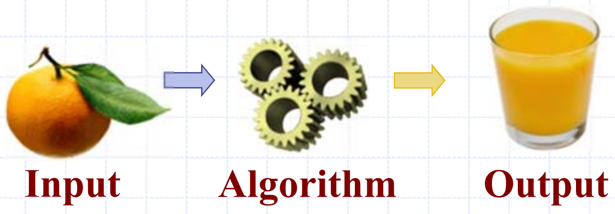
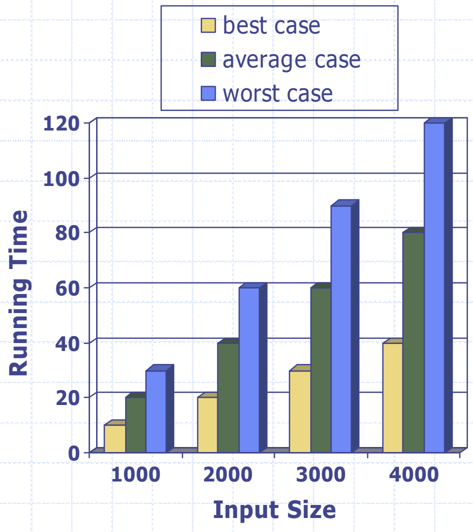
Presentation for use with the textbook Data Structures and Algorithms in Java, 6th edition, by M. T. Goodrich, R. Tamassia, and M. H. Goldwasser, Wiley, 2014

Analysis of Algorithms



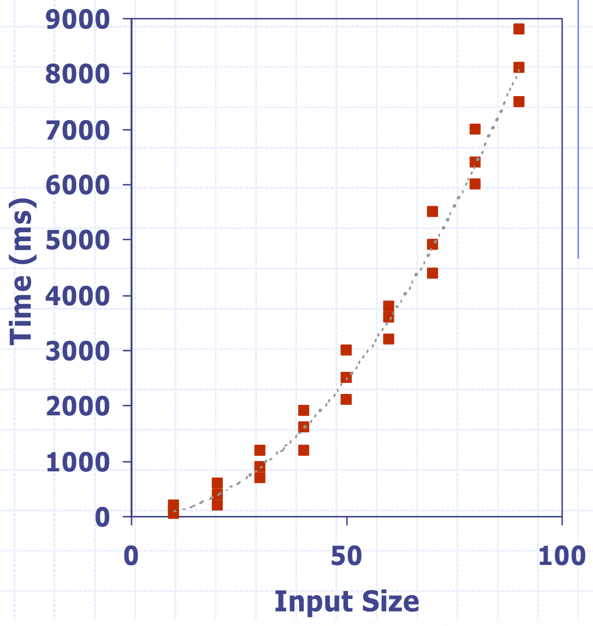
Running Time

- Most algorithms transform input objects into output objects.
- The running time of an algorithm typically grows with the ____________.
- Average case time is often difficult to determine.
- We focus on the _____ case running time.
  - Easier to analyze
  - Crucial to applications such as games, finance and robotics



Experimental Studies

- Write a program implementing the algorithm
- Run the program with inputs of varying size and composition, noting the time needed:
- Plot the results



```
long startTime = System.currentTimeMillis( );        // record the starting time
/∗ (run the algorithm) ∗/
long endTime = System.currentTimeMillis( );          // record the ending time
long elapsed = endTime − startTime;                  // compute the elapsed time
```

Limitations of Experiments

- It is necessary to implement the algorithm, which may be difficult
- Results may not be indicative of the running time on other inputs not included in the experiment.
- In order to compare two algorithms, the same hardware and software environments must be used

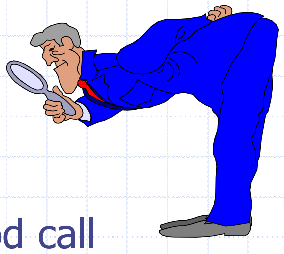
Theoretical Analysis

- Uses a high-level description of the algorithm instead of an implementation
- Characterizes running time as a function of the ______________, n
- Takes into account all possible inputs
- Allows us to evaluate the speed of an algorithm independent of the hardware/software environment

