On the Evolutionary Selection of Nash Equilibrium Components

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On the Evolutionary Selection of Nash Equilibrium Components[*]

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Abstract

It is well known for the common multi-population evolutionary dynamics applied to normal form games that a pure strategy combination is asymptotically stable if and only if it is a strict equilibrium point. We extend this result to sets as follows. For certain regular selection dynamics every connected and closed asymptotically stable set of rest points containing a pure strategy combination is a strict equilibrium set and hence a Nash equilibrium component. A converse statement holds for two person games, for convex strict equilibrium sets and for the standard replicator dynamic.

Keywords: evolutionary dynamics, replicator dynamic, regular selection dynamics, strict equilibrium set, Nash equilibrium component.

JEL *classification number*: C79.

1 Introduction

For the typical evolutionary dynamic of an asymmetric game a pure strategy combination is asymptotically stable if and only if it is a strict equilibrium point. This observation has been made repeatedly (see, for instance, Eshel and Akin (1983), Ritzberger and Vogelsberger (1989), Samuelson and Zhang (1992) and Ritzberger and Weibull (1996)).

Many games, in particular extensive-form games, have components of Nash equilibria rather than isolated equilibria. It is therefore important to investigate extentions of this result to sets of strategies. Ritzberger and Weibull (1996) provide necessary and sufficient condition for the asymptotic stability of faces. This is a weak selection criterion insofar as the population state is allowed to change when starting within a face with this property. In this paper we will consider evolutionary selection in a slightly narrower sense than Ritzberger and Weibull (1996) and limit attention to asymptotic stable sets that consist of rest points, i.e., where no movement takes place when the population state lies within the set. At the same time, we will investigate sets more general than faces in order to get a better understanding of the structural implications of the dynamic properties themselves. In particular, we do not restrict attention to convex sets.

The relevant static equilibrium concept for our results is that of a *strict equilibrium set* (short, *SE set*). Recall that a Nash equilibrium is called strict if every player strictly loses by unilaterally deviating from the equilibrium. Following Balkenborg (1994) a set of Nash equilibria is called a *strict equilibrium set (SE set)* if for any equilibrium in the set, a player either loses strictly by unilaterally deviating or if his deviation just leads to another equilibrium in the set. More precisely, if the strategy of a *single* player in an equilibrium belonging to the set is replaced with an alternative best reply of this player another equilibrium in the set is obtained. Thus the set of Nash equilibria as a whole is robust against deviations in the same sense as it is the case for a strict equilibrium.

We study multi-population evolutionary dynamics in normal form games. These normal form games are viewed as models for asymmetric evolutionary contests between as many populations as there are players in the game.[1] The individuals in each population are assumed to play pure strategies. The dynamics governing the change in play are in continuous time and have to be *regular selection dynamics* as defined in Samuelson and Zhang (1992) or Ritzberger and Weibull (1996). In particular, faces of the strategy simplex have to be invariant. Intuitively, such dynamics model processes where only mutation, not selection itself, can cause new strategies to be played. They have therefore been extensively studied in the literature. However, many processes which assume more rationality of the agents like the best response dynamics (see Hofbauer (1995b)), fictitious play or perturbed versions of these dynamics (see, e.g., Fudenberg and Levine (1998) and Hopkins (1999)) are ruled out. In addition we require that the dynamic *reinforces best replies* which consists of two conditions. (i) The growth rate of any best reply in a given population is positive unless all strategies currently played in this population are already best replies. (ii) When all strategies currently played in a population are best replies then the growth rate of non-best replies is negative. Included are all payoff positive and payoff monotonic dynamics, in particular the standard and the adjusted replicator dynamic, all

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[1] In contrast, the classic concept of an ESS applies to settings where a symmetric game is played within a single population.

regular selection dynamics discussed in Nachbar (1990), all sophisticated imitation dynamics in Hofbauer and Schlag (2000) and many of the imitation dynamics considered in Hofbauer (1995a) and Weibull (1995).

For such dynamics we show that every closed and connected asymptotically stable set of rest points which contains at least one pure strategy combination must be a SE set. For the standard replicator dynamic one can drop the assumption that the set contains a pure strategy combination. However, for other dynamics this additional condition can be crucial. For example, in Matching Pennies the singleton set containing the mixed equilibrium is not a SE set but it is well known that it is asymptotically stable in the adjusted replicator dynamics of Maynard Smith (1982).

We also obtain the converse statement (every SE set is an asymptotically stable set of rest points) (i) if each element of the SE set is a best response to its best responses, or (ii) if the underlying dynamic is the standard replicator dynamic. Condition (i) is always satisfied in two-player games or when the SE set is convex. The result with condition (ii) is due to Balkenborg (1994).

Summarizing, we obtain a tight characterization of connected asymptotically stable sets of rest points that satisfy one of the following additional conditions: (a) the set is convex and contains a pure strategy combination, (b) the set contains a pure strategy combination and the game is between two players only, and (c) the underlying dynamic is the standard replicator dynamic.

Our research complements that of Ritzberger and Weibull (1996). They show that a face is asymptotically stable if and only if it is *closed under better replies*. To be closed under better replies means that after replacing the strategy of one or several players by a better reply to the current state the resulting state also belongs to the set. We show that a convex SE set is a face that is closed under better replies. Thus we obtain a refined version of the characterization in Ritzberger and Weibull (1996) by adding the dynamic property that the face contains only rest points and the static property that it consists of Nash equilibria. At the same time our research broadens the understanding of asymptotic stability beyond Ritzberger and Weibull (1996) because our analysis allows for non-convex sets of equilibria which arise in simple applications (see Section 2.2).

Our results are also of relevance for recent research on the relation between the degree and the index of equilibrium components (see, for instance, Govindan and Wilson (1997) and DeMichelis and Germano (2000)). DeMichelis and Ritzberger (2000) show for asymptotically stable Nash equilibrium components that their degree must equal their Euler characteristic. They show, moreover, that the component must contain a strategically stable set as defined in Mertens (1989) if the Euler characteristic is not zero.[2] In the case of the standard replicator dynamic the asymptotically stable Nash equilibrium components are, by Theorem 12 of this paper, precisely the connected SE sets. This means that a SE set with a non-zero Euler characteristic contains a strategically stable set. Because a SE set is a union of faces its Euler characteristic is easy to calculate.

In Section 2 the basic concepts and terminology are introduced. Section 3 contains our results and some examples illustrating the necessity of our assumptions. Section 4

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[2]Notice that this condition is much weaker than requiring the component to be convex or contractible, which is a crucial assumption in Swinkels (1993).

concludes. The appendix shows that our results do not extend to asymmetric playing-the-field models.

2 Concepts

Except where indicated, we use the notations and definitions in Ritzberger and Weibull (1996).

