Математически анализ 2
Упражнения

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Съдържание

1 Упражнение към лекция 1

Задача 1.0.1. Да се покаже дали посочените редици $\{X_n\} = \{x_n, y_n\}$ са сходящи или разходящи. За сходящите да се намери границите им.

1. $x_n = 1 + \frac{1}{n}$, $y_n = 2 + \frac{\sin n}{n}$
2. $x_n = \left(1 + \frac{1}{n}\right)^n$, $y_n = 2 + n$
3. $x_n = (-1)^n$, $y_n = n$
4. $x_n = (-1)^n$, $y_n = \frac{1}{n}$
5. $x_n = \sin \frac{n\pi}{2}$, $y_n = (-1)^n$
6. $x_n = \sin n$, $y_n = \frac{(-1)^n}{n}$

Решение:

1. $\lim\limits_{n\to\infty} \frac{1}{n} = 0$, $\frac{|\sin n|}{n} \in \left[0, \frac{1}{n}\right] \implies \lim\limits_{n\to\infty} x_n = 1$, $\lim\limits_{n\to\infty} y_n = 2 \implies$ редицата е сходяща; точката (1,2) е нейна граница
2. $\lim\limits_{n\to\infty} x_n = e$, $\lim\limits_{n\to\infty} y_n = \infty \implies$ разходяща редица
3. $\lim\limits_{n\to\infty} x_n$не съществува, защото има две точки на сгъстяване., $\lim\limits_{n\to\infty} y_n = \infty \implies$ разходяща редица
4. $\lim\limits_{n\to\infty} x_n$не съществува, защото има две точки на сгъстяване., $\lim\limits_{n\to\infty} y_n = 0 \implies$ разходяща редица
5. $\lim\limits_{n\to\infty} x_n$не съществува, $\lim\limits_{n\to\infty} y_n = \infty \implies$ разходяща редица
6. $\lim\limits_{n\to\infty} x_n$не съществува, $\lim\limits_{n\to\infty} y_n = 0 \implies$ разходяща редица

2 Упражнение към лекция 2

Задача 2.0.1. Нека $D \subset \mathbb{R}^m$ и са разгледани няколко функции. Да се напишат дефиниционните им множества и да се даде пояснение.

1. $z(x,y) = x^2 + y^2$
2. $z(x,y) = \sqrt{y^2 - 2x}$
3. $z(x,y) = \ln\sqrt{y^2 - 2x}$
4. $z(x,y) = \dfrac{1}{\sqrt{-y^2 + 2x + 1}}$
5. $w(x,y,z) = \arccos(x^2 + y^2 + z^2)$
6. $f(n) = \begin{cases} 1, & x \in \mathbb{Q}^m \\ 0, & x \in \dfrac{\mathbb{R}^m}{\mathbb{Q}^m} \end{cases}$

Решение:

1. $z(x,y) = x^2 + y^2$
   $D = \mathbb{R}^2$
2. $z(x,y) = \sqrt{y^2 - 2x}$
   $D = \{(x,y) : y^2 - 2x \geq 0\} \subset \mathbb{R}^2, x \leq \dfrac{y^2}{2}$
3. $z(x,y) = \ln\sqrt{y^2 - 2x}$
   $D = \{(x,y) : y^2 - 2x > 0\} \subset \mathbb{R}^2, x < \dfrac{y^2}{2}$
4. $z(x,y) = \dfrac{1}{\sqrt{-y^2 + 2x + 1}}$
   $D = \{(x,y) : -y^2 + 2x + 1 > 0\} \subset \mathbb{R}^2, x > \dfrac{y^2 - 1}{2}$
5. $w(x,y,z) = \arccos(x^2 + y^2 + z^2)$
   $D = \{(x,y,z) : x^2 + y^2 + z^2 \leq \pi\} \subset \mathbb{R}^3,$
   Графиката е кълбо с център (0,0,0) и радиус$\sqrt{\pi}$
6. $D \subset \mathbb{R}^m$

Задача 2.0.2. Разгледаните по - долу функциите са дефинирани в $D = \mathbb{R}^2 \setminus \{(0,0)\}$. Кои от границите съществуват и колко са

$$A = \lim_{(x,y)\to(0,0)} f(x,y) \quad A_{1,2} = \lim_{y\to 0}\left(\lim_{x\to 0} f(x,y)\right) \quad A_{2,1} = \lim_{x\to 0}\left(\lim_{y\to 0} f(x,y)\right)$$

1. $f(x,y) = \dfrac{x-y}{x+y}$
2. $f(x,y) = \dfrac{x^2+y^2}{x^2y^2+(x-y)^2}$
3. $f(x,y) = \dfrac{xy^2}{x^2+y^4}$
4. $f(x,y) = (x+y)\sin\dfrac{1}{x}\cos\dfrac{1}{y}$
5. $f(x,y) = \dfrac{x^4+y^4}{x^2+y^2}$

Решение:

1.

$$f(x,y) = \frac{x-y}{x+y}$$

$$\lim_{x\to 0} f(x,y) = \frac{-y}{y} = -1 \qquad \lim_{y\to 0} f(x,y) = \frac{x}{x} = 1$$

$$A_{1,2} = \lim_{y\to 0}\left(\lim_{x\to 0} f(x,y)\right) = \lim_{y\to 0}(-1) = -1$$

$$A_{2,1} = \lim_{x\to 0}\left(\lim_{y\to 0} f(x,y)\right) = \lim_{x\to 0}(1) = 1$$

$A = \lim\limits_{(x,y)\to(0,0)} f(x,y)$ Не съществува, защото трябва $A_{1,2} = A_{2,1}$

2.

$$f(x,y) = \frac{x^2+y^2}{x^2y^2+(x-y)^2}$$
$$\lim_{x\to 0} f(x,y) = \frac{y^2}{(-y)^2} = 1 \qquad \lim_{y\to 0} f(x,y) = \frac{x^2}{x^2} = 1$$
$$\implies A_{1,2} = A_{2,1} = 1 \implies \exists A = \lim_{(x,y)\to(0,0)} f(x,y)$$
Редица: $(x_n, y_n) = \left(\frac{1}{n}, \frac{1}{n}\right) \to (0,0), f(x_n, y_n) = 1 \to 1$

Редица: $(x'_n, y'_n) = \left(\frac{1}{n}, \frac{-1}{n}\right) \to (0,0), f(x'_n, y'_n) = \frac{2n^2}{1+4n^2} \to \frac{1}{2} \neq 1$

$\implies f(x,y)$ няма граница при $(x,y) \to (0,0)$

3.

$$f(x,y) = \frac{xy^2}{x^2+y^4}$$
$$\lim_{x\to 0} f(x,y) = \frac{0}{y^4} = 0 \qquad \lim_{y\to 0} f(x,y) = \frac{0}{x^2} = 0$$
$$A_{1,2} = A_{2,1} = 0 \implies \exists A = \lim_{(x,y)\to(0,0)} f(x,y)$$
Редица: $(x_n, y_n) = \left(\frac{1}{n^2}, \frac{1}{n}\right) \to (0,0), f(x_n, y_n) = \frac{1}{2} \to \frac{1}{2} \neq 0$

$\implies f(x,y)$ няма граница при $(x,y) \to (0,0)$

4.

$$f(x,y) = (x+y)\sin\frac{1}{x}\cos\frac{1}{y}$$
$0 \leq |f(x,y)| \leq |x+y| \leq |x| + |y|$ и $|x|+|y| \to 0$

$A = 0$

$\lim_{x\to 0} \sin\frac{1}{x}$ - не съществува

$$\lim_{x\to 0} f(x,y) = y\cos\frac{1}{y}\lim_{x\to 0}\sin\frac{1}{x}$$

Аналогично и другата вътрешна граница не съществува. Но тогава и повторните граници $A_{1,2}$, $A_{2,1}$ не съществуват.

5.

$$f(x,y) = \frac{x^4+y^4}{x^2+y^2}$$
$$\lim_{x\to 0} f(x,y) = y^2 \qquad \lim_{y\to 0} f(x,y) = x^2$$
$$A_{1,2} = \lim_{y\to 0}\Big(\lim_{x\to 0} f(x,y)\Big) = \lim_{y\to 0}\left(y^2\right) = 0$$
$$A_{2,1} = \lim_{x\to 0}\left(\lim_{y\to 0} f(x,y)\right) = \lim_{x\to 0}\left(x^2\right) = 0$$
$$\implies A = A_{1,2} = A_{2,1} = 0$$

Задача 2.0.3. Нека A, B, C, D са подмножества на $\mathbb{R}^2$ дефинирани както следва
$A = \{(x,y) : x \geq 0, y \leq 1, y > x\}$
$B = \{(x,y) : x \leq 1, y \geq 0, y < x\}$
$C = \{(x,y) : x = y, 0 \leq x \leq 1\}$
$D = A \cup B \cup C$
и функцията $f : D \to \mathbb{R}$ зададена по следния начин
$$f(x,y) = \begin{cases} \dfrac{1}{y^2}, & (x,y) \in A \\ 0, & x = y \\ -\dfrac{1}{x^2}, & (x,y) \in B \end{cases}$$
Да се изследва непрекъснатостта на тази функция.
Решение:
Функцията f е непрекъсната в A, защото е частно на две функции със знаменател $y^2 \neq 0$, в A.
Аналогично е непрекъсната в B защото знаменателя е $x^2 \neq 0$.
Остана да се изследва поведението върху C.

$(x_0, y_0) = (x_0, x_0) \in C$
$R = \{(x_n, y_n)\},\ (x_n, y_n) \in A$
$\lim\limits_{n\to\infty} R = (x_0, y_0)$
$\lim\limits_{n\to\infty} f(x_n, y_n) = \dfrac{1}{y_0^2} = \dfrac{1}{x_0^2} \neq 0$
Ако $x_0 \neq 0,\ f(x_0, y_0) = 0$
$\Longrightarrow$ функцията е прекъсната в точката $(x_0, x_0) \neq (0,0)$
Ако $(x_n, y_n) \in B,\ \lim\limits_{n\to\infty} f(x_n, y_n) = -\dfrac{1}{x_0^2} \neq f(x_0, x_0) \neq 0.$
Ако $x_0 = 0,\ \lim\limits_{n\to\infty} f(x_n, y_n) = \infty(-\infty),\ f(0,0) = 0,$
$\Longrightarrow$ f е прекъсната в точката (0,0).

