and vice versa; but this possibility has not occurred. It is the complementarity between actions of the private sector and of the policymaker that resulted in the multiplicity. The difference from KW is that the equilibria in the LQ RE framework are non-symmetric. It is additionally important for their existence that the time horizon is infinite.

Both KW and this paper assume that the policymaker acts as a Stackelberg leader: when optimizing it takes the forecast of future policy which are implicit in private sector expectations as given, and chooses the best 'point' on the household's reaction function. Because the policymaker optimizes subject to the private sector's forecast, it can be said that it is 'trapped' by the private sector. The best equilibrium may therefore not prevail. A similar conclusion about the inability to choose the best equilibrium is reached in Albanesi et al. (2003), although for a different reason. Their non-linear model is essentially static where the sequence of actions by the agents is entirely dependent on the time when shock happens, but all events happen within a single period. Unlike our model, their model assumes intra-period Stackelberg leadership of the *private sector*, so the private sector is able to manipulate the policymaker and to trap it in a particular equilibrium *within the period*, and, by implication, within the whole time path.[35] The reason why we have similar results when *the policymaker leads* is because we have *infinite* time horizon and dynamic expectations.

6.7   Empirical Relevance

Finding multiple equilibria can help us to understand why policies fluctuate. Following a sunspot, the economy can switch from one equilibrium to another with an apparent change in volatility but without an apparent change in fundamentals. One example of fluctuating policies is documented in Davig and Leeper (2006) (henceforth, DL). The authors work with a model that is very close to the one we consider here.

DL identify four different monetary and fiscal policy regimes in the US in the post-war period. They assume that economic policy is formulated in terms of simple rules and each policymaker can implement either an 'active' or a 'passive' rule. As a result, there are four possible combinations of active/passive monetary policy (we label them as AM and PM correspondingly) and active/passive fiscal policy (labelled AF and PF). The authors estimate these rules and document the sequence of movements from one such regime to another in the post-war US history. Using our model and the resulting discretionary equilibria we can interpret these movements as switches between equilibria. The third panel in Figure 2 demonstrates this in stylized form.
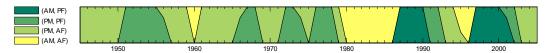
Although our fiscal feedback parameter λ does not directly correspond to the estimated fiscal rules in DL, we can still associate it with their estimated degree of fiscal feedback.[36] We can plot social welfare[37] where the two different equilibria are denoted by two separate branches against

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[35]See also Armenter (2007).

[36]DL use taxes as fiscal instrument and estimate more elaborate policy rules.

[37]We plot welfare as minus loss, hence negative numbers. The loss is measured in percentage of steady state consumption; the loss is small as we work with deterministic model, assuming that the only sourse of displacement is the higher debt level (1%) in the initial moment.

Panel I: The Four Policy Regimes in Post-war History of the US (as shown in Figure 5 in Davig and Leeper (2006))

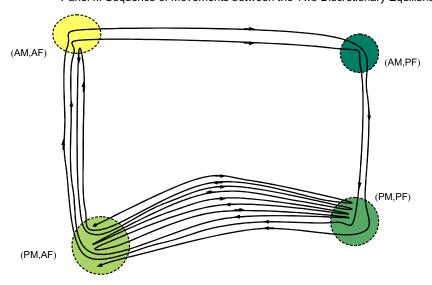Panel II: Sequence of Movements between the Two Discretionary Equilibria

Figure 3: Four Regimes and Two Discretionary Equilibria

the strength of fiscal feedback. The four labels show where the four policy regimes identified by DL might lie. Arrows show all realized movements between the states that happened in the post-war period in the US. The sequence of regimes can be seen in the first panel of Figure 3.

This picture is adapted from Figure 5 in DW. For every year the relative width of every color corresponds to probability that each regime prevails. For example, in 1956 we have approximately a 70% probability of (PM, PF) regime and a 30% probability of (PM, AF) while in 1985 it is a 100% probability of (AM, AF).

