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DIFFERENTIATING AMBIGUITY: A COMMENT

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Abstract

In [4] Ghirardato, Macheroni and Marinacci (GMM) propose a method for distinguishing between perceived ambiguity and the decision-maker's reaction to it. They study a general class of preferences which includes CEU and α-MEU and axiomatise a subclass of α-MEU preferences. We show that for Hurwicz preferences the proposed measure of ambiguity depends on parameters which intuitively reflect ambiguity-attitude. Furthermore, any α-MEU preferences which satisfy the CEU axioms, satisfy GMM's axioms *if and only if* $\alpha = 0$ or 1, that is, the capacity must be either convex or concave.

Keywords, Ambiguity, multiple priors, Hurwicz, Choquet expected utility.

JEL Classification: D81.

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DIFFERENTIATING AMBIGUITY: A COMMENT

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1 INTRODUCTION

In a recent paper Ghirardato, Macheroni and Marinacci [4] (henceforth GMM), consider a class of preferences that encompasses both the Choquet expected utility (henceforth CEU) model of Schmeidler [10] and the multiple prior model of Gilboa & Schmeidler [6]. For a given preference relation $\succsim$ from this class, they define the (generally) partial ordering $\succsim^*$ to be the maximal sub-relation of $\succsim$ that satisfies all the axioms of subjective expected utility (SEU) except completeness. They dub $\succsim^*$ the *unambiguous preference relation* and show that it admits a representation in the style of Bewley [1]: in particular, there is a utility $u(\cdot)$ defined on the set of outcomes X and a non-empty, compact and convex set of probability measures $\mathcal{D}$ defined on the state space S such that for any pair of acts f and g,

$$f \succsim^* g \text{ if and only if } \int_S u(f(s))\, dP(s) \geq \int_S u(g(s))\, dP(s), \text{ for all } P \in \mathcal{D}.$$

The relation $\succsim^*$ is complete if and only if $\mathcal{D}$ is a singleton in which case $\succsim$ equals $\succsim^*$ and is SEU.

Furthermore, they establish the existence of a function $\beta(\cdot)$ that maps each act f to a weight $\beta(f)$ in $[0,1]$, such that $\succsim$ can be represented by the functional

$$V(f) = \beta(f) \min_{P \in \mathcal{D}} \int_S u(f(s))\, dP(s) + (1 - \beta(f)) \max_{P \in \mathcal{D}} \int_S u(f(s))\, dP(s). \tag{1}$$

GMM interpret $\mathcal{D}$ as the decision maker's perception of ambiguity and the weight $\beta(f)$ (which in general depends on the act f, being evaluated) as an index of the decision-maker's aversion to that ambiguity, since the larger is $\beta(f)$, the greater is the weight the DM gives to the "pessimistic" evaluation of f given by $\min_{P\in\mathcal{D}} \int_S u(f(s))\, dP(s)$.

An interesting subclass of these preferences is one in which the ambiguity aversion index $\beta(\cdot)$ is constant. That is, the special case of their class of preferences for which we can find a β in $[0,1]$, such that $\succsim$ can be represented by the functional

$$V(f) = \beta \min_{P \in \mathcal{D}} \int_S u(f(s))\, dP(s) + (1 - \beta) \max_{P \in \mathcal{D}} \int_S u(f(s))\, dP(s). \tag{2}$$

This representation is a special case of the well-known α-MEU, which has the same functional representation

$$V(f) = \alpha \min_{P \in \mathcal{P}} \int_S u(f(s))\, dP(s) + (1-\alpha) \max_{P \in \mathcal{P}} \int_S u(f(s))\, dP(s), \tag{3}$$

except that $\mathcal{P}$ (a non-empty, compact and convex set of probability measures defined on the state space S) need not be equal to the set $\mathcal{D}$.

GMM provide an axiomatic characterization of their subclass by taking a preference relation $\succsim$ that admits a representation of the form given by (1) and showing that the associated ambiguity aversion index $\beta(\cdot)$ is constant if and only if $\succsim$ satisfies the following axiom.