Функцията е непрекъсната в D, с изключение на точките от C, където е прекъсната.

3 Упражнение към лекция 3

Задача 3.0.1. Да се намерят първите частни производни на следните функции

1. $f(x,y,z) = e^{4x+3y} + xy^2z^3 + 1111e^{\pi}$ за произволна точка $(x_0, y_0, z_0) \in \mathbb{R}^3$

2. $f(x,y) = |x+y|$ в точката (0,0)

3. $f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$ в равнината $\mathbb{R}^2$

Решение:

1.

$$
\begin{aligned}
&f(x,y,z) = e^{4x+3y} + xy^2z^3 + 1111e^{\pi} \\
&f(x,y_0,z_0) \implies f'_x(x_0,y_0,z_0) = 4e^{4x_0+3y_0} + y_0^2 z_0^3 \\
&f(x_0,y,z_0) \implies f'_y(x_0,y_0,z_0) = 3e^{4x_0+3y_0} + 2x_0y_0z_0^3 \\
&f(x_0,y_0,z) \implies f'_z(x_0,y_0,z_0) = 3x_0y_0^2z_0^2
\end{aligned}
$$

2.

$$
\begin{aligned}
&f(x,y) = |x+y| \\
&\frac{g(h)-g(0)}{h} = \frac{f(0+h,0)-f(0,0)}{h} \\
&\lim_{h\to 0}\frac{f(0+h,0)-f(0,0)}{h} = \lim_{h\to 0}\frac{|h|}{h} \text{ не съществува} \\
&\implies \nexists f'_x(0,0) (\text{Аналогично се получава за } f'_y(0,0))
\end{aligned}
$$

3.

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

$(x,y) \neq (0,0)$

$$f'_x(x,y) = \frac{y(y^2-x^2)}{(x^2+y^2)^2}$$

$$f'_y(x,y) = \frac{x(x^2-y^2)}{(x^2+y^2)^2}$$

$$\lim_{h\to 0} \frac{f(0+h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = \lim_{h\to 0} = 0$$

$$\lim_{k\to 0} \frac{f(0,0+k)-f(0,0)}{k} = \lim_{k\to 0} \frac{0-0}{k} = \lim_{k\to 0} = 0$$

$\implies$ Функцията има частни производни във всичко точки на равнината $\mathbb{R}^2$

Задача 3.0.2. $f(x,y) = x + (y-1)\arcsin\sqrt{\dfrac{x}{y}} \qquad f'_x(x,1) = ?$

Решение:

$$f'_x(a,b) = \lim_{h\to 0} \frac{f(a+h,b)-f(a,b)}{h} (\text{Ако съществува}) \implies$$

$$f'_x(x,1) = \lim_{h\to 0} \frac{f(x+h,1)-f(x,1)}{h} (\text{Ако съществува})$$

$$f(x+h,1) = x+h+(1-1)\arcsin\sqrt{\frac{x}{1}} = x+h+0\arcsin\sqrt{\frac{x}{1}} = x+h$$

$$f(x,1) = x+(1-1)\arcsin\sqrt{\frac{x}{1}} = x+0\arcsin\sqrt{\frac{x}{1}} = x \implies$$

$$\lim_{h\to 0} \frac{f(x+h,1)-f(x,1)}{h} = \lim_{h\to 0} \frac{x+h-x}{h} = \lim_{h\to 0} \frac{h}{h} = \lim_{h\to 0} 1 \implies f'_x(x,1) = 1$$

Задача 3.0.3. Да се докаже че функцията $f(x,y) = \begin{cases} \dfrac{x^3y}{x^6+y^2}, & (x,y) \neq (0,0) \\ 0, & x^2+y^2 = (0,0) \end{cases}$
е прекъсната в точката (0,0) но има частни производни в тази точка.

Решение:

Редица $(x_n, y_n) = \left(\frac{1}{n}, \frac{1}{n^3}\right)$

$$f(x_n, y_n) = \frac{\left(\frac{1}{n}\right)^3 \cdot \frac{1}{n^3}}{\left(\frac{1}{n}\right)^6 + \left(\frac{1}{n^3}\right)^3} = \frac{\frac{1}{n^6}}{\frac{2}{n^6}} = \frac{1}{2} \qquad \lim_{n\to\infty} f(x_n, y_n) = \frac{1}{2} \implies$$

$\lim_{x\to 0, y\to 0} f(x,y) \neq f(0,0) = 0 \implies f(x,y)$ е прекъсната в т. $(0,0)$.

$$f'_x(0,0) = \lim_{x\to 0} \frac{f(x,0) - f(0,0)}{x-0} = \frac{\frac{x^3 \cdot 0}{x^6+0} - 0}{x-0} = 0$$

$$f'_y(0,0) = \lim_{y\to 0} \frac{f(0,y) - f(0,0)}{y-0} = \frac{\frac{0^3 \cdot y}{0^6+y^2} - 0}{y-0} = 0$$

Задача 3.0.4. Да се намерят първите частни производни на следните функции:

1. $f(x,y) = \sin(2x+3) + 3e^{-x}e^{4y} - 11x^3 + 19e^{\pi}$
2. $f(x,y) = \sqrt{x^2+y^2} + \arctan\frac{y}{x}$
3. $f(x,y,z) = (xy)^z$
4. $\sqrt[3]{x^2+3y^2}e^{x^2-5y}$

Решение:

1.

$$\begin{aligned}
f(x,y) &= \sin(2x+3) + 3e^{-x}e^{4y} - 11x^3 + 19e^{\pi}\\
f'_x(x,y) &= (\sin(2x+3))'_x + (3e^{-x}e^{4y})'_x - (11x^3)'_x + (19e^{\pi})'_x\\
f'_x(x,y) &= \cos(2x+3)\cdot 2 + (-3e^{-x}e^{4y}) - (3\cdot 11x^2) + 0\\
f'_x(x,y) &= 2\cos(2x+3) - 3e^{-x}e^{4y} - 33x^2\\
f'_y(x,y) &= (\sin(2x+3))'_y + (3e^{-x}e^{4y})'_y - (11x^3)'_y + (19e^{\pi})'_y\\
f'_y(x,y) &= 0 + (3\cdot 4e^{-x}e^{4y}) - 0 + 0 = 12e^{-x}e^{4y}
\end{aligned}$$

2.

$$f(x,y) = \sqrt{x^2+y^2} + \arctan\frac{y}{x}$$
$$f'_x(x,y) = \frac{1}{2}(x^2)^{-\frac{1}{2}} \cdot 2x + \frac{1}{1+\frac{y^2}{x^2}} \cdot y \cdot (-\frac{1}{x^2})$$
$$f'_x(x,y) = \frac{x}{\sqrt{x^2+y^2}} - \frac{x^2y}{x^2+y^2} \cdot \frac{1}{x^2}$$
$$f'_x(x,y) = \frac{x}{\sqrt{x^2+y^2}} - \frac{xy}{x^2+y^2}$$
$$f'_y(x,y) = \frac{1}{2}(x^2)^{-\frac{1}{2}} \cdot 2y + \frac{1}{1+\frac{y^2}{x^2}} \cdot \frac{1}{x}$$
$$f'_y(x,y) = \frac{y}{\sqrt{x^2+y^2}} + \frac{x^2}{x^2+y^2} \cdot \frac{1}{x}$$
$$f'_y(x,y) = \frac{y}{\sqrt{x^2+y^2}} + \frac{x}{x^2+y^2}$$

3.

$$f(x,y,z) = (xy)^z$$
$$f'_x(x,y,z) = z(xy)^{z-1} \cdot (xy)'x = yz(xy)^{z-1}$$
$$f'_y(x,y,z) = z(xy)^{z-1} \cdot (xy)'y = xz(xy)^{z-1}$$
$$f'_z(x,y,z) = (xy)^z \ln{(xy)}$$

4.

$$\sqrt[3]{x^2+3y^2}e^{x^2-5y}$$

$$f'_x(x,y)=\left[\sqrt[3]{x^2+3y^2}\right]'_x\cdot e^{x^2-5y}+\sqrt[3]{x^2+3y^2}\cdot(e^{x^2-5y})'_x$$

$$f'_x(x,y)=\frac{1}{3}(x^2+3y^2)^{-\frac{2}{3}}\cdot 2x\cdot e^{x^2-5y}+\sqrt[3]{x^2+3y^2}\cdot 2xe^{x^2-5y}$$

$$f'_x(x,y)=\frac{2x}{3}\cdot\frac{e^{x^2-5y}}{\sqrt[3]{(x^2+3y^2)^2}}+2x\sqrt[3]{x^2+3y^2}\cdot e^{x^2-5y}$$

$$f'_x(x,y)=\frac{2x}{3}\cdot\frac{e^{x^2-5y}}{\sqrt[3]{(x^2+3y^2)^2}}\left[1+3(x^2+3y^2)\right]$$

$$f'_x(x,y)=\frac{2x}{3}(1+3x^2+9y^2)\frac{e^{x^2-5y}}{\sqrt[3]{(x^2+3y^2)^2}}$$

$$f'_y(x,y)=\left[\sqrt[3]{x^2+3y^2}\right]'_y\cdot e^{x^2-5y}+\sqrt[3]{x^2+3y^2}\cdot(e^{x^2-5y})'_y$$

$$f'_y(x,y)=\frac{1}{3}(x^2+3y^2)^{-\frac{2}{3}}\cdot 6y\cdot e^{x^2-5y}+\sqrt[3]{x^2+3y^2}\cdot(-5e^{x^2-5y})$$

$$f'_y(x,y)=2y\cdot\frac{1}{\sqrt[3]{(x^2+3y^2)^2}}\cdot e^{x^2-5y}-5\sqrt[3]{x^2+3y^2}\cdot e^{x^2-5y}$$

