Localization and Mapping

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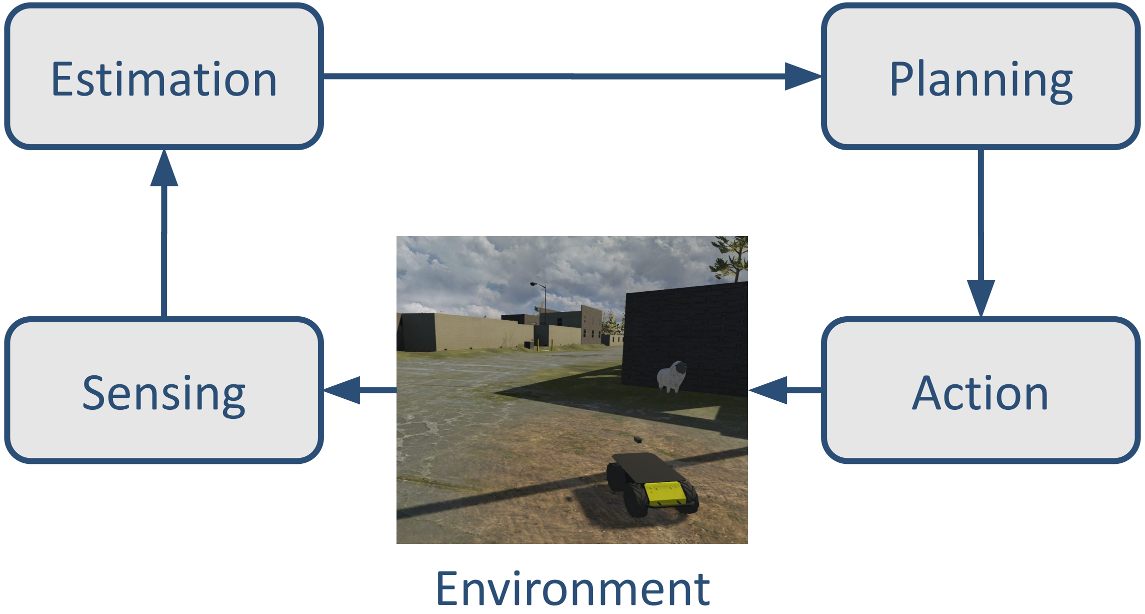
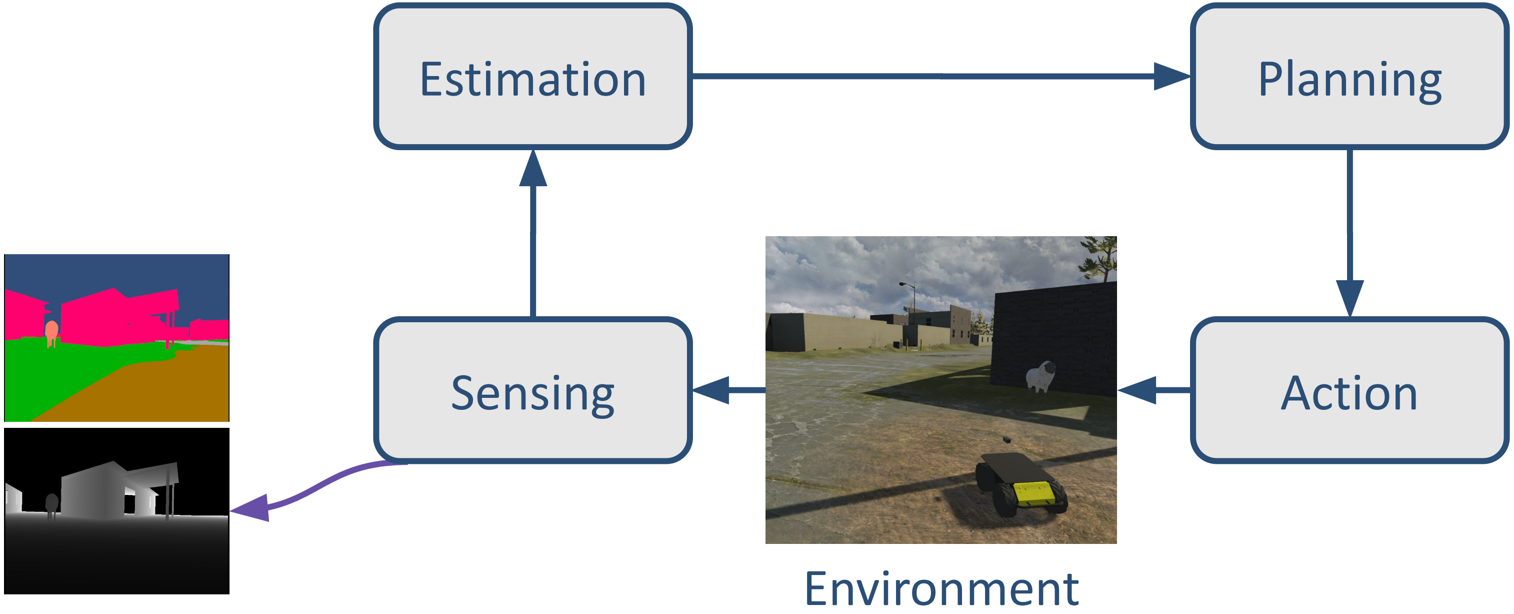
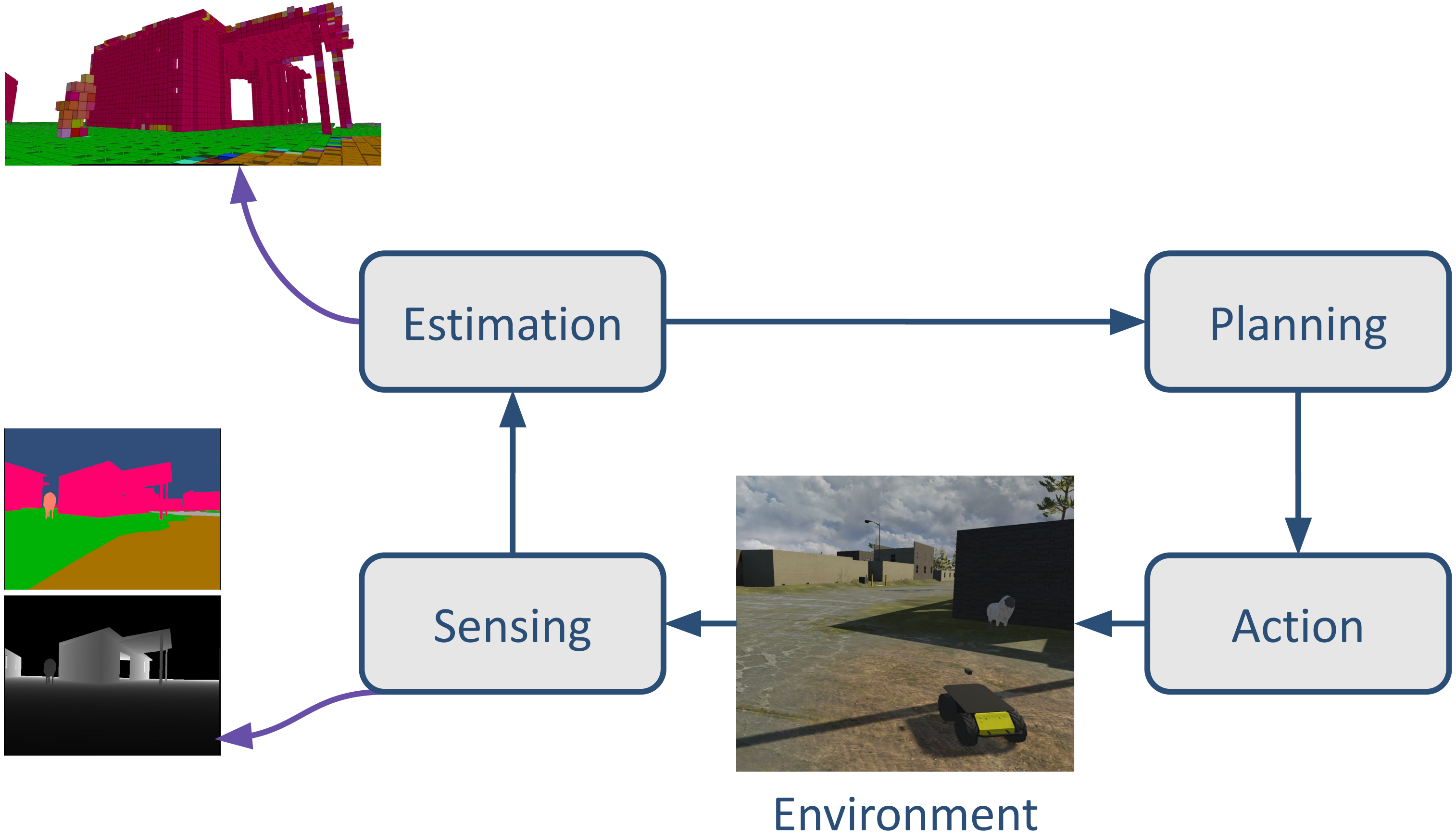
Robot Autonomy