2.1 Games

For a (finite) a *normal form game* Γ the finite set of players is denoted by $\mathcal{N} = \{1, 2, \ldots, n\}$. $\mathcal{S}_i$ is the finite set consisting of the K_i pure strategies s_i^k, $k = \{1, 2, \ldots, K_i\}$, of player $i \in \mathcal{N}$. $\mathcal{S} = \times_{i \in \mathcal{N}} \mathcal{S}_i$ is the set of pure strategy combinations with generic element $s = (s_1, s_2, \ldots, s_n)$. The set of *mixed strategies* of player i is the $(K_i - 1)$-dimensional unit simplex $\Delta_i = \left\{ \sigma_i \in \mathfrak{R}_+^{K_i} \mid \sum_{k=1}^{K_i} \sigma_i^k = 1 \right\}$. $\Delta = \times_{i \in \mathcal{N}} \Delta_i$ is the set of mixed strategy combinations with generic element $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_2)$. Pure strategies s_i^k are identified with the corresponding unit vectors $e_i^k \in \Delta_i$. The *support* of a mixed strategy $\sigma_i \in \Delta_i$ is denoted by $\operatorname{supp}(\sigma_i) = \left\{ s_i^k \in \mathcal{S}_i \mid \sigma_i^k > 0 \right\}$. Let $\operatorname{supp}(\sigma) := \times_{i \in \mathcal{N}} \operatorname{supp}(\sigma_i)$. The *face* $\mathfrak{F}(\sigma_i)$ generated by the mixed strategy σ_i is defined as the set of all strategies σ_i' with the same support as σ_i. The face generated by the strategy combination σ is $\mathfrak{F}(\sigma) := \times_{i \in \mathcal{N}} \mathfrak{F}(\sigma_i)$. A face of Δ is a face generated by some $\sigma \in \Delta$. The mapping $u : \mathcal{S} \to \mathfrak{R}^n$ defines the *payoff* for pure strategy combinations. The multilinear expected payoff function $U : \Delta \to \mathfrak{R}^n$ and the mixed *best reply correspondence* $\tilde{\beta} = \times_{i \in \mathcal{N}} \tilde{\beta}_i : \Delta \to \Delta$ are defined in the usual manner. The pure best reply correspondence $\beta = \times_{i \in \mathcal{N}} \beta_i : \Delta \to \mathcal{S}$ is defined by $\beta(\sigma) = \tilde{\beta}(\sigma) \cap \mathcal{S}$.

A set $W \subset \Delta$ is *closed under better replies* if $\sigma \in W$, $\rho \in \Delta$ and $U_i(\sigma_{-i}, \rho_i) \geq U_i(\sigma)$ for all i implies $\rho \in W$. Ritzberger and Weibull (1996) call a convex set W *closed under pure better replies* if the same holds with respect to all pure strategy combinations ρ.

2.2 Strict equilibrium sets

Definition 1 *i) A non-empty subset $G \subset \Delta$ is a* STRICT EQUILIBRIUM SET *(SE set) if for all $\sigma \in G$ and all $\tau_i \in \Delta_i$ the inequality*

$$U_i(\sigma_{-i}, \tau_i) \leq U_i(\sigma)$$

holds whereby equality implies $(\sigma_{-i}, \tau_i) \in G$.

Notice that a SE set consists of Nash equilibria. Moreover, a singleton set $\{\sigma\}$ is a SE set if and only if σ is a strict equilibrium point.

The following lemma generalizes the observation that every strict equilibrium point is a pure strategy combination.

Lemma 2 *For every strategy combination σ in a SE set G the face $\mathfrak{F}(\sigma)$ is contained in G. In particular, G is a finite union of faces and hence closed.*

Proof. By induction. For $j = 1, \ldots, n+1$ define

$$F_j := (\times_{i<j} \mathfrak{F}(\sigma_i)) \times \{(\sigma_i, \ldots, \sigma_n)\}.$$

$F_1 \subset G$ holds by assumption. Suppose $F_j \subset G$ holds for $j \leq n$. The strategy combination $\sigma' = (\sigma'_1, \ldots, \sigma'_{j-1}, \sigma_j, \ldots, \sigma_n) \in F_j$ is in G and hence a Nash-equilibrium point. Since σ_j is a best reply to σ', every strategy in the support of σ_j and hence every strategy $\sigma'_j \in \mathfrak{F}(\sigma_j)$ is a best reply to σ'. Therefore $(\sigma'_1, \ldots, \sigma'_{j-1}, \sigma'_j, \sigma_{j+1}, \ldots, \sigma_n) \in G$ for every $\sigma'_j \in \mathfrak{F}(\sigma_j)$, i.e. $F_{j+1} \subset G$. We conclude $\mathfrak{F}(\sigma) = F_{n+1} \subset G$. ∎

The following proposition is technically the core result in Balkenborg (1994):

Proposition 3 *Every strategy combination σ in a SE set has a neighborhood $V(\sigma)$ such that*

$$\sum_{i \in \mathcal{N}} U_i(\rho_{-i}, \sigma_i) \geq \sum_{i \in \mathcal{N}} U_i(\rho) \tag{1}$$

holds for all $\rho \in V(\sigma)$ whereby equality occurs if and only if ρ is in the SE set.

Corollary 4 *A SE set is a union of closed Nash equilibrium components.*

Proof. Let G be a SE set. G is closed by the lemma. For each $\sigma \in G$ we have to find a neighborhood $V(\sigma)$ such that all Nash equilibria in this neighborhood are in G. Choose $V(\sigma)$ as described in the previous proposition. Suppose ρ is a Nash equilibrium contained in $V(\sigma)$. Then

$$\sum_{i \in \mathcal{N}} U_i(\rho_{-i}, \sigma_i) \leq \sum_{i \in \mathcal{N}} U_i(\rho)$$

since ρ is a Nash equilibrium. Therefore inequality (1) holds with equality and hence ρ is in G. ∎

The following corollary helps to compare our results with those of Ritzberger and Weibull (1996).

Corollary 5 *The following statements are equivalent:*
  (i) G is a convex SE set.
  (ii) G is a face $\mathfrak{F}(\sigma)$ that is a SE set.
  (iii) G is a face $\mathfrak{F}(\sigma)$ consisting of Nash equilibria that is closed under pure better replies.
  (iv) G is a Nash equilibrium component closed under better replies.

Proof. A union of faces is convex if and only if it is face. Therefore (i) and (ii) are equivalent.

