Технически университет - София
Факултет по приложна математика и информатика

Курсова работа

Математическа Екология

Студент:
Кристиян Кръчмаров

Преподавател:
проф. дмн. Людмил Каранджулов

Съдържание

1 Задание

За математическия модел на съжителство на две популации

$$\left| \begin{array}{ll} \dot{N}_1 = (a - bN_1 - \sigma N_2)\, N_1 & \quad a, b, \sigma > 0 \\ \dot{N}_2 = (c - \nu N_1 - dN_2)\, N_2 & \quad c, d, \nu > 0 \end{array} \right. \tag{$*$}$$

са въведени следните означения

$$\Delta = \begin{pmatrix} b & \sigma \\ \nu & d \end{pmatrix} \qquad \Delta_1 = \begin{pmatrix} a & \sigma \\ c & d \end{pmatrix} \qquad \Delta_2 = \begin{pmatrix} b & a \\ \nu & c \end{pmatrix}$$

Изследвайте вида на особенните точки, фазова картина, компютърна реализация, съответни чертежи и биологични изводи, ако е изпълнено

$$\Delta > 0 \qquad \Delta_1 > 0 \qquad \Delta_2 > 0$$

2 Решение

2.1 Особенни точки

Особенните точки се получават като решение на системата

$$\left| \begin{array}{l} (a - bN_1 - \sigma N_2)\, N_1 = 0 \\ (c - \nu N_1 - dN_2)\, N_2 = 0 \end{array} \right.$$

2.1.1 Първи случай

$$\left| \begin{array}{l} N_1 = 0 \\ N_2 = 0 \end{array} \right. \tag{I}$$

2.1.2 Втори случай

$$\left| \begin{array}{l} N_1 = 0 \\ N_2 \neq 0 \end{array} \right. \Longrightarrow c - dN_2 = 0 \Longrightarrow \left| \begin{array}{l} N_1 = 0 \\ N_2 = \frac{c}{d} \end{array} \right. \tag{II}$$

2.1.3 Трети случай

$$\left| \begin{array}{l} N_1 \neq 0 \\ N_2 = 0 \end{array} \right. \Longrightarrow a - bN_1 = 0 \Longrightarrow \left| \begin{array}{l} N_1 = \frac{a}{b} \\ N_2 = 0 \end{array} \right. \tag{III}$$

2.1.4 Четвърти случай

$$\left| \begin{array}{l} N_1 \neq 0 \\ N_2 \neq 0 \end{array} \right. \Longrightarrow \left| \begin{array}{l} a - bN_1 - \sigma N_2 = 0 \\ c - \nu N_1 - dN_2 = 0 \end{array} \right. \Longleftrightarrow \left| \begin{array}{l} bN_1 + \sigma N_2 = a \\ \nu N_1 + dN_2 = c \end{array} \right. \Longrightarrow$$

$$\left| \begin{array}{l} N_1 = \frac{\Delta_1}{\Delta} \\ N_2 = \frac{\Delta_2}{\Delta} \end{array} \right. \quad \text{(Крамер)} \tag{IV}$$

2.2 Линеаризация

Линеаризацията се получава като се замести в $(*)$

$$\left| \begin{array}{l} N_1 - \alpha = y_1 \\ N_2 - \beta = y_2 \end{array} \right. \Longleftrightarrow \left| \begin{array}{l} N_1 = y_1 + \alpha \\ N_2 = y_2 + \beta \end{array} \right.$$

където (α, β) е особенна точка и се вземе линейната част за всяка една променлива y_1, y_2

2.2.1 Първи случай

$$\left|\begin{array}{l} N_1 - 0 = y_1 \\ N_2 - 0 = y_2 \end{array}\right. \Longleftrightarrow \left|\begin{array}{l} N_1 = y_1 \\ N_2 = y_2 \end{array}\right. \Longrightarrow \left|\begin{array}{l} \dot{y_1} = (a - by_1 - \sigma y_2)\, y_1 \\ \dot{y_2} = (c - \nu y_1 - dy_2)\, y_2 \end{array}\right. \Longleftrightarrow$$

$$\left|\begin{array}{l} \dot{y_1} = ay_1 - by_1^2 - \sigma y_1 y_2 \\ \dot{y_2} = cy_2 - \nu y_1 y_2 - dy_2^2 \end{array}\right. \Longrightarrow W = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix} \qquad (I)$$

