Presentation Documents

-

Selection of Heterogeneous Instruments in Partially Linear Fixed Effects Panel Regression

or the clickbait version:

Can You Select Heterogeneous Instruments in a Partially Linear Fixed Effects Panel Regression? A Machine Learning Approach

By: Eric Penner

September 19, 2018

Introduction pt.1

Basic Elements

- Linear in parameters fixed effects panel model
- Subset of regressors are endogenous
- Relevant instruments vary by cross section
- Relevant instruments are unknown subset of larger collection

Introduction pt.2

Applications / Motivation

- Spatial distance separating cross sections
- Lack of sufficient domain/out of sample information
- Growth and Foreign Aid Models: Burnside and Dollar (2000) AER
- Sensitivity to Set of Instruments: Leon-Gonzalez et. al. (2015) JoMa
- Explicitly incorporate model selection and battling overfitting
- Replacing ad hoc procedures

Initial Model

$$Y_{jt} = \beta_0 + Z'_{1jt}\beta_1 + Z'_{2jt}\beta_2 + e_j + \varepsilon_{jt} \tag{1}$$

$$Z_{1jdt} = \alpha_{jd0} + Z'_{2jt}\alpha_{1d} + W'_{jdt}\alpha_{2jd} + V_{jdt} \tag{2d}$$

$$E(\varepsilon_{jt}|Z_{1jt}, Z_{2jt}) = E(\varepsilon_{jt}|Z_{1jt}) \neq 0 \tag{3}$$

$$E(V_{jdt}|Z_{2jt}, W_{jt}) = 0 \tag{4}$$

- Y_{jt} is a scalar random variable
- $Z_{1jt} \in \mathbb{R}^{p_1}$, $Z_{2jt} \in \mathbb{R}^{p_2}$ are endogenous and exogenous variables respectively
- e_j is a fixed effect, α_{jd0} is a combined constant and fixed effect
- $W_t \in \mathbb{R}^w$ a vector of instrumental variables.
- $W_{jdt} \in \mathbb{R}^{w_{jd}}$ unknown subvector of $W_t \in \mathbb{R}^w$.

Primary Assumptions

i.) Exclusion Restriction: $E(\varepsilon_{jt}|Z_{1jt}) = E(\varepsilon_{jt}|Z_{2jt}, W_t, V_{jt}) = E(\varepsilon_{jt}|V_{jt})$

ii.) Control Function: $\varepsilon_{jt} = f_j(V_{jt}) + u_{jt}$

iii.) Orthogonality: $E[u_{jt}|Z_{1jt}, Z_{2jt}, W_t] = 0$

iv.) Additivity: $f_j(V_{jt}) = \sum_d f_{jd}(V_{jdt})$

v.) Panel Secondary Equation: $W'_{jdt}\alpha_{2jd} = W'_t M_{jd}\alpha_{2d}$ where $M_{jd} = diag(m_{jd})$ and

$$m_{jd} = \begin{bmatrix} 1\{W_{1t} \in W_{jdt}\} & 1\{W_{2t} \in W_{jdt}\} & \cdots & 1\{W_{wt} \in W_{jdt}\} \end{bmatrix}'$$

First Differenced Model

Let Δ be the first difference in operator, and assume $\{Y_{jt}, Z_{1jt}, Z_{2jt}, W_t\}_{t=1}^{n}$ is i.i.d

$$\Delta Y_{jt} = \Delta Z_{1jt}'\beta_1 + \Delta Z_{2jt}'\beta_2 + \sum_{d=1}^{p_1} \Delta f_{jd}(V_{jdt}) + \Delta u_{jt} \tag{5}$$

$$Z_{1jdt} = \alpha_{jd0} + Z_{2jt}'\alpha_{1d} + W_t' M_{jd}\alpha_{2d} + V_{jdt} \tag{2d}$$

$$E(\Delta u_{jt} | V_{jt}, V_{j,t-1}) = 0 \tag{7}$$

$$E(V_{jdt} | Z_{2jt}, W_{jt}) = 0 \tag{8}$$

Residuals on residuals regression: Manzan and Zerom (2005) SaPL

Projections and Notation

Density Ratio

$$\phi_{jt} = \frac{\prod_{d=1}^{p_1} p(V_{jdt}, V_{jd(t-1)})}{p(V_{jt}, V_{j(t-1)})}$$