Pseudocode

- High-level description of an algorithm
- More structured than English prose
- Less detailed than a program
- Preferred notation for describing algorithms ✓
- Hides program design issues

Pseudocode Details



- Control flow
  - **if** … **then** … [**else** …]
  - **while** … **do** …
  - **repeat** … **until** …
  - **for** … **do** …
  - Indentation replaces braces
- Method declaration

  Algorithm *method* (*arg* [, *arg*…])

  Input …

  Output …
- Method call

  method (*arg* [, *arg*…])
- Return value

  return *expression*
- Expressions:

  ← Assignment

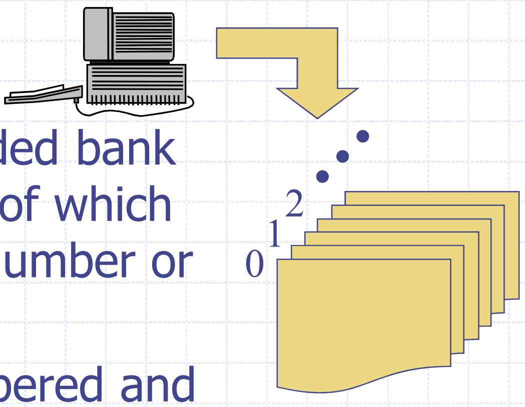
  = Equality testing

  n^2 Superscripts and other mathematical formatting allowed

The Random Access Machine (RAM) Model

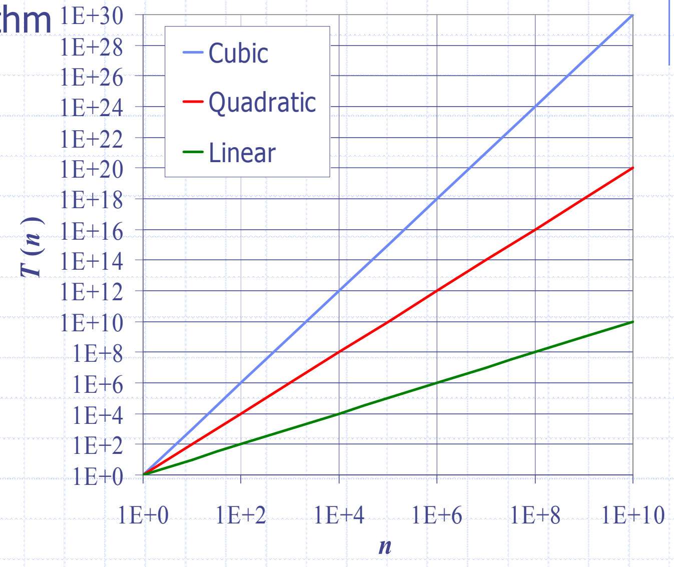
A RAM consists of

- A CPU
- An potentially unbounded bank of memory cells, each of which can hold an arbitrary number or character
- Memory cells are numbered and accessing any cell in memory takes unit time



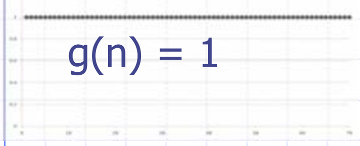
Seven Important Functions

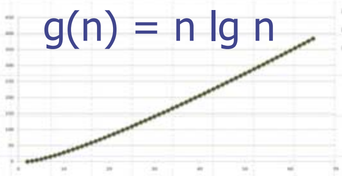
- Seven functions that often appear in algorithm analysis:
  - Constant ≈ 1
  - Logarithmic ≈ $\log n$
  - Linear ≈ n
  - N-Log-N ≈ $n \log n$
  - Quadratic ≈ n^2
  - Cubic ≈ n^3
  - Exponential ≈ 2^n
- In a log-log chart, the slope of the line corresponds to the growth rate

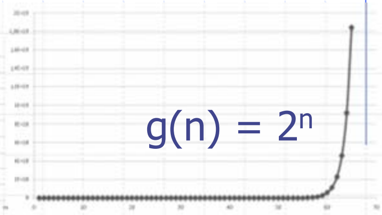


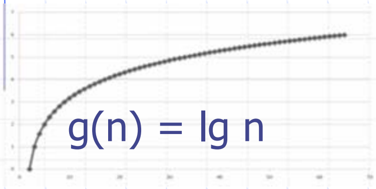
Functions Graphed Using "Normal" Scale

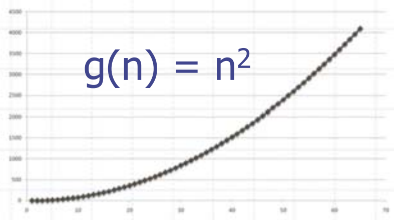
Slide by Matt Stallmann included with permission.

