

Математически анализ 2

Упражнения

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1 Упражнение към лекция 1

1.1 Задачи

Задача 1

Да се покаже дали посочените редици $\{X_n\} = \{x_n, y_n\}$ са сходящи или разходящи. За сходящите да се намери границите им.

1. $x_n = 1 + \frac{1}{n}, y_n = 2 + \frac{\sin n}{n}$

2. $x_n = \left(1 + \frac{1}{n}\right)^n, y_n = 2 + n$

3. $x_n = (-1)^n, y_n = n$

4. $x_n = (-1)^n, y_n = \frac{1}{n}$

5. $x_n = \sin \frac{n\pi}{2}, y_n = (-1)^n$

6. $x_n = \sin n, y_n = \frac{(-1)^n}{n}$

1.2 Решения

Задача 1

1. $\lim_{n \rightarrow \infty} \frac{1}{n} = 0, \frac{|\sin n|}{n} \in \left[0, \frac{1}{n}\right] \implies \lim_{n \rightarrow \infty} x_n = 1, \lim_{n \rightarrow \infty} y_n = 2 \implies$
редицата е сходяща; точката $(1,2)$ е нейна граница
2. $\lim_{n \rightarrow \infty} x_n = e, \lim_{n \rightarrow \infty} y_n = \infty \implies$ разходяща редица
3. $\lim_{n \rightarrow \infty} x_n$ не съществува, защото има две точки на сгъстяване., $\lim_{n \rightarrow \infty} y_n = \infty \implies$ разходяща редица
4. $\lim_{n \rightarrow \infty} x_n$ не съществува, защото има две точки на сгъстяване., $\lim_{n \rightarrow \infty} y_n = 0 \implies$ разходяща редица
5. $\lim_{n \rightarrow \infty} x_n$ не съществува, $\lim_{n \rightarrow \infty} y_n = \infty \implies$ разходяща редица
6. $\lim_{n \rightarrow \infty} x_n$ не съществува, $\lim_{n \rightarrow \infty} y_n = 0 \implies$ разходяща редица

2 Упражнение към лекция 2

2.1 Задачи

Задача 1

Нека $D \subset \mathbb{R}^m$ и са разгледани няколко функции. Да се напишат дефиниционните им множества и да се даде пояснение.

1. $z(x, y) = x^2 + y^2$

2. $z(x, y) = \sqrt{y^2 - 2x}$

3. $z(x, y) = \ln \sqrt{y^2 - 2x}$

4. $z(x, y) = \frac{1}{\sqrt{-y^2 + 2x + 1}}$

5. $w(x, y, z) = \arccos(x^2 + y^2 + z^2)$

6. $f(n) = \begin{cases} 1, & x \in \mathbb{Q}^m \\ 0, & x \in \overline{\mathbb{Q}^m} \end{cases}$

Задача 2

Разгледаните по - долу функциите са дефинирани в $D = \mathbb{R}^2 \setminus \{(0, 0)\}$.
Кои от границите съществуват и колко са

$$A = \lim_{(x,y) \rightarrow (0,0)} f(x, y) \quad A_{1,2} = \lim_{y \rightarrow 0} \left(\lim_{x \rightarrow 0} f(x, y) \right) \quad A_{2,1} = \lim_{x \rightarrow 0} \left(\lim_{y \rightarrow 0} f(x, y) \right)$$

1. $f(x, y) = \frac{x - y}{x + y}$

2. $f(x, y) = \frac{x^2 + y^2}{x^2 y^2 + (x - y)^2}$

3. $f(x, y) = \frac{xy^2}{x^2 + y^4}$

4. $f(x, y) = (x + y) \sin \frac{1}{x} \cos \frac{1}{y}$

5. $f(x, y) = \frac{x^4 + y^4}{x^2 + y^2}$

Задача 3

Нека A, B, C, D са подмножества на \mathbb{R}^2 дефинирани както следва

$$A = \{(x, y) : x \geq 0, y \leq 1, y > x\}$$

$$B = \{(x, y) : x \leq 1, y \geq 0, y < x\}$$

$$C = \{(x, y) : x = y, 0 \leq x \leq 1\}$$

$$D = A \cup B \cup C$$

и функцията $f : D \rightarrow \mathbb{R}$ зададена по следния начин

$$f(x, y) = \begin{cases} \frac{1}{y^2}, & (x, y) \in A \\ 0, & x = y \\ -\frac{1}{x^2}, & (x, y) \in B \end{cases}$$

Да се изследва непрекъснатостта на тази функция.

2.2 Решения

Задача 1

$$1. \quad z(x, y) = x^2 + y^2 \\ D = \mathbb{R}^2$$

$$2. \quad z(x, y) = \sqrt{y^2 - 2x} \\ D = \{(x, y) : y^2 - 2x \geq 0\} \subset \mathbb{R}^2, x \leq \frac{y^2}{2}$$

$$3. \quad z(x, y) = \ln \sqrt{y^2 - 2x} \\ D = \{(x, y) : y^2 - 2x > 0\} \subset \mathbb{R}^2, x < \frac{y^2}{2}$$

$$4. \quad z(x, y) = \frac{1}{\sqrt{-y^2 + 2x + 1}} \\ D = \{(x, y) : -y^2 + 2x + 1 > 0\} \subset \mathbb{R}^2, x > \frac{y^2 - 1}{2}$$

$$5. \quad w(x, y, z) = \arccos(x^2 + y^2 + z^2) \\ D = \{(x, y, z) : x^2 + y^2 + z^2 \leq \pi\} \subset \mathbb{R}^3, \\ \text{Графиката е кълбо с център } (0, 0, 0) \text{ и радиус } \sqrt{\pi}$$

$$6. \quad D \subset \mathbb{R}^m$$

Задача 2

1.

$$f(x, y) = \frac{x - y}{x + y}$$

$$\lim_{x \rightarrow 0} f(x, y) = \frac{-y}{y} = -1 \quad \lim_{y \rightarrow 0} f(x, y) = \frac{x}{x} = 1$$

$$A_{1,2} = \lim_{y \rightarrow 0} \left(\lim_{x \rightarrow 0} f(x, y) \right) = \lim_{y \rightarrow 0} (-1) = -1$$

$$A_{2,1} = \lim_{x \rightarrow 0} \left(\lim_{y \rightarrow 0} f(x, y) \right) = \lim_{x \rightarrow 0} (1) = 1$$

$$A = \lim_{(x,y) \rightarrow (0,0)} f(x, y) \text{ Не съществува, защото трябва } A_{1,2} = A_{2,1}$$

2.

$$f(x, y) = \frac{x^2 + y^2}{x^2 y^2 + (x - y)^2}$$

$$\lim_{x \rightarrow 0} f(x, y) = \frac{y^2}{(-y)^2} = 1 \quad \lim_{y \rightarrow 0} f(x, y) = \frac{x^2}{x^2} = 1$$

$$\implies A_{1,2} = A_{2,1} = 1 \implies \exists A = \lim_{(x,y) \rightarrow (0,0)} f(x, y)$$

$$\text{Редица: } (x_n, y_n) = \left(\frac{1}{n}, \frac{1}{n}\right) \rightarrow (0, 0), f(x_n, y_n) = 1 \rightarrow 1$$

$$\text{Редица: } (x'_n, y'_n) = \left(\frac{1}{n}, \frac{-1}{n}\right) \rightarrow (0, 0), f(x'_n, y'_n) = \frac{2n^2}{1 + 4n^2} \rightarrow \frac{1}{2} \neq 1$$

$$\implies f(x, y) \text{ няма граница при } (x, y) \rightarrow (0, 0)$$

3.

$$f(x, y) = \frac{xy^2}{x^2 + y^4}$$

$$\lim_{x \rightarrow 0} f(x, y) = \frac{0}{y^4} = 0 \quad \lim_{y \rightarrow 0} f(x, y) = \frac{0}{x^2} = 0$$

$$A_{1,2} = A_{2,1} = 0 \implies \exists A = \lim_{(x,y) \rightarrow (0,0)} f(x, y)$$

$$\text{Редица: } (x_n, y_n) = \left(\frac{1}{n^2}, \frac{1}{n}\right) \rightarrow (0, 0), f(x_n, y_n) = \frac{1}{2} \rightarrow \frac{1}{2} \neq 0$$

$$\implies f(x, y) \text{ няма граница при } (x, y) \rightarrow (0, 0)$$

4.

$$f(x, y) = (x + y) \sin \frac{1}{x} \cos \frac{1}{y}$$

$$0 \leq |f(x, y)| \leq |x + y| \leq |x| + |y| \text{ и } |x| + |y| \rightarrow 0$$

$$A = 0$$

$$\lim_{x \rightarrow 0} \sin \frac{1}{x} \text{ - не съществува}$$

$$\lim_{x \rightarrow 0} f(x, y) = y \cos \frac{1}{y} \lim_{x \rightarrow 0} \sin \frac{1}{x}$$

Аналогично и другата вътрешна граница не съществува. Но тогава и повторните граници $A_{1,2}$, $A_{2,1}$ не съществуват.

5.

$$\begin{aligned}
f(x, y) &= \frac{x^4 + y^4}{x^2 + y^2} \\
\lim_{x \rightarrow 0} f(x, y) &= y^2 \quad \lim_{y \rightarrow 0} f(x, y) = x^2 \\
A_{1,2} &= \lim_{y \rightarrow 0} \left(\lim_{x \rightarrow 0} f(x, y) \right) = \lim_{y \rightarrow 0} (y^2) = 0 \\
A_{2,1} &= \lim_{x \rightarrow 0} \left(\lim_{y \rightarrow 0} f(x, y) \right) = \lim_{x \rightarrow 0} (x^2) = 0 \\
\implies A &= A_{1,2} = A_{2,1} = 0
\end{aligned}$$

Задача 3

Функцията f е непрекъсната в A , защото е частно на две функции със знаменател $y^2 \neq 0$, в A .

Аналогично е непрекъсната в B защото знаменателя е $x^2 \neq 0$.

Остана да се изследва поведението върху C .

$$\begin{aligned}
(x_0, y_0) &= (x_0, x_0) \in C \\
R &= \{(x_n, y_n)\}, (x_n, y_n) \in A \\
\lim_{n \rightarrow \infty} R &= (x_0, y_0) \\
\lim_{n \rightarrow \infty} f(x_n, y_n) &= \frac{1}{y_0^2} = \frac{1}{x_0^2} \neq 0 \\
\text{Ако } x_0 &\neq 0, f(x_0, y_0) = 0 \\
\implies &\text{ функцията е прекъсната в точката } (x_0, x_0) \neq (0, 0) \\
\text{Ако } (x_n, y_n) &\in B, \lim_{n \rightarrow \infty} f(x_n, y_n) = -\frac{1}{x_0^2} \neq f(x_0, x_0) \neq 0. \\
\text{Ако } x_0 &= 0, \lim_{n \rightarrow \infty} f(x_n, y_n) = \infty(-\infty), f(0, 0) = 0, \\
\implies &f \text{ е прекъсната в точката } (0, 0).
\end{aligned}$$

Функцията е непрекъсната в D , с изключение на точките от C , където е прекъсната.

3 Упражнение към лекция 3

3.1 Задачи

Задача 1

Да се намерят първите частни производни на следните функции

1. $f(x, y, z) = e^{4x+3y} + xy^2z^3 + 1111e^\pi$ за произволна точка $(x_0, y_0, z_0) \in \mathbb{R}^3$

2. $f(x, y) = |x + y|$ в точката $(0, 0)$

3. $f(x, y) = \begin{cases} \frac{xy}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$ в равнината \mathbb{R}^2

Задача 2

$$f(x, y) = x + (y - 1) \arcsin \sqrt{\frac{x}{y}} \quad f'_x(x, 1) = ?$$

Задача 3

Да се докаже че функцията $f(x, y) = \begin{cases} \frac{x^3y}{x^6 + y^2}, & (x, y) \neq (0, 0) \\ 0, & x^2 + y^2 = (0, 0) \end{cases}$

е прекъсната в точката $(0, 0)$ но има частни производни в тази точка.

Задача 4

Да се намерят първите частни производни на следните функции:

1. $f(x, y) = \sin(2x + 3) + 3e^{-x}e^{4y} - 11x^3 + 19e^\pi$

2. $f(x, y) = \sqrt{x^2 + y^2} + \arctan \frac{y}{x}$

3. $f(x, y, z) = (xy)^z$

4. $\sqrt[3]{x^2 + 3y^2}e^{x^2-5y}$

3.2 Решения

Задача 1

1.

$$\begin{aligned}
 f(x, y, z) &= e^{4x+3y} + xy^2z^3 + 1111e^\pi \\
 f(x, y_0, z_0) &\implies f'_x(x_0, y_0, z_0) = 4e^{4x_0+3y_0} + y_0^2z_0^3 \\
 f(x_0, y, z_0) &\implies f'_y(x_0, y_0, z_0) = 3e^{4x_0+3y_0} + 2x_0y_0z_0^3 \\
 f(x_0, y_0, z) &\implies f'_z(x_0, y_0, z_0) = 3x_0y_0^2z_0^2
 \end{aligned}$$

2.

$$\begin{aligned}
 f(x, y) &= |x + y| \\
 \frac{g(h) - g(0)}{h} &= \frac{f(0 + h, 0) - f(0, 0)}{h} \\
 \lim_{h \rightarrow 0} \frac{f(0 + h, 0) - f(0, 0)}{h} &= \lim_{h \rightarrow 0} \frac{|h|}{h} \text{ не съществува} \\
 \implies \nexists f'_x(0, 0) & \text{ (Аналогично се получава за } f'_y(0, 0))
 \end{aligned}$$

3.

$$\begin{aligned}
 f(x, y) &= \begin{cases} \frac{xy}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases} \\
 (x, y) &\neq (0, 0) \\
 f'_x(x, y) &= \frac{y(y^2 - x^2)}{(x^2 + y^2)^2} \\
 f'_y(x, y) &= \frac{x(x^2 - y^2)}{(x^2 + y^2)^2} \\
 \lim_{h \rightarrow 0} \frac{f(0 + h, 0) - f(0, 0)}{h} &= \lim_{h \rightarrow 0} \frac{0 - 0}{h} = \lim_{h \rightarrow 0} 0 = 0 \\
 \lim_{k \rightarrow 0} \frac{f(0, 0 + k) - f(0, 0)}{k} &= \lim_{k \rightarrow 0} \frac{0 - 0}{k} = \lim_{k \rightarrow 0} 0 = 0 \\
 \implies \text{Функцията има частни производни във всяко точки на равнината } \mathbb{R}^2
 \end{aligned}$$

Задача 2

$$\begin{aligned}
f'_x(a, b) &= \lim_{h \rightarrow 0} \frac{f(a+h, b) - f(a, b)}{h} \text{ (Ако съществува) } \implies \\
f'_x(x, 1) &= \lim_{h \rightarrow 0} \frac{f(x+h, 1) - f(x, 1)}{h} \text{ (Ако съществува) } \\
f(x+h, 1) &= x+h + (1-1) \arcsin \sqrt{\frac{x}{1}} = x+h + 0 \arcsin \sqrt{\frac{x}{1}} = x+h \\
f(x, 1) &= x + (1-1) \arcsin \sqrt{\frac{x}{1}} = x + 0 \arcsin \sqrt{\frac{x}{1}} = x \implies \\
\lim_{h \rightarrow 0} \frac{f(x+h, 1) - f(x, 1)}{h} &= \lim_{h \rightarrow 0} \frac{x+h-x}{h} = \lim_{h \rightarrow 0} \frac{h}{h} = \lim_{h \rightarrow 0} 1 \implies f'_x(x, 1) = 1
\end{aligned}$$

Задача 3

$$\begin{aligned}
\text{Редица } (x_n, y_n) &= \left(\frac{1}{n}, \frac{1}{n^3} \right) \\
f(x_n, y_n) &= \frac{\left(\frac{1}{n} \right)^3 \cdot \frac{1}{n^3}}{\left(\frac{1}{n} \right)^6 + \left(\frac{1}{n^3} \right)^3} = \frac{\frac{1}{n^6}}{\frac{1}{n^6} + \frac{1}{n^9}} = \frac{1}{1 + \frac{1}{n^3}} \implies \lim_{n \rightarrow \infty} f(x_n, y_n) = \frac{1}{2} \implies \\
\lim_{x \rightarrow 0, y \rightarrow 0} f(x, y) &\neq f(0, 0) = 0 \implies f(x, y) \text{ е прекъсната в т. } (0, 0).
\end{aligned}$$

$$\begin{aligned}
f'_x(0, 0) &= \lim_{x \rightarrow 0} \frac{f(x, 0) - f(0, 0)}{x - 0} = \frac{\frac{x^3 \cdot 0}{x^6 + 0} - 0}{x - 0} = 0 \\
f'_y(0, 0) &= \lim_{y \rightarrow 0} \frac{f(0, y) - f(0, 0)}{y - 0} = \frac{\frac{0^3 \cdot y}{0^6 + y^2} - 0}{y - 0} = 0
\end{aligned}$$

Задача 4

1.

$$\begin{aligned}
f(x, y) &= \sin(2x + 3) + 3e^{-x}e^{4y} - 11x^3 + 19e^\pi \\
f'_x(x, y) &= (\sin(2x + 3))'_x + (3e^{-x}e^{4y})'_x - (11x^3)'_x + (19e^\pi)'_x \\
f'_x(x, y) &= \cos(2x + 3) \cdot 2 + (-3e^{-x}e^{4y}) - (3 \cdot 11x^2) + 0 \\
f'_x(x, y) &= 2\cos(2x + 3) - 3e^{-x}e^{4y} - 33x^2 \\
f'_y(x, y) &= (\sin(2x + 3))'_y + (3e^{-x}e^{4y})'_y - (11x^3)'_y + (19e^\pi)'_y \\
f'_y(x, y) &= 0 + (3 \cdot 4e^{-x}e^{4y}) - 0 + 0 = 12e^{-x}e^{4y}
\end{aligned}$$

2.

$$\begin{aligned}
f(x, y) &= \sqrt{x^2 + y^2} + \arctan \frac{y}{x} \\
f'_x(x, y) &= \frac{1}{2}(x^2)^{-\frac{1}{2}} \cdot 2x + \frac{1}{1 + \frac{y^2}{x^2}} \cdot y \cdot \left(-\frac{1}{x^2}\right) \\
f'_x(x, y) &= \frac{x}{\sqrt{x^2 + y^2}} - \frac{x^2 y}{x^2 + y^2} \cdot \frac{1}{x^2} \\
f'_x(x, y) &= \frac{x}{\sqrt{x^2 + y^2}} - \frac{xy}{x^2 + y^2} \\
f'_y(x, y) &= \frac{1}{2}(x^2)^{-\frac{1}{2}} \cdot 2y + \frac{1}{1 + \frac{y^2}{x^2}} \cdot \frac{1}{x} \\
f'_y(x, y) &= \frac{y}{\sqrt{x^2 + y^2}} + \frac{x^2}{x^2 + y^2} \cdot \frac{1}{x} \\
f'_y(x, y) &= \frac{y}{\sqrt{x^2 + y^2}} + \frac{x}{x^2 + y^2}
\end{aligned}$$

3.

$$\begin{aligned}
f(x, y, z) &= (xy)^z \\
f'_x(x, y, z) &= z(xy)^{z-1} \cdot (xy)'_x = yz(xy)^{z-1} \\
f'_y(x, y, z) &= z(xy)^{z-1} \cdot (xy)'_y = xz(xy)^{z-1} \\
f'_z(x, y, z) &= (xy)^z \ln(xy)
\end{aligned}$$

4.

$$\begin{aligned}
& \sqrt[3]{x^2 + 3y^2} e^{x^2 - 5y} \\
f'_x(x, y) &= \left[\sqrt[3]{x^2 + 3y^2} \right]'_x \cdot e^{x^2 - 5y} + \sqrt[3]{x^2 + 3y^2} \cdot (e^{x^2 - 5y})'_x \\
f'_x(x, y) &= \frac{1}{3} (x^2 + 3y^2)^{-\frac{2}{3}} \cdot 2x \cdot e^{x^2 - 5y} + \sqrt[3]{x^2 + 3y^2} \cdot 2x e^{x^2 - 5y} \\
f'_x(x, y) &= \frac{2x}{3} \cdot \frac{e^{x^2 - 5y}}{\sqrt[3]{(x^2 + 3y^2)^2}} + 2x \sqrt[3]{x^2 + 3y^2} \cdot e^{x^2 - 5y} \\
f'_x(x, y) &= \frac{2x}{3} \cdot \frac{e^{x^2 - 5y}}{\sqrt[3]{(x^2 + 3y^2)^2}} [1 + 3(x^2 + 3y^2)] \\
f'_x(x, y) &= \frac{2x}{3} (1 + 3x^2 + 9y^2) \frac{e^{x^2 - 5y}}{\sqrt[3]{(x^2 + 3y^2)^2}}
\end{aligned}$$

$$\begin{aligned}
f'_y(x, y) &= \left[\sqrt[3]{x^2 + 3y^2} \right]'_y \cdot e^{x^2 - 5y} + \sqrt[3]{x^2 + 3y^2} \cdot (e^{x^2 - 5y})'_y \\
f'_y(x, y) &= \frac{1}{3} (x^2 + 3y^2)^{-\frac{2}{3}} \cdot 6y \cdot e^{x^2 - 5y} + \sqrt[3]{x^2 + 3y^2} \cdot (-5e^{x^2 - 5y}) \\
f'_y(x, y) &= 2y \cdot \frac{1}{\sqrt[3]{(x^2 + 3y^2)^2}} \cdot e^{x^2 - 5y} - 5 \sqrt[3]{x^2 + 3y^2} \cdot e^{x^2 - 5y} \\
f'_y(x, y) &= e^{x^2 - 5y} \cdot \sqrt[3]{(x^2 + 3y^2)^2} (2y - 5(x^2 + 3y^2)) \\
f'_y(x, y) &= (2y - 5x^2 - 15y^2) \frac{e^{x^2 - 5y}}{\sqrt[3]{(x^2 + 3y^2)^2}}
\end{aligned}$$

4 Упражнение към лекция 4

4.1 Задачи

Задача 1

$$f(x, y) = \sqrt[3]{xy}$$

Изследвайте $f(x, y)$ за диференцируемост в $(0, 0)$.

$$f'_x(0, 0) = ?$$

$$f'_y(0, 0) = ?$$

Задача 2

$$f(x, y) = \sqrt[3]{x^3 + y^3}$$

Изследвайте $f(x, y)$ за диференцируемост в $(0, 0)$.

