The 1,2,4,8 Theorem: A Topological Viewpoint

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This article is devoted to review some basic knowledge on algebraic topology (in particular, characteristic classes and K-theory) and prove two famous 1,2,4,8-type results of Bott-Milnor-Kervaire and F.Adams, mentioned by Lei.Y in the first year's algebra course. The materials we present here are collected from various places, especially Hatcher's *Vector Bundles and K-Theory* [Hat17], Milnor-Stasheff's *Characteristic Classes* [MS16], and Hirzebruch's *Division Algebras and Topology* [Hir91]. These notes [Lan16] on characteristic classes and K-theory are also enlightening and helpful.

1 Characteristic Classes

Characteristic classes, generally speaking, can be viewed as a homology theoretic study of vector bundles. They are also closely related to differential geometry and algebraic geometry. According to our needs, we introduce the Stiefel-Whitney classes and Chern classes in the view of algebraic topology.

1.1 Stiefel-Whitney Classes

Given a real vector bundle $E \longrightarrow B$, recall that the Stiefel-Whithey classes $w_i(E) \in H^i(B;\mathbb{Z}_2)$ can be defined by some axioms and realized by the its corresponding projectivization bundle or the pullback of cohomology classes of the universal bundle. We list some useful properties of vector bundles and Stiefel-Whitney classes.

Lemma 1.1 (Splitting Principle) *Given a vector bundle $E \longrightarrow B$, there exists a $p : F(E) \longrightarrow B$ such that the pullback bundle $p^*E \longrightarrow F(E)$ splits as a direct sum of line bundles, and p induces an injection on cohomology rings.*

Lemma 1.2 (Product Formula) *For vector bundle $E \longrightarrow B$, recall the* ***total Stiefel-Whitney class*** *is defined by $w(E) = 1 + w_1(E) + w_2(E) + \cdots \in H^*(B;\mathbb{Z}_2)$. Let E_1, E_2 be vector bundles over B, then $w(E_1 \oplus E_2) = w(E_1) \cdot w(E_2)$.*

The next lemma shows that the first Stiefel-Whitney class is a homomorphism from the multiplication group of line bundles to cohomology groups.

Lemma 1.3 *Let L_1, L_2 be line bundles over B, then $w(L_1 \otimes L_2) = 1 + w_1(L_1) + w_1(L_2)$.*

Proof. Reduce to the universal bundle and notice that $\mathbb{RP}^\infty \times \mathbb{RP}^\infty$ and $\mathbb{RP}^\infty \vee \mathbb{RP}^\infty$ have the same H^1 and $\mathbb{RP}^\infty$ is a $K(\mathbb{Z}_2, 1)$. □

1.2 Chern Classes

The analogy of Stiefel-Whitney classes in the case of complex vector bundles are Chern classes. The constructions and main properties are entirely similar with the real case and therefore we omit them. We only mention the construction of Chern character.

Construction 1.4 *We define the **Chern character** $ch(E)$ for complex vector bundle $E \to B$ as follows. Firstly, for line bundle $L \to B$, let*

$$ch(L) = \exp(c_1(L)) \in H^{**}(B;\mathbb{Q}) := \prod_{i\geq 0} H^{2i}(B;\mathbb{Q}),$$

where the exponential should be viewed as formal series. Then we extend the definition of ch to direct sums of line bundles by forcing it to become additive, and to all the vector bundles by the splitting principle.

Intrinsic people might find uncomfortable with the definition given above, and here it is an alternative definition without the help of splitting principle, and therefore it makes the Chern character more computable. The philosophy is to pretend *all the vector bundles split*, as Brezhnev did, see the joke 8 or 41 in [Peo91]. More concretely, one can write the sum of powers as a polynomial of elementary symmetric polynomials, which is called the **Newton polynomial**. Let the graded Chern character

$$\mathrm{ch}_k(E) = \frac{1}{k!} s_k(c_1(E), c_2(E), \ldots, c_n(E))$$

and let $\mathrm{ch}(E)$ be the sum of them. The reader can verify the two definitions coincident.

Chern character connects K-theory with homology theory. The following lemma confirms this slogan, and will be used in the next section.

Lemma 1.5 *The Chern character is a natural homomorphism between K-rings and cohomology rings of even degree. In the case $B = S^{2n}$, it is a monomorphism.*

2 K-Theory

We state some basic results of topological K-theory. The more complicated constructions, Adams operations, are going to be introduced in section 5.