Suppose that $\mathfrak{F}(\sigma) = \times_{i \in \mathcal{N}} \mathfrak{F}(\sigma_i)$ is a SE set. Suppose that $\tau \in \mathfrak{F}(\sigma)$, $\rho \in \Delta$ and that $U_i(\tau_{-i}, \rho_i) \geq U_i(\tau_i)$ for all $i \in \mathcal{N}$. Then $\rho \in \tilde{\beta}(\tau)$, since τ is a Nash equilibrium, and $(\rho_{-i}, \tau_i) \in \times_{i \in \mathcal{N}} \mathfrak{F}(\sigma_i)$ for all $i \in \mathcal{N}$ since $\mathfrak{F}(\sigma)$ is a SE set. Therefore $\tau_i \in \mathfrak{F}(\sigma_i)$ for all

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$i \in \mathcal{N}$. Thus (ii) implies (iii) and, since $\mathfrak{F}(\sigma)$ is a Nash equilibrium component by the previous corollary, (ii) implies (iv).

Suppose next that $\mathfrak{F}(\sigma) = \times_{i \in \mathcal{N}} \mathfrak{F}(\sigma_i)$ satisfies iii). Suppose $\rho_i \in \tilde{\beta}(\tau)$ for $\tau \in \mathfrak{F}(\sigma)$. Then $U_i\left(\tau_{-i}, s_i^k\right) = U_i(\tau)$ for all pure strategies s_i^k in the support of ρ_i. Since $\mathfrak{F}(\sigma)$ is closed under better replies, $s_i^k \in \operatorname{supp}(\sigma_i)$ and hence $(\tau_{-i}, \rho_i) \in \mathfrak{F}(\sigma)$. Therefore iii) implies ii).

Suppose finally that G satisfies (iv). Then G is clearly a SE set and hence a union of faces. Suppose that $\mathfrak{F}(\sigma)$ is a face contained in G that is not a proper subset of another face contained in G. Suppose $\rho \in \tilde{\beta}(\tau)$ for $\tau \in \mathfrak{F}(\sigma)$. Then $\frac{1}{2}\rho + \frac{1}{2}\tau \in \tilde{\beta}(\tau)$. The definition of G implies that $\mathfrak{F}\left(\frac{1}{2}\rho + \frac{1}{2}\tau\right)$ is contained in G. Therefore $\mathfrak{F}\left(\frac{1}{2}\rho + \frac{1}{2}\tau\right) = \mathfrak{F}(\sigma)$ by the choice of σ and hence $\rho \in \mathfrak{F}(\sigma)$. It follows in particular that $\mathfrak{F}(\sigma)$ is a SE set and hence a Nash equilibrium component by the previous corollary. Hence $\mathfrak{F}(\sigma) = G$, so (iv) implies (ii). ∎

We now provide some examples of SE sets. It is easy to see that the set of all Pareto-efficient Nash equilibria in a common interest game as defined in Aumann and Sorin (1989) is a SE set. Similarly, the set of mixed strategy combinations maximimizing the potential in a weighted potential game as defined in Monderer and Shapley (1996) is a SE set. The following game analyzed in van Damme (1989) using forward induction does not belong to one of these two classes of games. Consider the repeated game where the following battle-of-the-sexes game is played twice:

	L	R
	1	0
T	3	0
	0	3
B	0	1

The repeated game has two connected SE sets. One consists of all strategy combinations inducing the play $((T, L), (B, R))$, the other consists of all strategy combinations inducing the play $((B, R), (T, L))$. It is immediate to verify that these two sets are indeed SE sets. To verify that there are no other SE sets (except for the union of the two sets) one uses that every SE set must contain a pure Nash equilibrium. Suppose, for instance, that a SE Set would contain a pure strategy equilibrium (σ_1, σ_2) inducing the play $((T, L), (T, L))$. Consider the strategy τ_1 of player 1 where he plays T in the first period and in the second period T after the history (T, L) and B after any other history. τ_1 is a best reply to σ_2 and hence (τ_1, σ_2) must be in the SE set. However, σ_2 is not a best reply against τ_1: Player 2 receives 1+1 with σ_2 while he would receive 0+3 if he would play R in the first and, regardless of the history, R in the second period. So there can be no SE set of the type assumed. Similarly, one checks all remaining possibilities. (For more on SE sets in repeated games see Balkenborg (1995)).

To provide an interesting example of a game with a non-convex SE set we consider the following action commitment game discussed in van Damme and Hurkens (1996).

Suppose the coordination game

	L	R
T	3, 3	0, 0
B	0, 0	1, 1

is played in two stages as follows: In the first stage each player may either wait or commit himself to one of his two actions in the coordination game. These first stage decisions are made simultaneously and independently. At the beginning of the second stage both players are informed which commitments, if any, have been made. If there are players who have not committed themselves in the first stage, they must then, again simultaneously and independently, choose one of their two actions. At the end of the second stage both players have selected one of their two actions of the coordination game, either by committing themselves or by choosing later, and they receive the corresponding payoffs.

This action commitment game (and, obviously, also the stage game on which it is based) is a game with identical interests, i.e., both players have identical payoffs in each strategy combination. Therefore it is a common interest game (or a potential game) and hence the set of all strategy combinations where both players receive the payoff 3 is a SE set. Moreover, it is easily seen to be the only SE set in the action commitment game.[3] Suppose, for instance, that player 1 commits himself to the action B and that this is part of a strategy combination in the SE set. A possible best reply of player 2 is to wait and to play in the second stage L if player 1 committed himself to T and to play R otherwise. Player 1's strategy together with player 2's best reply would have to be in the SE set, although it is not a Nash equilibrium.

In this example the SE set is not convex. After eliminating weakly dominated strategies in the reduced normal form of the action commitment game one is left with a 4×4-game with the payoffs

3, 3	3, 3	3, 3	0, 0
3, 3	3, 3	0, 0	1, 1
3, 3	0, 0	1, 1	1, 1
0, 0	1, 1	1, 1	1, 1

where the set of strategy combinations yielding payoff 3 is clearly not convex.

In the previous example the SE set is not convex, but it is still contractible. However, SE sets can also be non-contractible cyles. An example can be found in Hofbauer and Sigmund (1998), Section 8.6, where the asymptotically stable set for $\varepsilon = 0$ is a SE set.

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[3]This observation extends to any action commitment game based on a common-interest game.

2.3 Dynamics

A *regular selection dynamic* on Δ is a system of ordinary differential equations

$$\dot{\sigma}_i^k = f_i^k(\sigma)\,\sigma_i^k, \quad \forall k = 1, \ldots, K_i, \; \forall i \in \mathcal{N}$$

with Lipschitz continuous functions $f_i : \Delta \to \mathfrak{R}$ for $i \in \mathcal{N}$ satisfying $\sum_k f_i(\sigma) \cdot \sigma_i^k = 0$ for all $\sigma \in \Delta$. We consider only regular selection dynamics in this paper and often write briefly "dynamic" when we mean a regular selection dynamic. For every $\sigma^0 \in \Delta$ the system of differential equations has a unique solution $\{\sigma^t\}_{t \geq 0} \subset \Delta$ starting in σ^0. This trajectory satisfies $\mathrm{supp}\,(\sigma^t) = \mathrm{supp}\,(\sigma^0)$ for all $t \geq 0$. In particular, faces of Δ are invariant under the dynamic.