$$f'_y(x,y)=e^{x^2-5y}\cdot\sqrt[3]{(x^2+3y^2)^2}(2y-5(x^2+3y^2))$$

$$f'_y(x,y)=(2y-5x^2-15y^2)\frac{e^{x^2-5y}}{\sqrt[3]{(x^2+3y^2)^2}}$$

4 Упражнение към лекция 4

Задача 4.0.1. $f(x,y)=\sqrt[3]{xy}$
Изследвайте $f(x,y)$ за диференцируемост в $(0,0)$.
$f'_x(0,0)=?$
$f'_y(0,0)=?$
Решение:

$$f(x,0)-f(0,0)=\sqrt[3]{x0}-\sqrt[3]{0} \implies$$

$$\lim_{x\to 0}\frac{f(x,0)-f(0,0)}{x-0}=\lim_{x\to 0}\frac{0}{x}=0 f'_x(0,0)=\lim_{x\to 0}\frac{f(x,0)-f(0,0)}{x-0}=\lim_{x\to 0}\frac{0}{x}=0$$

$$f(0,y)-f(0,0)=\sqrt[3]{0y}-\sqrt[3]{0} \implies$$

$$f'_y(0,0)=\lim_{y\to 0}\frac{f(0,y)-f(0,0)}{y-0}=\lim_{y\to 0}\frac{0}{y}=0$$

Нека: $\lim\limits_{(x\to 0,y\to 0)}\varepsilon(x,y)\to 0, \rho(x,y)=\sqrt{x^2+y^2}$

Проверка за диференцируемост в (0,0):

$$f(x,y)-f(0,0)=f'_x(0,0)(x-0)+f'_y(0,0)(y-0)+\varepsilon(x,y)\rho(x,y)$$

$$\sqrt[3]{xy}-0=0x+0y+\varepsilon(x,y)\sqrt{x^2+y^2} \implies$$

$$\varepsilon(x,y)=\frac{\sqrt[3]{xy}}{\sqrt{x^2+y^2}}\to 0?$$

Разглеждаме редица с общ член $(x_n,y_n)=\left(\dfrac{1}{n^3},\dfrac{1}{n^3}\right)$ за която $(x_n,y_n)\to(0,0)$,

$$\varepsilon(x_n,y_n)=\frac{\frac{1}{n^2}}{\frac{\sqrt{2}}{n^3}}=\frac{n}{\sqrt{2}} \implies \lim_{(x,y)\to(0,0)}\varepsilon(x_n,y_n)\not\to 0 \implies$$

$f(x,y)$ не е диференцируема в т.$(0,0)$

Задача 4.0.2. $f(x,y)=\sqrt[3]{x^3+y^3}$
Изследвайте $f(x,y)$ за диференцируемост в $(0,0)$.

Решение:

$$f(x,0)-f(0,0)=\sqrt[3]{x^3}-0=x \implies$$
$$\lim_{x\to 0}\frac{f(x,0)-f(0,0)}{x-0}=\lim_{x\to 0}\frac{x}{x}=1 \implies \exists f'_x(0,0)=1$$
$$f(0,y)-f(0,0)=\sqrt[3]{y^3}-0=y \implies$$
$$\lim_{y\to 0}\frac{f(0,y)-f(0,0)}{y-0}=\lim_{y\to 0}\frac{y}{y}=1 \implies \exists f'_y(0,0)=1$$

Нека: $\lim\limits_{(x\to 0,y\to 0)}\varepsilon(x,y)\to 0, \rho(x,y)=\sqrt{x^2+y^2}$

Проверка за диференцируемост в (0,0):

$$f(x,y)-f(0,0)=f'_x(0,0)(x-0)+f'_y(0,0)(y-0)+\varepsilon(x,y)\rho(x,y)$$
$$\sqrt[3]{x^3+y^3}=x+y+\varepsilon(x,y)\sqrt{x^2+y^2}$$
$$\varepsilon(x,y)=\frac{\sqrt[3]{x^3+y^3}-x-y}{\sqrt{x^2+y^2}}$$
$$\lim_{(x\to 0,y\to 0)}\varepsilon(x,y)\to 0?$$

Разглеждаме редица с общ член $(x_n,y_n)=\left(\frac{1}{n},\frac{1}{n}\right)$ за която $(x_n,y_n)\to(0,0)$,

$$\varepsilon(x_n,y_n)=\frac{\frac{\sqrt[3]{2}}{n}-\frac{2}{n}}{\frac{\sqrt{2}}{n}}=\frac{\sqrt[3]{2}-2}{\sqrt{2}} \implies \lim_{(x\to 0,y\to 0)}\varepsilon(x,y)\not\to 0 \implies$$

$f(x,y)$ не е диференцируема в т.$(0,0)$

Задача 4.0.3. Да се изследвай за диференцируемост в $(0,0)$ функцията

$$f(x,y)=\begin{cases} e^{-\frac{1}{x^2+y^2}}, & x^2+y^2\neq 0\\ 0, & x^2+y^2=0\end{cases}$$

Решение:

$$f(x,0)-f(0,0)=e^{-\dfrac{1}{x^2}}-0=e^{-\dfrac{1}{x^2}}$$

$$\lim_{x\to 0}\frac{f(x,0)-f(0,0)}{x-0}=\lim_{x\to 0}\frac{e^{-\dfrac{1}{x^2}}}{x}=\left[\frac{0}{0}\right]$$

$$\lim_{x\to 0}\frac{e^{-\dfrac{1}{x^2}}}{x}=\lim_{x\to 0}\frac{\dfrac{1}{x}}{e^{\dfrac{1}{x^2}}}=\left[\frac{\infty}{\infty}\right]$$

$$\left(\frac{1}{x}\right)'=-\frac{1}{x^2}\qquad\left(e^{\dfrac{1}{x^2}}\right)'=-\frac{2}{x^3}e^{\dfrac{1}{x^2}}$$

$$\lim_{x\to 0}\frac{-\dfrac{1}{x^2}}{-\dfrac{2}{x^3}e^{\dfrac{1}{x^2}}}=\lim_{x\to 0}\frac{x}{2e^{\dfrac{1}{x^2}}}=\frac{0}{\infty}=0\implies f'_x(0,0)=0$$

Аналогично$f'_y(0,0)=0$

Нека : $\lim\limits_{(x\to 0,y\to 0)}\varepsilon(x,y)\to 0,\rho(x,y)=\sqrt{x^2+y^2}$

Проверка за диференцируемост в (0,0):

$$f(x,y)-f(0,0)=f'_x(0,0)(x-0)+f'_y(0,0)(y-0)+\varepsilon(x,y)\rho(x,y)$$

$$e^{-\dfrac{1}{x^2+y^2}}-0=0(x-0)+0(y-0)+\varepsilon(x,y)\sqrt{x^2+y^2}$$

$$e^{-\dfrac{1}{x^2+y^2}}=\varepsilon(x,y)\sqrt{x^2+y^2}$$

$$\varepsilon(x,y)=\frac{e^{-\dfrac{1}{x^2+y^2}}}{\sqrt{x^2+y^2}}$$

$$\lim_{(x\to 0,y\to 0)}\varepsilon(x,y)\to 0?$$

$$\rho(x,y)=\sqrt{x^2+y^2} \implies \lim_{(x\to 0, y\to 0)} \rho(x,y) \to 0$$

$$\lim_{(x\to 0, y\to 0)} \varepsilon(x,y) = \lim_{\rho\to 0} \frac{e^{-\frac{1}{\rho^2}}}{\rho} = \left[\frac{\infty}{\infty}\right]$$

$$\left(\frac{1}{\rho}\right)' = -\frac{1}{\rho^2} \qquad \left(e^{\frac{1}{\rho^2}}\right)' = -\frac{2}{\rho^3} e^{\frac{1}{\rho^2}}$$

$$\lim_{\rho\to 0} \frac{\rho}{2e^{\frac{1}{\rho^2}}} = \frac{0}{\infty} = 0 \implies$$

$$\lim_{(x\to 0, y\to 0)} \varepsilon(x,y) = \lim_{\rho\to 0} \frac{\frac{1}{\rho}}{e^{\frac{1}{\rho^2}}} = \lim_{\rho\to 0} \frac{\left(\frac{1}{\rho}\right)'}{\left(e^{\frac{1}{\rho^2}}\right)'} = 0 \implies$$

$$\lim_{(x\to 0, y\to 0)} \varepsilon(x,y) = 0 \implies f(x,y) \text{ е диференцируема в } (0,0)$$

Задача 4.0.4. $f(x,y)=x^2+3xy-8y^3+11, \quad df(0,1)=?$
$f(x,y,z)=x^2+3xy-8y^3-2e^{3z}x, \quad df(0,0,4)=?$
Решение:

$$df(x,y)=f'_x(x,y)dx+f'_y(x,y)dy$$
$$f'_x(x,y)=2x+3y \qquad f'_x(0,1)=3$$
$$f'_y(x,y)=3x-24y^2 \qquad f'_y(0,1)=-24$$
$$df(x,y)=(2x+3y)dx+(3x-24y^2)dy$$
$$df(0,1)=3dx-24dy$$

$$df(x,y,z)=f'_x(x,y,z)dx+f'_y(x,y,z)dy+f'_z(x,y,z)dz$$
$$f'_x(x,y,z)=2x+3y-2e^{3z} \qquad f'_x(0,0,4)=-2e^{12}$$
$$f'_y(x,y,z)=3x-24y^2 \qquad f'_y(0,0,4)=0$$
$$f'_z(x,y,z)=6xe^{3z} \qquad f'_z(0,0,4)=0$$
$$df(x,y,z)=(2x+3y-2e^{3z})dx+(3x-24y^2)dy+(6xe^{3z})dz$$
$$df(x,y,z)=-2e^{12}dx+0dy+0dz=-2e^{12}dx$$