2.1 Introduction

Roughly speaking, K-theory considers the classifying problem of vector bundles by operations on it, including the direct sum and tensor product.

In the complex case, we consider the set $\mathrm{Vect}_{\mathbb{C}}(X)$ of isomorphic classes of complex vector bundles over a compact Hausdorff space X. It has a structure of abelian semigroup under direct sum, and therefore we have an abelian group after the standard construction of Grothendieck K-group, denoted by $K(X)$. The kernel of $K(X) \to K(x_0)$, where $x_0 \in X$ is denoted by $\widetilde{K}(X)$, called the **reduced K-group**. We can also give a ring structure on $K(X)$ by considering tensor products. It is not hard to see that $K(\cdot), \widetilde{K}(\cdot)$ are contravariant functors. The real K-theory is an analogy to the complex K-theory, and we denote the real K-group by $KO(\cdot)$ and its reduced version by $\widetilde{KO}(\cdot)$.

K-theory can be viewed as a **generalized cohomology theory**, which means that it satisfies all the Eilenberg-Steenrod axioms but the dimension axiom, after define the graded structure by suspension and relative K-group. Therefore, many methods of (co)homology theory can be used in the study of K-theory. We can define the ring structure on $\widetilde{K}^*(X) := \widetilde{K}^0(X) \oplus WK^1(X)$, after the discussion on Bott's periodicity. As a further exemplification, we have splitting principle in K-theory. The proof of which is similar to the original splitting principle 1.1 in the study of characteristic classes: consider the projectivization bundle and prove by induction, see [May99].

Lemma 2.1 (Splitting Principle in K-Theory) *Given a vector bundle $E \to B$, where B is compact Hausdorff, there exists a $p : F(E) \to B$ such that $F(E)$ is also compact Hausdorff, and the pullback bundle $p^*E \to F(E)$ splits as a direct sum of line bundles, and p induces an injection on K-rings.*

2.2 Bott's Periodicity

Bott's periodicity is one of the most important results in K-theory and homotopy theory. Its K-theory version depends on the construction of external product. We give an explicit description in the complex case, and only state the main result in the real case. Firstly, the following product theorem is the technical hardcore in the proof of Bott's periodicity.

Proposition 2.2 *Let H be the canonical line bundle over $S^2 = \mathbb{CP}^1$, then in $K(S^2)$ we have $(H-1)^2 = 0$, which gives map $\mathbb{Z}[H]/(H-1)^2 \to K(S^2)$. It induces an isomorphism*

$$\mu : K(X) \otimes \mathbb{Z}[H]/(H-1)^2 \to K(X) \otimes K(S^2) \to K(X \times S^2).$$

Construction 2.3 (External Products) *In the unreduced version, recall that the external product is defined by*

$$K(X) \otimes K(Y) \to K(X \times Y),\ a \otimes b \mapsto a * b := p_1^* a \cdot p_2^* b,$$

*where p_1, p_2 are projections onto X and Y, respectively. In the reduced case, we consider the **smash product** $X \wedge Y := X \times Y / X \vee Y$ and the long exact sequence for the pair $(X \times Y, X \vee Y)$. The reduced external product $\widetilde{K}(X) \otimes \widetilde{K}(Y) \to \widetilde{K}(X \wedge Y)$ is obtained from the original external product by exuding the term $\widetilde{K}(X) \oplus \widetilde{K}(Y) \oplus \mathbb{Z}$.*

Theorem 2.4 (Bott's Periodicity) *The map*

$$\beta : \widetilde{K}(X) \to \widetilde{K}(S^2 X),\ a \mapsto (H-1) * a,$$

induced by the reduced external products, is an isomorphism for compact Hausdorff space X.

Corollary 2.5 *$\widetilde{K}(S^n)$ is $\mathbb{Z}$ for n even and 0 for n odd. $\widetilde{K}(S^2)$ is generated by $H-1$ and has trivial multiplication structure, and similar result holds for $\widetilde{K}(S^{2n})$.*

The consequence in the real case is similar except the period is 8 rather than 2.

Theorem 2.6 (Bott's Periodicity, Real Version) *The map*

$$\beta : \widetilde{KO}(X) \to \widetilde{KO}(S^8 X),\ a \mapsto (\rho_8 - 8) * a,$$

induced by the reduced external products, is an isomorphism for compact Hausdorff space X, where ρ_8 is the canonical bundle over $\mathbb{OP}^1 = S^8$.