Conditional Expectations, for $k \in \{1, 2\}$

$$H_{jd}(\Delta Z_{kjt}) = E[\phi_{jt} \Delta Z_{kjt} | V_{jdt}, V_{jd(t-1)}] \qquad H_{jd}(\Delta Y_{jt}) = E[\phi_{jt} \Delta Y_{jt} | V_{jdt}, V_{jd(t-1)}]$$

$$H_j(\Delta Z_{kjt}) = \sum_{d=1}^{p_1} H_{jd}(\Delta Z_{kjt}) \qquad H_j(\Delta Y_{jt}) = \sum_{d=1}^{p_1} H_{jd}(\Delta Y_{jt})$$

Vectors

$$H(\Delta Y_t) = [\ H_1(\Delta Y_{1t}) \ \ H_2(\Delta Y_{2t}) \ \ \cdots \ \ H_q(\Delta Y_{qt})\ ]'$$

$$H(\Delta Z_{kt}) = [\ H_1(\Delta Z_{k1t}) \ \ H_2(\Delta Z_{k2t}) \ \ \cdots \ \ H_q(\Delta Z_{kqt})\ ]'$$

$$\Delta Y_t = [\ \Delta Y_{1t} \ \ \Delta Y_{2t} \ \ \cdots \ \ \Delta Y_{qt}\ ]'$$

$$\Delta Z_{kt} = [\ \Delta Z_{k1t} \ \ \Delta Z_{k2t} \ \ \cdots \ \ \Delta Z_{kqt}\ ]'$$

Identification Lemma

Lemma: Identification

Let $\phi_t = diag\left(\{\phi_{jt}\}_{j=1}^{q}\right)$, and

$$\Delta Z_t = \left[\Delta Z_{1t}' \quad \Delta Z_{2t}'\right]' \qquad H(\Delta Z_t) = \left[H(\Delta Z_{1t})' \quad H(\Delta Z_{2t})'\right]'$$

Then if,

i.) for all $d \in \{1, 2, \ldots, p_1\}$, $j \in \{1, 2, \cdots, q\}$, and $t \in \{2, \ldots, T\}$

$$E\left[f_{jd}(V_{jdt})\right] = E\left[f_{jd}(V_{jd(t-1)})\right]$$

ii.) $E\Big([\Delta Z_t - H(\Delta Z_t)]'\phi_t[\Delta Z_t - H(\Delta Z_t)]\Big)$ is positive semi definite,

Then β_1 and β_2 are identified, in particular,

$$\begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = E\left([\Delta Z_t - H(\Delta Z_t)]'\phi_t[\Delta Z_t - H(\Delta Z_t)]\right)^{-1} E\left([\Delta Z_t - H(\Delta Z_t)]'\phi_t[\Delta Y_t - H(\Delta Y_t)]\right)$$

Secondary Equation Estimation pt.1

Must Estimate $\hat{V}_{jdt}$ as a first step

- $W_{jdt} \in \mathbb{R}^{w_{jd}}$ unknown subvector of $W_t \in \mathbb{R}^{w}$.
- Subset selection and coefficient estimation with Lasso

Why Subset / Model Selection?

- Overfitting $E(Z_{1jdt}|Z_{2jt}, W_t)$
- If $T << w$

Lasso: Is the treatment worse than the original problem?