GMM's Axiom 7 For all pairs of acts f, g, $C^*(f) = C^*(g) \Rightarrow f \sim g$, where

$$C^*(h) = \{x \in X : y \succcurlyeq^* h \Rightarrow y \succcurlyeq^* x \text{ and } h \succcurlyeq^* y \Rightarrow x \succcurlyeq^* y\}.$$

To understand what this axiom entails, notice first that the set $C^*(f)$ may be viewed as the set of certainty equivalents of f with respect to the relation $\succcurlyeq^*$. If $\succsim$ is not SEU then $\succcurlyeq^*$ is incomplete, which in turn means that in general an act has an interval of certainty equivalents. Furthermore, GMM show that

$$u(C^*(f)) = \left[\min_{P \in \mathcal{D}} \int_S u(f(s))\, dP(s), \max_{P \in \mathcal{D}} \int_S u(f(s))\, dP(s)\right].$$

Thus for a DM who satisfies this axiom, the set of certainty equivalents with respect to $\succcurlyeq^*$ contains *all* the information the DM uses in evaluating f. What their characterization result establishes is that this dependence on the range of utilities must be linear.

In the next section we first analyse the case of Hurwicz preferences, a special case of preferences that admit representations of the form given in (3) with $\mathcal{P}$ equal to the entire set of probability measures defined on the state space S. We show that for $\alpha \neq 0, 1$, the set $\mathcal{D}$ is a strict subset of $\mathcal{P}$. Furthermore, the only members of this class that admit a representation of the form (2), that is, satisfy GMM's Axiom 7, are ones for which $\beta = 0$ and $\beta = 1$.

We then proceed to the more general class of CEU preferences that admit an α-MEU representation in the section after that and show that again the only members of this class that admit a representation of the form (2), that is, satisfy GMM's Axiom 7, are ones for which $\beta = 0$ and $\beta = 1$.

2 HURWICZ PREFERENCES

For the purposes of this note we shall assume there is a finite set of states of nature S, with cardinality n. For simplicity we shall also assume that acts pay-off in utility terms, that is, an *act* is a function from S to $\mathbb{R}$. This is without any essential loss of generality since our analysis could also be conducted in a setting where representations involve a conventional utility function over outcomes.

For a given preference relation $\succsim$ from the GMM class of orderings, let $\mathcal{D}$ denote the set of probabilities associated with the derived unambiguous preference relation $\succsim^*$. For each act f, set $\underline{f} := \min_{P \in \mathcal{D}} \mathbf{E}_P [f]$ and $\bar{f} := \max_{P \in \mathcal{D}} \mathbf{E}_P [f]$. Thus the GMM representation (1) may be expressed as $V(f) = \beta(f) \underline{f} + (1 - \beta(f)) \bar{f}$.

Let $o_f : S \to \{1, ..., n\}$ be a one-to-one mapping that denotes the ordering of states from best to worst induced by the act f. That is, we have $f\left(o_f^{-1}(1)\right) \geq f\left(o_f^{-1}(2)\right) \geq ... \geq f\left(o_f^{-1}(n)\right)$. Define $\phi : \{1, ..., n\} \to \mathbb{R}$ by the rule, $\phi(i) := f \circ o_f^{-1}(i)$. For an act in which no two states have the same outcome, $\phi(i)$ is then the ith best outcome that can obtain. In particular, $\phi(1)$ is the best outcome that can arise under f and $\phi(n)$ is the worst.

We begin by considering the following class of preferences introduced by Hurwicz [8] and [7]:

$$f \succcurlyeq_\alpha g \iff \alpha \min_{s \in S} f(s) + (1 - \alpha) \max_{s \in S} f(s) \geqslant \alpha \min_{s \in S} g(s) + (1 - \alpha) \max_{s \in S} g(s). \quad (4)$$

Letting $V_a^H(\cdot)$ denote this functional representation of $\succsim_\alpha$, we have the following alternative ways to express this representation:

$$\begin{aligned} V_\alpha^H(f) &= \alpha\phi(n) + (1-\alpha)\phi(1) \\ &= \alpha \min_{P\in\Delta(S)} \mathbf{E}_P[f] + (1-\alpha) \max_{P\in\Delta(S)} \mathbf{E}_P[f], \end{aligned} \tag{5}$$

where $\Delta(S)$ denotes the set of probability measures on S and $\mathbf{E}_P[f] = \int_S f(s)\,dP(s)$ $= \sum_{i=1}^n P\left(\left\{o_f^{-1}(i)\right\}\right)\phi(i)$ denotes the expectation of the act f with respect to the probability measure P.

