Математически анализ 2
Упражнения

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Съдържание

1 Упражнение към лекция 1

1.1 Задачи

Задача 1

Да се покаже дали посочените редици $\{X_n\} = \{x_n, y_n\}$ са сходящи или разходящи. За сходящите да се намери границите им.

1. $x_n = 1 + \dfrac{1}{n}$, $y_n = 2 + \dfrac{\sin n}{n}$
2. $x_n = \left(1 + \dfrac{1}{n}\right)^n$, $y_n = 2 + n$
3. $x_n = (-1)^n$, $y_n = n$
4. $x_n = (-1)^n$, $y_n = \dfrac{1}{n}$
5. $x_n = \sin \dfrac{n\pi}{2}$, $y_n = (-1)^n$
6. $x_n = \sin n$, $y_n = \dfrac{(-1)^n}{n}$

1.2 Решения

Задача 1

1. $\lim\limits_{n\to\infty}\frac{1}{n}=0$, $\frac{|\sin n|}{n}\in\left[0,\frac{1}{n}\right] \implies \lim\limits_{n\to\infty}x_n=1$, $\lim\limits_{n\to\infty}y_n=2 \implies$ редицата е сходяща; точката (1,2) е нейна граница
2. $\lim\limits_{n\to\infty}x_n=e$, $\lim\limits_{n\to\infty}y_n=\infty \implies$ разходяща редица
3. $\lim\limits_{n\to\infty}x_n$не съществува, защото има две точки на сгъстяване., $\lim\limits_{n\to\infty}y_n=\infty \implies$ разходяща редица
4. $\lim\limits_{n\to\infty}x_n$не съществува, защото има две точки на сгъстяване., $\lim\limits_{n\to\infty}y_n=0 \implies$ разходяща редица
5. $\lim\limits_{n\to\infty}x_n$не съществува, $\lim\limits_{n\to\infty}y_n=\infty \implies$ разходяща редица
6. $\lim\limits_{n\to\infty}x_n$не съществува, $\lim\limits_{n\to\infty}y_n=0 \implies$ разходяща редица

2 Упражнение към лекция 2

2.1 Задачи

Задача 1

Нека $D \subset \mathbb{R}^m$ и са разгледани няколко функции. Да се напишат дефиниционните им множества и да се даде пояснение.

1. $z(x, y) = x^2 + y^2$
2. $z(x, y) = \sqrt{y^2 - 2x}$
3. $z(x, y) = \ln \sqrt{y^2 - 2x}$
4. $z(x, y) = \dfrac{1}{\sqrt{-y^2 + 2x + 1}}$
5. $w(x, y, z) = \arccos(x^2 + y^2 + z^2)$
6. $f(n) = \begin{cases} 1, & x \in \mathbb{Q}^m \\ 0, & x \in \dfrac{\mathbb{R}^m}{\mathbb{Q}^m} \end{cases}$

Задача 2

Разгледаните по - долу функциите са дефинирани в $D = \mathbb{R}^2 \setminus \{(0, 0)\}$. Кои от границите съществуват и колко са

$$A = \lim_{(x,y)\to(0,0)} f(x, y) \quad A_{1,2} = \lim_{y\to 0}\left(\lim_{x\to 0} f(x, y)\right) \quad A_{2,1} = \lim_{x\to 0}\left(\lim_{y\to 0} f(x, y)\right)$$

1. $f(x, y) = \dfrac{x - y}{x + y}$
2. $f(x, y) = \dfrac{x^2 + y^2}{x^2y^2 + (x - y)^2}$
3. $f(x, y) = \dfrac{xy^2}{x^2 + y^4}$
4. $f(x, y) = (x + y)\sin\dfrac{1}{x}\cos\dfrac{1}{y}$
5. $f(x, y) = \dfrac{x^4 + y^4}{x^2 + y^2}$

Задача 3

Нека A, B, C, D са подмножества на $\mathbb{R}^2$ дефинирани както следва
$A = \{(x, y) : x \geq 0, y \leq 1, y > x\}$
$B = \{(x, y) : x \leq 1, y \geq 0, y < x\}$
$C = \{(x, y) : x = y, 0 \leq x \leq 1\}$
$D = A \cup B \cup C$
и функцията $f : D \to \mathbb{R}$ зададена по следния начин

$$f(x, y) = \begin{cases} \dfrac{1}{y^2}, & (x, y) \in A \\ 0, & x = y \\ -\dfrac{1}{x^2}, & (x, y) \in B \end{cases}$$

Да се изследва непрекъснатостта на тази функция.

2.2 Решения

Задача 1

1. $z(x,y) = x^2 + y^2$
   $D = \mathbb{R}^2$

2. $z(x,y) = \sqrt{y^2 - 2x}$
   $D = \{(x,y) : y^2 - 2x \geq 0\} \subset \mathbb{R}^2, x \leq \dfrac{y^2}{2}$

3. $z(x,y) = \ln \sqrt{y^2 - 2x}$
   $D = \{(x,y) : y^2 - 2x > 0\} \subset \mathbb{R}^2, x < \dfrac{y^2}{2}$

4. $z(x,y) = \dfrac{1}{\sqrt{-y^2 + 2x + 1}}$
   $D = \{(x,y) : -y^2 + 2x + 1 > 0\} \subset \mathbb{R}^2, x > \dfrac{y^2 - 1}{2}$

5. $w(x,y,z) = \arccos(x^2 + y^2 + z^2)$
   $D = \{(x,y,z) : x^2 + y^2 + z^2 \leq \pi\} \subset \mathbb{R}^3,$
   Графиката е кълбо с център (0,0,0) и радиус$\sqrt{\pi}$

6. $D \subset \mathbb{R}^m$

Задача 2

1.

$$f(x,y) = \frac{x-y}{x+y}$$

$$\lim_{x\to 0} f(x,y) = \frac{-y}{y} = -1 \qquad \lim_{y\to 0} f(x,y) = \frac{x}{x} = 1$$

$$A_{1,2} = \lim_{y\to 0}\left(\lim_{x\to 0} f(x,y)\right) = \lim_{y\to 0}(-1) = -1$$

$$A_{2,1} = \lim_{x\to 0}\left(\lim_{y\to 0} f(x,y)\right) = \lim_{x\to 0}(1) = 1$$

$A = \lim\limits_{(x,y)\to(0,0)} f(x,y)$ Не съществува, защото трябва $A_{1,2} = A_{2,1}$

2.

$$f(x,y) = \frac{x^2+y^2}{x^2y^2+(x-y)^2}$$

$$\lim_{x\to 0} f(x,y) = \frac{y^2}{(-y)^2} = 1 \qquad \lim_{y\to 0} f(x,y) = \frac{x^2}{x^2} = 1$$

$$\implies A_{1,2} = A_{2,1} = 1 \implies \exists A = \lim_{(x,y)\to(0,0)} f(x,y)$$

Редица: $(x_n, y_n) = \left(\frac{1}{n}, \frac{1}{n}\right) \to (0,0), f(x_n, y_n) = 1 \to 1$

Редица: $(x'_n, y'_n) = \left(\frac{1}{n}, \frac{-1}{n}\right) \to (0,0), f(x'_n, y'_n) = \frac{2n^2}{1+4n^2} \to \frac{1}{2} \neq 1$

$\implies f(x,y)$ няма граница при $(x,y) \to (0,0)$

3.

$$f(x,y) = \frac{xy^2}{x^2+y^4}$$

$$\lim_{x\to 0} f(x,y) = \frac{0}{y^4} = 0 \qquad \lim_{y\to 0} f(x,y) = \frac{0}{x^2} = 0$$

$$A_{1,2} = A_{2,1} = 0 \implies \exists A = \lim_{(x,y)\to(0,0)} f(x,y)$$

Редица: $(x_n, y_n) = \left(\frac{1}{n^2}, \frac{1}{n}\right) \to (0,0), f(x_n, y_n) = \frac{1}{2} \to \frac{1}{2} \neq 0$

$\implies f(x,y)$ няма граница при $(x,y) \to (0,0)$

4.

$$f(x,y) = (x+y)\sin\frac{1}{x}\cos\frac{1}{y}$$

$0 \leq |f(x,y)| \leq |x+y| \leq |x| + |y|$ и $|x| + |y| \to 0$

$A = 0$

$\lim_{x\to 0} \sin\frac{1}{x}$ - не съществува

$$\lim_{x\to 0} f(x,y) = y\cos\frac{1}{y}\lim_{x\to 0}\sin\frac{1}{x}$$

Аналогично и другата вътрешна граница не съществува. Но тогава и повторните граници $A_{1,2}$, $A_{2,1}$ не съществуват.

5.

$$
\begin{aligned}
& f(x,y) = \frac{x^4+y^4}{x^2+y^2} \\
& \lim_{x\to 0} f(x,y) = y^2 \qquad \lim_{y\to 0} f(x,y) = x^2 \\
& A_{1,2} = \lim_{y\to 0}\left(\lim_{x\to 0} f(x,y)\right) = \lim_{y\to 0}\left(y^2\right) = 0 \\
& A_{2,1} = \lim_{x\to 0}\left(\lim_{y\to 0} f(x,y)\right) = \lim_{x\to 0}\left(x^2\right) = 0 \\
& \implies A = A_{1,2} = A_{2,1} = 0
\end{aligned}
$$

Задача 3

Функцията f е непрекъсната в A, защото е частно на две функции със знаменател $y^2 \neq 0$, в A.
Аналогично е непрекъсната в B защото знаменателя е $x^2 \neq 0$.
Остана да се изследва поведението върху C.

$$
\begin{aligned}
& (x_0,y_0) = (x_0,x_0) \in C \\
& R = \{(x_n,y_n)\},\ (x_n,y_n) \in A \\
& \lim_{n\to\infty} R = (x_0,y_0) \\
& \lim_{n\to\infty} f(x_n,y_n) = \frac{1}{y_0^2} = \frac{1}{x_0^2} \neq 0 \\
& \text{Ако } x_0 \neq 0,\ f(x_0,y_0) = 0 \\
& \implies \text{ функцията е прекъсната в точката } (x_0,x_0) \neq (0,0) \\
& \text{Ако } (x_n,y_n) \in B,\ \lim_{n\to\infty} f(x_n,y_n) = -\frac{1}{x_0^2} \neq f(x_0,x_0) \neq 0. \\
& \text{Ако } x_0 = 0,\ \lim_{n\to\infty} f(x_n,y_n) = \infty(-\infty),\ f(0,0) = 0, \\
& \implies \text{f е прекъсната в точката (0,0).}
\end{aligned}
$$

Функцията е непрексъната в D, с изключение на точките от C, където е прекъсната.

3 Упражнение към лекция 3

3.1 Задачи

Задача 1

Да се намерят първите частни производни на следните функции

1. $f(x, y, z) = e^{4x+3y} + xy^2z^3 + 1111e^{\pi}$ за произволна точка $(x_0, y_0, z_0) \in \mathbb{R}^3$
2. $f(x, y) = |x + y|$ в точката (0,0)
3. $f(x, y) = \begin{cases} \dfrac{xy}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$ в равнината $\mathbb{R}^2$

Задача 2

$$f(x, y) = x + (y - 1) \arcsin \sqrt{\frac{x}{y}} \qquad f'_x(x, 1) =?$$

Задача 3

Да се докаже че функцията $f(x, y) = \begin{cases} \dfrac{x^3 y}{x^6 + y^2}, & (x, y) \neq (0, 0) \\ 0, & x^2 + y^2 = (0, 0) \end{cases}$
е прекъсната в точката (0,0) но има частни производни в тази точка.

Задача 4

Да се намерят първите частни производни на следните функции:

1. $f(x, y) = \sin(2x + 3) + 3e^{-x}e^{4y} - 11x^3 + 19e^{\pi}$
2. $f(x, y) = \sqrt{x^2 + y^2} + \arctan \frac{y}{x}$
3. $f(x, y, z) = (xy)^z$
4. $\sqrt[3]{x^2 + 3y^2} e^{x^2 - 5y}$

3.2 Решения

Задача 1

1.

$$
\begin{aligned}
&f(x,y,z) = e^{4x+3y} + xy^2z^3 + 1111e^{\pi}\\
&f(x,y_0,z_0) \implies f'_x(x_0,y_0,z_0) = 4e^{4x_0+3y_0} + y_0^2z_0^3\\
&f(x_0,y,z_0) \implies f'_y(x_0,y_0,z_0) = 3e^{4x_0+3y_0} + 2x_0y_0z_0^3\\
&f(x_0,y_0,z) \implies f'_z(x_0,y_0,z_0) = 3x_0y_0^2z_0^2
\end{aligned}
$$

2.

$$
\begin{aligned}
&f(x,y) = |x+y|\\
&\frac{g(h)-g(0)}{h} = \frac{f(0+h,0)-f(0,0)}{h}\\
&\lim_{h\to 0}\frac{f(0+h,0)-f(0,0)}{h} = \lim_{h\to 0}\frac{|h|}{h} \text{ не съществува}\\
&\implies \nexists f'_x(0,0)(\text{Аналогично се получава за } f'_y(0,0))
\end{aligned}
$$

3.

$$
\begin{aligned}
&f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}\\
&(x,y)\neq(0,0)\\
&f'_x(x,y) = \frac{y(y^2-x^2)}{(x^2+y^2)^2}\\
&f'_y(x,y) = \frac{x(x^2-y^2)}{(x^2+y^2)^2}\\
&\lim_{h\to 0}\frac{f(0+h,0)-f(0,0)}{h} = \lim_{h\to 0}\frac{0-0}{h} = \lim_{h\to 0} = 0\\
&\lim_{k\to 0}\frac{f(0,0+k)-f(0,0)}{k} = \lim_{k\to 0}\frac{0-0}{k} = \lim_{k\to 0} = 0\\
&\implies \text{Функцията има частни производни във всичко точки на равнината } \mathbb{R}^2
\end{aligned}
$$

Задача 2

$$f'_x(a,b) = \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h} (\text{Ако съществува}) \implies$$
$$f'_x(x,1) = \lim_{h\to 0} \frac{f(x+h,1) - f(x,1)}{h} (\text{Ако съществува})$$
$$f(x+h,1) = x + h + (1-1)\arcsin\sqrt{\frac{x}{1}} = x + h + 0\arcsin\sqrt{\frac{x}{1}} = x + h$$
$$f(x,1) = x + (1-1)\arcsin\sqrt{\frac{x}{1}} = x + 0\arcsin\sqrt{\frac{x}{1}} = x \implies$$
$$\lim_{h\to 0} \frac{f(x+h,1) - f(x,1)}{h} = \lim_{h\to 0} \frac{x+h-x}{h} = \lim_{h\to 0} \frac{h}{h} = \lim_{h\to 0} 1 \implies f'_x(x,1) = 1$$

Задача 3

Редица $(x_n, y_n) = \left(\frac{1}{n}, \frac{1}{n^3}\right)$

$$f(x_n, y_n) = \frac{\left(\frac{1}{n}\right)^3 \cdot \frac{1}{n^3}}{\left(\frac{1}{n}\right)^6 + \left(\frac{1}{n^3}\right)^3} = \frac{\frac{1}{n^6}}{\frac{2}{n^6}} = \frac{1}{2} \qquad \lim_{n\to\infty} f(x_n, y_n) = \frac{1}{2} \implies$$
$$\lim_{x\to 0, y\to 0} f(x,y) \neq f(0,0) = 0 \implies f(x,y) \text{ е прекъсната в т. } (0,0).$$

$$f'_x(0,0) = \lim_{x\to 0} \frac{f(x,0) - f(0,0)}{x - 0} = \frac{\frac{x^3 \cdot 0}{x^6 + 0} - 0}{x - 0} = 0$$
$$f'_y(0,0) = \lim_{y\to 0} \frac{f(0,y) - f(0,0)}{y - 0} = \frac{\frac{0^3 \cdot y}{0^6 + y^2} - 0}{y - 0} = 0$$

Задача 4

1.

$$
\begin{aligned}
&f(x,y) = \sin{(2x+3)} + 3e^{-x}e^{4y} - 11x^3 + 19e^{\pi}\\
&f'_x(x,y) = (\sin{(2x+3)})'_x + (3e^{-x}e^{4y})'_x - (11x^3)'_x + (19e^{\pi})'_x\\
&f'_x(x,y) = \cos{(2x+3)}\cdot 2 + (-3e^{-x}e^{4y}) - (3\cdot 11x^2) + 0\\
&f'_x(x,y) = 2\cos{(2x+3)} - 3e^{-x}e^{4y} - 33x^2\\
&f'_y(x,y) = (\sin{(2x+3)})'_y + (3e^{-x}e^{4y})'_y - (11x^3)'_y + (19e^{\pi})'_y\\
&f'_y(x,y) = 0 + (3\cdot 4e^{-x}e^{4y}) - 0 + 0 = 12e^{-x}e^{4y}
\end{aligned}
$$

2.

$$
\begin{aligned}
&f(x,y) = \sqrt{x^2+y^2} + \arctan{\frac{y}{x}}\\
&f'_x(x,y) = \frac{1}{2}(x^2)^{-\dfrac{1}{2}}\cdot 2x + \frac{1}{1+\dfrac{y^2}{x^2}}\cdot y\cdot(-\frac{1}{x^2})\\
&f'_x(x,y) = \frac{x}{\sqrt{x^2+y^2}} - \frac{x^2y}{x^2+y^2}\cdot\frac{1}{x^2}\\
&f'_x(x,y) = \frac{x}{\sqrt{x^2+y^2}} - \frac{xy}{x^2+y^2}\\
&f'_y(x,y) = \frac{1}{2}(x^2)^{-\dfrac{1}{2}}\cdot 2y + \frac{1}{1+\dfrac{y^2}{x^2}}\cdot\frac{1}{x}\\
&f'_y(x,y) = \frac{y}{\sqrt{x^2+y^2}} + \frac{x^2}{x^2+y^2}\cdot\frac{1}{x}\\
&f'_y(x,y) = \frac{y}{\sqrt{x^2+y^2}} + \frac{x}{x^2+y^2}
\end{aligned}
$$

3.

$$
\begin{aligned}
&f(x,y,z) = (xy)^z\\
&f'_x(x,y,z) = z(xy)^{z-1}\cdot(xy)'x = yz(xy)^{z-1}\\
&f'_y(x,y,z) = z(xy)^{z-1}\cdot(xy)'y = xz(xy)^{z-1}\\
&f'_z(x,y,z) = (xy)^z\ln{(xy)}
\end{aligned}
$$

4.

$$\sqrt[3]{x^2+3y^2}e^{x^2-5y}$$

$$f'_x(x,y) = \left[\sqrt[3]{x^2+3y^2}\right]'_x \cdot e^{x^2-5y} + \sqrt[3]{x^2+3y^2} \cdot (e^{x^2-5y})'_x$$

$$f'_x(x,y) = \frac{1}{3}(x^2+3y^2)^{-\frac{2}{3}} \cdot 2x \cdot e^{x^2-5y} + \sqrt[3]{x^2+3y^2} \cdot 2xe^{x^2-5y}$$

$$f'_x(x,y) = \frac{2x}{3} \cdot \frac{e^{x^2-5y}}{\sqrt[3]{(x^2+3y^2)^2}} + 2x\sqrt[3]{x^2+3y^2} \cdot e^{x^2-5y}$$

$$f'_x(x,y) = \frac{2x}{3} \cdot \frac{e^{x^2-5y}}{\sqrt[3]{(x^2+3y^2)^2}} \left[1+3(x^2+3y^2)\right]$$

$$f'_x(x,y) = \frac{2x}{3}(1+3x^2+9y^2)\frac{e^{x^2-5y}}{\sqrt[3]{(x^2+3y^2)^2}}$$

$$f'_y(x,y) = \left[\sqrt[3]{x^2+3y^2}\right]'_y \cdot e^{x^2-5y} + \sqrt[3]{x^2+3y^2} \cdot (e^{x^2-5y})'_y$$

$$f'_y(x,y) = \frac{1}{3}(x^2+3y^2)^{-\frac{2}{3}} \cdot 6y \cdot e^{x^2-5y} + \sqrt[3]{x^2+3y^2} \cdot (-5e^{x^2-5y})$$

$$f'_y(x,y) = 2y \cdot \frac{1}{\sqrt[3]{(x^2+3y^2)^2}} \cdot e^{x^2-5y} - 5\sqrt[3]{x^2+3y^2} \cdot e^{x^2-5y}$$

$$f'_y(x,y) = e^{x^2-5y} \cdot \sqrt[3]{(x^2+3y^2)^2}(2y-5(x^2+3y^2))$$

$$f'_y(x,y) = (2y-5x^2-15y^2)\frac{e^{x^2-5y}}{\sqrt[3]{(x^2+3y^2)^2}}$$

4 Упражнение към лекция 4

4.1 Задачи

Задача 1

$f(x,y) = \sqrt[3]{xy}$
Изследвайте $f(x,y)$ за диференцируемост в $(0,0)$.
$f'_x(0,0) = ?$
$f'_y(0,0) = ?$

Задача 2

$f(x,y) = \sqrt[3]{x^3 + y^3}$
Изследвайте $f(x,y)$ за диференцируемост в $(0,0)$.