- Underfitting $E(Z_{1jdt}|Z_{2jt}, W_t)$
- Overfitting on the sample

Sample Splitting

- Similar to Chernozhukov et al. (2017) NBER working paper 23564
- Let $n_p \in \mathbb{N}$ be the number of partitions to be generated
- Generate Partition $\{\mathcal{I}_g\}_{g=1}^{n_p}$ of $\{1, 2, \cdots, T\}$ i.e.

$$\mathcal{I}_g \subset \{i\}_{i=1}^{T} \qquad \mathcal{I}_g \cap \mathcal{I}_{g'} = \emptyset \quad \text{and} \quad \bigcup_{g=1}^{n_p} \mathcal{I}_g = \{i\}_{i=1}^{T}$$

- Model Selection and Projection Estimation with Training Set: $\mathcal{I}_g^c$
- Estimation of $[\ \hat{\beta}_{g1}' \ \ \hat{\beta}_{g2}'\ ]'$ with Testing Set: $\mathcal{I}_g$
- Final Estimator:

$$[\ \hat{\beta}_1' \ \ \hat{\beta}_2'\ ]' = n_p^{-1} \sum_{g=1}^{n_p} [\ \hat{\beta}_{g1}' \ \ \hat{\beta}_{g2}'\ ]'$$

Secondary Equation Estimation pt.2

First Differenced Model

$$\Delta Z_{1jdt} = \Delta Z'_{2jt}\alpha_{1d} + \Delta W'_t M_{jd}\alpha_{2d} + \Delta V_{jdt}$$

Lasso Estimator with Training Set $\mathcal{I}_g^c$

$$(\hat{\alpha}_{1d}, \hat{\alpha}_{2jd}) = \arg\min \sum_{j=1}^{q} \sum_{i \in \mathcal{I}_g^c} \left(\Delta Z_{1jdi} - \Delta Z'_{2ji}\alpha_1 - \Delta W'_i \alpha_{2j}\right)^2 \quad \text{s.t.} \quad \sum_{l=1}^{w} |\alpha_{2j,l}| \leq \lambda$$

for all $1 \leq j \leq q$.

Secondary Equation Estimation pt.3

Next define for some $\varepsilon > 0$

$$\tilde{\alpha}_{2d,l} = \frac{\sum_{j=1}^{q} \hat{\alpha}_{2jd,l} 1\{|\hat{\alpha}_{2jd,l}| > \varepsilon\}}{\sum_{j=1}^{q} 1\{|\hat{\alpha}_{2jd,l}| > \varepsilon\} + 1\{\sum_{j=1}^{q} 1\{|\hat{\alpha}_{2jd,l}| > \varepsilon\} = 0\}}$$

Now let

$$\tilde{\alpha}_{2d} = [\ \tilde{\alpha}_{2d,1} \quad \tilde{\alpha}_{2d,2} \quad \cdots \quad \tilde{\alpha}_{2d,w}\ ]'$$

and

$$\tilde{m}_{jd} = \left[1\{|\hat{\alpha}_{2jd,1}| > \varepsilon\} \quad 1\{|\hat{\alpha}_{2jd,2}| > \varepsilon\} \quad \cdots \quad 1\{|\hat{\alpha}_{2jd,w}| > \varepsilon\}\right]'$$

so that, for all $t \in \{1, 2, \cdots, T\}$

$$\hat{V}_{jdt} = Z_{1jdt} - Z_{2jt}'\hat{\alpha}_{1d} - W_t'\tilde{M}_{jd}\tilde{\alpha}_{2d} - \#(\mathcal{I}_g^c)^{-1} \sum_{i \in \mathcal{I}_g^c} (Z_{1jdi} - Z_{2ji}'\hat{\alpha}_{1d} - W_i'\tilde{M}_{jd}\tilde{\alpha}_{2d})$$

where $\tilde{M}_{jd} = diag(\tilde{m}_{jd})$

Density Ratio Estimation

Density Estimation: for all $j \in \{1, 2, \cdots, q\}$, $d \in \{1, 2, \cdots, p_1\}$, and $t \in \{2, \cdots, T\}$