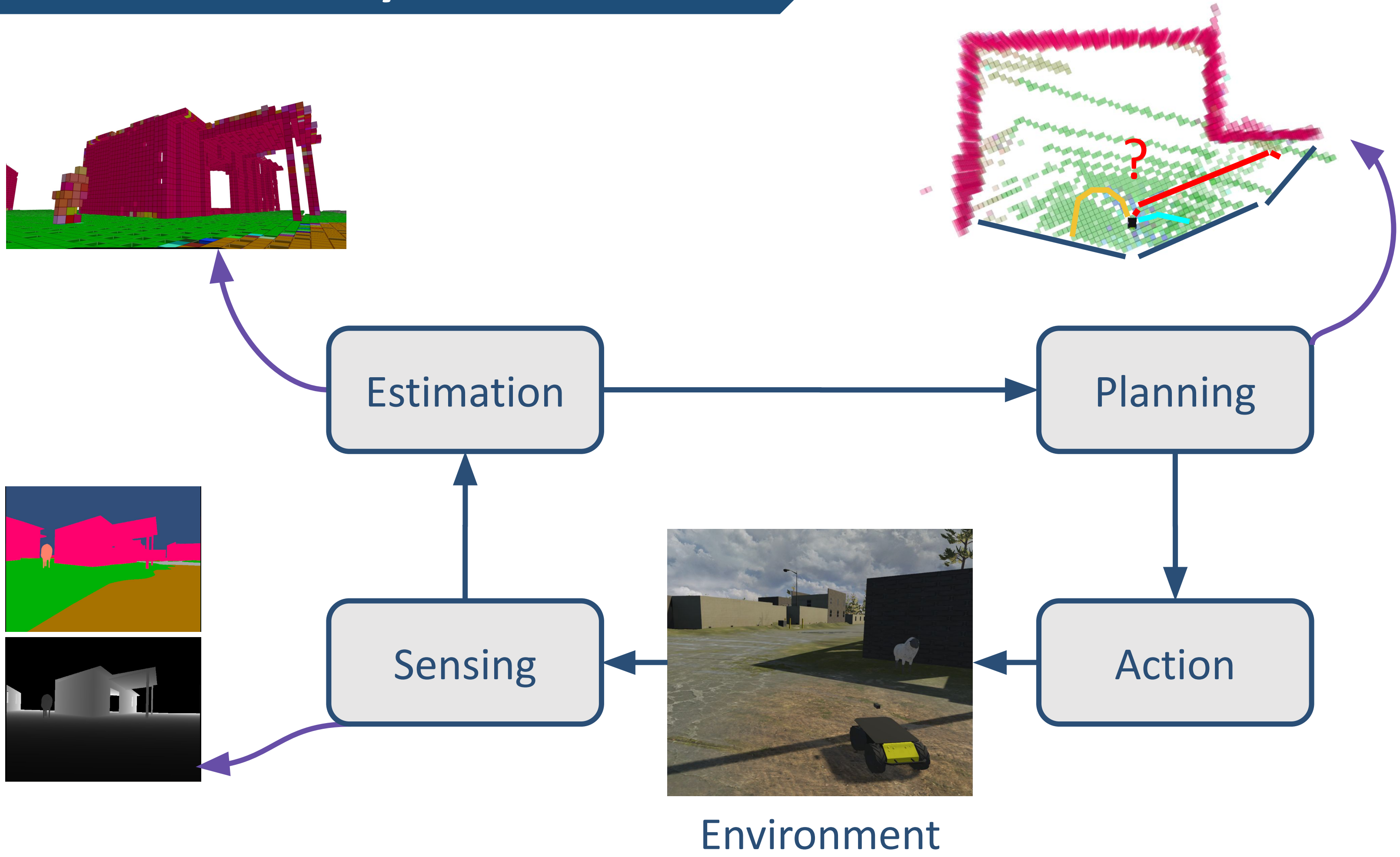
Robot Autonomy



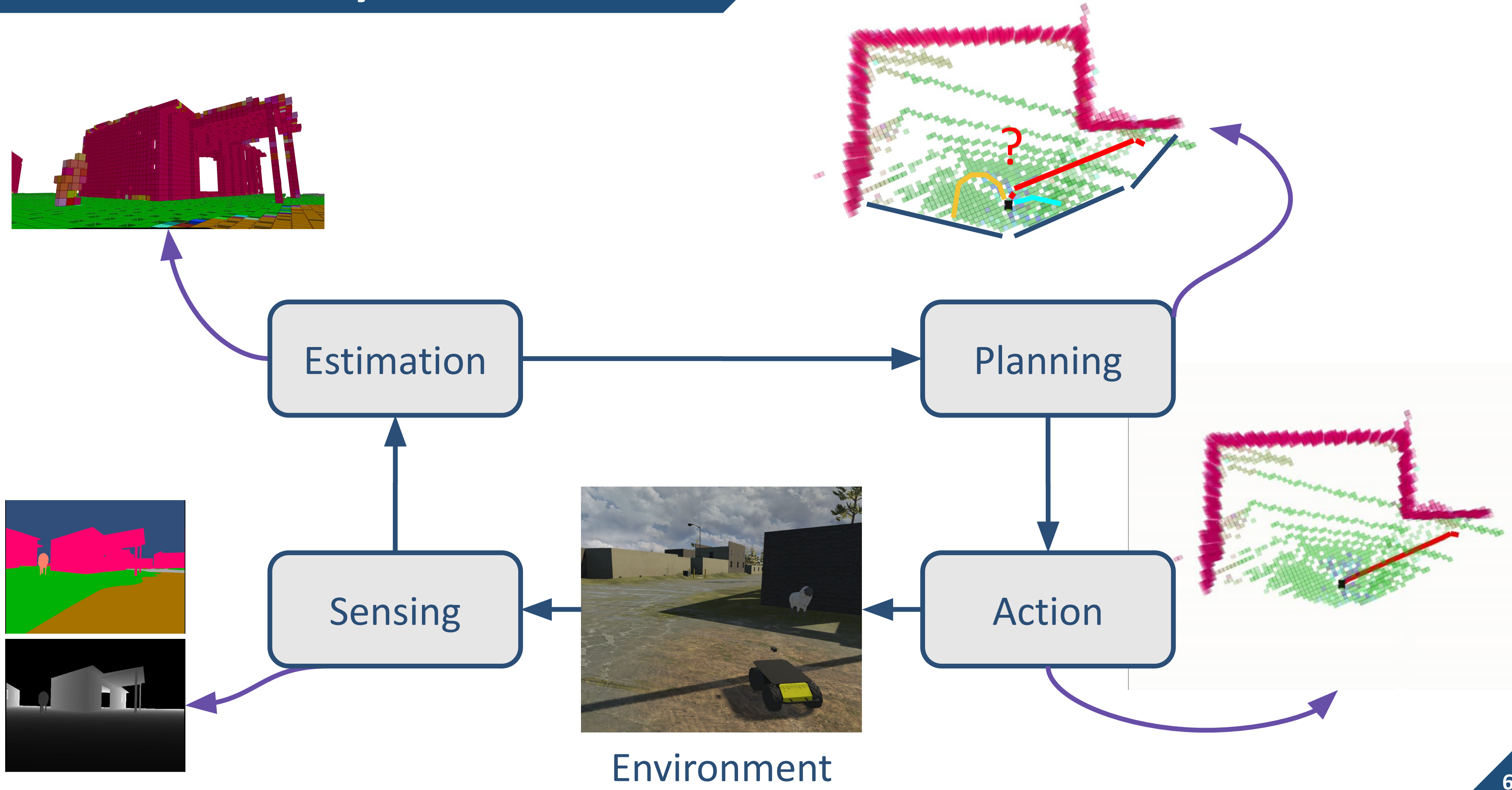
Robot Autonomy



Robot Autonomy

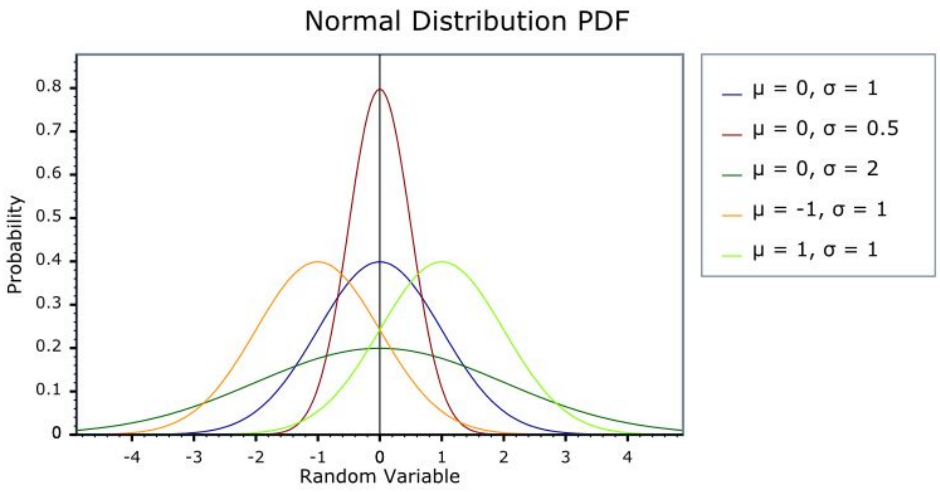


Robot Autonomy



Robot Autonomy

Probability Review



Probability Axioms

- For an event A in sample space □, we assume that the following is always true:

1. $P(A) \geq 0$
2. $P(\Omega) = 1$
3. If $A_i \cap A_j = \varnothing,\ \forall i \neq j$, then $P\left(\bigcup_i A_i\right) = \sum_i P(A_i)$

- Corollary:

1. $P(\varnothing) = 0$
2. $\max\{P(A), P(B)\} \leq P(A \cup B) = P(A) + P(B) - P(A \cap B) \leq P(A) + P(B)$
3. $A \subseteq B \Rightarrow P(A) \leq P(B)$

Probability Formulas

- Total Probability:

$$P(B) = \sum_{i=1}^{n} P\left(B \cap A_i\right),\ \text{if } \Omega = \bigcup_i A_i \text{ and } A_i \cap A_j = \varnothing,\ \forall i \neq j$$

- Conditional Probability:

$$P(A \cap B) = P(A|B)P(B)$$

- Bayes Theorem:

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)},\ P(B) \neq 0$$

- Corollary:

$$P\left(A_i|B\right) = \frac{P\left(B|A_i\right)P\left(A_i\right)}{\sum_{i=1}^{n} P\left(B|A_i\right)P\left(A_i\right)},\ \text{if } \Omega = \bigcup_i A_i \text{ and } A_i \cap A_j = \varnothing,\ \forall i \neq j$$

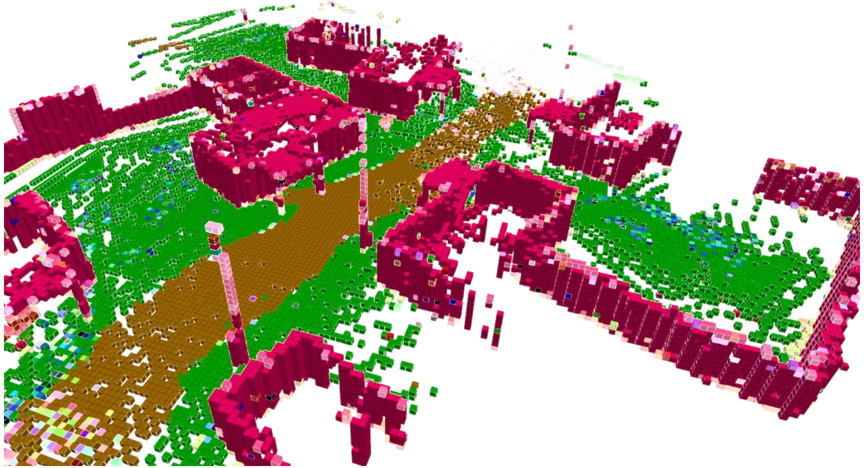
Independent Events

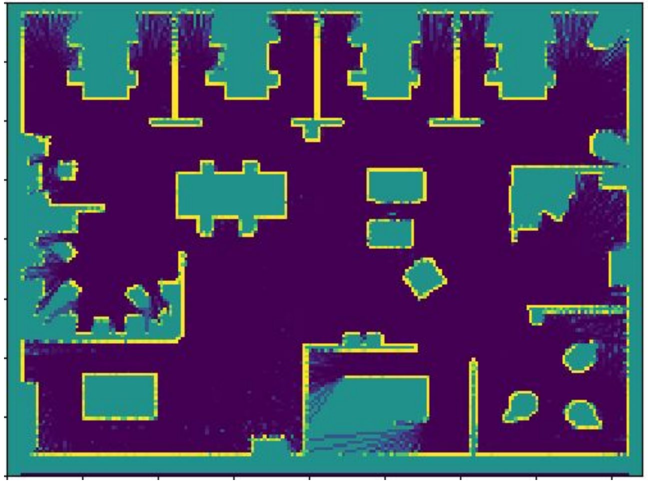
- Events A and B are independent if and only if:

$$\mathrm{P}(\mathrm{A}|\mathrm{B}) = \mathrm{P}(\mathrm{A}) \Leftrightarrow \mathrm{P}(\mathrm{A} \cap \mathrm{B}) = \mathrm{P}(\mathrm{A})\mathrm{P}(\mathrm{B})$$

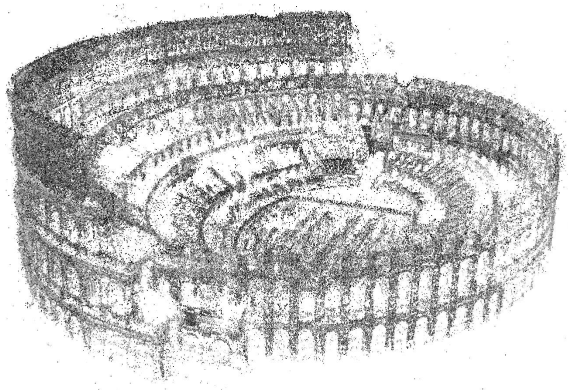
- Observing one does not give any information about another
- Disjoint events are always dependent (why?)

Mapping

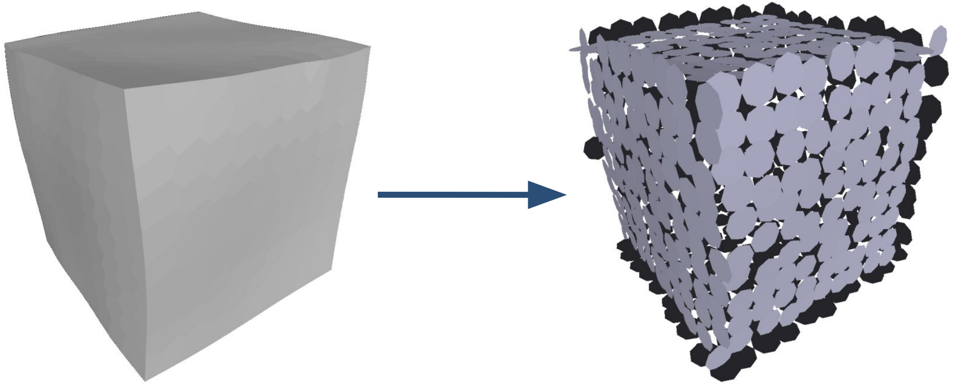




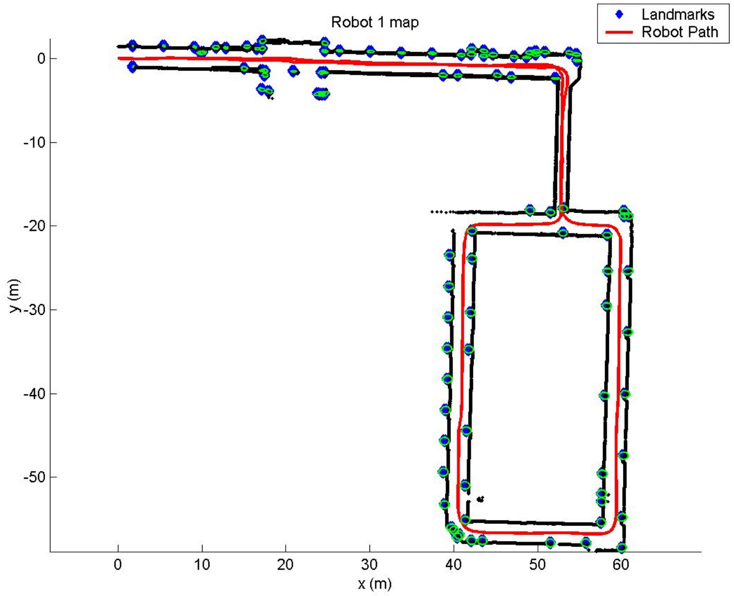
Map Representations (Sparse)



Point Cloud

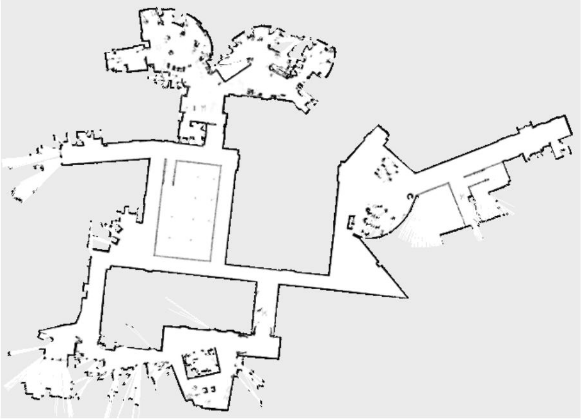


Surfels

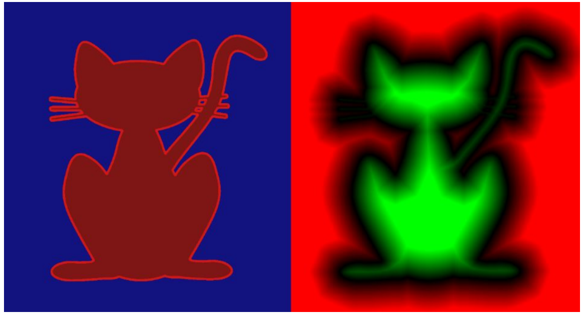


Landmark-based

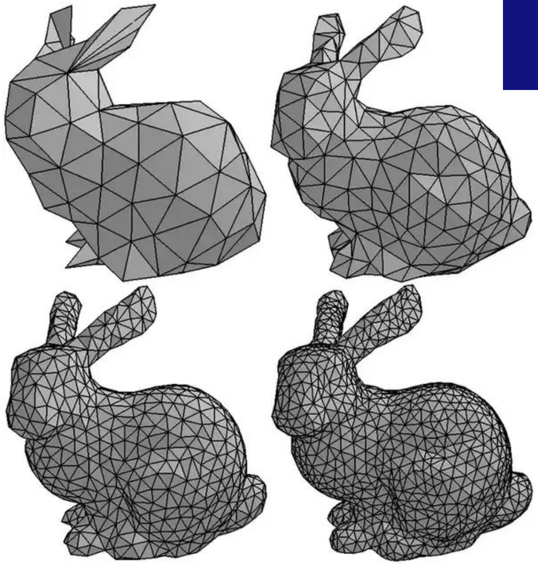
Map Representations (Dense)