2.3 Hopf Invariants

Construction 2.7 *Given a map $f : S^{2n-1} \to S^n$, we can view f as the gluing map of a $2n$-cell and construct a CW complex C_f. Then the long exact sequence of pair (C_f, S^n) gives isomorphisms $H^n(S^n) \cong H^n(C_f)$, $H^{2n}(S^{2n}) \cong H^{2n}(C_f, S^n) \cong H^{2n}(C_f)$. Assume that α, β are a generator of $H^{2n}(C_f)$, $H^n(C_f)$, respectively. Then there is an integer $H(f)$ such that $\beta^2 = H(f)\alpha$ up to sign, called the **Hopf invariant** of f. It is not hard to show that $H(f)$ only depends on the homotopy class of f.*

We give another description of Hopf invariants by K-theory. The pair (C_f, S^n), defined as above, induces the long exact sequence of K-groups. By Bott's periodicity, it implies that

$$0 \to \widetilde{K}(S^{2n}) \to \widetilde{K}(C_f) \to \widetilde{K}(S^n) \to 0$$

is exact. Let β be a preimage of the generator $(H-1) * (H-1) * \cdots * (H-1) \in \widetilde{K}(S^n)$ and α be the image of the generator $(H-1) * (H-1) * \cdots * (H-1) \in \widetilde{K}(S^{2n})$. By the ring structure of $\widetilde{K}(S^2)$, we have $\beta^2 = 0$, and therefore $\beta^2 = h\alpha$ for some integer h.

Proposition 2.8 *Suppose that n is even and choose the generator carefully, we have $h = H(f)$.*

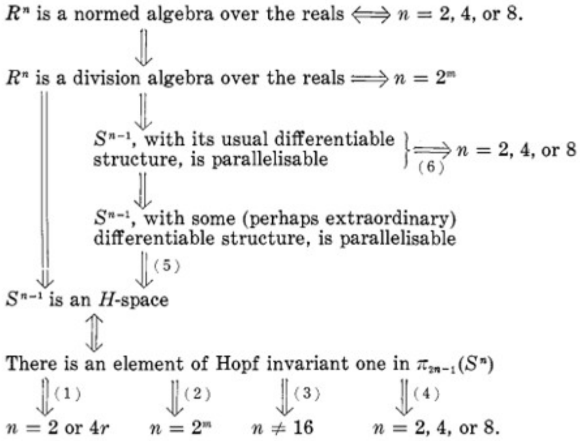
Proof. It is essentially from the following commutative diagram.

$$\begin{tikzcd}
0 \arrow[r] & \widetilde{K}(S^{4n}) \arrow[r] \arrow[d, "\mathrm{ch}"] & \widetilde{K}(C_f) \arrow[r] \arrow[d, "\mathrm{ch}"] & \widetilde{K}(S^{2n}) \arrow[r] \arrow[d, "\mathrm{ch}"] & 0 \\
0 \arrow[r] & H^{**}(S^{4n}; \mathbb{Q}) \arrow[r] & H^{**}(C_f; \mathbb{Q}) \arrow[r] & H^{**}(S^{2n}; \mathbb{Q}) \arrow[r] & 0
\end{tikzcd}$$

The second and fourth columns are injective by lemma 1.5, and therefore the middle column is injective by the five lemma. Then do simple diagram-chasing. □

3 Topological Preparations, and Elementary Results

Recall that a **division algebra** over k is a k-vector space V with a bilinear map $V \times V \to V$ such that $x \mapsto a \cdot x$, $x \mapsto x \cdot a$ are invertible for $a \neq 0$. Our problem is to find all n such that there exists a division algebra of dimension n over $\mathbb{R}$. This is a historical problem and related to many important topological phenomenons. The following figure shows the connection between those propositions, which is copied from Adams' famous paper [Ada60].



Our ultimate goal is to prove Adams' result (4) in section 5, and another theorem in section 4, which also answers the original problem. Notice that all the positive answer can be given by the example of $\mathbb{R}, \mathbb{C}$, quaternions $\mathbb{H}$ and octonions $\mathbb{O}$. In this section, we are going to introduce a series of topological observations and prove some elementary (negative) results.