Задача 4.0.5. $f(x,y) = x^6 - 7xy^2 + 14y$,
$f''_{xx} =?, f''_{yy} =?, f''_{xy} =?, d^2f(x,y) =?$
$f(x,y,z) = x^6 - 7xy + y^2 - xz + z^3$,
$f''_{xx} =?, f''_{xy} =?, f''_{xz} =? f''_{yx} =?, f''_{yy} =?, f''_{yz} =? f''_{zx} =?, f''_{zy} =?, f''_{zz} =?, d^2f(1,0,0)$

$$f'_x(x,y,z) = 6x^5 - 7y - z$$
$$f''_{xx}(x,y,z) = (6x^5 - 7y - z)'_x = 30x^4 \qquad f''_{xx}(1,0,0) = 30$$
$$f''_{xy}(x,y,z) = (6x^5 - 7y - z)'_y = -7 \qquad f''_{xy}(1,0,0) = -7$$
$$f''_{xz}(x,y,z) = (6x^5 - 7y - z)'_z = -1 \qquad f''_{xz}(1,0,0) = -1$$

$$f'_y(x,y,z) = -7x + 2y$$
$$f''_{yx}(x,y,z) = (-7x + 2y)'_x = -7 \qquad f''_{yx}(1,0,0) = -7$$
$$f''_{yy}(x,y,z) = (-7x + 2y)'_y = 2 \qquad f''_{yy}(1,0,0) = 2$$
$$f''_{yz}(x,y,z) = (-7x + 2y)'_z = 0 \qquad f''_{yz}(1,0,0) = 0$$

$$f'_z(x,y,z) = -x + 3z^2$$
$$f''_{zx}(x,y,z) = (-x + 3z^2)'_x = -1 \qquad f''_{zx}(1,0,0) = -1$$
$$f''_{zy}(x,y,z) = (-x + 3z^2)'_y = 0 \qquad f''_{zy}(1,0,0) = 0$$
$$f''_{zz}(x,y,z) = (-x + 3z^2)'_z = 6z \qquad f''_{zz}(1,0,0) = 0$$

$$d^2f = f''_{xx}dx^2 + 2f''_{xy}dxdy + f''_{yy}dy^2 + 2f''_{xz}dxdz + f''_{zz}dz^2 + f''_{yz}dydz$$
$$d^2f(x,y,z) = 30x^4dx^2 + 2\cdot(-7)dxdy + 2dy^2 + 2\cdot(-1)dxdz + 6zdz^2 + 2\cdot 0dydz$$
$$d^2f(1,0,0) = 30dx^2 - 14dxdy + 2dy^2 - 2dxdz + 0dz^2 + 0dydz$$
$$d^2f(1,0,0) = 30dx^2 + 2dy^2 - 14dxdy - 2dxdz$$

5 Упражнение към лекция 5

Задача 5.0.1. Да се намерят посочените частни производни на следните функции.

1. $u(x,y) = x^4 + 11x^2y^3, \qquad u''_{xx} =?,\ u''_{xy} =?$
2. $u(x,y) = \arctan\dfrac{x+y}{1-xy}, \qquad u''_{xx} =?,\ u''_{xy} =?,\ u''_{yy} =?$
3. $u(x,y) = \dfrac{1}{2}\ln(x^2+y^2), \qquad u''_{xx} =?,\ u''_{xy} =?,\ u''_{yx} =?,\ u''_{yy} =?$
4. $u(x,y) = \ln(x+2y), \qquad u'''_{xxy} =?$
5. $u(x,y,z) = e^{xy^2z^3}, \qquad u'''_{xyz} =?$

$$
\begin{aligned}
&u(x,y) = x^4 + 11x^2y^3\\
&u'_x = 4x^3 + 22xy^3\\
&u''_{xx} = 12x^2 + 22y^3\\
&u''_{xy} = 4x^3 + 66xy^2
\end{aligned}
$$

$$
\begin{aligned}
&u(x,y) = \arctan\frac{x+y}{1-xy}\\
&u'_x = \frac{1}{1+\left(\dfrac{x+y}{1-xy}\right)^2}\cdot\left(\frac{x+y}{1-xy}\right)'_x\\
&u'_y = \frac{1}{1+\left(\dfrac{x+y}{1-xy}\right)^2}\cdot\left(\frac{x+y}{1-xy}\right)'_y\\
&u''_{xx} = (u'_x)'_x\\
&u''_{xy} = (u'_x)'_y\\
&u''_{yy} = (u'_y)'_y
\end{aligned}
$$

$$A = \frac{1}{1+\left(\dfrac{x+y}{1-xy}\right)^2}. \quad B = \left(\frac{x+y}{1-xy}\right)'_x \implies u'_x = AB$$

$$A = \frac{1}{1+\left(\dfrac{x+y}{1-xy}\right)^2} = \frac{1}{1+\dfrac{(x+y)^2}{(1-xy)^2}} = \frac{(1-xy)^2}{(1-xy)^2+(x+y)^2}$$

$$A = \frac{(1-xy)^2}{1-2xy+x^2y^2+x^2+2xy+y^2} = \frac{(1-xy)^2}{1+x^2y^2+x^2+y^2}$$

$$A = \frac{(1-xy)^2}{(1+y^2)+x^2+x^2y^2} = \frac{(1-xy)^2}{(1+y^2)+x^2(1+y^2)} = \frac{(1-xy)^2}{(1+y^2)(1+x^2)}$$

$$B = \left(\frac{x+y}{1-xy}\right)'_x = \frac{1(1-xy)-(x+y)(-y)}{(1-xy)^2} = \frac{1-xy+xy+y^2}{(1-xy)^2} = \frac{1+y^2}{(1-xy)^2}$$

$$u'_x = AB = \frac{(1-xy)^2}{(1+y^2)(1+x^2)} \cdot \frac{1-y^2}{(1-xy)^2} = \frac{1}{1+x^2}$$

$$C = \left(\frac{x+y}{1-xy}\right)'_y \implies u'_y = AC$$

$$C = \frac{1(1-xy)-(x+y)(-x)}{(1-xy)^2} = \frac{1-xy+x^2+xy}{(1-xy)^2} = \frac{1+x^2}{(1-xy)^2}$$

$$u'_y = AC = \frac{(1-xy)^2}{(1+y^2)(1+x^2)} \cdot \frac{1+x^2}{(1-xy)^2} = \frac{1}{1+y^2}$$

$$u''_{xx} = \left(\frac{1}{1+x^2}\right)'_x = ((1+x^2)^{-1})'_x$$

$$u''_{yy} = -(1+x^2)^{-2}(1+x^2)'_x = -2x(1+x^2)^{-2} = \frac{-2x}{(1+x^2)^2}$$

$$u''_{xy} = \left(\frac{1}{1+x^2}\right)'_y = 0$$

$$u''_{yy} = \left(\frac{1}{1+y^2}\right)'_y = ((1+y^2)^{-1})'_y$$

$$u''_{yy} = -(1+y^2)^{-2}(1+y^2)'_y = -2y(1+y^2)^{-2} = \frac{-2y}{(1+y^2)^2}$$

$$u(x,y) = \frac{1}{2}\ln(x^2+y^2)$$
$$u'_x = \frac{1}{2(x^2+y^2)}\cdot(x^2+y^2)'_x = \frac{2x}{2(x^2+y^2)} = \frac{x}{x^2+y^2}$$
$$u'_y = \frac{1}{2(x^2+y^2)}\cdot(x^2+y^2)'_y = \frac{2y}{2(x^2+y^2)} = \frac{y}{x^2+y^2}$$
$$u''_{xx} = (u'_x)'_x = \left(\frac{x}{x^2+y^2}\right)'_x = \frac{1(x^2+y^2)-(2x)x}{(x^2+y^2)^2} = \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$
$$u''_{xy} = (u'_x)'_y = \left(\frac{x}{x^2+y^2}\right)'_y = \frac{-2xy}{(x^2+y^2)^2}$$
$$u''_{yy} = (u'_y)'_y = \left(\frac{y}{x^2+y^2}\right)'_y = \frac{1(x^2+y^2)-(2y)y}{(x^2+y^2)^2} = \frac{x^2+y^2-2y^2}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$$
$$u''_{yx} = (u'_y)'_x = \left(\frac{y}{x^2+y^2}\right)'_x = \frac{-2xy}{(x^2+y^2)^2}$$

$$u(x,y) = \ln(x+2y)$$
$$u'_x = \frac{1}{x+2y}$$
$$u''_{xx} = \left(\frac{1}{x+2y}\right)'_x = ((x+2y)^{-1})'_x = -(x+2y)^{-2}(x+2y)'_x = -\frac{1}{(x+2y)^2}$$
$$u'''_{xxy} = \left(-\frac{1}{(x+2y)^2}\right)'_y = -((x+2y)^{-2})'_y = 2((x+2y)^{-3})(x+2y)'_y = \frac{4}{(x+2y)^3}$$

$$u(x,y,z) = e^{xy^2z^3}$$
$$u'_x = e^{xy^2z^3}(xy^2z^3)'_x = y^2z^3e^{xy^2z^3}$$
$$u''_{xy} = (y^2z^3\cdot e^{xy^2z^3})'_y = (y^2z^3)'_y\cdot e^{xy^2z^3} + y^2z^3(e^{xy^2z^3})'_y$$
$$u''_{xy} = 2yz^3e^{xy^2z^3} + 2xy^3z^6e^{xy^2z^3} = 2yz^3e^{xy^2z^3}(1+xy^2z^3)$$