$$\hat{p}(\hat{V}_{jdt}, \hat{V}_{jd(t-1)}) = (n(t)h_1^2)^{-1} \sum_{i \in \mathcal{I}_g^c, i \neq t} k\left(\frac{\hat{V}_{jdi} - \hat{V}_{jdt}}{h_1}\right) k\left(\frac{\hat{V}_{jd(i-1)} - \hat{V}_{jd(t-1)}}{h_2}\right)$$

$$\hat{p}(\hat{V}_{jt}, \hat{V}_{j(t-1)}) = (n(t)h_2^{2p_1})^{-1} \sum_{i \in \mathcal{I}_g^c, i \neq t} \prod_{d=1}^{p_1} k\left(\frac{\hat{V}_{jdi} - \hat{V}_{jdt}}{h_2}\right) k\left(\frac{\hat{V}_{jd(i-1)} - \hat{V}_{jd(t-1)}}{h_2}\right)$$

where

$$n(t) = \begin{cases} \#(\mathcal{I}_g^c) & \text{if } t \in \mathcal{I}_g \\ \#(\mathcal{I}_g^c) - 1 & \text{if } t \in \mathcal{I}_g^c \end{cases}$$

Density Ratio Construction:

$$\hat{\phi}_{jt} = \frac{\prod_{d=1}^{p_1} \hat{p}(\hat{V}_{jt,d}, \hat{V}_{j(t-1),d})}{\hat{p}(\hat{V}_{jt}, \hat{V}_{j(t-1)})} \qquad \hat{\theta}_{jdt} = \frac{\prod_{l \neq d}^{p_1} \hat{p}(\hat{V}_{jlt}, \hat{V}_{jl(t-1)})}{\hat{p}(\hat{V}_{jt}, \hat{V}_{j(t-1)})}$$

H Function Estimation

H Function Estimation: for each $a \in \{1, 2\}$, $j \in \{1, 2, \cdots, q\}$, $d \in \{1, 2, \cdots, p_1\}$, $\ell \in \{1, 2, \cdots, p_a\}$, and $k \in \mathcal{I}_g$

$$\hat{H}_{jd}(\Delta Z_{aj\ell k}) = [n_{\mathcal{I}_g^c} h_3]^{-1} \sum_{i \in \mathcal{I}_g^c} k\left(\frac{\hat{V}_{jdi} - \hat{V}_{jdk}}{h_3}\right) k\left(\frac{\hat{V}_{jd(i-1)} - \hat{V}_{jd(k-1)}}{h_3}\right) \hat{\theta}_{jdi} \Delta Z_{aj\ell i}$$

and for each $j \in \{1, 2, \cdots, q\}$, $d \in \{1, 2, \cdots, p_1\}$, and $k \in \mathcal{I}_g$

$$\hat{H}_{jd}(\Delta Y_{jk}) = [n_{\mathcal{I}_g^c} h_3]^{-1} \sum_{i \in \mathcal{I}_g^c} k\left(\frac{\hat{V}_{jdi} - \hat{V}_{jdk}}{h_3}\right) k\left(\frac{\hat{V}_{jd(i-1)} - \hat{V}_{jd(k-1)}}{h_3}\right) \hat{\theta}_{jdi} \Delta Y_{ji}$$

then construct

$$\hat{H}_j(\Delta Z_{aj\ell k}) = \sum_{d=1}^{p_1} \hat{H}_{jd}(\Delta Z_{aj\ell k}) \qquad \hat{H}_j(\Delta Y_{jk}) = \sum_{d=1}^{p_1} \hat{H}_{jd}(\Delta Y_{jk})$$

the rest is book keeping.

Final Estimator for Testing Set $\mathcal{I}_g$

Let $\hat{\phi} = diag\left(\{\hat{\phi}_k\}_{k\in\mathcal{I}_g}\right)$

$$\begin{bmatrix}\hat{\beta}_{g1}\\ \hat{\beta}_{g2}\end{bmatrix} = \left([\Delta Z - \hat{H}(\Delta Z)]'\hat{\phi}[\Delta Z - \hat{H}(\Delta Z)]\right)^{-1}\left([\Delta Z - \hat{H}(\Delta Z)]'\hat{\phi}[\Delta Y - \hat{H}(\Delta Y)]\right)$$

Comparing the sampling distribution of $\hat{\beta}_1$ and $\hat{\beta}_2$.