Occupancy Grid Map

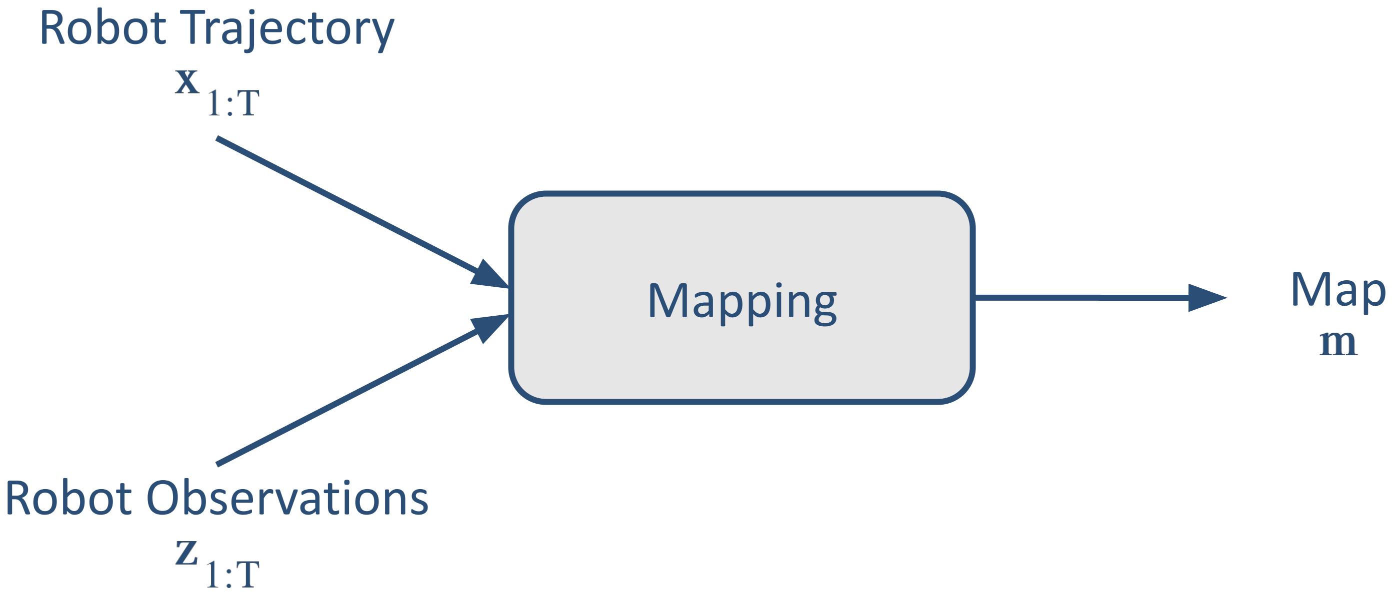


Signed Distance Field

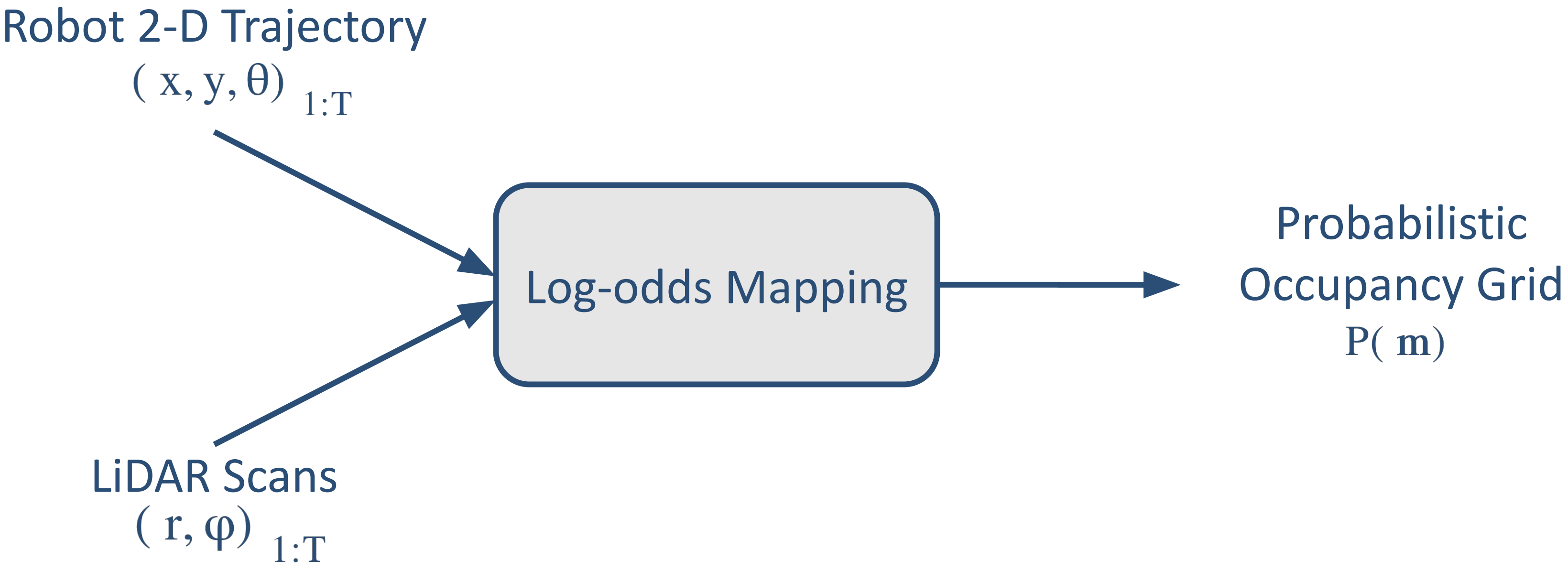


Polygon Mesh

Problem Formulation



Problem Formulation

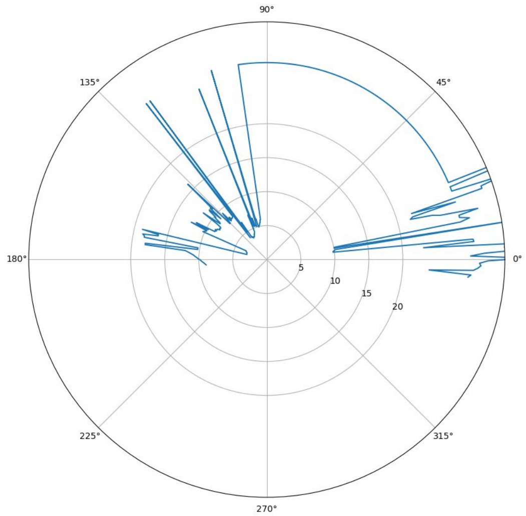


LiDAR Scanner

- LiDAR: Light Detection And Ranging

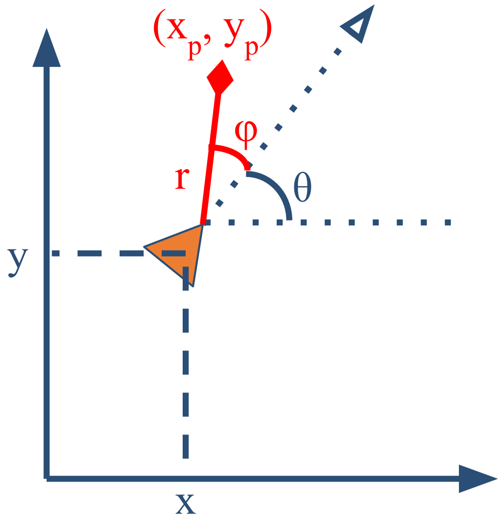
- Illuminates the scene with pulsed laser light and measures the return times and wavelengths of the reflected pulses

- Each LiDAR scan provides a collection of range measurements $\{r_1, \cdots, r_B\}$ over a set of known angles $\{\varphi_1, \cdots, \varphi_B\}$ where B is the number of rays