Definition 6 *A regular selection dynamic* REINFORCES BEST REPLIES *if the following holds for any player $i \in \mathcal{N}$, any strategy combination $\sigma \in \Delta$ and any pure strategy $s_i^k \in S_i^k$:*
   a) $s_i^k \in \beta_i(\sigma)$ implies $f_i^k(\sigma) \geq 0$,
   b) $s_i^k \in \beta_i(\sigma)$ and $\sigma_i \notin \tilde{\beta}(\sigma)$ imply $f_i^k(\sigma) > 0$,
   c) $s_i^k \notin \beta_i(\sigma)$ and $\sigma_i \in \tilde{\beta}(\sigma)$ imply $f_i^k(\sigma) < 0$.

   Notice that for non-trivial games a) is a consequence of b) and the continuity of the f_i^k. Condition (c) is a weak condition on what happens to non-best replies. It requires for a given population in which only best replies are played that non best replies yield negative growth rates. In the terminology of Ritzberger and Weibull (1996), any payoff positive selection dynamic and any sign preserving selection dynamic reinforces best replies. However, not every a weakly sign-preserving dynamics reinforces best replies. While these differences exist in theory, any explicitly formulated regular selection dynamics known to us in the literature either satisfies our conditions or is not even weakly sign preserving.

   The *standard replicator dynamic* is defined by $f_i^k(\sigma) := U_i(\sigma_{-i}, s_i^k) - U_i(\sigma)$.

   $\sigma \in \Delta$ is a *rest point* if $\dot{\sigma}_i^k = 0$ for all $i \in \mathcal{N}$ and $1 \leq k \leq K_i$. Under a dynamic that reinforces best replies, any Nash equilibrium is a rest point, but the converse does not have to be true.

   A set is *forward invariant* if all trajectories starting in this set remain in this set. Consider a non-empty closed set. The set is *(Lyapunov) stable* if for every neighborhood of the set trajectories starting sufficiently close to the set stay in this neighborhood. The set is an *attractor* if it has a neighborhood such that all ω-limit points of trajectories starting in the neighborhood are in the set. The set is *asymptotically stable* if it stable and an attractor. $\sigma \in \Delta$ is *(asymptotically) stable* if the singleton set $\{\sigma\}$ is (asymptotically) stable.

3 Results

For a regular selection dynamic reinforcing best replies any asymptotically stable set containing a pure strategy combination is a strict equilibrium. Our first result generalizes this observation to components of rest points.

Theorem 7 *For a regular selection dynamic reinforcing best replies any closed and connected asymptotically stable set of rest points containing a pure strategy combination is a SE set.*

In the proof below, the pure strategy combination acts like a virus spreading the properties of a SE set within the asymptotically stable set first to faces containing it, then to adjacent faces and so on until the entire asymptotically stable set is covered. Let s be the pure strategy combination and let ρ_i be a pure best reply for player i. The fact that the set, say G, is asymptotically stable and only contains rest points implies that $U_i(s_{-i}, \rho_i) = U_i(s)$ and hence the convex hull of s and (s_{-i}, ρ_i) is contained in G.

Proof. Let R be the given asymptotically stable set of rest points and let G be the union of faces contained in the set. We claim that G is a strict equilibrium set equal to R. Since R contains at least one pure strategy combination, G is not empty. Let $\mathfrak{F}$ be a face contained in R.

Step 1: We show first that every strategy combination $\sigma \in \mathfrak{F}$ is a Nash equilibrium.

Otherwise we could find a best reply s_i^k against σ satisfying $U_i(\sigma_{-i}, s_i^k) > U_i(\sigma)$ and therefore $f_i^k(\sigma) > 0$ because best replies are reinforced. Since faces are invariant under the dynamic we can assume without loss of generality that s_i^k is the only strategy in the game not belonging to $\mathfrak{F}$. By continuity we can choose a neighborhood $V(\sigma)$ of σ and a constant $c > 0$ such that $f_i^k(\rho) > c$ holds for all ρ in a some neighborhood $V(\sigma)$ of σ. Then $V(\sigma) \cap R \subset V(\sigma) \cap \{\rho_i^k = 0\}$. Using Lipschitz continuity and the fact that $\mathfrak{F}$ consists of rest points one can show that $\left| \rho_j^l f_j^l(\rho) \right| \le 2L\rho_i^k \le \frac{2L}{c}\rho_i^k f_i^k(\rho)$ holds for all pure strategies $s_j^l \ne s_i^k$ with some constant $L > 0$ in a neighborhood of σ, which we can assume again to be $V(\sigma)$. Hence one can construct a neighborhood $V'(\sigma) \subset V(\sigma)$ such that any trajectory starting in $V'(\sigma) \backslash \mathfrak{F}$ moves to a point in $V(\sigma)$ with $\rho_i^k \ge \alpha$, where $\alpha > 0$ is a constant.[4] Because R is stable it has a forward invariant neighborhood W contained in the complement of $V(\sigma) \cap \{\rho_i^k \ge \alpha\}$. However, any trajectory starting in $(V'(\sigma) \backslash \mathfrak{F}) \cap W$ will move into $V(\sigma) \cap \{\rho_i^k \ge \alpha\}$, which yields a contradiction.

Step 2: Suppose now that $U_i(\sigma_{-i}, \rho_i) = U_i(\sigma)$ for some $\sigma \in \mathfrak{F}$ and $i \in \mathcal{N}$. We can assume $i = 1$ without loss of generality. For each pure strategy combination $s \in \mathrm{supp}(\sigma)$ and $j \in \mathcal{N}$ we define the faces

$$F_s^j := \mathfrak{F}\left(\frac{1}{2}\sigma_1 + \frac{1}{2}\rho_1\right) \times \left(\times_{2 \le i \le j} \mathfrak{F}(\sigma_i)\right) \times \left(\times_{j < i} \{s_i\}\right)$$

We prove by induction on j that all faces F_s^j are contained in R and hence that $(\sigma_{-1}, \rho_1) \in F_s^n \subset G$.

We notice that $U_1(s_{-1}, s_1^k) \le U_1(s)$ holds for all $s \in \mathrm{supp}(\sigma)$ and for all $s_1^k \in \mathrm{supp}\left(\frac{1}{2}\sigma_1 + \frac{1}{2}\rho_1\right)$ since $s \in \mathfrak{F}$ is a Nash equilibrium by Step 1. Since $U_1(\sigma_{-1}, \rho_1) = U_1(\sigma)$ the multilinearity of the payoff function implies $U_1(s_{-1}, s_1^k) = U_1(s)$ for all $s \in \mathrm{supp}(\sigma)$ and $s_1^k \in \mathrm{supp}(\rho_1)$. We obtain $U_1(\sigma'_{-1}, \rho'_1) = U_1(\sigma')$ and hence $\rho'_1 \in \tilde{\beta}(\sigma')$ for all $\sigma' \in \mathfrak{F}(\sigma)$ and $\rho'_1 \in \mathfrak{F}\left(\frac{1}{2}\sigma_1 + \frac{1}{2}\rho_1\right)$.