2.2.2 Втори случай

$$\left|\begin{array}{l} N_1 - 0 = y_1 \\ N_2 - \frac{c}{d} = y_2 \end{array}\right. \Longleftrightarrow \left|\begin{array}{l} N_1 = y_1 \\ N_2 = y_2 + \frac{c}{d} \end{array}\right. \Longrightarrow \left|\begin{array}{l} \dot{y_1} = \left[a - by_1 - \sigma\left(y_2 + \frac{c}{d}\right)\right] y_1 \\ \dot{y_2} = \left[c - \nu y_1 - d\left(y_2 + \frac{c}{d}\right)\right]\left(y_2 + \frac{c}{d}\right) \end{array}\right. \Longleftrightarrow$$

$$\left|\begin{array}{l} \dot{y_1} = ay_1 - by_1^2 - \sigma y_1 y_2 - \frac{\sigma c}{d} y_1 \\ \dot{y_2} = [c - \nu y_1 - dy_2 - c]\left(y_2 + \frac{c}{d}\right) \end{array}\right. \Longleftrightarrow \left|\begin{array}{l} \dot{y_1} = ay_1 - by_1^2 - \sigma y_1 y_2 - \frac{\sigma c}{d} y_1 \\ \dot{y_2} = \nu y_1 y_2 - \frac{\nu c}{d} y_1 - dy_2^2 - cy_2 \end{array}\right. \Longrightarrow$$

$$W = \begin{pmatrix} a - \frac{\sigma c}{d} & 0 \\ -\frac{\nu c}{d} & -c \end{pmatrix} \qquad (II)$$

2.2.3 Трети случай

$$\left|\begin{array}{l} N_1 - \frac{a}{b} = y_1 \\ N_2 - 0 = y_2 \end{array}\right. \Longleftrightarrow \left|\begin{array}{l} N_1 = y_1 + \frac{a}{b} \\ N_2 = y_2 \end{array}\right. \Longrightarrow \left|\begin{array}{l} \dot{y_1} = \left[a - b\left(y_1 + \frac{a}{b}\right) - \sigma y_2\right]\left(y_1 + \frac{a}{b}\right) \\ \dot{y_2} = \left[c - \nu\left(y_1 + \frac{a}{b}\right) - dy_2\right] y_2 \end{array}\right. \Longleftrightarrow$$

$$\left|\begin{array}{l} \dot{y_1} = [a - by_1 - a - \sigma y_2]\left(y_1 + \frac{a}{b}\right) \\ \dot{y_2} = cy_2 - \nu y_1 y_2 - \frac{\nu a}{b} - dy_2^2 \end{array}\right. \Longleftrightarrow \left|\begin{array}{l} \dot{y_1} = -by_1^2 - ay_1 - \sigma y_1 y_2 - \frac{\sigma a}{b} \\ \dot{y_2} = cy_2 - \nu y_1 y_2 - \frac{\nu a}{b} - dy_2^2 \end{array}\right. \Longrightarrow$$

$$W = \begin{pmatrix} -a & -\frac{\sigma a}{b} \\ 0 & c - \frac{\nu a}{b} \end{pmatrix} \qquad (III)$$