3.1 Topological Preparations

Lemma 3.1 *Assume that there exists a division algebra of dimension n over $\mathbb{R}$, then S^{n-1} and $\mathbb{RP}^{n-1}$ are parallelizable. Or equivalently, the tangent bundle of S^{n-1} and $\mathbb{RP}^{n-1}$ are trivial.*

Proof. Take a standard orthogonal basis of $\mathbb{R}^n$, say, $e_1, e_2, \ldots, e_n$. For $y \in S^{n-1}$, $e_1 \cdot y, e_2 \cdot y, \ldots, e_n \cdot y$ are linearly independent by the definition of division algebra. Run the Gram-Schmidt process and we get $w_1(y) = e_1 \cdot y/||e_1 \cdot y||, w_2(y), \ldots, w_n(y)$. The map $y \mapsto w_1(y)$ is bijective from S^{n-1} to itself. Therefore the other w_is give the vector fields we need. Furthermore, they are stable under the antipodal transformation. □

Recall that there is a natural map $[S^{n-1}, \mathrm{GL}(n, \mathbb{R})] \to \mathrm{Vect}^k(S^n)$. More concretely, the sphere S^n can be divided into two semisphere on which the vector bundle is trivial. Thus, the structure of the bundle is determined by the gluing function on the equator S^{n-1}. Let $f : S^{n-1} \to \mathrm{GL}(n, \mathbb{R})$, we denote the corresponding vector bundle by E_f. If f is derived from the algebra structure, then we call E_f the **Hopf bundle** of the algebra. By the way, the natural map become a one-to-one correspondence in the complex case.

Lemma 3.2 *Assume that there exists a division algebra of dimension n over $\mathbb{R}$, then there exists a vector bundle E of rank n over S^n such that $w_n(E) \neq 0$.*

Proof. The structure of division algebra on $\mathbb{R}^n$ restrict to a map $f : S^{n-1} \to \mathrm{GL}(n, \mathbb{R})$ by left multiplication. It gives a vector bundle E_f by clutching function. Then it can be verified that $w_n(E_f) = 1$, see [Mil58]. □

We say a space X is an **H-space** if there exists a continuous map $X \times X \to X$, denoted by multiplication, with a two-sided identity $e \in X$, i.e. $e \cdot x = x \cdot e = x$.

Lemma 3.3 *If S^{n-1} is parallelizable, then it is an H-space.*

Proof. Assume that $v_1, v_2, \ldots, v_{n-1}$ are everywhere linearly independent vector fields. By the Gram-Schmidt process again, we can assume that they form a normal orthogonal basis. We can assume also that $v_1(e_1), \ldots, v_{n-1}(e_1)$ are $e_2, \ldots, e_n$. For $x \in S^{n-1}$. Let $\alpha_x \in \mathrm{SO}(n)$ send the standard basis to the basis corresponds to x, and $(x, y) \mapsto \alpha_x(y)$ gives an H-space structure on S^{n-1} with identity e_1. □

Construction 3.4 *Given a map $f : S^{n-1} \times S^{n-1} \to S^{n-1}$, we construct its associated map $\tilde{f} : S^{2n-1} \to S^n$ as follows. Regard S^{2n-1} as $\partial D^{2n} \cong \partial(D^n \times D^n) = \partial D^n \times D^n \cup D^n \times \partial D^n$, and map the first part to the north semisphere of S^n by $(x, y) \mapsto |y|g(x, y/|y|)$ and the second part to the south semisphere by $(x, y) \mapsto |x|g(x/|x|, y)$. It is easy to verify this map is well-defined.*

Lemma 3.5 *If $g : S^{2n-1} \times S^{2n-1} \to S^{2n-1}$ is an H-space multiplication, then its associated map $\tilde{g} : S^{4n-1} \to S^{2n}$ has Hopf invariant ± 1.*

Proof. Let $e \in S^{2n-1}$ be the identity and let $f = \tilde{g}$. Consider the characteristic map Φ of the $4n$-cell of C_f as a map $(D^{2n} \times D^{2n}, \partial(D^{2n} \times D^{2n}) \to (C_f, S^{2n})$. By the definition of f, Φ restricts to a homeomorphism from $D^{2n} \times \{e\}$ to D^{2n}_- and from $\{e\} \times D^{2n}$ to D^{2n}_+. Then gaze at the following diagram, where the diagonal map is the external product and it is an isomorphism from Bott's periodicity.