Задача 3

Да се изследвай за диференцируемост в $(0, 0)$ функцията

$$f(x, y) = \begin{cases} e^{-\frac{1}{x^2 + y^2}}, & x^2 + y^2 \neq 0 \\ 0, & x^2 + y^2 = 0 \end{cases}$$

Задача 4

$$f(x, y) = x^2 + 3xy - 8y^3 + 11, \quad df(0, 1) = ?$$

$$f(x, y, z) = x^2 + 3xy - 8y^3 - 2e^{3z}x, \quad df(0, 0, 4) = ?$$

Задача 5

$$f(x, y) = x^6 - 7xy^2 + 14y,$$

$$f''_{xx} = ?, f''_{yy} = ?, f''_{xy} = ?, d^2f(x, y) = ?$$

$$f(x, y, z) = x^6 - 7xy + y^2 - xz + z^3,$$

$$f''_{xx} = ?, f''_{xy} = ?, f''_{xz} = ?, f''_{yx} = ?, f''_{yy} = ?, f''_{yz} = ?, f''_{zx} = ?, f''_{zy} = ?, f''_{zz} = ?, d^2f(1, 0, 0)$$

4.2 Решения

Задача 1

$$f(x, 0) - f(0, 0) = \sqrt[3]{x0} - \sqrt[3]{0} \implies$$

$$\lim_{x \rightarrow 0} \frac{f(x, 0) - f(0, 0)}{x - 0} = \lim_{x \rightarrow 0} \frac{0}{x} = 0 f'_x(0, 0) = \lim_{x \rightarrow 0} \frac{f(x, 0) - f(0, 0)}{x - 0} = \lim_{x \rightarrow 0} \frac{0}{x} = 0$$

$$f(0, y) - f(0, 0) = \sqrt[3]{0y} - \sqrt[3]{0} \implies$$

$$f'_y(0, 0) = \lim_{y \rightarrow 0} \frac{f(0, y) - f(0, 0)}{y - 0} = \lim_{y \rightarrow 0} \frac{0}{y} = 0$$

$$\text{Нека: } \lim_{(x \rightarrow 0, y \rightarrow 0)} \varepsilon(x, y) \rightarrow 0, \rho(x, y) = \sqrt{x^2 + y^2}$$

Проверка за диференцируемост в $(0, 0)$:

$$f(x, y) - f(0, 0) = f'_x(0, 0)(x - 0) + f'_y(0, 0)(y - 0) + \varepsilon(x, y)\rho(x, y)$$

$$\sqrt[3]{xy} - 0 = 0x + 0y + \varepsilon(x, y)\sqrt{x^2 + y^2} \implies$$

$$\varepsilon(x, y) = \frac{\sqrt[3]{xy}}{\sqrt{x^2 + y^2}} \rightarrow 0?$$

Разглеждаме редица с общ член $(x_n, y_n) = \left(\frac{1}{n^3}, \frac{1}{n^3}\right)$ за която $(x_n, y_n) \rightarrow (0, 0)$,

$$\varepsilon(x_n, y_n) = \frac{\frac{1}{n^3}}{\frac{n^2}{\sqrt{2}}} = \frac{n}{\sqrt{2}} \implies \lim_{(x, y) \rightarrow (0, 0)} \varepsilon(x_n, y_n) \not\rightarrow 0 \implies$$

$f(x, y)$ не е диференцируема в т. $(0, 0)$

Задача 2

$$f(x, 0) - f(0, 0) = \sqrt[3]{x^3} - 0 = x \implies$$

$$\lim_{x \rightarrow 0} \frac{f(x, 0) - f(0, 0)}{x - 0} = \lim_{x \rightarrow 0} \frac{x}{x} = 1 \implies \exists f'_x(0, 0) = 1$$

$$f(0, y) - f(0, 0) = \sqrt[3]{y^3} - 0 = y \implies$$

$$\lim_{y \rightarrow 0} \frac{f(0, y) - f(0, 0)}{y - 0} = \lim_{y \rightarrow 0} \frac{y}{y} = 1 \implies \exists f'_y(0, 0) = 1$$

$$\text{Нека: } \lim_{(x \rightarrow 0, y \rightarrow 0)} \varepsilon(x, y) \rightarrow 0, \rho(x, y) = \sqrt{x^2 + y^2}$$

Проверка за диференцируемост в $(0, 0)$:

$$f(x, y) - f(0, 0) = f'_x(0, 0)(x - 0) + f'_y(0, 0)(y - 0) + \varepsilon(x, y)\rho(x, y)$$

$$\sqrt[3]{x^3 + y^3} = x + y + \varepsilon(x, y)\sqrt{x^2 + y^2}$$

$$\varepsilon(x, y) = \frac{\sqrt[3]{x^3 + y^3} - x - y}{\sqrt{x^2 + y^2}}$$

$$\lim_{(x \rightarrow 0, y \rightarrow 0)} \varepsilon(x, y) \rightarrow 0?$$

Разглеждаме редица с общ член $(x_n, y_n) = \left(\frac{1}{n}, \frac{1}{n}\right)$ за която $(x_n, y_n) \rightarrow (0, 0)$,

$$\varepsilon(x_n, y_n) = \frac{\frac{\sqrt[3]{2}}{n} - \frac{2}{n}}{\frac{\sqrt{2}}{n}} = \frac{\sqrt[3]{2} - 2}{\sqrt{2}} \implies \lim_{(x \rightarrow 0, y \rightarrow 0)} \varepsilon(x, y) \not\rightarrow 0 \implies$$

$f(x, y)$ не е диференцируема в т. $(0, 0)$

Задача 3

$$f(x, 0) - f(0, 0) = e^{-\frac{1}{x^2}} - 0 = e^{-\frac{1}{x^2}}$$

$$\lim_{x \rightarrow 0} \frac{f(x, 0) - f(0, 0)}{x - 0} = \lim_{x \rightarrow 0} \frac{e^{-\frac{1}{x^2}}}{x} = \left[\frac{0}{0} \right]$$

$$\lim_{x \rightarrow 0} \frac{e^{-\frac{1}{x^2}}}{x} = \lim_{x \rightarrow 0} \frac{\frac{1}{x}}{\frac{1}{e x^2}} = \left[\frac{\infty}{\infty} \right]$$

$$\left(\frac{1}{x} \right)' = -\frac{1}{x^2} \quad \left(\frac{1}{e x^2} \right)' = -\frac{2}{x^3} e^{-\frac{1}{x^2}}$$

$$\lim_{x \rightarrow 0} \frac{-\frac{1}{x^2}}{-\frac{2}{x^3} e^{-\frac{1}{x^2}}} = \lim_{x \rightarrow 0} \frac{x}{2 e^{-\frac{1}{x^2}}} = \frac{0}{\infty} = 0 \implies f'_x(0, 0) = 0$$

Аналогично $f'_y(0, 0) = 0$

Нека : $\lim_{(x \rightarrow 0, y \rightarrow 0)} \varepsilon(x, y) \rightarrow 0, \rho(x, y) = \sqrt{x^2 + y^2}$

Проверка за дифференцируемость в $(0, 0)$:

$$f(x, y) - f(0, 0) = f'_x(0, 0)(x - 0) + f'_y(0, 0)(y - 0) + \varepsilon(x, y)\rho(x, y)$$

$$e^{-\frac{1}{x^2 + y^2}} - 0 = 0(x - 0) + 0(y - 0) + \varepsilon(x, y)\sqrt{x^2 + y^2}$$

$$e^{-\frac{1}{x^2 + y^2}} = \varepsilon(x, y)\sqrt{x^2 + y^2}$$

$$\varepsilon(x, y) = \frac{e^{-\frac{1}{x^2 + y^2}}}{\sqrt{x^2 + y^2}}$$

$$\lim_{(x \rightarrow 0, y \rightarrow 0)} \varepsilon(x, y) \rightarrow 0?$$

$$\rho(x, y) = \sqrt{x^2 + y^2} \implies \lim_{(x \rightarrow 0, y \rightarrow 0)} \rho(x, y) \rightarrow 0$$

$$\lim_{(x \rightarrow 0, y \rightarrow 0)} \varepsilon(x, y) = \lim_{\rho \rightarrow 0} \frac{e^{-\frac{1}{\rho^2}}}{\rho} = \left[\frac{\infty}{\infty} \right]$$

$$\left(\frac{1}{\rho} \right)' = -\frac{1}{\rho^2} \quad \left(e^{\frac{1}{\rho^2}} \right)' = -\frac{2}{\rho^3} e^{\frac{1}{\rho^2}}$$

$$\lim_{\rho \rightarrow 0} \frac{\rho}{\frac{1}{2e\rho^2}} = \frac{0}{\infty} = 0 \implies$$

$$\lim_{(x \rightarrow 0, y \rightarrow 0)} \varepsilon(x, y) = \lim_{\rho \rightarrow 0} \frac{\frac{1}{\rho}}{\frac{1}{e\rho^2}} = \lim_{\rho \rightarrow 0} \frac{\left(\frac{1}{\rho} \right)'}{\left(\frac{1}{e\rho^2} \right)'} = 0 \implies$$

$$\lim_{(x \rightarrow 0, y \rightarrow 0)} \varepsilon(x, y) = 0 \implies f(x, y) \text{ е диференцируема в } (0, 0)$$

Задача 4

$$df(x, y) = f'_x(x, y)dx + f'_y(x, y)dy$$

$$f'_x(x, y) = 2x + 3y \quad f'_x(0, 1) = 3$$

$$f'_y(x, y) = 3x - 24y^2 \quad f'_y(0, 1) = -24$$

$$df(x, y) = (2x + 3y)dx + (3x - 24y^2)dy$$

$$df(0, 1) = 3dx - 24dy$$

$$df(x, y, z) = f'_x(x, y, z)dx + f'_y(x, y, z)dy + f'_z(x, y, z)dz$$

$$f'_x(x, y, z) = 2x + 3y - 2e^{3z} \quad f'_x(0, 0, 4) = -2e^{12}$$

$$f'_y(x, y, z) = 3x - 24y^2 \quad f'_y(0, 0, 4) = 0$$

$$f'_z(x, y, z) = 6xe^{3z} \quad f'_z(0, 0, 4) = 0$$

$$df(x, y, z) = (2x + 3y - 2e^{3z})dx + (3x - 24y^2)dy + (6xe^{3z})dz$$

$$df(x, y, z) = -2e^{12}dx + 0dy + 0dz = -2e^{12}dx$$

Задача 5

$$\begin{aligned}
f'_x(x, y, z) &= 6x^5 - 7y - z \\
f''_{xx}(x, y, z) &= (6x^5 - 7y - z)'_x = 30x^4 & f''_{xx}(1, 0, 0) &= 30 \\
f''_{xy}(x, y, z) &= (6x^5 - 7y - z)'_y = -7 & f''_{xy}(1, 0, 0) &= -7 \\
f''_{xz}(x, y, z) &= (6x^5 - 7y - z)'_z = -1 & f''_{xz}(1, 0, 0) &= -1
\end{aligned}$$

$$\begin{aligned}
f'_y(x, y, z) &= -7x + 2y \\
f''_{yx}(x, y, z) &= (-7x + 2y)'_x = -7 & f''_{yx}(1, 0, 0) &= -7 \\
f''_{yy}(x, y, z) &= (-7x + 2y)'_y = 2 & f''_{yy}(1, 0, 0) &= 2 \\
f''_{yz}(x, y, z) &= (-7x + 2y)'_z = 0 & f''_{yz}(1, 0, 0) &= 0
\end{aligned}$$

$$\begin{aligned}
f'_z(x, y, z) &= -x + 3z^2 \\
f''_{zx}(x, y, z) &= (-x + 3z^2)'_x = -1 & f''_{zx}(1, 0, 0) &= -1 \\
f''_{zy}(x, y, z) &= (-x + 3z^2)'_y = 0 & f''_{zy}(1, 0, 0) &= 0 \\
f''_{zz}(x, y, z) &= (-x + 3z^2)'_z = 6z & f''_{zz}(1, 0, 0) &= 0
\end{aligned}$$

$$\begin{aligned}
d^2f &= f''_{xx}dx^2 + 2f''_{xy}dxdy + f''_{yy}dy^2 + 2f''_{xz}dxdz + f''_{zz}dz^2 + f''_{yz}dydz \\
d^2f(x, y, z) &= 30x^4dx^2 + 2 \cdot (-7)dxdy + 2dy^2 + 2 \cdot (-1)dxdz + 6zdz^2 + 2 \cdot 0dydz \\
d^2f(1, 0, 0) &= 30dx^2 - 14dxdy + 2dy^2 - 2dxdz + 0dz^2 + 0dydz \\
d^2f(1, 0, 0) &= 30dx^2 + 2dy^2 - 14dxdy - 2dxdz
\end{aligned}$$

5 Упражнение към лекция 5

5.1 Задачи

Задача 1

Да се намерят посочените частни производни на следните функции.

1. $u(x, y) = x^4 + 11x^2y^3$, $u''_{xx} = ?$, $u''_{xy} = ?$
2. $u(x, y) = \arctan \frac{x+y}{1-xy}$, $u''_{xx} = ?$, $u''_{xy} = ?$, $u''_{yy} = ?$
3. $u(x, y) = \frac{1}{2} \ln(x^2 + y^2)$, $u''_{xx} = ?$, $u''_{xy} = ?$, $u''_{yx} = ?$, $u''_{yy} = ?$
4. $u(x, y) = \ln(x + 2y)$, $u'''_{xxy} = ?$
5. $u(x, y, z) = e^{xy^2z^3}$, $u'''_{xyz} = ?$

Задача 2

Дали са верни равенствата:

- Ако $z = y \ln(x^2 + y^2)$ то $\frac{1}{x}z'_x + \frac{1}{y}z'_y = \frac{z}{y^2}$
- Ако $u = \ln(x^3 + y^3 + z^3 - 3xyz)$ то $u'_x + u'_y + u'_z = \frac{3}{x+y+z}$

Задача 3

Да се докаже, че функцията: $z(x, y) = \arctan \left(\frac{x+y}{x-y} \right)$ удовлетворява

тържеството: $z'_x + z'_y = \frac{x-y}{x^2+y^2}$

Задача 4

Да се провери тържеството на Ойлер за следните функции: $z(x, y) = \frac{1}{(x^2 + y^2)^2}$ $u(x, y, z) = \sqrt{x^2 + y^2 + z^2} \cdot \ln \left(\frac{y}{x} \right)$

Тържество на Ойлер ($f : D \rightarrow R, D \subset \mathbb{R}^m$)

$$x_1 f'_{x_1} + x_2 f'_{x_2} + \dots + x_m f'_{x_m} = m f$$

5.2 Решения

Задача 1

$$u(x, y) = x^4 + 11x^2y^3$$

$$u'_x = 4x^3 + 22xy^3$$

$$u''_{xx} = 12x^2 + 22y^3$$

$$u''_{xy} = 4x^3 + 66xy^2$$

$$u(x, y) = \arctan \frac{x+y}{1-xy}$$

$$u'_x = \frac{1}{1 + \left(\frac{x+y}{1-xy}\right)^2} \cdot \left(\frac{x+y}{1-xy}\right)'_x$$

$$u'_y = \frac{1}{1 + \left(\frac{x+y}{1-xy}\right)^2} \cdot \left(\frac{x+y}{1-xy}\right)'_y$$

$$u''_{xx} = (u'_x)'_x$$

$$u''_{xy} = (u'_x)'_y$$

$$u''_{yy} = (u'_y)'_y$$

$$A = \frac{1}{1 + \left(\frac{x+y}{1-xy}\right)^2} \quad B = \left(\frac{x+y}{1-xy}\right)'_x \implies u'_x = AB$$

$$A = \frac{1}{1 + \left(\frac{x+y}{1-xy}\right)^2} = \frac{1}{1 + \frac{(x+y)^2}{(1-xy)^2}} = \frac{(1-xy)^2}{(1-xy)^2 + (x+y)^2}$$

$$A = \frac{(1-xy)^2}{1 - 2xy + x^2y^2 + x^2 + 2xy + y^2} = \frac{(1-xy)^2}{1 + x^2y^2 + x^2 + y^2}$$

$$A = \frac{(1-xy)^2}{(1+y^2) + x^2 + x^2y^2} = \frac{(1-xy)^2}{(1+y^2) + x^2(1+y^2)} = \frac{(1-xy)^2}{(1+y^2)(1+x^2)}$$

$$B = \left(\frac{x+y}{1-xy}\right)'_x = \frac{1(1-xy) - (x+y)(-y)}{(1-xy)^2} = \frac{1-xy+xy+y^2}{(1-xy)^2} = \frac{1+y^2}{(1-xy)^2}$$

$$u'_x = AB = \frac{(1-xy)^2}{(1+y^2)(1+x^2)} \cdot \frac{1+y^2}{(1-xy)^2} = \frac{1}{1+x^2}$$

$$C = \left(\frac{x+y}{1-xy}\right)'_y \implies u'_y = AC$$

$$C = \frac{1(1-xy) - (x+y)(-x)}{(1-xy)^2} = \frac{1-xy+x^2+xy}{(1-xy)^2} = \frac{1+x^2}{(1-xy)^2}$$

$$u'_y = AC = \frac{(1-xy)^2}{(1+y^2)(1+x^2)} \cdot \frac{1+x^2}{(1-xy)^2} = \frac{1}{1+y^2}$$

$$u''_{xx} = \left(\frac{1}{1+x^2}\right)'_x = ((1+x^2)^{-1})'_x$$

$$u''_{yy} = -(1+x^2)^{-2}(1+x^2)'_x = -2x(1+x^2)^{-2} = \frac{-2x}{(1+x^2)^2}$$

$$u''_{xy} = \left(\frac{1}{1+x^2}\right)'_y = 0$$

$$u''_{yy} = \left(\frac{1}{1+y^2}\right)'_y = ((1+y^2)^{-1})'_y$$

$$u''_{yy} = -(1+y^2)^{-2}(1+y^2)'_y = -2y(1+y^2)^{-2} = \frac{-2y}{(1+y^2)^2}$$

$$\begin{aligned}
u(x, y) &= \frac{1}{2} \ln(x^2 + y^2) \\
u'_x &= \frac{1}{2(x^2 + y^2)} \cdot (x^2 + y^2)'_x = \frac{2x}{2(x^2 + y^2)} = \frac{x}{x^2 + y^2} \\
u'_y &= \frac{1}{2(x^2 + y^2)} \cdot (x^2 + y^2)'_y = \frac{2y}{2(x^2 + y^2)} = \frac{y}{x^2 + y^2} \\
u''_{xx} &= (u'_x)'_x = \left(\frac{x}{x^2 + y^2} \right)'_x = \frac{1(x^2 + y^2) - (2x)x}{(x^2 + y^2)^2} = \frac{x^2 + y^2 - 2x^2}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2} \\
u''_{xy} &= (u'_x)'_y = \left(\frac{x}{x^2 + y^2} \right)'_y = \frac{-2xy}{(x^2 + y^2)^2} \\
u''_{yy} &= (u'_y)'_y = \left(\frac{y}{x^2 + y^2} \right)'_y = \frac{1(x^2 + y^2) - (2y)y}{(x^2 + y^2)^2} = \frac{x^2 + y^2 - 2y^2}{(x^2 + y^2)^2} = \frac{x^2 - y^2}{(x^2 + y^2)^2} \\
u''_{yx} &= (u'_y)'_x = \left(\frac{y}{x^2 + y^2} \right)'_x = \frac{-2xy}{(x^2 + y^2)^2}
\end{aligned}$$

$$\begin{aligned}
u(x, y) &= \ln(x + 2y) \\
u'_x &= \frac{1}{x + 2y} \\
u''_{xx} &= \left(\frac{1}{x + 2y} \right)'_x = ((x + 2y)^{-1})'_x = -(x + 2y)^{-2}(x + 2y)'_x = -\frac{1}{(x + 2y)^2} \\
u'''_{xxy} &= \left(-\frac{1}{(x + 2y)^2} \right)'_y = -((x + 2y)^{-2})'_y = 2((x + 2y)^{-3})(x + 2y)'_y = \frac{4}{(x + 2y)^3}
\end{aligned}$$

$$\begin{aligned}
u(x, y, z) &= e^{xy^2z^3} \\
u'_x &= e^{xy^2z^3} (xy^2z^3)'_x = y^2z^3 e^{xy^2z^3} \\
u''_{xy} &= (y^2z^3 \cdot e^{xy^2z^3})'_y = (y^2z^3)'_y \cdot e^{xy^2z^3} + y^2z^3 (e^{xy^2z^3})'_y \\
u''_{xy} &= 2yz^3 e^{xy^2z^3} + 2xy^3z^6 e^{xy^2z^3} = 2yz^3 e^{xy^2z^3} (1 + xy^2z^3)
\end{aligned}$$

$$\begin{aligned}
u'''_{xyz} &= \left[2yz^3 e^{xy^2 z^3} (1 + xy^2 z^3) \right]'_z = (2yz^3 e^{xy^2 z^3})'_z (1 + xy^2 z^3) + 2yz^3 e^{xy^2 z^3} (1 + xy^2 z^3)'_z \\
&= \left[(2yz^3)'_z \cdot e^{xy^2 z^3} + 2yz^3 \cdot (e^{xy^2 z^3})'_z \right] (1 + xy^2 z^3) + 2yz^3 e^{xy^2 z^3} (1 + xy^2 z^3)'_z \\
u'''_{xyz} &= \left[6yz^2 e^{xy^2 z^3} + 2yz^3 e^{xy^2 z^3} 3xy^2 z^2 \right] (1 + xy^2 z^3) + 2yz^3 e^{xy^2 z^3} (3xy^2 z^2) \\
u'''_{xyz} &= \left[6yz^2 e^{xy^2 z^3} + 6xy^3 z^5 e^{xy^2 z^3} \right] (1 + xy^2 z^3) + 6xy^3 z^5 e^{xy^2 z^3} \\
u'''_{xyz} &= \left[6yz^2 e^{xy^2 z^3} + 6yz^2 e^{xy^2 z^3} xy^2 z^3 + 6xy^3 z^5 e^{xy^2 z^3} + 6xy^3 z^5 e^{xy^2 z^3} xy^2 z^3 \right] + 6xy^3 z^5 e^{xy^2 z^3} \\
u'''_{xyz} &= \left[6yz^2 e^{xy^2 z^3} + 6xy^3 z^5 e^{xy^2 z^3} + 6xy^3 z^5 e^{xy^2 z^3} + 6x^2 y^5 z^8 e^{xy^2 z^3} \right] + 6xy^3 z^5 e^{xy^2 z^3} \\
u'''_{xyz} &= 6yz^2 e^{xy^2 z^3} + 6xy^3 z^5 e^{xy^2 z^3} + 6xy^3 z^5 e^{xy^2 z^3} + 6x^2 y^5 z^8 e^{xy^2 z^3} + 6xy^3 z^5 e^{xy^2 z^3} \\
u'''_{xyz} &= 6yz^2 e^{xy^2 z^3} + 18xy^3 z^5 e^{xy^2 z^3} + 6x^2 y^5 z^8 e^{xy^2 z^3} \\
u'''_{xyz} &= 6yz^2 e^{xy^2 z^3} [1 + 3xy^2 z^3 + x^2 y^4 z^6]
\end{aligned}$$

Задача 2

$$\begin{aligned}
z &= y \ln(x^2 + y^2) \\
z'_x &= y \frac{1}{x^2 + y^2} 2x = \frac{2xy}{x^2 + y^2} \\
z'_y &= \ln(x^2 + y^2) + y \frac{1}{x^2 + y^2} - 2y = \ln(x^2 + y^2) - \frac{2y^2}{x^2 + y^2} \\
\frac{1}{x} z'_x + \frac{1}{y} z'_y &= \frac{1}{x} \cdot \frac{2xy}{x^2 + y^2} + \frac{1}{y} \cdot \left[\ln(x^2 + y^2) - \frac{2y^2}{x^2 + y^2} \right] = \\
&= \frac{2y}{x^2 + y^2} + \frac{\ln(x^2 + y^2)}{y} - \frac{2y}{x^2 + y^2} = \frac{\ln(x^2 + y^2)}{y} \\
\frac{z}{y^2} &= \frac{y \ln(x^2 + y^2)}{y^2} = \frac{\ln(x^2 + y^2)}{y} \implies \text{Равенството е вярно.}
\end{aligned}$$

$$\begin{aligned}
u &= \ln(x^3 + y^3 + z^3 - 3xyz) \\
u'_x &= \frac{(x^3 + y^3 + z^3 - 3xyz)'_x}{x^3 + y^3 + z^3 - 3xyz} = \frac{3x^2 - 3yz}{x^3 + y^3 + z^3 - 3xyz} \\
u'_y &= \frac{(x^3 + y^3 + z^3 - 3xyz)'_y}{x^3 + y^3 + z^3 - 3xyz} = \frac{3y^2 - 3xz}{x^3 + y^3 + z^3 - 3xyz} \\
u'_z &= \frac{(x^3 + y^3 + z^3 - 3xyz)'_z}{x^3 + y^3 + z^3 - 3xyz} = \frac{3z^2 - 3xy}{x^3 + y^3 + z^3 - 3xyz} \\
u'_x + u'_y + u'_z &= \frac{3x^2 - 3yz}{x^3 + y^3 + z^3 - 3xyz} + \frac{3y^2 - 3xz}{x^3 + y^3 + z^3 - 3xyz} + \frac{3z^2 - 3xy}{x^3 + y^3 + z^3 - 3xyz} = \\
&= \frac{3x^2 - 3yz + 3y^2 - 3xz + 3z^2 - 3xy}{x^3 + y^3 + z^3 - 3xyz} = \frac{3(x^2 - yz + y^2 - xz + z^2 - xy)}{x^3 + y^3 + z^3 - 3xyz} = \\
&= \frac{3(x^2 + y^2 + z^2 - xy - xz - yz)}{x^3 + y^3 + z^3 - 3xyz} \cdot \frac{x + y + z}{x + y + z} = \frac{3(x^3 + y^3 + z^3 - 3xyz)}{(x^3 + y^3 + z^3 - 3xyz)(x + y + z)} = \\
&= \frac{3}{x + y + z} \implies \text{Равенството е вярно.}
\end{aligned}$$

Задача 3

$$\begin{aligned}
z'_x &= \frac{1}{1 + \left(\frac{x+y}{x-y}\right)^2} \cdot \left(\frac{x+y}{x-y}\right)'_x = \frac{1}{\frac{(x-y)^2 + (x+y)^2}{(x-y)^2}} \cdot \frac{x-y-x-y}{(x-y)^2} = \\
z'_x &= \frac{(x-y)^2}{(x-y)^2 + (x+y)^2} \cdot \frac{-2y}{(x-y)^2} = \frac{-2y}{x^2 - 2xy + y^2 + x^2 + 2xy + y^2} \\
z'_x &= \frac{-2y}{2(x^2 + y^2)} = -\frac{y}{x^2 + y^2} \\
z'_y &= \frac{1}{1 + \left(\frac{x+y}{x-y}\right)^2} \cdot \left(\frac{x+y}{x-y}\right)'_y = \frac{1}{\frac{(x-y)^2 + (x+y)^2}{(x-y)^2}} \cdot \frac{x-y+x+y}{(x-y)^2} = \\
z'_y &= \frac{(x-y)^2}{(x-y)^2 + (x+y)^2} \cdot \frac{2x}{(x-y)^2} = \frac{2x}{x^2 - 2xy + y^2 + x^2 + 2xy + y^2} = \frac{x}{x^2 + y^2} \\
z'_x + z'_y &= -\frac{y}{x^2 + y^2} + \frac{x}{x^2 + y^2} = \frac{x-y}{x^2 + y^2} \implies \text{тържеството е вярно}
\end{aligned}$$