The third panel in Figure 2 locates these 'regimes' in our framework, and it is interesting to look at the switches between the equilibria and the order in which they happen. The sequence of events can be schematically presented with a graph going through four possible states, that are represented as 'corners' in Figure 3, Panel II. In 1945 the US economy starts at (PM, AF) and it arrives to the same state in 2005.

Our framework suggests that a policymaker who wishes to implement a policy change needs the private sector to believe in the success of such a change. This can happen in two cases: Either the private sector decides that there are fundamental reasons why the policy change will

be successful, or there is a sunspot that changes the private sector expectations about the future course of the policy. We now make several observations.

First, the majority of movements are from state (PM, AF) to (PM, PF) and back. In our framework this implies that the economy remains in the worst equilibrium. Note also that (PM, PF) is the worst state in terms of social welfare among the four possibilities and the biggest gain in welfare can be obtained by a 'jump' to (AM, AF).

Second, the best equilibrium has been attained only in 'jumps' *from* the worst state (PM, AF). There are three such 'jumps' from (PM, AF) to (AM, AF), in 1959, 1980 and in 1994, one of which resulted in a 'fall' back. This 'fall' back has happened almost immediately: the economy stayed in (AM, AF) for less than a couple of years.

Third, after the two relatively long periods of being in the best equilibrium the economy 'falls' into the worst equilibrium. It moves from (AM, PF) to (PM, PF) and this route is with the minimal loss in welfare as the two branches in the third panel of Figure 2 are closest to each other. These happened in 1991 and 2001, both post-recession years in which the FED lowered interest rates.

This sequence of events might suggest that if the economy is 'stuck' in the worst equilibrium, all agents are only able to coordinate on the better equilibrium once it is *substantially* different from the current one in welfare terms *and* there is a risk of a 'slide down' into even worse and disconnected part of the worst equilibrium. For example, adopting an additional aggressiveness in monetary policy could work as a 'sunspot' to signal to the private sector to coordinate their expectations in the desired way.

This interpretation also suggests that it might be difficult for all agents to see full welfare consequences of moving from (AM, PF) to (PM, PF) should the economy reach (AM, PF). Agents should have happily coordinated on a sunspot expecting temporarily less active monetary policy without incurring big welfare losses. The evidence suggests that the latter has never been the case: the economy has always slid back down towards the inferior (PM, AF) state.

To summarize, the evidence presented in DL can be interpreted consistently within the framework of multiple discretionary equilibria. Some of the changes in 'regimes' can be seen as switches between the two equilibria that follow a sunspot. In the context of our model, a sunspot can activate a change of monetary policy even if fiscal policy remains 'active'. We do not discuss what exact signal can be interpreted as a sunspot in each particular case, but note instead the very particular sequence of 'jumps' and 'falls' between the two equilibria. The observed regularity invites new research to explain the pattern, but further explorations are beyond the scope of this paper.

Finally, note that it is generally difficult to identify whether the economy was 'stuck' in a particular equilibrium as we only observe realized jumps conditional on realized change in expectations. In other words, we cannot 'detect' sunspots that keep the economy in a particular equilibrium if fundamentals change, as the same sunspot does not allow a change in monetary policy. The evidence in DL suggests there is an overlap of the two equilibrium branches in the third panel in Figure 2. Better policy design can help to either minimize the area of overlap or to drive the economy further away from this area and so avoid sunspot-driven fluctuations.

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7 Conclusions

The main contribution of this paper is to show that infinite-horizon linear quadratic RE models of discretionary policy not only can have multiple equilibria, but in some situations do. This result means that papers applying these methods may not have discovered all of the equilibria to their problem and that researchers should check that they either have a unique equilibrium or dealt with the multiplicity issue for their problem. We describe some general properties of discretionary equilibria in LQ RE models and demonstrate one way how all equilibria can be obtained for a simple model.