$$\begin{tikzcd}
\widetilde{K}(C_f)\otimes\widetilde{K}(C_f) \arrow[rr, "\text{product}"] & & \widetilde{K}(C_f) \\
\widetilde{K}(C_f, D^{2n}_-)\otimes\widetilde{K}(C_f, D^{2n}_+) \arrow[u, "\cong"] \arrow[rr, "\text{product}"] \arrow[d, "\Phi^*\otimes\Phi^*"] & & \widetilde{K}(C_f, S^{2n}) \arrow[u] \arrow[d, "\Phi^*"', "\cong"] \\
\widetilde{K}(D^{2n}\times D^{2n}, \partial D^{2n}\times D^{2n})\otimes\widetilde{K}(D^{2n}\times D^{2n}, D^{2n}\times \partial D^{2n}) \arrow[rr] \arrow[d, "\cong"] & & \widetilde{K}(D^{2n}\times D^{2n}, \partial(D^{2n}\times D^{2n})) \\
\widetilde{K}(D^{2n}\times\{e\}, \partial D^{2n}\times\{e\})\times\widetilde{K}(\{e\}\times D^{2n}, \{e\}\times\partial D^{2n}) \arrow[urr, "\cong"'] & &
\end{tikzcd}$$

□

3.2 Partial Results

Proposition 3.6 (Hopf) *Assume that there exists a division algebra of dimension n over $\mathbb{R}$, then n is a power of 2.*

Proof. By lemma 3.1, the tangent bundle $T\mathbb{RP}^{n-1}$ is trivial. On the other hand, we have the following lemma.

Lemma 3.7 *$w(T\mathbb{RP}^{n-1}) = (1+h)^n$, where h is the generator of $H^1(\mathbb{RP}^{n-1};\mathbb{Z}_2)$.*

Proof. Consider the tautological line bundle ℓ over $\mathbb{RP}^{n-1}$. Then $T\mathbb{RP}^{n-1} = \mathrm{Hom}(\ell, \ell^\perp)$. Since $\ell\oplus\ell^\perp$ is the trivial bundle ϵ^n, we have $\mathrm{Hom}(\ell,\epsilon^n) = \mathrm{Hom}(\ell,\ell)\oplus\mathrm{Hom}(\ell, T\mathbb{RP}^{n-1}) \cong \epsilon^1\oplus T\mathbb{RP}^{n-1}$. Therefore $w(T\mathbb{RP}^{n-1}) = \mathrm{Hom}(\ell,\epsilon^n) = (w(\ell^*))^n = (1+h)^n$. □

Therefore we have $1 = w(T\mathbb{RP}^{n-1}) = (1+h)^n$. It implies $\binom{n}{k}$ is even for $0<k<n$. Elementary number theory gives the conclusion. □

Proposition 3.8 *Assume that S^n is an H-space, then $n=0$ or n is odd.*

Proof. Suppose the contrast that $n = 2k > 0$ and $\mu: S^{2k}\times S^{2k}\longrightarrow S^{2k}$ gives an H-space structure. Apply the functor $K(\cdot)$ we have an induced homomorphism

$$\mu^*:\mathbb{Z}[\gamma]/(\gamma^2)\longrightarrow\mathbb{Z}[\alpha,\beta]/(\alpha^2,\beta^2).$$

Composing with the inclusion $i: S^{2k}\longrightarrow S^{2k}\times S^{2k}$, $x\mapsto(x,e)$, i^* maps α to γ and β to 0. Thus, the coefficient of α in $\mu^*(\gamma)$ must be 1, and the similar argument holds for β. Therefore $\mu^*(\gamma^2)$ must contain a nonzero term $2\alpha\beta$, which contradicts to the fact that $\gamma^2 = 0$. □

4 Atiyah-Hirzebruch's Proof, Using Real K-Theory

In this section we are going to prove the 1,2,4,8 theorem, following [AH61].