Задача 3

Да се изследвай за диференцируемост в $(0,0)$ функцията

$$f(x,y) = \begin{cases} e^{-\dfrac{1}{x^2+y^2}}, & x^2+y^2 \neq 0 \\ 0, & x^2+y^2 = 0 \end{cases}$$

Задача 4

$f(x,y) = x^2 + 3xy - 8y^3 + 11, \quad df(0,1) = ?$
$f(x,y,z) = x^2 + 3xy - 8y^3 - 2e^{3z}x, \quad df(0,0,4) = ?$

Задача 5

$f(x,y) = x^6 - 7xy^2 + 14y,$
$f''_{xx} = ?, f''_{yy} = ?, f''_{xy} = ?, d^2 f(x,y) = ?$
$f(x,y,z) = x^6 - 7xy + y^2 - xz + z^3,$
$f''_{xx} = ?, f''_{xy} = ?, f''_{xz} = ? f''_{yx} = ?, f''_{yy} = ?, f''_{yz} = ? f''_{zx} = ?, f''_{zy} = ?, f''_{zz} = ?, d^2 f(1,0,0)$

4.2 Решения

Задача 1

$$f(x,0)-f(0,0)=\sqrt[3]{x0}-\sqrt[3]{0} \implies$$
$$\lim_{x\to 0}\frac{f(x,0)-f(0,0)}{x-0}=\lim_{x\to 0}\frac{0}{x}=0 f'_x(0,0)=\lim_{x\to 0}\frac{f(x,0)-f(0,0)}{x-0}=\lim_{x\to 0}\frac{0}{x}=0$$
$$f(0,y)-f(0,0)=\sqrt[3]{0y}-\sqrt[3]{0} \implies$$
$$f'_y(0,0)=\lim_{y\to 0}\frac{f(0,y)-f(0,0)}{y-0}=\lim_{y\to 0}\frac{0}{y}=0$$

Нека: $\lim\limits_{(x\to 0,y\to 0)}\varepsilon(x,y)\to 0, \rho(x,y)=\sqrt{x^2+y^2}$

Проверка за диференцируемост в (0,0):

$$f(x,y)-f(0,0)=f'_x(0,0)(x-0)+f'_y(0,0)(y-0)+\varepsilon(x,y)\rho(x,y)$$
$$\sqrt[3]{xy}-0=0x+0y+\varepsilon(x,y)\sqrt{x^2+y^2} \implies$$
$$\varepsilon(x,y)=\frac{\sqrt[3]{xy}}{\sqrt{x^2+y^2}}\to 0?$$

Разглеждаме редица с общ член $(x_n,y_n)=\left(\frac{1}{n^3},\frac{1}{n^3}\right)$ за която $(x_n,y_n)\to(0,0)$,

$$\varepsilon(x_n,y_n)=\frac{\frac{1}{n^2}}{\frac{\sqrt{2}}{n^3}}=\frac{n}{\sqrt{2}} \implies \lim_{(x,y)\to(0,0)}\varepsilon(x_n,y_n)\not\to 0 \implies$$

$f(x,y)$ не е диференцируема в т.$(0,0)$

Задача 2

$f(x,0) - f(0,0) = \sqrt[3]{x^3} - 0 = x \implies$

$\lim\limits_{x\to 0} \dfrac{f(x,0) - f(0,0)}{x - 0} = \lim\limits_{x\to 0} \dfrac{x}{x} = 1 \implies \exists f'_x(0,0) = 1$

$f(0,y) - f(0,0) = \sqrt[3]{y^3} - 0 = y \implies$

$\lim\limits_{y\to 0} \dfrac{f(0,y) - f(0,0)}{y - 0} = \lim\limits_{y\to 0} \dfrac{y}{y} = 1 \implies \exists f'_y(0,0) = 1$

Нека: $\lim\limits_{(x\to 0, y\to 0)} \varepsilon(x,y) \to 0, \rho(x,y) = \sqrt{x^2 + y^2}$

Проверка за диференцируемост в (0,0):

$f(x,y) - f(0,0) = f'_x(0,0)(x-0) + f'_y(0,0)(y-0) + \varepsilon(x,y)\rho(x,y)$

$\sqrt[3]{x^3 + y^3} = x + y + \varepsilon(x,y)\sqrt{x^2 + y^2}$

$\varepsilon(x,y) = \dfrac{\sqrt[3]{x^3 + y^3} - x - y}{\sqrt{x^2 + y^2}}$

$\lim\limits_{(x\to 0, y\to 0)} \varepsilon(x,y) \to 0?$

Разглеждаме редица с общ член $(x_n, y_n) = \left(\dfrac{1}{n}, \dfrac{1}{n}\right)$ за която $(x_n, y_n) \to (0,0)$,

$\varepsilon(x_n, y_n) = \dfrac{\dfrac{\sqrt[3]{2}}{n} - \dfrac{2}{n}}{\dfrac{\sqrt{2}}{n}} = \dfrac{\sqrt[3]{2} - 2}{\sqrt{2}} \implies \lim\limits_{(x\to 0, y\to 0)} \varepsilon(x,y) \not\to 0 \implies$

$f(x,y)$ не е диференцируема в т.$(0,0)$

Задача 3

$$f(x,0)-f(0,0)=e^{-\frac{1}{x^2}}-0=e^{-\frac{1}{x^2}}$$

$$\lim_{x\to 0}\frac{f(x,0)-f(0,0)}{x-0}=\lim_{x\to 0}\frac{e^{-\frac{1}{x^2}}}{x}=\left[\frac{0}{0}\right]$$

$$\lim_{x\to 0}\frac{e^{-\frac{1}{x^2}}}{x}=\lim_{x\to 0}\frac{\frac{1}{x}}{e^{\frac{1}{x^2}}}=\left[\frac{\infty}{\infty}\right]$$

$$\left(\frac{1}{x}\right)'=-\frac{1}{x^2}\qquad\left(e^{\frac{1}{x^2}}\right)'=-\frac{2}{x^3}e^{\frac{1}{x^2}}$$

$$\lim_{x\to 0}\frac{-\frac{1}{x^2}}{-\frac{2}{x^3}e^{\frac{1}{x^2}}}=\lim_{x\to 0}\frac{x}{2e^{\frac{1}{x^2}}}=\frac{0}{\infty}=0\implies f'_x(0,0)=0$$

Аналогично$f'_y(0,0)=0$

Нека : $\lim\limits_{(x\to 0,y\to 0)}\varepsilon(x,y)\to 0,\rho(x,y)=\sqrt{x^2+y^2}$

Проверка за диференцируемост в (0,0):

$$f(x,y)-f(0,0)=f'_x(0,0)(x-0)+f'_y(0,0)(y-0)+\varepsilon(x,y)\rho(x,y)$$

$$e^{-\frac{1}{x^2+y^2}}-0=0(x-0)+0(y-0)+\varepsilon(x,y)\sqrt{x^2+y^2}$$

$$e^{-\frac{1}{x^2+y^2}}=\varepsilon(x,y)\sqrt{x^2+y^2}$$

$$\varepsilon(x,y)=\frac{e^{-\frac{1}{x^2+y^2}}}{\sqrt{x^2+y^2}}$$

$$\lim_{(x\to 0,y\to 0)}\varepsilon(x,y)\to 0?$$

$$\rho(x,y)=\sqrt{x^2+y^2}\implies \lim_{(x\to 0,y\to 0)}\rho(x,y)\to 0$$

$$\lim_{(x\to 0,y\to 0)}\varepsilon(x,y)=\lim_{\rho\to 0}\frac{e^{-\dfrac{1}{\rho^2}}}{\rho}=\left[\frac{\infty}{\infty}\right]$$

$$\left(\frac{1}{\rho}\right)'=-\frac{1}{\rho^2}\qquad\left(e^{\dfrac{1}{\rho^2}}\right)'=-\frac{2}{\rho^3}e^{\dfrac{1}{\rho^2}}$$

$$\lim_{\rho\to 0}\frac{\rho}{2e^{\dfrac{1}{\rho^2}}}=\frac{0}{\infty}=0\implies$$

$$\lim_{(x\to 0,y\to 0)}\varepsilon(x,y)=\lim_{\rho\to 0}\frac{\dfrac{1}{\rho}}{e^{\dfrac{1}{\rho^2}}}=\lim_{\rho\to 0}\frac{\left(\dfrac{1}{\rho}\right)'}{\left(e^{\dfrac{1}{\rho^2}}\right)'}=0\implies$$

$$\lim_{(x\to 0,y\to 0)}\varepsilon(x,y)=0\implies f(x,y)\text{ е диференцируема в }(0,0)$$

Задача 4

$$df(x,y)=f'_x(x,y)dx+f'_y(x,y)dy$$
$$f'_x(x,y)=2x+3y\qquad f'_x(0,1)=3$$
$$f'_y(x,y)=3x-24y^2\qquad f'_y(0,1)=-24$$
$$df(x,y)=(2x+3y)dx+(3x-24y^2)dy$$
$$df(0,1)=3dx-24dy$$

$$df(x,y,z)=f'_x(x,y,z)dx+f'_y(x,y,z)dy+f'_z(x,y,z)dz$$
$$f'_x(x,y,z)=2x+3y-2e^{3z}\qquad f'_x(0,0,4)=-2e^{12}$$
$$f'_y(x,y,z)=3x-24y^2\qquad f'_y(0,0,4)=0$$
$$f'_z(x,y,z)=6xe^{3z}\qquad f'_z(0,0,4)=0$$
$$df(x,y,z)=(2x+3y-2e^{3z})dx+(3x-24y^2)dy+(6xe^{3z})dz$$
$$df(x,y,z)=-2e^{12}dx+0dy+0dz=-2e^{12}dx$$

Задача 5

$$f'_x(x,y,z) = 6x^5 - 7y - z$$
$$f''_{xx}(x,y,z) = (6x^5 - 7y - z)'_x = 30x^4 \qquad f''_{xx}(1,0,0) = 30$$
$$f''_{xy}(x,y,z) = (6x^5 - 7y - z)'_y = -7 \qquad f''_{xy}(1,0,0) = -7$$
$$f''_{xz}(x,y,z) = (6x^5 - 7y - z)'_z = -1 \qquad f''_{xz}(1,0,0) = -1$$

$$f'_y(x,y,z) = -7x + 2y$$
$$f''_{yx}(x,y,z) = (-7x + 2y)'_x = -7 \qquad f''_{yx}(1,0,0) = -7$$
$$f''_{yy}(x,y,z) = (-7x + 2y)'_y = 2 \qquad f''_{yy}(1,0,0) = 2$$
$$f''_{yz}(x,y,z) = (-7x + 2y)'_z = 0 \qquad f''_{yz}(1,0,0) = 0$$

$$f'_z(x,y,z) = -x + 3z^2$$
$$f''_{zx}(x,y,z) = (-x + 3z^2)'_x = -1 \qquad f''_{zx}(1,0,0) = -1$$
$$f''_{zy}(x,y,z) = (-x + 3z^2)'_y = 0 \qquad f''_{zy}(1,0,0) = 0$$
$$f''_{zz}(x,y,z) = (-x + 3z^2)'_z = 6z \qquad f''_{zz}(1,0,0) = 0$$

$$d^2f = f''_{xx}dx^2 + 2f''_{xy}dxdy + f''_{yy}dy^2 + 2f''_{xz}dxdz + f''_{zz}dz^2 + f''_{yz}dydz$$
$$d^2f(x,y,z) = 30x^4dx^2 + 2\cdot(-7)dxdy + 2dy^2 + 2\cdot(-1)dxdz + 6zdz^2 + 2\cdot 0dydz$$
$$d^2f(1,0,0) = 30dx^2 - 14dxdy + 2dy^2 - 2dxdz + 0dz^2 + 0dydz$$
$$d^2f(1,0,0) = 30dx^2 + 2dy^2 - 14dxdy - 2dxdz$$

5 Упражнение към лекция 5

5.1 Задачи

Задача 1

Да се намерят посочените частни производни на следните функции.

1. $u(x,y) = x^4 + 11x^2y^3, \qquad u''_{xx} =?,\ u''_{xy} =?$
2. $u(x,y) = \arctan\dfrac{x+y}{1-xy}, \qquad u''_{xx} =?,\ u''_{xy} =?,\ u''_{yy} =?$
3. $u(x,y) = \dfrac{1}{2}\ln(x^2+y^2), \qquad u''_{xx} =?,\ u''_{xy} =?,\ u''_{yx} =?,\ u''_{yy} =?$
4. $u(x,y) = \ln(x+2y), \qquad u'''_{xxy} =?$
5. $u(x,y,z) = e^{xy^2z^3}, \qquad u'''_{xyz} =?$

Задача 2

Дали са верни равенствата:

- Ако $z = y\ln(x^2+y^2)$ то $\quad \dfrac{1}{x}z'_x + \dfrac{1}{y}z'_y = \dfrac{z}{y^2}$
- Ако $u = \ln(x^3+y^3+z^3-3xyz)$ то $\quad u'_x + u'_y + u'_z = \dfrac{3}{x+y+z}$

Задача 3

Да се докаже, че функцията: $z(x,y) = \arctan\left(\dfrac{x+y}{x-y}\right)$ удовлетворява тъждеството: $z'_x + z'_y = \dfrac{x-y}{x^2+y^2}$

Задача 4

Да се провери тъждеството на Ойлер за следните функции: $z(x,y) = \dfrac{1}{(x^2+y^2)^2} \qquad u(x,y,z) = \sqrt{x^2+y^2+z^2}\cdot\ln\left(\dfrac{y}{x}\right)$

Тъждество на Ойлер($f : D \to R, D \subset \mathbb{R}^m$)

$$x_1f'_{x_1} + x_2f'_{x_2} + ... + x_mf'_{x_m} = mf$$

5.2 Решения

Задача 1

$$u(x,y) = x^4 + 11x^2y^3$$
$$u'_x = 4x^3 + 22xy^3$$
$$u''_{xx} = 12x^2 + 22y^3$$
$$u''_{xy} = 4x^3 + 66xy^2$$

$$u(x,y) = \arctan\frac{x+y}{1-xy}$$
$$u'_x = \frac{1}{1+\left(\frac{x+y}{1-xy}\right)^2}\cdot\left(\frac{x+y}{1-xy}\right)'_x$$
$$u'_y = \frac{1}{1+\left(\frac{x+y}{1-xy}\right)^2}\cdot\left(\frac{x+y}{1-xy}\right)'_y$$
$$u''_{xx} = (u'_x)'_x$$
$$u''_{xy} = (u'_x)'_y$$
$$u''_{yy} = (u'_y)'_y$$

$$A = \frac{1}{1 + \left(\dfrac{x+y}{1-xy}\right)^2}. \quad B = \left(\frac{x+y}{1-xy}\right)'_x \implies u'_x = AB$$

$$A = \frac{1}{1 + \left(\dfrac{x+y}{1-xy}\right)^2} = \frac{1}{1 + \dfrac{(x+y)^2}{(1-xy)^2}} = \frac{(1-xy)^2}{(1-xy)^2 + (x+y)^2}$$

$$A = \frac{(1-xy)^2}{1 - 2xy + x^2y^2 + x^2 + 2xy + y^2} = \frac{(1-xy)^2}{1 + x^2y^2 + x^2 + y^2}$$

$$A = \frac{(1-xy)^2}{(1+y^2) + x^2 + x^2y^2} = \frac{(1-xy)^2}{(1+y^2) + x^2(1+y^2)} = \frac{(1-xy)^2}{(1+y^2)(1+x^2)}$$

$$B = \left(\frac{x+y}{1-xy}\right)'_x = \frac{1(1-xy) - (x+y)(-y)}{(1-xy)^2} = \frac{1 - xy + xy + y^2}{(1-xy)^2} = \frac{1+y^2}{(1-xy)^2}$$

$$u'_x = AB = \frac{(1-xy)^2}{(1+y^2)(1+x^2)} \cdot \frac{1-y^2}{(1-xy)^2} = \frac{1}{1+x^2}$$

$$C = \left(\frac{x+y}{1-xy}\right)'_y \implies u'_y = AC$$

$$C = \frac{1(1-xy) - (x+y)(-x)}{(1-xy)^2} = \frac{1 - xy + x^2 + xy}{(1-xy)^2} = \frac{1+x^2}{(1-xy)^2}$$

$$u'_y = AC = \frac{(1-xy)^2}{(1+y^2)(1+x^2)} \cdot \frac{1+x^2}{(1-xy)^2} = \frac{1}{1+y^2}$$

$$u''_{xx} = \left(\frac{1}{1+x^2}\right)'_x = ((1+x^2)^{-1})'_x$$

$$u''_{yy} = -(1+x^2)^{-2}(1+x^2)'_x = -2x(1+x^2)^{-2} = \frac{-2x}{(1+x^2)^2}$$

$$u''_{xy} = \left(\frac{1}{1+x^2}\right)'_y = 0$$

$$u''_{yy} = \left(\frac{1}{1+y^2}\right)'_y = ((1+y^2)^{-1})'_y$$

$$u''_{yy} = -(1+y^2)^{-2}(1+y^2)'_y = -2y(1+y^2)^{-2} = \frac{-2y}{(1+y^2)^2}$$

$$u(x,y) = \frac{1}{2}\ln(x^2+y^2)$$
$$u'_x = \frac{1}{2(x^2+y^2)} \cdot (x^2+y^2)'_x = \frac{2x}{2(x^2+y^2)} = \frac{x}{x^2+y^2}$$
$$u'_y = \frac{1}{2(x^2+y^2)} \cdot (x^2+y^2)'_y = \frac{2y}{2(x^2+y^2)} = \frac{y}{x^2+y^2}$$
$$u''_{xx} = (u'_x)'_x = \left(\frac{x}{x^2+y^2}\right)'_x = \frac{1(x^2+y^2)-(2x)x}{(x^2+y^2)^2} = \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$
$$u''_{xy} = (u'_x)'_y = \left(\frac{x}{x^2+y^2}\right)'_y = \frac{-2xy}{(x^2+y^2)^2}$$
$$u''_{yy} = (u'_y)'_y = \left(\frac{y}{x^2+y^2}\right)'_y = \frac{1(x^2+y^2)-(2y)y}{(x^2+y^2)^2} = \frac{x^2+y^2-2y^2}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$$
$$u''_{yx} = (u'_y)'_x = \left(\frac{y}{x^2+y^2}\right)'_x = \frac{-2xy}{(x^2+y^2)^2}$$

$$u(x,y) = \ln(x+2y)$$
$$u'_x = \frac{1}{x+2y}$$
$$u''_{xx} = \left(\frac{1}{x+2y}\right)'_x = ((x+2y)^{-1})'_x = -(x+2y)^{-2}(x+2y)'_x = -\frac{1}{(x+2y)^2}$$
$$u'''_{xxy} = \left(-\frac{1}{(x+2y)^2}\right)'_y = -((x+2y)^{-2})'_y = 2((x+2y)^{-3})(x+2y)'_y = \frac{4}{(x+2y)^3}$$

$$u(x,y,z) = e^{xy^2z^3}$$
$$u'_x = e^{xy^2z^3}(xy^2z^3)'_x = y^2z^3e^{xy^2z^3}$$
$$u''_{xy} = (y^2z^3 \cdot e^{xy^2z^3})'_y = (y^2z^3)'_y \cdot e^{xy^2z^3} + y^2z^3(e^{xy^2z^3})'_y$$
$$u''_{xy} = 2yz^3e^{xy^2z^3} + 2xy^3z^6e^{xy^2z^3} = 2yz^3e^{xy^2z^3}(1+xy^2z^3)$$

$$u'''_{xyz} = \left[2yz^3e^{xy^2z^3}(1+xy^2z^3)\right]'_z = (2yz^3e^{xy^2z^3})'_z(1+xy^2z^3) + 2yz^3e^{xy^2z^3}(1+xy^2z^3)'_z$$
$$= \left[(2yz^3)'_z \cdot e^{xy^2z^3} + 2yz^3 \cdot (e^{xy^2z^3})'_z\right](1+xy^2z^3) + 2yz^3e^{xy^2z^3}(1+xy^2z^3)'_z$$
$$u'''_{xyz} = \left[6yz^2e^{xy^2z^3} + 2yz^3e^{xy^2z^3}3xy^2z^2\right](1+xy^2z^3) + 2yz^3e^{xy^2z^3}(3xy^2z^2)$$
$$u'''_{xyz} = \left[6yz^2e^{xy^2z^3} + 6xy^3z^5e^{xy^2z^3}\right](1+xy^2z^3) + 6xy^3z^5e^{xy^2z^3}$$
$$u'''_{xyz} = \left[6yz^2e^{xy^2z^3} + 6yz^2e^{xy^2z^3}xy^2z^3 + 6xy^3z^5e^{xy^2z^3} + 6xy^3z^5e^{xy^2z^3}xy^2z^3\right] + 6xy^3z^5e^{xy^2z^3}$$
$$u'''_{xyz} = \left[6yz^2e^{xy^2z^3} + 6xy^3z^5e^{xy^2z^3} + 6xy^3z^5e^{xy^2z^3} + 6x^2y^5z^8e^{xy^2z^3}\right] + 6xy^3z^5e^{xy^2z^3}$$
$$u'''_{xyz} = 6yz^2e^{xy^2z^3} + 6xy^3z^5e^{xy^2z^3} + 6xy^3z^5e^{xy^2z^3} + 6x^2y^5z^8e^{xy^2z^3} + 6xy^3z^5e^{xy^2z^3}$$
$$u'''_{xyz} = 6yz^2e^{xy^2z^3} + 18xy^3z^5e^{xy^2z^3} + 6x^2y^5z^8e^{xy^2z^3}$$
$$u'''_{xyz} = 6yz^2e^{xy^2z^3}\left[1 + 3xy^2z^3 + x^2y^4z^6\right]$$

Задача 2

$$z = y\ln(x^2+y^2)$$
$$z'_x = y\frac{1}{x^2+y^2}2x = \frac{2xy}{x^2+y^2}$$
$$z'_y = \ln(x^2+y^2) + y\frac{1}{x^2+y^2} - 2y = \ln(x^2+y^2) - \frac{2y^2}{x^2+y^2}$$
$$\frac{1}{x}z'_x + \frac{1}{y}z'_y = \frac{1}{x}\cdot\frac{2xy}{x^2+y^2} + \frac{1}{y}\cdot\left[\ln(x^2+y^2) - \frac{2y^2}{x^2+y^2}\right] =$$
$$\frac{2y}{x^2+y^2} + \frac{\ln(x^2+y^2)}{y} - \frac{2y}{x^2+y^2} = \frac{\ln(x^2+y^2)}{y}$$
$$\frac{z}{y^2} = \frac{y\ln(x^2+y^2)}{y^2} = \frac{\ln(x^2+y^2)}{y} \implies \text{Равенството е вярно.}$$

$$u = \ln\left(x^3 + y^3 + z^3 - 3xyz\right)$$
$$u'_x = \frac{(x^3 + y^3 + z^3 - 3xyz)'_x}{x^3 + y^3 + z^3 - 3xyz} = \frac{3x^2 - 3yz}{x^3 + y^3 + z^3 - 3xyz}$$
$$u'_y = \frac{(x^3 + y^3 + z^3 - 3xyz)'_y}{x^3 + y^3 + z^3 - 3xyz} = \frac{3y^2 - 3xz}{x^3 + y^3 + z^3 - 3xyz}$$
$$u'_z = \frac{(x^3 + y^3 + z^3 - 3xyz)'_z}{x^3 + y^3 + z^3 - 3xyz} = \frac{3z^2 - 3xy}{x^3 + y^3 + z^3 - 3xyz}$$
$$u'_x + u'_y + u'_z = \frac{3x^2 - 3yz}{x^3 + y^3 + z^3 - 3xyz} + \frac{3y^2 - 3xz}{x^3 + y^3 + z^3 - 3xyz} + \frac{3z^2 - 3xy}{x^3 + y^3 + z^3 - 3xyz} =$$
$$\frac{3x^2 - 3yz + 3y^2 - 3xz + 3z^2 - 3xy}{x^3 + y^3 + z^3 - 3xyz} = \frac{3(x^2 - yz + y^2 - xz + z^2 - xy)}{x^3 + y^3 + z^3 - 3xyz} =$$
$$\frac{3(x^2 + y^2 + z^2 - xy - xz - yz)}{x^3 + y^3 + z^3 - 3xyz} \cdot \frac{x + y + z}{x + y + z} = \frac{3(x^3 + y^3 + z^3 - 3xyz)}{(x^3 + y^3 + z^3 - 3xyz)(x + y + z)} =$$
$$\frac{3}{x + y + z} \implies \text{Равенството е вярно.}$$