Point Cloud Position

- Given robot pose (x, y, θ) and a range measurement (r, φ), the location (x_p, y_p) of the observed point can be found by polar to Cartesian transformation:



$$x_p = x + r\cos(\phi + \theta)$$

$$y_p = y + r\sin(\phi + \theta)$$

Demo

1. Visit the wiki page of the workshop
2. Navigate to the Localization and Mapping page
3. Run Demo 1

Point Cloud Representation

- Pros:
  - Does not require ray-tracing
  - Light-weight computation

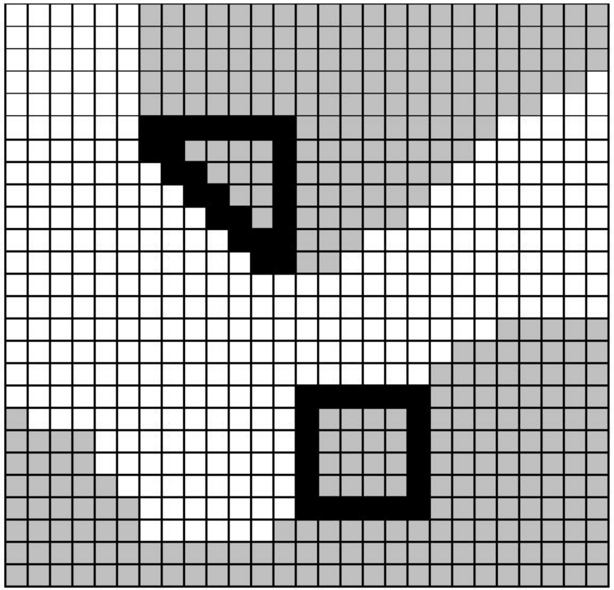
- Cons:
  - Memory inefficient: map size grows with the number of observations
  - Not suitable for collision avoidance

Occupancy Grid Mapping

- One of the simplest yet widely used map representations

- The environment is divided into a regular grid of N cells

- The value of each cell indicates its occupancy (1=occupied and 0=free)



Log-odds Mapping

- We represent the map $\mathbf{m}$ as a vector where an element m_i is a cell

- Due to the measurement noise and the error in robot location, we need to maintain a probability distribution over the map as $P(\mathbf{m}|\mathbf{x}_{1:T}, \mathbf{z}_{1:T})$

- For simplicity, we assume that map cells are independent:

$$P\left(\mathbf{m}|\mathbf{x}_{1:T}, \mathbf{z}_{1:T}\right) = \prod_{i=1}^{N} P\left(m_i|\mathbf{x}_{1:T}, \mathbf{z}_{1:T}\right)$$

- Therefore, we can compute occupancy probability for each cell m_i separately

Log-odds Mapping

- Given robot trajectory $\mathbf{x}_{1:T}$ and observations $\mathbf{z}_{1:T}$, let $\boldsymbol{\gamma}_{i,T}$ be the probability that m_i is occupied, i.e.:

$$\gamma_{i,T} = P\left(m_i = 1 | \mathbf{x}_{1:T}, \mathbf{z}_{1:T}\right)$$

$$\overset{\text{Bayes Rule}}{=} \frac{1}{\eta_T} P\left(\mathbf{z}_T | m_i = 1, \mathbf{x}_T\right) P\left(m_i = 1 | \mathbf{x}_{1:T-1}, \mathbf{z}_{1:T-1}\right) = \frac{1}{\eta_T} P\left(\mathbf{z}_T | m_i = 1, \mathbf{x}_T\right) \gamma_{i,T-1}$$

$$1 - \gamma_{i,T} = P\left(m_i = 0 | \mathbf{x}_{1:T}, \mathbf{z}_{1:T}\right)$$

$$\overset{\text{Bayes Rule}}{=} \frac{1}{\eta_T} P\left(\mathbf{z}_T | m_i = 0, \mathbf{x}_T\right) \left(1 - \gamma_{i,T-1}\right)$$

$$\Rightarrow \boxed{\frac{\gamma_{i,T}}{1 - \gamma_{i,T}} = \frac{P\left(\mathbf{z}_T | m_i = 1, \mathbf{x}_T\right)}{P\left(\mathbf{z}_T | m_i = 0, \mathbf{x}_T\right)} \frac{\gamma_{i,T-1}}{1 - \gamma_{i,T-1}}}$$

Odds Ratio

Inverse Observation Model

- Using Bayes rule once more, we have:

$$\frac{P(\mathbf{z}_T | m_i = 1, \mathbf{x}_T)}{P(\mathbf{z}_T | m_i = 0, \mathbf{x}_T)} = \underbrace{\frac{P(m_i = 1 | \mathbf{z}_T, \mathbf{x}_T)}{P(m_i = 0 | \mathbf{z}_T, \mathbf{x}_T)}}_{\text{inverse observation model}} \underbrace{\frac{P(m_i = 0)}{P(m_i = 1)}}_{\text{map prior}}$$

- The inverse observation model simply encodes our trust in the sensors:

$$\frac{P(m_i = 1 | \mathbf{z}_T, \mathbf{x}_T)}{P(m_i = 0 | \mathbf{z}_T, \mathbf{x}_T)} = \begin{cases} TP/FP & \mathbf{z}_T \text{ indicates occupied} \\ FN/TN & \mathbf{z}_T \text{ indicates free} \end{cases}$$

Inverse Observation Model

- A "good" sensor has high TP and TN, and small FP and FN

$$\frac{P(m_i = 1 | \mathbf{z}_T, \mathbf{x}_T)}{P(m_i = 0 | \mathbf{z}_T, \mathbf{x}_T)} = \begin{cases} TP/FP & \mathbf{z}_T \text{ indicates occupied} \\ FN/TN & \mathbf{z}_T \text{ indicates free} \end{cases}$$

$$\frac{P(m_i = 1 | \mathbf{z}_T, \mathbf{x}_T)}{P(m_i = 0 | \mathbf{z}_T, \mathbf{x}_T)} = \boxed{\begin{cases} 10 \\ 0.1 \end{cases}} \quad \boxed{\begin{cases} 0.4 \\ 0.4 \end{cases}} \quad \boxed{\begin{cases} 1 \\ 1 \end{cases}} \quad \boxed{\begin{cases} 10 \\ 10 \end{cases}} \quad \begin{matrix} \mathbf{z}_T \text{ indicates occupied} \\ \mathbf{z}_T \text{ indicates free} \end{matrix}$$

Good Sensor | Bad Sensor | Bad Sensor | Bad Sensor

Log-odds Mapping

- Going back to the odds ratio formula:

$$\frac{\gamma_{i,T}}{1-\gamma_{i,T}} = \frac{P(\mathbf{z}_T | m_i = 1, \mathbf{x}_T)}{P(\mathbf{z}_T | m_i = 0, \mathbf{x}_T)} \frac{\gamma_{i,T-1}}{1-\gamma_{i,T-1}} = \frac{P(m_i = 1 | \mathbf{z}_T, \mathbf{x}_T)}{P(m_i = 0 | \mathbf{z}_T, \mathbf{x}_T)} \frac{P(m_i = 0)}{P(m_i = 1)} \frac{\gamma_{i,T-1}}{1-\gamma_{i,T-1}}$$

- Taking log of both sides yields:

$$\underbrace{\log\left(\frac{\gamma_{i,T}}{1-\gamma_{i,T}}\right)}_{\lambda_{i,T}} = \underbrace{\log\left(\frac{P(m_i = 1 | \mathbf{z}_T, \mathbf{x}_T)}{P(m_i = 0 | \mathbf{z}_T, \mathbf{x}_T)}\right)}_{\Delta\lambda_{i,T}(\mathbf{z}_T)} + \underbrace{\log\left(\frac{\gamma_{i,T-1}}{1-\gamma_{i,T-1}}\right)}_{\lambda_{i,T-1}} - \underbrace{\log\left(\frac{P(m_i = 1)}{P(m_i = 0)}\right)}_{\lambda_{i,0}}$$

- Therefore, the map update can be expressed as accumulating log of odds ratios over time; hence the name log-odds mapping:

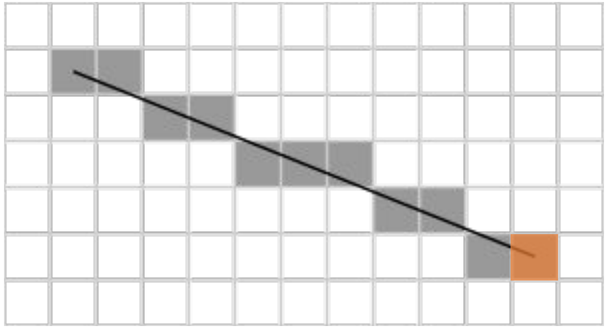
$$\boxed{\lambda_{i,T} = \Delta\lambda_{i,T}(\mathbf{z}_T) + \lambda_{i,T-1} - \lambda_{i,0}}$$

Log-odds Mapping

- In order to recover cell occupancy probability from log-odds value, we use the logistic sigmoid function:

$$P\left(m_i = 1 \mid \mathbf{x}_{1:T}, \mathbf{z}_{1:T}\right) = \gamma_{i,T} = \sigma\left(\lambda_{i,T}\right) = \frac{\exp\left(\lambda_{i,T}\right)}{1 + \exp\left(\lambda_{i,T}\right)}$$

- For occupancy grid mapping using LiDAR, we need to determine the cells visited by each ray



- This process is called line rasterization

Log-odds Mapping (Summary)

- Log-odds occupancy grid mapping is equivalent to maintaining a grid of log-odds values $\lambda_{i,T}$ for each cell