- Oracle v.s. Known Subset v.s. Unknown Subset v.s. Lasso
- Varying total number of instruments available w
- Varying number of cross sections q
- Varying number of time periods T

Equivalences and Covariances

Let

- $n_{tp} \equiv T$ be the total number of time periods.
- $n_{end} \equiv p_1$ be the number of endogneous regressors
- $n_{exo} \equiv p_2$ be the number of exogenous regressors
- $n_{tinst} \equiv w$ be the total number of available instruments
- $n_{cinst} \equiv w_j$ the number of instruments relevant to each crossection

$$\rho_{er} = \begin{bmatrix} \rho_{er,1} & \rho_{er,2} & \cdots & \rho_{er,n_{end}} \end{bmatrix}$$

$$\rho_{inst} = \begin{bmatrix} \rho_{inst,1} & \rho_{inst,2} & \cdots & \rho_{inst,n_{inst}-1} \end{bmatrix}$$

$$\rho_{ex} = \begin{bmatrix} \rho_{ex,1} & \rho_{ex,2} & \cdots & \rho_{ex,n_{ex}-1} \end{bmatrix}$$

be vectors of covariances.

Error Covariance Matrix

For each cross section

$$V_{er} = \begin{bmatrix} 1 & \rho_{er,1} & \rho_{er,2} & \cdots & \rho_{er,n_{end}} \\ \rho_{er,1} & 1 & \rho_{er,1} & \cdots & \rho_{er,n_{end}-1} \\ \rho_{er,2} & \rho_{er,1} & 1 & \cdots & \rho_{er,n_{end}-2} \\ \vdots & & & \ddots & \\ \rho_{er,n_{end}} & \rho_{er,n_{end}-1} & \rho_{er,n_{end}-2} & \cdots & 1 \end{bmatrix}$$

For all cross sections

$$CV_{er} = \begin{bmatrix} V_{er} & \mathbf{0}_{(n_{end}+1 \times n_{end}+1)} & \cdots & \mathbf{0}_{(n_{end}+1 \times n_{end}+1)} \\ \mathbf{0}_{(n_{end}+1 \times n_{end}+1)} & V_{er} & \cdots & \mathbf{0}_{(n_{end}+1 \times n_{end}+1)} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0}_{(n_{end}+1 \times n_{end}+1)} & \mathbf{0}_{(n_{end}+1 \times n_{end}+1)} & \cdots & V_{er} \end{bmatrix}$$

Exogenous Variable Covariance Matrix

For each cross section

$$V_{ex} = \begin{bmatrix} 1 & \rho_{ex,1} & \rho_{ex,2} & \cdots & \rho_{ex,n_{ex}-1} \\ \rho_{ex,1} & 1 & \rho_{ex,1} & \cdots & \rho_{ex,n_{ex}-2} \\ \rho_{ex,2} & \rho_{ex,1} & 1 & \cdots & \rho_{ex,n_{ex}-3} \\ \vdots & & & \ddots & \\ \rho_{ex,n_{ex}-1} & \rho_{ex,n_{ex}-2} & \rho_{ex,n_{ex}-2} & \cdots & 1 \end{bmatrix}$$

For all cross sections

$$CV_{ex} = \begin{bmatrix} V_{ex} & \mathbf{0}_{(n_{ex} \times n_{ex})} & \cdots & \mathbf{0}_{(n_{ex} \times n_{ex})} \\ \mathbf{0}_{(n_{ex} \times n_{ex})} & V_{ex} & \cdots & \mathbf{0}_{(n_{ex} \times n_{ex})} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0}_{(n_{ex} \times n_{ex})} & \mathbf{0}_{(n_{ex} \times n_{ex})} & \cdots & V_{ex} \end{bmatrix}$$