Notice that this is a polar example of the α-MEU representation given in (3), with $\mathcal{P} = \Delta(S)$. Indeed, Hurwicz introduced these preferences for situations in which the DM has no information about the process that will determine the state of nature. Intuitively, we would expect that her ambiguity is represented by the set of all probability distributions over S, $\Delta(S)$, and her ambiguity-attitude is measured by α with higher values of α corresponding to greater ambiguity-aversion.[1] Notice that in expression (5) the weight α is allocated to the least favourable probability distribution for the act, namely the (degenerate) distribution that places probability one on the state in which the worst outcome in the range of the act obtains (i.e. state $o_f^{-1}(n)$ on which the outcome $\phi(n)$ obtains). The remaining weight $(1-\alpha)$ is allocated to the most favourable probability distribution for this act, the (degenerate) distribution that places probability one on the state in which the best outcome in the range of the act obtains (i.e. the state $o_f^{-1}(1)$ on which the outcome $\phi(1)$ obtains).

According to GMM, however, the perceived ambiguity of the Hurwicz preference relation $\succsim_\alpha$ is represented by the set $\mathcal{D}$, which is the convex hull of the following set of $n(n-1)$ probability measures,

$$\left\{P_{st}^\alpha \in \Delta(S),\, s,t \in S, s \neq t : P_{st}^\alpha(\{s\}) = \alpha,\, P_{st}^\alpha(\{t\}) = 1-\alpha, P_{st}^\alpha(\{\omega\}) = 0, \omega \neq s,t\right\}.$$

[1] All the formal theories of ambiguity-aversion, which we are aware of, would agree that an increase in α would correspond to an increase in ambiguity-aversion, see [3] or [5].

In figure 1, we provide an illustration of the set $\mathcal{D}$ for the case of $n = 3$ and $\alpha < 1/2$.

Figure 1. The set of priors $\mathcal{D}$, the convex hull of the set

$$\{P_{st}^{\alpha} \in \Delta(S), s,t \in S, s \neq t : P_{st}^{\alpha}(\{s\}) = \alpha, P_{st}^{\alpha}(\{t\}) = 1 - \alpha\}.$$

for the case $n = 3$ and $\alpha < 1/2$.

To see why this set of probabilities constitutes $\mathcal{D}$, notice that for any three acts f, g and h and any $\lambda \in (0,1)$, it is immediate that if $\mathbf{E}_{P_{st}^{\alpha}}[f] \geq \mathbf{E}_{P_{st}^{\alpha}}[g]$ for all $s,t \in S$, $s \neq t$, then $\mathbf{E}_{P_{st}^{\alpha}}[\lambda f + (1-\lambda) h] \geq \mathbf{E}_{P_{st}^{\alpha}}[\lambda g + (1-\lambda) h]$ for all $s,t \in S$, $s \neq t$. In this case, for the pair of acts f and g not only is $f \succsim_{\alpha} g$, but we also have $\lambda f + (1-\lambda) h \succsim_{\alpha} \lambda g + (1-\lambda) h$, for all h and all $\lambda \in (0,1)$. Full independence holds for this particular pair of acts. What takes a little more work to show formally, but is readily apparent, is that if for a pair of acts $\hat{f}$ and $\hat{g}$, we have $\hat{f} \succsim_{\alpha} \hat{g}$ but for some pair of states s and t, $s \neq t$, $\mathbf{E}_{P_{st}^{\alpha}}[f] < \mathbf{E}_{P_{st}^{\alpha}}[g]$, then there exists an act $\hat{h}$ and a $\hat{\lambda} \in (0,1)$ for which $\hat{\lambda}\hat{g} + \left(1 - \hat{\lambda}\right)\hat{h} \succ_{\alpha} \hat{\lambda}\hat{f} + \left(1 - \hat{\lambda}\right)\hat{h}$. That is, for this pair of acts we can find a violation of independence. To express it using GMM terminology, for the Hurwicz preference relation $\succsim_{\alpha}$ we have f is *unambiguously* preferred to g (i.e.