Theorem 4.1 (Bott-Milnor-Kervaire) *Assume that there exists a division algebra of dimension n over $\mathbb{R}$, then $n = 1, 2, 4, 8$.*

Thanks to lemma 3.2, it is enough to prove

Theorem 4.2 *There is an vector bundle $E \longrightarrow S^n$ of rank n with $w_n(E) \neq 0$ only if $n = 1, 2, 4, 8$.*

Proof. The total Stiefel-Whitney class does not change if we replace E by $E - n \in \widetilde{KO}(S^n)$. Therefore it is enough to prove $w(c) = 1$ for $c \in \widetilde{KO}(S^n)$ whenever $n \neq 1, 2, 4, 8$. Bott's periodicity for real K-group shows it is true for $3, 5, 6, 7$ since the reduced K-group is trivial in these cases. For $n > 8$, let $m = n - 8$ and we can write $c = a(\rho_8 - 8)$ by Bott isomorphism, where $a \in \widetilde{KO}(S^m)$ and ρ_8 is the canonical bundle over $\mathbb{OP}^1 = S^8$. Write $a = E - F$, where E, F are vector bundles over S^m with the same dimension. We can assume that the dimension of E and F is even in addition by add 1 simultaneously. Then $c = (E - F)(\rho_8 - 8)$ holds in $KO(S^m \times S^8)$, which implies

$$w(c) = w(E \cdot \rho_8) \cdot w(F \cdot \rho_8)^{-1} \cdot w(8 \cdot E)^{-1} \cdot w(8 \cdot F).$$

We show that every term above is 1, which comes from the following lemma.

Lemma 4.3 *Let E_1, E_2 be even-dimensional vector bundles over B with $w(E_1) = 1 + w_r(E_1)$, $w(E_2) = 1 + w_s(E_2)$, r or s even and $w_r(E_1)^2 = w_s(E_2)^2 = 0$, then $w(E_1 \otimes E_2) = 1$.*

Proof of the Lemma. Use the splitting principle twice, we can assume that E_1, E_2 splits to direct sums of line bundles. Assume that r is even. Let

$$E_1 = L_1 \oplus L_2 \cdots \oplus L_m\,,\ E_2 = L_1' \oplus L_2' \cdots \oplus L_n'\,,$$

and $x_i = w_1(L_i),\ y_j = w_1(L_j')$. Then

$$w(E_1) = \prod_{1 \le i \le m} (1 + x_i) = 1 + w_r(E_1),$$

and therefore

$$\prod_{1 \le i \le m} (t + x_i) = t^m + w_r(E_1) t^{m-r},\ t \in H^*(B; \mathbb{Z}_2).$$

Similar formula holds for y_j. Therefore we have

$$\begin{aligned}
w(E_1 \otimes E_2) &= \prod_{i,j} w(L_i \otimes L_j') = \prod_{i,j} (1 + x_i + y_j) \\
&= \prod_j ((1 + y_j)^m + w_r(E_1)(1 + y_j)^{m-r}) \\
&= (\prod_j (1 + y_j))^{m-r} \cdot \prod_j ((1 + y_j)^r + w_r(E_1)) \\
&= (1 + w_s(E_2))^{m-r} \cdot [(1 + w_s(E_2))^r + w_r(E_1)(n + (\sum_j y_j)^r)] \\
&= 1.
\end{aligned}$$

□

Let $B = S^m \times S^8$ and let E_1 be the pullback of E, E_2 be the pullback of ρ_8, and use the fact that $E \cdot \rho_8$ is corresponding to the bundle $E_1 \otimes E_2$, by the construction of external product. The lemma implies $w(E_1 \otimes E_2) = 1$. The other terms are proved in the same way, and the proof have been done.

□

5 Adams' Proof, Using His Operations

In this section we prove the following result, which implies theorem 4.1 immediately by lemma 3.5 and proposition 3.8.

Theorem 5.1 (Adams) *Suppose that $f : S^{4n-1} \longrightarrow S^{2n}$ has Hopf invariant ± 1, then $n = 1, 2, 4$.*

The proof of the theorem depends on the so-called Adams operations, which can be characterized as follows.

Proposition 5.2 *There exists a series of homomorphism of rings $\psi^k : K(X) \longrightarrow K(X)$, where X is a compact Hausdorff space, satisfying the following properties:*

(a) $\psi^k f^* = f^* \psi^k$, i.e. they are endomorphisms of the functor $K : CHTop \longrightarrow Ring$.

(b) $\psi^k \circ \psi^l = \psi^{kl}$.

(c) $\psi^k(\ell) = \ell^k$ where ℓ is a line bundle.

(d) $\psi^p(\alpha) \equiv \alpha^p \pmod p$ for p prime, i.e. there exists $\beta \in K(X)$ such that $\psi^p(\alpha) - \alpha^p = p\beta$.