Задача 3

$$z'_x = \frac{1}{1 + \left(\dfrac{x + y}{x - y}\right)^2} \cdot \left(\frac{x + y}{x - y}\right)'_x = \frac{1}{\dfrac{(x - y)^2 + (x + y)^2}{(x - y)^2}} \cdot \frac{x - y - x - y}{(x - y)^2}$$
$$z'_x = \frac{(x - y)^2}{(x - y)^2 + (x + y)^2} \cdot \frac{-2y}{(x - y)^2} \frac{-2y}{x^2 - 2xy + y^2 + x^2 + 2xy + y^2}$$
$$z'_x = \frac{-2y}{2(x^2 + y^2)} = -\frac{y}{x^2 + y^2}$$
$$z'_y = \frac{1}{1 + \left(\dfrac{x + y}{x - y}\right)^2} \cdot \left(\frac{x + y}{x - y}\right)'_y \frac{1}{\dfrac{(x - y)^2 + (x + y)^2}{(x - y)^2}} \cdot \frac{x - y + x + y}{(x - y)^2} =$$
$$z'_y = \frac{(x - y)^2}{(x - y)^2 + (x + y)^2} \cdot \frac{2x}{(x - y)^2} = \frac{2x}{x^2 - 2xy + y^2 + x^2 + 2xy + y^2} = \frac{x}{x^2 + y^2}$$
$$z'_x + z'_y = -\frac{y}{x^2 + y^2} + \frac{x}{x^2 + y^2} = \frac{x - y}{x^2 + y^2} \implies \text{тъжеството е вярно}$$

Задача 4

$$z(x,y) = \frac{1}{(x^2+y^2)^2}$$
$$xz'_x + yz'_y = 2z$$
$$z'_x = \left(\frac{1}{(x^2+y^2)^2}\right)'_x = \left((x^2+y^2)^{-2}\right)'_x = -2(x^2+y^2)^{-3}(x^2+y^2)'_x = -\frac{4x}{(x^2+y^2)^3}$$
$$z'_y = \left(\frac{1}{(x^2+y^2)^2}\right)'_y = \left((x^2+y^2)^{-2}\right)'_y = -2(x^2+y^2)^{-3}(x^2+y^2)'_x = -\frac{4y}{(x^2+y^2)^3}$$
$$xz'_x + yz'_y = x \cdot \left(-\frac{4x}{(x^2+y^2)^3}\right) + y \cdot \left(-\frac{4y}{(x^2+y^2)^3}\right) = -\frac{4x^2}{(x^2+y^2)^3} - \frac{4y^2}{(x^2+y^2)^3} =$$
$$\frac{-4(x^2+y^2)}{(x^2+y^2)^3} = -\frac{4}{(x^2+y^2)^2}$$
$$2z = \frac{2}{(x^2+y^2)^2}$$
$$-\frac{4}{(x^2+y^2)^2} \neq \frac{2}{(x^2+y^2)^2} \implies \text{Твърдението не е изпълнено.}$$

$$u(x,y,z) = \sqrt{x^2+y^2+z^2} \cdot \ln\left(\frac{y}{x}\right)$$
$$xu'_x + yu'_y + zu'_z = 3z$$
$$u'_x = \left(\sqrt{x^2+y^2+z^2}\right)'_x \ln\left(\frac{y}{x}\right) + \sqrt{x^2+y^2+z^2}\left(\ln\left(\frac{y}{x}\right)\right)'_x$$
$$u'_y = \left(\sqrt{x^2+y^2+z^2}\right)'_y \ln\left(\frac{y}{x}\right) + \sqrt{x^2+y^2+z^2}\left(\ln\left(\frac{y}{x}\right)\right)'_y$$
$$u'_z = \left(\sqrt{x^2+y^2+z^2}\right)'_z \ln\left(\frac{y}{x}\right) + \sqrt{x^2+y^2+z^2}\left(\ln\left(\frac{y}{x}\right)\right)'_z$$

$$u'_x = \left(\sqrt{x^2+y^2+z^2}\right)'_x \ln\left(\frac{y}{x}\right) + \sqrt{x^2+y^2+z^2}\left(\ln\left(\frac{y}{x}\right)\right)'_x$$
$$u'_x = \frac{x\ln\left(\frac{y}{x}\right)}{\sqrt{x^2+y^2+z^2}} - \frac{\sqrt{x^2+y^2+z^2}}{x} = \frac{x\ln\left(\frac{y}{x}\right)x - \left(\sqrt{x^2+y^2+z^2}\right)^2}{x\sqrt{x^2+y^2+z^2}}$$
$$u'_x = \frac{x^2\ln\left(\frac{y}{x}\right) - x^2 - y^2 - z^2}{x\sqrt{x^2+y^2+z^2}}$$

$$u'_y = \left(\sqrt{x^2+y^2+z^2}\right)'_y \ln\left(\frac{y}{x}\right) + \sqrt{x^2+y^2+z^2}\left(\ln\left(\frac{y}{x}\right)\right)'_y$$
$$u'_y = \frac{y\ln\left(\frac{y}{x}\right)}{\sqrt{x^2+y^2+z^2}} + \frac{\sqrt{x^2+y^2+z^2}}{y} = \frac{y\ln\left(\frac{y}{x}\right)y + \left(\sqrt{x^2+y^2+z^2}\right)^2}{y\sqrt{x^2+y^2+z^2}}$$
$$u'_y = \frac{y^2\ln\left(\frac{y}{x}\right) + x^2+y^2+z^2}{y\sqrt{x^2+y^2+z^2}}$$

$$u'_z = \left(\sqrt{x^2+y^2+z^2}\right)'_z \ln\left(\frac{y}{x}\right) + \sqrt{x^2+y^2+z^2}\left(\ln\left(\frac{y}{x}\right)\right)'_z$$
$$u'_z = \frac{z\ln\left(\frac{y}{x}\right)}{\sqrt{x^2+y^2+z^2}} + 0\cdot\sqrt{x^2+y^2+z^2} = \frac{z\ln\left(\frac{y}{x}\right)}{\sqrt{x^2+y^2+z^2}}$$

$$xu'_x + yu'_y + zu'_z = 3z, \quad A = xu'_x + yu'_y + zu'_z, \quad B = 3u$$
$$A = x\cdot\frac{x^2\ln\left(\frac{y}{x}\right) - x^2-y^2-z^2}{x\sqrt{x^2+y^2+z^2}} + y\cdot\frac{y^2\ln\left(\frac{y}{x}\right)+x^2+y^2+z^2}{y\sqrt{x^2+y^2+z^2}} + z\cdot\frac{z\ln\left(\frac{y}{x}\right)}{\sqrt{x^2+y^2+z^2}}$$
$$A = \frac{x^2\ln\left(\frac{y}{x}\right) - x^2-y^2-z^2}{\sqrt{x^2+y^2+z^2}} + \frac{y^2\ln\left(\frac{y}{x}\right)+x^2+y^2+z^2}{\sqrt{x^2+y^2+z^2}} + \frac{z^2\ln\left(\frac{y}{x}\right)}{\sqrt{x^2+y^2+z^2}}$$
$$A = \frac{x^2\ln\left(\frac{y}{x}\right) - x^2-y^2-z^2 + y^2\ln\left(\frac{y}{x}\right) + x^2+y^2+z^2 + z^2\ln\left(\frac{y}{x}\right)}{\sqrt{x^2+y^2+z^2}}$$
$$A = \frac{x^2\ln\left(\frac{y}{x}\right)\ln\left(\frac{y}{x}\right) + z^2\ln\left(\frac{y}{x}\right)}{\sqrt{x^2+y^2+z^2}} = \frac{\ln\left(\frac{y}{x}\right)(x^2+y^2+z^2)}{\sqrt{x^2+y^2+z^2}} = \sqrt{x^2+y^2+z^2}\cdot\ln\left(\frac{y}{x}\right)$$
$$B = 3u = 3\sqrt{x^2+y^2+z^2}\cdot\ln\left(\frac{y}{x}\right) \implies A \neq B \implies \text{Твърдението не е изпълнено.}$$

6 Упражнение към лекция 6

6.1 Задачи

Задача 1

Дадени са функцията $z(x,y) = \varphi(x+y) + \psi(x-y)$, където φ, ψ - непрекъснато диференцируеми Да се намерят първите частни производни.

Задача 2

Да се провери дали $w(x,y,z)$ удволетворява тъждествено равенството:

$$xw_x + yw_y + zw_z = w + \frac{xy}{z}$$

Ако $w = \dfrac{xy}{z} + \ln x + x \cdot \varphi\left(\dfrac{y}{x}, \dfrac{z}{x}\right)$, φ е непрекъснато диференцируема.

Задача 3

Дадени са функциите и точката M(2,1). Да се пресметне gradf(M) и $\|gradf(M)\|$

1. $f(x,y) = x^2 + 11y^2 - 3$
2. $f(x,y) = x^2 - y^2$
3. $f(x,y) = \ln(x^2 + y^2)$

Задача 4

Дадени са функциите и точката M(2,1).
Да се пресметне $\dfrac{\partial f(M)}{\partial \nu}, \nu = \left(\dfrac{\sqrt{3}}{2}, \dfrac{1}{2}\right)$

1. $f(x,y) = x^2 + 11y^2 - 3$
2. $f(x,y) = x^2 - y^2$
3. $f(x,y) = \ln(x^2 + y^2)$

Задача 5

Да се определи ъгъла между градиентите на функцията

$$u = x^2 + y^2 + z^2 - 111$$

в точките A$(\varepsilon, 0, 0)$ и B $(0, \varepsilon, 0), \varepsilon > 0$

Задача 6

Да се намери y', y'' на неявната функция $y = f(x)$, дефинирана от уравнението

$$x^2 - 2xy + 5y^2 + 4y = 2x + 9$$

Да се пресметнат $y'(0), y''(0)$, ако $y(0) = 1$

6.2 Решения

Задача 1

$$z(x,y) = \varphi(x+y) + \psi(x-y)$$
$$z'_x = \varphi'(x+y)(x+y)'_x + \psi'(x-y)(x-y)'_x = \varphi'(x+y)1 + \psi'(x-y)1$$
$$z'_x = \varphi'(x+y) + \psi'(x-y)$$
$$z'_y = \varphi'(x+y)(x+y)'_y + \psi'(x-y)(x-y)'_y = \varphi'(x+y)1 + \psi'(x-y)(-1)$$
$$z'_y = \varphi'(x+y) - \psi'(x-y)$$

Задача 2

$$u = \frac{y}{x} \qquad v = \frac{z}{x}$$
$$u'_x = -\frac{y}{x^2} \qquad u'_y = \frac{1}{x} \qquad u'_z = 0$$
$$v'_x = -\frac{z}{x^2} \qquad v'_y = 0 \qquad v'_z = \frac{1}{x}$$
$$w'_x = \frac{y}{z}\ln x + \frac{xy}{z}\cdot\frac{1}{x} + \varphi\left(\frac{y}{x},\frac{z}{x}\right) + x(\varphi'_u u'_x + \varphi'_v v_x) =$$
$$w'_x = \frac{y}{z}\ln x + \frac{y}{z} + \varphi\left(\frac{y}{x},\frac{z}{x}\right) - \frac{y}{x}\varphi'_u - \frac{z}{x}\varphi'_v$$
$$w'_y = \frac{x}{z}\ln x + x(\varphi'_u u'_y + \varphi'_v v_y) = \frac{x}{z}\ln x + \varphi'_u$$
$$w'_z = -\frac{xy}{z^2}\ln x + x(\varphi'_u u'_z + \varphi'_v v_z) = -\frac{xy}{z^2}\ln x + \varphi'_v$$
$$xw_x + yw_y + zw_z =$$
$$= \frac{xy}{z}\ln x + \frac{xy}{z} + x\varphi\left(\frac{y}{x},\frac{z}{x}\right) - y\varphi'_u - z\varphi'_v + \frac{xy}{z}\ln x + y\varphi'_u + -\frac{xy}{z}\ln x + z\varphi'_v =$$
$$= \frac{xy}{z} + \ln x + x\cdot\varphi\left(\frac{y}{x},\frac{z}{x}\right) + \frac{xy}{z} = w + \frac{xy}{z}$$

Задача 3

$$gradf = (f'_x, f'_y)$$

$$f(x,y) = x^2 + 11y^2 - 3$$
$$f'_x = 2x \qquad f'_y = 22y$$
$$gradf(x,y) = (2x, 22y)$$
$$gradf(M) = (2\cdot 2, 22\cdot 1) = (4, 22)$$
$$\|gradf(M)\| = \sqrt{4^2 + 22^2} = \sqrt{500} = 10\sqrt{5}$$

$$
\begin{aligned}
&f(x,y) = x^2 - y^2\\
&f'_x = 2x \qquad f'_y = -2y\\
&gradf(x,y) = (2x, -2y)\\
&gradf(M) = (2\cdot 2, -2\cdot 1) = (4, -2)\\
&\|gradf(M)\| = \sqrt{4^2 + (-2)^2} = \sqrt{20} = 2\sqrt{5}
\end{aligned}
$$

$$
\begin{aligned}
&f(x,y) = \ln\left(x^2 + y^2\right)\\
&f'_x = \frac{2x}{x^2+y^2} \qquad f'_y = \frac{2y}{x^2+y^2}\\
&gradf(x,y) = \left(\frac{2x}{x^2+y^2}, \frac{2y}{x^2+y^2}\right)\\
&gradf(M) = \left(\frac{2\cdot 2}{2^2+1^2}, \frac{2\cdot 1}{2^2+1^2}\right) = \left(\frac{4}{5}, \frac{2}{5}\right)\\
&\|gradf(M)\| = \sqrt{\left(\frac{4}{5}\right)^2 + \left(\frac{2}{5}\right)^2} = \sqrt{\frac{20}{25}} = \frac{2}{\sqrt{5}}
\end{aligned}
$$

Задача 4

$$
\frac{\partial f(M)}{\partial \nu} = (gradf, \nu)
$$

$$
\begin{aligned}
&f(x,y) = x^2 + 11y^2 - 3\\
&gradf(M) = (2\cdot 2, 22\cdot 1) = (4, 22)\\
&\frac{\partial f(M)}{\partial \nu} = 4\cdot\frac{\sqrt{3}}{2} + 22\cdot\frac{1}{2} = 2\sqrt{3} + 11
\end{aligned}
$$

$$
\begin{aligned}
&f(x,y) = x^2 - y^2\\
&gradf(M) = (2\cdot 2, -2\cdot 1) = (4, -2)\\
&\frac{\partial f(M)}{\partial \nu} = 4\cdot\frac{\sqrt{3}}{2} - 2\cdot\frac{1}{2} = 2\sqrt{3} - 1
\end{aligned}
$$

$$f(x,y) = \ln(x^2+y^2)$$
$$gradf(M) = \left(\frac{2\cdot 2}{2^2+1^2}, \frac{2\cdot 1}{2^2+1^2}\right) = \left(\frac{4}{5}, \frac{2}{5}\right)$$
$$\frac{\partial f(M)}{\partial \nu} = \frac{4}{5}\cdot\frac{\sqrt{3}}{2} + \frac{2}{5}\cdot\frac{1}{2} = \frac{4\sqrt{3}}{10} + \frac{1}{5} = \frac{4\sqrt{3}+2}{10}$$

Задача 5

$$u'_x = 2x \qquad u'_y = 2y \qquad u'_z = 2z$$
$$gradu(A) = (2\varepsilon, 0, 0) \qquad gradu(B) = (0, 2\varepsilon, 0)$$
$$(gradu(A), gradu(B)) = 2\varepsilon\cdot 0 + 0\cdot 2\varepsilon + 0\cdot 0 = 0$$
$$(gradu(A), gradu(B)) = \|u(A)\|\cdot\|u(B)\|\cdot\cos\alpha$$
$$\cos\alpha = 0 \Leftrightarrow \alpha = \frac{\pi}{2}$$

Задача 6

$$F(x,y) = x^2 - 2xy + 5y^2 + 4y = 2x + 9$$
$$F'_y = -2x + 10y + 4 \neq 0$$
$$F'_x(x,y) = 2x - 2y - 2$$
$$F'_y(0,1) = -2\cdot 0 + 10\cdot 1 + 4 \neq 0$$
$$y'(x) = -\frac{F'_x(x,y)}{F'_y(x,y)} = -\frac{2x-2y-2}{-2x+10y+4} = -\frac{x-y-1}{-x+5y+2}$$
$$y'(0) = -\frac{0-1-1}{-0+5\cdot 1+2} = -\frac{-2}{7} = \frac{2}{7}$$
$$y''(x) = -\frac{F''_{xx}(x,y) + 2F''_{xy}y' + F''_{yy}(x,y)y'^2}{F'_y(x,y)}$$
$$F''_{xx} = 2, \quad F''_{yy} = 10, \quad F''_{xy} = -2$$
$$F''_{xx}(0,1) = 2, \quad F''_{yy}(0,1) = 10, \quad F''_{xy}(0,1) = -2$$
$$y''(x) = -\frac{2 + 2\cdot(-2)y' + 10y'^2}{-2x+10y+4}$$
$$y''(x) = -\frac{2 + -4y' + 10y'^2}{-2x+10y+4}$$
$$y''(0) = -\frac{2 + -4\cdot\frac{2}{7} + 10\cdot\left(\frac{2}{7}\right)^2}{-2\cdot 0 + 10\cdot 1 + 4}$$
$$y''(0) = -\frac{2 + -\frac{8}{7} + \frac{40}{49}}{14}$$
$$y''(0) = -\frac{\frac{98-56+40}{49}}{14} = -\frac{\frac{82}{49}}{14} = \frac{82}{49}\cdot\frac{1}{14} = \frac{41}{343}$$

7 Упражнение към лекция 7

7.1 Задачи

Задача 1

Да се намерят локалните екстремуми на функциите

- $z = \sin x + \sin y + \sin(x+y) \quad (0 < x < \frac{\pi}{2}, 0 < y < \frac{\pi}{2})$
- $z = x^4 + y^4 - 4xy$

Задача 2

Да се намерят локалните екстремуми на функциите

- $u = x^2 + y^2 + z^2 + 2x + 4y - 6z$
- $u = x^3 + y^2 + z^2 - 3x + 6y - 2z$
- $u = x^3 + y^2 + z^2 - 3x - 2y$

Задача 3

Да се намерят $y'(0), y''(0)$ ако $y(0) = 2$ на неявната функция $y = f(x)$ дефинирана от уравнението

$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$

Задача 4

Да се покаже, че функцията $z = f(x, y)$ дефинирана неявно от уравнението

$$z = x\varphi(\frac{z}{y})$$

φ - непрекъснато диференцируема, удовелетворява тъждествено уравнението

$$xz'_x + yz'_y = z$$

7.2 Решения

Задача 1

$$z = \sin x + \sin y + \sin(x+y) \quad (0 < x < \frac{\pi}{2}, 0 < y < \frac{\pi}{2})$$

$$z'_x = \cos x + \cos(x+y) \quad z'_y = \cos y + \cos(x+y)$$

$$\left| \begin{array}{l} \cos x + \cos(x+y) = 0 \\ \cos y + \cos(x+y) = 0 \end{array} \right. \Leftrightarrow \left| \begin{array}{l} 2\cos\frac{2x+y}{2}\cos\frac{y}{2} = 0 \\ 2\cos\frac{x+2y}{2}\cos\frac{x}{2} = 0 \end{array} \right. \Leftrightarrow \left| \begin{array}{l} \frac{2x+y}{2} = \frac{\pi}{2} \\ \frac{x+2y}{2} = \frac{\pi}{2} \end{array} \right.$$

$$\frac{y}{2} = \frac{\pi}{2}, \quad \frac{x}{2} = \frac{\pi}{2} \implies x = y = \pi \notin (0 < x, y < \frac{\pi}{2})$$

$$x_0 = y_0 = \frac{\pi}{3} \implies M_0\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$$

$$z''_{xx} = -\sin x - \sin(x+y) \quad z''_{yy} = -\sin y - \sin(x+y) \quad z''_{xy} = -\sin(x+y)$$

$$z''_{xx}(M_0) = -\frac{2\sqrt{3}}{2} = -\sqrt{3} = \Delta_1 \quad z''_{yy}(M_0) = -\frac{2\sqrt{3}}{2} = -\sqrt{3} \quad z''_{xy}(M_0) = -\frac{\sqrt{3}}{2}$$

$$\begin{pmatrix} z''_{xx}(M_0) & z''_{xy}(M_0) \\ z''_{yx}(M_0) & z''_{yy}(M_0) \end{pmatrix} = \begin{pmatrix} -\sqrt{3} & -\frac{2\sqrt{3}}{2} \\ -\frac{2\sqrt{3}}{2} & -\sqrt{3} \end{pmatrix}$$

$$\Delta_1 = -\sqrt{3} < 0 \qquad \Delta_2 = 3 - \frac{3}{4} > 0$$

$$\implies \exists \text{ локален максимум}, z_{max} = z(M_0) - \frac{3\sqrt{3}}{2}$$

$$z = x^4 + y^4 - 4xy$$

$$z'_x = 4x^3 - 4y \quad z'_y = 4y^3 - 4x$$

$$\left| \begin{array}{l} 4x^3 - 4y = 0 \\ 4y^3 - 4x = 0 \end{array} \right. \Leftrightarrow \left| \begin{array}{l} y = x^3 \\ x^9 - x = 0 \end{array} \right. \Leftrightarrow \left| \begin{array}{l} x(x^2-1)(x^2+1)(x^4+1) = 0 \\ y = x^3 \end{array} \right. \implies$$

$$M_0(0,0) \quad M_1(1,1) \quad M_2(-1,1)$$

$$z''_{xx} = 12x^2 \quad z''_{yy} = 12y^2 \quad z''_{xy} = -4$$

$$d^2z = \begin{pmatrix} z''_{xx}(M_0) & z''_{xy}(M_0) \\ z''_{yx}(M_0) & z''_{yy}(M_0) \end{pmatrix} = \begin{pmatrix} 12x^2 & -4 \\ -4 & 12y^2 \end{pmatrix}$$

$$d^2z(M_0) = \begin{pmatrix} 0 & -4 \\ -4 & 0 \end{pmatrix} \implies \Delta = \begin{vmatrix} 0 & -4 \\ -4 & 0 \end{vmatrix} = -16 < 0 \implies \text{няма лок. екстремум в } M_0$$

$$d^2z(M_1) = \begin{pmatrix} 12 & -4 \\ -4 & 12 \end{pmatrix} \implies \Delta_1 = 12 > 0 \Delta_2 = \begin{vmatrix} 0 & -4 \\ -4 & 0 \end{vmatrix} 144 - 16 > 0 \implies$$

z има локален минимум

$z_{min} = z(M_1) = 1^4 + 1^4 - 4 \cdot 1 \cdot 1 = -2$

Аналогично и за M_2 има лок. мин $z_{min} = -2$

Задача 2

$u = x^2 + y^2 + z^2 + 2x + 4y - 6z$

$u'_x = 2x + 2 \quad u'_y = 2y + 4 \quad u'_z = 2z - 6$

$$\left| \begin{array}{l} x + 1 = 0 \\ y + 2 = 0 \implies M_0(-1, -2, 3) \\ z - 3 = 0 \end{array} \right.$$