Each face F_s^1 is invariant under the dynamic with the opponent's behavior fixed at s_{-i}. Because the dynamic reinforces best replies it follows from the previous argument

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[4] We skip the detailed arguments for the last two statements because they are similar to, although simpler than, the arguments in the proof of Theorem 10.

that $f_1^k(\sigma') = 0$ for all $s_1^k \in \text{supp}(\sigma_1) \cup \text{supp}(\rho_1)$ and $\sigma' \in F_s^1$. Therefore F_s^1 consists of rest points. Since $s \in F_s^1 \cap R$ and since R is a closed asymptotically stable set of rest points we must have $F_s^1 \subset R$. (Otherwise there would be rest points arbitrarily close to R which are not contained in R.)

Now let $j \geq 2$ and suppose $F_s^{j-1} \subset R$ for all $s \in \text{supp}(\sigma)$. By Step 1 each F_s^{j-1} is a set of Nash equilibria. Let $\sigma' \in F_s^{j-1}$. Then $(\sigma'_{-j}, s_j^h) \in F_{(s_{-j}, s_j^h)}^{j-1}$ is a Nash equilibrium for all $s_j^h \in \text{supp}(\sigma_j)$. Therefore each $\rho_j \in \mathfrak{F}(\sigma_j)$ is a best reply to σ'. For each player $i \neq j$ σ'_i is a best reply to all (σ'_{-j}, s_j^h), $s_j^h \in \text{supp}(\sigma_j)$, and hence also to (σ'_{-j}, ρ_j). It follows that F_s^j is a set of Nash equilibria and therefore a set of rest points contained in R.

Step 2 implies that G is a SE set. It remains to show that $G = R$. For $\sigma \in R$ we can find a neighborhood $V(\sigma)$ such that $\text{supp}(\sigma) \subset \text{supp}(\rho)$ for all $\rho \in V(\sigma)$ and such that Proposition 3 applies. For a rest point $\rho \in V(\sigma)$ we have $U_i(\rho_{-i}, s_i) = U_i(\rho)$ for all $i \in \mathcal{N}$ and $s_i \in \text{supp}(\sigma_i)$. Hence $\sum_{i \in \mathcal{N}} U_i(\rho_{-i}, \sigma_i) = \sum_{i \in \mathcal{N}} U_i(\rho)$ and $\rho \in G$. Since G is compact and R a connected set of rest points we conclude $G = R$.[5] ∎

Since every Nash equilibrium is a rest point, we obtain

Corollary 8 *For a regular selection dynamic reinforcing best replies, any any asymptotically stable Nash equilibrium component containing a pure strategy combination is a SE set.*
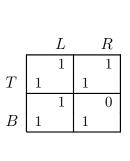
Theorem 7 requires the asymptotically stable set of rest points to contain a pure strategy combination. This requirement is necessary because a set consisting of a single Nash equilibrium in completely mixed strategies can be asymptotically stable, although it is not a SE set. An example is the mixed strategy equilibrium of the game "matching pennies" which is asymptotically stable under the adjusted replicator dynamics (see Maynard Smith (1982), Appendix J, or Weibull (1995), p. 199 ff) and under sophisticated imitation dynamics (Hofbauer and Schlag (2000)).

The restriction to regular selection dynamics is also crucial. This assumption rules out important dynamics as the best reply dynamics, fictitious play or the dynamics discussed in (DeMichelis and Ritzberger (2000)) which have exactly the Nash equilibria as fixed points. To see why, consider the simple 2×2 game in Figure 1. The best reply dynamics (see e.g. Hofbauer and Sigmund (1998) for this game is

$$\dot{x}_1 = 0$$
$$\dot{x}_2 = -x_2.$$

where $x_1 = \text{prob}(B)$, $x_2 = \text{prob}(R)$. The set $G = \{x_1 = 0\} \cup \{x_2 = 0\}$ is an asymptotically stable set for this dynamics. In contrast, for a regular selection dynamic reinforcing best replies all points on the line segment $\{x_2 = 1\}$ would be rest points and consequently G would not be a closed asymptotically stable set of rest points.

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[5]Note that $G = R$ can be deduced for the special case of two-player games without using Proposition 3. By Proposition 10 below, G is asymptotically stable and hence $G = R$ as R is connected.

$$
\begin{array}{c|c|c|}
 & L & R \\
\hline
T & \begin{smallmatrix}1\\1\end{smallmatrix}\, 1 & 1 \\
\end{array}
$$

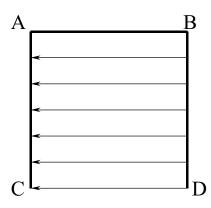|	L	R
T	1 1	1 1
B	1 1	1 0

A   B

C   D

Figure 1: For regular selection dynamics reinforcing best replies the set of rest points in this example is the union of the line segments AB, BC and CD.

We can prove a converse statement to Theorem 7 only under an additional restriction that is, however, always satisfied in the case of two-player games or for convex SE sets. It is also satisfied for the cyclical, non-convex SE set of the three-player game in Hofbauer and Sigmund (1998), Section 8.6, with $\varepsilon = 0$.

Lemma 9 *Suppose G is a SE set in a two-player game or a convex SE set. Then G satisfies the property:*

$$\sigma \in G \text{ and } \rho \in \tilde{\beta}(\sigma) \text{ implies } \sigma \in \tilde{\beta}(\rho) \tag{*}$$

Proof. Consider $\sigma \in G$ and $\rho \in \tilde{\beta}(\sigma)$. Assume that there are only two players. Then $(\sigma_{-i}, \rho_i) \in G$ for $i = 1, 2$ and hence $\sigma \in \tilde{\beta}(\rho)$. Alternatively, assume that G is convex. Then Corollary 5 implies that G is a face, that $\rho \in G$ and hence that $\sigma \in \tilde{\beta}(\rho)$. ∎

Theorem 10 *Every SE set satisfying (*) is an asymptotically stable set of Lyapunov stable rest points for any regular selection dynamic reinforcing best replies.*

In order to outline the idea of the proof we consider the dynamics restricted to the invariant subset $\tilde{\beta}(\sigma)$ where σ is an element of the SE set G. As best replies are reinforced, (*) implies that the frequency of every pure strategy in the support of σ cannot decrease over time on this invariant subspace. This implies that σ is Lyapunov stable on $\tilde{\beta}(\sigma)$. In the case where the dynamics is differentiable linearization around σ shows that all difference vectors $\tau - \sigma$ with $\tau \in \tilde{\beta}(\sigma)$ are Eigenvectors to the Eigenvalue 0 while every pure strategy which is not a best reply to σ generates an Eigenvector with a real and negative Eigenvalue. It follows that $\tilde{\beta}(\sigma)$ is a center manifold. Under this circumstances it is well-known (see e.g, Arnold, Afrajmovich, Il'yashenko, and Shilnikov (1999)) that Lyapunov stability on $\tilde{\beta}(\sigma)$ implies Lyapunov stability for the unrestricted dynamic. However, we do not want to assume differentiabilty and hence give a direct proof. To show asymptotic stability of the SE set one needs that there are no rest points close to the sets. This can easily be shown for two-player games. Otherwise one has to rely on Proposition 3.