- There is a one-to-one relationship between log-odds value $\lambda_{i,T}$ and occupancy probability $\boldsymbol{\gamma}_{i,T}$, namely the logistic sigmoid function

- Upon each LiDAR measurement $\mathbf{z}_T$, follow the below steps:
  - For each ray, find the set of visited cells using line rasterization
  - Use the log-odds formula in order to update the map:
    - If cell i is observed free:

$$\lambda_{i,T} = \Delta\lambda_{i,T}(free) + \lambda_{i,T-1} - \lambda_{i,0}$$

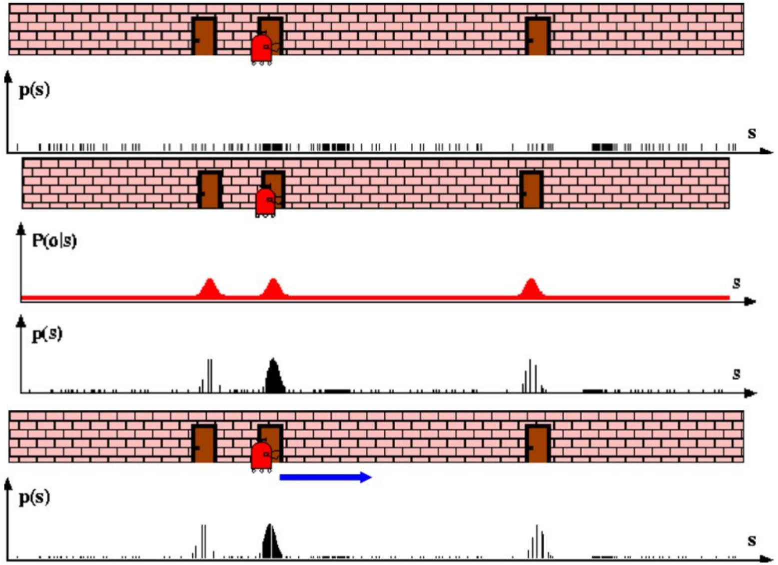
    - If cell i is observed occupied:

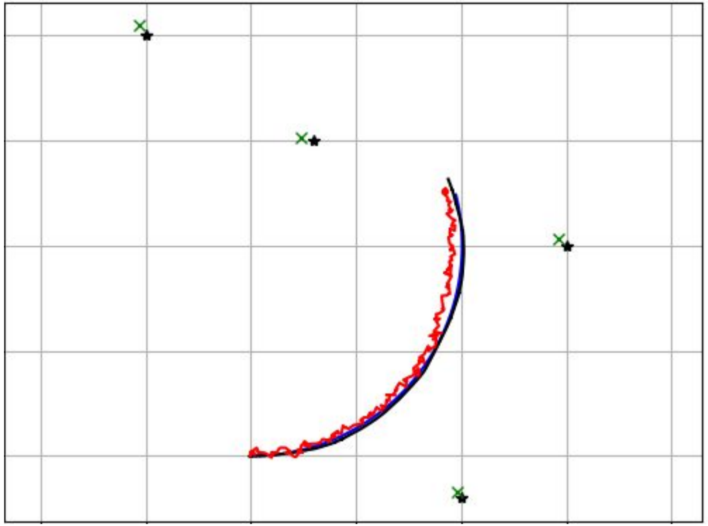
$$\lambda_{i,T} = \Delta\lambda_{i,T}(occupied) + \lambda_{i,T-1} - \lambda_{i,0}$$

Demo

1. Visit the wiki page of the workshop
2. Navigate to the Localization and Mapping page
3. Run Demo 2

Localization



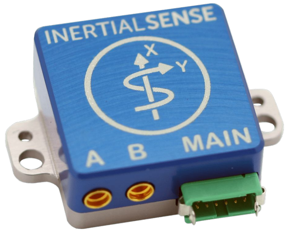


Robot Localization

- Odometry: The use of motion sensor data to estimate change of robot pose

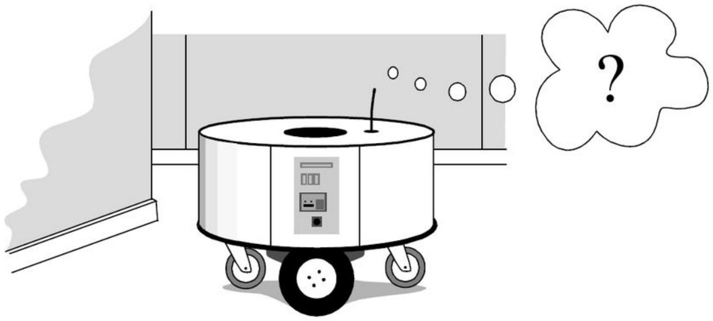
- Motion sensors usually have small measurement noise:
  - Accurate between t and t+1
  - Error accumulates during long time horizon

- Therefore, simply relying on odometry leads to large error in robot pose estimation



IMU

Wheel Encoder

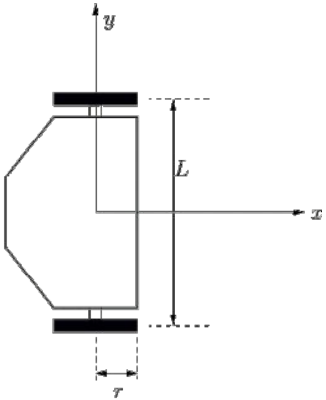


Differential-drive Model

- Consider the differential-drive motion model

- The velocity of left and right wheels control the motion of the robot:
  - Sum of wheels velocities: Robot linear velocity
  - Difference of wheels velocity: Robot angular velocity

- Therefore, the control inputs are linear velocity v and angular velocity ω



$$x_{t+1} = x_t + \tau\, v_t \cos(\theta_t)$$
$$y_{t+1} = y_t + \tau\, v_t \sin(\theta_t)$$
$$\theta_{t+1} = \theta_t + \tau\, \omega_t$$

Demo

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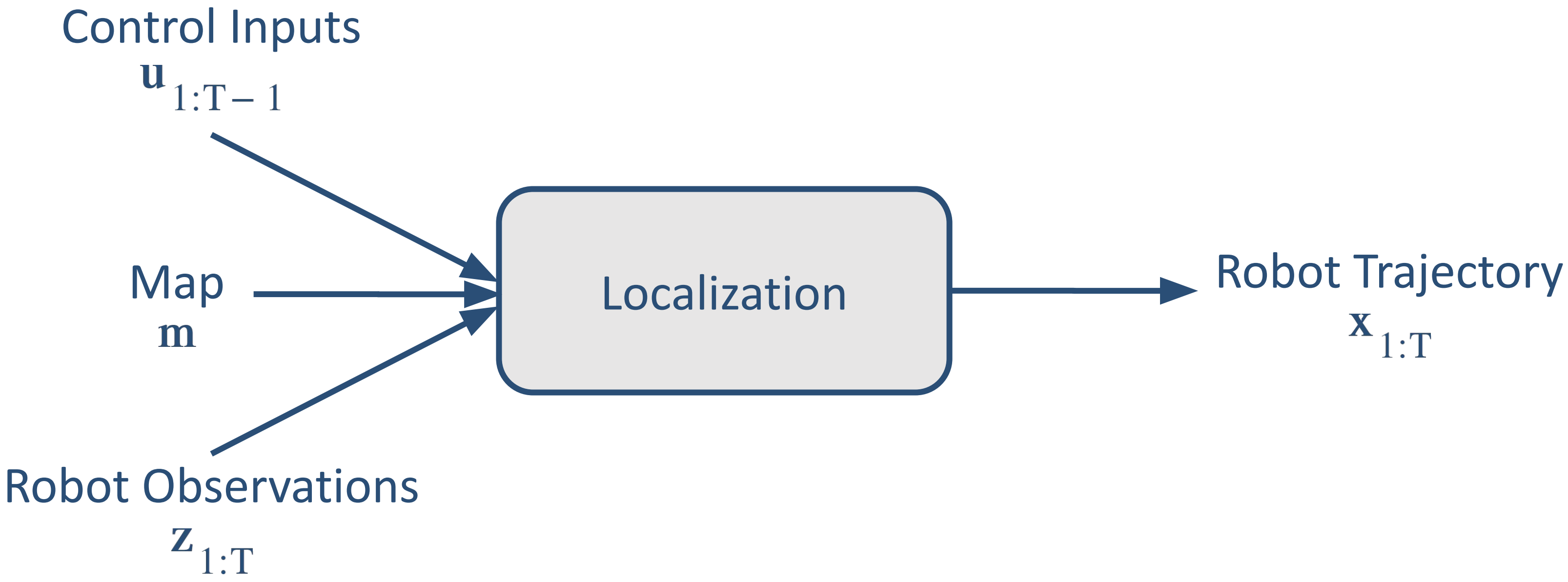
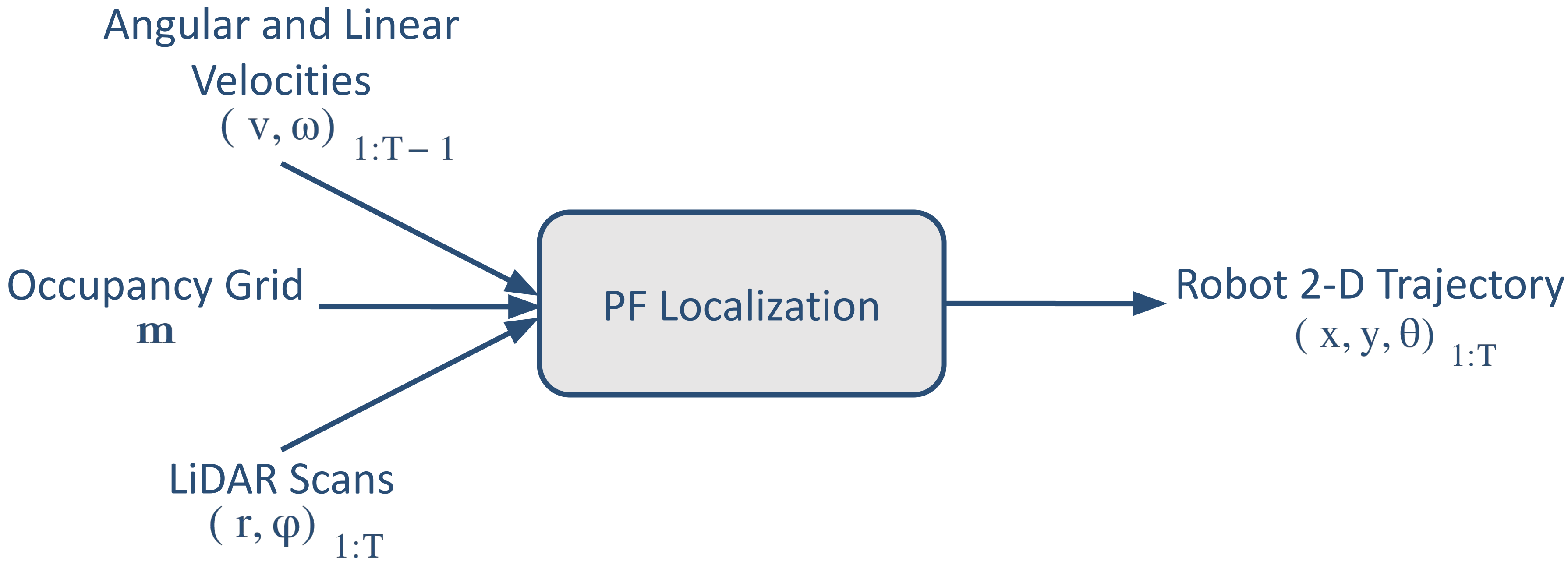
2. Navigate to the Localization and Mapping page
3. Run Demo 3