2.2.4 Четвърти случай

$$\left| \begin{array}{l} N_1 - \frac{\Delta_1}{\Delta} = y_1 \\ N_2 - \frac{\Delta_2}{\Delta} = y_2 \end{array} \right. \iff \left| \begin{array}{l} N_1 = y_1 + \frac{\Delta_1}{\Delta} \\ N_2 = y_2 + \frac{\Delta_2}{\Delta} \end{array} \right. \implies \left| \begin{array}{l} \dot{y_1} = \left[a - b\left(y_1 + \frac{\Delta_1}{\Delta}\right) - \sigma\left(y_2 + \frac{\Delta_2}{\Delta}\right)\right]\left(y_1 + \frac{\Delta_1}{\Delta}\right) \\ \dot{y_2} = \left[c - \nu\left(y_1 + \frac{\Delta_1}{\Delta}\right) - d\left(y_2 + \frac{\Delta_2}{\Delta}\right)\right]\left(y_2 + \frac{\Delta_2}{\Delta}\right) \end{array} \right.$$

$$\left| \begin{array}{l} \dot{y_1} = \left[a - by_1 - \frac{b\Delta_1}{\Delta} - \sigma y_2 - \frac{\sigma\Delta_2}{\Delta}\right]\left(y_1 + \frac{\Delta_1}{\Delta}\right) \\ \dot{y_2} = \left[c - \nu y_1 - \frac{\nu\Delta_1}{\Delta} - dy_2 - \frac{d\Delta_2}{\Delta}\right]\left(y_2 + \frac{\Delta_2}{\Delta}\right) \end{array} \right. \iff$$

$$\left| \begin{array}{l} \dot{y_1} = ay_1 + \frac{a\Delta_1}{\Delta} - by_1^2 - \frac{b\Delta_1}{\Delta}y_1 - \frac{b\Delta_1}{\Delta}y_1 - b\left(\frac{\Delta_1}{\Delta}\right)^2 - \sigma y_1 y_2 - \frac{\sigma\Delta_1}{\Delta}y_1 - \frac{\sigma\Delta_2}{\Delta}y_1 - \frac{\sigma\Delta_1\Delta_2}{\Delta^2} \\ \dot{y_2} = cy_2 + \frac{c\Delta_2}{\Delta} - \nu y_1 y_2 - \frac{\nu\Delta_2}{\Delta}y_1 - \frac{\nu\Delta_1}{\Delta}y_2 - \frac{\nu\Delta_1\Delta_2}{\Delta^2} - dy_2^2 - \frac{d\Delta_2}{\Delta}y_2 - \frac{d\Delta_2}{\Delta}y_2 - d\left(\frac{\Delta_2}{\Delta}\right)^2 \end{array} \right. \implies$$

$$W = \begin{pmatrix} a - \frac{2b\Delta_1}{\Delta} - \frac{\sigma\Delta_2}{\Delta} & -\frac{\sigma\Delta_1}{\Delta} \\ -\frac{\nu\Delta_2}{\Delta} & c - \frac{2d\Delta_2}{\Delta} - \frac{\nu\Delta_1}{\Delta} \end{pmatrix}$$

$$W = \frac{1}{\Delta}\begin{pmatrix} \Delta a - 2b\Delta_1 - \sigma\Delta_2 & -\sigma\Delta_1 \\ -\nu\Delta_2 & \Delta c - 2d\Delta_2 - \nu\Delta_1 \end{pmatrix}$$

$\Delta a - 2b\Delta_1 - \sigma\Delta_2 = (bd - \nu\sigma)a - 2b(ad - c\sigma) - \sigma(bc - a\nu) =$
$abd - a\nu\sigma - 2abd + 2bc\sigma - bc\sigma + a\nu\sigma = bc\sigma - abd = b(c\sigma - ad) = -b(ad - c\sigma) = -b\Delta_1$

$\Delta c - 2d\Delta_2 - \nu\Delta_1 = (bd - \nu\sigma)c - 2d(bc - a\nu) - \nu(ad - c\sigma) =$
$bcd - c\nu\sigma - 2bcd + 2ad\nu - ad\nu + c\nu\sigma = ad\nu - bcd = d(a\nu - bc) = -d(bc - a\nu) = -d\Delta_2$

$$\implies W = -\frac{1}{\Delta}\begin{pmatrix} b\Delta_1 & \sigma\Delta_1 \\ \nu\Delta_2 & d\Delta_2 \end{pmatrix} \qquad (IV)$$