Our results can improve our understanding of why macroeconomic policies fluctuate. The multiplicity of discretionary equilibria imply that, depending on the mutual beliefs about the future course of macroeconomic policy the economy can either switch to a new equilibrium with no apparent reason or remain in a particular equilibrium overly long. We present one example of how this might happen.

Our results can have important implications for the two strands of the theoretical literature on monetary policy. First, the discovery of multiple equilibria calls for future research on equilibrium selection: applications of dynamic games and learning can bring important results. Second, the 'theory of optimal delegation' might need to be revisited as it is built on examples with unique equilibrium. Policy implications can be highly misleading as a change of policy objective can provoke a switch to a different equilibrium than desired.

Finally, our example shows that even a simple three-equation dynamic model can generate multiple discretionary equilibria. We suspect that such multiple equilibria are likely to be a frequent feature of most moderately complicated models routinely employed for policy forecasts, as such models typically incorporate capital and other state variables that generate additional persistence. In a world where precommitment is not possible, policymakers should be aware of the possibility of multiple equilibria.

A Proof of Proposition 2

Formulae (12) suggests $K = A_{22}^{-1} B_2$ so it does not depend on N. It follows also that R^* does not depend on N and, generally speaking, is non-singular.

Equation (16) can be written as

$$A_{22}N - NA_{11} = A_{21} - B_2F, \tag{CLE}$$

so there is no quadratic term in N. (CLE) is in the form of a continuous-time Lyapunov equation, a linear equation in N. Similarly, equation (POLICY) collapses to

$$F = (R^*)^{-1} \left(\left(B_2' A_{22}^{-1\prime} \mathcal{Q}_{22}' - \mathcal{P}_2' \right) \left(N + A_{22}^{-1} B_2 F \right) - \left(\mathcal{Q}_{12} A_{22}^{-1} B_2 - \mathcal{P}_1 \right)' \right), \tag{44}$$

and hence S does not affect F. Both equations together constitute a linear in coefficients of F and N system of $(k+n_2)n_1$ equations (after applying the vec-operator). If the system is non-singular, the solution is unique. Having determined F we can find corresponding S from equation (DARE), which always has a unique symmetric solution, as we discuss in Section 3.1.

B Solutions of (DARE)

This appendix states conditions under which a solution to (DARE) exists and is unique.

We obtained (DARE) from the first order conditions to the optimization problem, assuming that we know the response of the private sector N. By substituting out the rational response of the private sector, we reduce our problem to a well-known engineering one, and simply apply that analysis. Explicitly, solving the following *standard* optimal control problem is equivalent to solving (DARE) and (POLICY). The problem is to stabilize the following linear system, given the initial conditions:

$$w_{t+1} = \beta^{\frac{1}{2}} \left(A^* w_t - B^* v_t \right), \quad w_0 = \bar{w}, \tag{45}$$

and given the positive semi-definite cost functional:

$$J(w_0, v) = \sum_{t=0}^{\infty} \begin{bmatrix} w_t' & v_t' \end{bmatrix} \begin{bmatrix} Q^* & P^* \\ P^{*\prime} & R^* \end{bmatrix} \begin{bmatrix} w_t \\ v_t \end{bmatrix}. \tag{46}$$

It is well known (Kwakernaak and Sivan (1972, Ch. 6)) that if the pair matrices $(\beta^{\frac{1}{2}} A^*, \beta^{\frac{1}{2}} B^*)$ in (45) is controllable (i.e. if the $k \times n_1$ controllability matrix $[\beta^{\frac{1}{2}} B^*, \beta^{\frac{2}{2}} A^* B^*, \beta^{\frac{3}{2}} A^{*2} B^*, ..., \beta^{\frac{n_1}{2}} A^{*n_1-1} B^*]$ has rank n_1 or full row rank) then there exists a unique optimal control v_t that minimizes (46). The system under this control evolves following:

$$w_{t+1} = \beta^{\frac{1}{2}} \left(A^* - B^* (R^* + \beta B^{*\prime} S B^*)^{-1} \left(P^{*\prime} + \beta B^{*\prime} S A^* \right) \right) w_t = \beta^{\frac{1}{2}} \Omega w_t,$$

and all eigenvalues of matrix $\beta^{\frac{1}{2}} \Omega$ are strictly inside the unit circle. Obviously, the controllability of $(\beta^{\frac{1}{2}} A^*, \beta^{\frac{1}{2}} B^*)$ is equivalent to the controllability of (A^*, B^*). It follows, therefore, that *the solution pair $\{S, F\}$ to (DARE) and (POLICY) exists and unique if (A^*, B^*) is controllable.* It is also a textbook result that matrix S is symmetric and positive semi-definite if $\hat{R}^* = \begin{bmatrix} Q^* & P^* \\ P^{*\prime} & R^* \end{bmatrix}$ is symmetric and positive semi-definite.

One can easily demonstrate that $\hat{R}^* = (\mathcal{C}\Psi)' \mathcal{Q} (\mathcal{C}\Psi)$, where $\Psi = \begin{bmatrix} I & 0 \\ -J & -K \\ 0 & I \end{bmatrix}$ and $\mathcal{C}$ is defined in Section 2.1. Because $\mathcal{Q}$ is symmetric and positive semi-definite then $\hat{R}^*$ has the same properties. Hence S is symmetric and positive semi-definite.

C Solutions of (NCARE)

This appendix discusses existence and uniqueness of a solution to (NCARE).

C.1 Existence of solution of (NCARE).

What does the general solution to (NCARE) look like? Suppose matrix C in equation (5) can be diagonalized as[38] $C = M^{-1} \Lambda M$. Matrix M is the matrix of left eigenvectors which correspond

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[38] In what follows we always assume that matrix C is *simple*, i.e. all its eigenvalues are of geometric multiplicity one, and the column rank of M is equal to n. This case is of practical interest; but Freiling (2002) discusses

to eigenvalues Λ. Arrange the eigenvalues so that Λ_u is a diagonal matrix of size n_2 and Λ_s a diagonal matrix of size $n_1 = n - n_2$. Rearrange similarly M and partition it to give

$$\Lambda = \begin{bmatrix} \Lambda_s & 0 \\ 0 & \Lambda_u \end{bmatrix} \text{ and } M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}.$$

Now, construct:

$$N = M_{22}^{-1} M_{21}. \tag{47}$$

Matrix M_{22} needs to be invertible; this is the additional condition, but it is unlikely to be restrictive. It has been shown in the literature (see e.g. Medanic (1982) Theorem 1) that *any solution of (NCARE) can be represented in this way for some adequate Jordan basis of C. If all eigenvalues of C are simple then there are at most $\binom{n}{n_2}$ of different solutions N.* Note that we did not make any assumptions about matrices Λ_s and Λ_u apart from assuming that they are of particular size.

C.2 Uniqueness of a stable solution to (NCARE).

Can we choose a particular matrix N? Suppose Λ_u collects all eigenvalues that are greater than ρ in modulus and suppose there are $m \le n_2$ of them, so Λ_u might also have $n_2 - m$ eigenvalues that are not greater than ρ in modulus. Clearly, if $x_t = -M_{22}^{-1} M_{21} y_t = -N y_t$ then system (5) collapses to:

$$y_{t+1} = \left(C_{11} - C_{12} M_{22}^{-1} M_{21} \right) y_t. \tag{48}$$

The following theorem was proved in e.g. Freiling (2002), Blake (2004).