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Proof. Fix a strategy combination σ in the SE set G and let $\mathfrak{F}(\sigma) = \times_i \mathfrak{F}(\sigma_i) \subseteq G$ be the face generated by σ.

We define for $\rho \in \Delta$

$$\gamma(\rho) := \sum_{i \in \mathcal{N}} \sum_{s_i^k \notin \beta_i(\sigma)} \rho_i^k \qquad \text{and} \qquad \dot{\gamma}(\rho) := \sum_{i \in \mathcal{N}} \sum_{s_i^k \notin \beta_i(\sigma)} \rho_i^k f_i^k(\rho).$$

With these definitions $\frac{d\gamma(\rho^t)}{dt} = \dot{\gamma}(\rho^t)$ holds along every trajectory $\{\rho^t\}$. Since the dynamic reinforces best replies we have $f_i^k(\sigma) < 0$ for all $s_i^k \notin \beta_i(\sigma)$. Because each $f_i^k(\sigma)$ is continuous there exists a constant $c > 0$ such that $f_i^k(\rho) < -c$ holds for all $i \in \mathcal{N}$ and all $s_i^k \notin \beta_i(\sigma)$ in a sufficiently small neighborhood of σ. Therefore

$$\dot{\gamma}(\rho) \leq -c\gamma(\rho) \leq 0 \tag{2}$$

holds in this neighborhood. Since the dynamic is Lipschitz continuous we can find a constant $L > 0$ such that the inequality

$$\sum_{i \in \mathcal{N}} \sum_{s_i^k \in \mathcal{S}_i} \left| \rho_i^k f_i^k(\rho) - \tau_i^k f_i^k(\tau) \right| \leq L \sum_{i \in \mathcal{N}} \sum_{s_i^k \in \mathcal{S}_i} \left| \rho_i^k - \tau_i^k \right|$$

holds for all $\rho, \tau \in \Delta$ sufficiently close to σ. We can represent each component ρ_i of $\rho \in \Delta$ as a convex combination $\rho_i = (1 - \lambda_i) \tau_i + \lambda_i \nu_i$ where $0 \leq \lambda_i \leq 1$, $\tau_i \in \tilde{\beta}_i(\sigma)$ and $\nu_i \in \Delta_i$, $\text{supp}(\nu_i) \cap \beta_i(\sigma) = \emptyset$. For this representation

$$\sum_{i \in \mathcal{N}} \sum_{s_i^k \in \mathcal{S}_j} \left| \rho_i^k - \tau_i^k \right| = 2 \sum_{i \in \mathcal{N}} \lambda_i = 2\gamma(\rho). \tag{3}$$

Assumption (*) implies for $\tau = (\tau_i)_{i \in \mathcal{N}} \in \tilde{\beta}(\sigma)$ that each $s_i^k \in \text{supp}(\sigma_i)$, $i \in \mathcal{N}$, is a best reply to τ. Hence $f_i^k(\tau) \geq 0$ holds since the dynamic reinforces best replies and therefore

$$-\rho_i^k f_i^k(\rho) \leq -\rho_i^k f_i^k(\rho) + \tau_i^k f_i^k(\tau) \leq L \sum_{j \in \mathcal{N}} \sum_{s_j^l \in \mathcal{S}_j} \left| \rho_j^l - \tau_j^l \right| = 2L\gamma(\rho).$$

Together with (2) we obtain

$$\rho_i^k f_i^k(\rho) \geq \frac{2L}{c} \dot{\gamma}(\rho)$$

for all $s_i^k \in \text{supp}(\sigma_i)$, $i \in \mathcal{N}$, and all ρ sufficiently close to σ.

It can be easily seen that the sets

$$V_\delta(\sigma) = \left\{ \rho \in \Delta \,\middle|\, \begin{array}{l} \gamma(\rho) < \delta \quad \text{and} \\ \rho_i^k > \sigma_i^k + \frac{2L}{c}\gamma(\rho) - \frac{2L}{c}\delta \quad \text{for all } i \in \mathcal{N}, \, s_i^k \in \text{supp}(\sigma_i) \end{array} \right\}$$

form for varying $\delta > 0$ a basis of open neighborhoods of σ.[6] These neighborhoods are forward invariant for all sufficiently small $\delta > 0$: Let $\{\rho(t)\}_{t \geq 0}$ be a trajectory with

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[6] Notice that for $\rho \in V_\delta(\sigma)$ and all $i \in \mathcal{N}$

$$\sum_{s_i^k \neq \beta(\sigma)} \rho_i^k < a\delta$$

for some constant $a > 0$.

$\rho(0) \in V_\delta(\sigma)$. Let $T > 0$ be the first time where the trajectory leaves $V_\delta(\sigma)$. We must have $\gamma(\rho(T)) < \delta$ since $\dot{\gamma}(\rho) \leq 0$ holds for all ρ in the neighborhood. Moreover, for all $s_i^k \in \operatorname{supp}(\sigma_i)$, $i \in \mathcal{N}$,

$$\rho_i^k(T) - \rho_i^k(0) = \int_0^T \rho_i^k(t) f_i^k(\rho(t))\, dt \geq \frac{2L}{c} \int_0^T \dot{\gamma}(\rho(t))\, dt = \frac{2L}{c}(\gamma(\rho(T)) - \gamma(\rho(0)))$$

and hence

$$\rho_i^k(T) \geq \rho_i^k(0) + \frac{2L}{c}\gamma(\rho(T)) - \frac{2L}{c}\gamma(\rho(0)) > \sigma_i^k + \frac{2L}{c}\gamma(\rho(T)) - \frac{2L}{c}\delta$$

so that $\rho(T) \in V_\delta(\sigma)$, a contradiction.