Задача 4

$$\begin{aligned}
z(x, y) &= \frac{1}{(x^2 + y^2)^2} \\
xz'_x + yz'_y &= 2z \\
z'_x &= \left(\frac{1}{(x^2 + y^2)^2} \right)'_x = ((x^2 + y^2)^{-2})'_x = -2(x^2 + y^2)^{-3}(x^2 + y^2)'_x = -\frac{4x}{(x^2 + y^2)^3} \\
z'_y &= \left(\frac{1}{(x^2 + y^2)^2} \right)'_y = ((x^2 + y^2)^{-2})'_y = -2(x^2 + y^2)^{-3}(x^2 + y^2)'_y = -\frac{4y}{(x^2 + y^2)^3} \\
xz'_x + yz'_y &= x \cdot \left(-\frac{4x}{(x^2 + y^2)^3} \right) + y \cdot \left(-\frac{4y}{(x^2 + y^2)^3} \right) = -\frac{4x^2}{(x^2 + y^2)^3} - \frac{4y^2}{(x^2 + y^2)^3} = \\
&= \frac{-4(x^2 + y^2)}{(x^2 + y^2)^3} = -\frac{4}{(x^2 + y^2)^2} \\
2z &= \frac{2}{(x^2 + y^2)^2} \\
-\frac{4}{(x^2 + y^2)^2} &\neq \frac{2}{(x^2 + y^2)^2} \implies \text{Тъждението не е изпълнено.}
\end{aligned}$$

$$\begin{aligned}
u(x, y, z) &= \sqrt{x^2 + y^2 + z^2} \cdot \ln \left(\frac{y}{x} \right) \\
xu'_x + yu'_y + zu'_z &= 3z \\
u'_x &= \left(\sqrt{x^2 + y^2 + z^2} \right)'_x \ln \left(\frac{y}{x} \right) + \sqrt{x^2 + y^2 + z^2} \left(\ln \left(\frac{y}{x} \right) \right)'_x \\
u'_y &= \left(\sqrt{x^2 + y^2 + z^2} \right)'_y \ln \left(\frac{y}{x} \right) + \sqrt{x^2 + y^2 + z^2} \left(\ln \left(\frac{y}{x} \right) \right)'_y \\
u'_z &= \left(\sqrt{x^2 + y^2 + z^2} \right)'_z \ln \left(\frac{y}{x} \right) + \sqrt{x^2 + y^2 + z^2} \left(\ln \left(\frac{y}{x} \right) \right)'_z
\end{aligned}$$

$$\begin{aligned}
u'_x &= \left(\sqrt{x^2 + y^2 + z^2} \right)'_x \ln \left(\frac{y}{x} \right) + \sqrt{x^2 + y^2 + z^2} \left(\ln \left(\frac{y}{x} \right) \right)'_x \\
u'_x &= \frac{x \ln \left(\frac{y}{x} \right)}{\sqrt{x^2 + y^2 + z^2}} - \frac{\sqrt{x^2 + y^2 + z^2}}{x} = \frac{x \ln \left(\frac{y}{x} \right) x - \left(\sqrt{x^2 + y^2 + z^2} \right)^2}{x \sqrt{x^2 + y^2 + z^2}} \\
u'_x &= \frac{x^2 \ln \left(\frac{y}{x} \right) - x^2 - y^2 - z^2}{x \sqrt{x^2 + y^2 + z^2}}
\end{aligned}$$

$$\begin{aligned}
u'_y &= \left(\sqrt{x^2 + y^2 + z^2} \right)'_y \ln \left(\frac{y}{x} \right) + \sqrt{x^2 + y^2 + z^2} \left(\ln \left(\frac{y}{x} \right) \right)'_y \\
u'_y &= \frac{y \ln \left(\frac{y}{x} \right)}{\sqrt{x^2 + y^2 + z^2}} + \frac{\sqrt{x^2 + y^2 + z^2}}{y} = \frac{y \ln \left(\frac{y}{x} \right) y + \left(\sqrt{x^2 + y^2 + z^2} \right)^2}{y \sqrt{x^2 + y^2 + z^2}} \\
u'_y &= \frac{y^2 \ln \left(\frac{y}{x} \right) + x^2 + y^2 + z^2}{y \sqrt{x^2 + y^2 + z^2}}
\end{aligned}$$

$$\begin{aligned}
u'_z &= \left(\sqrt{x^2 + y^2 + z^2} \right)'_z \ln \left(\frac{y}{x} \right) + \sqrt{x^2 + y^2 + z^2} \left(\ln \left(\frac{y}{x} \right) \right)'_z \\
u'_z &= \frac{z \ln \left(\frac{y}{x} \right)}{\sqrt{x^2 + y^2 + z^2}} + 0 \cdot \sqrt{x^2 + y^2 + z^2} = \frac{z \ln \left(\frac{y}{x} \right)}{\sqrt{x^2 + y^2 + z^2}}
\end{aligned}$$

$$\begin{aligned}
xu'_x + yu'_y + zu'_z &= 3z, \quad A = xu'_x + yu'_y + zu'_z, \quad B = 3u \\
A &= x \cdot \frac{x^2 \ln \left(\frac{y}{x} \right) - x^2 - y^2 - z^2}{x \sqrt{x^2 + y^2 + z^2}} + y \cdot \frac{y^2 \ln \left(\frac{y}{x} \right) + x^2 + y^2 + z^2}{y \sqrt{x^2 + y^2 + z^2}} + z \cdot \frac{z \ln \left(\frac{y}{x} \right)}{\sqrt{x^2 + y^2 + z^2}} \\
A &= \frac{x^2 \ln \left(\frac{y}{x} \right) - x^2 - y^2 - z^2}{\sqrt{x^2 + y^2 + z^2}} + \frac{y^2 \ln \left(\frac{y}{x} \right) + x^2 + y^2 + z^2}{\sqrt{x^2 + y^2 + z^2}} + \frac{z^2 \ln \left(\frac{y}{x} \right)}{\sqrt{x^2 + y^2 + z^2}} \\
A &= \frac{x^2 \ln \left(\frac{y}{x} \right) - x^2 - y^2 - z^2 + y^2 \ln \left(\frac{y}{x} \right) + x^2 + y^2 + z^2 + z^2 \ln \left(\frac{y}{x} \right)}{\sqrt{x^2 + y^2 + z^2}} \\
A &= \frac{x^2 \ln \left(\frac{y}{x} \right) \ln \left(\frac{y}{x} \right) + z^2 \ln \left(\frac{y}{x} \right)}{\sqrt{x^2 + y^2 + z^2}} = \frac{\ln \left(\frac{y}{x} \right) (x^2 + y^2 + z^2)}{\sqrt{x^2 + y^2 + z^2}} = \sqrt{x^2 + y^2 + z^2} \cdot \ln \left(\frac{y}{x} \right) \\
B &= 3u = 3\sqrt{x^2 + y^2 + z^2} \cdot \ln \left(\frac{y}{x} \right) \implies A \neq B \implies \text{Тъждението не е изпълнено.}
\end{aligned}$$

6 Упражнение към лекция 6

6.1 Задачи

Задача 1

Дадени са функцията $z(x, y) = \varphi(x + y) + \psi(x - y)$, където φ, ψ - непрекъснато диференцируеми. Да се намерят първите частни производни.

Задача 2

Да се провери дали $w(x, y, z)$ удовлетворява тъждествено равенството:

$$xw_x + yw_y + zw_z = w + \frac{xy}{z}$$

Ако $w = \frac{xy}{z} + \ln x + x \cdot \varphi\left(\frac{y}{x}, \frac{z}{x}\right)$, φ е непрекъснато диференцируема.

Задача 3

Дадени са функциите и точката $M(2,1)$. Да се пресметне $\text{grad}f(M)$ и $\|\text{grad}f(M)\|$

1. $f(x, y) = x^2 + 11y^2 - 3$
2. $f(x, y) = x^2 - y^2$
3. $f(x, y) = \ln(x^2 + y^2)$

Задача 4

Дадени са функциите и точката $M(2,1)$.

Да се пресметне $\frac{\partial f(M)}{\partial \nu}, \nu = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

1. $f(x, y) = x^2 + 11y^2 - 3$
2. $f(x, y) = x^2 - y^2$
3. $f(x, y) = \ln(x^2 + y^2)$

Задача 5

Да се определи ъгъла между градиентите на функцията

$$u = x^2 + y^2 + z^2 - 111$$

в точките $A(\varepsilon, 0, 0)$ и $B(0, \varepsilon, 0), \varepsilon > 0$

Задача 6

Да се намери y', y'' на неявната функция $y = f(x)$, дефинирана от уравнението

$$x^2 - 2xy + 5y^2 + 4y = 2x + 9$$

Да се пресметнат $y'(0), y''(0)$, ако $y(0) = 1$

6.2 Решения

Задача 1

$$\begin{aligned}
 z(x, y) &= \varphi(x + y) + \psi(x - y) \\
 z'_x &= \varphi'(x + y)(x + y)'_x + \psi'(x - y)(x - y)'_x = \varphi'(x + y)1 + \psi'(x - y)1 \\
 z'_x &= \varphi'(x + y) + \psi'(x - y) \\
 z'_y &= \varphi'(x + y)(x + y)'_y + \psi'(x - y)(x - y)'_y = \varphi'(x + y)1 + \psi'(x - y)(-1) \\
 z'_y &= \varphi'(x + y) - \psi'(x - y)
 \end{aligned}$$

Задача 2

$$\begin{aligned}
 u &= \frac{y}{x} & v &= \frac{z}{x} \\
 u'_x &= -\frac{y}{x^2} & u'_y &= \frac{1}{x} & u'_z &= 0 \\
 v'_x &= -\frac{z}{x^2} & v'_y &= 0 & v'_z &= \frac{1}{x} \\
 w'_x &= \frac{y}{z} \ln x + \frac{xy}{z} \cdot \frac{1}{x} + \varphi\left(\frac{y}{x}, \frac{z}{x}\right) + x(\varphi'_u u'_x + \varphi'_v v_x) = \\
 w'_x &= \frac{y}{z} \ln x + \frac{y}{z} + \varphi\left(\frac{y}{x}, \frac{z}{x}\right) - \frac{y}{x} \varphi'_u - \frac{z}{x} \varphi'_v \\
 w'_y &= \frac{x}{z} \ln x + x(\varphi'_u u'_y + \varphi'_v v_y) = \frac{x}{z} \ln x + \varphi'_u \\
 w'_z &= -\frac{xy}{z^2} \ln x + x(\varphi'_u u'_z + \varphi'_v v_z) = -\frac{xy}{z^2} \ln x + \varphi'_v \\
 xw_x + yw_y + zw_z &= \\
 &= \frac{xy}{z} \ln x + \frac{xy}{z} + x\varphi\left(\frac{y}{x}, \frac{z}{x}\right) - y\varphi'_u - z\varphi'_v + \frac{xy}{z} \ln x + y\varphi'_u + -\frac{xy}{z} \ln x + z\varphi'_v = \\
 &= \frac{xy}{z} + \ln x + x \cdot \varphi\left(\frac{y}{x}, \frac{z}{x}\right) + \frac{xy}{z} = w + \frac{xy}{z}
 \end{aligned}$$

Задача 3

$$\operatorname{grad} f = (f'_x, f'_y)$$

$$\begin{aligned}
 f(x, y) &= x^2 + 11y^2 - 3 \\
 f'_x &= 2x & f'_y &= 22y \\
 \operatorname{grad} f(x, y) &= (2x, 22y) \\
 \operatorname{grad} f(M) &= (2 \cdot 2, 22 \cdot 1) = (4, 22) \\
 \|\operatorname{grad} f(M)\| &= \sqrt{4^2 + 22^2} = \sqrt{500} = 10\sqrt{5}
 \end{aligned}$$

$$\begin{aligned}
f(x, y) &= x^2 - y^2 \\
f'_x &= 2x \quad f'_y = -2y \\
\text{grad}f(x, y) &= (2x, -2y) \\
\text{grad}f(M) &= (2 \cdot 2, -2 \cdot 1) = (4, -2) \\
\|\text{grad}f(M)\| &= \sqrt{4^2 + (-2)^2} = \sqrt{20} = 2\sqrt{5}
\end{aligned}$$

$$\begin{aligned}
f(x, y) &= \ln(x^2 + y^2) \\
f'_x &= \frac{2x}{x^2 + y^2} \quad f'_y = \frac{2y}{x^2 + y^2} \\
\text{grad}f(x, y) &= \left(\frac{2x}{x^2 + y^2}, \frac{2y}{x^2 + y^2} \right) \\
\text{grad}f(M) &= \left(\frac{2 \cdot 2}{2^2 + 1^2}, \frac{2 \cdot 1}{2^2 + 1^2} \right) = \left(\frac{4}{5}, \frac{2}{5} \right) \\
\|\text{grad}f(M)\| &= \sqrt{\left(\frac{4}{5}\right)^2 + \left(\frac{2}{5}\right)^2} = \sqrt{\frac{20}{25}} = \frac{2}{\sqrt{5}}
\end{aligned}$$

Задача 4

$$\frac{\partial f(M)}{\partial \nu} = (\text{grad}f, \nu)$$

$$\begin{aligned}
f(x, y) &= x^2 + 11y^2 - 3 \\
\text{grad}f(M) &= (2 \cdot 2, 22 \cdot 1) = (4, 22) \\
\frac{\partial f(M)}{\partial \nu} &= 4 \cdot \frac{\sqrt{3}}{2} + 22 \cdot \frac{1}{2} = 2\sqrt{3} + 11
\end{aligned}$$

$$\begin{aligned}
f(x, y) &= x^2 - y^2 \\
\text{grad}f(M) &= (2 \cdot 2, -2 \cdot 1) = (4, -2) \\
\frac{\partial f(M)}{\partial \nu} &= 4 \cdot \frac{\sqrt{3}}{2} - 2 \cdot \frac{1}{2} = 2\sqrt{3} - 1
\end{aligned}$$

$$\begin{aligned}
 f(x, y) &= \ln(x^2 + y^2) \\
 \operatorname{grad} f(M) &= \left(\frac{2 \cdot 2}{2^2 + 1^2}, \frac{2 \cdot 1}{2^2 + 1^2} \right) = \left(\frac{4}{5}, \frac{2}{5} \right) \\
 \frac{\partial f(M)}{\partial \nu} &= \frac{4}{5} \cdot \frac{\sqrt{3}}{2} + \frac{2}{5} \cdot \frac{1}{2} = \frac{4\sqrt{3}}{10} + \frac{1}{5} = \frac{4\sqrt{3} + 2}{10}
 \end{aligned}$$

Задача 5

$$\begin{aligned}
 u'_x &= 2x & u'_y &= 2y & u'_z &= 2z \\
 \operatorname{gradu}(A) &= (2\varepsilon, 0, 0) & \operatorname{gradu}(B) &= (0, 2\varepsilon, 0) \\
 (\operatorname{gradu}(A), \operatorname{gradu}(B)) &= 2\varepsilon \cdot 0 + 0 \cdot 2\varepsilon + 0 \cdot 0 = 0 \\
 (\operatorname{gradu}(A), \operatorname{gradu}(B)) &= \|u(A)\| \cdot \|u(B)\| \cdot \cos \alpha \\
 \cos \alpha = 0 &\Leftrightarrow \alpha = \frac{\pi}{2}
 \end{aligned}$$

Задача 6

$$F(x, y) = x^2 - 2xy + 5y^2 + 4y = 2x + 9$$

$$F'_y = -2x + 10y + 4 \neq 0$$

$$F'_x(x, y) = 2x - 2y - 2$$

$$F'_y(0, 1) = -2 \cdot 0 + 10 \cdot 1 + 4 \neq 0$$

$$y'(x) = -\frac{F'_x(x, y)}{F'_y(x, y)} = -\frac{2x - 2y - 2}{-2x + 10y + 4} = -\frac{x - y - 1}{-x + 5y + 2}$$

$$y'(0) = -\frac{0 - 1 - 1}{-0 + 5 \cdot 1 + 2} = -\frac{-2}{7} = \frac{2}{7}$$

$$y''(x) = -\frac{F''_{xx}(x, y) + 2F''_{xy}y' + F''_{yy}(x, y)y'^2}{F'_y(x, y)}$$

$$F''_{xx} = 2, \quad F''_{yy} = 10, \quad F''_{xy} = -2$$

$$F''_{xx}(0, 1) = 2, \quad F''_{yy}(0, 1) = 10, \quad F''_{xy}(0, 1) = -2$$

$$y''(x) = -\frac{2 + 2 \cdot (-2)y' + 10y'^2}{-2x + 10y + 4}$$

$$y''(x) = -\frac{2 + -4y' + 10y'^2}{-2x + 10y + 4}$$

$$y''(0) = -\frac{2 + -4 \cdot \frac{2}{7} + 10 \cdot \left(\frac{2}{7}\right)^2}{-2 \cdot 0 + 10 \cdot 1 + 4}$$

$$y''(0) = -\frac{2 + -\frac{8}{7} + \frac{40}{49}}{14}$$

$$y''(0) = -\frac{98 - 56 + 40}{49 \cdot 14} = -\frac{82}{49 \cdot 14} = \frac{82}{49} \cdot \frac{1}{14} = \frac{41}{343}$$

7 Упражнение към лекция 7

7.1 Задачи

Задача 1

Да се намерят локалните екстремуми на функциите

- $z = \sin x + \sin y + \sin(x + y) \quad (0 < x < \frac{\pi}{2}, 0 < y < \frac{\pi}{2})$
- $z = x^4 + y^4 - 4xy$

Задача 2

Да се намерят локалните екстремуми на функциите

- $u = x^2 + y^2 + z^2 + 2x + 4y - 6z$
- $u = x^3 + y^2 + z^2 - 3x + 6y - 2z$
- $u = x^3 + y^2 + z^2 - 3x - 2y$

Задача 3

Да се намерят $y'(0), y''(0)$ ако $y(0) = 2$ на неявната функция $y = f(x)$ дефинирана от уравнението

$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$

Задача 4

Да се покаже, че функцията $z = f(x, y)$ дефинирана неявно от уравнението

$$z = x\varphi\left(\frac{z}{y}\right)$$

φ - непрекъснато диференцируема, удовлетворява тъждествено уравнението

$$xz'_x + yz'_y = z$$

7.2 Решения

Задача 1

$$z = \sin x + \sin y + \sin(x + y) \quad (0 < x < \frac{\pi}{2}, 0 < y < \frac{\pi}{2})$$

$$z'_x = \cos x + \cos(x + y) \quad z'_y = \cos y + \cos(x + y)$$

$$\begin{cases} \cos x + \cos(x + y) = 0 \\ \cos y + \cos(x + y) = 0 \end{cases} \Leftrightarrow \begin{cases} 2 \cos \frac{2x+y}{2} \cos \frac{y}{2} = 0 \\ 2 \cos \frac{x+2y}{2} \cos \frac{x}{2} = 0 \end{cases} \Leftrightarrow \begin{cases} \frac{2x+y}{2} = \frac{\pi}{2} \\ \frac{x+2y}{2} = \frac{\pi}{2} \end{cases}$$

$$\frac{y}{2} = \frac{\pi}{2}, \quad \frac{x}{2} = \frac{\pi}{2} \implies x = y = \pi \notin (0 < x, y < \frac{\pi}{2})$$

$$x_0 = y_0 = \frac{\pi}{3} \implies M_0 \left(\frac{\pi}{3}, \frac{\pi}{3} \right)$$

$$z''_{xx} = -\sin x - \sin(x + y) \quad z''_{yy} = -\sin y - \sin(x + y) \quad z''_{xy} = -\sin(x + y)$$

$$z''_{xx}(M_0) = -\frac{2\sqrt{3}}{2} = -\sqrt{3} = \Delta_1 \quad z''_{yy}(M_0) = -\frac{2\sqrt{3}}{2} = -\sqrt{3} \quad z''_{xy}(M_0) = -\frac{\sqrt{3}}{2}$$

$$\begin{pmatrix} z''_{xx}(M_0) & z''_{xy}(M_0) \\ z''_{yx}(M_0) & z''_{yy}(M_0) \end{pmatrix} = \begin{pmatrix} -\sqrt{3} & -\frac{2\sqrt{3}}{2} \\ -\frac{2\sqrt{3}}{2} & -\sqrt{3} \end{pmatrix}$$

$$\Delta_1 = -\sqrt{3} < 0 \quad \Delta_2 = 3 - \frac{3}{4} > 0$$

$$\implies \exists \text{ локален максимум, } z_{max} = z(M_0) - \frac{3\sqrt{3}}{2}$$

$$z = x^4 + y^4 - 4xy$$

$$z'_x = 4x^3 - 4y \quad z'_y = 4y^3 - 4x$$

$$\begin{cases} 4x^3 - 4y = 0 \\ 4y^3 - 4x = 0 \end{cases} \Leftrightarrow \begin{cases} y = x^3 \\ x^9 - x = 0 \end{cases} \Leftrightarrow \begin{cases} x(x^2 - 1)(x^2 + 1)(x^4 + 1) = 0 \\ y = x^3 \end{cases} \implies$$

$$M_0(0, 0) \quad M_1(1, 1) \quad M_2(-1, 1)$$

$$z''_{xx} = 12x^2 \quad z''_{yy} = 12y^2 \quad z''_{xy} = -4$$

$$d^2z = \begin{pmatrix} z''_{xx}(M_0) & z''_{xy}(M_0) \\ z''_{yx}(M_0) & z''_{yy}(M_0) \end{pmatrix} = \begin{pmatrix} 12x^2 & -4 \\ -4 & 12y^2 \end{pmatrix}$$

$$d^2z(M_0) = \begin{pmatrix} 0 & -4 \\ -4 & 0 \end{pmatrix} \Rightarrow \Delta = \begin{vmatrix} 0 & -4 \\ -4 & 0 \end{vmatrix} = -16 < 0 \Rightarrow \text{няма лок. екстремум в } M_0$$

$$d^2z(M_1) = \begin{pmatrix} 12 & -4 \\ -4 & 12 \end{pmatrix} \Rightarrow \Delta_1 = 12 > 0 \Delta_2 = \begin{vmatrix} 0 & -4 \\ -4 & 0 \end{vmatrix} 144 - 16 > 0 \Rightarrow$$

z има локален минимум

$$z_{min} = z(M_1) = 1^4 + 1^4 - 4 \cdot 1 \cdot 1 = -2$$

Аналогично и за M_2 има лок. мин $z_{min} = -2$

Задача 2

$$u = x^2 + y^2 + z^2 + 2x + 4y - 6z$$

$$u'_x = 2x + 2 \quad u'_y = 2y + 4 \quad u'_z = 2z - 6$$

$$\begin{cases} x + 1 = 0 \\ y + 2 = 0 \\ z - 3 = 0 \end{cases} \Rightarrow M_0(-1, -2, 3)$$

$$u''_{xx} = 2 \quad u''_{yy} = 2 \quad u''_{zz} = 2$$

$$u''_{xy} = u''_{xz} = u''_{yx} = u''_{yz} = u''_{zx} = u''_{zy} = 0$$

$$d^2u(M_0) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$\Delta_1 = u''_{xx} = 2 > 0 \quad \Delta_2 = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4 > 0 \quad \Delta_3 = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 8 > 0 \Rightarrow$$

d^2u е положително дефинитна квадратична форма

$$u \text{ има лок. минимум } u_{min} = u(M_0) = 1 + 4 + 9 - 2 - 4 \cdot 2 - 18 = -14$$

$$u = x^3 + y^2 + z^2 - 3x + 6y - 2z$$

$$u'_x = 3x^2 + 2 \quad u'_y = 2y + 6 \quad u'_z = 2z - 2$$

$$\begin{cases} 3x^2 + 2 = 0 \\ 2y + 6 = 0 \\ 2z - 2 = 0 \end{cases} \implies M_0(1, -3, 1) \quad M_1(-1, -3, 1)$$

$$u''_{xx} = 6x \quad u''_{yy} = 2 \quad u''_{zz} = 2$$

$$u''_{xy} = u''_{xz} = u''_{yx} = u''_{yz} = u''_{zx} = u''_{zy} = 0$$

$$d^2u(M_0) = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \implies$$

$$\Delta_1 = 6 > 0 \quad \Delta_2 = \begin{vmatrix} 6 & 0 \\ 0 & 2 \end{vmatrix} = 12 > 0 \quad \Delta_3 = \begin{vmatrix} 6 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 24 > 0 \implies$$

d^2u е положително дефинитна квадратична форма

и има лок. минимум $u_{min} = u(M_0) = 1 + 9 + 1 - 3 - 18 - 2 = -12$

$$d^2u(M_1) = \begin{pmatrix} -6 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \implies$$

$$\Delta_1 = -6 < 0 \quad \Delta_2 = \begin{vmatrix} -6 & 0 \\ 0 & 2 \end{vmatrix} = -12 < 0 \quad \Delta_3 = \begin{vmatrix} -6 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix} = -24 < 0 \implies$$

d^2u е не е дефинитна квадратична форма \implies няма лок. екстремуми

$$u = x^3 + y^2 + z^2 - 3x - 2y$$

$$u'_x = 3x^2 - 3 \quad u'_y = 2y - 2 \quad u'_z = 2z$$

$$\begin{cases} 3x^2 - 3 = 0 \\ 2y - 2 = 0 \\ 2z = 0 \end{cases} \implies M_0(1, 1, 0) \quad M_1(-1, 0, 0)$$

$$u''_{xx} = 6x \quad u''_{yy} = 2 \quad u''_{zz} = 2$$

$$u''_{xy} = u''_{xz} = u''_{yx} = u''_{yz} = u''_{zx} = u''_{zy} = 0$$

$$d^2u(M_0) = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \implies$$

$$\Delta_1 = 6 > 0 \quad \Delta_2 = \begin{vmatrix} 6 & 0 \\ 0 & 2 \end{vmatrix} = 12 > 0 \quad \Delta_3 = \begin{vmatrix} 6 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 24 > 0 \implies$$

d^2u е положително дефинитна квадратична форма

и има лок. минимум $u_{min} = u(M_0) = 1 + 1 - 3 - 2 = -3$

$$d^2u(M_1) = \begin{pmatrix} -6 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \implies$$

$$\Delta_1 = -6 < 0 \quad \Delta_2 = \begin{vmatrix} -6 & 0 \\ 0 & 2 \end{vmatrix} = -12 < 0 \quad \Delta_3 = \begin{vmatrix} -6 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix} = -24 < 0 \implies$$

d^2u е не е дефинитна квадратична форма \implies няма лок. екстремуми

Задача 3

$$F(x, y) = \frac{x^2}{9} + \frac{y^2}{4} - 1 \quad M_0(0, 2)$$

$$F'_y = \frac{2y}{4} = \frac{y}{2} \neq 0 \quad y'(x) = -\frac{F'_x}{F'_y} \quad y''(x) = -\frac{F'''_{xx} + 2F''_{xy}y' + F''_{yy}(y')^2}{F'_y}$$

$$F'_x = \frac{2}{9}x \quad F'_y = \frac{2y}{4} \quad F''_{xx} = \frac{2}{9} \quad F''_{xy} = F''_{yx} = 0 \quad F''_{yy} = \frac{1}{2}$$

$$F'_x(0, 2) = 0 \quad F'_y(0, 2) = 1 \quad F''_{xx}(0, 2) = \frac{2}{9} \quad F''_{xy}(0, 2) = F''_{yx}(0, 2) = 0 \quad F''_{yy}(0, 2) = \frac{1}{2}$$

$$y'(0) = -\frac{0}{1} = 0 \quad y''(0) = -\frac{\frac{2}{9} + 2 \cdot 0 \cdot 0 + \frac{1}{2} \cdot 0^2}{1} = -\frac{2}{9}$$