Theorem 2 (The eigenvalues of matrix C) *The eigenvalues of $C_{11} - C_{12} M_{22}^{-1} M_{21}$ are Λ_s and the eigenvalues of $C_{11} + C_{22} M_{22}^{-1} M_{21}$ are Λ_u.*

Theorem 2 demonstrates that eigenvalues of $C_{11} - C_{12} M_{22}^{-1} M_{21}$ are all not greater than ρ and so the system (48) has solutions that do not grow faster than ρ^t. Given y_0 we completely determine the path y_t, $t > 0$. It is clear that if $m = n_2$ then Λ_u is uniquely determined and (47) is a unique solution (as in Blanchard and Kahn (1980)). If $m < n_2$ we can construct Λ_u in at most $\binom{n-m}{n_2-m}$ ways, adding different eigenvalues that are smaller than ρ in modulus to Λ_u.[39] We have proved Point 3 in Proposition 4.

D Proof of Proposition 5

System (DARE)-(NCARE), (6)-(12) is a polynomial system of equations for unknown coefficients of ten matrices N, F, S, J, K, Q^*, P^*, R^*, A^*, and B^*. Any polynomial system of order l can either have a finite number of real and locally isolated solutions, or it has a continuum of solutions. The second case happens if and only if the determinant of the Jacobian is equal to zero *identically.*

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implications of higher geometric multiplicity.

[39]By construction, $C_{11} + C_{22} M_{22}^{-1} M_{21}$ is invertible if M_{22} is invertible.

We can compute the Jacobian (denote it $\mathfrak{J}$) of system (DARE)-(NCARE), (6)-(12) and demonstrate that:[40]

$$\det \mathfrak{J} = (\det \Lambda)^{n_1+k} (\det \Theta)^{n_1} \det \left(\beta \Omega' \otimes \Omega' - I_{n_1^2} \right) \times \mathcal{J}$$

where $\mathcal{J}$ is defined by formula (22). The assumption of that at least one solution can be found implies $\det \Theta \neq 0$, $\det \Lambda \neq 0$ and all eigenvalues of Ω are strictly less than $1/\sqrt{\beta}$ in modulus so $\det \left(\beta \Omega' \otimes \Omega' - I_{n_1^2} \right) \neq 0$. As $\mathcal{J} \neq 0$ then $\det \mathfrak{J} \neq 0$ and so the polynomial system (DARE)-(NCARE), (6)-(12) can only have a finite number of locally isolated solutions.

E   Calibration

One period is taken as equal to one quarter of a year. All behavioral parameters below are taken from Rotemberg and Woodford (1997). A more detailed derivation is given in Blake and Kirsanova (2008a).

Assumed parameters			
$\beta = 0.99$	Household discount rate	$\psi = 2.0$	Labour elasticity
$\sigma = 0.5$	Intertemporal elasticity of substitution	$\chi = \frac{B}{Y} = 0.1$	Steady state level of debt to output ratio
$\varepsilon = 5.0$	Elasticity of substitution between domestic goods	$\theta = \frac{C}{Y} = 0.75$	Steady state consumption to output ratio
$\gamma = 0.75$	Probability of that price remains unchanged		
Derived parameters			
$\kappa_c = (1-\gamma\beta)(1-\gamma)\psi/ \left(\gamma\sigma \left(\psi+\epsilon\right)\right)$	$a_c = \psi(1-\gamma\beta)\left(1-\gamma\right)\theta/ \left((\epsilon+\psi)\gamma\epsilon\sigma\right)$		
$\kappa_y = (1-\gamma\beta)(1-\gamma)/\left(\gamma\left(\psi+\epsilon\right)\right)$	$a_g = \psi(1-\gamma\beta)\left(1-\gamma\right)\left(1-\theta\right)/ \left((\epsilon+\psi)\gamma\epsilon\sigma\right)$		
	$a_y = \psi(1-\gamma\beta)\left(1-\gamma\right)/ \left((\epsilon+\psi)\gamma\epsilon\psi\right)$		

The tax rate, τ, is found from the steady state condition: $\chi = \frac{1}{\beta}\left(\chi + (1-\theta) - \tau\right)$. When computing losses we assume that in the initial moment debt is 1% higher than it is in the steady state.

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[40]The proof is tedious but straightforward and can be found in the working paper version of this paper, Blake and Kirsanova (2008a).

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