$$u'''_{xyz} = \left[2yz^3 e^{xy^2z^3}(1+xy^2z^3)\right]'_z = (2yz^3 e^{xy^2z^3})'_z(1+xy^2z^3) + 2yz^3 e^{xy^2z^3}(1+xy^2z^3)'_z$$
$$= \left[(2yz^3)'_z \cdot e^{xy^2z^3} + 2yz^3 \cdot (e^{xy^2z^3})'_z\right](1+xy^2z^3) + 2yz^3 e^{xy^2z^3}(1+xy^2z^3)'_z$$
$$u'''_{xyz} = \left[6yz^2 e^{xy^2z^3} + 2yz^3 e^{xy^2z^3} 3xy^2z^2\right](1+xy^2z^3) + 2yz^3 e^{xy^2z^3}(3xy^2z^2)$$
$$u'''_{xyz} = \left[6yz^2 e^{xy^2z^3} + 6xy^3z^5 e^{xy^2z^3}\right](1+xy^2z^3) + 6xy^3z^5 e^{xy^2z^3}$$
$$u'''_{xyz} = \left[6yz^2 e^{xy^2z^3} + 6yz^2 e^{xy^2z^3} xy^2z^3 + 6xy^3z^5 e^{xy^2z^3} + 6xy^3z^5 e^{xy^2z^3} xy^2z^3\right] + 6xy^3z^5 e^{xy^2z^3}$$
$$u'''_{xyz} = \left[6yz^2 e^{xy^2z^3} + 6xy^3z^5 e^{xy^2z^3} + 6xy^3z^5 e^{xy^2z^3} + 6x^2y^5z^8 e^{xy^2z^3}\right] + 6xy^3z^5 e^{xy^2z^3}$$
$$u'''_{xyz} = 6yz^2 e^{xy^2z^3} + 6xy^3z^5 e^{xy^2z^3} + 6xy^3z^5 e^{xy^2z^3} + 6x^2y^5z^8 e^{xy^2z^3} + 6xy^3z^5 e^{xy^2z^3}$$
$$u'''_{xyz} = 6yz^2 e^{xy^2z^3} + 18xy^3z^5 e^{xy^2z^3} + 6x^2y^5z^8 e^{xy^2z^3}$$
$$u'''_{xyz} = 6yz^2 e^{xy^2z^3}\left[1 + 3xy^2z^3 + x^2y^4z^6\right]$$

Задача 5.0.2. Дали са верни равенствата:

- Ако $z = y\ln(x^2+y^2)$ то $\quad \dfrac{1}{x}z'_x + \dfrac{1}{y}z'_y = \dfrac{z}{y^2}$
- Ако $u = \ln(x^3+y^3+z^3-3xyz)$ то $\quad u'_x + u'_y + u'_z = \dfrac{3}{x+y+z}$

$$z = y\ln(x^2+y^2)$$
$$z'_x = y\frac{1}{x^2+y^2}2x = \frac{2xy}{x^2+y^2}$$
$$z'_y = \ln(x^2+y^2) + y\frac{1}{x^2+y^2} - 2y = \ln(x^2+y^2) - \frac{2y^2}{x^2+y^2}$$
$$\frac{1}{x}z'_x + \frac{1}{y}z'_y = \frac{1}{x}\cdot\frac{2xy}{x^2+y^2} + \frac{1}{y}\cdot\left[\ln(x^2+y^2) - \frac{2y^2}{x^2+y^2}\right] =$$
$$\frac{2y}{x^2+y^2} + \frac{\ln(x^2+y^2)}{y} - \frac{2y}{x^2+y^2} = \frac{\ln(x^2+y^2)}{y}$$
$$\frac{z}{y^2} = \frac{y\ln(x^2+y^2)}{y^2} = \frac{\ln(x^2+y^2)}{y} \implies \text{Равенството е вярно.}$$

$$u = \ln\left(x^3+y^3+z^3-3xyz\right)$$
$$u'_x = \frac{(x^3+y^3+z^3-3xyz)'_x}{x^3+y^3+z^3-3xyz} = \frac{3x^2-3yz}{x^3+y^3+z^3-3xyz}$$
$$u'_y = \frac{(x^3+y^3+z^3-3xyz)'_y}{x^3+y^3+z^3-3xyz} = \frac{3y^2-3xz}{x^3+y^3+z^3-3xyz}$$
$$u'_z = \frac{(x^3+y^3+z^3-3xyz)'_z}{x^3+y^3+z^3-3xyz} = \frac{3z^2-3xy}{x^3+y^3+z^3-3xyz}$$
$$u'_x+u'_y+u'_z = \frac{3x^2-3yz}{x^3+y^3+z^3-3xyz}+\frac{3y^2-3xz}{x^3+y^3+z^3-3xyz}+\frac{3z^2-3xy}{x^3+y^3+z^3-3xyz} =$$
$$\frac{3x^2-3yz+3y^2-3xz+3z^2-3xy}{x^3+y^3+z^3-3xyz} = \frac{3(x^2-yz+y^2-xz+z^2-xy)}{x^3+y^3+z^3-3xyz} =$$
$$\frac{3(x^2+y^2+z^2-xy-xz-yz)}{x^3+y^3+z^3-3xyz}\cdot\frac{x+y+z}{x+y+z} = \frac{3(x^3+y^3+z^3-3xyz)}{(x^3+y^3+z^3-3xyz)(x+y+z)} =$$
$$\frac{3}{x+y+z} \implies \text{Равенството е вярно.}$$

Задача 5.0.3. Да се докаже, че функцията: $z(x,y) = \arctan\left(\dfrac{x+y}{x-y}\right)$ удовлетворява тъждеството: $z'_x + z'_y = \dfrac{x-y}{x^2+y^2}$

$$z'_x = \frac{1}{1+\left(\dfrac{x+y}{x-y}\right)^2}\cdot\left(\frac{x+y}{x-y}\right)'_x = \frac{1}{\dfrac{(x-y)^2+(x+y)^2}{(x-y)^2}}\cdot\frac{x-y-x-y}{(x-y)^2}$$
$$z'_x = \frac{(x-y)^2}{(x-y)^2+(x+y)^2}\cdot\frac{-2y}{(x-y)^2}\frac{-2y}{x^2-2xy+y^2+x^2+2xy+y^2}$$
$$z'_x = \frac{-2y}{2(x^2+y^2)} = -\frac{y}{x^2+y^2}$$
$$z'_y = \frac{1}{1+\left(\dfrac{x+y}{x-y}\right)^2}\cdot\left(\frac{x+y}{x-y}\right)'_y\frac{1}{\dfrac{(x-y)^2+(x+y)^2}{(x-y)^2}}\cdot\frac{x-y+x+y}{(x-y)^2} =$$
$$z'_y = \frac{(x-y)^2}{(x-y)^2+(x+y)^2}\cdot\frac{2x}{(x-y)^2} = \frac{2x}{x^2-2xy+y^2+x^2+2xy+y^2} = \frac{x}{x^2+y^2}$$
$$z'_x+z'_y = -\frac{y}{x^2+y^2}+\frac{x}{x^2+y^2} = \frac{x-y}{x^2+y^2} \implies \text{тъжеството е вярно}$$

Задача 5.0.4. Да се провери тъждеството на Ойлер за следните функции: $z(x,y) = \dfrac{1}{(x^2+y^2)^2}$ $\qquad u(x,y,z) = \sqrt{x^2+y^2+z^2}\cdot\ln\left(\dfrac{y}{x}\right)$

Тъждество на Ойлер($f : D \to R, D \subset \mathbb{R}^m$)

$$x_1 f'_{x_1} + x_2 f'_{x_2} + ... + x_m f'_{x_m} = mf$$

$$z(x,y) = \frac{1}{(x^2+y^2)^2}$$

$$xz'_x + yz'_y = 2z$$

$$z'_x = \left(\frac{1}{(x^2+y^2)^2}\right)'_x = \left((x^2+y^2)^{-2}\right)'_x = -2(x^2+y^2)^{-3}(x^2+y^2)'_x = -\frac{4x}{(x^2+y^2)^3}$$

$$z'_y = \left(\frac{1}{(x^2+y^2)^2}\right)'_y = \left((x^2+y^2)^{-2}\right)'_y = -2(x^2+y^2)^{-3}(x^2+y^2)'_x = -\frac{4y}{(x^2+y^2)^3}$$

$$xz'_x + yz'_y = x \cdot \left(-\frac{4x}{(x^2+y^2)^3}\right) + y \cdot \left(-\frac{4y}{(x^2+y^2)^3}\right) = -\frac{4x^2}{(x^2+y^2)^3} - \frac{4y^2}{(x^2+y^2)^3} =$$

$$\frac{-4(x^2+y^2)}{(x^2+y^2)^3} = -\frac{4}{(x^2+y^2)^2}$$

$$2z = \frac{2}{(x^2+y^2)^2}$$

$-\frac{4}{(x^2+y^2)^2} \neq \frac{2}{(x^2+y^2)^2} \implies$ Тъждението не е изпълнено.

$$u(x,y,z) = \sqrt{x^2+y^2+z^2} \cdot \ln\left(\frac{y}{x}\right)$$

$$xu'_x + yu'_y + zu'_z = 3z$$

$$u'_x = \left(\sqrt{x^2+y^2+z^2}\right)'_x \ln\left(\frac{y}{x}\right) + \sqrt{x^2+y^2+z^2}\left(\ln\left(\frac{y}{x}\right)\right)'_x$$

$$u'_y = \left(\sqrt{x^2+y^2+z^2}\right)'_y \ln\left(\frac{y}{x}\right) + \sqrt{x^2+y^2+z^2}\left(\ln\left(\frac{y}{x}\right)\right)'_y$$

$$u'_z = \left(\sqrt{x^2+y^2+z^2}\right)'_z \ln\left(\frac{y}{x}\right) + \sqrt{x^2+y^2+z^2}\left(\ln\left(\frac{y}{x}\right)\right)'_z$$

$$u'_x = \left(\sqrt{x^2+y^2+z^2}\right)'_x \ln\left(\frac{y}{x}\right) + \sqrt{x^2+y^2+z^2}\left(\ln\left(\frac{y}{x}\right)\right)'_x$$

$$u'_x = \frac{x\ln\left(\frac{y}{x}\right)}{\sqrt{x^2+y^2+z^2}} - \frac{\sqrt{x^2+y^2+z^2}}{x} = \frac{x\ln\left(\frac{y}{x}\right)x - \left(\sqrt{x^2+y^2+z^2}\right)^2}{x\sqrt{x^2+y^2+z^2}}$$

$$u'_x = \frac{x^2\ln\left(\frac{y}{x}\right) - x^2 - y^2 - z^2}{x\sqrt{x^2+y^2+z^2}}$$