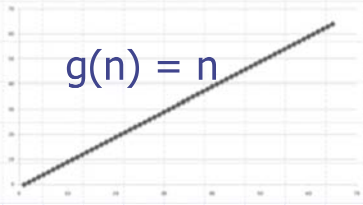


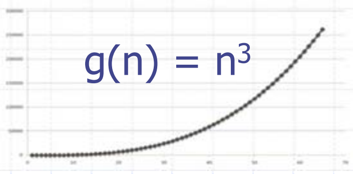












Primitive Operations

- Basic computations performed by an algorithm
- Identifiable in pseudocode
- Largely independent from the programming language
- Exact definition not important (we will see why later)
- Assumed to take a constant amount of time in the RAM model

- Examples:
  - Evaluating an expression
  - Assigning a value to a variable
  - Indexing into an array
  - Calling a method
  - Returning from a method

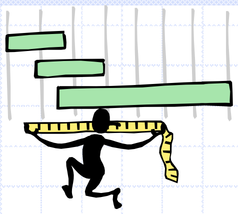
Counting Primitive Operations

- By inspecting the pseudocode, we can determine the maximum number of primitive operations executed by an algorithm, as a function of the input size

```
1  /** Returns the maximum value of a nonempty array of numbers. */
2  public static double arrayMax(double[ ] data) {
3    int n = data.length;
4    double currentMax = data[0];        // assume first entry is biggest (for now)
5    for (int j=1; j < n; j++)           // consider all other entries
6      if (data[j] > currentMax)         // if data[j] is biggest thus far...
7        currentMax = data[j];           // record it as the current max
8    return currentMax;
9  }
```

- Step 3: 2 ops, 4: 2 ops, 5: 2n ops,
  6: ______ ops, 7: 0 to _____ ops, 8: 1 op

Estimating Running Time



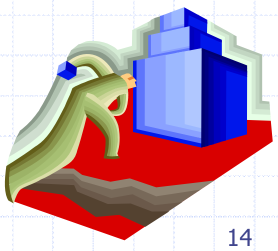
- Algorithm arrayMax executes $5n + 2$ primitive operations in the worst case, $4n + 3$ in the best case. Define:
  - a = Time taken by the fastest primitive operation
  - b = Time taken by the slowest primitive operation
- Let $T(n)$ be worst-case time of arrayMax. Then

$$a\,(4n + 3) \le T(n) \le b(5n + 2)$$

- Hence, the running time $T(n)$ is bounded by two linear functions

Growth Rate of Running Time

- Changing the hardware/ software environment
  - Affects $T(n)$ by a constant factor, but
  - Does not alter the growth rate of $T(n)$
- The linear growth rate of the running time $T(n)$ is an intrinsic property of algorithm arrayMax



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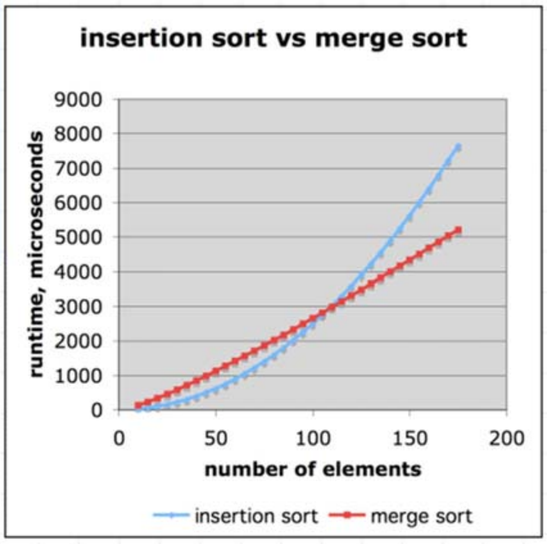
Comparison of Two Algorithms



insertion sort is
$n^2 / 4$

merge sort is
$2\, n \lg n$

sort a million items?