In particular, σ is Lyapunov stable. In order to show that G is asymptotically stable we show for all sufficiently small $\delta > 0$ that all trajectories $\{\rho(t)\}_{t \geq 0}$ starting in $V_\delta(\sigma)$ have ω-limit points only in G. Let $\hat{\rho}$ be a limit point of the trajectory. $\gamma(\rho(t))$ is decreasing along the trajectory and hence (see Hofbauer and Sigmund (1998) Theorem 2.6.1) $\dot{\gamma}(\hat{\rho}) = 0$. Because $\dot{\gamma}(\rho) < 0$ for all $\rho \in V_\delta(\sigma) \setminus \tilde{\beta}(\sigma)$ we conclude $\hat{\rho} \in \tilde{\beta}(\sigma)$ and therefore that $f_i^k(\hat{\rho}) \geq 0$ holds for all $i \in \mathcal{N}$ and $s_i^k \in \operatorname{supp}(\sigma) \subseteq \operatorname{supp}(\hat{\rho})$. If $f_i^k(\hat{\rho}) > 0$ then $\rho_i^k(t)$ would be increasing for large t. Again using Theorem 2.6.1 in Hofbauer and Sigmund (1998) we would obtain the contradiction $\hat{\rho}_i^k f_i^k(\hat{\rho}) = 0$ with $\hat{\rho}_i^k \neq 0$. Hence $f_i^k(\hat{\rho}) = 0$ for all $i \in \mathcal{N}$ and $s_i^k \in \operatorname{supp}(\sigma)$. Since each $s_i^k \in \operatorname{supp}(\sigma)$ is a best reply to $\hat{\rho}$ and since the dynamic reinforces best replies it follows that $U_i(\hat{\rho}_{-i}, \sigma_i) = U_i(\hat{\rho})$ for all $i \in \mathcal{N}$. Therefore, if we have chosen $V_\delta(\sigma)$ such that Proposition 3 applies, we can conclude that $\sigma \in G$.[7] ∎

The following example is an SE set in a $2 \times 2 \times 2$-game not satisfying Property (*). We construct a dynamics where not all strategy combinations in the set are Lyapunov stable.

Let $x_1 = \operatorname{prob}(D)$, $x_2 = \operatorname{prob}(R)$ and $x_3 = \operatorname{prob}(B)$.

The set $G = \{x_1 = 1\} \cup \{x_3 = 1\}$ is a SE set of the game in Figure 2 which does not satisfy property (*). For instance, (U, L, T) is a best reply to $(D, L, B) \in G$, but (D, R, B) is the unique best reply to (U, L, T).

Consider the following payoff monotonic dynamic:

$$\begin{aligned}
\dot{x}_1 &= x_1(1 - x_1)(1 - x_3)^2 \\
\dot{x}_2 &= x_2(1 - x_2)(1 - x_1)(1 - x_3) \\
\dot{x}_3 &= x_3(1 - x_3)(1 - x_1)^2
\end{aligned}$$

It is obtained from the standard replicator equations by squaring the terms $(1 - x_3)$ and, respectively, $(1 - x_1)$ in the equations for $\dot{x}_1$ and $\dot{x}_3$ so that the speed of a trajectory in the x_1 and x_3 direction is slowed down while the speed in the x_2 direction is unaltered. It is not difficult to see that G is an asymptotically stable set of rest points. We claim that the points $(1, x_2, 1) \in G$ with $x_2 < 1$ are not Lyapunov stable for $x_2 < 1$.

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[7]In the case of two-player games one can directly argue as follows: Since $\hat{\rho} \in \tilde{\beta}(\sigma)$ we have $(\hat{\rho}_1, \sigma_2) \in G$. But then $U_2(\hat{\rho}_1, \sigma_2) = U_2(\hat{\rho})$ means that $\hat{\rho}_2$ is a best reply to $(\hat{\rho}_1, \sigma_2)$ and hence $\hat{\rho} = (\hat{\rho}_1, \hat{\rho}_2) \in G$.
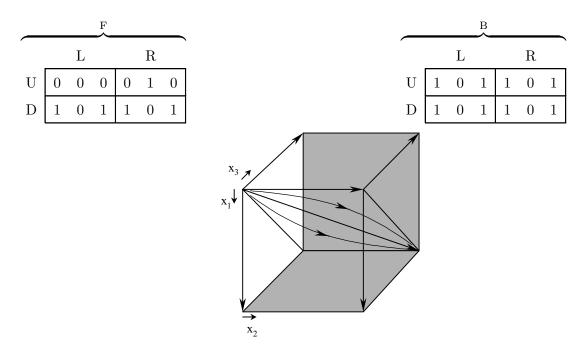
Figure 2: A game with an SE set not satisfying Property (*) where not all Nash equilibria are Lyapunov stable.

By symmetry, the plane $\{x_1 = x_3\}$ is invariant. Consider any trajectory $x(t)$ starting in the relative interior of this plane. Then $\lim_{t\to\infty} x_1(t) = \lim_{t\to\infty} x_3(t) = 1$. Separation of variables shows that

$$x_2(t) = \frac{cx_1(t)}{1 - x_1(t) + cx_1(t)}$$

holds for an appropriate constant $c > 0$. Therefore, $\lim_{t\to\infty} x_2(t) = 1$. Thus we can find trajectories starting arbitrarily close to any $(1, x_2, 1)$ which converge to $(1, 1, 1)$.

We do not know whether an a SE set not satisfying Property (*) is always asymptotically stable. If this were true, multilinearity of the payoff functions would have to be essential in a proof. Without multilinearity asymptotic stability does not have to hold, as we show in the appendix.

Bringing together Theorem 7 and Theorem 10 we obtain:

Corollary 11 *Consider a regular selection dynamic reinforcing best replies and a closed and connected set G containing at least one pure strategy combination. Suppose that there are only two players or that G is convex. Then G is an asymptotically stable set of rest points if and only if G is a strict equilibrium set.*

The strongest result can be obtained for the standard replicator dynamic (the "if" statement here is due to Balkenborg (1994)):

Theorem 12 *A non-empty and closed set is an asymptotically stable set of rest points under the standard replicator dynamics if and only if it is a strict equilibrium set.*

Proof. Suppose G is an asymptotically stable set of rest points. Because the standard replicator dynamic is volume preserving, every connected component of G contains a pure strategy combination (see Ritzberger and Weibull (1996), Proposition 6). By Proposition 7 every connected component of G is a SE set. Since the union of SE sets is a SE set, G is a SE set. Conversely, using Proposition 5 and the Lypunov function

$$L\left(\rho\right) = \sum_{i\in\mathcal{N}} \sum_{s_i^k\in\mathcal{S}_i} \sigma_i^k \cdot \ln\left(\rho_i^k\right)$$

for $\sigma \in G$ it is straightforward to show that a SE set is an asymptotically stable set of stable rest points for the standard replicator dynamics (see Balkenborg (1994) for details). ∎

4 Conclusion

Static game theoretic concepts as necessary or sufficient conditions for dynamic stability are extremely useful because explicit dynamics are often difficult to handle. The concept of an Evolutionarily Stable Strategy (ESS) has been popular as it is sufficient for asymptotic stability under the replicator dynamics in single population contests. As ESS often do not exist, set-valued generalizations such as that of an Evolutionarily Stable Set (ES Set, Thomas (1985)) have been proposed (see also Balkenborg and Schlag (2000) and Weibull (1995)). In multi-population contests based on normal form games, a population state has to be a strict equilibrium point in order to be asymptotically stable for a large variety of evolutionary dynamics. A strict equilibrium is hence the counterpart of an ESS for this setting. Similar to ESS, strict equilibria often fail to exist, in particular in extensive-form games.