$u''_{xx} = 2 \quad u''_{yy} = 2 \quad u''_{zz} = 2$

$u''_{xy} = u''_{xz} = u''_{yx} = u''_{yz} = u''_{zx} = u''_{zy} = 0$

$$d^2u(M_0) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$\Delta_1 = u''_{xx} = 2 > 0 \quad \Delta_2 = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4 > 0 \quad \Delta_3 = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 8 > 0 \implies$$

d^2u е положително дефинитна квадратична форма

u има лок. минимум $\quad u_{min} = u(M_0) = 1 + 4 + 9 - 2 - 4 \cdot 2 - 18 = -14$

$u = x^3 + y^2 + z^2 - 3x + 6y - 2z$

$u'_x = 3x^2 + 2 \quad u'_y = 2y + 6 \quad u'_z = 2z - 2$

$$\left|\begin{array}{l} 3x^2 + 2 = 0 \\ 2y + 6 = 0 \\ 2z - 2 = 0 \end{array}\right. \Longrightarrow M_0(1, -3, 1)\, M_1(-1, -3, 1)$$

$u''_{xx} = 6x \quad u''_{yy} = 2 \quad u''_{zz} = 2$

$u''_{xy} = u''_{xz} = u''_{yx} = u''_{yz} = u''_{zx} = u''_{zy} = 0$

$$d^2u(M_0) = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \Longrightarrow$$

$$\Delta_1 = 6 > 0 \quad \Delta_2 = \begin{vmatrix} 6 & 0 \\ 0 & 2 \end{vmatrix} = 12 > 0 \quad \Delta_3 = \begin{vmatrix} 6 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 24 > 0 \Longrightarrow$$

d^2u е положително дефинитна квадратична форма

u има лок. минимум $\quad u_{min} = u(M_0) = 1 + 9 + 1 - 3 - 18 - 2 = -12$

$$d^2u(M_1) = \begin{pmatrix} -6 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \Longrightarrow$$

$$\Delta_1 = -6 < 0 \quad \Delta_2 = \begin{vmatrix} -6 & 0 \\ 0 & 2 \end{vmatrix} = -12 < 0 \quad \Delta_3 = \begin{vmatrix} -6 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix} = -24 < 0 \Longrightarrow$$

d^2u е не е дефинитна квадратична форма $\Longrightarrow$ няма лок. екстремуми

$$u = x^3 + y^2 + z^2 - 3x - 2y$$
$$u'_x = 3x^2 - 3 \quad u'_y = 2y - 2 \quad u'_z = 2z$$
$$\left| \begin{array}{l} 3x^2 - 3 = 0 \\ 2y - 2 = 0 \\ 2z = 0 \end{array} \right. \Longrightarrow M_0(1,1,0)\, M_1(-1,0,0)$$
$$u''_{xx} = 6x \quad u''_{yy} = 2 \quad u''_{zz} = 2$$
$$u''_{xy} = u''_{xz} = u''_{yx} = u''_{yz} = u''_{zx} = u''_{zy} = 0$$
$$d^2u(M_0) = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \Longrightarrow$$
$$\Delta_1 = 6 > 0 \quad \Delta_2 = \begin{vmatrix} 6 & 0 \\ 0 & 2 \end{vmatrix} = 12 > 0 \quad \Delta_3 = \begin{vmatrix} 6 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 24 > 0 \Longrightarrow$$

d^2u е положително дефинитна квадратична форма

u има лок. минимум $\quad u_{min} = u(M_0) = 1 + 1 - 3 - 2 = -3$

$$d^2u(M_1) = \begin{pmatrix} -6 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \Longrightarrow$$
$$\Delta_1 = -6 < 0 \quad \Delta_2 = \begin{vmatrix} -6 & 0 \\ 0 & 2 \end{vmatrix} = -12 < 0 \quad \Delta_3 = \begin{vmatrix} -6 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix} = -24 < 0 \Longrightarrow$$

d^2u е не е дефинитна квадратична форма $\Longrightarrow$ няма лок. екстремуми

Задача 3

$$F(x,y) = \frac{x^2}{9} + \frac{y^2}{4} - 1 \quad M_0(0,2)$$
$$F'_y = \frac{2y}{4} = \frac{y}{2} \neq 0 y'(x) = -\frac{F'_x}{F'_y} \qquad y''(x) = -\frac{F\text{''}_{xx} + 2F''_{xy}y' + F''_{yy}(y')^2}{F'_y}$$
$$F'_x = \frac{2}{9}x \quad F'_y = \frac{2y}{4} \quad F''_{xx} = \frac{2}{9} \quad F''_{xy} = F''_{yx} = 0 \quad F''_{yy} = \frac{1}{2}$$
$$F'_x(0,2) = 0 \quad F'_y(0,2) = 1 \quad F''_{xx}(0,2) = \frac{2}{9} \quad F''_{xy}(0,2) = F''_{yx}(0,2) = 0 \quad F''_{yy}(0,2) = \frac{1}{2}$$
$$y'(0) = -\frac{0}{1} = 0 \qquad y''(0) = -\frac{\frac{2}{9} + 2 \cdot 0 \cdot 0 + \frac{1}{2} \cdot 0^2}{1} = -\frac{2}{9}$$

Задача 4

Решение. Означаваме

$$F(x,y,z) = z - x\,\varphi(z/y), \qquad (3)$$

откъдето извеждаме условието за $\exists!$ неявна функция:

$$F'_z = 1 - x\,\varphi'\left(\frac{z}{y}\right)\cdot\frac{1}{y} \neq 0. \qquad (4)$$

По-нататък:

$$F'_x = -\varphi(z/y)\,; \quad F'_y = -x\,\varphi'(z/y)\cdot\left(-\frac{z}{y^2}\right), \text{ т.е}$$

$$F'_x = -\varphi(z/y)\,; \quad F'_y = \frac{xz}{y^2}\,\varphi'(z/y) \Rightarrow$$

$$z'_x = -\frac{F'_x}{F'_z} = -\frac{-\varphi(z/y)}{1 - x\,\varphi'(z/y)\cdot\frac{1}{y}} = \frac{\varphi(z/y)}{1 - \frac{x}{y}\varphi'(z/y)}$$

$$z'_y = -\frac{F'_y}{F'_z} = -\frac{\frac{xz}{y^2}\varphi'(z/y)}{1 - \frac{x}{y}\varphi'(z/y)} \Rightarrow$$

$$x z'_x + y z'_y = \frac{\overbrace{x\cdot\varphi(z/y)}^{z\,=}}{1 - \frac{x}{y}\varphi'(z/y)} - \frac{\frac{xz}{y}\varphi'(z/y)}{1 - \frac{x}{y}\varphi'(z/y)} =$$

$$= z\cdot\frac{\cancel{1 - \frac{x}{y}\varphi'(z/y)}}{\cancel{1 - \frac{x}{y}\varphi'(z/y)}} = z.$$

8 Упражнение към лекция 8

8.1 Задачи

Задача 1

Да се изследва за локален екстремум следната функция.

$$z = 1 - \sqrt{x^2 - y^2}$$

Задача 2

Намерете точките на условен екстремум и екстремумите на следните функции.

- $z = x^2 + y^2$, ако $x + y = 1$
- $u = x^2 + y^2 - 12x + 16y$, ако $x^2 + y^2 = 25$
- $u = x + y + z$, ако $z = 1$ и $x^2 + y^2 = 1$

Задача 3

Да се изследва функцията $u = xy + yz$ за условен екстремум, при ограничения.

$$x^2 + y^2 = 2$$
$$y + z = 2$$

Задача 4

Да се изследва функцията $z = x + y$ за условен екстремум, при ограничения.

$$xy = 1$$

Задача 5

Да се намери дефиниционното множество на функциите.

- $z = \sqrt{1 - x^2 - y^2 + 2x}$
- $z = \frac{x^2 y}{2x+y}$
- $z = \arcsin(x + y)$
- $w = \frac{1}{\sqrt{xy}}$

Задача 6

Да се намерят границите ако съществуват.

- $\lim\limits_{(x,y)\to(0,0)} \frac{\tan(xy)}{xy}$
- $\lim\limits_{(x,y)\to(0,0)} \frac{y}{\sin(xy)}$
- $\lim\limits_{(x,y)\to(0,0)} \frac{1-\sqrt{1-xy}}{xy}$

Задача 7

Да се провери дали уравнението удовлетворява посочената функция.

- $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}, z(x,y) = \ln(x^2+y^2+1)$
- $\frac{x}{y}\frac{\partial z}{\partial x} + \frac{1}{\ln x}\frac{\partial z}{\partial y} = 2z, z(x,y) = x^y$
- $2\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} = 0, z(x,y) = 2\cos^2(y-\frac{x}{2})$
- $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 1, u(x,y,z) = x + \frac{x-y}{y-z}$

Задача 8

Да се изследват за локален екстремум следните функции.

- $z = x^4 + y^4 - x^2 - 2xy - y^2$
- $z = xy(1-x-y)$
- $z = x^3 - y^3 - 3x + 3y + 2$
- $u = x^3 + y^2 + z^2 + 12xy + 2z$

Задача 9

Да се изследват за локален екстремум следните неявно зададени функции.

- $x^3 + y^3 = 3xy, y = y(x)$
- $y^2 - 3y - \sin(x) = 0, y = y(x)$
- $x^2 + y^2 + z^2 - xz - yz + 2x + 2y + 2z - 2 = 0, z = z(x,y)$
- $2x^2 + 2y^2 + z^2 + 8xz - 8yz + 8 = 0, z = z(x,y)$

Задача 10

Да се изследва за условен екстремум

- $z = xy$, ако $2x + y = 1$
- $z = x^2 + y^2$, ако $x - y = 1$
- $u = x^2 + y^2 + z^2$, ако $\frac{x^2}{16} + \frac{y^2}{9} + \frac{z^2}{4} = 1$
- $u = xyz$, ако $x + y + z = 5, xy + yz + zx = 8$

Задача 11

Намерете точките на условен екстремум и екстремумите на следните функции.

- $u = x^2 + y^2 + z^2 + 2x + 4y - 6z$, ако $x^2 + y^2 + z^2 = 14$
- $u = x^2 + y^2 + z^2 + 2x + 4y$, ако $x^2 + y^2 = 20$
- $u = x^2 + y^2 + z^2 + 6x - 2y + 4z$, ако $x^2 + y^2 + z^2 = 56$

Задача 12

Намерете абсолютните екстремуми на следните функции и определете вида им (условен, локален, минимум, максимум)

- $u = x^2 + y^2 - 12x + 16y$, ако $x^2 + y^2 \leq 25, x^2 + y^2 \leq 400, x^2 + y^2 \leq 100$
- $u = x^2 + y^2 + z^2 + 2x + 4y - 6$, ако $x^2 + y^2 + z^2 \leq 9$
- $u = x^2 + 2y^2 + 3z^2$, ако $x^2 + y^2 + z^2 \leq 100$

8.2 Решения

Задача 1

$$z(\Delta x,\Delta y)-z(0,0)=1-\sqrt{\Delta x^2-\Delta y^2}-1=-\sqrt{\Delta x^2-\Delta y^2}<0$$

Имаме строг локален максимум в $z(0,0)=1$

Задача 2

$$z=x^2+y^2,\quad x+y=1$$
$$F(x,y\lambda)=x^2+y^2+\lambda(x+y-1),\quad \lambda\neq 0$$
$$F'_x=2x+\lambda \qquad F'_y=2y+\lambda$$
$$\left|\begin{array}{l}2x+\lambda=0\\2y+\lambda=0\\x+y=1\end{array}\right.\Leftrightarrow\left|\begin{array}{l}x=-\frac{\lambda}{2}\\y=-\frac{\lambda}{2}\\-\frac{\lambda}{2}-\frac{\lambda}{2}=1\end{array}\right.\Leftrightarrow\left|\begin{array}{l}x=\frac{1}{2}\\y=\frac{1}{2}\\\lambda=-1\end{array}\right.\implies M_0\left(\frac{1}{2},\frac{1}{2},-1\right)$$
$$F"_{xx}=2\quad F"_{yy}=2\qquad F"_{xy}=F"_{yx}=0$$
$$\Delta=F''_{xx}F_{yy}-F''_{xy}F''_{yx}=4\Big|_{M_0}=4>0$$
$$F''_{xx}\Big|_{M_0}=2>0\implies z_{min}=z\left(\frac{1}{2},\frac{1}{2}\right)=\frac{1}{4}+\frac{1}{4}=\frac{1}{2}$$

$$u=x^2+y^2-12x+16y,\quad x^2+y^2=25$$
$$F(x,y,\lambda)=x^2+y^2-12x-16x+\lambda(x^2+y^2-25)\quad \lambda\neq 0$$
$$F'_x=2x+16+2\lambda x \qquad F'_y=2y-12+2\lambda y$$
$$\left|\begin{array}{l}2x+16+2\lambda x=0\\2y-12+2\lambda y=0\\x^2+y^2=25\end{array}\right.\Leftrightarrow\left|\begin{array}{l}x=\frac{6}{1+\lambda}\\y=-\frac{8}{1+\lambda}\\\frac{36}{(1+\lambda)^2}+\frac{64}{(1+\lambda)^2}=25\end{array}\right.\Leftrightarrow\left|\begin{array}{l}x=-3\wedge x=3\\y=4\wedge y=-4\\\lambda=-3\wedge\lambda=1\end{array}\right.$$
$$\implies M_1(-3,4,-3)\wedge M_2(3,-4,1)$$
$$F''_{xx}=2+2\lambda\quad F''_{yy}=2+2\lambda\quad F''_{xy}=F"_{yx}=0$$
$$\Delta=F''_{xx}F_{yy}-F''_{xy}F''_{yx}=4(1+\lambda)^2$$

$$\Delta\Big|_{M_1} = 4(1+(-3))^2 = 16 > 0$$
$$F''_{xx}\Big|_{M_1} = 2 + 2\cdot -3 = -4 < 0 \implies$$
$$u_{max} = u(-3,4) = (-3)^2 + 4^2 - 12(-3) + 16(4) = 125$$
$$\Delta\Big|_{M_2} = 4(1+1)^2 = 16$$
$$F''_{xx}\Big|_{M_2} = 2 + 2\cdot 1 = 4 > 0 \implies$$
$$u_{min} = u(3,-4) = 3^2 + (-4)^2 - 12\cdot 3 + 16(-4) = -75$$

$$u = x + y + z, \qquad z = 1,\ x^2 + y^2 = 1$$
$$F(x,y,z,\lambda,\nu) = x + y + z + \lambda(z-1) + \nu(x^2+y^2-1), \quad \lambda,\nu \neq 0$$
$$F'_x = 1 + 2\nu x \qquad F'_y = 1 + 2\nu y \qquad F'_z = 1 + \lambda$$
$$\left|\begin{array}{l} 1+2\nu x = 0 \\ 1+2\nu y = 0 \\ 1+\lambda = 0 \\ z = 1 \\ x^2+y^2 = 1 \end{array}\right. \Leftrightarrow \left|\begin{array}{l} x = -\frac{1}{2\nu} \\ y = -\frac{1}{2\nu} \\ z = 1 \\ \lambda = -1 \\ \frac{1}{4\nu^2} + \frac{1}{4\nu^2} = 1 \end{array}\right. \Leftrightarrow \left|\begin{array}{l} x = -\frac{1}{\sqrt{2}} \wedge \frac{1}{\sqrt{2}} \\ y = -\frac{1}{\sqrt{2}} \wedge \frac{1}{\sqrt{2}} \\ z = 1 \\ \lambda = -1 \\ \nu = \frac{1}{\sqrt{2}} \wedge -\frac{1}{\sqrt{2}} \end{array}\right. \implies$$
$$M_1\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 1, -1, \frac{1}{\sqrt{2}}\right) \wedge M_2\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 1, -1, -\frac{1}{\sqrt{2}}\right)$$
$$F''_{xx} = 2\nu \quad F"_{yy} = 2\nu \quad F''_{zz} = F''_{xy} = F''_{xz} = F''_{yx} = F''_{yz} = F''_{zx} = F''_{zy} = 0$$
$$d^2F = \begin{pmatrix} 2\nu & 0 & 0 \\ 0 & 2\nu & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$\Delta_1 = F''_{xx} = 2\mu\Big|_{M_1} = \frac{2}{\sqrt{2}} > 0 \quad \Delta_2 = 4\nu^2\Big|_{M_1} = 2 > 0 \quad \Delta_3 = 0\Big|_{M_1} = 0$$
$$\Delta_1 = F''_{xx} = 2\mu\Big|_{M_2} = -\frac{2}{\sqrt{2}} < 0 \quad \Delta_2 = 4\nu^2\Big|_{M_2} = 2 > 0 \quad \Delta_3 = 0\Big|_{M_2} = 0$$
$$d^2F = 2\nu\, dx^2 + 2\nu\, dy^2$$
$$x^2 + y^2 = 1 \implies x\,dx + y\,dy = 0 (x = y \in M_1, M_2) \implies dx + dy = 0 \implies dy = -dx$$
$$d^2F = 2\nu\, dx^2 + 2\nu\,(-dx)^2 = 4\nu\, dx^2$$
$$d^2F\Big|_{M_1} > 0 \implies u_{min} = u\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = -\sqrt{2} + 1$$
$$d^2F\Big|_{M_2} < 0 \implies u_{max} = u\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \sqrt{2} + 1$$

Задача 3

$$u = xy + yz \quad x^2 + y^2 = 2,\ y + z = 2$$
$$F(x,y,z,\lambda,\nu) = xy + yz + \lambda(x^2 + y^2 - 2) + \nu(y + z - 2), \qquad \lambda,\nu \neq 0$$
$$F'_x = 2\lambda x + y \qquad F'_y = 2\lambda y + \nu + x + z \qquad F'_z = y + \nu$$
$$\left|\begin{array}{l} 2\lambda x + y = 0 \\ 2\lambda y + \nu + x + z = 0 \\ y + \nu = 0 \\ x^2 + y^2 = 2 \\ y + z = 2 \end{array}\right. \Leftrightarrow \left|\begin{array}{l} x = 1 \wedge x = -1 \\ y = 1 \\ z = 1 \\ \lambda = -\frac{1}{2} \wedge \lambda = \frac{1}{2} \\ \nu = -1 \end{array}\right. \implies M\left(1,1,1,-\frac{1}{2},-1\right) \wedge N\left(-1,1,1,\frac{1}{2},-1\right)$$
$$F''_{xx} = 2\lambda \qquad F''_{yy} = 2\lambda \qquad F''_{zz} = 0$$
$$F''_{xy} = F''_{yx} = F''_{yz} = F''_{zy} = 1$$
$$F''_{xz} = F''_{zx} = 0$$
$$d^2F = \begin{pmatrix} 2\lambda & 1 & 0 \\ 1 & 2\lambda & 1 \\ 0 & 1 & 0 \end{pmatrix} \implies d^2F = 2\lambda\, dx^2 + 2\lambda\, dy^2 + 2\, dx\, dy + 2\, dy\, dz$$
$$2x\, dx + 2y\, dy = 0 \qquad dy + dz = 0$$
$$d^2L(M) = -dx^2 - dy^2 + 2\, dx\, dy + 2\, dy\, dz \quad (dx + dy = 0, dy + dz = 0, dx + dz = -2\, dy)$$
$$d^2L(M) = -dx^2 - dy^2 + 2(dx + dz)dy = -dx^2 - dy^2 - 2dy^2 < 0 \implies$$
$$u_{max} = u(1,1,1) = 1 \cdot 1 + 1 \cdot 1 = 2$$
$$d^2(N) = dx^2 + dy^2 + 2(dx + dz)dy \quad (-dx + dy = 0, dy + dz = 0, dx + dz = 0)$$
$$d^2(N) = dx^2 dy^2 - 2(dx + dz)dy = dx^2 + dy^2 > 0 \implies$$
$$u_{min} = u(-1,1,1) = -1 \cdot 1 + 1 \cdot 1 = 0$$

Задача 4

$$z = x + y \qquad xy = 1$$
$$F = x + y + \lambda(xy - 1) \implies F'_x = 1 + \lambda y \quad F'_y = 1 + \lambda x$$
$$\left|\begin{array}{l} 1 + \lambda y = 0 \\ 1 + \lambda x = 0 \\ xy = 1 \end{array}\right. \Leftrightarrow \left|\begin{array}{l} x = 1 \wedge x = -1 \\ y = 1 \wedge y = -1 \\ \lambda = -1 \wedge \lambda = 1 \end{array}\right. \implies M(1,1,-1) \wedge N(-1,-1,1)$$
$$F''_{xx} = F''_{yy} = 0, \qquad F''_{xy} = F''_{yx} = \lambda \implies d^2F = 2\lambda\, dx\, dy$$
$$x\, dx + y\, dy = 0 \implies dx + dy = 0 (x, y \in M, N) \implies d^2F = -2\lambda\, dx^2$$
$$d^2F(M) = 2dx^2 > 0 \implies u_{min} = u(1,1) = 1 + 1 = 2$$
$$d^2F(N) = -2dx^2 < 0 \implies u_{max} = u(-1,-1) = -1 + (-1) = -2$$

Задача 5

- $z = \sqrt{1 - x^2 - y^2 + 2x}$

$$D : (x, y \in \mathbb{R} : x^2 + y^2 \leq 2x + 1)$$

- $z = \frac{x^2 y}{2x+y}$

$$D : (x, y \in \mathbb{R} : 2x + y \neq 0)$$

- $z = \arcsin{(x + y)}$

$$D : (x, y \in \mathbb{R} : -1 \leq x + y \leq 1)$$

- $w = \frac{1}{\sqrt{xy}}$

$$D : (x, y \in \mathbb{R} \setminus 0 : xy > 0)$$

Задача 6

- $\lim\limits_{(x,y)\to(0,0)} \frac{\tan(xy)}{xy}$
  При заместване се получават недефинирани форми от вида $\left[\frac{0}{0}\right]$

$$\lim_{(x,y)\to(0,0)} \frac{\tan(xy)}{xy} = \lim_{(x,y)\to(0,0)} \frac{\sin(xy)}{xy\cos(xy)} = \lim_{(x,y)\to(0,0)} \frac{\sin(xy)}{xy} = \lim_{(x,y)\to(0,0)} \frac{y\cos(xy)}{y} = \lim_{(x,y)\to(0,0)} \cos(xy) = 1$$

- $\lim\limits_{(x,y)\to(0,0)} \frac{y}{\sin(xy)}$

$$\lim_{y\to 0} \frac{y}{\sin(xy)} = \left[\frac{0}{0}\right] \implies \lim_{y\to 0} \frac{1}{x\cos(xy)} = \frac{1}{x}$$

$$\lim_{x\to 0} \frac{1}{x} = \frac{1}{0} = \infty$$

$$\lim_{x\to 0} \frac{y}{\sin xy} = \left[\frac{0}{0}\right] \implies \lim_{x\to 0} \frac{0}{y} = 0$$

$$\lim_{y\to 0} 0 = 0$$

$0 \neq \infty \implies$ Няма граница.