$$u'_y = \left(\sqrt{x^2+y^2+z^2}\right)'_y \ln\left(\frac{y}{x}\right) + \sqrt{x^2+y^2+z^2}\left(\ln\left(\frac{y}{x}\right)\right)'_y$$

$$u'_y = \frac{y\ln\left(\frac{y}{x}\right)}{\sqrt{x^2+y^2+z^2}} + \frac{\sqrt{x^2+y^2+z^2}}{y} = \frac{y\ln\left(\frac{y}{x}\right)y + \left(\sqrt{x^2+y^2+z^2}\right)^2}{y\sqrt{x^2+y^2+z^2}}$$

$$u'_y = \frac{y^2\ln\left(\frac{y}{x}\right) + x^2+y^2+z^2}{y\sqrt{x^2+y^2+z^2}}$$

$$u'_z = \left(\sqrt{x^2+y^2+z^2}\right)'_z \ln\left(\frac{y}{x}\right) + \sqrt{x^2+y^2+z^2}\left(\ln\left(\frac{y}{x}\right)\right)'_z$$

$$u'_z = \frac{z\ln\left(\frac{y}{x}\right)}{\sqrt{x^2+y^2+z^2}} + 0\cdot\sqrt{x^2+y^2+z^2} = \frac{z\ln\left(\frac{y}{x}\right)}{\sqrt{x^2+y^2+z^2}}$$

$$xu'_x + yu'_y + zu'_z = 3z, \quad A = xu'_x + yu'_y + zu'_z, \quad B = 3u$$

$$A = x\cdot\frac{x^2\ln\left(\frac{y}{x}\right) - x^2 - y^2 - z^2}{x\sqrt{x^2+y^2+z^2}} + y\cdot\frac{y^2\ln\left(\frac{y}{x}\right) + x^2+y^2+z^2}{y\sqrt{x^2+y^2+z^2}} + z\cdot\frac{z\ln\left(\frac{y}{x}\right)}{\sqrt{x^2+y^2+z^2}}$$

$$A = \frac{x^2\ln\left(\frac{y}{x}\right) - x^2 - y^2 - z^2}{\sqrt{x^2+y^2+z^2}} + \frac{y^2\ln\left(\frac{y}{x}\right) + x^2+y^2+z^2}{\sqrt{x^2+y^2+z^2}} + \frac{z^2\ln\left(\frac{y}{x}\right)}{\sqrt{x^2+y^2+z^2}}$$

$$A = \frac{x^2\ln\left(\frac{y}{x}\right) - x^2 - y^2 - z^2 + y^2\ln\left(\frac{y}{x}\right) + x^2+y^2+z^2 + z^2\ln\left(\frac{y}{x}\right)}{\sqrt{x^2+y^2+z^2}}$$

$$A = \frac{x^2\ln\left(\frac{y}{x}\right)\ln\left(\frac{y}{x}\right) + z^2\ln\left(\frac{y}{x}\right)}{\sqrt{x^2+y^2+z^2}} = \frac{\ln\left(\frac{y}{x}\right)(x^2+y^2+z^2)}{\sqrt{x^2+y^2+z^2}} = \sqrt{x^2+y^2+z^2}\cdot\ln\left(\frac{y}{x}\right)$$

$B = 3u = 3\sqrt{x^2+y^2+z^2}\cdot\ln\left(\frac{y}{x}\right) \implies A \neq B \implies$ Твърдението не е изпълнено.

6 Упражнение към лекция 6

Задача 6.0.1. Дадени са функцията $z(x, y) = \varphi(x+y) + \psi(x-y)$, където φ, ψ - непрекъснато диференцируеми Да се намерят първите частни производни.

$$
\begin{aligned}
&z(x, y) = \varphi(x+y) + \psi(x-y)\\
&z'_x = \varphi'(x+y)(x+y)'_x + \psi'(x-y)(x-y)'_x = \varphi'(x+y)1 + \psi'(x-y)1\\
&z'_x = \varphi'(x+y) + \psi'(x-y)\\
&z'_y = \varphi'(x+y)(x+y)'_y + \psi'(x-y)(x-y)'_y = \varphi'(x+y)1 + \psi'(x-y)(-1)\\
&z'_y = \varphi'(x+y) - \psi'(x-y)
\end{aligned}
$$

Задача 6.0.2. Да се провери дали $w(x, y, z)$ удволетворява тъждествено равенството:

$$xw_x + yw_y + zw_z = w + \frac{xy}{z}$$

Ако $w = \dfrac{xy}{z} + \ln x + x \cdot \varphi\left(\dfrac{y}{x}, \dfrac{z}{x}\right)$, φ е непрекъснато диференцируема.
Решение:

$$
\begin{aligned}
&u = \frac{y}{x} \qquad v = \frac{z}{x}\\
&u'_x = -\frac{y}{x^2} \qquad u'_y = \frac{1}{x} \qquad u'_z = 0\\
&v'_x = -\frac{z}{x^2} \qquad v'_y = 0 \qquad v'_z = \frac{1}{x}\\
&w'_x = \frac{y}{z}\ln x + \frac{xy}{z}\cdot\frac{1}{x} + \varphi\left(\frac{y}{x}, \frac{z}{x}\right) + x(\varphi'_u u'_x + \varphi'_v v_x) =\\
&w'_x = \frac{y}{z}\ln x + \frac{y}{z} + \varphi\left(\frac{y}{x}, \frac{z}{x}\right) - \frac{y}{x}\varphi'_u - \frac{z}{x}\varphi'_v\\
&w'_y = \frac{x}{z}\ln x + x(\varphi'_u u'_y + \varphi'_v v_y) = \frac{x}{z}\ln x + \varphi'_u\\
&w'_z = -\frac{xy}{z^2}\ln x + x(\varphi'_u u'_z + \varphi'_v v_z) = -\frac{xy}{z^2}\ln x + \varphi'_v\\
&xw_x + yw_y + zw_z =\\
&= \frac{xy}{z}\ln x + \frac{xy}{z} + x\varphi\left(\frac{y}{x}, \frac{z}{x}\right) - y\varphi'_u - z\varphi'_v + \frac{xy}{z}\ln x + y\varphi'_u + -\frac{xy}{z}\ln x + z\varphi'_v =\\
&= \frac{xy}{z} + \ln x + x\cdot\varphi\left(\frac{y}{x}, \frac{z}{x}\right) + \frac{xy}{z} = w + \frac{xy}{z}
\end{aligned}
$$

Задача 6.0.3. Дадени са функциите и точката M(2,1). Да се пресметне gradf(M) и $\|gradf(M)\|$

1. $f(x,y) = x^2 + 11y^2 - 3$
2. $f(x,y) = x^2 - y^2$
3. $f(x,y) = \ln(x^2 + y^2)$

Решение grad f $= (f'_x, f'_y)$

$$
\begin{aligned}
&f(x,y) = x^2 + 11y^2 - 3\\
&f'_x = 2x \qquad f'_y = 22y\\
&gradf(x,y) = (2x, 22y)\\
&gradf(M) = (2 \cdot 2, 22 \cdot 1) = (4, 22)\\
&\|gradf(M)\| = \sqrt{4^2 + 22^2} = \sqrt{500} = 10\sqrt{5}
\end{aligned}
$$

$$
\begin{aligned}
&f(x,y) = x^2 - y^2\\
&f'_x = 2x \qquad f'_y = -2y\\
&gradf(x,y) = (2x, -2y)\\
&gradf(M) = (2 \cdot 2, -2 \cdot 1) = (4, -2)\\
&\|gradf(M)\| = \sqrt{4^2 + (-2)^2} = \sqrt{20} = 2\sqrt{5}
\end{aligned}
$$

$$
\begin{aligned}
&f(x,y) = \ln(x^2 + y^2)\\
&f'_x = \frac{2x}{x^2 + y^2} \qquad f'_y = \frac{2y}{x^2 + y^2}\\
&gradf(x,y) = \left(\frac{2x}{x^2 + y^2}, \frac{2y}{x^2 + y^2}\right)\\
&gradf(M) = \left(\frac{2 \cdot 2}{2^2 + 1^2}, \frac{2 \cdot 1}{2^2 + 1^2}\right) = \left(\frac{4}{5}, \frac{2}{5}\right)\\
&\|gradf(M)\| = \sqrt{\left(\frac{4}{5}\right)^2 + \left(\frac{2}{5}\right)^2} = \sqrt{\frac{20}{25}} = \frac{2}{\sqrt{5}}
\end{aligned}
$$

Задача 6.0.4. Дадени са функциите и точката M(2,1).
Да се пресметне $\dfrac{\partial f(M)}{\partial \nu}, \nu = \left(\dfrac{\sqrt{3}}{2}, \dfrac{1}{2}\right)$

1. $f(x,y) = x^2 + 11y^2 - 3$

2. $f(x, y) = x^2 - y^2$

3. $f(x, y) = \ln(x^2 + y^2)$

Решение: $\dfrac{\partial f(M)}{\partial \nu} = (gradf, \nu)$

$$f(x, y) = x^2 + 11y^2 - 3$$
$$gradf(M) = (2 \cdot 2, 22 \cdot 1) = (4, 22)$$
$$\frac{\partial f(M)}{\partial \nu} = 4 \cdot \frac{\sqrt{3}}{2} + 22 \cdot \frac{1}{2} = 2\sqrt{3} + 11$$

$$f(x, y) = x^2 - y^2$$
$$gradf(M) = (2 \cdot 2, -2 \cdot 1) = (4, -2)$$
$$\frac{\partial f(M)}{\partial \nu} = 4 \cdot \frac{\sqrt{3}}{2} - 2 \cdot \frac{1}{2} = 2\sqrt{3} - 1$$

$$f(x, y) = \ln(x^2 + y^2)$$
$$gradf(M) = \left(\frac{2 \cdot 2}{2^2 + 1^2}, \frac{2 \cdot 1}{2^2 + 1^2}\right) = \left(\frac{4}{5}, \frac{2}{5}\right)$$
$$\frac{\partial f(M)}{\partial \nu} = \frac{4}{5} \cdot \frac{\sqrt{3}}{2} + \frac{2}{5} \cdot \frac{1}{2} = \frac{4\sqrt{3}}{10} + \frac{1}{5} = \frac{4\sqrt{3} + 2}{10}$$

