Expedient Distance Transforms for Gaussian Object Representations

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Abstract

It is shown that the expected distance to surface represented by a sum of Gaussians can be expediently approximated by taking the log of Gaussians with slightly modified covariance. The main idea is to use the heat method with *probabilistic* boundary conditions given by the sum of Gaussians.

1 Introduction

Suppose we are given a representation of obstacle $\Omega \subset \mathbb{R}^D$ as surface containment probability $p_{\partial\Omega}(\mathbf{x})$:

$$p_{\partial\Omega}(\mathbf{x}) = \mathbb{P}(\mathbf{x} \in \partial\Omega). \tag{1}$$

In other words, $p_{\partial\Omega}(\mathbf{x})$ is the probability that $\mathbf{x}$ is in $\partial\Omega$. In practice, $p_{\partial\Omega}$ could be built as an occupancy map (discrete or continuous), a Gaussian mixture model, or Gaussian splats[1]. Here, we consider the special case where $p_{\Omega}(\mathbf{x})$ is a sum of Gaussians:

$$p_{\Omega}(\mathbf{x}) = \sum_i w_i G(\mathbf{x}; \mu_i, \Sigma_i) \tag{2}$$

Our **objective** is to characterise the unsigned distance function $d_{\Omega}(\mathbf{x})$ given $p_{\partial\Omega}(\mathbf{x})$. To do so, we invoke the Varadhan's formula for heat equation[2]:

Theorem 1 (Varadhan's formula)**.** *Consider the solution to the heat equation:*

$$\frac{\partial u}{\partial t} = \frac{1}{2}\nabla^2 u \tag{3}$$

with boundary condition $u(\mathbf{x}, 0) = 1$ on $\partial\Omega$, and 0 otherwise. Then, we have:

$$\lim_{t\to 0} -2t \log u(\mathbf{x}, t) = d^2_{\Omega}(\mathbf{x}). \tag{4}$$

[1] We may treat Ω as a *random compact set*

[2] Log-GPIS uses screened Poisson equation instead of heat equation to avoid taking the square-root of log, but here it will be more convenient to use the heat equation.

The key insight is that we can solve the heat PDE (3) with respect to a *probabilistic* boundary condition given by $p_\Omega(\mathbf{x})$. First, consider the determinstic case when $\partial\Omega$ is known. To solve boundary conditions, it is convenient to compute the *kernel* or the Green function:

$$\begin{aligned} \frac{\partial G(\mathbf{x},\mathbf{y},t)}{\partial t} &= \nabla^2_{\mathbf{x}} G(\mathbf{x},\mathbf{y},t), \\ G(\mathbf{x},\mathbf{y},0) &= \delta(\mathbf{x},\mathbf{y}). \end{aligned} \tag{5}$$

For the heat equation, this is:

$$G(\mathbf{x},\mathbf{y},t) = (2\pi t)^{-N/2} \exp\left(-\frac{||\mathbf{x}-\mathbf{y}||^2}{2t}\right). \tag{6}$$

Given the kernel $G(\mathbf{x},\mathbf{y},t)$, the heat equation for a deterministic $\partial\Omega$ is:

$$u(\mathbf{x},t) = \int G(\mathbf{x},\mathbf{y}) 1_{\partial\Omega}(\mathbf{y}) d\mathbf{y}, \tag{7}$$

where $1_{\partial\Omega}(\mathbf{x})$ is the indicator function of $\partial\Omega$.

Taking the expectations of both sides with respect to random Ω:

$$\begin{aligned} \mathbb{E}_{\partial\Omega}[u(\mathbf{x},t)] &= \mathbb{E}_{\partial\Omega} \int G(\mathbf{x},\mathbf{y}) 1_{\partial\Omega}(\mathbf{y}) d\mathbf{y}, \\ &= \int G(\mathbf{x},\mathbf{y}) \mathbb{E}_{\partial\Omega}[1_{\partial\Omega}(\mathbf{y})] d\mathbf{y}, \\ &= \int G(\mathbf{x},\mathbf{y}) p_{\partial\Omega}(\mathbf{y}) d\mathbf{y}. \end{aligned} \tag{8}$$

In other words, the expectation of $u(\mathbf{x},t)$ is given by a convolution between $G(\mathbf{x},\mathbf{y})$ with $p_{\partial\Omega}$. Conveniently, since G is Gaussian, the convolution takes a closed form if $p_{\partial\Omega}$ is a sum of Gaussian:

$$\int G(\mathbf{x},\mathbf{y}) p_{\partial\Omega}(\mathbf{y}) d\mathbf{y} = \sum w_i G(\mathbf{x}; \mu_i, \Sigma + tI), \tag{9}$$

So, we have:

$$\begin{aligned} \mathbb{E}[d^2_\Omega(\mathbf{x})] &\approx_{t\to 0} -2t\mathbb{E}[\log u(\mathbf{x},t)], \\ &\geq -2t \log \mathbb{E}[u(\mathbf{x},t)], \\ &= -2t \log\left(\sum w_i G(\mathbf{x}; \mu_i, \Sigma + tI)\right), \end{aligned} \tag{10}$$

where the second line makes use of Jensen's inequality. Using the third line, we can compute an approximate expected square distance to Ω.