- $\lim\limits_{(x,y)\to(0,0)} \frac{1-\sqrt{1-xy}}{xy}$

$$\lim_{y\to 0}\frac{1-\sqrt{1-xy}}{xy} = \left[\frac{0}{0}\right] \implies \lim_{y\to 0}\frac{x}{2\sqrt{1-xy}}\cdot\frac{1}{x} = \lim_{y\to 0}\frac{1}{2\sqrt{1-xy}} = \frac{1}{2}$$
$$\lim_{x\to 0}\frac{1}{2} = \frac{1}{2}$$
$$\lim_{x\to 0}\frac{1-\sqrt{1-xy}}{xy} = \left[\frac{0}{0}\right] \implies \lim_{x\to 0}\frac{y}{2\sqrt{1-xy}}\cdot\frac{1}{y} = \lim_{x\to 0}\frac{1}{2\sqrt{1-xy}} = \frac{1}{2}$$
$$\lim_{y\to 0}\frac{1}{2} = \frac{1}{2}$$

Задача 7

- $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}, z(x,y) = \ln(x^2+y^2+1)$

$$z'_x = \frac{2x}{x^2+y^2+1} \qquad z'_y = \frac{2y}{x^2+y^2+1}$$
$$z''_{xy} = -\frac{4xy}{(x^2+y^2+1)^2} \qquad z''_{yx} = -\frac{4xy}{(x^2+y^2+1)^2} \implies$$
$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x} \text{ е вярно}$$

- $\frac{x}{y}\frac{\partial z}{\partial x} + \frac{1}{\ln x}\frac{\partial z}{\partial y} = 2z, z(x,y) = x^y$

$$z'_x = \frac{x^y y}{x} \qquad z'_y = x^y \ln(x)$$
$$A = \frac{x}{y}\cdot z'_x = \frac{x}{y}\cdot\frac{x^y y}{x} = x^y$$
$$B = \frac{1}{\ln(x)}\cdot z'_y = \frac{1}{\ln(x)}\cdot x^y\ln(x) = x^y$$
$$A + B = 2x^y \qquad 2z = 2x^y \implies$$
$$\frac{x}{y}\frac{\partial z}{\partial x} + \frac{1}{\ln x}\frac{\partial z}{\partial y} = 2z \text{ е вярно}$$

- $2\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial x\partial y}=0, z(x,y)=2\cos^2(y-\frac{x}{2})$

$$z'_x=-2\cos\left(-y+\frac{x}{2}\right)\sin\left(-y+\frac{x}{2}\right)$$
$$z''_{xx}=-2\cos^2\left(-y+\frac{x}{2}\right)+1 \qquad z''_{xy}=4\cos^2\left(-y+\frac{x}{2}\right)-2$$
$$2z''_{xx}=-4\cos^2\left(-y+\frac{x}{2}\right)+2$$
$$2z''_{xx}+z''_{xy}=-4\cos^2\left(-y+\frac{x}{2}\right)+2+4\cos^2\left(-y+\frac{x}{2}\right)-2=0 \implies$$
$$2\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial x\partial y}=0 \text{ е вярно}$$

- $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=1, u(x,y,z)=x+\frac{x-y}{y-z}$

$$u'_x=\frac{y-z+1}{y-z} \qquad u'_y=\frac{z-x}{(y-x)^2} \qquad u'_z=\frac{x-y}{(y-z)^2}$$
$$u'_x+u'_y+u'_z=\frac{y-z+1}{y-z}+\frac{z-x}{(y-x)^2}+\frac{x-y}{(y-z)^2}=$$
$$\frac{(y-z+1)(y-z)+z-x+x-y}{(y-z)^2}=\frac{y^2-yz-yz+z^2+y-z+z-y}{(y-z)^2}=$$
$$=\frac{y^2-2yz+z^2}{(y-z)^2}=1 \implies$$
$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=1 \text{ е вярно}$$

Задача 8

- $z=x^4+y^4-x^2-2xy-y^2$

$$z'_x=4x^3+2x-2y \qquad z'_y=4y^3-2y-2x$$
$$\left|\begin{array}{l}4x^3+2x-2y=0\\4y^3-2y-2x=0\end{array}\right.\Leftrightarrow\left|\begin{array}{l}x=0\wedge x=1\wedge x=-1\\y=0\wedge y=1\wedge y=-1\end{array}\right.\implies L(0,0),\ M(1,1),\ N(-1,-1)$$
$$z''_{xx}=12x^2+2 \qquad z''_{yy}=12y^2-2 \qquad z'_{xy}=z'_{yx}=-2$$
$$\Delta=z''_{xx}z''_{yy}-z''_{xy}z''_{yx}=(12x^2-2)(12y^2-2)-4$$
$$z''_{xx}(L)=-2<0 \qquad \Delta(L)=0 \implies \text{Няма екстремум.}$$
$$z''_{xx}(M)=10>0 \qquad \Delta(M)=96>0 \implies$$
$$z'_{min1}=z(1,1)=-2$$
$$z''_{xx}(N)=10>0 \qquad \Delta(N)=96>0 \implies$$
$$z'_{min2}=z(-1,-1)=-2$$

- $z = xy(1-x-y)$

$$z'_x = y(1-x-y) - xy \qquad z'_y = x(1-x-y) - xy$$

$$\left| \begin{array}{l} y(1-x-y) - xy = 0 \\ x(1-x-y) - xy = 0 \end{array} \right. \Leftrightarrow \left| \begin{array}{l} x = 0 \wedge x = 0 \wedge x = 1 \wedge x = \frac{1}{3} \\ y = 0 \wedge y = 1 \wedge y = 0 \wedge y = \frac{1}{3} \end{array} \right. \Longrightarrow$$

$$A(0,0),\ B(0,1),\ C(1,0),\ D\left(\frac{1}{3}, \frac{1}{3}\right)$$

$$z''_{xx} - 2y = \qquad z''_{yy} = -2y \qquad z''_{xy} = z''_{yx} = 1 - 2x - 2y$$

$$\Delta = z''_{xx} z''_{yy} - z''_{xy} z''_{yx} = 4xy - (1-2x-2y)^2$$

$z''_{xx}(A) = 0 \qquad \Delta(A) = -1 \implies$ Няма екстремум.

$z''_{xx}(B) = -2 \qquad \Delta(A) = -1 \implies$ Няма екстремум.

$z''_{xx}(C) = 0 \qquad \Delta(C) = -1 \implies$ Няма екстремум.

$$z''_{xx}(D) = -\frac{2}{3} \qquad \Delta(D) = \frac{1}{3} \implies z_{max} = z\left(\frac{1}{3}, \frac{1}{3}\right) = \frac{1}{27}$$

- $z = x^3 - y^3 - 3x + 3y + 2$

$$z'_x = 3x^2 - 3 \qquad z'_y = -3y^2 + 3$$

$$\left| \begin{array}{l} 3x^2 - 3 = 0 \\ -3y^2 + 3 = 0 \end{array} \right. \Leftrightarrow \left| \begin{array}{l} x = 1 \wedge x = -1 \\ x = 1 \wedge x = -1 \end{array} \right. \Longrightarrow A(1,1),\ B(1,-1),\ C(-1,1),\ D(-1,-1)$$

$$z''_{xx} 6x = \qquad z''_{yy} = -6y \qquad z''_{xy} = z''_{yx} = 0$$

$$\Delta = z''_{xx} z''_{yy} - z''_{xy} z''_{yx} = -36xy$$

$z''_{xx}(A) = 6 > 0 \qquad \Delta(A) = -36 < 0 \implies$ Няма екстремум.

$z''_{xx}(B) = 6 > 0 \qquad \Delta(A) = 36 > 0 \implies z_{min} = z(1,-1) = -2$

$z''_{xx}(C) = -6 < 0 \qquad \Delta(A) = 36 > 0 \implies z_{max} = z(-1,1) = 6$

$z''_{xx}(D) = -6 < 0 \qquad \Delta(A) = -36 < 0 \implies$ Няма екстремум.

- $u = x^3 + y^2 + z^2 + 12xy + 2z$

$$u'_x = 3x^2 - 12y \qquad u'_y = 3y^2 - 12x \qquad u'_z = 2z^2 + 2$$

$$\left| \begin{array}{l} 3x^2 - 12y = 0 \\ 3y^2 - 12x = 0 \\ 2z^2 + 2 = 0 \end{array} \right. \Leftrightarrow \left| \begin{array}{l} x = 0 \wedge x = 24 \\ y = 0 \wedge y = -144 \\ z = -1 \end{array} \right. \implies A(0,0,-1),\ B(24,-144,-1)$$

$$u''_{xx} = 6x \qquad u''_{xy} = 12 \qquad u''_{xz} = 0$$

$$u''_{yx} = 12 \qquad u''_{yy} = 2 \qquad u''_{yz} = 0$$

$$u''_{zx} = 0 \qquad u''_{zy} = 0 \qquad u''_{zz} = 2$$

$$\Delta_1 = u''_{xx} = 6x$$

$$\Delta_2 = \begin{vmatrix} u''_{xx} & u''_{xy} \\ u''_{yx} & u''_{yy} \end{vmatrix} = \begin{vmatrix} 6x & 12 \\ 12 & 2 \end{vmatrix} = 12x - 144$$

$$\Delta_3 = \begin{vmatrix} u''_{xx} & u''_{xy} & u''_{xz} \\ u''_{yx} & u''_{yy} & u''_{yz} \\ u''_{zx} & u''_{zy} & u''_{zz} \end{vmatrix} = \begin{vmatrix} 6x & 12 & 0 \\ 12 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 24x - 288$$

$$\Delta_1(A) = 0 \qquad \Delta_2(A) = -144 \qquad \Delta_3(A) = -288 \implies \text{Няма екстремум.}$$

$$\Delta_1(B) = 144 > 0 \qquad \Delta_2(B) = 144 > 0 \qquad \Delta_3(B) = 288 > 0 \implies$$

$$u_{min} = u(24, -144, -1) = -6913$$

Задача 9

- $x^3 + y^3 = 3xy, y = y(x)$

$$F = x^3 + y^3 - 3xy$$

$$F'_x = 3x^2 - 3y \qquad F'_y = 3y^2 - 3x$$

$$y' = -\frac{f'_x}{f'_y} = -\frac{3x^2 - 3y}{3y^2 - 3x}$$

$$\left| \begin{array}{l} -\frac{3x^2-3y}{3y^2-3x} = 0 \\ x^3 + y^3 = 3xy \\ 3y^2 - 3x \neq 0 \end{array} \right. \Leftrightarrow \left| \begin{array}{l} x = \sqrt[3]{2} \\ y = \sqrt[3]{2^2} \end{array} \right. \implies A = (\sqrt[3]{2}, \sqrt[3]{2^2})$$

$$F''_{xx} = 6x$$

$$y'' = -\frac{f''_{xx}}{f'_y} = -\frac{6x}{3y^2 - 3x}$$

$$y''(A) = -2 < 0 \implies y_{max} = y(\sqrt[3]{2}) = \sqrt[3]{2^2}$$

- $y^2 - 3y - \sin(x) = 0, y = y(x)$

$$F = y^2 - 3y - \sin(x)$$
$$F'_x = -\cos(x) \qquad F'_y = 2y - 3$$
$$y' = -\frac{f'_x}{f'_y} = \frac{\cos(x)}{2y-3}$$
$$\left| \begin{array}{l} \frac{\cos(x)}{2y-3} = 0 \\ y^2 - 3y - \sin(x) = 0 \\ 2y - 3 \neq 0 \end{array} \right. \Leftrightarrow \left| \begin{array}{l} x = \frac{\pi}{2} \wedge x = -\frac{\pi}{2} \\ y = \frac{3+\sqrt{13}}{2} \wedge y = \frac{3-\sqrt{5}}{2} \end{array} \right. \implies$$
$$A = \left(\frac{\pi}{2} + 2k\pi, \frac{3+\sqrt{13}}{2}\right), B\left(-\frac{\pi}{2} + 2k\pi, \frac{3-\sqrt{5}}{2}\right)$$
$$F''_{xx} = \sin(x)$$
$$y'' = -\frac{f''_{xx}}{f'_y} = -\frac{\sin(x)}{2y-3}$$
$$y''(A) = -\frac{\sqrt{13}}{13} < 0 \implies y_{max,k} = y\left(\frac{\pi}{2} + 2k\pi\right) = \frac{3+\sqrt{13}}{2}$$
$$y''(B) = -\frac{\sqrt{5}}{5} < 0 \implies y_{min,k} = y\left(-\frac{\pi}{2} + 2k\pi\right) = \frac{3-\sqrt{5}}{2}$$

(Панева, как тва е минимум беее)

- $x^2 + y^2 + z^2 - xz - yz + 2x + 2y + 2z - 2 = 0, z = z(x, y)$

$$F = x^2 + y^2 + z^2 - xz - yz + 2x + 2y + 2z - 2$$
$$F'_x = 2x - z + 2 \qquad F'_y = 2y - z + 2 \qquad F'_z = -x - y + 2z + 2$$
$$z'_x = -\frac{F'_x}{F'_z} = -\frac{2x - z + 2}{-x - y + 2z + 2} \qquad z'_y = -\frac{F'_y}{F'_z} = -\frac{2y - z + 2}{-x - y + 2z + 2}$$
$$\left| \begin{array}{l} -\frac{2x-z+2}{-x-y+2z+2} = 0 \\ -\frac{2y-z+2}{-x-y+2z+2} = 0 \\ x^2 + y^2 + z^2 - xz - yz + 2x + 2y + 2z - 2 = 0 \\ -x - y + 2z + 2 \neq 0 \end{array} \right. \Leftrightarrow \left| \begin{array}{l} x = -3 + \sqrt{6} \\ y = -3 + \sqrt{6} \\ z = -4 + 2\sqrt{6} \end{array} \right. \implies$$
$$A(-3 + \sqrt{6}, -3 + \sqrt{6}, -4 + 2\sqrt{6})$$

$$F''_{xx} = 2 \qquad F''_{xy} = 0 \qquad F''_{yx} = 0 \qquad F''_{yy} = 2$$

$$\Delta = \begin{vmatrix} -\frac{F''_{xx}}{F'_z} & -\frac{F''_{xy}}{F'_z} \\ -\frac{F''_{yx}}{F'_z} & -\frac{F''_{yy}}{F'_z} \end{vmatrix} = \begin{vmatrix} -\frac{2}{-x-y+2z+2} & 0 \\ 0 & -\frac{2}{-x-y+2z+2} \end{vmatrix} = \frac{4}{(-x-y+2z+2)^2}$$

$$-\frac{F''_{xx}}{F'_z}(A) = -\frac{\sqrt{6}}{6} \qquad \Delta(A) = \frac{1}{6} \implies z_{max} = z(-3+\sqrt{6}, -3+\sqrt{6}) = -4 + 2\sqrt{6}$$

от къде дойде втората точка в решението на Панева

- $2x^2 + 2y^2 + z^2 + 8xz - 8yz + 8 = 0, z = z(x, y)$

$$F = 2x^2 + 2y^2 + z^2 + 8xz - 8yz + 8$$

$$F'_x = 4x + 8z \qquad F'_y = 4y - 8z \qquad F'_z = 8x - 8y + 2z$$

$$z'_x = -\frac{F'_x}{F'_z} = -\frac{4x+8z}{8x-8y+2z} \qquad z'_y = -\frac{F'_y}{F'_z} = -\frac{4y-8z}{8x-8y+2z}$$

$$\left| \begin{array}{l} -\frac{4x+8z}{8x-8y+2z} = 0 \\ -\frac{4y-8z}{8x-8y+2z} = 0 \\ 2x^2 + 2y^2 + z^2 + 8xz - 8yz + 8 = 0 \\ 8x - 8y + 2z \neq 0 \end{array} \right. \Leftrightarrow \left| \begin{array}{l} x = -\frac{4\sqrt{30}}{15} \\ y = \frac{4\sqrt{30}}{15} \\ z = \frac{2\sqrt{30}}{15} \end{array} \right. \implies A\left(-\frac{4\sqrt{30}}{15}, \frac{4\sqrt{30}}{15}, \frac{2\sqrt{30}}{15}\right)$$

$$F''_{xx} = 4 \qquad F''_{xy} = 0 \qquad F''_{yx} = 0 \qquad F''_{yy} = 4$$

$$\Delta = \begin{vmatrix} -\frac{F''_{xx}}{F'_z} & -\frac{F''_{xy}}{F'_z} \\ -\frac{F''_{yx}}{F'_z} & -\frac{F''_{yy}}{F'_z} \end{vmatrix} = \begin{vmatrix} -\frac{4}{8x-8y+2z} & 0 \\ 0 & -\frac{4}{8x-8y+2z} \end{vmatrix} = \frac{16}{(8x-8y+2z)^2}$$

$$-\frac{F''_{xx}}{F'_z}(A) = \frac{\sqrt{30}}{30} > 0 \qquad \Delta(A) = \frac{1}{30} > 0 \implies z_{min} = z\left(-\frac{4\sqrt{30}}{15}, \frac{4\sqrt{30}}{15}\right) = \frac{2\sqrt{30}}{15}$$

Задача 10

- $z = xy$, ако $2x + y = 1$

$$F(x, y, \lambda) = xy + \lambda(2x + y - 1) \quad \lambda \neq 0$$

$$F'_x = 2\lambda + y \qquad F'_y = \lambda + x$$

$$\left| \begin{array}{l} 2\lambda + y = 0 \\ \lambda + x = 0 \\ 2x + y = 1 \end{array} \right. \Leftrightarrow \left| \begin{array}{l} x = \frac{1}{4} \\ y = \frac{1}{2} \\ \lambda = -\frac{1}{4} \end{array} \right. \implies A\left(\frac{1}{4}, \frac{1}{2}, -\frac{1}{4}\right)$$

$$F''_{xx} = 0 \qquad F''_{xy} = 1 \qquad F''_{yx} = 1 \qquad F''_{yy} = 0$$

$$\Delta = F''_{xx}F''_{yy} - F''_{xy}F''_{yx} = -1$$

- $z = x^2 + y^2$, ако $x - y = 1$

$$F(x,y,\lambda) = x^2 + y^2 + \lambda(x - y - 1) \quad \lambda \neq 0$$
$$F'_x = \lambda + 2x \qquad F'_y = 2y - \lambda$$
$$\left|\begin{array}{l} \lambda + 2x = 0 \\ 2y - \lambda = 0 \\ x - y = 1 \end{array}\right. \Leftrightarrow \left|\begin{array}{l} x = \frac{1}{2} \\ y = -\frac{1}{2} \\ \lambda = -1 \end{array}\right. \Longrightarrow A\left(\frac{1}{2}, -\frac{1}{2}, -1\right)$$
$$F''_{xx} = 2 \qquad F''_{xy} = 0 \qquad F''_{yx} = 0 \qquad F''_{yy} = 2$$
$$\Delta = F''_{xx}F''_{yy} - F''_{xy}F''_{yx} = 4$$
$$F''_{xx}(A) = 2 > 0 \qquad \Delta(A) = 4 > 0 \Longrightarrow z_{min} = z\left(\frac{1}{2}, -\frac{1}{2}\right) = \frac{1}{2}$$

- $u = x^2 + y^2 + z^2$, ако $\frac{x^2}{16} + \frac{y^2}{9} + \frac{z^2}{4} = 1$

$$F(x,y,z,\lambda) = x^2 + y^2 + z^2 + \lambda\left(\frac{x^2}{16} + \frac{y^2}{9} + \frac{z^2}{4} - 1\right) \quad \lambda \neq 0$$
$$F'_x = \frac{1}{8}\lambda x + 2x \qquad F'_y = \frac{2}{9}\lambda y + 2y \qquad F'_z = \frac{1}{2}\lambda z + 2z$$
$$\left|\begin{array}{l} \frac{1}{8}\lambda x + 2x = 0 \\ \frac{2}{9}\lambda y + 2y = 0 \\ \frac{1}{2}\lambda z + 2z = 0 \\ \frac{x^2}{16} + \frac{y^2}{9} + \frac{z^2}{4} = 1 \end{array}\right. \Leftrightarrow \left|\begin{array}{l} x = 4 \wedge x = -4 \wedge x = 0 \\ y = 0 \wedge y = 3 \wedge y = -3 \\ z = 0 \wedge z = 2 \wedge z = -2 \\ \lambda = -16 \wedge \lambda = -9 \wedge \lambda = -4 \end{array}\right. \Longrightarrow$$
$$A(4,0,0,-16),\ B(-4,0,0,-16),\ C(0,3,0,-9),$$
$$D(0,-3,0,-9)\ E(0,0,2,-4),\ F(0,0,-2,-4)$$

$$F''_{xx} = \frac{1}{8}\lambda + 2 \qquad F''_{yy} = \frac{2}{9}\lambda + 2 \qquad F''_{zz} = \frac{1}{2}\lambda + 2$$
$$F''_{xy} = F''_{xz} = F''_{yx} = F''_{yz} = F''_{zx} = F''_{zy} = 0$$
$$\Delta_1 = F''_{xx} = \frac{1}{8}\lambda$$
$$\Delta_2 = \begin{vmatrix} F''_{xx} & F''_{xy} \\ F''_{yx} & F''_{yy} \end{vmatrix} = \begin{vmatrix} \frac{1}{8}\lambda + 2 & 0 \\ 0 & \frac{2}{9}\lambda + 2 \end{vmatrix} = \left(\frac{1}{8}\lambda + 2\right)\left(\frac{2}{9}\lambda + 2\right)$$
$$\Delta_3 = \begin{vmatrix} F''_{xx} & F''_{xy} & F''_{xz} \\ F''_{yx} & F''_{yy} & F''_{yz} \\ F''_{zx} & F''_{zy} & F''_{zz} \end{vmatrix} = \begin{vmatrix} \frac{1}{8}\lambda + 2 & 0 & 0 \\ 0 & \frac{2}{9}\lambda + 2 & 0 \\ 0 & 0 & \frac{1}{2}\lambda + 2 \end{vmatrix} = \left(\frac{1}{8}\lambda + 2\right)\left(\frac{2}{9}\lambda + 2\right)\left(\frac{1}{2}\lambda + 2\right)$$

$$\begin{array}{llll}
\Delta_1(A)=0 & \Delta_2(A)=0 & \Delta_3(A)=0 \implies & \text{Няма екстремум.}\\
\Delta_1(B)=0 & \Delta_2(B)=0 & \Delta_3(B)=0 \implies & \text{Няма екстремум.}\\
\Delta_1(C)=0 & \Delta_2(C)=0 & \Delta_3(C)=0 \implies & \text{Няма екстремум.}\\
\Delta_1(D)=0 & \Delta_2(D)=0 & \Delta_3(D)=0 \implies & \text{Няма екстремум.}\\
\Delta_1(E)=0 & \Delta_2(E)=0 & \Delta_3(E)=0 \implies & \text{Няма екстремум.}\\
\Delta_1(F)=0 & \Delta_2(F)=0 & \Delta_3(F)=0 \implies & \text{Няма екстремум.}
\end{array}$$

Панева кво стааа?!