Задача 6.0.5. Да се определи ъгъла между градиентите на функцията

$$u = x^2 + y^2 + z^2 - 111$$

в точките A$(\varepsilon, 0, 0)$ и B $(0, \varepsilon, 0), \varepsilon > 0$
Решение:

$$u'_x = 2x \qquad u'_y = 2y \qquad u'_z = 2z$$
$$gradu(A) = (2\varepsilon, 0, 0) \qquad gradu(B) = (0, 2\varepsilon, 0)$$
$$(gradu(A), gradu(B)) = 2\varepsilon \cdot 0 + 0 \cdot 2\varepsilon + 0 \cdot 0 = 0$$
$$(gradu(A), gradu(B)) = \|u(A)\| \cdot \|u(B)\| \cdot \cos\alpha$$
$$\cos\alpha = 0 \Leftrightarrow \alpha = \frac{\pi}{2}$$

Задача 6.0.6. Да се намери y', y'' на неявната функция $y = f(x)$, дефинирана от уравнението

$$x^2 - 2xy + 5y^2 + 4y = 2x + 9$$

Да се пресметнат $y'(0), y''(0)$, ако $y(0)=1$
Решение:

$$F(x,y)=x^2-2xy+5y^2+4y=2x+9$$
$$F'_y=-2x+10y+4\neq 0$$
$$F'_x(x,y)=2x-2y-2$$
$$F'_y(0,1)=-2\cdot 0+10\cdot 1+4\neq 0$$
$$y'(x)=-\frac{F'_x(x,y)}{F'_y(x,y)}=-\frac{2x-2y-2}{-2x+10y+4}=-\frac{x-y-1}{-x+5y+2}$$
$$y'(0)=-\frac{0-1-1}{-0+5\cdot 1+2}=-\frac{-2}{7}=\frac{2}{7}$$
$$y''(x)=-\frac{F''_{xx}(x,y)+2F''_{xy}y'+F''_{yy}(x,y)y'^2}{F'_y(x,y)}$$
$$F''_{xx}=2,\quad F''_{yy}=10,\quad F''_{xy}=-2$$
$$F''_{xx}(0,1)=2,\quad F''_{yy}(0,1)=10,\quad F''_{xy}(0,1)=-2$$
$$y''(x)=-\frac{2+2\cdot(-2)y'+10y'^2}{-2x+10y+4}$$
$$y''(x)=-\frac{2+-4y'+10y'^2}{-2x+10y+4}$$
$$y''(0)=-\frac{2+-4\cdot\frac{2}{7}+10\cdot\left(\frac{2}{7}\right)^2}{-2\cdot 0+10\cdot 1+4}$$
$$y''(0)=-\frac{2+-\frac{8}{7}+\frac{40}{49}}{14}$$
$$y''(0)=-\frac{\frac{98-56+40}{49}}{14}=-\frac{\frac{82}{49}}{14}=\frac{82}{49}\cdot\frac{1}{14}=\frac{41}{343}$$

7 Упражнение към лекция 7

Задача 7.0.1. Да се намерят локалните екстремуми на функциите

- $z = \sin x + \sin y + \sin(x+y) \quad (0 < x < \frac{\pi}{2}, 0 < y < \frac{\pi}{2})$
- $z = x^4 + y^4 - 4xy$

$$z = \sin x + \sin y + \sin(x+y) \quad (0 < x < \frac{\pi}{2}, 0 < y < \frac{\pi}{2})$$

$$z'_x = \cos x + \cos(x+y) \quad z'_y = \cos y + \cos(x+y)$$

$$\left| \begin{array}{l} \cos x + \cos(x+y) = 0 \\ \cos y + \cos(x+y) = 0 \end{array} \right. \Leftrightarrow \left| \begin{array}{l} 2\cos\frac{2x+y}{2}\cos\frac{y}{2} = 0 \\ 2\cos\frac{x+2y}{2}\cos\frac{x}{2} = 0 \end{array} \right. \Leftrightarrow \left| \begin{array}{l} \frac{2x+y}{2} = \frac{\pi}{2} \\ \frac{x+2y}{2} = \frac{\pi}{2} \end{array} \right.$$

$$\frac{y}{2} = \frac{\pi}{2}, \quad \frac{x}{2} = \frac{\pi}{2} \implies x = y = \pi \notin (0 < x, y < \frac{\pi}{2})$$

$$x_0 = y_0 = \frac{\pi}{3} \implies M_0\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$$

$$z''_{xx} = -\sin x - \sin(x+y) \quad z''_{yy} = -\sin y - \sin(x+y) \quad z''_{xy} = -\sin(x+y)$$

$$z''_{xx}(M_0) = -\frac{2\sqrt{3}}{2} = -\sqrt{3} = \Delta_1 \quad z''_{yy}(M_0) = -\frac{2\sqrt{3}}{2} = -\sqrt{3} \quad z''_{xy}(M_0) = -\frac{\sqrt{3}}{2}$$

$$\begin{pmatrix} z''_{xx}(M_0) & z''_{xy}(M_0) \\ z''_{yx}(M_0) & z''_{yy}(M_0) \end{pmatrix} = \begin{pmatrix} -\sqrt{3} & -\frac{2\sqrt{3}}{2} \\ -\frac{2\sqrt{3}}{2} & -\sqrt{3} \end{pmatrix}$$

$$\Delta_1 = -\sqrt{3} < 0 \qquad \Delta_2 = 3 - \frac{3}{4} > 0$$

$$\implies \exists \text{ локален максимум}, z_{max} = z(M_0) - \frac{3\sqrt{3}}{2}$$

$$z = x^4 + y^4 - 4xy$$

$$z'_x = 4x^3 - 4y \quad z'_y = 4y^3 - 4x$$

$$\left| \begin{array}{l} 4x^3 - 4y = 0 \\ 4y^3 - 4x = 0 \end{array} \right. \Leftrightarrow \left| \begin{array}{l} y = x^3 \\ x^9 - x = 0 \end{array} \right. \Leftrightarrow \left| \begin{array}{l} x(x^2-1)(x^2+1)(x^4+1) = 0 \\ y = x^3 \end{array} \right. \implies$$

$$M_0(0,0) \quad M_1(1,1) \quad M_2(-1,1)$$

$$z''_{xx} = 12x^2 \quad z''_{yy} = 12y^2 \quad z''_{xy} = -4$$

$$d^2 z = \begin{pmatrix} z''_{xx}(M_0) & z''_{xy}(M_0) \\ z''_{yx}(M_0) & z''_{yy}(M_0) \end{pmatrix} = \begin{pmatrix} 12x^2 & -4 \\ -4 & 12y^2 \end{pmatrix}$$

$$d^2z(M_0) = \begin{pmatrix} 0 & -4 \\ -4 & 0 \end{pmatrix} \implies \Delta = \begin{vmatrix} 0 & -4 \\ -4 & 0 \end{vmatrix} = -16 < 0 \implies \text{няма лок. екстремум в } M_0$$

$$d^2z(M_1) = \begin{pmatrix} 12 & -4 \\ -4 & 12 \end{pmatrix} \implies \Delta_1 = 12 > 0 \Delta_2 = \begin{vmatrix} 0 & -4 \\ -4 & 0 \end{vmatrix} 144 - 16 > 0 \implies$$

z има локален минимум

$z_{min} = z(M_1) = 1^4 + 1^4 - 4 \cdot 1 \cdot 1 = -2$

Аналогично и за M_2 има лок. мин $z_{min} = -2$

Задача 7.0.2. Да се намерят локалните екстремуми на функциите

- $u = x^2 + y^2 + z^2 + 2x + 4y - 6z$
- $u = x^3 + y^2 + z^2 - 3x + 6y - 2z$
- $u = x^3 + y^2 + z^2 - 3x - 2y$

$u = x^2 + y^2 + z^2 + 2x + 4y - 6z$

$u'_x = 2x + 2 \quad u'_y = 2y + 4 \quad u'_z = 2z - 6$

$$\left| \begin{array}{l} x + 1 = 0 \\ y + 2 = 0 \implies M_0(-1, -2, 3) \\ z - 3 = 0 \end{array} \right.$$

$u''_{xx} = 2 \quad u''_{yy} = 2 \quad u''_{zz} = 2$

$u''_{xy} = u''_{xz} = u''_{yx} = u''_{yz} = u''_{zx} = u''_{zy} = 0$

$$d^2u(M_0) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$\Delta_1 = u''_{xx} = 2 > 0 \quad \Delta_2 = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4 > 0 \quad \Delta_3 = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 8 > 0 \implies$$

d^2u е положително дефинитна квадратична форма

u има лок. минимум $\quad u_{min} = u(M_0) = 1 + 4 + 9 - 2 - 4 \cdot 2 - 18 = -14$

$u = x^3 + y^2 + z^2 - 3x + 6y - 2z$

$u'_x = 3x^2 + 2 \quad u'_y = 2y + 6 \quad u'_z = 2z - 2$

$$\left| \begin{array}{l} 3x^2 + 2 = 0 \\ 2y + 6 = 0 \\ 2z - 2 = 0 \end{array} \right. \Longrightarrow M_0(1, -3, 1)\, M_1(-1, -3, 1)$$