insertion sort takes roughly 70 hours

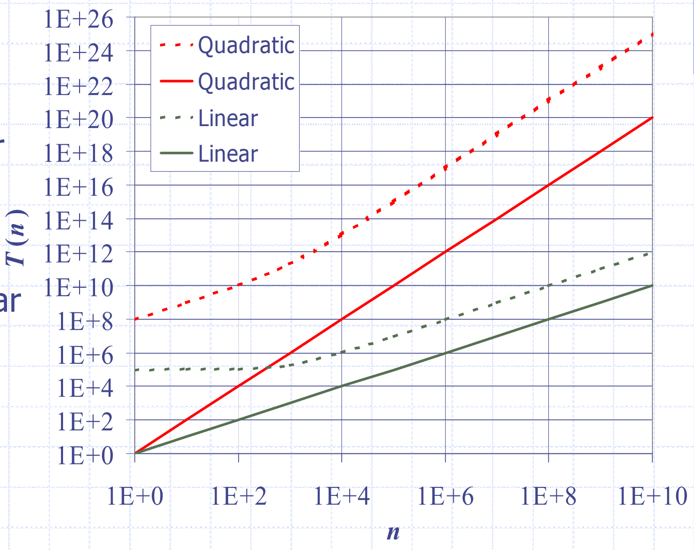
while

merge sort takes roughly 40 seconds

This is a slow machine, but if 100 x as fast then it's 40 minutes versus less than 0.5 seconds

Constant Factors

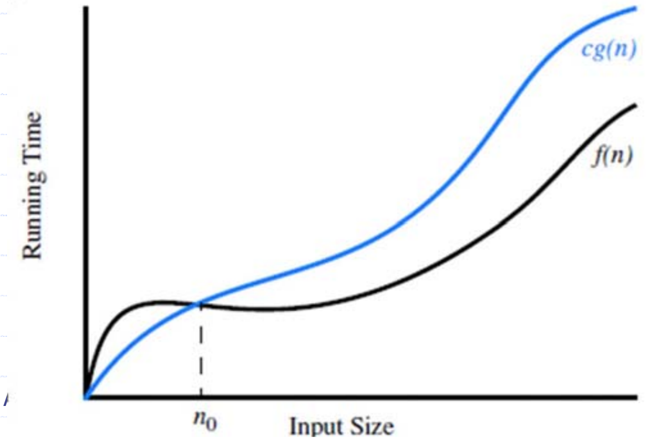
- The growth rate is not affected by
  - constant factors or
  - ___________ terms
- Examples
  - $10^2 n + 10^5$ is a linear function
  - $10^5 n^2 + 10^8 n$ is a quadratic function



Big-Oh Notation

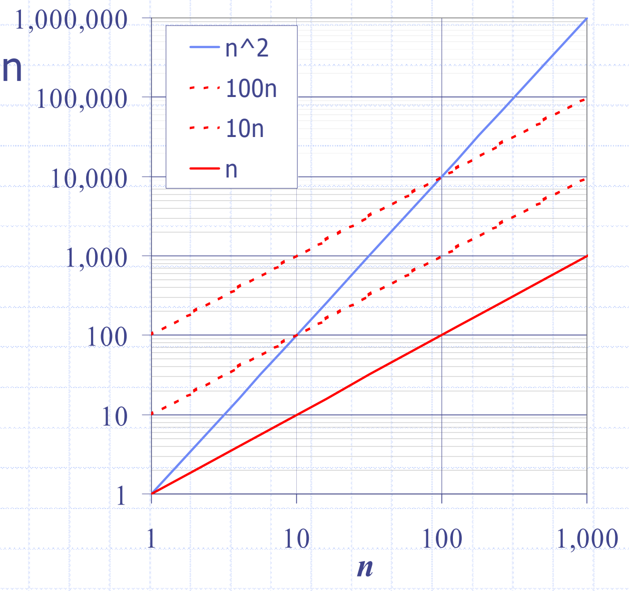
- Given functions $f(n)$ and $g(n)$, we say that $f(n)$ is $O(g(n))$ if there are positive constants c and n_0 such that