Hence Balkenborg (1994) introduces the set-valued concept of a SE set which is a generalization of a strict equilibrium in the same spirit as an ES Set generalizes an ESS. In his unpublished manuscript Balkenborg shows that SE Sets are asymptotically stable under the multi-population standard replicator dynamic. Doing this he uncovers the observation stated in Proposition 3 that we build on in the proof of Proposition 10.[8] We complement his research by showing, conversely, that only SE Sets can be asymptotically stable under this dynamic.

Both in the single- and the multi-population setting Swinkels (1992) has proposed alternative notions for the evolutionary stability of sets called equilibrium evolutionarily stable sets (EES sets). Their dynamic interpretation is, however, far from obvious. For symmetric two-player games Balkenborg and Schlag (2000) show that every ES set is a EES set. Using their results and those in Balkenborg (1994) it is not difficult to show that every connected SE set of a two-player game is a EES set as defined by Swinkels in the two-population setting.

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[8]For two player games our main findings are, however, independent of his his results.

The standard replicator dynamic is a particularly important example of a regular selection dynamic as it results from many different models of individual learning or imitation (see, e.g., Borgers and Sarin (1997), Gale, Binmore, and Samuelson (1995), Schlag (1998)). But alternative regular selection dynamics have emerged from individual learning models (e.g., Schlag (1999), Hofbauer and Schlag (2000)) and the theoretical analysis has focused on properties which hold in general classes of such dynamics (see, e.g., Samuelson and Zhang (1992), Ritzberger and Weibull (1996)). In this paper we investigate a broad class of regular selection dynamics that covers both sign-preserving and payoff positive dynamics and incorporates all explicit regular selection dynamics we have found in the literature. We show that the SE set concept is closely linked to asymptotic stability for such dynamics. It is hence an important tool for analyzing evolutionary dynamics in asymmetric games.

While this paper is more concerned with the development of tools than with immediate applications we obtain direct implications in two examples. It follows that the forward induction outcome for the twice repeated battle-of-the-sexes game determined in van Damme (1989) is asymptotically stable under a wide variety of evolutionary dynamics. The same holds for the set of efficient equilibria in an action commitment game studied by van Damme and Hurkens (1996). Because we do not restrict attention to convex sets of strategy combinations our approach yields stability results directly for the set of efficient equilibria whereas the solution set identified by van Damme and Hurkens (1996) also contains inefficient strategy combinations which are not even Nash equilibria. For further applications of SE sets we refer to Balkenborg (1995) for repeated games and to Schlag (1994) for cheap talk games.

Our paper complements the findings of Ritzberger and Weibull (1996) who study the asymptotic stability for faces while we are concerned with the asymptotic stability of components of rest points containing a pure strategy combination. It is natural to concentrate on (components of) rest points because they describe states where learning or evolution has come to a stand-still. However, our analysis says little about asymptotically stable interior rest points or components of rest points. Here we refer to Hofbauer and Hopkins (2000) who obtain necessary and sufficient conditions for the asymptotic stability of interior Nash equilibria in two-player games under perturbed best reponse dynamics. These are, of course, not regular selection dynamics.

For sign-preserving as well as many other dynamics rest points are Nash equilibria in the game restricted to strategies in the support of this rest point. Thus it is not too surprising that the asymptotically stable sets of rest points we consider turn out to be Nash equilibrium components. A more interesting finding in this context is that all elements of a convex asymptotically stable set of rest points containing a pure strategy combination are necessarily Lyapunov stable. As we demonstrate in Section 3 (see Figure 3) this is not necessarily true when the asymptotically stable set is not convex. The difficulties revealed by this example and by the negative result in the appendix are also the reason why our analysis remains unsatisfactory for games with more than two players. We do not know whether Theorem 10 continues to hold without assuming Property (*).

For several important classes of extensive-form games components of rest points can only be asymptotically stable with respect to trajectories starting in the interior of the space of mixed strategy combinations (see, e.g., Cressman (1996), Cressman and Schlag

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(1997) and Binmore and Samuelson (1994)). Our restriction to components which are asymptotically stable with respect to all trajectories is hence severe. It is an open question whether a general game theoretic characterisation of components which are asymptotically stable in the weaker sense is feasible.

A   An Example for playing the field models

It is an open question whether SE sets which do not satisfy property (*) are stable for any dynamic reinforcing best replies. If so, this must be due to the multilinearity of the expected payoff function for a normal form game. As shown for the example below, once we allow for the possibility that the expected payoff function is not multilinear in the opponent's mixed strategies SE sets do not have to be asymptotically stable or even Lyapunov stable. Such games are of interest as they constitute the n-population analogue of playing the field models based on a single population (see Maynard Smith (1982) and Hammerstein and Selten (1994)). A detailed study of such models can be found in Cressman (1992). Nash equilibria, SE sets etc. are still defined for these games.

Consider the $2 \times 2 \times 2$ playing-the-field model with the payoff functions

$$U_i\left(x_1, x_2, x_3\right) := x_i\left(x_j - a\left(x_j + x_k\right)\right)\left(x_k - a\left(x_j + x_k\right)\right)$$

where $\frac{1}{2} \leq a < 1$ and where (i, j, k) is a permutation of $(1, 2, 3)$. The game is not multilinear because the utility depends quadratically on x_j and on x_k. The set G consisting of the three line segments $L_i := \{x_j = x_k = 0\}$, $i \in \{1, 2, 3\}$ is a SE set. For $a = \frac{1}{2}$ the diagonal $\delta = \{(x, x, x) \mid 0 \leq x \leq 1\}$ consists of Nash equilibria. G is not a closed isolated equilibrium component since $G \cap \delta = \{(0, 0, 0)\}$. In particular G is not asymptotically stable under the standard replicator dynamic. For $a > \frac{1}{2}$ G is not even stable. It has a "leak" at $(0, 0, 0)$ since trajectories starting on the diagonal δ close to this point converge to $(1, 1, 1)$.
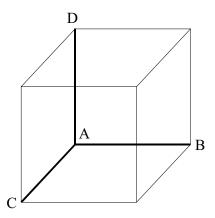

Figure 3: The SE set described is here indicated by the line segments AB, AC and AD.

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