Задача 4

Решение. Означаваме

$$F(x, y, z) = z - x \varphi(z/y), \quad (3)$$

откъдето извеждаме условието за $\exists!$ невяна функция:

$$F'_z = 1 - x \varphi'(z/y) \cdot \frac{1}{y} \neq 0. \quad (4)$$

по-нататък:

$$F'_x = -\varphi(z/y); \quad F'_y = -x \varphi'(z/y) \cdot \left(-\frac{z}{y^2}\right), \text{ т.е.}$$

$$F'_x = -\varphi(z/y); \quad F'_y = \frac{xz}{y^2} \varphi'(z/y) \Rightarrow$$

$$z'_x = -\frac{F'_x}{F'_z} = -\frac{-\varphi(z/y)}{1 - x \varphi'(z/y) \cdot \frac{1}{y}} = \frac{\varphi(z/y)}{1 - \frac{x}{y} \varphi'(z/y)}$$

$$z'_y = -\frac{F'_y}{F'_z} = -\frac{\frac{xz}{y^2} \varphi'(z/y)}{1 - \frac{x}{y} \varphi'(z/y)} \Rightarrow$$

$$xz'_x + yz'_y = \frac{\overset{z}{x} \cdot \varphi(z/y)}{1 - \frac{x}{y} \varphi'(z/y)} - \frac{\cancel{\frac{xz}{y}} \varphi'(z/y)}{1 - \frac{x}{y} \varphi'(z/y)} =$$

$$= z \cdot \frac{1 - \cancel{\frac{x}{y}} \varphi'(z/y)}{1 - \cancel{\frac{x}{y}} \varphi'(z/y)} = z.$$

8 Упражнение към лекция 8

8.1 Задачи

Задача 1

Да се изследва за локален екстремум следната функция.

$$z = 1 - \sqrt{x^2 - y^2}$$

Задача 2

Намерете точките на условен екстремум и екстремумите на следните функции.

- $z = x^2 + y^2$, ако $x + y = 1$
- $u = x^2 + y^2 - 12x + 16y$, ако $x^2 + y^2 = 25$
- $u = x + y + z$, ако $z = 1$ и $x^2 + y^2 = 1$

Задача 3

Да се изследва функцията $u = xy + yz$ за условен екстремум, при ограничения.

$$\begin{aligned} x^2 + y^2 &= 2 \\ y + z &= 2 \end{aligned}$$

Задача 4

Да се изследва функцията $z = x + y$ за условен екстремум, при ограничения.

$$xy = 1$$

Задача 5

Да се намери дефиниционното множество на функциите.

- $z = \sqrt{1 - x^2 - y^2 + 2x}$
- $z = \frac{x^2 y}{2x + y}$
- $z = \arcsin(x + y)$
- $w = \frac{1}{\sqrt{xy}}$

Задача 6

Да се намерят границите ако съществуват.

- $\lim_{(x,y) \rightarrow (0,0)} \frac{\tan(xy)}{xy}$
- $\lim_{(x,y) \rightarrow (0,0)} \frac{y}{\sin(xy)}$
- $\lim_{(x,y) \rightarrow (0,0)} \frac{1 - \sqrt{1 - xy}}{xy}$

Задача 7

Да се провери дали уравнението удовлетворява посочената функция.

- $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}, z(x, y) = \ln(x^2 + y^2 + 1)$
- $\frac{x}{y} \frac{\partial z}{\partial x} + \frac{1}{\ln x} \frac{\partial z}{\partial y} = 2z, z(x, y) = x^y$
- $2 \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} = 0, z(x, y) = 2 \cos^2(y - \frac{x}{2})$
- $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 1, u(x, y, z) = x + \frac{x-y}{y-z}$

Задача 8

Да се изследват за локален екстремум следните функции.

- $z = x^4 + y^4 - x^2 - 2xy - y^2$
- $z = xy(1 - x - y)$
- $z = x^3 - y^3 - 3x + 3y + 2$
- $u = x^3 + y^2 + z^2 + 12xy + 2z$

Задача 9

Да се изследват за локален екстремум следните неявно зададени функции.

- $x^3 + y^3 = 3xy, y = y(x)$
- $y^2 - 3y - \sin(x) = 0, y = y(x)$
- $x^2 + y^2 + z^2 - xz - yz + 2x + 2y + 2z - 2 = 0, z = z(x, y)$
- $2x^2 + 2y^2 + z^2 + 8xz - 8yz + 8 = 0, z = z(x, y)$

Задача 10

Да се изследва за условен екстремум

- $z = xy$, ако $2x + y = 1$
- $z = x^2 + y^2$, ако $x - y = 1$
- $u = x^2 + y^2 + z^2$, ако $\frac{x^2}{16} + \frac{y^2}{9} + \frac{z^2}{4} = 1$
- $u = xyz$, ако $x + y + z = 5, xy + yz + zx = 8$

Задача 11

Намерете точките на условен екстремум и екстремумите на следните функции.

- $u = x^2 + y^2 + z^2 + 2x + 4y - 6z$, ако $x^2 + y^2 + z^2 = 14$
- $u = x^2 + y^2 + z^2 + 2x + 4y$, ако $x^2 + y^2 = 20$
- $u = x^2 + y^2 + z^2 + 6x - 2y + 4z$, ако $x^2 + y^2 + z^2 = 56$

Задача 12

Намерете абсолютните екстремуми на следните функции и определете вида им (условен, локален, минимум, максимум)

- $u = x^2 + y^2 - 12x + 16y$, ако $x^2 + y^2 \leq 25, x^2 + y^2 \leq 400, x^2 + y^2 \leq 100$
- $u = x^2 + y^2 + z^2 + 2x + 4y - 6$, ако $x^2 + y^2 + z^2 \leq 9$
- $u = x^2 + 2y^2 + 3z^2$, ако $x^2 + y^2 + z^2 \leq 100$

8.2 Решения

Задача 1

$$z(\Delta x, \Delta y) - z(0, 0) = 1 - \sqrt{\Delta x^2 - \Delta y^2} - 1 = -\sqrt{\Delta x^2 - \Delta y^2} < 0$$

Имаме строг локален максимум в $z(0, 0) = 1$

Задача 2

$$z = x^2 + y^2, \quad x + y = 1$$

$$F(x, y, \lambda) = x^2 + y^2 + \lambda(x + y - 1), \quad \lambda \neq 0$$

$$F'_x = 2x + \lambda \quad F'_y = 2y + \lambda$$

$$\begin{cases} 2x + \lambda = 0 \\ 2y + \lambda = 0 \\ x + y = 1 \end{cases} \Leftrightarrow \begin{cases} x = -\frac{\lambda}{2} \\ y = -\frac{\lambda}{2} \\ -\frac{\lambda}{2} - \frac{\lambda}{2} = 1 \end{cases} \Leftrightarrow \begin{cases} x = \frac{1}{2} \\ y = \frac{1}{2} \\ \lambda = -1 \end{cases} \Rightarrow M_0\left(\frac{1}{2}, \frac{1}{2}, -1\right)$$

$$F''_{xx} = 2 \quad F''_{yy} = 2 \quad F''_{xy} = F''_{yx} = 0$$

$$\Delta = F''_{xx}F''_{yy} - F''_{xy}F''_{yx} = 4 \Big|_{M_0} = 4 > 0$$

$$F''_{xx} \Big|_{M_0} = 2 > 0 \Rightarrow z_{min} = z\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

$$u = x^2 + y^2 - 12x + 16y, \quad x^2 + y^2 = 25$$

$$F(x, y, \lambda) = x^2 + y^2 - 12x - 16y + \lambda(x^2 + y^2 - 25) \quad \lambda \neq 0$$

$$F'_x = 2x + 16 + 2\lambda x \quad F'_y = 2y - 12 + 2\lambda y$$

$$\begin{cases} 2x + 16 + 2\lambda x = 0 \\ 2y - 12 + 2\lambda y = 0 \\ x^2 + y^2 = 25 \end{cases} \Leftrightarrow \begin{cases} x = \frac{6}{1+\lambda} \\ y = -\frac{8}{1+\lambda} \\ \frac{36}{(1+\lambda)^2} + \frac{64}{(1+\lambda)^2} = 25 \end{cases} \Leftrightarrow \begin{cases} x = -3 \wedge x = 3 \\ y = 4 \wedge y = -4 \\ \lambda = -3 \wedge \lambda = 1 \end{cases}$$

$$\Rightarrow M_1(-3, 4, -3) \wedge M_2(3, -4, 1)$$

$$F''_{xx} = 2 + 2\lambda \quad F''_{yy} = 2 + 2\lambda \quad F''_{xy} = F''_{yx} = 0$$

$$\Delta = F''_{xx}F''_{yy} - F''_{xy}F''_{yx} = 4(1 + \lambda)^2$$

$$\begin{aligned}
\Delta|_{M_1} &= 4(1 + (-3))^2 = 16 > 0 \\
F''_{xx}|_{M_1} &= 2 + 2 \cdot -3 = -4 < 0 \implies \\
u_{max} &= u(-3, 4) = (-3)^2 + 4^2 - 12(-3) + 16(4) = 125 \\
\Delta|_{M_2} &= 4(1 + 1)^2 = 16 \\
F''_{xx}|_{M_2} &= 2 + 2 \cdot 1 = 4 > 0 \implies \\
u_{min} &= u(3, -4) = 3^2 + (-4)^2 - 12 \cdot 3 + 16(-4) = -75
\end{aligned}$$

$$\begin{aligned}
u &= x + y + z, \quad z = 1, \quad x^2 + y^2 = 1 \\
F(x, y, z, \lambda, \nu) &= x + y + z + \lambda(z - 1) + \nu(x^2 + y^2 - 1), \quad \lambda, \nu \neq 0 \\
F'_x &= 1 + 2\nu x \quad F'_y = 1 + 2\nu y \quad F'_z = 1 + \lambda \\
\begin{cases} 1 + 2\nu x = 0 \\ 1 + 2\nu y = 0 \\ 1 + \lambda = 0 \\ z = 1 \\ x^2 + y^2 = 1 \end{cases} &\Leftrightarrow \begin{cases} x = -\frac{1}{2\nu} \\ y = -\frac{1}{2\nu} \\ z = 1 \\ \lambda = -1 \\ \frac{1}{4\nu^2} + \frac{1}{4\nu^2} = 1 \end{cases} \Leftrightarrow \begin{cases} x = -\frac{1}{\sqrt{2}} \wedge \frac{1}{\sqrt{2}} \\ y = -\frac{1}{\sqrt{2}} \wedge \frac{1}{\sqrt{2}} \\ z = 1 \\ \lambda = -1 \\ \nu = \frac{1}{\sqrt{2}} \wedge -\frac{1}{\sqrt{2}} \end{cases} \implies \\
M_1 \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 1, -1, \frac{1}{\sqrt{2}} \right) &\wedge M_2 \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 1, -1, -\frac{1}{\sqrt{2}} \right) \\
F''_{xx} &= 2\nu \quad F''_{yy} = 2\nu \quad F''_{zz} = F''_{xy} = F''_{xz} = F''_{yz} = F''_{zx} = F''_{zy} = 0 \\
d^2F &= \begin{pmatrix} 2\nu & 0 & 0 \\ 0 & 2\nu & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
\Delta_1 = F''_{xx} = 2\mu|_{M_1} &= \frac{2}{\sqrt{2}} > 0 \quad \Delta_2 = 4\nu^2|_{M_1} = 2 > 0 \quad \Delta_3 = 0|_{M_1} = 0 \\
\Delta_1 = F''_{xx} = 2\mu|_{M_2} &= -\frac{2}{\sqrt{2}} < 0 \quad \Delta_2 = 4\nu^2|_{M_2} = 2 > 0 \quad \Delta_3 = 0|_{M_2} = 0 \\
d^2F &= 2\nu dx^2 + 2\nu dy^2 \\
x^2 + y^2 = 1 &\implies x dx + y dy = 0 \quad (x = y \in M_1, M_2) \implies dx + dy = 0 \implies dy = -dx \\
d^2F &= 2\nu dx^2 + 2\nu (-dx)^2 = 4\nu dx^2 \\
d^2F|_{M_1} &> 0 \implies u_{min} = u \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right) = -\sqrt{2} + 1 \\
d^2F|_{M_2} &< 0 \implies u_{max} = u \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) = \sqrt{2} + 1
\end{aligned}$$

Задача 3

$$u = xy + yz \quad x^2 + y^2 = 2, \quad y + z = 2$$

$$F(x, y, z, \lambda, \nu) = xy + yz + \lambda(x^2 + y^2 - 2) + \nu(y + z - 2), \quad \lambda, \nu \neq 0$$

$$F'_x = 2\lambda x + y \quad F'_y = 2\lambda y + \nu + x + z \quad F'_z = y + \nu$$

$$\left\{ \begin{array}{l} 2\lambda x + y = 0 \\ 2\lambda y + \nu + x + z = 0 \\ y + \nu = 0 \\ x^2 + y^2 = 2 \\ y + z = 2 \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} x = 1 \wedge x = -1 \\ y = 1 \\ z = 1 \\ \lambda = -\frac{1}{2} \wedge \lambda = \frac{1}{2} \\ \nu = -1 \end{array} \right. \Rightarrow M\left(1, 1, 1, -\frac{1}{2}, -1\right) \wedge N\left(-1, 1, 1, \frac{1}{2}, -1\right)$$

$$F''_{xx} = 2\lambda \quad F''_{yy} = 2\lambda \quad F''_{zz} = 0$$

$$F''_{xy} = F''_{yx} = F''_{yz} = F''_{zy} = 1$$

$$F''_{xz} = F''_{zx} = 0$$

$$d^2F = \begin{pmatrix} 2\lambda & 1 & 0 \\ 1 & 2\lambda & 1 \\ 0 & 1 & 0 \end{pmatrix} \Rightarrow d^2F = 2\lambda dx^2 + 2\lambda dy^2 + 2dx dy + 2dy dz$$

$$2x dx + 2y dy = 0 \quad dy + dz = 0$$

$$d^2L(M) = -dx^2 - dy^2 + 2dx dy + 2dy dz \quad (dx + dy = 0, dy + dz = 0, dx + dz = -2dy)$$

$$d^2L(M) = -dx^2 - dy^2 + 2(dx + dz)dy = -dx^2 - dy^2 - 2dy^2 < 0 \Rightarrow$$

$$u_{max} = u(1, 1, 1) = 1 \cdot 1 + 1 \cdot 1 = 2$$

$$d^2(N) = dx^2 + dy^2 + 2(dx + dz)dy \quad (-dx + dy = 0, dy + dz = 0, dx + dz = 0)$$

$$d^2(N) = dx^2 dy^2 - 2(dx + dz)dy = dx^2 + dy^2 > 0 \Rightarrow$$

$$u_{min} = u(-1, 1, 1) = -1 \cdot 1 + 1 \cdot 1 = 0$$

Задача 4

$$z = x + y \quad xy = 1$$

$$F = x + y + \lambda(xy - 1) \Rightarrow F'_x = 1 + \lambda y \quad F'_y = 1 + \lambda x$$

$$\left\{ \begin{array}{l} 1 + \lambda y = 0 \\ 1 + \lambda x = 0 \\ xy = 1 \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} x = 1 \wedge x = -1 \\ y = 1 \wedge y = -1 \\ \lambda = -1 \wedge \lambda = 1 \end{array} \right. \Rightarrow M(1, 1, -1) \wedge N(-1, -1, 1)$$

$$F''_{xx} = F''_{yy} = 0, \quad F''_{xy} = F''_{yx} = \lambda \Rightarrow d^2F = 2\lambda dx dy$$

$$x dx + y dy = 0 \Rightarrow dx + dy = 0 (x, y \in M, N) \Rightarrow d^2F = -2\lambda dx^2$$

$$d^2F(M) = 2dx^2 > 0 \Rightarrow u_{min} = u(1, 1) = 1 + 1 = 2$$

$$d^2F(N) = -2dx^2 < 0 \Rightarrow u_{max} = u(-1, -1) = -1 + (-1) = -2$$

Задача 5

- $z = \sqrt{1 - x^2 - y^2 + 2x}$

$$D : (x, y \in \mathbb{R} : x^2 + y^2 \leq 2x + 1)$$

- $z = \frac{x^2 y}{2x + y}$

$$D : (x, y \in \mathbb{R} : 2x + y \neq 0)$$

- $z = \arcsin(x + y)$

$$D : (x, y \in \mathbb{R} : -1 \leq x + y \leq 1)$$

- $w = \frac{1}{\sqrt{xy}}$

$$D : (x, y \in \mathbb{R} \setminus 0 : xy > 0)$$

Задача 6

- $\lim_{(x,y) \rightarrow (0,0)} \frac{\tan(xy)}{xy}$

При заместване се получават недефинирани форми от вида $\left[\frac{0}{0} \right]$

$$\begin{aligned} \lim_{(x,y) \rightarrow (0,0)} \frac{\tan(xy)}{xy} &= \lim_{(x,y) \rightarrow (0,0)} \frac{\sin(xy)}{xy \cos(xy)} = \lim_{(x,y) \rightarrow (0,0)} \frac{\sin(xy)}{xy} = \lim_{(x,y) \rightarrow (0,0)} \frac{y \cos(xy)}{y} = \lim_{(x,y) \rightarrow (0,0)} \cos(xy) = 1 \end{aligned}$$

- $\lim_{(x,y) \rightarrow (0,0)} \frac{y}{\sin(xy)}$

$$\lim_{y \rightarrow 0} \frac{y}{\sin(xy)} = \left[\frac{0}{0} \right] \implies \lim_{y \rightarrow 0} \frac{1}{x \cos(xy)} = \frac{1}{x}$$

$$\lim_{x \rightarrow 0} \frac{1}{x} = \frac{1}{0} = \infty$$

$$\lim_{x \rightarrow 0} \frac{y}{\sin xy} = \left[\frac{0}{0} \right] \implies \lim_{x \rightarrow 0} \frac{0}{y} = 0$$

$$\lim_{y \rightarrow 0} 0 = 0$$

$$0 \neq \infty \implies \text{Няма граница.}$$

$$\bullet \lim_{(x,y) \rightarrow (0,0)} \frac{1 - \sqrt{1 - xy}}{xy}$$

$$\lim_{y \rightarrow 0} \frac{1 - \sqrt{1 - xy}}{xy} = \left[\frac{0}{0} \right] \Rightarrow \lim_{y \rightarrow 0} \frac{x}{2\sqrt{1 - xy}} \cdot \frac{1}{x} = \lim_{y \rightarrow 0} \frac{1}{2\sqrt{1 - xy}} = \frac{1}{2}$$

$$\lim_{x \rightarrow 0} \frac{1}{2} = \frac{1}{2}$$

$$\lim_{x \rightarrow 0} \frac{1 - \sqrt{1 - xy}}{xy} = \left[\frac{0}{0} \right] \Rightarrow \lim_{x \rightarrow 0} \frac{y}{2\sqrt{1 - xy}} \cdot \frac{1}{y} = \lim_{x \rightarrow 0} \frac{1}{2\sqrt{1 - xy}} = \frac{1}{2}$$

$$\lim_{y \rightarrow 0} \frac{1}{2} = \frac{1}{2}$$

Задача 7

$$\bullet \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}, z(x, y) = \ln(x^2 + y^2 + 1)$$

$$z'_x = \frac{2x}{x^2 + y^2 + 1} \quad z'_y = \frac{2y}{x^2 + y^2 + 1}$$

$$z''_{xy} = -\frac{4xy}{(x^2 + y^2 + 1)^2} \quad z''_{yx} = -\frac{4xy}{(x^2 + y^2 + 1)^2} \Rightarrow$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x} \text{ е вярно}$$

$$\bullet \frac{x}{y} \frac{\partial z}{\partial x} + \frac{1}{\ln x} \frac{\partial z}{\partial y} = 2z, z(x, y) = x^y$$

$$z'_x = \frac{x^y y}{x} \quad z'_y = x^y \ln(x)$$

$$A = \frac{x}{y} \cdot z'_x = \frac{x}{y} \cdot \frac{x^y y}{x} = x^y$$

$$B = \frac{1}{\ln(x)} \cdot z'_y = \frac{1}{\ln(x)} \cdot x^y \ln(x) = x^y$$

$$A + B = 2x^y \quad 2z = 2x^y \Rightarrow$$

$$\frac{x}{y} \frac{\partial z}{\partial x} + \frac{1}{\ln x} \frac{\partial z}{\partial y} = 2z \text{ е вярно}$$

- $2\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} = 0, z(x, y) = 2 \cos^2(y - \frac{x}{2})$

$$z'_x = -2 \cos\left(-y + \frac{x}{2}\right) \sin\left(-y + \frac{x}{2}\right)$$

$$z''_{xx} = -2 \cos^2\left(-y + \frac{x}{2}\right) + 1 \quad z''_{xy} = 4 \cos^2\left(-y + \frac{x}{2}\right) - 2$$

$$2z''_{xx} = -4 \cos^2\left(-y + \frac{x}{2}\right) + 2$$

$$2z''_{xx} + z''_{xy} = -4 \cos^2\left(-y + \frac{x}{2}\right) + 2 + 4 \cos^2\left(-y + \frac{x}{2}\right) - 2 = 0 \implies$$

$$2\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} = 0 \text{ е вярно}$$

- $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 1, u(x, y, z) = x + \frac{x-y}{y-z}$

$$u'_x = \frac{y-z+1}{y-z} \quad u'_y = \frac{z-x}{(y-x)^2} \quad u'_z = \frac{x-y}{(y-z)^2}$$

$$u'_x + u'_y + u'_z = \frac{y-z+1}{y-z} + \frac{z-x}{(y-x)^2} + \frac{x-y}{(y-z)^2} =$$

$$\frac{(y-z+1)(y-z) + z-x + x-y}{(y-z)^2} = \frac{y^2 - yz - yz + z^2 + y - z + z - y}{(y-z)^2} =$$

$$= \frac{y^2 - 2yz + z^2}{(y-z)^2} = 1 \implies$$

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 1 \text{ е вярно}$$

Задача 8

- $z = x^4 + y^4 - x^2 - 2xy - y^2$

$$z'_x = 4x^3 + 2x - 2y \quad z'_y = 4y^3 - 2y - 2x$$

$$\begin{cases} 4x^3 + 2x - 2y = 0 \\ 4y^3 - 2y - 2x = 0 \end{cases} \Leftrightarrow \begin{cases} x = 0 \wedge x = 1 \wedge x = -1 \\ y = 0 \wedge y = 1 \wedge y = -1 \end{cases} \implies L(0, 0), M(1, 1), N(-1, -1)$$

$$z''_{xx} = 12x^2 + 2 \quad z''_{yy} = 12y^2 - 2 \quad z'_{xy} = z'_{yx} = -2$$

$$\Delta = z''_{xx}z''_{yy} - z''_{xy}z''_{yx} = (12x^2 + 2)(12y^2 - 2) - 4$$

$$z''_{xx}(L) = -2 < 0 \quad \Delta(L) = 0 \implies \text{Няма екстремум.}$$

$$z''_{xx}(M) = 10 > 0 \quad \Delta(M) = 96 > 0 \implies$$

$$z'_{min1} = z(1, 1) = -2$$

$$z''_{xx}(N) = 10 > 0 \quad \Delta(N) = 96 > 0 \implies$$

$$z'_{min2} = z(-1, -1) = -2$$

- $z = xy(1 - x - y)$

$$\begin{aligned}
z'_x &= y(1 - x - y) - xy & z'_y &= x(1 - x - y) - xy \\
\left| \begin{array}{l} y(1 - x - y) - xy = 0 \\ x(1 - x - y) - xy = 0 \end{array} \right. &\Leftrightarrow \left| \begin{array}{l} x = 0 \wedge x = 0 \wedge x = 1 \wedge x = \frac{1}{3} \\ y = 0 \wedge y = 1 \wedge y = 0 \wedge y = \frac{1}{3} \end{array} \right. \Rightarrow \\
&A(0, 0), B(0, 1), C(1, 0), D\left(\frac{1}{3}, \frac{1}{3}\right) \\
z''_{xx} - 2y &= & z''_{yy} &= -2y & z''_{xy} = z''_{yx} &= 1 - 2x - 2y \\
\Delta &= z''_{xx}z''_{yy} - z''_{xy}z''_{yx} = 4xy - (1 - 2x - 2y)^2 \\
z''_{xx}(A) &= 0 & \Delta(A) &= -1 \Rightarrow \text{Няма екстремум.} \\
z''_{xx}(B) &= -2 & \Delta(A) &= -1 \Rightarrow \text{Няма екстремум.} \\
z''_{xx}(C) &= 0 & \Delta(C) &= -1 \Rightarrow \text{Няма екстремум.} \\
z''_{xx}(D) &= -\frac{2}{3} & \Delta(D) &= \frac{1}{3} \Rightarrow z_{max} = z\left(\frac{1}{3}, \frac{1}{3}\right) = \frac{1}{27}
\end{aligned}$$