$f \succsim_\alpha^* g$) if and only if

$$\alpha f(s) + (1-\alpha) f(t) \geq \alpha g(s) + (1-\alpha) g(t), \forall s, t \in S, s \neq t.$$

We argued above that ambiguity was intuitively represented by $\Delta(S)$. Yet $\mathcal{D}$ only coincides with $\Delta(S)$ if $\alpha = 0$ or 1. One would think intuitively that α represents the attitude towards ambiguity of the Hurwicz preference relation $\succsim_\alpha$ and yet $\mathcal{D}$ depends on α.

Let us next consider how GMM's index of ambiguity attitude applies to Hurwicz preferences. We shall first consider the case $n > 2$. We shall treat the setting in which there is a two-element state space separately below.

Fix a non-constant act f (that is, $\phi(1) > \phi(n)$). There are two cases to consider: when α is greater than or equal to 1/2 and when α is less than 1/2.

1. Suppose $\alpha \geq 1/2$. Then $\underline{f} = \alpha\phi(n) + (1-\alpha)\phi(n-1)$ and $\bar{f} = \alpha\phi(1) + (1-\alpha)\phi(2)$. To represent the preferences in the GMM form given in (1) we require,

$$\beta(f)\underline{f} + (1-\beta(f))\bar{f} = \alpha\phi(n) + (1-\alpha)\phi(1).$$

Thus,

$$\begin{aligned} & \beta(f)[\alpha\phi(n) + (1-\alpha)\phi(n-1)] + [1-\beta(f)][\alpha\phi(1) + (1-\alpha)\phi(2)] \\ = \; & \alpha\phi(n) + (1-\alpha)\phi(1), \end{aligned}$$

which yields

$$\begin{aligned} \beta(f) &= \frac{\alpha(\phi(1)-\phi(n)) - (1-\alpha)(\phi(1)-\phi(2))}{\alpha(\phi(1)-\phi(n)) + (1-\alpha)(\phi(2)-\phi(n-1))} \\ &= \frac{\alpha - (1-\alpha)(1-\gamma)}{\alpha + (1-\alpha)\delta}, \text{ where } \gamma = \frac{\phi(2)-\phi(n)}{\phi(1)-\phi(n)} \text{ and } \delta = \frac{\phi(2)-\phi(n-1)}{\phi(1)-\phi(n)}. \end{aligned}$$

In particular, if $\alpha = 1/2$, then we have $\beta(f) = \gamma/(1+\delta)$.

2. Suppose $\alpha < 1/2$. Then $\underline{f} = (1-\alpha)\phi(n) + \alpha\phi(n-1)$ and $\bar{f} = (1-\alpha)\phi(1) + \alpha\phi(2)$. Now we require

$$\begin{aligned} & \beta(f)[(1-\alpha)\phi(n) + \alpha\phi(n-1)] + [1-\beta(f)][(1-\alpha)\phi(1) + \alpha\phi(2)] \\ = \; & \alpha\phi(n) + (1-\alpha)\phi(1), \end{aligned}$$

which yields

$$\begin{aligned}\beta(f) &= \frac{\alpha(\phi(2)-\phi(n))}{\alpha(\phi(2)-\phi(n-1))+(1-\alpha)(\phi(1)-\phi(n))} \\ &= \frac{\alpha\gamma}{\alpha\delta+(1-\alpha)}.\end{aligned}$$

While $\beta(f)$ depends on α, which intuitively measures ambiguity-attitude, it also depends on γ and δ. The variable γ is a measure of the difference between the second-best outcome and the worst outcome relative to the difference between the best and worst outcomes, while δ is a measure of the difference between the second-best and second-worst outcomes relative to the difference between the best and worst outcomes. It does not seem at all obvious to us why such variables should be relevant for an ambiguity-attitude index.

What we can also glean from these expressions for $\beta(f)$ is that $\beta(\cdot)$ is a constant function only if $\alpha = 1$ or if $\alpha = 0$. This means that if there are more than two states then the only Hurwicz preference relations that satisfy GMM's axiom 7 are $\succsim_1$ and $\succsim_0$, that is, the cases of extreme pessimism and extreme optimism.