Problem Formulation



Problem Formulation



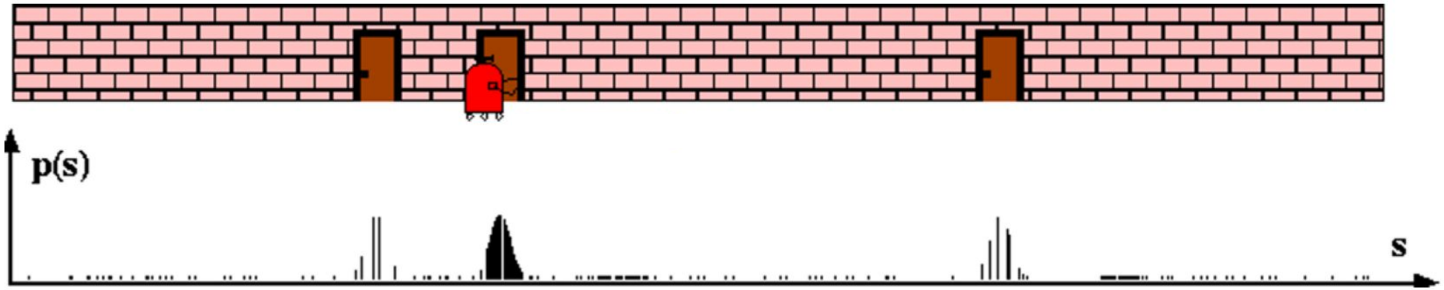
Particle Filtering

- We maintain a probability distribution over robot pose via a set of hypotheses (particles) with pose $\{\boldsymbol{\mu}^{(k)}\}_k$ and their respective weight $\{\alpha^{(k)}\}_k$:

$$P\left(\mathbf{x}_T \middle| \mathbf{z}_{1:T}, \mathbf{u}_{1:T-1}\right) = \begin{cases} \alpha^{(k)} & \mathbf{x}_T = \boldsymbol{\mu}^{(k)} \\ 0 & \text{otherwise} \end{cases}$$

- The weights $\alpha^{(k)}$ represent the probability of each particle, hence:

$$\sum_{k=1}^{N} \alpha^{(k)} = 1$$



Prediction Step

- Given a particle set $\{(\boldsymbol{\mu}_{T|T}^{(k)}, \alpha_{T|T}^{(k)})\}_k$ at time T, we are interested to obtain the predicted particle set $\{(\boldsymbol{\mu}_{T+1|T}^{(k)}, \alpha_{T+1|T}^{(k)})\}_k$ after applying the control $\mathbf{u}_T$
- We can simply plug each $\boldsymbol{\mu}_{T|T}^{(k)}$ to the motion model
- However, we also want to model the motion sensor noise
- In other words, we want our particles be diverse, since each of them represent a hypothesis over robot pose
- Therefore, we should add a small noise to our motion model when we apply the motion model to each particle

Prediction Step

- For a differential-drive motion model, each particle represents a 2-D robot pose:

$$\boldsymbol{\mu}_{T|T}^{(k)} = \left(\mu_{x,\,T|T}^{(k)},\ \mu_{y,\,T|T}^{(k)},\ \mu_{\theta,\,T|T}^{(k)}\right)$$

- Prediction step:

$$\begin{aligned}
\mu_{x,\,T+1|T}^{(k)} &= \mu_{x,\,T|T}^{(k)} + \tau\left(v_T + n_v^{(k)}\right)\cos\left(\mu_{\theta,\,T|T}^{(k)}\right)\\
\mu_{y,\,T+1|T}^{(k)} &= \mu_{y,\,T|T}^{(k)} + \tau\left(v_T + n_v^{(k)}\right)\sin\left(\mu_{\theta,\,T|T}^{(k)}\right)\\
\mu_{\theta,\,T+1|T}^{(k)} &= \mu_{\theta,\,T|T}^{(k)} + \tau\left(\omega_T + n_\omega^{(k)}\right)
\end{aligned}$$

- Note that the particle weights $\{\alpha^{(k)}\}_k$ do not change during the prediction step

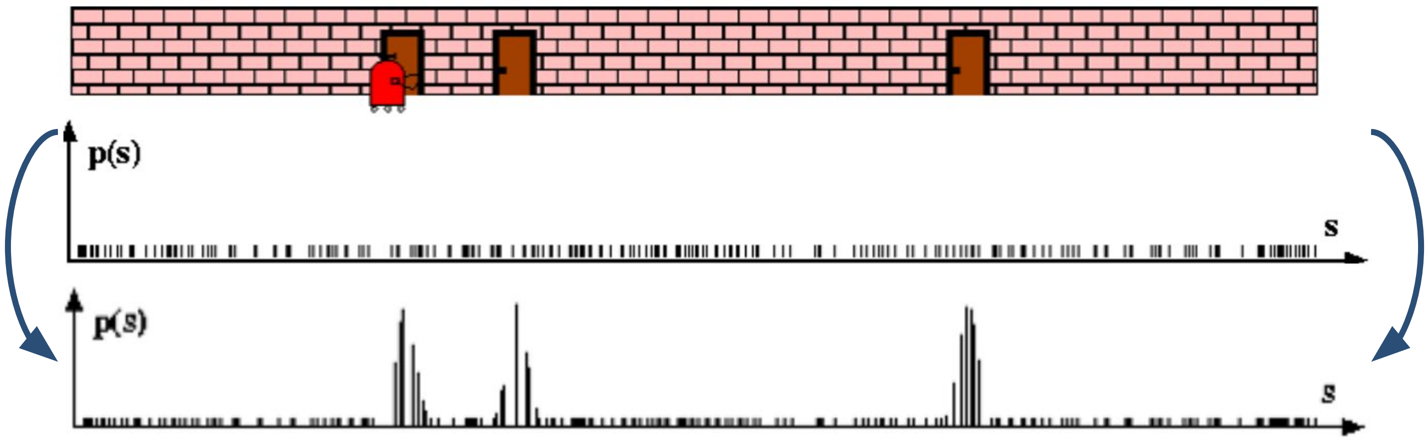
- We denote the predicted particle set as $\{(\boldsymbol{\mu}_{T+1|T}{}^{(k)}, \alpha_{T+1|T}{}^{(k)})\}_k$

Update Step

- Given a predicted particle set $\{(\boldsymbol{\mu}_{T+1|T}{}^{(k)}, \alpha_{T+1|T}{}^{(k)})\}_k$, we want to update the weight of each particle according to the sensor observation $\mathbf{z}_{T+1}$

- Here, we are interested to know if a hypothesis $\boldsymbol{\mu}_{T+1|T}{}^{(k)}$ agrees with an observation $\mathbf{z}_{T+1}$

- If it agrees (disagrees), increase (decrease) the particle's weight $\alpha_{T+1|T}{}^{(k)}$