- $z = x^3 - y^3 - 3x + 3y + 2$

$$\begin{aligned}
z'_x &= 3x^2 - 3 & z'_y &= -3y^2 + 3 \\
\left| \begin{array}{l} 3x^2 - 3 = 0 \\ -3y^2 + 3 = 0 \end{array} \right. &\Leftrightarrow \left| \begin{array}{l} x = 1 \wedge x = -1 \\ x = 1 \wedge x = -1 \end{array} \right. \Rightarrow A(1, 1), B(1, -1), C(-1, 1), D(-1, -1) \\
z''_{xx} 6x &= & z''_{yy} &= -6y & z''_{xy} = z''_{yx} &= 0 \\
\Delta &= z''_{xx}z''_{yy} - z''_{xy}z''_{yx} = -36xy \\
z''_{xx}(A) &= 6 > 0 & \Delta(A) &= -36 < 0 \Rightarrow \text{Няма екстремум.} \\
z''_{xx}(B) &= 6 > 0 & \Delta(A) &= 36 > 0 \Rightarrow z_{min} = z(1, -1) = -2 \\
z''_{xx}(C) &= -6 < 0 & \Delta(A) &= 36 > 0 \Rightarrow z_{max} = z(-1, 1) = 6 \\
z''_{xx}(D) &= -6 < 0 & \Delta(A) &= -36 < 0 \Rightarrow \text{Няма екстремум.}
\end{aligned}$$

- $u = x^3 + y^2 + z^2 + 12xy + 2z$

$$\begin{aligned} u'_x &= 3x^2 - 12y & u'_y &= 3y^2 - 12x & u'_z &= 2z^2 + 2 \\ \left| \begin{array}{l} 3x^2 - 12y = 0 \\ 3y^2 - 12x = 0 \\ 2z^2 + 2 = 0 \end{array} \right| & \Leftrightarrow \left| \begin{array}{l} x = 0 \wedge x = 24 \\ y = 0 \wedge y = -144 \\ z = -1 \end{array} \right| & \Rightarrow & A(0, 0, -1), B(24, -144, -1) \end{aligned}$$

$$u''_{xx} = 6x \quad u''_{xy} = 12 \quad u''_{xz} = 0$$

$$u''_{yx} = 12 \quad u''_{yy} = 2 \quad u''_{yz} = 0$$

$$u''_{zx} = 0 \quad u''_{zy} = 0 \quad u''_{zz} = 2$$

$$\Delta_1 = u''_{xx} = 6x$$

$$\Delta_2 = \begin{vmatrix} u''_{xx} & u''_{xy} \\ u''_{yx} & u''_{yy} \end{vmatrix} = \begin{vmatrix} 6x & 12 \\ 12 & 2 \end{vmatrix} = 12x - 144$$

$$\Delta_3 = \begin{vmatrix} u''_{xx} & u''_{xy} & u''_{xz} \\ u''_{yx} & u''_{yy} & u''_{yz} \\ u''_{zx} & u''_{zy} & u''_{zz} \end{vmatrix} = \begin{vmatrix} 6x & 12 & 0 \\ 12 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 24x - 288$$

$$\Delta_1(A) = 0 \quad \Delta_2(A) = -144 \quad \Delta_3(A) = -288 \Rightarrow \text{Няма екстремум.}$$

$$\Delta_1(B) = 144 > 0 \quad \Delta_2(B) = 144 > 0 \quad \Delta_3(B) = 288 > 0 \Rightarrow$$

$$u_{min} = u(24, -144, -1) = -6913$$

Задача 9

- $x^3 + y^3 = 3xy, y = y(x)$

$$F = x^3 + y^3 - 3xy$$

$$F'_x = 3x^2 - 3y \quad F'_y = 3y^2 - 3x$$

$$y' = -\frac{f'_x}{f'_y} = -\frac{3x^2 - 3y}{3y^2 - 3x}$$

$$\left| \begin{array}{l} -\frac{3x^2-3y}{3y^2-3x} = 0 \\ x^3 + y^3 = 3xy \\ 3y^2 - 3x \neq 0 \end{array} \right| \Leftrightarrow \left| \begin{array}{l} x = \sqrt[3]{2} \\ y = \sqrt[3]{2^2} \end{array} \right| \Rightarrow A = (\sqrt[3]{2}, \sqrt[3]{2^2})$$

$$F''_{xx} = 6x$$

$$y'' = -\frac{f''_{xx}}{f'_y} = -\frac{6x}{3y^2 - 3x}$$

$$y''(A) = -2 < 0 \Rightarrow y_{max} = y(\sqrt[3]{2}) = \sqrt[3]{2^2}$$

- $y^2 - 3y - \sin(x) = 0, y = y(x)$

$$F = y^2 - 3y - \sin(x)$$

$$F'_x = -\cos(x) \quad F'_y = 2y - 3$$

$$y' = -\frac{f'_x}{f'_y} = \frac{\cos(x)}{2y - 3}$$

$$\left| \begin{array}{l} \frac{\cos(x)}{2y-3} = 0 \\ y^2 - 3y - \sin(x) = 0 \\ 2y - 3 \neq 0 \end{array} \right| \Leftrightarrow \left| \begin{array}{l} x = \frac{\pi}{2} \wedge x = -\frac{\pi}{2} \\ y = \frac{3+\sqrt{13}}{2} \wedge y = \frac{3-\sqrt{5}}{2} \end{array} \right| \Rightarrow$$

$$A = \left(\frac{\pi}{2} + 2k\pi, \frac{3+\sqrt{13}}{2} \right), B = \left(-\frac{\pi}{2} + 2k\pi, \frac{3-\sqrt{5}}{2} \right)$$

$$F''_{xx} = \sin(x)$$

$$y'' = -\frac{f''_{xx}}{f'_y} = -\frac{\sin(x)}{2y - 3}$$

$$y''(A) = -\frac{\sqrt{13}}{13} < 0 \Rightarrow y_{\max,k} = y\left(\frac{\pi}{2} + 2k\pi\right) = \frac{3+\sqrt{13}}{2}$$

$$y''(B) = -\frac{\sqrt{5}}{5} < 0 \Rightarrow y_{\min,k} = y\left(-\frac{\pi}{2} + 2k\pi\right) = \frac{3-\sqrt{5}}{2}$$

(Панева, как тва е минимум беее)

- $x^2 + y^2 + z^2 - xz - yz + 2x + 2y + 2z - 2 = 0, z = z(x, y)$

$$F = x^2 + y^2 + z^2 - xz - yz + 2x + 2y + 2z - 2$$

$$F'_x = 2x - z + 2 \quad F'_y = 2y - z + 2 \quad F'_z = -x - y + 2z + 2$$

$$z'_x = -\frac{F'_x}{F'_z} = -\frac{2x - z + 2}{-x - y + 2z + 2} \quad z'_y = -\frac{F'_y}{F'_z} = -\frac{2y - z + 2}{-x - y + 2z + 2}$$

$$\left| \begin{array}{l} -\frac{2x-z+2}{-x-y+2z+2} = 0 \\ -\frac{2y-z+2}{-x-y+2z+2} = 0 \\ x^2 + y^2 + z^2 - xz - yz + 2x + 2y + 2z - 2 = 0 \\ -x - y + 2z + 2 \neq 0 \end{array} \right| \Leftrightarrow \left| \begin{array}{l} x = -3 + \sqrt{6} \\ y = -3 + \sqrt{6} \\ z = -4 + 2\sqrt{6} \end{array} \right| \Rightarrow$$

$$A(-3 + \sqrt{6}, -3 + \sqrt{6}, -4 + 2\sqrt{6})$$

$$\begin{aligned}
F''_{xx} &= 2 & F''_{xy} &= 0 & F''_{yx} &= 0 & F''_{yy} &= 2 \\
\Delta &= \begin{vmatrix} -\frac{F''_{xx}}{F'_x} & -\frac{F''_{xy}}{F'_y} \\ -\frac{F''_{yx}}{F'_x} & -\frac{F''_{yy}}{F'_y} \end{vmatrix} = \begin{vmatrix} -\frac{2}{-x-y+2z+2} & 0 \\ 0 & -\frac{2}{-x-y+2z+2} \end{vmatrix} = \frac{4}{(-x-y+2z+2)^2} \\
-\frac{F''_{xx}}{F'_x}(A) &= -\frac{\sqrt{6}}{6} & \Delta(A) &= \frac{1}{6} \implies z_{max} = z(-3 + \sqrt{6}, -3 + \sqrt{6}) = -4 + 2\sqrt{6}
\end{aligned}$$

от къде дойде втората точка в решението на Панева

$$\bullet \quad 2x^2 + 2y^2 + z^2 + 8xz - 8yz + 8 = 0, z = z(x, y)$$

$$\begin{aligned}
F &= 2x^2 + 2y^2 + z^2 + 8xz - 8yz + 8 \\
F'_x &= 4x + 8z & F'_y &= 4y - 8z & F'_z &= 8x - 8y + 2z \\
z'_x &= -\frac{F'_x}{F'_z} = -\frac{4x + 8z}{8x - 8y + 2z} & z'_y &= -\frac{F'_y}{F'_z} = -\frac{4y - 8z}{8x - 8y + 2z} \\
\begin{cases} -\frac{4x+8z}{8x-8y+2z} = 0 \\ -\frac{4y-8z}{8x-8y+2z} = 0 \\ 2x^2 + 2y^2 + z^2 + 8xz - 8yz + 8 = 0 \\ 8x - 8y + 2z \neq 0 \end{cases} &\Leftrightarrow \begin{cases} x = -\frac{4\sqrt{30}}{15} \\ y = \frac{4\sqrt{30}}{15} \\ z = \frac{2\sqrt{30}}{15} \end{cases} \implies A \left(-\frac{4\sqrt{30}}{15}, \frac{4\sqrt{30}}{15}, \frac{2\sqrt{30}}{15} \right) \\
F''_{xx} &= 4 & F''_{xy} &= 0 & F''_{yx} &= 0 & F''_{yy} &= 4 \\
\Delta &= \begin{vmatrix} -\frac{F''_{xx}}{F'_x} & -\frac{F''_{xy}}{F'_y} \\ -\frac{F''_{yx}}{F'_x} & -\frac{F''_{yy}}{F'_y} \end{vmatrix} = \begin{vmatrix} -\frac{4}{8x-8y+2z} & 0 \\ 0 & -\frac{4}{8x-8y+2z} \end{vmatrix} = \frac{16}{(8x-8y+2z)^2} \\
-\frac{F''_{xx}}{F'_x}(A) &= \frac{\sqrt{30}}{30} > 0 & \Delta(A) &= \frac{1}{30} > 0 \implies z_{min} = z \left(-\frac{4\sqrt{30}}{15}, \frac{4\sqrt{30}}{15} \right) = \frac{2\sqrt{30}}{15}
\end{aligned}$$

Задача 10

$$\bullet \quad z = xy, \text{ ако } 2x + y = 1$$

$$\begin{aligned}
F(x, y, \lambda) &= xy + \lambda(2x + y - 1) \quad \lambda \neq 0 \\
F'_x &= 2\lambda + y & F'_y &= \lambda + x \\
\begin{cases} 2\lambda + y = 0 \\ \lambda + x = 0 \\ 2x + y = 1 \end{cases} &\Leftrightarrow \begin{cases} x = \frac{1}{4} \\ y = \frac{1}{2} \\ \lambda = -\frac{1}{4} \end{cases} \implies A \left(\frac{1}{4}, \frac{1}{2}, -\frac{1}{4} \right) \\
F''_{xx} &= 0 & F''_{xy} &= 1 & F''_{yx} &= 1 & F''_{yy} &= 0 \\
\Delta &= F''_{xx}F''_{yy} - F''_{xy}F''_{yx} = -1
\end{aligned}$$

- $z = x^2 + y^2$, ако $x - y = 1$

$$F(x, y, \lambda) = x^2 + y^2 + \lambda(x - y - 1) \quad \lambda \neq 0$$

$$F'_x = \lambda + 2x \quad F'_y = 2y - \lambda$$

$$\begin{cases} \lambda + 2x = 0 \\ 2y - \lambda = 0 \\ x - y = 1 \end{cases} \Leftrightarrow \begin{cases} x = \frac{1}{2} \\ y = -\frac{1}{2} \\ \lambda = -1 \end{cases} \Rightarrow A\left(\frac{1}{2}, -\frac{1}{2}, -1\right)$$

$$F''_{xx} = 2 \quad F''_{xy} = 0 \quad F''_{yx} = 0 \quad F''_{yy} = 2$$

$$\Delta = F''_{xx}F''_{yy} - F''_{xy}F''_{yx} = 4$$

$$F''_{xx}(A) = 2 > 0 \quad \Delta(A) = 4 > 0 \Rightarrow z_{min} = z\left(\frac{1}{2}, -\frac{1}{2}\right) = \frac{1}{2}$$

- $u = x^2 + y^2 + z^2$, ако $\frac{x^2}{16} + \frac{y^2}{9} + \frac{z^2}{4} = 1$

$$F(x, y, z, \lambda) = x^2 + y^2 + z^2 + \lambda\left(\frac{x^2}{16} + \frac{y^2}{9} + \frac{z^2}{4} - 1\right) \quad \lambda \neq 0$$

$$F'_x = \frac{1}{8}\lambda x + 2x \quad F'_y = \frac{2}{9}\lambda y + 2y \quad F'_z = \frac{1}{2}\lambda z + 2z$$

$$\begin{cases} \frac{1}{8}\lambda x + 2x = 0 \\ \frac{2}{9}\lambda y + 2y = 0 \\ \frac{1}{2}\lambda z + 2z = 0 \\ \frac{x^2}{16} + \frac{y^2}{9} + \frac{z^2}{4} = 1 \end{cases} \Leftrightarrow \begin{cases} x = 4 \wedge x = -4 \wedge x = 0 \\ y = 0 \wedge y = 3 \wedge y = -3 \\ z = 0 \wedge z = 2 \wedge z = -2 \\ \lambda = -16 \wedge \lambda = -9 \wedge \lambda = -4 \end{cases} \Rightarrow$$

$$A(4, 0, 0, -16), B(-4, 0, 0, -16), C(0, 3, 0, -9),$$

$$D(0, -3, 0, -9), E(0, 0, 2, -4), F(0, 0, -2, -4)$$

$$F''_{xx} = \frac{1}{8}\lambda + 2 \quad F''_{yy} = \frac{2}{9}\lambda + 2 \quad F''_{zz} = \frac{1}{2}\lambda + 2$$

$$F''_{xy} = F''_{xz} = F''_{yx} = F''_{yz} = F''_{zx} = F''_{zy} = 0$$

$$\Delta_1 = F''_{xx} = \frac{1}{8}\lambda$$

$$\Delta_2 = \begin{vmatrix} F''_{xx} & F''_{xy} \\ F''_{yx} & F''_{yy} \end{vmatrix} = \begin{vmatrix} \frac{1}{8}\lambda + 2 & 0 \\ 0 & \frac{2}{9}\lambda + 2 \end{vmatrix} = \left(\frac{1}{8}\lambda + 2\right)\left(\frac{2}{9}\lambda + 2\right)$$

$$\Delta_3 = \begin{vmatrix} F''_{xx} & F''_{xy} & F''_{xz} \\ F''_{yx} & F''_{yy} & F''_{yz} \\ F''_{zx} & F''_{zy} & F''_{zz} \end{vmatrix} = \begin{vmatrix} \frac{1}{8}\lambda + 2 & 0 & 0 \\ 0 & \frac{2}{9}\lambda + 2 & 0 \\ 0 & 0 & \frac{1}{2}\lambda + 2 \end{vmatrix} = \left(\frac{1}{8}\lambda + 2\right)\left(\frac{2}{9}\lambda + 2\right)\left(\frac{1}{2}\lambda + 2\right)$$

$$\begin{array}{lll}
\Delta_1(A) = 0 & \Delta_2(A) = 0 & \Delta_3(A) = 0 \implies \text{Няма екстремум.} \\
\Delta_1(B) = 0 & \Delta_2(B) = 0 & \Delta_3(B) = 0 \implies \text{Няма екстремум.} \\
\Delta_1(C) = 0 & \Delta_2(C) = 0 & \Delta_3(C) = 0 \implies \text{Няма екстремум.} \\
\Delta_1(D) = 0 & \Delta_2(D) = 0 & \Delta_3(D) = 0 \implies \text{Няма екстремум.} \\
\Delta_1(E) = 0 & \Delta_2(E) = 0 & \Delta_3(E) = 0 \implies \text{Няма екстремум.} \\
\Delta_1(F) = 0 & \Delta_2(F) = 0 & \Delta_3(F) = 0 \implies \text{Няма екстремум.}
\end{array}$$

Панева кво стааа?!

- $u = xyz$, ако $x + y + z = 5, xy + yz + zx = 8$

$$\begin{aligned}
F(x, y, z, \lambda, \mu) &= xyz + \lambda(x + y + z - 5) + \mu(xy + yz + zx - 8) \quad \lambda, \mu \neq 0 \\
F'_x &= \lambda + \mu(y + z) + yz & F'_y &= \lambda + \mu(x + z) + xz & F'_z &= \lambda + \mu(y + x) + yx \\
\left\{ \begin{array}{l} \lambda + \mu(y + z) + yz = 0 \\ \lambda + \mu(x + z) + xz = 0 \\ \lambda + \mu(y + x) + yx = 0 \\ x + y + z = 5 \\ xy + yz + zx = 8 \end{array} \right. &\Leftrightarrow \left\{ \begin{array}{l} x = 1 \wedge x = 2 \wedge x = \frac{4}{3} \wedge x = \frac{7}{3} \\ y = 2 \wedge y = 1 \wedge y = \frac{4}{3} \wedge y = \frac{7}{3} \\ z = 2 \wedge z = 1 \wedge z = \frac{4}{3} \\ \lambda = 4 \wedge \lambda = \frac{16}{9} \\ \mu = -2 \wedge \mu = -\frac{4}{3} \end{array} \right. \implies \\
A(1, 2, 2, 4, -2), B(2, 2, 1, 4, -2), C(2, 1, 2, 4, -2) \\
D\left(\frac{4}{3}, \frac{7}{3}, \frac{4}{3}, \frac{16}{9}, -\frac{4}{3}\right), E\left(\frac{7}{3}, \frac{4}{3}, \frac{4}{3}, \frac{16}{9}, -\frac{4}{3}\right), F\left(\frac{4}{3}, \frac{4}{3}, \frac{7}{3}, \frac{16}{9}, -\frac{4}{3}\right)
\end{aligned}$$

$$\begin{aligned}
F''_{xx} &= F''_{yy} = F''_{zz} = 0 \\
F''_{xy} &= F''_{yx} = \mu + z \\
F''_{xz} &= F''_{zx} = \mu + y \\
F''_{yz} &= F''_{zy} = \mu + x \\
\Delta_1 &= F''_{xx} = 0 \\
\Delta_2 &= \begin{vmatrix} F''_{xx} & F''_{xy} \\ F''_{yx} & F''_{yy} \end{vmatrix} = \begin{vmatrix} 0 & \mu + z \\ \mu + z & 0 \end{vmatrix} = -(\mu + z)^2 \\
\Delta_3 &= \begin{vmatrix} F''_{xx} & F''_{xy} & F''_{xz} \\ F''_{yx} & F''_{yy} & F''_{yz} \\ F''_{zx} & F''_{zy} & F''_{zz} \end{vmatrix} = \begin{vmatrix} 0 & \mu + z & \mu + y \\ \mu + z & 0 & \mu + x \\ \mu + y & \mu + x & 0 \end{vmatrix} = 2(\mu + z)(\mu + y)(\mu + x)
\end{aligned}$$

$$\begin{aligned}
\Delta_1(A) = 0 \quad \Delta_2(A) = 0 \quad \Delta_3(A) = 0 &\implies \text{Няма екстремум.} \\
\Delta_1(B) = 0 \quad \Delta_2(B) = -1 \quad \Delta_3(B) = 0 &\implies \text{Няма екстремум.} \\
\Delta_1(C) = 0 \quad \Delta_2(C) = 0 \quad \Delta_3(C) = 0 &\implies \text{Няма екстремум.} \\
\Delta_1(D) = 0 \quad \Delta_2(D) = 0 \quad \Delta_3(D) = 0 &\implies \text{Няма екстремум.} \\
\Delta_1(E) = 0 \quad \Delta_2(E) = 0 \quad \Delta_3(E) = 0 &\implies \text{Няма екстремум.} \\
\Delta_1(F) = 0 \quad \Delta_2(F) = -1 \quad \Delta_3(F) = 0 &\implies \text{Няма екстремум.}
\end{aligned}$$

Панева кво стааа?!