For the case of a two-element state space, we obtain $\beta(f) = 1$ if $\alpha > 1/2$ and $\beta(f) = 0$ if $\alpha < 1/2$. For the two-element state-space, in the GMM interpretation, α determines the degree of perceived ambiguity which is greater the further away α is from $1/2$. The attitude towards ambiguity is dichotomous, either extreme pessimism or extreme optimism depending on whether α is strictly greater than or strictly less than $1/2$. The preference relation $\succsim_{1/2}$ is ambiguity neutral since it corresponds to the preference relation of an expected value maximiser who thinks each state is equally likely.

The two-element state-space for Hurwicz preferences also illustrates GMM's Proposition 20. If the preference relation satisfies axiom 7 and so admits a representation of the form in (2) then among all possible α-MEU type representations, GMM's one is the only one yielding a set of probability measures which represent $\succsim^*$, and it yields the smallest set of probability measures. However, as the analysis of Hurwicz preferences for the case in which there are more than two states illustrates, the existence of an α-MEU representation does not guarantee that GMM's axiom 7 is satisfied, and if axiom 7 does not hold, then a representation of the form given in (2) fails to exist. In this case, although one can always find the representation of the form given in (1), this may not provide the most intuitive separation of perceived degree of ambiguity and attitude towards ambiguity.

3 $\alpha-$MEU and CEU

Our analysis of Hurwicz preferences in the previous section demonstrates that not all α-MEU preferences satisfy GMM's axioms. In this section we explore when GMM's axiom 7 holds. To do this we shall consider the family of preferences which admit both CEU and α-MEU representations.

For CEU preferences, the decision weights over states used in evaluating an act are derived from capacities.

Definition 3.1 *A* capacity *on S is a real-valued function ν on the subsets of S which satisfies the following properties:,*

1. $E \subseteq E' \Rightarrow \nu(E) \leqslant \nu(E')$;

2. $\nu(\varnothing) = 0,\ \nu(S) = 1.$

Its conjugate, denoted $\tilde{\nu}(\cdot)$, is the capacity defined as $\tilde{\nu}(E) = 1 - \nu(E^c)$. The capacity is said to be convex (resp. concave) if for all $E, E' \subseteq S$, $\nu(E \cup E') + \nu(E \cap E') \geq$ (resp. $\leq$) $\nu(E) + \nu(E')$.[2]

Given a capacity on S, the expected value of a given act with respect to that capacity can be found using the Choquet integral.

[2]Notice that if the capacity ν is convex (resp. concave) then its conjugate $\tilde{\nu}$ is concave (resp. convex).

Definition 3.2 *The* Choquet expected value *of f with respect to capacity ν, which we shall denote by $\int f d\nu$ is given by*

$$\phi(1)\,\nu\left(\left\{o_f^{-1}(1)\right\}\right)+\sum_{i=2}^{n}\phi(i)\left[\nu\left(\left\{o_f^{-1}(1),\ldots,o_f^{-1}(i)\right\}\right)-\nu\left(\left\{o_f^{-1}(1),\ldots,o_f^{-1}(i-1)\right\}\right)\right].$$

Let $\hat{P}(f)$ be the probability measure for which $\hat{P}(f)\left[\left\{o_f^{-1}(1)\right\}\right]=\nu\left(\left\{o_f^{-1}(1)\right\}\right)$, and $\hat{P}(f)\left[\left\{o_f^{-1}(i)\right\}\right]=\nu\left(\left\{o_f^{-1}(1),\ldots,o_f^{-1}(i)\right\}\right)-\nu\left(\left\{o_f^{-1}(1),\ldots,o_f^{-1}(i-1)\right\}\right)$, for $i=2,\ldots,n$. It readily follows that $\int f dv=\mathbf{E}_{\hat{P}(f)}[f]=\sum_{i=1}^{n}\hat{P}(f)\left[\left\{o_f^{-1}(i)\right\}\right]\phi(i)$. Furthermore, for any other act g that is comonotonic with f, that is, $[g(s)-g(t)][f(s)-f(t)]$ ≥ 0, for all $s,t\in S$, as is well-known, $\int g dv=\mathbf{E}_{\hat{P}(f)}[g]=\sum_{i=1}^{n}\hat{P}(f)\left[\left\{o_f^{-1}(i)\right\}\right]g\left(o_f^{-1}(i)\right)$.