2.3 Собствени стойностти

Собствените стойностти на матрицата W се получават от уравнението

$$\det(W - \lambda I) = 0$$

където I е единичната матрица

2.3.1 Първи случай

$$\det(W - \lambda I) = \begin{vmatrix} a - \lambda & 0 \\ 0 & c - \lambda \end{vmatrix} = (a - \lambda)(c - \lambda) = 0 \implies$$

$$\left| \begin{array}{l} \lambda_1 = a > 0 \\ \lambda_2 = c > 0 \end{array} \right. \implies \text{неустойчив възел} \qquad (I)$$

2.3.2 Втори случай

$$\det(W-\lambda I) = \begin{vmatrix} a-\frac{\sigma c}{d}-\lambda & 0 \\ -\frac{\nu c}{d} & -c-\lambda \end{vmatrix} = \left(\frac{ad-\sigma c}{d}-\lambda\right)(-c-\lambda) = 0 \implies$$

$$\left| \begin{array}{l} \lambda_1 = \frac{ad-\sigma c}{d} = \frac{\Delta_1}{d} > 0 \\ \lambda_2 = -c < 0 \end{array} \right. \implies \text{седло} \qquad (II)$$

2.3.3 Трети случай

$$\det(W-\lambda I) = \begin{vmatrix} -a-\lambda & -\frac{\sigma a}{b} \\ 0 & c-\frac{\nu a}{b}-\lambda \end{vmatrix} = (-a-\lambda)\left(\frac{bc-\nu a}{b}-\lambda\right) = 0 \implies$$

$$\left| \begin{array}{l} \lambda_1 = \frac{bc-\nu a}{b} = \frac{\Delta_2}{d} > 0 \\ \lambda_2 = -a < 0 \end{array} \right. \implies \text{седло} \qquad (III)$$

2.3.4 Четвърти случай

$$\det(W-\lambda I) = \begin{vmatrix} b\Delta_1-\lambda & \sigma\Delta_1 \\ \nu\Delta_2 & d\Delta_2-\lambda \end{vmatrix} = (b\Delta_1-\lambda)(d\Delta_2-\lambda) - \nu\sigma\Delta_1\Delta_2 = 0 \iff$$

$$bd\Delta_1\Delta_2 - b\Delta_1\lambda - d\Delta_2\lambda + \lambda^2 - \nu\sigma\Delta_1\Delta_2 = 0 \iff$$

$$\lambda^2 - \lambda(b\Delta_1 + d\Delta_2) + \Delta_1\Delta_2(bd-\nu\sigma) = 0 \iff \lambda^2 - \lambda(b\Delta_1 + d\Delta_2) + \Delta\Delta_1\Delta_2 = 0$$

Формули на Виет : $p\lambda^2 + q\lambda + r$

$$\left| \begin{array}{l} \lambda_1 + \lambda_2 = -\frac{q}{p} = -(b\Delta_1 + d\Delta_2) < 0 \\ \lambda_1 \cdot \lambda_2 = \frac{r}{p} = \Delta\Delta_1\Delta_2 > 0 \end{array} \right.$$

1. $\lambda_1 = \lambda_2 < 0, \lambda_{1,2} \in \mathbb{R} \implies$ устойчив възел
2. $\lambda_1 = \alpha + \beta i; \lambda_2 = \alpha - \beta i; \lambda_{1,2} \in \mathbb{C} \implies$ устойчив фокус

2.4 Компютърна реализация

Следния код реализира модела на съжителство в Matlab

```
function ecologyGraph(a,b,sigma,c,nu,d,N0,i)
% Дефиниране интервала на времето
tspan = [0 100];

% Дефиниране на функцията, която описва системата
ode = @(t, N) [ (a - b*N(1) - sigma*N(2)) * N(1);
                (c - nu*N(1) - d*N(2)) * N(2)];

% Решаване на системата от диференциални уравнения
[t, N] = ode45(ode, tspan, N0);

% Визуализация на резултатите
figure(i),plot(t, N(:,1), 'r-', t, N(:,2), 'b--');
legend('N_1', 'N_2', 'Location' , 'best');
xlabel('Time');
ylabel('Population');
end
```