Задача 11

- $u = x^2 + y^2 + z^2 + 2x + 4y - 6z$, ако $x^2 + y^2 + z^2 = 14$

$$\begin{aligned}
F &= x^2 + y^2 + z^2 + 2x + 4y - 6z + \lambda(x^2 + y^2 + z^2 - 14) \quad \lambda \neq 0 \\
F'_x &= 2\lambda x + 2x + 2 \quad F'_y = 2\lambda y + 2y + 4 \quad F'_z = 2\lambda z + 2z - 6 \\
\left\{ \begin{array}{l} 2\lambda x + 2x + 2 = 0 \\ 2\lambda y + 2y + 4 = 0 \\ 2\lambda z + 2z - 6 = 0 \\ x^2 + y^2 + z^2 = 14 \end{array} \right. &\Leftrightarrow \left\{ \begin{array}{l} x = -1 \wedge x = 1 \\ y = -2 \wedge y = 2 \\ z = 3 \wedge z = -3 \\ \lambda = 0 \wedge \lambda = -2 \end{array} \right. \implies A(-1, -2, 3, 0), B(1, 2, -3, -2)
\end{aligned}$$

$$\begin{aligned}
F''_{xx} &= F''_{yy} = F''_{zz} = 2\lambda + 2 \\
F''_{xy} &= F''_{xz} = F''_{yx} = F''_{yz} = F''_{zx} = F''_{zy} = 0 \\
\Delta_1 &= F''_{xx} = 2\lambda + 2 \\
\Delta_2 &= \begin{vmatrix} F''_{xx} & F''_{xy} \\ F''_{yx} & F''_{yy} \end{vmatrix} = \begin{vmatrix} 2\lambda + 2 & 0 \\ 0 & 2\lambda + 2 \end{vmatrix} = (2\lambda + 2)^2 \\
\Delta_3 &= \begin{vmatrix} F''_{xx} & F''_{xy} & F''_{xz} \\ F''_{yx} & F''_{yy} & F''_{yz} \\ F''_{zx} & F''_{zy} & F''_{zz} \end{vmatrix} = \begin{vmatrix} 2\lambda + 2 & 0 & 0 \\ 0 & 2\lambda + 2 & 0 \\ 0 & 0 & 2\lambda + 2 \end{vmatrix} = (2\lambda + 2)^3
\end{aligned}$$

$$\begin{aligned}
\Delta_1(A) &= 2 > 0 \quad \Delta_2(A) = 4 > 0 \quad \Delta_3(A) = 8 > 0 \implies \\
u_{min} &= u(-1, -2, 3) = -14 \\
\Delta_1(B) &= -2 < 0 \quad \Delta_2(B) = 4 > 0 \quad \Delta_3(B) = -8 < 0 \implies \\
u_{max} &= u(1, 2, -3) = 42
\end{aligned}$$

- $u = x^2 + y^2 + z^2 + 2x + 4y$, ако $x^2 + y^2 = 20$

$$\begin{aligned}
 F &= x^2 + y^2 + z^2 + 2x + 4y + \lambda(x^2 + y^2 - 20) \quad \lambda \neq 0 \\
 F'_x &= 2\lambda x + 2x + 2 \quad F'_y = 2\lambda y + 2y + 4 \quad F'_z = 2z \\
 \begin{cases} 2\lambda x + 2x + 2 = 0 \\ 2\lambda y + 2y + 4 = 0 \\ 2z = 0 \\ x^2 + y^2 = 20 \end{cases} &\Leftrightarrow \begin{cases} x = 2 \wedge x = -2 \\ y = 4 \wedge y = -4 \\ z = 0 \\ \lambda = -\frac{3}{2} \wedge \lambda = -\frac{1}{2} \end{cases} \Rightarrow A\left(2, 4, 0, -\frac{3}{2}\right), B\left(-2, -4, 0, -\frac{1}{2}\right)
 \end{aligned}$$

$$\begin{aligned}
 F''_{xx} &= F''_{yy} = 2\lambda + 2 \quad F''_{zz} = 2 \\
 F''_{xy} &= F''_{xz} = F''_{yx} = F''_{yz} = F''_{zx} = F''_{zy} = 0 \\
 \Delta_1 &= F''_{xx} = 2\lambda + 2 \\
 \Delta_2 &= \begin{vmatrix} F''_{xx} & F''_{xy} \\ F''_{yx} & F''_{yy} \end{vmatrix} = \begin{vmatrix} 2\lambda + 2 & 0 \\ 0 & 2\lambda + 2 \end{vmatrix} = (2\lambda + 2)^2 \\
 \Delta_3 &= \begin{vmatrix} F''_{xx} & F''_{xy} & F''_{xz} \\ F''_{yx} & F''_{yy} & F''_{yz} \\ F''_{zx} & F''_{zy} & F''_{zz} \end{vmatrix} = \begin{vmatrix} 2\lambda + 2 & 0 & 0 \\ 0 & 2\lambda + 2 & 0 \\ 0 & 0 & 2 \end{vmatrix} = (2\lambda + 2)(4\lambda + 4)
 \end{aligned}$$

$$\begin{aligned}
 \Delta_1(A) &= -1 \quad \Delta_2(A) = 1 > 0 \quad \Delta_3(A) = 2 > 0 \Rightarrow \text{Няма екстремум.} \\
 \Delta_1(B) &= 1 > 0 \quad \Delta_2(B) = 1 > 0 \quad \Delta_3(B) = 2 > 0 \Rightarrow u_{\min} = u(-2, -4, 0) = 0
 \end{aligned}$$

- $u = x^2 + y^2 + z^2 + 6x - 2y + 4z$, ако $x^2 + y^2 + z^2 = 56$

$$\begin{aligned}
 F &= x^2 + y^2 + z^2 + 6x - 2y + 4z + \lambda(x^2 + y^2 + z^2 - 56) \quad \lambda \neq 0 \\
 F'_x &= 2\lambda x + 2x + 6 \quad F'_y = 2\lambda y + 2y - 2 \quad F'_z = 2\lambda z + 2z + 4 \\
 \begin{cases} 2\lambda x + 2x + 6 = 0 \\ 2\lambda y + 2y - 2 = 0 \\ 2\lambda z + 2z + 4 = 0 \\ x^2 + y^2 + z^2 = 56 \end{cases} &\Leftrightarrow \begin{cases} x = 6 \wedge x = -6 \\ y = -2 \wedge y = 2 \\ z = 4 \wedge z = -4 \\ \lambda = -\frac{3}{2} \wedge \lambda = -\frac{1}{2} \end{cases} \Rightarrow A\left(6, -2, 4, -\frac{3}{2}\right), B\left(-6, 2, -4, -\frac{1}{2}\right)
 \end{aligned}$$

$$\begin{aligned}
F''_{xx} &= F''_{yy} = F''_{zz} = 2\lambda + 2 \\
F''_{xy} &= F''_{xz} = F''_{yx} = F''_{yz} = F''_{zx} = F''_{zy} = 0 \\
\Delta_1 &= F''_{xx} = 2\lambda + 2 \\
\Delta_2 &= \begin{vmatrix} F''_{xx} & F''_{xy} \\ F''_{yx} & F''_{yy} \end{vmatrix} = \begin{vmatrix} 2\lambda + 2 & 0 \\ 0 & 2\lambda + 2 \end{vmatrix} = (2\lambda + 2)^2 \\
\Delta_3 &= \begin{vmatrix} F''_{xx} & F''_{xy} & F''_{xz} \\ F''_{yx} & F''_{yy} & F''_{yz} \\ F''_{zx} & F''_{zy} & F''_{zz} \end{vmatrix} = \begin{vmatrix} 2\lambda + 2 & 0 & 0 \\ 0 & 2\lambda + 2 & 0 \\ 0 & 0 & 2\lambda + 2 \end{vmatrix} = (2\lambda + 2)^3
\end{aligned}$$

$$\begin{aligned}
\Delta_1(A) &= -1 < 0 \quad \Delta_2(A) = 1 > 0 \quad \Delta_3(A) = -1 < 0 \implies \\
u_{max} &= u(6, -2, 4) = 112 \\
\Delta_1(B) &= 1 > 0 \quad \Delta_2(B) = 1 > 0 \quad \Delta_3(B) = 1 > 0 \implies \\
u_{min} &= u(-6, 2, -4) = 0
\end{aligned}$$

Задача 12

- $u = x^2 + y^2 - 12x + 16y$, ако $x^2 + y^2 \leq 25, x^2 + y^2 \leq 400, x^2 + y^2 \leq 100$
- $u = x^2 + y^2 + z^2 + 2x + 4y - 6$, ако $x^2 + y^2 + z^2 \leq 9$
- $u = x^2 + 2y^2 + 3z^2$, ако $x^2 + y^2 + z^2 \leq 100$

9 Упражнение към лекция 9

9.1 Задачи

Задача 1

Да се пресметнат интегралите

- $I = \iint_D xy \, dx \, dy, \quad D : \begin{cases} 0 \leq x \leq 1 \\ 0 \leq y \leq 2 \end{cases}$
- $I = \iint_D xy \, dx \, dy$, ако D е оградена от $\begin{cases} xy = 1 \\ x + y = \frac{5}{2} \end{cases}$
- $I = \iint_D dx \, dy$, ако D е оградена от кривите $D : \begin{cases} 4y = x^2 - 4x \\ x - y - 3 = 0 \end{cases}$

Задача 2

Да се определят границите на интегриране и да се пресметне интеграла

$$\iint_D (x^2 + y^2) \, dx \, dy, \quad D : \begin{cases} y = x, & y = 2 \\ y = x + 2, & y = 6 \end{cases}$$

Задача 3

Да се пресметне интеграла

$$\iint_D (x + y) \, dx \, dy, \quad \partial D : \begin{cases} y^2 = 2x \\ x + y = 4 \\ x + y = 12 \end{cases}$$

Задача 4

Да се пресметне интеграла

$$\iint_D (x + y) \, dx \, dy$$

Където D е триъгълник $\triangle ABO$ с върхове $A(1, 0)$, $B = (1, 1)$, $O = (0, 0)$

9.2 Решения

Задача 1

$$\bullet I = \iint_D xy \, dx \, dy, \quad D : \begin{cases} 0 \leq x \leq 1 \\ 0 \leq y \leq 2 \end{cases}$$

$$I = \int_0^1 x \int_0^2 y \, dy \, dx = \int_0^1 x \left[\frac{y^2}{2} \right]_0^2 dx = \int_0^1 2x \, dx = \left[x^2 \right]_0^1 = 1$$

$$I = 2 \int_0^1 x \, dx = 2 \left[\frac{x^2}{2} \right]_0^1 = 1$$

$$\bullet I = \iint_D xy \, dx \, dy, \text{ ако } D \text{ е оградена от } \begin{cases} xy = 1 \\ x + y = \frac{5}{2} \end{cases}$$

$$\left| \begin{matrix} xy = 1 \\ x + y = \frac{5}{2} \end{matrix} \right| \Leftrightarrow \left| \begin{matrix} y = \frac{5}{2} - x \\ x \left(\frac{5}{2} - x \right) = 1 \end{matrix} \right| \Rightarrow D : \begin{cases} \frac{1}{2} \leq x \leq 2 \\ \frac{1}{x} \leq y \leq \frac{5}{2} - x \end{cases}$$

$$I = \int_{\frac{1}{2}}^2 x \int_{\frac{1}{x}}^{\frac{5}{2}-x} y \, dy \, dx$$

$$\int_{\frac{1}{x}}^{\frac{5}{2}-x} y \, dy = \left[\frac{y^2}{2} \right]_{\frac{1}{x}}^{\frac{5}{2}-x} = \frac{1}{2} \left[\left(\frac{5}{2} - x \right)^2 - \frac{1}{x^2} \right]$$

$$I = \frac{1}{2} \int_{\frac{1}{2}}^2 x \left[\left(\frac{5}{2} - x \right)^2 - \frac{1}{x^2} \right] dx = \frac{1}{2} \int_{\frac{1}{2}}^2 x \left(\frac{25}{4} - 5x + x^2 - \frac{1}{x^2} \right) dx =$$

$$= \frac{1}{2} \int_{\frac{1}{2}}^2 \left(\frac{25}{4}x - 5x^2 + x^3 - \frac{1}{x} \right) dx = \frac{1}{2} \left[\frac{25}{8}x^2 - \frac{5}{3}x^3 + \frac{x^4}{4} - \ln|x| \right]_{\frac{1}{2}}^2 = \frac{165}{128} - \ln(2)$$

$$\bullet I = \iint_D dx \, dy, \text{ ако } D \text{ е оградена от кривите } D : \begin{cases} 4y = x^2 - 4x \\ x - y - 3 = 0 \end{cases}$$

$$\begin{cases} 4y = x^2 - 4x \\ x - y - 3 = 0 \end{cases} \Leftrightarrow \begin{cases} y = x - 3 \\ 4(x - 3) = x^2 - 4x \end{cases} \Rightarrow D : \begin{cases} 2 \leq x \leq 6 \\ \frac{x^2}{4} - x \leq y \leq x - 3 \end{cases}$$

$$I = \int_2^6 \int_{\frac{x^2}{4} - x}^{x-3} dy \, dx$$

$$\int_{\frac{x^2}{4} - x}^{x-3} dy = y \Big|_{\frac{x^2}{4} - x}^{x-3} = \left[x - 3 - \left(\frac{x^2}{4} - x \right) \right]$$

$$I = \int_2^6 \left[x - 3 - \left(\frac{x^2}{4} - x \right) \right] dx = \int_2^6 \left[2x - 3 - \frac{x^2}{4} \right] dx = \left[x^2 - 3x - \frac{x^3}{12} \right] \Big|_2^6 = \frac{8}{3}$$

Задача 2

$$I = \iint_D (x^2 + y^2) dx \, dy, \quad D : \begin{cases} y = x, & y = 2 \\ y = x + 2, & y = 6 \end{cases}$$

$$D_1 : \begin{cases} 0 \leq x \leq 2 \\ 2 \leq y \leq x + 2 \end{cases} \quad D_2 : \begin{cases} 2 \leq x \leq 4 \\ x \leq y \leq x + 2 \end{cases} \quad D_3 : \begin{cases} 4 \leq x \leq 6 \\ x \leq y \leq 6 \end{cases}$$

$$I_1 = \int_0^2 x^2 \int_2^{x+2} y^2 dy \, dx \quad I_2 = \int_2^4 x^2 \int_x^{x+2} y^2 dy \, dx \quad I_3 = \int_4^6 x^2 \int_x^6 y^2 dy \, dx$$

$$I_1 = \int_0^2 x^2 \int_2^{x+2} y^2 dy \, dx$$

$$\int_2^{x+2} y^2 dy \Big|_2^{x+2} = \frac{(x+2)^3}{3} - \frac{8}{3}$$

$$I_1 = \int_0^2 x^2 \left(\frac{(x+2)^3}{3} - \frac{8}{3} \right) dx = \int_0^2 \left(\frac{x^2(x+2)^3}{3} - \frac{8x^2}{3} \right) dx = \frac{56}{3}$$

$$\begin{aligned}
I_2 &= \int_2^4 x^2 \int_x^{x+2} y^2 dy dx \\
&\int_x^{x+2} y^2 dy \frac{y^3}{3} \Big|_x^{x+2} = \frac{(x+2)^3}{3} - \frac{x^3}{3} \\
I_2 &= \int_2^4 x^2 \left(\frac{(x+2)^3}{3} - \frac{x^3}{3} \right) dx = \int_0^2 \left(\frac{x^2(x+2)^3}{3} - \frac{x^5}{3} \right) dx = 104
\end{aligned}$$

$$\begin{aligned}
I_3 &= \int_4^6 x^2 \int_x^6 y^2 dy dx \\
&\int_x^6 y^2 dy = \frac{y^3}{3} \Big|_x^6 = 72 - \frac{x^3}{3} \\
I_3 &= \int_4^6 x^2 \left(72 - \frac{x^3}{3} \right) dx = \int_4^6 \left(72x^2 - \frac{x^5}{3} \right) dx = \frac{304}{3} \\
I &= I_1 + I_2 + I_3 = \frac{56}{3} + 104 + \frac{304}{3} = 224
\end{aligned}$$

Задача 3

$$\iint_D (x+y) dx dy, \quad \partial D : \begin{cases} y^2 = 2x \\ x+y = 4 \\ x+y = 12 \end{cases}$$

$$\begin{aligned}
\left| \begin{array}{l} y^2 = 2x \\ x+y = 4 \end{array} \right| &\Leftrightarrow \left| \begin{array}{l} y = 4-x \\ (4-x)^2 = 2x \end{array} \right| \Rightarrow D_1 = \begin{cases} 2 \leq x \leq 8 \\ 4-x \leq y \leq \sqrt{2x} \end{cases} \\
\left| \begin{array}{l} y^2 = 2x \\ x+y = 12 \end{array} \right| &\Leftrightarrow \left| \begin{array}{l} y = 12-x \\ (12-x)^2 = 2x \end{array} \right| \Rightarrow D_2 = \begin{cases} 8 \leq x \leq 18 \\ -\sqrt{2x} \leq y \leq 12-x \end{cases} \\
I_1 &= \int_2^8 \int_{4-x}^{\sqrt{2x}} (x+y) dy dx & I_2 &= \int_8^{18} \int_{-\sqrt{2x}}^{12-x} (x+y) dy dx
\end{aligned}$$

$$\begin{aligned}
I_1 &= \int_2^8 \int_{4-x}^{\sqrt{2x}} (x+y) dy dx \\
\int_{4-x}^{\sqrt{2x}} (x+y) dy &= xy + \frac{y^2}{2} \Big|_{4-x}^{\sqrt{2x}} = \sqrt{2}x^{\frac{3}{2}} + \frac{x^2}{2} + x - 8 \\
I_1 &= \int_2^8 \sqrt{2}x^{\frac{3}{2}} + \frac{x^2}{2} + x - 8 dx = \frac{826}{5} \\
I_2 &= \int_8^{18} \int_{-\sqrt{2x}}^{12-x} (x+y) dy dx \\
\int_{-\sqrt{2x}}^{12-x} (x+y) dy &= xy + \frac{y^2}{2} \Big|_{-\sqrt{2x}}^{12-x} = \sqrt{2}x^{\frac{3}{2}} - \frac{x^2}{2} - x + 72 \\
I_2 &= \int_8^{18} \sqrt{2}x^{\frac{3}{2}} - \frac{x^2}{2} - x + 72 dx = \frac{5678}{15} \\
I &= I_1 + I_2 = \frac{8156}{15}
\end{aligned}$$

Задача 4

$$\iint_D (x+y) dx dy$$

Където D е триъгълник $\triangle ABO$ с върхове $A(1,0)$, $B = (1,1)$, $O = (0,0)$

$$D = \begin{cases} 0 \leq x \leq 1 \\ 0 \leq y \leq x \end{cases}$$

$$I = \int_0^1 \int_0^x (x + y) \, dy \, dx$$

$$\int_0^x (x + y) \, dy = xy + \frac{y^2}{2} \Big|_0^x = \frac{3x^2}{2}$$

$$I = \frac{3}{2} \int_0^1 x^2 \, dx = \frac{x^3}{2} \Big|_0^1 = \frac{1}{2} - 0 = \frac{1}{2}$$

10 Упражнение към лекция 10

10.1 Задачи

10.2 Решения

11 Упражнение към лекция 11

11.1 Задачи

Задача 1

Да се пресметнат интегралите

- $I = \iint_D \arctan \frac{y}{x} dx dy, \quad D = \begin{cases} 1 \leq x^2 + y^2 \leq 9 \\ y \geq \frac{x}{\sqrt{3}} & y \leq \sqrt{3}x \end{cases}$
- $I = \iint_D dx dy, \quad D = \{(x, y) : x^2 + y^2 \leq 25\}$
- $I = \iint_D dx dy, \quad D$ е оградена от лемниската $(x^2 + y^2)^2 = 2a^2xy$
- $I = \iint_D \sqrt{1 + \frac{x^2}{9} + \frac{y^2}{16}} dx dy, \quad D : \frac{x^2}{9} + \frac{y^2}{16} \leq 1$

Задача 2

Да се пресметнат интегралите

- $I = \iint_D dx dy, \quad \partial D : \begin{cases} y^2 = 2x, & xy = 1 \\ y^2 = 3x, & xy = 2 \end{cases}$
- $I = \iint_D (2x - y) dx dy, \quad \partial D : \begin{cases} x + y = 1, & x + y = 2 \\ 2x - y = 1, & 2x - y = 3 \end{cases}$

11.2 Решения

Задача 1

$$\bullet I = \iint_D \arctan \frac{y}{x} dx dy, \quad D = \begin{cases} 1 \leq x^2 + y^2 \leq 9 \\ y \geq \frac{x}{\sqrt{3}} \quad y \leq \sqrt{3}x \end{cases}$$

Полярна смяна

$$x = \rho \cos \varphi \quad y = \rho \sin \varphi \quad (\rho > 0, \quad 0 < \varphi < 2\pi)$$

За допълнителните условия имаме

$$1 \leq x^2 + y^2 = \rho^2 \cos^2 \varphi + \rho^2 \sin^2 \varphi \leq 9 \implies 1 \leq \rho^2 \leq 9 \implies 1 \leq \rho \leq 3$$

$y = \frac{x}{\sqrt{3}}$ е права сключваща ъгъл $\frac{\pi}{6}$ с положителната посока на Ox , а

$$y = \sqrt{3}x - \frac{\pi}{3} \implies$$

$$\tan \varphi \geq \frac{1}{\sqrt{3}} \quad \tan \varphi \leq \sqrt{3}$$

$$\varphi \in \left[\frac{\pi}{6}, \frac{\pi}{2}\right) \quad \varphi \in \left(-\frac{\pi}{2}, \frac{\pi}{3}\right] \implies \varphi \in \left[\frac{\pi}{6}, \frac{\pi}{3}\right]$$

$$\arctan \frac{y}{x} = \arctan \frac{\rho \sin \varphi}{\rho \cos \varphi} = \arctan \tan \varphi = \varphi$$

$$\Delta = \rho \implies dx dy = \rho d\rho d\varphi$$

$$I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \int_1^3 \varphi \cdot \rho d\rho d\varphi = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \varphi \int_1^3 \rho d\rho d\varphi$$

$$\int_1^3 \rho d\rho = \frac{\rho^2}{2} \Big|_1^3 = \frac{1}{2}(9 - 1) = 4$$

$$I = 4 \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \varphi = 4 \frac{\varphi^2}{2} \Big|_{\frac{\pi}{6}}^{\frac{\pi}{3}} = 2 \left(\frac{\pi^2}{9} - \frac{\pi^2}{36} \right) = 2 \frac{3\pi^2}{36} = \frac{\pi^2}{6}$$

$$\bullet I = \iint_D dx \, dy, \quad D = \{(x, y) : x^2 + y^2 \leq 25\}$$

Полярна смяна

$$x = \rho \cos \varphi \quad y = \rho \sin \varphi \quad (\rho > 0, \quad 0 < \varphi < 2\pi)$$

За допълнителните условия имаме

$$x^2 + y^2 = \rho^2 \cos^2 \varphi + \rho^2 \sin^2 \varphi \leq 25 \implies 0 \leq \rho^2 \leq 25 \implies 0 \leq \rho \leq 5$$

За φ не се появяват допълнителни ограничения

$$\text{Якобианът} = \Delta = \rho$$

$$I = \int_0^{2\pi} \int_0^5 \rho \, d\rho \, d\varphi$$

$$\int_0^5 \rho \, d\rho = \left. \frac{\rho^2}{2} \right|_0^5 = \frac{25}{2}$$

$$I = \frac{25}{2} \int_0^{2\pi} d\varphi = \frac{25}{2} \cdot 2\pi = 25\pi$$

$$\bullet I = \iint_D dx \, dy, \quad D \text{ е оградена от лемниската } (x^2 + y^2)^2 = 2a^2 xy$$

Полярна смяна

$$x = \rho \cos \varphi \quad y = \rho \sin \varphi \quad (\rho > 0, \quad 0 < \varphi < 2\pi)$$

$$xy > 0 \implies \cos \varphi > 0 \quad \sin \varphi > 0$$

$$\varphi \in \left(0, \frac{\pi}{2}\right) \quad \varphi \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$$

$$\bullet I = \iint_D \sqrt{1 + \frac{x^2}{9} + \frac{y^2}{16}} dx dy, \quad D : \frac{x^2}{9} + \frac{y^2}{16} \leq 1$$

Полярна смяна

$$x = 3\rho \cos \varphi \quad y = 4\rho \sin \varphi \quad (\rho > 0, \quad 0 < \varphi < 2\pi)$$

Якобианът на смяната: $\Delta = 12\rho$

$$\sqrt{1 + \frac{x^2}{9} + \frac{y^2}{16}} = \sqrt{1 + \frac{9\rho^2 \cos^2 \varphi}{9} + \frac{16\rho^2 \sin^2 \varphi}{16}} = \sqrt{1 + \rho^2(\cos^2 \varphi + \sin^2 \varphi)} \implies$$

$$\sqrt{1 + \frac{x^2}{9} + \frac{y^2}{16}} = \sqrt{1 + \rho^2}$$

$$\frac{9\rho^2 \cos^2 \varphi}{9} + \frac{16\rho^2 \sin^2 \varphi}{16} \leq 1 \implies \rho^2 \leq 1 \implies 0 \leq \rho \leq 1$$

$$I = 12 \int_0^{2\pi} \int_0^1 \sqrt{1 + \rho^2} d\rho d\varphi$$

$$\int_0^1 \sqrt{1 + \rho^2} d\rho = \frac{1}{2} \int_0^1 (1 + \rho^2)^{\frac{1}{2}} d(\rho^2 + 1) = \frac{1}{2} \cdot \frac{(\rho^2 + 1)^{\frac{3}{2}}}{\frac{3}{2}} \Big|_0^1 = \frac{1}{2} \cdot \frac{2^{\frac{3}{2}} - 1}{\frac{3}{2}}$$

$$I = \frac{12}{2} \cdot \frac{2}{3} \int_0^{2\pi} 2^{\frac{3}{2}} - 1 d\varphi = \frac{12}{3} (2\sqrt{2} - 1) \cdot 2\pi = \frac{8\pi}{3} (2\sqrt{2} - 1)$$

Задача 2

$$\bullet I = \iint_D dx dy, \quad \partial D : \begin{cases} y^2 = 2x, & xy = 1 \\ y^2 = 3x, & xy = 2 \end{cases}$$

Нека

$$u = \frac{y^2}{x} \quad v = xy$$

$$2 \leq u \leq 3 \quad 1 \leq v \leq 2$$

$$\text{Якобианът на смяната е : } \Delta = \frac{D(x, y)}{D(u, v)} = \frac{1}{\frac{D(u, v)}{D(x, y)}}$$

$$\text{Освен това } \frac{D(u, v)}{\partial(x, y)} = \begin{vmatrix} u'_x & u'_y \\ v'_x & v'_y \end{vmatrix} = \begin{vmatrix} -\frac{y^2}{x^2} & \frac{2y}{x} \\ y & x \end{vmatrix} = -\frac{y^2}{x} - \frac{2y^2}{x} = -\frac{3y^2}{x} = 3u$$

$$\Delta = -\frac{1}{3u} \implies |\Delta| = \frac{1}{3u}$$

$$I = \int_1^2 \int_2^3 \frac{1}{3u} du dv$$

$$\int_2^3 \frac{1}{3u} du = \frac{1}{3} \int_2^3 \frac{1}{u} = \frac{1}{3} \ln u \Big|_2^3 = \frac{1}{3} \ln \frac{3}{2}$$

$$I = \int_1^2 \frac{1}{3} \ln \frac{3}{2} dv = \frac{1}{3} \ln \frac{3}{2} \int_1^2 dv = \frac{1}{3} \ln \frac{3}{2} v \Big|_1^2 = \frac{1}{3} \ln \frac{3}{2}$$

$$\bullet I = \iint_D (2x - y) dx dy, \quad \partial D : \begin{cases} x + y = 1, & x + y = 2 \\ 2x - y = 1, & 2x - y = 3 \end{cases}$$