- $u = xyz$, ако $x+y+z=5, xy+yz+zx=8$

$$F(x,y,z,\lambda,\mu) = xyz + \lambda(x+y+z-5) + \mu(xy+yz+zx-8) \quad \lambda,\mu \neq 0$$

$$F'_x = \lambda + \mu(y+z) + yz \qquad F'_y = \lambda + \mu(x+z) + xz \qquad F'_z = \lambda + \mu(y+x) + yx$$

$$\left|\begin{array}{l}
\lambda + \mu(y+z) + yz = 0\\
\lambda + \mu(x+z) + xz = 0\\
\lambda + \mu(y+x) + yx = 0\\
x+y+z=5\\
xy+yz+zx=8
\end{array}\right. \Leftrightarrow
\left|\begin{array}{l}
x=1 \wedge x=2 \wedge x=\frac{4}{3} \wedge x=\frac{7}{3}\\
y=2 \wedge y=1 \wedge y=\frac{4}{3} \wedge y=\frac{7}{3}\\
z=2 \wedge z=1 \wedge z=\frac{4}{3}\\
\lambda=4 \wedge \lambda=\frac{16}{9}\\
\mu=-2 \wedge \mu=-\frac{4}{3}
\end{array}\right. \implies$$

$$A\,(1,2,2,4,-2)\,,\ B\,(2,2,1,4,-2)\,,\ C\,(2,1,2,4,-2)$$

$$D\left(\frac{4}{3},\frac{7}{3},\frac{4}{3},\frac{16}{9},-\frac{4}{3}\right),\ E\left(\frac{7}{3},\frac{4}{3},\frac{4}{3},\frac{16}{9},-\frac{4}{3}\right),\ F\left(\frac{4}{3},\frac{4}{3},\frac{7}{3},\frac{16}{9},-\frac{4}{3}\right)$$

$$F''_{xx} = F''_{yy} = F''_{zz} = 0$$

$$F''_{xy} = F''_{yx} = \mu + z$$

$$F''_{xz} = F''_{zx} = \mu + y$$

$$F''_{yz} = F''_{zy} = \mu + x$$

$$\Delta_1 = F''_{xx} = 0$$

$$\Delta_2 = \begin{vmatrix} F''_{xx} & F''_{xy} \\ F''_{yx} & F''_{yy} \end{vmatrix} = \begin{vmatrix} 0 & \mu+z \\ \mu+z & 0 \end{vmatrix} = -(\mu+z)^2$$

$$\Delta_3 = \begin{vmatrix} F''_{xx} & F''_{xy} & F''_{xz} \\ F''_{yx} & F''_{yy} & F''_{yz} \\ F''_{zx} & F''_{zy} & F''_{zz} \end{vmatrix} = \begin{vmatrix} 0 & \mu+z & \mu+y \\ \mu+z & 0 & \mu+x \\ \mu+y & \mu+x & 0 \end{vmatrix} = 2(\mu+z)(\mu+y)(\mu+x)$$

$\Delta_1(A) = 0 \qquad \Delta_2(A) = 0 \qquad \Delta_3(A) = 0 \implies$ Няма екстремум.
$\Delta_1(B) = 0 \qquad \Delta_2(B) = -1 \qquad \Delta_3(B) = 0 \implies$ Няма екстремум.
$\Delta_1(C) = 0 \qquad \Delta_2(C) = 0 \qquad \Delta_3(C) = 0 \implies$ Няма екстремум.
$\Delta_1(D) = 0 \qquad \Delta_2(D) = 0 \qquad \Delta_3(D) = 0 \implies$ Няма екстремум.
$\Delta_1(E) = 0 \qquad \Delta_2(E) = 0 \qquad \Delta_3(E) = 0 \implies$ Няма екстремум.
$\Delta_1(F) = 0 \qquad \Delta_2(F) = -1 \qquad \Delta_3(F) = 0 \implies$ Няма екстремум.
Панева кво стааа?!

Задача 11

- $u = x^2 + y^2 + z^2 + 2x + 4y - 6z$, ако $x^2 + y^2 + z^2 = 14$

$$F = x^2 + y^2 + z^2 + 2x + 4y - 6z + \lambda(x^2 + y^2 + z^2 - 14) \quad \lambda \neq 0$$

$$F'_x = 2\lambda x + 2x + 2 \qquad F'_y = 2\lambda y + 2y + 4 \qquad F'_z = 2\lambda z + 2z - 6$$

$$\left| \begin{array}{l} 2\lambda x + 2x + 2 = 0 \\ 2\lambda y + 2y + 4 = 0 \\ 2\lambda z + 2z - 6 = 0 \\ x^2 + y^2 + z^2 = 14 \end{array} \right. \Leftrightarrow \left| \begin{array}{l} x = -1 \wedge x = 1 \\ y = -2 \wedge y = 2 \\ z = 3 \wedge z = -3 \\ \lambda = 0 \wedge \lambda = -2 \end{array} \right. \implies A(-1,-2,3,0),\ B(1,2,-3,-2)$$

$$F''_{xx} = F''_{yy} = F''_{zz} = 2\lambda + 2$$

$$F''_{xy} = F''_{xz} = F''_{yx} = F''_{yz} = F''_{zx} = F''_{zy} = 0$$

$$\Delta_1 = F''_{xx} = 2\lambda + 2$$

$$\Delta_2 = \begin{vmatrix} F''_{xx} & F''_{xy} \\ F''_{yx} & F''_{yy} \end{vmatrix} = \begin{vmatrix} 2\lambda + 2 & 0 \\ 0 & 2\lambda + 2 \end{vmatrix} = (2\lambda + 2)^2$$

$$\Delta_3 = \begin{vmatrix} F''_{xx} & F''_{xy} & F''_{xz} \\ F''_{yx} & F''_{yy} & F''_{yz} \\ F''_{zx} & F''_{zy} & F''_{zz} \end{vmatrix} = \begin{vmatrix} 2\lambda + 2 & 0 & 0 \\ 0 & 2\lambda + 2 & 0 \\ 0 & 0 & 2\lambda + 2 \end{vmatrix} = (2\lambda + 2)^3$$

$\Delta_1(A) = 2 > 0 \qquad \Delta_2(A) = 4 > 0 \qquad \Delta_3(A) = 8 > 0 \implies$
$u_{min} = u(-1,-2,3) = -14$
$\Delta_1(B) = -2 < 0 \qquad \Delta_2(B) = 4 > 0 \qquad \Delta_3(B) = -8 < 0 \implies$
$u_{max} = u(1,2,-3) = 42$

- $u = x^2 + y^2 + z^2 + 2x + 4y$, ако $x^2 + y^2 = 20$

$$F = x^2 + y^2 + z^2 + 2x + 4y + \lambda(x^2 + y^2 - 20) \quad \lambda \neq 0$$
$$F'_x = 2\lambda x + 2x + 2 \qquad F'_y = 2\lambda y + 2y + 4 \qquad F'_z = 2z$$
$$\left| \begin{array}{l} 2\lambda x + 2x + 2 = 0 \\ 2\lambda y + 2y + 4 = 0 \\ 2z = 0 \\ x^2 + y^2 = 20 \end{array} \right. \Leftrightarrow \left| \begin{array}{l} x = 2 \wedge x = -2 \\ y = 4 \wedge y = -4 \\ z = 0 \\ \lambda = -\frac{3}{2} \wedge \lambda = -\frac{1}{2} \end{array} \right. \Longrightarrow A\left(2, 4, 0, -\frac{3}{2}\right), B\left(-2, -4, 0, -\frac{1}{2}\right)$$

$$F''_{xx} = F''_{yy} = 2\lambda + 2 \qquad F''_{zz} = 2$$
$$F''_{xy} = F''_{xz} = F''_{yx} = F''_{yz} = F''_{zx} = F''_{zy} = 0$$
$$\Delta_1 = F''_{xx} = 2\lambda + 2$$
$$\Delta_2 = \begin{vmatrix} F''_{xx} & F''_{xy} \\ F''_{yx} & F''_{yy} \end{vmatrix} = \begin{vmatrix} 2\lambda + 2 & 0 \\ 0 & 2\lambda + 2 \end{vmatrix} = (2\lambda + 2)^2$$
$$\Delta_3 = \begin{vmatrix} F''_{xx} & F''_{xy} & F''_{xz} \\ F''_{yx} & F''_{yy} & F''_{yz} \\ F''_{zx} & F''_{zy} & F''_{zz} \end{vmatrix} = \begin{vmatrix} 2\lambda + 2 & 0 & 0 \\ 0 & 2\lambda + 2 & 0 \\ 0 & 0 & 2 \end{vmatrix} = (2\lambda + 2)(4\lambda + 4)$$

$\Delta_1(A) = -1 \qquad \Delta_2(A) = 1 > 0 \qquad \Delta_3(A) = 2 > 0 \Longrightarrow$ Няма екстремум.
$\Delta_1(B) = 1 > 0 \qquad \Delta_2(B) = 1 > 0 \qquad \Delta_3(B) = 2 > 0 \Longrightarrow u_{min} = u(-2, -4, 0) = 0$

- $u = x^2 + y^2 + z^2 + 6x - 2y + 4z$, ако $x^2 + y^2 + z^2 = 56$

$$F = x^2 + y^2 + z^2 + 6x - 2y + 4z + \lambda(x^2 + y^2 + z^2 - 56) \quad \lambda \neq 0$$
$$F'_x = 2\lambda x + 2x + 6 \qquad F'_y = 2\lambda y + 2y - 2 \qquad F'_z = 2\lambda z + 2z + 4$$
$$\left| \begin{array}{l} 2\lambda x + 2x + 6 = 0 \\ 2\lambda y + 2y - 2 = 0 \\ 2\lambda z + 2z + 4 = 0 \\ x^2 + y^2 + z^2 = 56 \end{array} \right. \Leftrightarrow \left| \begin{array}{l} x = 6 \wedge x = -6 \\ y = -2 \wedge y = 2 \\ z = 4 \wedge z = -4 \\ \lambda = -\frac{3}{2} \wedge \lambda = -\frac{1}{2} \end{array} \right. \Longrightarrow A\left(6, -2, 4, -\frac{3}{2}\right), B\left(-6, 2, -4, -\frac{1}{2}\right)$$

$$F''_{xx} = F''_{yy} = F''_{zz} = 2\lambda + 2$$
$$F''_{xy} = F''_{xz} = F''_{yx} = F''_{yz} = F''_{zx} = F''_{zy} = 0$$
$$\Delta_1 = F''_{xx} = 2\lambda + 2$$
$$\Delta_2 = \begin{vmatrix} F''_{xx} & F''_{xy} \\ F''_{yx} & F''_{yy} \end{vmatrix} = \begin{vmatrix} 2\lambda+2 & 0 \\ 0 & 2\lambda+2 \end{vmatrix} = (2\lambda+2)^2$$
$$\Delta_3 = \begin{vmatrix} F''_{xx} & F''_{xy} & F''_{xz} \\ F''_{yx} & F''_{yy} & F''_{yz} \\ F''_{zx} & F''_{zy} & F''_{zz} \end{vmatrix} = \begin{vmatrix} 2\lambda+2 & 0 & 0 \\ 0 & 2\lambda+2 & 0 \\ 0 & 0 & 2\lambda+2 \end{vmatrix} = (2\lambda+2)^3$$

$$\Delta_1(A) = -1 < 0 \qquad \Delta_2(A) = 1 > 0 \qquad \Delta_3(A) = -1 < 0 \implies$$
$$u_{max} = u(6, -2, 4) = 112$$
$$\Delta_1(B) = 1 > 0 \qquad \Delta_2(B) = 1 > 0 \qquad \Delta_3(B) = 1 > 0 \implies$$
$$u_{min} = u(-6, 2, -4) = 0$$

Задача 12

- $u = x^2 + y^2 - 12x + 16y$, ако $x^2 + y^2 \leq 25, x^2 + y^2 \leq 400, x^2 + y^2 \leq 100$
- $u = x^2 + y^2 + z^2 + 2x + 4y - 6$, ако $x^2 + y^2 + z^2 \leq 9$
- $u = x^2 + 2y^2 + 3z^2$, ако $x^2 + y^2 + z^2 \leq 100$

9 Упражнение към лекция 9

9.1 Задачи

Задача 1

Да се пресметнат интегралите

- $I = \iint\limits_D xy\,dx\ dy, \qquad D: \begin{cases} 0 \le x \le 1 \\ 0 \le y \le 2 \end{cases}$
- $I = \iint\limits_D xy\,dx\ dy$, ако D е оградена от $\begin{cases} xy = 1 \\ x + y = \frac{5}{2} \end{cases}$
- $I = \iint\limits_D \ dx\ dy$, ако D е оградена от кривите $D: \begin{cases} 4y = x^2 - 4x \\ x - y - 3 = 0 \end{cases}$

Задача 2

Да се определят границите на интегриране и да се пресметне интеграла

$$\iint\limits_D (x^2 + y^2)\,dx\ dy, \qquad D: \begin{cases} y = x, \quad y = 2 \\ y = x + 2, \quad y = 6 \end{cases}$$

Задача 3

Да се пресметне интеграла

$$\iint\limits_D (x + y)\,dx\ dy, \qquad \partial D: \begin{cases} y^2 = 2x \\ x + y = 4 \\ x + y = 12 \end{cases}$$

Задача 4

Да се пресметне интеграла

$$\iint\limits_D (x + y)\,dx\ dy$$

Където D е триъгълник $\triangle ABO$ с върхове $A(1, 0)$, $B = (1, 1)$, $O = (0, 0)$

9.2 Решения

Задача 1

- $I = \iint\limits_D xy\,dx\;dy, \qquad D: \begin{cases} 0 \le x \le 1 \\ 0 \le y \le 2 \end{cases}$

$$I = \int\limits_0^1 x \int\limits_0^2 y\,dy\;dx \quad \int\limits_0^2 y\,dy = \frac{y^2}{2}\Big|_0^2 = 2$$

$$I = 2\int\limits_0^1 x\,dx = 2\frac{x^2}{2}\Big|_0^1 = 1$$

- $I = \iint\limits_D xy\,dx\;dy$, ако D е оградена от $\begin{cases} xy = 1 \\ x + y = \frac{5}{2} \end{cases}$

$$\left| \begin{array}{l} xy = 1 \\ x + y = \frac{5}{2} \end{array} \right. \Leftrightarrow \left| \begin{array}{l} y = \frac{5}{2} - x \\ x\left(\frac{5}{2} - x\right) = 1 \end{array} \right. \implies D: \begin{cases} \frac{1}{2} \le x \le 2 \\ \frac{1}{x} \le y \le \frac{5}{2} - x \end{cases}$$

$$I = \int\limits_{\frac{1}{2}}^2 x \int\limits_{\frac{1}{x}}^{\frac{5}{2}-x} y\,dy\;dx$$

$$\int\limits_{\frac{1}{x}}^{\frac{5}{2}-x} y\,dy = \frac{y^2}{2}\Big|_{\frac{1}{x}}^{\frac{5}{2}-x} = \frac{1}{2}\left[\left(\frac{5}{2} - x\right)^2 - \frac{1}{x^2}\right]$$

$$I = \frac{1}{2}\int\limits_{\frac{1}{2}}^2 x\left[\left(\frac{5}{2} - x\right)^2 - \frac{1}{x^2}\right] dx = \frac{1}{2}\int\limits_{\frac{1}{2}}^2 x\left(\frac{25}{4} - 5x + x^2 - \frac{1}{x^2}\right) dx =$$

$$= \frac{1}{2}\int\limits_{\frac{1}{2}}^2 \left(\frac{25}{4}x - 5x^2 + x^3 - \frac{1}{x}\right) dx = \frac{1}{2}\left[\frac{x^4}{4} - \frac{5x^3}{3} + \frac{25x^2}{8} - \ln|x|\right]\Big|_{\frac{1}{2}}^2 = \frac{165}{128} - \ln(2)$$

- $I = \iint\limits_D dx\ dy$, ако D е оградена от кривите $D : \begin{cases} 4y = x^2 - 4x \\ x - y - 3 = 0 \end{cases}$

$$\left|\begin{array}{l} 4y = x^2 - 4x \\ x - y - 3 = 0 \end{array}\right. \Leftrightarrow \left|\begin{array}{l} y = x - 3 \\ 4(x-3) = x^2 - 4x \end{array}\right. \implies D : \begin{cases} 2 \le x \le 6 \\ \frac{x^2}{4} - x \le y \le x - 3 \end{cases}$$

$$I = \int\limits_2^6 \int\limits_{\frac{x^2}{4}-x}^{x-3} dy\ dx$$

$$\int\limits_{\frac{x^2}{4}-x}^{x-3} dy = y\Big|_{\frac{x^2}{4}-x}^{x-3} = \left[x - 3 - \left(\frac{x^2}{4} - x\right)\right]$$

$$I = \int\limits_2^6 \left[x - 3 - \left(\frac{x^2}{4} - x\right)\right] dx = \int\limits_2^6 \left[2x - 3 - \frac{x^2}{4}\right] dx = \left[x^2 - 3x - \frac{x^3}{12}\right]\Big|_2^6 = \frac{8}{3}$$

Задача 2

$$I = \iint\limits_D (x^2 + y^2)\, dx\ dy, \qquad D : \begin{cases} y = x, \quad y = 2 \\ y = x + 2, \quad y = 6 \end{cases}$$

$$D_1 : \begin{cases} 0 \le x \le 2 \\ 2 \le y \le x + 2 \end{cases} \qquad D_2 : \begin{cases} 2 \le x \le 4 \\ x \le y \le x + 2 \end{cases} \qquad D_3 : \begin{cases} 4 \le x \le 6 \\ x \le y \le 6 \end{cases}$$

$$I_1 = \int\limits_0^2 x^2 \int\limits_2^{x+2} y^2\, dy\ dx \qquad I_2 = \int\limits_2^4 x^2 \int\limits_x^{x+2} y^2\, dy\ dx \qquad I_3 = \int\limits_4^6 x^2 \int\limits_x^6 y^2\, dy\ dx$$

$$I_1 = \int\limits_0^2 x^2 \int\limits_2^{x+2} y^2\, dy\ dx$$

$$\int\limits_2^{x+2} y^2\, dy \frac{y^3}{3}\Big|_2^{x+2} = \frac{(x+2)^3}{3} - \frac{8}{3}$$

$$I_1 = \int\limits_0^2 x^2 \left(\frac{(x+2)^3}{3} - \frac{8}{3}\right) = \int\limits_0^2 \left(\frac{x^2(x+2)^3}{3} - \frac{8x^2}{3}\right) = \frac{56}{3}$$

$$I_2 = \int\limits_2^4 x^2 \int\limits_x^{x+2} y^2\,dy\;dx$$

$$\int\limits_x^{x+2} y^2\,dy \frac{y^3}{3}\bigg|_x^{x+2} = \frac{(x+2)^3}{3} - \frac{x^3}{3}$$

$$I_2 = \int\limits_2^4 x^2 \left(\frac{(x+2)^3}{3} - \frac{x^3}{3}\right) = \int\limits_0^2 \left(\frac{x^2(x+2)^3}{3} - \frac{x^5}{3}\right) = 104$$

$$I_3 = \int\limits_4^6 x^2 \int\limits_x^6 y^2\,dy\;dx$$

$$\int\limits_x^6 y^2\,dy = \frac{y^3}{3}\bigg|_x^6 = 72 - \frac{x^3}{3}$$

$$I_3 = \int\limits_4^6 x^2 \left(72 - \frac{x^3}{3}\right)\,dx = \int\limits_4^6 \left(72x^2 - \frac{x^5}{3}\right)\,dx = \frac{304}{3}$$

$$I = I_1 + I_2 + I_3 = \frac{56}{3} + 104 + \frac{304}{3} = 224$$

Задача 3

$$\iint\limits_D (x+y)\,dx\;dy, \qquad \partial D : \begin{cases} y^2 = 2x \\ x+y=4 \\ x+y=12 \end{cases}$$

$$\left| \begin{array}{l} y^2 = 2x \\ x+y=4 \end{array} \right. \Leftrightarrow \left| \begin{array}{l} y = 4-x \\ (4-x)^2 = 2x \end{array} \right. \implies D_1 = \begin{cases} 2 \le x \le 8 \\ 4-x \le y \le \sqrt{2x} \end{cases}$$

$$\left| \begin{array}{l} y^2 = 2x \\ x+y=12 \end{array} \right. \Leftrightarrow \left| \begin{array}{l} y = 12-x \\ (12-x)^2 = 2x \end{array} \right. \implies D_2 = \begin{cases} 8 \le x \le 18 \\ -\sqrt{2x} \le y \le 12-x \end{cases}$$

$$I_1 = \int\limits_2^8 \int\limits_{4-x}^{\sqrt{2x}} (x+y)\,dy\;dx \qquad I_2 = \int\limits_8^{18} \int\limits_{-\sqrt{2x}}^{12-x} (x+y)\,dy\;dx$$

$$I_1 = \int\limits_2^8 \int\limits_{4-x}^{\sqrt{2x}} (x+y)\,dy\ dx$$

$$\int\limits_{4-x}^{\sqrt{2x}} (x+y)\,dy = xy + \frac{y^2}{2}\Big|_{4-x}^{\sqrt{2x}} = \sqrt{2}x^{\frac{3}{2}} + \frac{x^2}{2} + x - 8$$

$$I_1 = \int\limits_2^8 \sqrt{2}x^{\frac{3}{2}} + \frac{x^2}{2} + x - 8\,dx = \frac{826}{5}$$

$$I_2 = \int\limits_8^{18} \int\limits_{-\sqrt{2x}}^{12-x} (x+y)\,dy\ dx$$

$$\int\limits_{-\sqrt{2x}}^{12-x} (x+y)\,dy = xy + \frac{y^2}{2}\Big|_{-\sqrt{2x}}^{12-x} = \sqrt{2}x^{\frac{3}{2}} - \frac{x^2}{2} - x + 72$$

$$I_2 = \int\limits_8^{18} \sqrt{2}x^{\frac{3}{2}} - \frac{x^2}{2} - x + 72\,dx = \frac{5678}{15}$$

$$I = I_1 + I_2 = \frac{8156}{15}$$

Задача 4

$$\iint\limits_D (x+y)\,dx\ dy$$

Където D е триъгълник $\triangle ABO$ с върхове $A(1,0)$, $B=(1,1)$, $O=(0,0)$

$$D = \begin{cases} 0 \le x \le 1 \\ 0 \le y \le x \end{cases}$$

$$I = \int\limits_0^1 \int\limits_0^x (x+y)\,dy\;dx$$

$$\int\limits_0^x (x+y)\,dy = xy + \frac{y^2}{2}\bigg|_0^x = \frac{3x^2}{2}$$

$$I = \frac{3}{2}\int\limits_0^1 x^2\,dx = \frac{x^3}{2}\bigg|_0^1 = \frac{1}{2} - 0 = \frac{1}{2}$$