$u''_{xx} = 6x \quad u''_{yy} = 2 \quad u''_{zz} = 2$

$u''_{xy} = u''_{xz} = u''_{yx} = u''_{yz} = u''_{zx} = u''_{zy} = 0$

$$d^2u(M_0) = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \Longrightarrow$$

$$\Delta_1 = 6 > 0 \quad \Delta_2 = \begin{vmatrix} 6 & 0 \\ 0 & 2 \end{vmatrix} = 12 > 0 \quad \Delta_3 = \begin{vmatrix} 6 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 24 > 0 \Longrightarrow$$

d^2u е положително дефинитна квадратична форма

u има лок. минимум $\quad u_{min} = u(M_0) = 1 + 9 + 1 - 3 - 18 - 2 = -12$

$$d^2u(M_1) = \begin{pmatrix} -6 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \Longrightarrow$$

$$\Delta_1 = -6 < 0 \quad \Delta_2 = \begin{vmatrix} -6 & 0 \\ 0 & 2 \end{vmatrix} = -12 < 0 \quad \Delta_3 = \begin{vmatrix} -6 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix} = -24 < 0 \Longrightarrow$$

d^2u е не е дефинитна квадратична форма $\Longrightarrow$ няма лок. екстремуми

$u = x^3 + y^2 + z^2 - 3x - 2y$

$u'_x = 3x^2 - 3 \quad u'_y = 2y - 2 \quad u'_z = 2z$

$$\left| \begin{array}{l} 3x^2 - 3 = 0 \\ 2y - 2 = 0 \\ 2z = 0 \end{array} \right. \implies M_0(1,1,0)\, M_1(-1,0,0)$$

$u''_{xx} = 6x \quad u''_{yy} = 2 \quad u''_{zz} = 2$

$u''_{xy} = u''_{xz} = u''_{yx} = u''_{yz} = u''_{zx} = u''_{zy} = 0$

$$d^2u(M_0) = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \implies$$

$$\Delta_1 = 6 > 0 \quad \Delta_2 = \begin{vmatrix} 6 & 0 \\ 0 & 2 \end{vmatrix} = 12 > 0 \quad \Delta_3 = \begin{vmatrix} 6 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 24 > 0 \implies$$

d^2u е положително дефинитна квадратична форма

u има лок. минимум $\quad u_{min} = u(M_0) = 1 + 1 - 3 - 2 = -3$

$$d^2u(M_1) = \begin{pmatrix} -6 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \implies$$

$$\Delta_1 = -6 < 0 \quad \Delta_2 = \begin{vmatrix} -6 & 0 \\ 0 & 2 \end{vmatrix} = -12 < 0 \quad \Delta_3 = \begin{vmatrix} -6 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix} = -24 < 0 \implies$$

d^2u е не е дефинитна квадратична форма $\implies$ няма лок. екстремуми

Задача 7.0.3. Да се намерят $y'(0), y''(0)$ ако $y(0) = 2$на неявната функция $y = f(x)$ дефинирана от уравнението

$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$

$$F(x,y) = \frac{x^2}{9} + \frac{y^2}{4} - 1 \quad M_0(0,2)$$

$$F'_y = \frac{2y}{4} = \frac{y}{2} \neq 0 y'(x) = -\frac{F'_x}{F'_y} \qquad y''(x) = -\frac{F"_{xx} + 2F''_{xy}y' + F''_{yy}(y')^2}{F'_y}$$

$$F'_x = \frac{2}{9}x \quad F'_y = \frac{2y}{4} \quad F''_{xx} = \frac{2}{9} \quad F''_{xy} = F''_{yx} = 0 \quad F''_{yy} = \frac{1}{2}$$

$$F'_x(0,2) = 0 \quad F'_y(0,2) = 1 \quad F''_{xx}(0,2) = \frac{2}{9} \quad F''_{xy}(0,2) = F''_{yx}(0,2) = 0 \quad F''_{yy}(0,2) = \frac{1}{2}$$

$$y'(0) = -\frac{0}{1} = 0 \qquad y''(0) = -\frac{\frac{2}{9} + 2 \cdot 0 \cdot 0 + \frac{1}{2} \cdot 0^2}{1} = -\frac{2}{9}$$

Задача 7.0.4. Да се покаже, че функцията $z = f(x,y)$ дефинирана неявно от уравнението

$$z = x\varphi(\frac{z}{y})$$

φ - непрекъснато диференцируема, удовелетворява тъждествено уравнението

$$xz'_x + yz'_y = z$$

Решение. Означаваме

$$F(x,y,z) = z - x\,\varphi(z/y), \qquad (3)$$

откъдето извеждаме условието за ∃! неявна функция:

$$F'_z = 1 - x\,\varphi'\left(\frac{z}{y}\right)\cdot\frac{1}{y} \neq 0. \qquad (4)$$

по-нататък:

$$F'_x = -\varphi(z/y)\,; \quad F'_y = -x\,\varphi'(z/y)\cdot\left(-\frac{z}{y^2}\right), \text{ т.е}$$

$$F'_x = -\varphi(z/y)\,; \quad F'_y = \frac{xz}{y^2}\,\varphi'(z/y) \Rightarrow$$

$$z'_x = -\frac{F'_x}{F'_z} = -\frac{-\varphi(z/y)}{1 - x\,\varphi'(z/y)\cdot\frac{1}{y}} = \frac{\varphi(z/y)}{1 - \frac{x}{y}\varphi'(z/y)}$$

$$z'_y = -\frac{F'_y}{F'_z} = -\frac{\frac{xz}{y^2}\varphi'(z/y)}{1 - \frac{x}{y}\varphi'(z/y)} \Rightarrow$$

$$xz'_x + yz'_y = \frac{\overset{z\,\backslash\backslash}{x\cdot\varphi(z/y)}}{1 - \frac{x}{y}\varphi'(z/y)} - \frac{\frac{xz}{y}\varphi'(z/y)}{1 - \frac{x}{y}\varphi'(z/y)} =$$

$$= z\cdot\frac{\cancel{1 - \frac{x}{y}\varphi'(z/y)}}{\cancel{1 - \frac{x}{y}\varphi'(z/y)}} = z.$$

8 Упражнение към лекция 8

Задача 8.0.1. Да се изследва за локален екстремум следната функция.

$$z = 1 - \sqrt{x^2 - y^2}$$

$$z(\Delta x, \Delta y) - z(0,0) = 1 - \sqrt{\Delta x^2 - \Delta y^2} - 1 = -\sqrt{\Delta x^2 - \Delta y^2} < 0$$

Имаме строг локален максимум в $z(0,0) = 1$

Задача 8.0.2. Намерете точките на условен екстремум и екстремумите на следните функции.

- $z = x^2 + y^2$, ако $x + y = 1$
- $u = x^2 + y^2 - 12x + 16y$, ако $x^2 + y^2 = 25$
- $u = x + y + z$, ако $z = 1$ и $x^2 + y^2 = 1$

Задача 8.0.3. Да се изследва функцията $u = xy + yz$ за условен екстремум, при ограничения.

$$x^2 + y^2 = 2$$

$$y + z = 2$$

Задача 8.0.4. Да се изследва функцията $z = x+y$ за условен екстремум, при ограничения.

$$xy = 1$$

Задача 8.0.5. Да се намери дефиниционното множество на функциите.

- $z = \sqrt{1 - x^2 - y^2 + 2x}$
- $z = \frac{x^2 y}{2x+y}$
- $z = \arcsin(x + y)$
- $w = \frac{1}{\sqrt{xy}}$

Задача 8.0.6. Да се намерят границите ако съществуват.

- $\lim\limits_{(x,y)\to(0,0)} \frac{\tan(xy)}{xy}$
- $\lim\limits_{(x,y)\to(0,0)} \frac{y}{\sin(xy)}$
- $\lim\limits_{(x,y)\to(0,0)} \frac{1-\sqrt{1-xy}}{xy}$

Задача 8.0.7. Да се провери дали уравнението удовлетворява посочената функция.

- $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial z}, z(x,y) = \ln(x^2+y^2+1)$
- $\frac{x}{y}\frac{\partial z}{\partial x} + \frac{1}{\ln x}\frac{\partial z}{\partial y} = 2z, z(x,y) = x^y$
- $2\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} = 0, z(x,y) = 2\cos^2(y - \frac{x}{2})$
- $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 1, u(x,y,z) = x + \frac{x-y}{y-z}$

Задача 8.0.8. Да се изследват за локален екстремум следните функции.

- $z = x^4 + y^4 - x^2 - 2xy - y^2$
- $z = xy(1-x-y)$
- $z = x^3 - y^3 - 3x + 3y + 2$
- $u = x^3 + y^3 + z^2 - 12xy + 2z$

Задача 8.0.9. Да се изследват за локален екстремум следните неявно зададени функции.

- $x^3 + y^3 = 3xy, y = y(x)$
- $y^2 - 3y - \sin(x) = 0, y = y(x)$
- $x^2 + y^2 + z^2 - xz - yz + 2x + 2y + 2z - 2 = 0, z = z(x,y)$
- $2x^2 + 2y^2 + z^2 + 8xz - 8yz + 8 = 0, z = z(x,y)$

Задача 8.0.10. Да се изследва за условен екстремум

- $z = xy$, ако $2x + y = 1$
- $z = x^2 + y^2$, ако $x - y = 1$
- $u = x^2 + y^2 + z^2$, ако $\frac{x^2}{16} + \frac{y^2}{9} + \frac{z^2}{4} = 1$
- $u = xyz$, ако $x + y + z = 5, xy + yz + zx = 8$

Задача 8.0.11. Намерете точките на условен екстремум и екстремумите на следните функции.

- $u = x^2 + y^2 + z^2 + 2x + 4y - 6z$, ако $x^2 + y^2 + z^2 = 14$
- $u = x^2 + y^2 + z^2 + 2x + 4y$, ако $x^2 + y^2 = 20$

- $u = x^2 + y^2 + z^2 + 6x - 2y + 4z$, ако $x^2 + y^2 + z^2 = 56$

Задача 8.0.12. Намерете абсолютните екстремуми на следните функции и определете вида им (условен, локален, минимум, максимум)

- $u = x^2 + y^2 - 12x + 16y$, ако $x^2 + y^2 \leq 25, x^2 + y^2 \leq 400, x^2 + y^2 \leq 100$
- $u = x^2 + y^2 + z^2 + 2x + 4y - 6$, ако $x^2 + y^2 + z^2 \leq 9$
- $u = x^2 + 2y^2 + 3z^2$, ако $x^2 + y^2 + z^2 \leq 100$