- Example: $2n + 10$ is $O(n)$
  - $2n + 10 \leq cn$
  - $(c - 2)\, n \geq 10$
  - $n \geq 10/(c - 2)$
  - Pick $c = 3$ and $n_0 = 10$

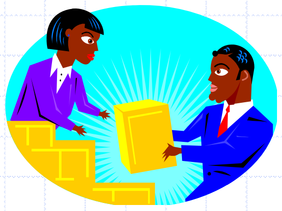


Big-Oh Example

- Example: the function n^2 is not $O(n)$
  - $n^2 \leq cn$
  - $n \leq c$
  - The above inequality cannot be satisfied since c must be a constant



More Big-Oh Examples



- 7n - 2

  7n-2 is O(n)
  need $c > 0$ and $n_0 \geq 1$ such that $7n - 2 \leq c\,n$ for $n \geq n_0$
  this is true for $c = 7$ and $n_0 = 1$

- $3n^3 + 20n^2 + 5$

  $3n^3 + 20n^2 + 5$ is $O(n^3)$
  need $c > 0$ and $n_0 \geq 1$ such that $3n^3 + 20n^2 + 5 \leq c\,n^3$ for $n \geq n_0$
  this is true for $c = 4$ and $n_0 = 21$

- 3 log n + 5

  3 log n + 5 is O(log n)
  need $c > 0$ and $n_0 \geq 1$ such that $3 \log n + 5 \leq c \log n$ for $n \geq n_0$
  this is true for $c = 8$ and $n_0 = 2$

Big-Oh and Growth Rate

- The big-Oh notation gives an ____________ on the growth rate of a function
- The statement "$f(n)$ is $O(g(n))$" means that the growth rate of $f(n)$ is no more than the growth rate of $g(n)$
- We can use the big-Oh notation to rank functions according to their growth rate

	$f(n)$ is $O(g(n))$	$g(n)$ is $O(f(n))$
$g(n)$ grows more	Yes	No
$f(n)$ grows more	No	Yes
Same growth	Yes	Yes

Big-Oh Rules

- If is $f(n)$ a polynomial of degree d, then $f(n)$ is $O(n^d)$, i.e.,
  1. Drop lower-order terms
  2. Drop constant factors
- Use the smallest possible class of functions
  - Say "$2n$ is $O(n)$" instead of "$2n$ is $O(n^2)$"
- Use the simplest expression of the class
  - Say "$3n + 5$ is $O(n)$" instead of "$3n + 5$ is $O(3n)$"

Asymptotic Algorithm Analysis

- The asymptotic analysis of an algorithm determines the running time in big-Oh notation
- To perform the asymptotic analysis
  - We find the worst-case number of primitive operations executed as a function of the input size
  - We express this function with big-Oh notation
- Example:
  - We say that algorithm arrayMax "runs in $O(n)$ time"
- Since constant factors and lower-order terms are eventually dropped anyhow, we can disregard them when counting primitive operations

Computing Prefix Averages

- We further illustrate asymptotic analysis with two algorithms for prefix averages
- The i-th prefix average of an array X is average of the first $(i + 1)$ elements of X:

$$A[i] = (X[0] + X[1] + \ldots + X[i])/(i+1)$$

- Computing the array A of prefix averages of another array X has applications to financial analysis

Prefix Averages (Quadratic)

- The following algorithm computes prefix averages in quadratic time by applying the definition

```
Algorithm prefixAverages1(X, n)
    Input array X of n integers
    Output array A of prefix averages of X
    A ← new array of n integers
    for i ← 0 to n − 1 do
        s ← 0
        for j ← 0 to i do
            s ← s + X[j]
        A[i] ← s / (i + 1)
    return A
```

prefixAverages1 runs in $O(n^2)$ time

Arithmetic Progression

- The running time of prefixAverage1 is $O(1 + 2 + \ldots + n)$
- The sum of the first n integers is $n(n + 1) / 2$
  - There is a simple visual proof of this fact
- Thus, algorithm prefixAverage1 runs in $O(n^2)$ time