Нека

$$u = x + y \quad v = 2x - y$$

$$1 \leq u \leq 2 \quad 1 \leq v \leq 3$$

$$\text{Якобианът на смяната е : } \Delta = \frac{D(x, y)}{D(u, v)} = \frac{1}{\frac{D(u, v)}{D(x, y)}}$$

$$\text{Освен това } \frac{D(u, v)}{\partial(x, y)} = \begin{vmatrix} u'_x & u'_y \\ v'_x & v'_y \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} = -3$$

$$\Delta = -\frac{1}{3} \implies |\Delta| = \frac{1}{3}$$

$$I = \int_1^2 \int_1^3 \frac{1}{3} v dv du$$

$$\int_1^3 \frac{1}{3} v dv = \frac{1}{3} \int_1^3 v dv = \frac{1}{3} \frac{v^2}{2} \Big|_1^3 = \frac{1}{3} \left(\frac{9}{2} - \frac{1}{2} \right) = \frac{4}{3}$$

$$I = \int_1^2 \frac{4}{3} du = \frac{4}{3} \int_1^2 du = \frac{4}{3} u \Big|_1^2 = \frac{4}{3}$$

12 Упражнение към лекция 12

12.1 Задачи

Задача 1

Пресметнете интегралите

- $\iiint_T (x^2 + y^2) \, dx \, dy \, dz, T : \begin{cases} x^2 + y^2 = 2z \\ z = 2 \end{cases}$
- $\iiint_T (x^2 + y^2) \, dx \, dy \, dz, T : \begin{cases} z = 6 - x^2 - y^2 \\ z^2 = x^2 + y^2 \\ z \geq 0 \end{cases}$

Задача 2

Да се пресметне обемът на тялото T

- $T : x^2 + y^2 + z^2 \leq R^2$
- $T : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1$

$$R > 0 \quad a > 0 \quad b > 0 \quad c > 0 \quad a, b, c, R = \text{const}$$

Задача 3

Да се намери лицето на частта от повърхнината

$$S : x^2 + y^2 = 2az, \quad a > 0$$

заклучена в цилиндъра

$$(x^2 + y^2)^2 = 2a^2xy$$

12.2 Решения

Задача 1

$$\bullet \iint_T (x^2 + y^2) \, dx \, dy \, dz, T : \begin{cases} x^2 + y^2 = 2z \\ z = 2 \end{cases}$$

$$D : \begin{cases} x^2 + y^2 = 2z \\ z = 2 \end{cases} \implies D : x^2 + y^2 \leq 4$$

Цилиндрична смяна

$$\begin{aligned} x &= \rho \cos \varphi & y &= \rho \sin \varphi & z &= z \implies |\Delta| = \rho \\ \rho^2 \cos^2 \varphi + \rho^2 \sin^2 \varphi &\leq 4 \implies \rho^2 \leq 4 \implies 0 \leq \rho \leq 2 \end{aligned}$$

$$\frac{1}{2}(x^2 + y^2) \leq z \leq 2 \implies \frac{\rho^2}{2} \leq z \leq 2$$

$$I = \int_0^{2\pi} \int_0^2 \int_{\frac{\rho^2}{2}}^2 \rho^2 \cdot \rho \, dz \, d\rho \, d\varphi = \int_0^{2\pi} \int_0^2 \int_{\frac{\rho^2}{2}}^2 \rho^3 \, dz \, d\rho \, d\varphi$$

$$\int_{\frac{\rho^2}{2}}^2 \rho^3 \, dz = \rho^3 \int_{\frac{\rho^2}{2}}^2 \rho^3 \, dz = \rho^3 \cdot z \Big|_{\frac{\rho^2}{2}}^2 = \rho^3 \cdot \left(2 - \frac{\rho^2}{2}\right) = 2\rho^3 - \frac{\rho^5}{2}$$

$$I = \int_0^{2\pi} \int_0^2 \left(2\rho^3 - \frac{\rho^5}{2}\right) d\rho \, d\varphi$$

$$\int_0^2 \left(2\rho^3 - \frac{\rho^5}{2}\right) d\rho = \int_0^2 2\rho^3 \, d\rho - \int_0^2 \frac{\rho^5}{2} \, d\rho =$$

$$\frac{2\rho^4}{4} \Big|_0^2 - \frac{\rho^6}{12} \Big|_0^2 = \frac{2 \cdot 16}{4} - \frac{64}{12} = 8 - \frac{16}{3} = \frac{24 - 16}{3} = \frac{8}{3}$$

$$I = \int_0^{2\pi} \frac{8}{3} \, d\varphi = \frac{8}{3} \int_0^{2\pi} d\varphi = \frac{8}{3} \varphi \Big|_0^{2\pi} = \frac{16\pi}{3}$$

$$\bullet \iint_T (x^2 + y^2) \, dx \, dy \, dz, T : \begin{cases} z = 6 - x^2 - y^2 \\ z^2 = x^2 + y^2 \\ z \geq 0 \end{cases}$$

Цилиндрична смяна

$$x = \rho \cos \varphi \quad y = \rho \sin \varphi \quad z = z \implies |\Delta| = \rho$$

$$\begin{cases} z = 6 - x^2 - y^2 \\ z = \sqrt{x^2 + y^2} \end{cases} \implies 6 - x^2 - y^2 = \sqrt{x^2 + y^2}$$

$$\implies 6 - \rho^2 = \rho \implies \rho = -3, \rho > 0, \rho = 2 \implies$$

$$\begin{cases} 0 \leq \rho \leq 2 \\ 0 \leq \varphi \leq 2\pi \end{cases}$$

$$I = \int_0^{2\pi} \int_0^2 \int_{\rho}^{6-\rho^2} \rho^3 \, dz \, d\rho \, d\varphi$$

$$\int_{\rho}^{6-\rho^2} \rho^3 \, dz = 6\rho^3 - \rho^5 - \rho^4$$

$$I = \int_0^{2\pi} \int_0^2 (6\rho^3 - \rho^5 - \rho^4) \, d\rho \, d\varphi$$

$$\int_0^2 (6\rho^3 - \rho^5 - \rho^4) \, d\rho = \left. \frac{6\rho^4}{4} \right|_0^2 - \left. \frac{\rho^6}{6} \right|_0^2 - \left. \frac{\rho^5}{5} \right|_0^2 = \frac{104}{15}$$

$$I = \int_0^{2\pi} \frac{104}{15} \, d\varphi = \frac{104}{15} \int_0^{2\pi} d\varphi = \frac{208\pi}{15}$$

Задача 2

$$R > 0 \quad a > 0 \quad b > 0 \quad c > 0 \quad a, b, c, R = \text{const}$$

$$V_T = \iiint_T dx \, dy \, dz$$

$$\bullet \quad T : x^2 + y^2 + z^2 \leq R^2$$

Сферична смяна

$$x = \rho \cos \varphi \sin \theta \quad y = \rho \sin \varphi \sin \theta \quad z = \rho \cos \theta \quad |\Delta| = \rho^2 \sin \theta$$

$$\rho^2 \cos^2 \varphi \sin^2 \theta + \rho^2 \sin^2 \varphi \sin^2 \theta + \rho^2 \cos^2 \theta \leq R^2$$

$$\rho^2 \sin^2 \theta (\cos^2 \varphi + \sin^2 \varphi) + \rho^2 \cos^2 \theta \leq R^2$$

$$\rho^2 \sin^2 \theta + \rho^2 \cos^2 \theta \leq R^2 \implies \rho^2 \leq R^2 \implies 0 \leq \rho \leq R$$

$$V_T = \int_0^{2\pi} \int_0^R \int_0^\pi \rho^2 \sin \theta \, d\theta \, d\rho \, d\varphi$$

$$\int_0^\pi \rho^2 \sin \theta \, d\theta = \rho^2 \int_0^\pi \sin \theta \, d\theta = \rho^2 (-\cos \theta) \Big|_0^\pi = 2\rho^2$$

$$V_T = \int_0^{2\pi} \int_0^R 2\rho^2 \, d\rho \, d\varphi = 2 \int_0^{2\pi} \int_0^R \rho^2 \, d\rho \, d\varphi$$

$$\int_0^R \rho^2 \, d\rho = \frac{\rho^3}{3} \Big|_0^R = \frac{R^3}{3}$$

$$2 \int_0^{2\pi} \frac{R^3}{3} \, d\varphi = \frac{2R^3}{3} \int_0^{2\pi} d\varphi = \frac{2R^3}{3} \varphi \Big|_0^{2\pi} = \frac{4\pi R^3}{3}$$

$$\bullet \quad T : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1$$

Сферична смяна

$$x = a\rho \cos \varphi \sin \theta \quad y = b\rho \sin \varphi \sin \theta \quad z = c\rho \cos \theta \quad |\Delta| = abc\rho^2 \sin \theta$$

$$\frac{a^2 \rho^2 \cos^2 \varphi \sin^2 \theta}{a^2} + \frac{b^2 \rho^2 \sin^2 \varphi \sin^2 \theta}{b^2} + \frac{c^2 \rho^2 \cos^2 \theta}{c^2} \leq 1$$

$$\rho^2 \cos^2 \varphi \sin^2 \theta + \rho^2 \sin^2 \varphi \sin^2 \theta + \rho^2 \cos^2 \theta \leq 1$$

$$\rho^2 \sin^2 \theta (\cos^2 \varphi + \sin^2 \varphi) + \rho^2 \cos^2 \theta \leq 1$$

$$\rho^2 \sin^2 \theta + \rho^2 \cos^2 \theta \leq R^2 \implies \rho^2 \leq 1 \implies 0 \leq \rho \leq 1$$

$$V_T = \int_0^{2\pi} \int_0^1 \int_0^\pi abc \rho^2 \sin \theta \, d\theta \, d\rho \, d\varphi = abc \int_0^{2\pi} \int_0^1 \int_0^\pi \rho^2 \sin \theta \, d\theta \, d\rho \, d\varphi$$

$$\int_0^\pi \rho^2 \sin \theta \, d\theta = \rho^2 \int_0^\pi \sin \theta \, d\theta = \rho^2 (-\cos \theta) \Big|_0^\pi = 2\rho^2$$

$$V_T = abc \int_0^{2\pi} \int_0^1 2\rho^2 \, d\rho \, d\varphi = 2abc \int_0^{2\pi} \int_0^1 \rho^2 \, d\rho \, d\varphi$$

$$\int_0^1 \rho^2 \, d\rho = \frac{\rho^3}{3} \Big|_0^1 = \frac{1}{3}$$

$$2abc \int_0^{2\pi} \frac{1}{3} \, d\varphi = \frac{2abc}{3} \int_0^{2\pi} d\varphi = \frac{2abc}{3} \varphi \Big|_0^{2\pi} = \frac{4}{3} \pi abc$$

Задача 3

$$S : x^2 + y^2 = 2az, \quad a > 0$$

$$(x^2 + y^2)^2 = 2a^2 xy$$

$$\sigma = \iint_D \sqrt{1 + (z'_x)^2 + (z'_y)^2} \, dx \, dy$$

$$D : (x^2 + y^2)^2 \leq 2a^2 xy$$

$$S : z = \frac{1}{2a^2} (x^2 + y^2) \implies z'_x = \frac{x}{a}, z'_y = \frac{y}{a}$$

$$1 + (z'_x)^2 + (z'_y)^2 = \frac{a^2 + x^2 + y^2}{a^2} \implies \sigma = \frac{4}{a} \iint_D \sqrt{a^2 + x^2 + y^2} \, dx \, dy$$

Полярна смяна

$$x = \rho \cos \varphi \quad y = \rho \sin \varphi \implies$$

$$0 \leq \varphi \leq \frac{\pi}{4} \quad 0 \leq \rho \leq a\sqrt{2 \sin \varphi \cos \varphi} = a\sqrt{\sin 2\varphi}$$

$$\sigma = \frac{4}{a} \int_0^{\frac{\pi}{4}} \int_0^{a\sqrt{\sin 2\varphi}} \sqrt{a^2 + \rho^2} \rho \, d\rho \, d\varphi = \frac{4}{a^2} \int_0^{\frac{\pi}{4}} \int_0^{a\sqrt{\sin 2\varphi}} (a^2 + \rho^2)^{\frac{1}{2}} \, d\rho^2 \, d\varphi =$$

$$\frac{2}{a} \int_0^{\frac{\pi}{4}} \frac{(a^2 + \rho^2)^{\frac{3}{2}}}{\frac{3}{2}} \Big|_0^{a\sqrt{\sin 2\varphi}} \, d\varphi = \frac{4}{3a} \int_0^{\frac{\pi}{4}} (a^2 + a^2 \sin 2\varphi)^{\frac{3}{2}} - (a^2)^{\frac{3}{2}} \, d\varphi =$$

$$\frac{4a^3}{3a} \int_0^{\frac{\pi}{4}} (1 + \sin 2\varphi)^{\frac{3}{2}} - 1 \, d\varphi$$

$$1 + \sin 2\varphi = \sin^2 \varphi + \cos^2 \varphi + 2 \sin \varphi \cos \varphi = (\sin \varphi + \cos \varphi)^2$$

$$\sigma = \frac{4a^2}{3} \int_0^{\frac{\pi}{4}} ((\sin \varphi + \cos \varphi)^2)^{\frac{3}{2}} - 1 \, d\varphi = \frac{4a^2}{3} \int_0^{\frac{\pi}{4}} (\sin \varphi + \cos \varphi)^3 - 1 \, d\varphi =$$

$$\frac{4a^2}{3} \int_0^{\frac{\pi}{4}} \sin^3 \varphi + 3 \sin^2 \varphi \cos \varphi + 3 \sin \varphi \cos^2 \varphi + \cos^3 \varphi - 1 \, d\varphi$$

$$\sigma = \frac{4a^2}{3} \left[\int_0^{\frac{\pi}{4}} (\sin^2 \varphi \sin \varphi + 3 \sin^2 \varphi \cos \varphi) \, d\varphi + \int_0^{\frac{\pi}{4}} (\cos^2 \varphi \cos \varphi + 3 \cos^2 \varphi \sin \varphi - 1) \, d\varphi \right]$$

$$\int_0^{\frac{\pi}{4}} (\sin^2 \varphi \sin \varphi + 3 \sin^2 \varphi \cos \varphi) \, d\varphi = \int_0^{\frac{\pi}{4}} ((1 - \cos^2 \varphi) \sin \varphi + 3 \sin^2 \varphi \cos \varphi) \, d\varphi =$$

$$\int_0^{\frac{\pi}{4}} (\sin \varphi - \cos^2 \varphi \sin \varphi + 3 \sin^2 \varphi \cos \varphi) \, d\varphi$$

$$\int_0^{\frac{\pi}{4}} (\cos^2 \varphi \cos \varphi + 3 \cos^2 \varphi \sin \varphi - 1) \, d\varphi = \int_0^{\frac{\pi}{4}} ((1 - \sin^2 \varphi) \cos \varphi + 3 \cos^2 \varphi \sin \varphi - 1) \, d\varphi =$$

$$\int_0^{\frac{\pi}{4}} (\cos \varphi - \sin^2 \varphi \cos \varphi + 3 \cos^2 \varphi \sin \varphi - 1) \, d\varphi$$

$$\sin \varphi - \cos^2 \varphi \sin \varphi + 3 \sin^2 \varphi \cos \varphi + \cos \varphi - \sin^2 \varphi \cos \varphi + 3 \cos^2 \varphi \sin \varphi - 1 = \\ \sin \varphi + \cos \varphi + 2 \sin^2 \varphi \cos \varphi + 2 \sin \varphi \cos^2 \varphi - 1$$

$$\sigma = \frac{4a^2}{3} \left[\int_0^{\frac{\pi}{4}} \sin \varphi + \cos \varphi + 2 \sin^2 \varphi \cos \varphi + 2 \sin \varphi \cos^2 \varphi - 1 \, d\varphi \right] = \\ \frac{4a^2}{3} \left[\int_0^{\frac{\pi}{4}} \sin \varphi \, d\varphi + \int_0^{\frac{\pi}{4}} \cos \varphi \, d\varphi + 2 \int_0^{\frac{\pi}{4}} \sin^2 \varphi \cos \varphi \, d\varphi + 2 \int_0^{\frac{\pi}{4}} \sin \varphi \cos^2 \varphi \, d\varphi - \int_0^{\frac{\pi}{4}} 1 \, d\varphi \right]$$

$$I_1 = \int_0^{\frac{\pi}{4}} \sin \varphi \, d\varphi = -\cos \varphi \Big|_0^{\frac{\pi}{4}} = -\left(\frac{\sqrt{2}}{2} - 1\right) = 1 - \frac{\sqrt{2}}{2}$$

$$I_2 = \int_0^{\frac{\pi}{4}} \cos \varphi \, d\varphi = \sin \varphi \Big|_0^{\frac{\pi}{4}} = \frac{\sqrt{2}}{2} - 0 = \frac{\sqrt{2}}{2}$$

$$I_3 = \int_0^{\frac{\pi}{4}} \sin^2 \varphi \cos \varphi \, d\varphi = \frac{\sin^3 \varphi}{3} \Big|_0^{\frac{\pi}{4}} = \frac{1}{3} \left[\left(\frac{\sqrt{2}}{2}\right)^3 - 0 \right] = \frac{1}{3} \cdot \frac{2\sqrt{2}}{8} = \frac{\sqrt{2}}{12}$$

$$I_4 = \int_0^{\frac{\pi}{4}} \sin \varphi \cos^2 \varphi \, d\varphi = -\frac{2 \cos^3 \varphi}{3} \Big|_0^{\frac{\pi}{4}} = -\frac{1}{3} \left[\left(\frac{\sqrt{2}}{2}\right)^3 - 1 \right] = -\frac{\sqrt{2}}{12} + \frac{1}{3} = \frac{1}{3} - \frac{\sqrt{2}}{12}$$

$$I_5 = \int_0^{\frac{\pi}{4}} 1 \, d\varphi = \varphi \Big|_0^{\frac{\pi}{4}} = \frac{\pi}{4}$$

$$\sigma = \frac{4a^2}{3} \left[1 - \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} + \frac{2\sqrt{2}}{12} + \frac{2}{3} - \frac{2\sqrt{2}}{12} - \frac{\pi}{4} \right] = \frac{4a^2}{3} \left[1 + \frac{2}{3} - \frac{\pi}{4} \right]$$

$$\sigma = \frac{4a^2}{3} \left[\frac{5}{3} - \frac{\pi}{4} \right] = \frac{a^2}{9} [20 - 3\pi]$$

13 Упражнение към лекция 13

13.1 Задачи

Задача 1

Да се пресметне

$$I = \oint_{\Gamma} (x + y) \, ds$$

където Γ е затворена начупена линия $OABC$, $O(0, 0)$, $A(1, 0)$, $B(0, 1)$.

Задача 2

Да се пресметне

$$I = \oint_C \sqrt{x^2 + y^2} \, ds$$

където

$$C \equiv \{x = a(\cos t + t \sin t), y = a(\sin t - t \cos t), 0 \leq t \leq 2\pi\}$$

Задача 3

Да се пресметне

$$I = \int_C \frac{1}{x + y} \, ds$$

ако интегрирането е извършено по отреза на правата, свързваща точките $A(2, 4)$ и $B(1, 3)$

Задача 4

Да се пресметне чрез криволинеен интеграл от първи род лицето на цилиндрична повърхнина $y^2 = \frac{4}{9}(x-1)^3$ ограничена отдолу от равнината $z = 0$, а отгоре от повърхнината $z = 2 - \sqrt{x}$.

Задача 5

Да се намери стойността на

$$I = \int_C (x^2 + y^2 + z^2) \, ds$$

където C е част от винтовата линия с параметрични уравнения

$$x = a \cos t \quad y = a \sin t \quad z = bt \quad 0 \leq t \leq 2\pi \quad a > 0, b > 0$$

Задача 6

Да се пресметне

$$\oint_C (x^2 - y^2) dx + (x^2 + y^2) dy$$

в положителна посока на описване на елипсата

$$\frac{x^2}{a^2} + \frac{y^2}{b} = 1$$

започвайки от точката $A(a, 0)$, $a > 0, b > 0$.

Задача 7

1. Да се покаже, че ако пътят на интегриране не пресича ординатната ос, то интегралът

$$\int_{(1,2)}^{(2,1)} \frac{y dx - x dy}{x^2}$$

не зависи от пътя на интегриране, и да се пресметне.

2. Да се пресметне стойността на интеграла от същата функция по затворен контур, непресичащ ординатната ос.

13.2 Решения

Задача 1

$$I = \oint_{\Gamma} (x + y) \, ds \quad \Gamma = AOBO \quad O(0,0), A(1,0), B(0,1)$$

$$\Gamma = OA + AB + BO \implies I = \int_{OA} (x + y) \, ds + \int_{AB} (x + y) \, ds + \int_{OB} (x + y) \, ds$$

$$I_1 = \int_{OA} (x + y) \, ds$$

$$I_1 : \begin{cases} y(t) = 0 \\ x(t) = t \\ 0 \leq t \leq 1 \end{cases} \implies y'(t) = 0 \quad x'(t) = 1 \implies ds = \sqrt{0^2 + 1^2} dt = \sqrt{1} dt$$

$$I_1 = \int_0^1 (t + 0) \sqrt{1} \, dt = \int_0^1 t \, dt = \left. \frac{t^2}{2} \right|_0^1 = \frac{1}{2}$$

$$I_2 = \int_{AB} (x + y) \, ds$$

$$I_2 : AB : x + y = 1 \Leftrightarrow AB : y = 1 - x, 0 \leq x \leq 1 \\ \implies y' = -1 \implies ds = \sqrt{(-1)^2 + 1^2} dx = \sqrt{2} dx$$

$$I_2 = \sqrt{2} \int_0^1 dx = \sqrt{2}$$

$$I_3 = \int_{OB} (x + y) \, ds$$

$$I_3 : OB : \begin{cases} x(t) = 0 \\ y(t) = t \end{cases} \implies x'(t) = 0 \quad y'(t) = 1 \implies ds = \sqrt{0^2 + 1^2} dt = \sqrt{1} dt$$

$$I_3 = \int_0^1 (0 + t) \sqrt{1} \, dt = \int_0^1 t \, dt = \left. \frac{t^2}{2} \right|_0^1 = \frac{1}{2}$$

$$I = \frac{1}{2} + \sqrt{2} + \frac{1}{2} = \sqrt{2} + 1$$

Задача 2

$$\begin{aligned}
I &= \oint_C \sqrt{x^2 + y^2} \, ds \quad C \equiv \{x = a(\cos t + t \sin t), y = a(\sin t - t \cos t), 0 \leq t \leq 2\pi\} \\
x' &= a(-\sin t + \sin t + t \cos t) \implies x' = at \cos t \\
y' &= a(-\cos t + \cos t + t \sin t) \implies y' = at \sin t \\
ds &= \sqrt{x'^2 + y'^2} dt = \sqrt{a^2 t^2 \cos^2 t + a^2 t^2 \sin^2 t} dt = \\
&= \sqrt{a^2 t^2 (\cos^2 t + \sin^2 t)} dt = \sqrt{a^2 t^2} dt = at dt \\
x^2 + y^2 &= a^2 (\cos t + t \sin t)^2 + a^2 (\sin t - t \cos t)^2 = \\
&= a^2 [\cos^2 t + t^2 \sin^2 t + 2t \sin t \cos t + \sin^2 t + t^2 \cos^2 t - 2t \sin t \cos t] = \\
&= a^2 [(\cos^2 t + \sin^2 t) + t^2 (\cos^2 t + \sin^2 t)] = a^2 [1 + t^2] \implies \\
\sqrt{x^2 + y^2} &= a(1 + t^2)^{\frac{1}{2}} \\
I &= \int_0^{2\pi} a(1 + t^2)^{\frac{1}{2}} at \, dt = a^2 \int_0^{2\pi} (1 + t^2)^{\frac{1}{2}} t \, dt = a^2 \frac{1}{2} \int_0^{2\pi} (1 + t^2)^{\frac{1}{2}} d(t^2 + 1) = \\
&= \frac{a^2}{2} \cdot \frac{(1 + t^2)^{\frac{3}{2}}}{\frac{3}{2}} \Big|_0^{2\pi} = \frac{a^2}{3} \left[(1 + 4\pi^2)^{\frac{3}{2}} - 1 \right]
\end{aligned}$$