Common Instrument Covariance Matrix

$$V_{inst} = \begin{bmatrix} 1 & \rho_{inst,1} & \rho_{inst,2} & \cdots & \rho_{inst,n_{tinst}-1} \\ \rho_{inst,1} & 1 & \rho_{inst,1} & \cdots & \rho_{inst,n_{tinst}-2} \\ \rho_{inst,2} & \rho_{tinst,1} & 1 & \cdots & \rho_{inst,n_{tinst}-3} \\ \vdots & & & \ddots & \\ \rho_{inst,n_{tinst}-1} & \rho_{inst,n_{tinst}-2} & \rho_{inst,n_{tinst}-3} & \cdots & 1 \end{bmatrix}$$

Exogenous Variable Generation

Let

$$Z_{2jt} = \begin{bmatrix} Z_{2jt,1} & Z_{2jt,2} & \cdots & Z_{2jt,n_{ex}} \end{bmatrix}'$$

$$\tilde{V}_{jt} = \begin{bmatrix} \varepsilon_j & V_{j1t} & V_{j2t} & \cdots & V_{jn_{end}t} \end{bmatrix}'$$

Then generate $\{Z_{1t}, Z_{2t}, W_t\}_{t=1}^{ntp}$ from the following distributions

$$\begin{bmatrix} W_{1t} & W_{2t} & \cdots & W_{t,n_{inst}} \end{bmatrix} \sim N(\mathbf{0}_{n_{inst}\times 1}, CV_{inst})$$

$$\begin{bmatrix} Z'_{21t} & Z'_{22t} & \cdots & Z'_{2n_{cs}t} \end{bmatrix}' \sim N(\mathbf{0}_{n_{cs}\cdot n_{exo}\times 1}, CV_{ex})$$

$$\begin{bmatrix} \tilde{V}'_{1t} & \tilde{V}'_{2t} & \cdots & \tilde{V}'_{n_{cs}t} \end{bmatrix}' \sim N(\mathbf{0}_{n_{cs}\cdot(n_{end}+1)\times 1}, CV_{er})$$

Endogenous Variable Generation

- Randomly Draw $\alpha_{1d} \in [1, -1]^{n_{exo}}$ for each $d \in \{1, 2, \cdots, n_{end}\}$
- Randomly Draw $\alpha_{2d} \in [1, -1]^{n_{tinst}}$ for each $d \in \{1, 2, \cdots, n_{end}\}$
- Randomly draw a set of integers from $C_{n_{cinst}}^{n_{tinst}}$ ways that that you can choose n_{cinst} instruments from n_{tinst} total instrument, for each $j \in \{1, 2, \cdots, n_{cs}\}$
- Map that set of integers to a binary vector m_j indicating the integers drawn above.
- Let $M_j = \mathrm{diag}(m_j)$, and generate the following

$$Z_{1jd} = \alpha_{0jd} + Z_{2jt}'\alpha_{1d} + W_t' M_j \alpha_{2d} + V_{jt,d} \quad \text{where} \quad \alpha_{0jd} = 1/2 + j/2$$

Regressand Generation

- Draw the coefficient vector $[\,\beta_1' \;\; \beta_2'\,]' \in [1,-1]^{n_{end}+n_{end}}$, and generate the following

$$Y_{jt} = \beta_0 + Z_{1jt}'\beta_1 + Z_{2jt}'\beta_2 + e_j + \varepsilon_{jt} \quad \text{where} \quad e_j = 1 + j/2$$

Note due the manner that the error vector is generated,

$$E(\varepsilon_{jt}|V_{jt}) = \rho_{er}' V_{er(2,2)}^{-1} \begin{bmatrix} V_{j1t} \\ V_{j2t} \\ \vdots \\ V_{jqt} \end{bmatrix}$$

where

$$V_{er} = \begin{bmatrix} 1 & \rho_{er}' \\ \rho_{er} & V_{er(22)} \end{bmatrix}$$