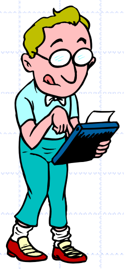
Prefix Averages (Linear)

- The following algorithm computes prefix averages in linear time by keeping a running sum

```
Algorithm prefixAverages2(X, n)
    Input array X of n integers
    Output array A of prefix averages of X
    A ← new array of n integers
    s ← 0
    for i ← 0 to n − 1 do
        s ← s + X[i]
        A[i] ← s / (i + 1)
    return A
```

- Algorithm *prefixAverages2* runs in $O(n)$ time

Math you need to Review



- Summations
- Powers
- Logarithms
- Proof techniques
- Basic probability

- Properties of powers:
  - $a^{(b+c)} = a^b a^c$
  - $a^{bc} = (a^b)^c$
  - $a^b / a^c = a^{(b-c)}$
  - $b = a^{\log_a b}$
  - $b^c = a^{c*\log_a b}$
- Properties of logarithms:
  - $\log_b(xy) = \log_b x + \log_b y$
  - $\log_b (x/y) = \log_b x - \log_b y$
  - $\log_b xa = a\log_b x$
  - $\log_b a = \log_x a/\log_x b$

Relatives of Big-Oh

big-Omega

- f(n) is $\Omega(g(n))$ if there is a constant $c > 0$ and an integer constant $n_0 \geq 1$ such that

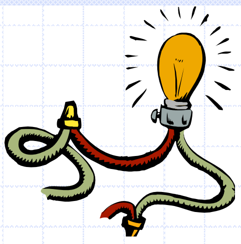
$$f(n) \geq c\, g(n) \text{ for } n \geq n_0$$

big-Theta

- f(n) is $\Theta(g(n))$ if there are constants $c' > 0$ and $c'' > 0$ and an integer constant $n_0 \geq 1$ such that

$$c'g(n) \leq f(n) \leq c''g(n) \text{ for } n \geq n_0$$

Intuition for Asymptotic Notation



big-Oh
- f(n) is O(g(n)) if f(n) is asymptotically less than or equal to g(n)

big-Omega
- f(n) is Ω(g(n)) if f(n) is asymptotically greater than or equal to g(n)

big-Theta
- f(n) is Θ(g(n)) if f(n) is asymptotically equal to g(n)

Example Uses of the Relatives of Big-Oh

- **$5n^2$ is $\Omega(n^2)$**

  $f(n)$ is $\Omega(g(n))$ if there is a constant $c > 0$ and an integer constant $n_0 \geq 1$ such that $f(n) \geq c\, g(n)$ for $n \geq n_0$

  let $c = 5$ and $n_0 = 1$
- **$5n^2$ is $\Omega(n)$**

  $f(n)$ is $\Omega(g(n))$ if there is a constant $c > 0$ and an integer constant $n_0 \geq 1$ such that $f(n) \geq c\, g(n)$ for $n \geq n_0$

  let $c = 1$ and $n_0 = 1$
- **$5n^2$ is $\Theta(n^2)$**

  $f(n)$ is $\Theta(g(n))$ if it is $\Omega(n^2)$ and $O(n^2)$. We have already seen the former, for the latter recall that $f(n)$ is $O(g(n))$ if there is a constant $c > 0$ and an integer constant $n_0 \geq 1$ such that $f(n) \leq c\, g(n)$ for $n \geq n_0$

  Let $c = 5$ and $n_0 = 1$

BIG IDEA! Asymptotic Analysis

1. Ignore machine dependent constants
2. Look at growth of running time T(n) as n -> infinity

Let n be size of program's input.

Let f(n) be function for running time

Let g(n) be another function -- preferably simple.

Let f and g are non-negative functions over non-negative integers.

Definition: [***Big-Oh***] $f(n) = O(g(n))$ iff there exist positive constants c and n_0 such that $f(n) \leq cg(n)$ for all n, $n \geq n_0$.

Example:

The function $T(n) = 3n^3 + 2n^2$ is $O(n^3)$.

To see this, let $n_0 = 0$ and $c = 5$. $3n^3 + 2n^2 \leq 5n^3$

You need to show that for all $n \geq 0$.