With these preliminaries now in place, consider the following CEU preference relation with a capacity ν that can be written in the form $\nu=\alpha\mu+(1-\alpha)\tilde{\mu}$, where μ is a convex capacity (and hence its conjugate, $\tilde{\mu}$, is concave). Jaffray and Philippe [9] (hereafter JP) show that such a preference relation may be represented by the function

$$V(f)=\alpha\min_{P\in\mathcal{C}}\mathbf{E}_P[f]+(1-\alpha)\max_{P\in\mathcal{C}}\mathbf{E}_P[f],$$

where $\mathcal{C}$ is the core of μ, that is the set of probability measures that dominate μ. That is,

$$\mathcal{C}=\{P\in\Delta(S):P(E)\geq\mu(E)\,,\text{ for all }E\subseteq S\}.$$

This family of preferences may be viewed as a generalization of the Hurwicz preferences analysed in the previous section. To see that Hurwicz preferences are a special case, consider the unanimity capacity μ^0, for which $\mu^0(E)=0$, if $E\neq S$. Let μ^1 denote the conjugate capacity of μ^0. By definition it follows that for all $E\neq\varnothing$, $\mu^1(E)=1$. We shall take the Hurwicz capacity with parameter α in $[0,1]$ to be the capacity given by $\nu_\alpha=\alpha\mu^0+(1-\alpha)\mu^1$. Straightforward calculation yields

$$\begin{aligned}\int f dv_\alpha &= \alpha\int f d\mu^0+(1-\alpha)\int f d\mu^1\\ &= \alpha\min_{s\in S}f(s)+(1-\alpha)\max_{s\in S}f(s)=V_\alpha^H(f).\end{aligned}$$

For the more general JP-form $\nu = \alpha\mu + (1-\alpha)\tilde{\mu}$, where $\mathcal{C} \subset \Delta(S)$, we can interpret this as a situation in which the individual now has some information about the process determining the state of nature, as represented by the probability distributions in $\mathcal{C}$. The parameter α again measures the DM's ambiguity-attitude since s/he allocates weight α (resp. $1-\alpha$) to the least (resp. most) favourable probability distribution in $\mathcal{C}$. But as our next result shows, for this subclass of α-MEU preferences, if $n > 2$ then we find that it is possible that axiom 7 is satisfied only if $\alpha = 1$ or 0. More generally, for the class of CEU preferences, axiom 7 holds if and only if ν is either convex or concave.

Proposition 3.1 *Suppose $n > 2$. Let $\succsim$ admit a CEU representation with associated capacity ν. Then $\succsim$ satisfies axiom 7 if and only if ν is either convex or concave.*

Proof. Let $\mathcal{O}$ denote the set of orderings of the state space, that is, the set of one-to-one mappings from S to $\{1,\ldots,n\}$. For each ordering of states $o \in \mathcal{O}$, let P^o be the probability measure for which $P^o\left(\left\{o^{-1}(i)\right\}\right) = \nu\left(\left\{o^{-1}(1),\ldots,o^{-1}(i)\right\}\right) - \nu\left(\left\{o^{-1}(1),\ldots,o^{-1}(i-1)\right\}\right)$, for $i = 2,\ldots,n$. GMM show that for a CEU preference relation $\mathcal{D}$ is the convex hull of the set $\{P^o : o \in \mathcal{O}\}$.

Fix an act f. Let $\underline{P}(f) = \operatorname{argmin}_{P\in\mathcal{D}} \mathbf{E}_P[f]$, let $\bar{P}(f) = \operatorname{argmax}_{P\in\mathcal{D}} \mathbf{E}_P[f]$. Since S is finite, there are a finite number of extremal points of $\mathcal{D}$. Hence for a generic act f, the functions $\underline{P}(f)$ and $\bar{P}(f)$ are constant on a neighbourhood of f.