Задача 3

$$\begin{aligned}
I &= \int_C \frac{1}{x+y} \, ds \quad C = AB \quad A(2, 4), B(1, 3) \\
y - y_1 &= k(x - x_1) \implies k = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4 - 3}{2 - 1} = 1 \implies \\
AB : y - 3 &= 1(x - 1); \quad x - 1 \implies y = 3 \quad x = 2 \implies y = 4 \implies \\
AB : y &= x + 2; 1 \leq x \leq 2 \\
ds &= \sqrt{1 + y'^2} dx = \sqrt{1 + 1} dx = \sqrt{2} dx \\
x + y &= x + x + 2 = 2x + 2 = 2(x + 1) \\
I &= \int_1^2 \frac{\sqrt{2}}{2(x+y)} \, dx = \frac{\sqrt{2}}{2} \int_1^2 \frac{1}{x+1} \, dx = \frac{\sqrt{2}}{2} \int_1^2 \frac{1}{x+1} \, d(x+1) = \frac{\sqrt{2}}{2} \ln |x+1| \Big|_1^2 = \\
&= \frac{\sqrt{2}}{2} (\ln 3 - \ln 2) = \frac{\sqrt{2}}{2} \ln \left(\frac{3}{2} \right)
\end{aligned}$$

Задача 4

$$\begin{aligned} \sigma &= \int_C z \, dl \quad 2 - \sqrt{x}, z = 0 \quad C : y^2 = \frac{4}{9}(x-1)^3, x \geq 1 \\ y &= \pm \frac{2}{3}(x-1)^{\frac{3}{2}} \implies y' = \pm \frac{2}{3} \cdot \frac{3}{2}(x-1)^{\frac{1}{2}} = y' = \pm(x-1)^{\frac{1}{2}} \implies y'^2 = x-1 \\ dl &= \sqrt{1+x-1} dx = \sqrt{x} dx \\ 0 \leq z &\leq 2 - \sqrt{x} \implies 2 - \sqrt{x} \geq 0 \implies x \leq 4 \implies 1 \leq x \leq 4 \\ \sigma &= 2 \int_1^4 (2 - \sqrt{x}) \sqrt{x} \, dx = 2 \int_1^4 (2\sqrt{x} - x) \, dx = \\ &2 \left[2 \cdot \frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^2}{2} \right] \Big|_1^4 = 2 \left[\frac{4}{3} x^{\frac{3}{2}} - \frac{x^2}{2} \right] \Big|_1^4 = 2 \left[\frac{4}{3} (\sqrt{4^3} - 1) - \frac{1}{2} (16 - 1) \right] = \\ &2 \left[\frac{4}{3} (7) - \frac{1}{2} (15) \right] = 2 \left[\frac{28}{3} - \frac{15}{2} \right] = 2 \left[\frac{56 - 45}{6} \right] = \frac{11}{3} \end{aligned}$$

Задача 5

$$\begin{aligned} I &= \int_C (x^2 + y^2 + z^2) \, ds \\ x &= a \cos t \quad y = a \sin t \quad z = bt \quad 0 \leq t \leq 2\pi \quad a > 0, b > 0 \\ x' &= -a \sin t \quad y' = a \cos t \quad z' = b \implies \\ x'^2 + y'^2 + z'^2 &= a^2 \sin^2 t + a^2 \cos^2 t + b^2 = a^2 + b^2 \implies dl = \sqrt{a^2 + b^2} dt \\ x^2 + y^2 + z^2 &= a^2 \cos^2 t + a^2 \sin^2 t + b^2 t^2 = a^2 + b^2 t^2 \\ I &= \int_0^{2\pi} (a^2 + b^2 t^2) \sqrt{a^2 + b^2} \, dt = \sqrt{a^2 + b^2} \int_0^{2\pi} (a^2 + b^2 t^2) \, dt = \\ &\sqrt{a^2 + b^2} \left[a^2 t + \frac{b^2 t^3}{3} \right] \Big|_0^{2\pi} = \sqrt{a^2 + b^2} \left[a^2 2\pi + \frac{b^2 (2\pi)^3}{3} \right] = \frac{\sqrt{a^2 + b^2}}{3} [a^2 2 \cdot 3\pi + 8\pi^3 b^2] = \\ &\frac{\sqrt{a^2 + b^2}}{3} [6\pi a^2 + 8\pi^3 b^2] = \frac{2\pi}{3} \sqrt{a^2 + b^2} [3a^2 + 4\pi^2 b^2] = \end{aligned}$$

Задача 6

$$\oint_C (x^2 - y^2) dx + (x^2 + y^2) dy \quad \frac{x^2}{a^2} + \frac{y^2}{b} = 1$$

$$C : x = a \cos t \quad y = b \sin t \quad 0 \leq t \leq 2\pi$$

$$x^2 - y^2 = a^2 \cos^2 t - b^2 \sin^2 t \quad dx = -a \sin t dt$$

$$(x^2 - y^2)dx = (a^2 \cos^2 t - b^2 \sin^2 t)(-a \sin t)dt$$

$$x^2 + y^2 = a^2 \cos^2 t + b^2 \sin^2 t \quad dy = b \cos t dt$$

$$(x^2 + y^2)dy = (a^2 \cos^2 t + b^2 \sin^2 t)(b \cos t)dt$$

$$I = \int_0^{2\pi} (a^2 \cos^2 t - b^2 \sin^2 t)(-a \sin t) dt + (a^2 \cos^2 t + b^2 \sin^2 t)(b \cos t) dt$$

$$I = \int_0^{2\pi} -a^3 \cos^2 t + ab^2 \sin^3 t dt + a^2 b \cos^3 t + b^3 \sin^2 t dt$$

$$I = -a^3 \int_0^{2\pi} \cos^2 t \sin t dt + ab^2 \int_0^{2\pi} \sin^3 t dt + a^2 b \int_0^{2\pi} \cos^3 t dt + b^3 \int_0^{2\pi} \sin^2 t \cos t dt$$

$$I = -a^3 I_1 + ab^2 I_2 + a^2 b I_3 + b^3 I_4$$

$$I_1 = \int_0^{2\pi} \cos^2 t \sin t dt = - \int_0^{2\pi} \cos^2 t d(\cos t) = - \frac{\cos^3 t}{3} \Big|_0^{2\pi} = 0$$

$$I_2 = \int_0^{2\pi} \sin^3 t dt = \int_0^{2\pi} (1 - \cos^2 t) d(\cos t) = 0$$

$$I_3 = \int_0^{2\pi} \cos^3 t dt = - \int_0^{2\pi} (1 - \sin^2 t) d(\sin t) = 0$$

$$I_4 = \int_0^{2\pi} \sin^2 t \cos t dt = \int_0^{2\pi} \sin^2 t d(\sin t) = \frac{\sin^3 t}{3} \Big|_0^{2\pi} = 0$$

$$I = 0$$

Задача 7

1.

$$\int_{(1,2)}^{(2,1)} \frac{y \, dx - x \, dy}{x^2}$$

Кривата не пресича Oy

$$P(x) = \frac{y}{x^2} \quad Q(x) = -\frac{1}{x} \implies x \neq 0$$

$$\frac{\partial P}{\partial y} = \frac{1}{x^2} \quad \frac{\partial Q}{\partial x} = \frac{1}{x^2} \implies \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

1) Интегралът I не зависи от пътя на интегриране

$$2) \exists u = u(x, y) : u'_x = \frac{y}{x^2} \quad u'_y = -\frac{1}{x}$$

$$u'_x = \frac{y}{x^2} \implies u(x, y) = \int \frac{y}{x^2} \, dx \implies$$

$$u(x, y) = y \int \frac{1}{x^2} \, dx = y \int x^{-2} \, dx = y \frac{x^{-1}}{-1} + \varphi(y) = -\frac{y}{x} + \varphi(y)$$

$$u'_y = -\frac{1}{x} \quad u'_y = -\frac{1}{x} + \varphi'(y) \implies -\frac{1}{x} = -\frac{1}{x} + \varphi'(y) \implies$$

$$\varphi'(y) = 0 \implies \varphi(y) = \text{const} = c \implies$$

$$u(x, y) = -\frac{y}{x} + cI = \int_{(1,2)}^{(2,1)} \frac{y \, dx - x \, dy}{x^2} = -\frac{y}{x} \Big|_{(1,2)}^{(2,1)} \implies$$

$$I = -\left[\frac{2}{1} - \frac{1}{2}\right] = -\frac{3}{2}$$

2.

$$\int_{\gamma} P \, dx + Q \, dy = ?$$

$$\gamma_1 = ACB \quad \gamma_2 = ADB \quad \gamma_2^- = BDA$$

$$\int_{\gamma_1} P \, dx + Q \, dy = \int_{\gamma_2} P \, dx + Q \, dy \implies$$

$$\int_{\gamma_1} P \, dx + Q \, dy - \int_{\gamma_2} P \, dx + Q \, dy = 0 \implies$$

$$\int_{\gamma_1} P \, dx + Q \, dy + \int_{\gamma_2^-} P \, dx + Q \, dy = 0 \implies$$

$$\int_{ACBDA} P \, dx + Q \, dy = 0$$

14 Примерен изпит

14.1 Задачи

Задача 1

Да се провери изпълнено ли е равенството, за такива стойности на независимите променливи, за които частните производни на функцията z съществува

$$\frac{1}{x}z'_x + \frac{1}{y}z'_y = \frac{z}{x^2} \quad z = 2x \arcsin(y^2 - x^2)$$

Задача 2

Дадени за функциите и точката

$$f(x, y) = x^4 + y^4 - 14xy - 7x^2 - 7y^2 + 24x + 24y \quad F(x, y, z) = f(x, y) + e^{2x^2 + 3y^2 + 4z^2} \quad A(1, 1, 1)$$

Да се намери

1. Локалните екстремуми на f и вида им.
2. $g = \text{grad}F(A)$, $\|g\|$, $\frac{\partial F(A)}{\partial g}$.

Задача 3

Нека y е неявна функция, която се дефинира с уравнението $f(x, y) = 0$, където

$$f(x, y) = \sin(x - y) + e^{xy} - 1 \quad y(0) = 0$$

Да се намери $y'(0)$.

Задача 4

Да се пресметне интеграла

$$\iint_D (6x^2 + y) \, dx \, dy$$

където D е оградена от кривите с уравнение $y = 2x$ и $y = x^2 - x$.

Задача 5

Пресметнете криволинейния интеграл от първи род

$$\int_C (x^2 + y^2 - 2z) \, dl \quad C : \{x = 4 \cos 2t, y = 4 \sin 2t, z = 6t, t \in [0, 3\pi]\}$$

Задача 6

Да се пресметне криволинейния интеграл от втори род

$$\int_C 2x \, dx + 3yx \, dy \quad C = \{x = 4 \cos 2t, y = 3 \sin 2t, t \in [0, 2\pi]\}$$

1. непосредствено
2. с формулата на Грийн.

Да се сравнят резултатите.

Задача 7

Да се намери лицето на частта от повърхнината

$$S : z = -\sqrt{x^2 + y^2}$$

заградена от цилиндъра

$$x^2 + y^2 = 36$$

Задача 8

Намерете обема на тялото

$$T = \{z \leq 2 - x^2 - y^2, z^2 \geq x^2 + y^2 \quad (z \geq 0)\}$$

14.2 Решения

Задача 1

$$\begin{aligned}
\frac{1}{x}z'_x + \frac{1}{y}z'_y &= \frac{z}{x^2} & z &= 2x \arcsin(y^2 - x^2) \\
z'_x &= 2 \arcsin(y^2 - x^2) + \frac{2x}{\sqrt{1 + (y^2 - x^2)^2}} \cdot -2x = 2 \arcsin(y^2 - x^2) - \frac{4x^2}{\sqrt{1 + (y^2 - x^2)^2}} = \\
z'_x &= \frac{2\sqrt{1 + (y^2 - x^2)^2} \arcsin(y^2 - x^2) - 4x^2}{\sqrt{1 + (y^2 - x^2)^2}} \\
z'_y &= 0 \arcsin(y^2 - x^2) + \frac{2x}{\sqrt{1 + (y^2 - x^2)^2}} \cdot 2y = \frac{4xy}{\sqrt{1 + (y^2 - x^2)^2}} \\
\frac{1}{x}z'_x &= \frac{\frac{2\sqrt{1 + (y^2 - x^2)^2} \arcsin(y^2 - x^2) - 4x^2}{\sqrt{1 + (y^2 - x^2)^2}}}{x} = \frac{2\sqrt{1 + (y^2 - x^2)^2} \arcsin(y^2 - x^2) - 4x^2}{x\sqrt{1 + (y^2 - x^2)^2}} \cdot \frac{1}{x} = \\
&= \frac{2\sqrt{1 + (y^2 - x^2)^2} \arcsin(y^2 - x^2) - 4x^2}{x\sqrt{1 + (y^2 - x^2)^2}} \\
\frac{1}{y}z'_y &= \frac{1}{y} \cdot \frac{4xy}{\sqrt{1 + (y^2 - x^2)^2}} = \frac{4x}{\sqrt{1 + (y^2 - x^2)^2}} \\
\frac{1}{x}z'_x + \frac{1}{y}z'_y &= \frac{2\sqrt{1 + (y^2 - x^2)^2} \arcsin(y^2 - x^2) - 4x^2}{x\sqrt{1 + (y^2 - x^2)^2}} + \frac{4x}{\sqrt{1 + (y^2 - x^2)^2}} = \\
&= \frac{2\sqrt{1 + (y^2 - x^2)^2} \arcsin(y^2 - x^2) - 4x^2 + 4x^2}{x\sqrt{1 + (y^2 - x^2)^2}} = \frac{2\sqrt{1 + (y^2 - x^2)^2} \arcsin(y^2 - x^2)}{x\sqrt{1 + (y^2 - x^2)^2}} \\
\frac{1}{x}z'_x + \frac{1}{y}z'_y &= \frac{2 \arcsin(y^2 - x^2)}{x} \\
\frac{z}{x^2} &= \frac{2x \arcsin(y^2 - x^2)}{x^2} = \frac{2 \arcsin(y^2 - x^2)}{x} \implies \text{Равенството е изпълнено}
\end{aligned}$$

Задача 2

$$f(x, y) = x^4 + y^4 - 14xy - 7x^2 - 7y^2 + 24x + 24y \quad F(x, y, z) = f(x, y) + e^{2x^2 + 3y^2 + 4z^2} \quad A(1, 1, 1)$$

1. Локалните екстремуми на f и вида им.

$$\begin{aligned}
 f'_x &= 4x^3 - 14x - 14y + 24 & f'_y &= 4y^3 - 14x - 14y + 24 \\
 \left| \begin{aligned} 4x^3 - 14x - 14y + 24 &= 0 \\ 4y^3 - 14x - 14y + 24 &= 0 \end{aligned} \right. &\Leftrightarrow \left| \begin{aligned} 4x^3 - 4y^3 &= 0 \\ 4y^3 - 14x - 14y + 24 &= 0 \end{aligned} \right. \\
 4x^3 - 4y^3 &= 0 \Leftrightarrow x^3 - y^3 = 0 \Leftrightarrow (x - y)(x^2 + xy + y^2) = 0 \\
 f''_{xx} &= 12x^2 - 14 & f''_{yy} &= 12y^2 - 14 & f''_{xy} &= f''_{yx} = -14 \\
 \Delta &= f''_{xx}f''_{yy} - f''_{xy}f''_{yx} = (12x^2 - 14)(12y^2 - 14) - 196
 \end{aligned}$$

2. $g = \text{grad}F(A), \|g\|, \frac{\partial F(A)}{\partial g}$.

Задача 3

Задача 4

$$I = \iint_D (6x^2 + y) \, dx \, dy \quad D : \begin{cases} y = 2x \\ y = x^2 - x \end{cases}$$

$$\left| \begin{aligned} y &= 2x \\ y &= x^2 - x \end{aligned} \right. \Leftrightarrow 2x = x^2 - x \Leftrightarrow x^2 - 3x = 0 \implies 0 \leq x \leq 3 \implies \left| \begin{aligned} 0 &\leq x \leq 3 \\ x^2 - x &\leq y \leq 2x \end{aligned} \right.$$

$$I = \int_0^3 \int_{x^2-x}^{2x} (6x^2 + y) \, dy \, dx$$

$$\int_{x^2-x}^{2x} (6x^2 + y) \, dy = 6x^2y + \frac{1}{2}y^2 \Big|_{x^2-x}^{2x} =$$

$$\begin{aligned}
 6x^2 \cdot 2x + \frac{4x^2}{2} - 6x^2(x^2 - x) - \frac{(x^2 - x)^2}{2} &= 12x^3 + 2x^2 - 6x^4 + 6x^3 - \frac{x^4 - 2x^3 + x^2}{2} = \\
 \frac{24x^3 + 4x^2 - 12x^4 + 12x^3 - x^4 + 2x^3 - x^2}{2} &= \frac{-13x^4 + 38x^3 + 3x^2}{2} = -\frac{13}{2}x^4 + 19x^3 + \frac{3}{2}x^2
 \end{aligned}$$

$$I = \int_0^3 \left(-\frac{13}{2}x^4 + 19x^3 + \frac{3}{2}x^2 \right) dx = -\frac{13x^5}{2 \cdot 5} + \frac{19x^4}{4} + \frac{3x^3}{2 \cdot 3} = -\frac{13}{10}x^5 + \frac{19}{4}x^4 + \frac{1}{2}x^3 \Big|_0^3$$

$$I = \frac{(-13 \cdot 2 \cdot 3^5) + (19 \cdot 5 \cdot 3^4) + (1 \cdot 10 \cdot 3^3)}{20} = \frac{1647}{20}$$

Задача 5

$$I = \int_C (x^2 + y^2 - 2z) \, dl \quad C : \{x = 4 \cos 2t, y = 4 \sin 2t, z = 6t, t \in [0, 3\pi]\}$$

$$I = \int_C f(x, y, z) \, dl \quad C : \{x = \varphi(t), y = \psi(t), z = \chi(t), t \in [t_1, t_2]\} \implies$$

$$I = \int_{t_1}^{t_2} f(\varphi(t), \psi(t), \chi(t)) \sqrt{\varphi'^2(t) + \psi'^2(t) + \chi'^2(t)} \, dt$$

$$\varphi' = -8 \sin 2t \quad \psi' = 8 \cos 2t \quad \chi' = 6$$

$$\varphi'^2 + \psi'^2 + \chi'^2 = 64 \sin^2 2t + 64 \cos^2 2t + 36 = 64(\sin^2 2t + \cos^2 2t) + 36 = 64 + 36 = 100$$

$$\sqrt{\varphi'^2 + \psi'^2 + \chi'^2} = 10$$

$$dl = \sqrt{\varphi'^2 + \psi'^2 + \chi'^2} dt = 10 dt$$

$$x^2 + y^2 - 2z = 16 \cos^2 2t + 16 \sin^2 2t - 12t = 16(\cos^2 2t + \sin^2 2t) - 12t = 16 - 12t$$

$$I = \int_0^{3\pi} (16 - 12t) \cdot 10 \, dt = \int_0^{3\pi} 160 - 120t \, dt = 160t - 60t^2 \Big|_0^{3\pi} = 160 \cdot 3\pi - 60 \cdot 9\pi^2 = 480\pi - 540\pi^2$$

Задача 6

$$I = \int_C 2x \, dx + 3yx \, dy \quad C = \{x = 4 \cos 2t, y = 3 \sin 2t, t \in [0, 2\pi]\}$$

1. непосредствено

$$\begin{aligned}
 P(t) &= 8 \cos 2t \quad Q(t) = 36 \cos 2t \sin 2t \quad x' = -8 \sin 2t \quad y' = 6 \cos 2t \implies \\
 I &= \int_0^{2\pi} P x' + Q y' dt = \int_0^{2\pi} -64 \cos 2t \sin 2t + 216 \cos^2 2t \sin 2t dt = \\
 &= \int_0^{2\pi} -64 \cos 2t \sin 2t dt + \int_0^{2\pi} 216 \cos^2 2t \sin 2t dt = \\
 &= -\frac{64}{2} \int_0^{2\pi} \cos 2t d(\cos 2t) + \frac{216}{2} \int_0^{2\pi} \cos^2 2t d(\cos 2t) = \\
 &= -32 \left(-\frac{\cos^2 2t}{2} \right) + 108 \left(-\frac{\cos^3 2t}{3} \right) \Big|_0^{2\pi} = \\
 &= 16 \cos^2 4\pi - 36 \cos^3 4\pi - 16 \cos^2 0 - 36 \cos^3 0 = 0
 \end{aligned}$$

2. с формулата на Грийн.

$$\begin{aligned}
 P &= 2x \quad Q = 3yx \\
 \frac{\partial P}{\partial y} &= P'_y = 0 \quad \frac{\partial Q}{\partial x} = Q'_x = 3y \\
 I &= \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy = \iint_D (Q'_x - P'_y) dx dy = \iint_D 3y dx dy
 \end{aligned}$$

Да се сравнят резултатите.

Задача 7

Да се намери лицето на частта от повърхнината

$$S: z = -\sqrt{x^2 + y^2}$$

заградена от цилиндъра

$$x^2 + y^2 = 36$$

$$\sigma = \iint_C \sqrt{1 + z_x'^2 + z_y'^2} \, dx \, dy \quad C : \{x^2 + y^2 = 36\}$$

$$z'_x = -\frac{x}{\sqrt{x^2 + y^2}} \quad z'_y = -\frac{y}{\sqrt{x^2 + y^2}}$$

$$1 + z_x'^2 + z_y'^2 = 1 + \frac{x^2}{x^2 + y^2} + \frac{y^2}{x^2 + y^2} = \frac{x^2 + y^2 + x^2 + y^2}{x^2 + y^2} = \frac{2(x^2 + y^2)}{x^2 + y^2} = 2$$

$$\sqrt{1 + z_x'^2 + z_y'^2} = \sqrt{2}$$

$$\sigma = \iint_C \sqrt{2} \, dx \, dy$$

Полярна смяна: $x = a \cos(t) \quad y = a \sin t \quad |\Delta| = a \quad a \geq 0, t \in [0, 2\pi]$

$$C = a^2 \cos^2 t + a^2 \sin^2 t = 36 \Leftrightarrow a^2 = 36 \implies a \in [0, 6]$$

$$\sigma = \int_0^{2\pi} \int_0^6 \sqrt{2} a \, da \, dt = \int_0^{2\pi} \sqrt{2} a \Big|_0^6 \, dt = \sqrt{2} \frac{a^2}{2} \Big|_0^6 = \frac{36}{2} \sqrt{2} = 18\sqrt{2}$$

$$\sigma = 18\sqrt{2} \int_0^{2\pi} dt = 18\sqrt{2} \cdot 2\pi = 36\sqrt{2}\pi$$

Задача 8

Намерете обема на тялото

$$T : z \leq 2 - x^2 - y^2, z^2 \geq x^2 + y^2 \quad (z \geq 0)$$

$$V_T = \iiint_T dx \, dy \, dz$$

$$\begin{cases} z \leq 2 - x^2 - y^2 \\ z^2 \geq x^2 + y^2 \\ z \geq 0 \end{cases}$$

Цилиндрична смяна

$$x = t \cos a \quad y = t \sin a \quad z = z \quad (t \geq 0, a \in [0, 2\pi])$$

$$\implies |\Delta| = t$$

$$2 - x^2 - y^2 = \sqrt{x^2 + y^2} \Leftrightarrow 2 - t^2 = t$$

$$t^2 + t - 2 = 0 \Leftrightarrow (t + 2)(t - 1) = 0 \implies t = 1 \implies t \in [0, 1] \quad z \in [0, 2 - t^2]$$

$$\int_0^{2\pi} \int_0^1 \int_0^{2-t^2} t \, dz \, dt \, da = \int_0^{2\pi} \int_0^1 t(2 - t^2) \, dt \, da = \int_0^{2\pi} \left. \frac{2t^2}{2} - \frac{t^4}{4} \right|_0^1 da =$$

$$\int_0^{2\pi} \left. \frac{3}{4} \right|_0^1 da = \frac{3 \cdot 2\pi}{4} = \frac{3}{2}\pi$$

2ра опция

$$z^2 \geq x^2 + y^2 \implies z \geq \pm \sqrt{x^2 + y^2}, z \geq 0 \implies z \geq \sqrt{x^2 + y^2} \implies$$

$$\sqrt{x^2 + y^2} \leq z \leq 2 - x^2 - y^2$$

$$D : \{(x, y), \sqrt{x^2 + y^2} \leq 2 - x^2 - y^2\}$$

Цилиндрична смяна

$$x = t \cos a \quad y = t \sin a \quad z = z \quad (t \geq 0, a \in [0, 2\pi]) \implies |\Delta| = t$$

$$2 - x^2 - y^2 \leq \sqrt{x^2 + y^2} \Leftrightarrow 2 - t^2 = t$$

$$t^2 + t - 2 = 0 \Leftrightarrow (t + 2)(t - 1) = 0 \implies t = 1 \implies t \in [0, 1] \quad z \in [t, 2 - t^2]$$

$$\int_0^{2\pi} \int_0^1 \int_t^{2-t^2} t \, dz \, dt \, da = \int_0^{2\pi} \int_0^1 t(2 - t^2 - t) \, dt \, da = \int_0^{2\pi} \left. \frac{2t^2}{2} - \frac{t^4}{4} - \frac{t^3}{3} \right|_0^1 da =$$

$$\int_0^{2\pi} \left. \frac{5}{12} \right|_0^1 da = \frac{5 \cdot 2\pi}{12} = \frac{5}{6}\pi$$