By (a), ψ^k restricts to an operation on reduced K-theory and commutes with the external product since it is defined by pullback and usual product.

Lemma 5.3 *$\psi^k : \widetilde{K}(S^{2n}) \longrightarrow \widetilde{K}(S^{2n})$ is multiplication by k^n.*

Proof. First assume that $n = 1$. Its enough to prove for the generator $H - 1$ of $\widetilde{K}(S^2)$ by the additivity. In this case, $\psi^k(H-1) = H^k - 1 = (1 + (H-1))^k - 1 = k(H-1)$. The last "=" is from corollary 2.5. In the general case, consider the isomorphism given by external product $\widetilde{K}(S^2) \otimes \widetilde{K}(S^{2n-2}) \longrightarrow \widetilde{K}(S^{2n})$ and it is easy to prove by induction. □

From this computation and the above properties we can prove Adams' theorem by some functional equation style's tricks. From the proof we will see that the condition can be relax as $H(f)$ is odd.

Proof of Theorem 5.1. Consider the exact sequence

$$0 \longrightarrow \widetilde{K}(S^{4n}) \longrightarrow \widetilde{K}(C_f) \longrightarrow \widetilde{K}(S^{2n}) \longrightarrow 0$$

and the (pre)image $\alpha, \beta \in \widetilde{K}(C_f)$ as in section 2.3. Then $\psi^k(\alpha) = k^{2n}\alpha$ by naturality and (c). Similarly, $\psi^k(\beta) = k^n\beta + \mu_k(\alpha)$ for some $\mu_k \in \mathbb{Z}$. Therefore by (b),

$$\psi^{kl}(\beta) = \psi^k \circ \psi^l(\beta) = \psi^k(l^n\beta + \mu_l\alpha) = k^n l^n \beta + (k^{2n}\mu_l + l^n\mu_k)\alpha.$$

The relation holds after switching k and l, therefore $k^{2n}\mu_l + l^n\mu_k = l^{2n}\mu_k + k^n\mu_l$, or

$$(k^{2n} - k^n)\mu_l = (l^{2n} - l^n)\mu_k.$$

We have

$$\mu_2\alpha \equiv \psi^2(\beta) \equiv \beta^2 = H(f)\alpha \equiv \alpha \pmod 2$$

by (d), and therefore μ_2 is odd, since α is a generator of $\widetilde{K}(S^{4n})$. Take $k = 3, l = 2$, we have $2^n \mid 3^n - 1$, which implies $n = 1, 2, 4$ by elementary number theory. □

We now describe the construction of Adams operations and verify that they satisfy the expected properties.

Construction 5.4 (Adams Operations) *The construction is somewhat similar to the Chern character. Firstly, for line bundle, the definition is clear by (c). By forcing it to become additive and using the splitting principle in K-theory, it can be easily extended to all vector bundles. The verification is quite simple by the construction, except the naturality (a). To prove this, we give an alternative and more concretely construction.*

Recall that for vector spaces we have exterior powers, which can be directly extended to vector bundles as a functional construction. For vector bundle E, we denote its exterior powers by $\lambda^k(E)$. Let s_k be Newton polynomials, that is, $s_k(\sigma_1, \sigma_2, \ldots, \sigma_k) = x_1^k + x_2^k + \cdots + x_n^k$, where σ_i is the i-th elementary symmetric polynomial of $x_1, x_2, \ldots, x_n$. Let

$$\psi^k(E) = s_k(\lambda^1(E), \ldots, \lambda^k(E)).$$

To verify two definition coincident, we notice that for $E = L_1 \oplus \cdots \oplus L_n$ we have

$$\lambda^k(E) = \sigma_k(L_1, \ldots, L_n).$$

And therefore the naturality of ψ^k comes from the functority of λ^k.

At the end of the article, we introduce a more general results of Adams in [Ada62], which entirely answer the question of the maximum number of everywhere linearly independent vector fields on the sphere, using more highly sophisticated K-theory.

Theorem 5.5 *Let n be a positive integer, write n as $u \cdot 2^{4\alpha+\beta}$, where u is odd and $0 \leq \beta \leq 3$, then the maximum number of everywhere linearly independent vector fields on S^{n-1} is the **Radon-Hurwitz number** $\rho(n) = 8\alpha + 2^\beta - 1$.*

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