10 Упражнение към лекция 10

10.1 Задачи

10.2 Решения

11 Упражнение към лекция 11

11.1 Задачи

Задача 1

Да се пресметнат интегралите

- $I = \iint\limits_D \arctan \frac{y}{x}\, dx\ dy, \qquad D = \begin{cases} 1 \le x^2 + y^2 \le 9 \\ y \ge \frac{x}{\sqrt{3}} \qquad y \le \sqrt{3}x \end{cases}$
- $I = \iint\limits_D\ dx\ dy, \qquad D = \{(x, y) : x^2 + y^2 \le 25\}$
- $I = \iint\limits_D\ dx\ dy,$ D е оградена от лемниската $(x^2 + y^2)^2 = 2a^2 xy$
- $I = \iint\limits_D \sqrt{1 + \frac{x^2}{9} + \frac{y^2}{16}}\, dx\ dy, \qquad D : \frac{x^2}{9} + \frac{y^2}{16} \le 1$

Задача 2

Да се пресметнат интегралите

- $I = \iint\limits_D\ dx\ dy, \qquad \partial D : \begin{cases} y^2 = 2x, & xy = 1 \\ y^2 = 3x, & xy = 2 \end{cases}$
- $I = \iint\limits_D (2x - y)\, dx\ dy, \qquad \partial D : \begin{cases} x + y = 1, & x + y = 2 \\ 2x - y = 1, & 2x - y = 3 \end{cases}$

11.2 Решения

Задача 1

- $I = \iint\limits_D \arctan\frac{y}{x}\,dx\;dy, \qquad D = \begin{cases} 1 \le x^2+y^2 \le 9 \\ y \ge \frac{x}{\sqrt{3}} \qquad y \le \sqrt{3}x \end{cases}$

Полярна смяна

$x = \rho\cos\varphi \qquad y = \rho\sin\varphi \qquad (\rho > 0, \qquad 0 < \varphi < 2\pi)$

За допълнителните условия имаме

$1 \le x^2+y^2 = \rho^2\cos^2\varphi + \rho^2\sin^2\varphi \le 9 \implies 1 \le \rho^2 \le 9 \implies 1 \le \rho \le 3$

$y = \dfrac{x}{\sqrt{3}}$ е права сключваща ъгъл $\dfrac{\pi}{6}$ с положителната посока на Ox, а

$y = \sqrt{3}x - \dfrac{\pi}{3} \implies$

$\tan\varphi \ge \dfrac{1}{\sqrt{3}} \qquad \tan\varphi \le \sqrt{3}$

$\varphi \in \left[\dfrac{\pi}{6}, \dfrac{\pi}{2}\right) \qquad \varphi \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{3}\right] \implies \varphi \in \left[\dfrac{\pi}{6}, \dfrac{\pi}{3}\right]$

$\arctan\dfrac{y}{x} = \arctan\dfrac{\rho\sin\varphi}{\rho\cos\varphi} = \arctan\tan\varphi = \varphi$

$\Delta = \rho \implies dxdy = \rho d\rho d\varphi$

$$I = \int\limits_{\frac{\pi}{6}}^{\frac{\pi}{3}} \int\limits_1^3 \varphi\cdot\rho\,d\rho\;d\varphi = \int\limits_{\frac{\pi}{6}}^{\frac{\pi}{3}} \varphi \int\limits_1^3 \rho\,d\rho\;d\varphi$$

$$\int\limits_1^3 \rho\,d\rho = \frac{\rho^2}{2}\Big|_1^3 = \frac{1}{2}(9-1) = 4$$

$$I = 4\int\limits_{\frac{\pi}{6}}^{\frac{\pi}{3}} \varphi = 4\frac{\varphi^2}{2}\Big|_{\frac{\pi}{6}}^{\frac{\pi}{3}} = 2\left(\frac{\pi^2}{9} - \frac{\pi^2}{36}\right) = 2\frac{3\pi^2}{36} = \frac{\pi^2}{6}$$

- $I = \iint\limits_D dx\ dy, \qquad D = \{(x,y) : x^2 + y^2 \leq 25\}$

  Полярна смяна
  $x = \rho\cos\varphi \qquad y = \rho\sin\varphi \qquad (\rho > 0, \qquad 0 < \varphi < 2\pi)$
  За допълнителните условия имаме
  $x^2 + y^2 = \rho^2\cos^2\varphi + \rho^2\sin^2\varphi \leq 25 \implies 0 \leq \rho^2 \leq 25 \implies 0 \leq \rho \leq 5$
  За φ не се появяват допълнителни ограничения
  Якобианът $= \Delta = \rho$

  $$I = \int\limits_0^{2\pi}\int\limits_0^5 \rho\, d\rho\ d\varphi$$

  $$\int\limits_0^5 \rho\, d\rho = \frac{\rho^2}{2}\Big|_0^5 = \frac{25}{2}$$

  $$I = \frac{25}{2}\int\limits_0^{2\pi} d\varphi = \frac{25}{2}\cdot 2\pi = 25\pi$$

- $I = \iint\limits_D dx\ dy, \qquad$ D е оградена от лемниската $(x^2 + y^2)^2 = 2a^2xy$

  Полярна смяна
  $x = \rho\cos\varphi \qquad y = \rho\sin\varphi \qquad (\rho > 0, \qquad 0 < \varphi < 2\pi)$
  $xy > 0 \implies \cos\varphi > 0 \qquad \sin\varphi > 0$

  $$\varphi \in \left(0, \frac{\pi}{2}\right) \qquad \varphi \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$$

- $I = \iint\limits_D \sqrt{1+\frac{x^2}{9}+\frac{y^2}{16}}\,dx\ dy, \qquad D: \frac{x^2}{9}+\frac{y^2}{16} \le 1$

  Полярна смяна
  $x = 3\rho\cos\varphi \qquad y = 4\rho\sin\varphi \qquad (\rho > 0, \qquad 0 < \varphi < 2\pi)$
  Якобианът на смяната:$\Delta = 12\rho$
  $$\sqrt{1+\frac{x^2}{9}+\frac{y^2}{16}} = \sqrt{1+\frac{9\rho^2\cos^2\varphi}{9}+\frac{16\rho^2\sin^2\varphi}{16}} = \sqrt{1+\rho^2(\cos^2\varphi+\sin^2\varphi)} \implies$$
  $$\sqrt{1+\frac{x^2}{9}+\frac{y^2}{16}} = \sqrt{1+\rho^2}$$
  $$\frac{9\rho^2\cos^2\varphi}{9}+\frac{16\rho^2\sin^2\varphi}{16} \le 1 \implies \rho^2 \le 1 \implies 0 \le \rho \le 1$$
  $$I = 12\int\limits_0^{2\pi}\int\limits_0^1 \sqrt{1+\rho^2}\,d\rho\ d\varphi$$
  $$\int\limits_0^1 \sqrt{1+\rho^2}\,d\rho = \frac{1}{2}\int\limits_0^1 (1+\rho^2)^{\frac{1}{2}}\,d(\rho^2+1) = \frac{1}{2}\cdot\frac{(\rho^2+1)^{\frac{3}{2}}}{\frac{3}{2}}\Big|_0^1 = \frac{1}{2}\cdot\frac{2^{\frac{3}{2}}-1}{\frac{3}{2}}$$
  $$I = \frac{12}{2}\cdot\frac{2}{3}\int\limits_0^{2\pi} 2^{\frac{3}{2}}-1\,d\varphi = \frac{12}{3}(2\sqrt{2}-1)\cdot 2\pi = \frac{8\pi}{3}(2\sqrt{2}-1)$$

Задача 2

- $I = \iint\limits_D dx\ dy, \qquad \partial D: \begin{cases} y^2 = 2x, & xy = 1 \\ y^2 = 3x, & xy = 2 \end{cases}$

  Нека
  $u = \frac{y^2}{x} \qquad v = xy$
  $2 \le u \le 3 \qquad 1 \le v \le 2$
  Якобианът на смяната е : $\Delta = \dfrac{D(x,y)}{D(u,v)} = \dfrac{1}{\frac{D(u,v)}{D(x,y)}}$
  Освен това$\dfrac{D(u,v)}{\partial(x,y)} = \begin{vmatrix} u'_x & u'_y \\ v'_x & v'_y \end{vmatrix} = \begin{vmatrix} -\frac{y^2}{x^2} & \frac{2y}{x} \\ y & x \end{vmatrix} = -\dfrac{y^2}{x} - \dfrac{2y^2}{x} = -\dfrac{3y^2}{x} = 3u$
  $\Delta = -\dfrac{1}{3u} \implies |\Delta| = \dfrac{1}{3u}$

$$I = \int_1^2 \int_2^3 \frac{1}{3u}\,du\;dv$$

$$\int_2^3 \frac{1}{3u}\,du = \frac{1}{3}\int_2^3 \frac{1}{u} = \frac{1}{3}\ln u\Big|_2^3 = \frac{1}{3}\ln\frac{3}{2}$$

$$I = \int_1^2 \frac{1}{3}\ln\frac{3}{2}\,dv = \frac{1}{3}\ln\frac{3}{2}\int_1^2 dv = \frac{1}{3}\ln\frac{3}{2}v\Big|_1^2 = \frac{1}{3}\ln\frac{3}{2}$$

- $I = \iint\limits_D (2x-y)\,dx\;dy, \qquad \partial D : \begin{cases} x+y=1, & x+y=2 \\ 2x-y=1, & 2x-y=3 \end{cases}$

  Нека

  $u = x+y \qquad v = 2x-y$

  $1 \le u \le 2 \qquad 1 \le v \le 3$

  Якобианът на смяната е : $\Delta = \dfrac{D(x,y)}{D(u,v)} = \dfrac{1}{\frac{D(u,v)}{D(x,y)}}$

  Освен това $\dfrac{D(u,v)}{\partial(x,y)} = \begin{vmatrix} u'_x & u'_y \\ v'_x & v'_y \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} = -3$

  $\Delta = -\dfrac{1}{3} \implies |\Delta| = \dfrac{1}{3}$

$$I = \int_1^2 \int_1^3 \frac{1}{3}v\,dv\;du$$

$$\int_1^3 \frac{1}{3}v\,dv = \frac{1}{3}\int_1^3 v\,dv = \frac{1}{3}\frac{v^2}{2}\Big|_1^3 = \frac{1}{3}\left(\frac{9}{2}-\frac{1}{2}\right) = \frac{4}{3}$$

$$I = \int_1^2 \frac{4}{3}\,du = \frac{4}{3}\int_1^2 du = \frac{4}{3}u\Big|_1^2 = \frac{4}{3}$$

12 Упражнение към лекция 12

12.1 Задачи

Задача 1

Пресметнете интегралите

- $\iint\limits_T (x^2 + y^2)\ dx\ dy\ dz, T : \begin{cases} x^2 + y^2 = 2z \\ z = 2 \end{cases}$
- $\iint\limits_T (x^2 + y^2)\ dx\ dy\ dz, T : \begin{cases} z = 6 - x^2 - y^2 \\ z^2 = x^2 + y^2 \\ z \geq 0 \end{cases}$

Задача 2

Да се пресметне обемът на тялото T

- $T : x^2 + y^2 + z^2 \leq R^2$
- $T : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1$

$$R > 0 \qquad a > 0 \qquad b > 0 \qquad c > 0 \qquad a, b, c, R = const$$

Задача 3

Да се намери лицето на частта от повърхнината

$$S : x^2 + y^2 = 2az, \qquad a > 0$$

заключена в цилиндъра

$$(x^2 + y^2)^2 = 2a^2xy$$

12.2 Решения

Задача 1

- $\iiint\limits_T (x^2+y^2)\ dx\ dy\ dz, T : \begin{cases} x^2+y^2=2z \\ z=2 \end{cases}$

$$D : \begin{cases} x^2+y^2=2z \\ z=2 \end{cases} \implies D : x^2+y^2 \le 4$$

Цилиндрична смяна

$x=\rho\cos\varphi \qquad y=\rho\sin\varphi \qquad z=z \implies |\Delta|=\rho$

$\rho^2\cos^2\varphi+\rho^2\sin^2\varphi \le 4 \implies \rho^2 \le 4 \implies 0 \le \rho \le 2$

$\frac{1}{2}(x^2+y^2) \le z \le 2 \implies \frac{\rho^2}{2} \le z \le 2$

$$I = \int\limits_0^{2\pi}\int\limits_0^2\int\limits_{\frac{\rho^2}{2}}^2 \rho^2\cdot\rho\ dz\ d\rho\ d\varphi = \int\limits_0^{2\pi}\int\limits_0^2\int\limits_{\frac{\rho^2}{2}}^2 \rho^3\ dz\ d\rho\ d\varphi$$

$$\int\limits_{\frac{\rho^2}{2}}^2 \rho^3\ dz = \rho^3\int\limits_{\frac{\rho^2}{2}}^2 \rho^3\ dz = \rho^3\cdot z\Big|_{\frac{\rho^2}{2}}^2 = \rho^3\cdot\left(2-\frac{\rho^2}{2}\right) = 2\rho^3-\frac{\rho^5}{2}$$

$$I = \int\limits_0^{2\pi}\int\limits_0^2 2\rho^3-\frac{\rho^5}{2}\ d\rho\ d\varphi$$

$$\int\limits_0^2 2\rho^3-\frac{\rho^5}{2}\ d\rho = \int\limits_0^2 2\rho^3\ d\rho - \int\limits_0^2 \frac{\rho^5}{2}\ d\rho =$$

$$\frac{2\rho^4}{4}\Big|_0^2 - \frac{\rho^6}{12}\Big|_0^2 = \frac{2\cdot 16}{4} - \frac{64}{12} = 8-\frac{16}{3} = \frac{24-16}{3} = \frac{8}{3}$$

$$I = \int\limits_0^{2\pi}\frac{8}{3}\ d\varphi = \frac{8}{3}\int\limits_0^{2\pi}\ d\varphi = \frac{8}{3}\varphi\Big|_0^{2\pi} = \frac{16\pi}{3}$$

- $\iiint\limits_T (x^2+y^2)\ dx\ dy\ dz, T : \begin{cases} z=6-x^2-y^2 \\ z^2=x^2+y^2 \\ z \ge 0 \end{cases}$

Цилиндрична смяна

$x = \rho\cos\varphi \qquad y = \rho\sin\varphi \qquad z = z \implies |\Delta| = \rho$

$$\left|\begin{array}{l} z = 6 - x^2 - y^2 \\ z = \sqrt{x^2+y^2} \end{array}\right. \implies 6 - x^2 - y^2 = \sqrt{x^2+y^2}$$

$\implies 6 - \rho^2 = \rho \implies \rho = -3, \rho > 0, \rho = 2 \implies$

$$\begin{cases} 0 \le \rho \le 2 \\ 0 \le \varphi \le 2\pi \end{cases}$$

$$I = \int\limits_0^{2\pi} \int\limits_0^2 \int\limits_{\rho}^{6-\rho^2} \rho^3 \; dz \; d\rho \; d\varphi$$

$$\int\limits_{\rho}^{6-\rho^2} \rho^3 \; dz = 6\rho^3 - \rho^5 - \rho^4$$

$$I = \int\limits_0^{2\pi} \int\limits_0^2 6\rho^3 - \rho^5 - \rho^4 \; d\rho \; d\varphi$$

$$\int\limits_0^2 6\rho^3 - \rho^5 - \rho^4 \; d\rho = \frac{6\rho^4}{4}\Big|_0^2 - \frac{\rho^6}{6}\Big|_0^2 - \frac{\rho^5}{5}\Big|_0^2 = \frac{104}{15}$$

$$I = \int\limits_0^{2\pi} \frac{104}{15} \; d\varphi = \frac{104}{15} \int\limits_0^{2\pi} \; d\varphi = \frac{208\pi}{15}$$

Задача 2

$$R > 0 \qquad a > 0 \qquad b > 0 \qquad c > 0 \qquad a, b, c, R = const$$

$$V_T = \iiint\limits_T dx\ dy\ dz$$

- $T : x^2 + y^2 + z^2 \leq R^2$

Сферична смяна

$x = \rho\cos\varphi\sin\theta \qquad y = \rho\sin\varphi\sin\theta \qquad z = \rho\cos\theta \qquad |\Delta| = \rho^2\sin\theta$

$\rho^2\cos^2\varphi\sin^2\theta + \rho^2\sin^2\varphi\sin^2\theta + \rho^2\cos^2\theta \leq R^2$

$\rho^2\sin^2\theta(\cos^2\varphi + \sin^2\varphi) + \rho^2\cos^2\theta \leq R^2$

$\rho^2\sin^2\theta + \rho^2\cos^2\theta \leq R^2 \implies \rho^2 \leq R^2 \implies 0 \leq \rho \leq R$

$$V_T = \int\limits_0^{2\pi}\int\limits_0^{R}\int\limits_0^{\pi} \rho^2\sin\theta\ d\theta\ d\rho\ d\varphi$$

$$\int\limits_0^{\pi} \rho^2\sin\theta\ d\theta = \rho^2\int\limits_0^{\pi}\sin\theta\ d\theta = \rho^2(-\cos\theta)\Big|_0^{\pi} = 2\rho^2$$

$$V_T = \int\limits_0^{2\pi}\int\limits_0^{R} 2\rho^2\ d\rho\ d\varphi = 2\int\limits_0^{2\pi}\int\limits_0^{R}\rho^2\ d\rho\ d\varphi$$

$$\int\limits_0^{R}\rho^2\ d\rho = \frac{\rho^3}{3}\Big|_0^{R} = \frac{R^3}{3}$$

$$2\int\limits_0^{2\pi}\frac{R^3}{3}\ d\varphi = \frac{2R^3}{3}\int\limits_0^{2\pi}\ d\varphi = \frac{2R^3}{3}\varphi\Big|_0^{2\pi} = \frac{4\pi R^3}{3}$$

- $T : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1$

Сферична смяна

$x = a\rho\cos\varphi\sin\theta \qquad y = b\rho\sin\varphi\sin\theta \qquad z = c\rho\cos\theta \qquad |\Delta| = abc\rho^2\sin\theta$

$$\frac{a^2\rho^2\cos^2\varphi\sin^2\theta}{a^2} + \frac{b^2\rho^2\sin^2\varphi\sin^2\theta}{b^2} + \frac{c^2\rho^2\cos^2\theta}{c^2} \le 1$$

$$\rho^2\cos^2\varphi\sin^2\theta + \rho^2\sin^2\varphi\sin^2\theta + \rho^2\cos^2\theta \le 1$$

$$\rho^2\sin^2\theta(\cos^2\varphi + \sin^2\varphi) + \rho^2\cos^2\theta \le 1$$

$$\rho^2\sin^2\theta + \rho^2\cos^2\theta \le R^2 \implies \rho^2 \le 1 \implies 0 \le \rho \le 1$$

$$V_T = \int\limits_0^{2\pi}\int\limits_0^1\int\limits_0^{\pi} abc\rho^2\sin\theta\ d\theta\ d\rho\ d\varphi = abc\int\limits_0^{2\pi}\int\limits_0^1\int\limits_0^{\pi} \rho^2\sin\theta\ d\theta\ d\rho\ d\varphi$$

$$\int\limits_0^{\pi} \rho^2\sin\theta\ d\theta = \rho^2\int\limits_0^{\pi}\sin\theta\ d\theta = \rho^2(-\cos\theta)\Big|_0^{\pi} = 2\rho^2$$

$$V_T = abc\int\limits_0^{2\pi}\int\limits_0^1 2\rho^2\ d\rho\ d\varphi = 2abc\int\limits_0^{2\pi}\int\limits_0^1 \rho^2\ d\rho\ d\varphi$$

$$\int\limits_0^1 \rho^2\ d\rho = \frac{\rho^3}{3}\Big|_0^1 = \frac{1}{3}$$

$$2abc\int\limits_0^{2\pi}\frac{1}{3}\ d\varphi = \frac{2abc}{3}\int\limits_0^{2\pi} d\varphi = \frac{2abc}{3}\varphi\Big|_0^{2\pi} = \frac{4}{3}\pi abc$$

Задача 3

$$S : x^2 + y^2 = 2az, \qquad a > 0$$
$$(x^2 + y^2)^2 = 2a^2xy$$

$$\sigma = \iint\limits_D \sqrt{1 + (z'_x)^2 + (z'_y)^2}\ dx\ dy$$

$$D : (x^2 + y^2)^2 \le 2a^2xy$$

$$S : z = \frac{1}{2a^2}(x^2 + y^2) \implies z'_x = \frac{x}{a}, z'_y = \frac{y}{a}$$

$$1 + (z'_x)^2 + (z'_y)^2 = \frac{a^2 + x^2 + y^2}{a^2} \implies \sigma = \frac{4}{a}\iint\limits_D \sqrt{a^2 + x^2 + y^2}\ dx\ dy$$