Update Step

- In order to incorporate a new observation $\mathbf{z}_{T+1}$ to the particle set, we need to use the Bayes rule:

$$
\begin{aligned}
& P\left(\mathbf{x}_{T+1}=\boldsymbol{\mu}_{T+1|T}^{(k)} \middle| \mathbf{z}_{1:T+1}, \mathbf{u}_{1:T}\right)=\alpha_{T+1|T+1}^{(k)} \\
& \overset{\text{Bayes Rule}}{=} \frac{1}{\eta_{T+1}} P\left(\mathbf{z}_{T+1} \middle| \mathbf{x}_{T+1}=\boldsymbol{\mu}_{T+1|T}^{(k)}\right) P\left(\mathbf{x}_{T+1}=\boldsymbol{\mu}_{T+1|T}^{(k)} \middle| \mathbf{z}_{1:T}, \mathbf{u}_{1:T}\right) \\
& =\frac{1}{\eta_{T+1}} \underbrace{P\left(\mathbf{z}_{T+1} \middle| \mathbf{x}_{T+1}=\boldsymbol{\mu}_{T+1|T}^{(k)}\right)}_{\text{forward observation model}} \alpha_{T+1|T}^{(k)}
\end{aligned}
$$

- We still need expressions for the forward observation model and the denominator η_{T+1}

Laser Correlation Model

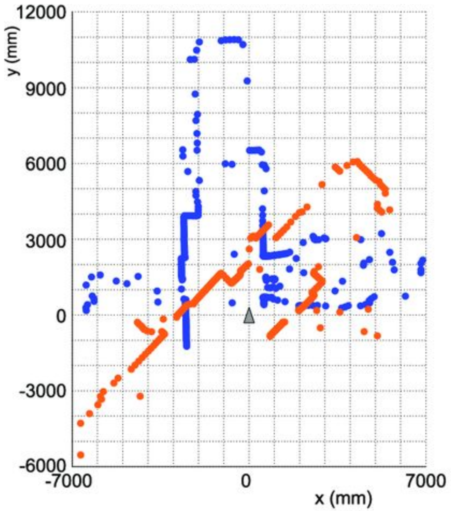
- We define the forward observation model as a function of the correlation between map $\mathbf{m}$ and LiDAR scan $\mathbf{z}$ obtained from robot pose $\mathbf{x}$:

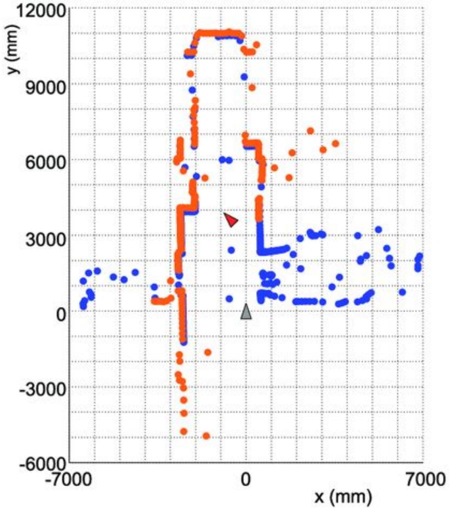
$$P(\mathbf{z}|\mathbf{x}) \propto \exp(\mathrm{corr}(\mathrm{r}(\mathbf{z}, \mathbf{x}), \mathbf{m}))$$

  where $\mathrm{r}(\mathbf{z}, \mathbf{x})$ is the LiDAR point cloud from pose $\mathbf{x}$ and:

$$\mathrm{corr}(\mathbf{r}, \mathbf{m}) = \sum_{i} \mathbb{1}\left\{\mathrm{r}_i = \mathrm{m}_i\right\}$$

- The correlation is large if $\mathrm{r}(\mathbf{z}, \mathbf{x})$ and the map $\mathbf{m}$ agree





Update Step

- Going back to the weight update:

$$\alpha^{(k)}_{T+1|T+1} = \frac{1}{\eta_{T+1}} \exp\left(\mathrm{corr}\left(\mathrm{r}\left(\mathbf{z}_{T+1}, \boldsymbol{\mu}^{(k)}_{T+1|T}\right), \mathbf{m}\right)\right) \alpha^{(k)}_{T+1|T}$$

- Furthermore, the denominator η_{T+1} is constant among all particles

- Therefore, we can eliminate it by normalization:

$$\text{for } k = 1 \ldots N:$$

$$\alpha^{(k)}_{T+1|T+1} \leftarrow \exp\left(\mathrm{corr}\left(\mathrm{r}\left(\mathbf{z}_{T+1}, \boldsymbol{\mu}^{(k)}_{T+1|T}\right), \mathbf{m}\right)\right) \alpha^{(k)}_{T+1|T}$$

$$\eta_{T+1} = \sum_{k=1}^{N} \alpha^{(k)}_{T+1|T+1}$$

$$\text{for } k = 1 \ldots N:$$

$$\alpha^{(k)}_{T+1|T+1} \leftarrow \frac{\alpha^{(k)}_{T+1|T+1}}{\eta_{T+1}}$$

Resampling

- Particle depletion: a situation in which most of the particle weights become close to zero

- This means that only a minority of the particles can be good candidates for the true robot pose while the rest are irrelevant

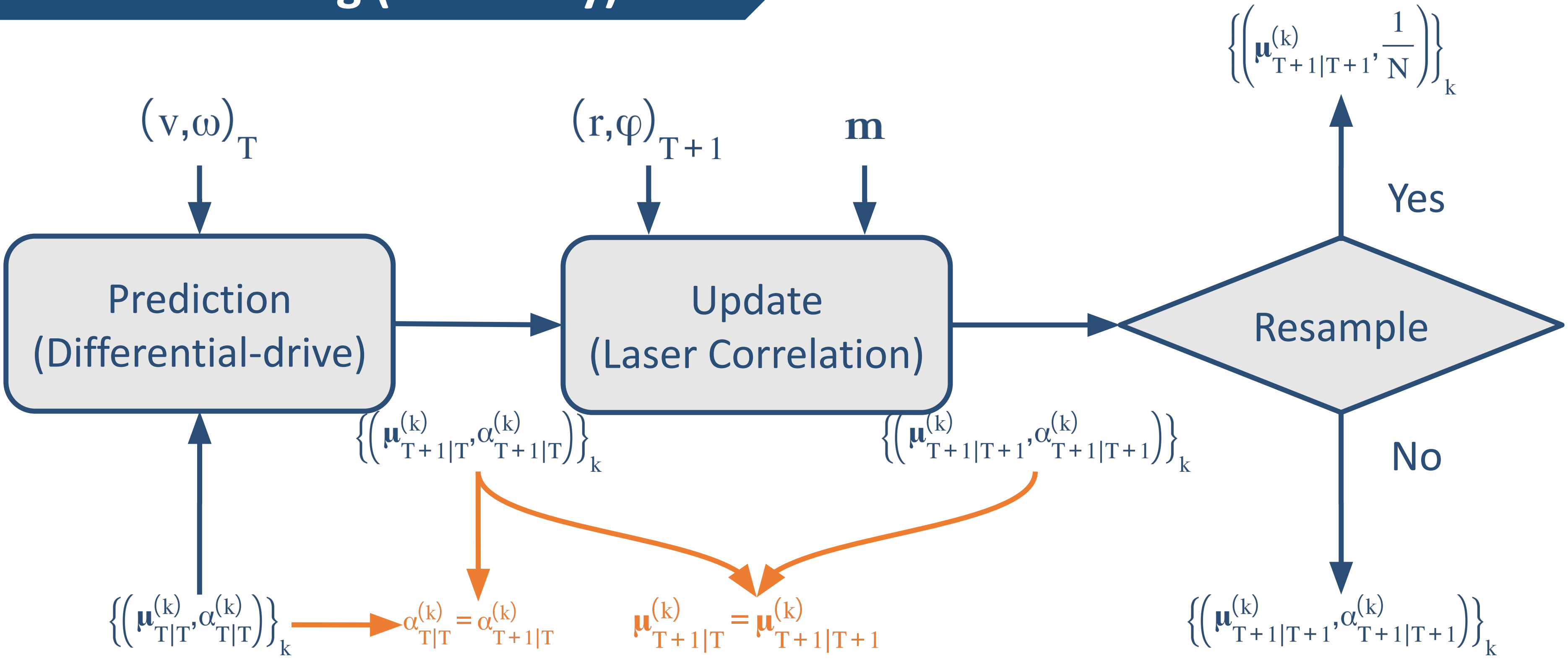
- Resampling is a procedure to avoid particle depletion

- Given a particle set $\{(\boldsymbol{\mu}_{T|T}^{(k)}, \alpha_{T|T}^{(k)})\}_k$, create a new particle set with equal weights $\alpha_{T|T}^{(k)} = 1/N$ by adding many particles to the locations with high weights and few particles to the locations with small weights

- Resampling can be triggered whenever the effective number of particles N_{eff} is less than a threshold:

$$N_{eff} = \frac{1}{\sum_k \left(\alpha_{T|T}^{(k)}\right)^2} < N_{thr}$$

Particle Filtering (Summary)



Demo

1. Visit the wiki page of the workshop
2. Navigate to the Localization and Mapping page
3. Run Demo 4

Thank you!

Scan to visit our website!

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