Consider an extremal point $\tilde{P}$ of $\mathcal{D}$. Then $\tilde{P} = \hat{P}(f)$ for all f in a set of comonotonic acts. Since a set of comonotonic acts contains an open set, we may find one such act f which is generic. Since axiom 7 is satisfied,

$$\mathbf{E}_{\hat{P}(f)}[f] = \beta\mathbf{E}_{\underline{P}(f)}[f] + (1-\beta)\,\mathbf{E}_{\bar{P}(f)}[f].$$

That is,

$$\begin{aligned}
&\sum_{i=1}^{n} \hat{P}(f)\left[\left\{o_f^{-1}(i)\right\}\right] \phi(i) \\
= \;& \beta\left(\sum_{i=1}^{n} \underline{P}(f)\left[\left\{o_f^{-1}(i)\right\}\right] \phi(i)\right)+(1-\beta)\left(\sum_{i=1}^{n} \bar{P}(f)\left[\left\{o_f^{-1}(i)\right\}\right] \phi(i)\right)
\end{aligned}$$

Let $g = f + \epsilon y$, where $\|y\| = 1$. For ϵ sufficiently small, $\hat{P}(f) = \hat{P}(g)$, $\underline{P}(f) = \underline{P}(g)$ and $\bar{P}(f) = \bar{P}(g)$. Thus

$$\mathbf{E}_{\hat{P}(f)}[g] = \beta \mathbf{E}_{\underline{P}(f)}[g] + (1-\beta)\, \mathbf{E}_{\bar{P}(f)}[g].$$

That is,

$$\begin{aligned}
&\sum_{i=1}^{n} \hat{P}(f)\left[\left\{o_f^{-1}(i)\right\}\right] g\left(o_f^{-1}(i)\right) \\
= \;& \beta\left(\sum_{i=1}^{n} \underline{P}(f)\left[\left\{o_f^{-1}(i)\right\}\right] g\left(o_f^{-1}(i)\right)\right)+(1-\beta)\left(\sum_{i=1}^{n} \bar{P}(f)\left[\left\{o_f^{-1}(i)\right\}\right] g\left(o_f^{-1}(i)\right)\right).
\end{aligned}$$

Since this holds for all g in a set of full rank,

$$\hat{P}(f) = \beta \underline{P}(f) + (1-\beta)\, \bar{P}(f).$$

However $\hat{P}(f)$ is an extremal point of $\mathcal{D}$, which implies that $\beta = 1$ or $\beta = 0$. If $\beta = 1$ (resp. 0) then ν is convex (resp. concave). ■

4 CONCLUSION

This paper has shown that there are no preferences in the intersection of the CEU and α-MEU models which satisfy GMM's axiomatisation of α-MEU for $0 < \alpha < 1$. The cases $\alpha = 0$ and $\alpha = 1$ have already been axiomatised. However CEU preferences have some properties not shared by α-MEU preferences. For instance CEU preferences cannot display a strict preference for randomisation while α-MEU preferences can, (see [2]). Thus it remains an open question which preferences satisfy GMM's axioms for $0 < \alpha < 1$.

References

[1] T. F. Bewley. Knightian decision theory: Part 1. Discussion Paper 807, Cowles Foundation, 1986.

[2] J. Eichberger and D. Kelsey. Uncertainty aversion and preference for randomisation. *Journal of Economic Theory*, 71:31–43, 1996.

[3] L. G. Epstein. A definition of uncertainty aversion. *Review of Economic Studies*, 66:579–606, 1999.

[4] P. Ghirardato, F. Maccheroni, and M. Marinacci. Differentiating ambiguity and ambiguity attitude. *Journal of Economic Theory*, 118:133–173, 2004.

[5] P. Ghirardato and M. Marinacci. Ambiguity made precise: A comparative foundation. *Journal of Economic Theory*, 102:251–289, 2002.

[6] I. Gilboa and D. Schmeidler. Maxmin expected utility with a non-unique prior. *Journal of Mathematical Economics*, 18:141–153, 1989.

[7] L. Hurwicz. Optimiality criteria for decision making under ignorance. *Discussion paper 370, Cowles Comission*, 1951.

[8] L. Hurwicz. Some specification problems and application to econometric models. *Econometrica*, 19:343–344, 1951.

[9] J.-Y. Jaffray and F. Philippe. On the existence of subjective upper and lower probabilities. *Mathematics of Operations Research*, 22:165–185, 1997.

[10] D. Schmeidler. Subjective probability and expected utility without additivity. *Econometrica*, 57:571–587, 1989.