$3n^3 + 2n^2 \leq 5n^3$ => $n^2 \leq n^3$

if $n = 0$, $0 \leq 0$

otherwise, $1 \leq n$. $\therefore\ 3n^3 + 2n^2 = O(n^3)$ □

Example:

$T(n) = 3^n$ is not $O(2^n)$.

Suppose that there were constants n_0 and c such that for all $n \geq n_0$, we had $3^n \leq c2^n$. Then $c \geq (3/2)^n$ for any $n \geq n_0$. But $\lim_{n \to \infty} (3/2)^n = \infty$. Therefore, no constant c can exceed $(3/2)^n$ for all n. □

- The statement $f(n) = O(g(n))$ only states that $g(n)$ is an ***<u>upper bound</u>*** on the growth rate of $f(n)$ for all n, $n \geq n_0$.

Set definition:

$O(g(n))$ = { f(n): there are positive constants c and n_0 such that $f(n) \le cg(n)$ for all $n \geq n_0$}

Example:

The $n = O(n^2)$, $n = O(n^{2.5})$, $n = O(n^3)$, $n = O(2^n)$.

In order for the statement *f(n)* = *O(g(n))* to be informative, *g(n)* should be ***as small a function of n as*** one can come up with for which *f(n)* = *O(g(n))*.

$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$ means g(n) is asymptotically bigger than f(n) and f(n) is asymptotically smaller than g(n).

$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty$ means vice versa.

To prove $f(n) = O(g(n))$, we only need to show $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$ or Constant C

Example)

$$\lim_{n\to\infty}\frac{nlog_2 n}{n^2}=\frac{1}{\ln 2}\lim_{n\to\infty}\frac{n\ln n}{n^2}=\frac{1}{\ln 2}\lim_{n\to\infty}\frac{\ln n+1}{2n}$$
$$=\frac{1}{\ln 2}(\lim_{n\to\infty}\frac{\ln n}{2n}+\lim_{n\to\infty}\frac{1}{2n})=\frac{1}{\ln 2}\lim_{n\to\infty}\frac{1}{2n}=0$$

Therefore, $nlog_2 n=O(n^2)$.

Summary

- Programs can be evaluated by comparing their Big-Oh functions.
  - Drop low-order terms and ignore leading constants.

    Ex) $3n^2+nlog_2 n=O(n^2)$

- The common Big-Oh functions provide a "yardstick" for classifying different algorithms. Algorithms of the same Big-Oh can be considered as equally good.

  For example, a program with $O(\log n)$ is better than one with $O(n)$.

- Warnings!

- Warnings!

1. Fallacious proof:

$n^2 = O(n)$ Proof: Choose c = n, $n_0 = 1$. Then $n^2 \le n^2$.

--> WRONG! c must be constant.

2. "$e^{3n} = O(e^n)$ because constant factors don't matter" -->

Bigger by a factor of e^{2n}

"$10^n = O(2^n)$ ==> WRONG! --> Bigger by a factor of 5^n

3. Big-Oh notation doesn't always tell whole story.

Big-Oh is meaningful only when n is sufficiently large ($n \ge n_0$).

This implies that we only care about large size problems.

$T(n) = nlog_2n$, $U(n) = 100n$

T(n) dominates U(n) asymptotically.

But, if $\log_2 n < 50$ in practice, what algorithm would you like to choose?

Definition: [*Big-Theta*] $f(n) = \Theta(g(n))$ iff there exist positive constants c_1 and c_2 and an n_0 such that $c_1 g(n) \le f(n) \le c_2 g(n)$ for all n, $n \ge n_0$.

Example: $3n^2 + 4n = \Theta(n^2)$

Proof) choose $c_1 = 3$ and $c_2 = 7$ and $n_0 = 0$. We have $3n^2 \le 3n^2 + 4n \le 7n^2$ for all n, $n \ge n_0$.

- Ω(f(n)): Big Omega

Definition: [*Big-Omega*] $f(n) = \Omega(g(n))$ iff there exist positive constants c and n_0 such that $f(n) \ge cg(n)$ for all n, $n \ge n_0$.

Example: f(n) = 3n+2 >= 3n for all n, so $f(n) = \Omega(n)$