Полярна смяна

$x = \rho\cos\varphi \qquad y = \rho\sin\varphi \implies$

$0 \le \varphi \le \dfrac{\pi}{4} \qquad 0 \le \rho \le a\sqrt{2\sin\varphi\cos\varphi} = a\sqrt{\sin 2\varphi}$

$$\sigma = \frac{4}{a}\int\limits_0^{\frac{\pi}{4}}\int\limits_0^{a\sqrt{\sin 2\varphi}} \sqrt{a^2+\rho^2}\rho\ d\rho\ d\varphi = \frac{4}{a2}\int\limits_0^{\frac{\pi}{4}}\int\limits_0^{a\sqrt{\sin 2\varphi}} (a^2+\rho^2)^{\frac{1}{2}}\ d\rho^2\ d\varphi =$$

$$\frac{2}{a}\int\limits_0^{\frac{\pi}{4}} \frac{(a^2+\rho^2)^{\frac{3}{2}}}{\frac{3}{2}}\Big|_0^{a\sqrt{\sin 2\varphi}}\ d\varphi = \frac{4}{3a}\int\limits_0^{\frac{\pi}{4}} (a^2+a^2\sin 2\varphi)^{\frac{3}{2}} - (a^2)^{\frac{3}{2}}\ d\varphi =$$

$$\frac{4a^3}{3a}\int\limits_0^{\frac{\pi}{4}} (1+\sin 2\varphi)^{\frac{3}{2}} - 1\ d\varphi$$

$1+\sin 2\varphi = \sin^2\varphi + \cos^2\varphi + 2\sin\varphi\cos\varphi = (\sin\varphi+\cos\varphi)^2$

$$\sigma = \frac{4a^2}{3}\int\limits_0^{\frac{\pi}{4}} ((\sin\varphi+\cos\varphi)^2)^{\frac{3}{2}} - 1\ d\varphi = \frac{4a^2}{3}\int\limits_0^{\frac{\pi}{4}} (\sin\varphi+\cos\varphi)^3 - 1\ d\varphi =$$

$$\frac{4a^2}{3}\int\limits_0^{\frac{\pi}{4}} \sin^3\varphi + 3\sin^2\varphi\cos\varphi + 3\sin\varphi\cos^2\varphi + \cos^3\varphi - 1\ d\varphi$$

$$\sigma = \frac{4a^2}{3}\left[\int\limits_0^{\frac{\pi}{4}} \left(\sin^2\varphi\sin\varphi + 3\sin^2\varphi\cos\varphi\right)\ d\varphi + \int\limits_0^{\frac{\pi}{4}} \left(\cos^2\varphi\cos\varphi + 3\cos^2\varphi\sin\varphi - 1\right)\ d\varphi\right]$$

$$\int\limits_0^{\frac{\pi}{4}} \left(\sin^2\varphi\sin\varphi + 3\sin^2\varphi\cos\varphi\right)\ d\varphi = \int\limits_0^{\frac{\pi}{4}} \left((1-\cos^2\varphi)\sin\varphi + 3\sin^2\varphi\cos\varphi\right)\ d\varphi =$$

$$\int\limits_0^{\frac{\pi}{4}} \left(\sin\varphi - \cos^2\varphi\sin\varphi + 3\sin^2\varphi\cos\varphi\right)\ d\varphi$$

$$\int\limits_0^{\frac{\pi}{4}} \left(\cos^2\varphi\cos\varphi + 3\cos^2\varphi\sin\varphi - 1\right)\ d\varphi = \int\limits_0^{\frac{\pi}{4}} \left((1-\sin^2\varphi)\cos\varphi + 3\cos^2\varphi\sin\varphi - 1\right)\ d\varphi =$$

$$\int\limits_0^{\frac{\pi}{4}} \left(\cos\varphi - \sin^2\varphi\cos\varphi + 3\cos^2\varphi\sin\varphi - 1\right)\ d\varphi$$

$\sin\varphi - \cos^2\varphi\sin\varphi + 3\sin^2\varphi\cos\varphi + \cos\varphi - \sin^2\varphi\cos\varphi + 3\cos^2\varphi\sin\varphi - 1 =$
$\sin\varphi + \cos\varphi + 2\sin^2\varphi\cos\varphi + 2\sin\varphi\cos^2\varphi - 1$

$$\sigma = \frac{4a^2}{3}\left[\int\limits_0^{\frac{\pi}{4}} \sin\varphi + \cos\varphi + 2\sin^2\varphi\cos\varphi + 2\sin\varphi\cos^2\varphi - 1\ d\varphi\right] =$$

$$\frac{4a^2}{3}\left[\int\limits_0^{\frac{\pi}{4}} \sin\varphi\ d\varphi + \int\limits_0^{\frac{\pi}{4}} \cos\varphi\ d\varphi + 2\int\limits_0^{\frac{\pi}{4}} \sin^2\varphi\cos\varphi\ d\varphi + 2\int\limits_0^{\frac{\pi}{4}} \sin\varphi\cos^2\varphi\ d\varphi - \int\limits_0^{\frac{\pi}{4}} 1\ d\varphi\right]$$

$$I_1 = \int\limits_0^{\frac{\pi}{4}} \sin\varphi\ d\varphi = -\cos\varphi\Big|_0^{\frac{\pi}{4}} = -\left(\frac{\sqrt{2}}{2} - 1\right) = 1 - \frac{\sqrt{2}}{2}$$

$$I_2 = \int\limits_0^{\frac{\pi}{4}} \cos\varphi\ d\varphi = \sin\varphi\Big|_0^{\frac{\pi}{4}} = \frac{\sqrt{2}}{2} - 0 = \frac{\sqrt{2}}{2}$$

$$I_3 = \int\limits_0^{\frac{\pi}{4}} \sin^2\varphi\cos\varphi\ d\varphi = \frac{\sin^3\varphi}{3}\Big|_0^{\frac{\pi}{4}} = \frac{1}{3}\left[\left(\frac{\sqrt{2}}{2}\right)^3 - 0\right] = \frac{1}{3}\cdot\frac{2\sqrt{2}}{8} = \frac{\sqrt{2}}{12}$$

$$I_4 = \int\limits_0^{\frac{\pi}{4}} \sin\varphi\cos^2\varphi\ d\varphi = -\frac{2\cos^3\varphi}{3}\Big|_0^{\frac{\pi}{4}} = -\frac{1}{3}\left[\left(\frac{\sqrt{2}}{2}\right)^3 - 1\right] = -\frac{\sqrt{2}}{12} + \frac{1}{3} = \frac{1}{3} - \frac{\sqrt{2}}{12}$$

$$I_5 = \int\limits_0^{\frac{\pi}{4}} 1\ d\varphi = \varphi\Big|_0^{\frac{\pi}{4}} = \frac{\pi}{4}$$

$$\sigma = \frac{4a^2}{3}\left[1 - \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} + \frac{2\sqrt{2}}{12} + \frac{2}{3} - \frac{2\sqrt{2}}{12} - \frac{\pi}{4}\right] = \frac{4a^2}{3}\left[1 + \frac{2}{3} - \frac{\pi}{4}\right]$$

$$\sigma = \frac{4a^2}{3}\left[\frac{5}{3} - \frac{\pi}{4}\right] = \frac{a^2}{9}\left[20 - 3\pi\right]$$

13 Упражнение към лекция 13

13.1 Задачи

Задача 1

Да се пресметне

$$I = \oint\limits_{\Gamma} (x+y)\ ds$$

където Γ е затворена начупена линия $OABC, O(0,0), A(1,0), B(0,1)$.

Задача 2

Да се пресметне

$$I = \oint\limits_{C} \sqrt{x^2+y^2}\ ds$$

където

$$C \equiv \{x = a(\cos t + t \sin t), y = a(\sin t - t\cos t), 0 \leq t \leq 2\pi\}$$

Задача 3

Да се пресметне

$$I = \int\limits_{C} \frac{1}{x+y}\ ds$$

ако интегрирането е извършено по отреза на правата, свързваща точките $A(2,4)$ и $B(1,3)$

Задача 4

Да се пресметне чрез криволинеен интеграл от първи род лицето на цилиндрична повърхнина $y^2 = \frac{4}{9}(x-1)^3$ ограничена отдолу от равнината $z = 0$, а отгоре от повърхнината $z = 2 - \sqrt{x}$.

Задача 5

Да се намери стойността на

$$I = \int\limits_{C} (x^2+y^2+z^2)\ ds$$

където C е част от винтовата линия с параметрични уравнения

$$x = a\cos t \quad y = a\sin t \quad z = bt \qquad 0 \le t \le 2\pi \qquad a > 0, b > 0$$

Задача 6

Да се пресметне

$$\oint\limits_C (x^2 - y^2)\ dx + (x^2 + y^2)\ dy$$

в положителна посока на описване на елипсата

$$\frac{x^2}{a^2} + \frac{y^2}{b} = 1$$

започвайки от точката $A(a, 0), a > 0, b > 0$.

Задача 7

1. Да се покаже, че ако пътят на интегриране не пресича ординатната ос, то интегралът

$$\int\limits_{(1,2)}^{(2,1)} \frac{y\ dx - x\ dy}{x^2}$$

   не зависи от пътя на интегриране, и да се пресметне.

2. Да се пресметне стойността на интеграла от същата функция по затворен контур, непресичащ ординатната ос.

13.2 Решения

Задача 1

$$I = \oint\limits_{\Gamma} (x+y)\ ds \qquad \Gamma = AOBO \qquad O(0,0), A(1,0), B(0,1)$$

$$\Gamma = OA + AB + BO \implies I = \int\limits_{OA} (x+y)\ ds + \int\limits_{AB} (x+y)\ ds + \int\limits_{OB} (x+y)\ ds$$

$$I_1 = \int\limits_{OA} (x+y)\ ds$$

$$I_1 : \begin{cases} y(t) = 0 \\ x(t) = t \\ 0 \le t \le 1 \end{cases} \implies y'(t) = 0 \quad x'(t) = 1 \implies ds = \sqrt{0^2+1^2}dt = \sqrt{1}dt$$

$$I_1 = \int\limits_0^1 (t+0)\sqrt{1}\ dt = \int\limits_0^1 t\ dt = \frac{t^2}{2}\Big|_0^1 = \frac{1}{2}$$

$$I_2 = \int\limits_{AB} (x+y)\ ds$$

$$I_2 : AB : x+y = 1 \Leftrightarrow AB : y = 1-x, 0 \le x \le 1$$

$$\implies y' = -1 \implies ds = \sqrt{(-1)^2+1^2}dx = \sqrt{2}dx$$

$$I_2 = \sqrt{2}\int\limits_0^1\ dx = \sqrt{2}$$

$$I_3 = \int\limits_{OB} (x+y)\ ds$$

$$I_3 : OB : \begin{cases} x(t) = 0 \\ y(t) = t \end{cases} \implies x'(t) = 0 \quad y'(t) = 1 \implies ds = \sqrt{0^2+1^2}dt = \sqrt{1}dt$$

$$I_3 = \int\limits_0^1 (0+t)\sqrt{1}\ dt = \int\limits_0^1 t\ dt = \frac{t^2}{2}\Big|_0^1 = \frac{1}{2}$$

$$I = \frac{1}{2} + \sqrt{2} + \frac{1}{2} = \sqrt{2} + 1$$

Задача 2

$$I = \oint\limits_C \sqrt{x^2+y^2}\ ds \qquad C \equiv \{x = a(\cos t + t\sin t), y = a(\sin t - t\cos t), 0 \le t \le 2\pi\}$$

$$x' = a(-\sin t + \sin t + t\cos t) \implies x' = at\cos t$$

$$y' = a(-\cos t + \cos t + t\sin t) \implies y' = at\sin t$$

$$ds = \sqrt{x'^2 + y'^2}dt = \sqrt{a^2t^2\cos^2 t + a^2t^2\sin^2 t}dt =$$

$$\sqrt{a^2t^2(\cos^2 t + \sin^2 t)}dt = \sqrt{a^2t^2}dt = atdt$$

$$x^2 + y^2 = a^2(\cos t + t\sin t)^2 + a^2(\sin t - t\cos t)^2 =$$

$$a^2\left[\cos^2 t + t^2\sin^2 t + 2t\sin t\cos t + \sin^2 t + t^2\cos^2 t - 2t\sin t\cos t\right] =$$

$$a^2\left[(\cos^2 t + \sin^2 t) + t^2(\cos^2 t + \sin^2 t)\right] = a^2\left[1 + t^2\right] \implies$$

$$\sqrt{x^2+y^2} = a(1+t^2)^{\frac{1}{2}}$$

$$I = \int\limits_0^{2\pi} a(1+t^2)^{\frac{1}{2}}at\ dt = a^2\int\limits_0^{2\pi}(1+t^2)^{\frac{1}{2}}t\ dt = a^2\frac{1}{2}\int\limits_0^{2\pi}(1+t^2)^{\frac{1}{2}}d(t^2+1) =$$

$$\frac{a^2}{2}\cdot\frac{(1+t^2)^{\frac{3}{2}}}{\frac{3}{2}}\Bigg|_0^{2\pi} = \frac{a^2}{3}\left[(1+4\pi^2)^{\frac{3}{2}} - 1\right]$$

Задача 3

$$I = \int\limits_C \frac{1}{x+y}\ ds \qquad C = AB \quad A(2,4), B(1,3)$$

$$y - y_1 = k(x - x_1) \implies k = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4-3}{2-1} = 1 \implies$$

$$AB: y - 3 = 1(x-1); \qquad x - 1 \implies y = 3 \qquad x = 2 \implies y = 4 \implies$$

$$AB: y = x + 2; 1 \le x \le 2$$

$$ds = \sqrt{1 + y'^2}dx = \sqrt{1+1}dx = \sqrt{2}dx$$

$$x + y = x + x + 2 = 2x + 2 = 2(x+1)$$

$$I = \int\limits_1^2 \frac{\sqrt{2}}{2(x+y)}\ dx = \frac{\sqrt{2}}{2}\int\limits_1^2 \frac{1}{x+1}\ dx = \frac{\sqrt{2}}{2}\int\limits_1^2 \frac{1}{x+1}\ d(x+1) = \frac{\sqrt{2}}{2}\ln|x+1|\Big|_1^2 =$$

$$\frac{\sqrt{2}}{2}(\ln 3 - \ln 2) = \frac{\sqrt{2}}{2}\ln\left(\frac{3}{2}\right)$$

Задача 4

$$\sigma = \int\limits_C z\ dl \qquad 2-\sqrt{x}, z=0 \qquad C: y^2=\frac{4}{9}(x-1)^3, x\geq 1$$

$$y=\pm\frac{2}{3}(x-1)^{\frac{3}{2}} \implies y'=\pm\frac{2}{3}\cdot\frac{3}{2}(x-1)^{\frac{1}{2}} = y' = \pm(x-1)^{\frac{1}{2}} \implies y'^2 = x-1$$

$$dl=\sqrt{1+x-1}dx=\sqrt{x}dx$$

$$0\leq z\leq 2-\sqrt{x} \implies 2-\sqrt{x}\geq 0 \implies x\leq 4 \implies 1\leq x\leq 4$$

$$\sigma = 2\int\limits_1^4 (2-\sqrt{x})\sqrt{x}\ dx = 2\int\limits_1^4 (2\sqrt{x}-x)\ dx =$$

$$2\left[2\cdot\frac{x^{\frac{3}{2}}}{\frac{3}{2}}-\frac{x^2}{2}\right]\Big|_1^4 = 2\left[\frac{4}{3}x^{\frac{3}{2}}-\frac{x^2}{2}\right]\Big|_1^4 = 2\left[\frac{4}{3}\left(\sqrt{4^3}-1\right)-\frac{1}{2}(16-1)\right] =$$

$$2\left[\frac{4}{3}(7)-\frac{1}{2}(15)\right] = 2\left[\frac{28}{3}-\frac{15}{2}\right] = 2\left[\frac{56-45}{6}\right] = \frac{11}{3}$$

Задача 5

$$I=\int\limits_C (x^2+y^2+z^2)\ ds$$

$$x=a\cos t \quad y=a\sin t \quad z=bt \qquad 0\leq t\leq 2\pi \qquad a>0, b>0$$

$$x'=-a\sin t \qquad y'=a\cos t \qquad z=b \implies$$

$$x'^2+y'^2+z'^2=a^2\sin^2 t+a^2\cos^2 t+b^2=a^2+b^2 \implies dl=\sqrt{a^2+b^2}dt$$

$$x^2+y^2+z^2=a^2\cos^2 t+a^2\sin^2 t+b^2t^2=a^2+b^2t^2$$

$$I=\int\limits_0^{2\pi}(a^2+b^2t^2)\sqrt{a^2+b^2}\ dt = \sqrt{a^2+b^2}\int\limits_0^{2\pi}(a^2+b^2t^2)\ dt =$$

$$\sqrt{a^2+b^2}\left[a^2t+\frac{b^2t^3}{3}\right]\Big|_0^{2\pi} = \sqrt{a^2+b^2}\left[a^2 2\pi+\frac{b^2(2\pi)^3}{3}\right] = \frac{\sqrt{a^2+b^2}}{3}[a^2 2\cdot 3\pi+8\pi^3b^2] =$$

$$\frac{\sqrt{a^2+b^2}}{3}[6\pi a^2+8\pi^3b^2] = \frac{2\pi}{3}\sqrt{a^2+b^2}[3a^2+4\pi^2b^2] =$$

Задача 6

$$\oint\limits_C (x^2 - y^2)\ dx + (x^2 + y^2)\ dy \qquad \frac{x^2}{a^2} + \frac{y^2}{b} = 1$$

$C : x = a\cos t \qquad y = b\sin t \qquad 0 \le t \le 2\pi$

$x^2 - y^2 = a^2\cos^2 t - b^2\sin^2 t \qquad dx = -a\sin t dt$

$(x^2 - y^2)dx = (a^2\cos^2 t - b^2\sin^2 t)(-a\sin t)dt$

$x^2 + y^2 = a^2\cos^2 t + b^2\sin^2 t \qquad dy = b\cos t dt$

$(x^2 + y^2)dy = (a^2\cos^2 t + b^2\sin^2 t)(b\cos t)dt$

$$I = \int\limits_0^{2\pi} (a^2\cos^2 t - b^2\sin^2 t)(-a\sin t)\ dt + (a^2\cos^2 t + b^2\sin^2 t)(b\cos t)\ dt$$

$$I = \int\limits_0^{2\pi} -a^3\cos^2 t + ab^2\sin^3 t dt + a^2 b\cos^3 t + b^3\sin^2 t\ dt$$

$$I = -a^3\int\limits_0^{2\pi} \cos^2 t\sin t\ dt + ab^2\int\limits_0^{2\pi} sin^3 t\ dt + a^2 b\int\limits_0^{2\pi} \cos^3 t\ dt + b^3\int\limits_0^{2\pi} \sin^2 t\cos t\ dt$$

$I = -a^3 I_1 + ab^2 I_2 + a^2 b I_3 + b^3 I_4$

$$I_1 = \int\limits_0^{2\pi} \cos^2 t\sin t\ dt = -\int\limits_0^{2\pi} \cos^2 t\ d(\cos t) = -\frac{\cos^3 t}{3}\Big|_0^{2\pi} = 0$$

$$I_2 = \int\limits_0^{2\pi} sin^3 t\ dt = \int\limits_0^{2\pi} (1 - \cos^2 t)\ d(\cos t) = 0$$

$$I_3 = \int\limits_0^{2\pi} \cos^3 t\ dt = -\int\limits_0^{2\pi} (1 - \sin^2 t)\ d(\sin t) = 0$$

$$I_4 = \int\limits_0^{2\pi} \sin^2 t\cos t\ dt = \int\limits_0^{2\pi} \sin^2 t\ d(\sin t) = \frac{\sin^3 t}{3}\Big|_0^{2\pi} = 0$$

$I = 0$

Задача 7

1.

$$\int\limits_{(1,2)}^{(2,1)} \frac{y\ dx - x\ dy}{x^2}$$

Кривата не пресича Oy

$P(x) = \dfrac{y}{x^2} \qquad Q(x) = -\dfrac{1}{x} \implies x \neq 0$

$\dfrac{\partial P}{\partial y} = \dfrac{1}{x^2} \qquad \dfrac{\partial Q}{\partial x} = \dfrac{1}{x^2} \implies \dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$

1)Интегралът I не зависи от пътя на интегриране

2)$\exists u = u(x,y) : u'_x = \dfrac{y}{x^2} \quad u'_y = -\dfrac{1}{x}$

$u'_x = \dfrac{y}{x^2} \implies u(x,y) = \displaystyle\int \frac{y}{x^2}\ dx \implies$

$u(x,y) = y\displaystyle\int \frac{1}{x^2}\ dx y\int x^{-2}\ dx = y\frac{x^{-1}}{-1} + \varphi(y) = -\frac{y}{x} + \varphi(y)$

$u'_y = -\dfrac{1}{x} \qquad u'_y = -\dfrac{1}{x} + \varphi'(y) \implies -\dfrac{1}{x} = -\dfrac{1}{x} + \varphi'(y) \implies$

$\varphi'(y) = 0 \implies \varphi(y) = const = c \implies$

$u(x,y) = -\dfrac{y}{x} + cI = \displaystyle\int\limits_{(1,2)}^{(2,1)} \frac{y\ dx - x\ dy}{x^2} = -\frac{y}{x}\bigg|_{(1,2)}^{(2,1)} \implies$

$I = -\left[\dfrac{2}{1} - \dfrac{1}{2}\right] = -\dfrac{3}{2}$

2.

$$\int\limits_{\gamma} P\ dx + Q\ dy = ?$$

$$\gamma_1 = ACB \qquad \gamma_2 = ADB \qquad \gamma_2^- = BDA$$

$$\int\limits_{\gamma_1} P\ dx + Q\ dy = \int\limits_{\gamma_2} P\ dx + Q\ dy \implies$$

$$\int\limits_{\gamma_1} P\ dx + Q\ dy - \int\limits_{\gamma_2} P\ dx + Q\ dy = 0 \implies$$

$$\int\limits_{\gamma_1} P\ dx + Q\ dy + \int\limits_{\gamma_2^-} P\ dx + Q\ dy = 0 \implies$$

$$\int\limits_{ACBDA} P\ dx + Q\ dy = 0$$

14 Упражнение към лекция 14

14.1 Задачи

14.2 Решения

15 Упражнение към лекция 15

15.1 Задачи

15.2 Решения