Techincal University - Sofia
Faculty of Applied Mathematics and Informatics

Project 1 - Topics of Algebra

Solution for version 4

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Project 1 - Topics of Algebra

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Problem 1. Let be given the following real matrix:

$$A = \begin{pmatrix} 1 & -3 & -1 \\ 3 & 7 & 5 \\ 2 & -2 & 4 \end{pmatrix}$$

Then

a) find its **LU** decomposition;

b) explain rigorously what is a **LU** decomposition.

Problem 2. Let be given the following real matrix:

$$A = \begin{bmatrix} 5 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & 2 & 1 \\ 7 & 4 & 2 \end{bmatrix}$$

Then

a) find its **QR** decomposition and **LQ** decomposition;

b) explain rigorously what is a **QR** decomposition and a **LQ** decomposition.

Problem 3. Let be given the following real matrix:

$$A = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 2 \end{pmatrix}$$

Then

a) prove by pivoting that A is positive defined;

b) find its **Cholesky-Banachiewicz** and **LDL**^{T} decompositions.

Problem 4. Let be given the following real matrix:

$$A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix}$$

Then

a) find its **SVD** decomposition;

b) explain rigorously what is a **SVD** decomposition.

1 Problem 1

$$A = \begin{pmatrix} 1 & -3 & -1 \\ 3 & 7 & 5 \\ 2 & -2 & 4 \end{pmatrix}$$

1.1 Solution for 1a

Lets start with L being the identity matrix

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We perform the following operations on matrix A:

$$\left| \begin{array}{l} R_2 = R_2 - 3R_1 \\ R_3 = R_3 - 2R_1 \end{array} \right.$$

Where R_i is the ith row of the matrix.
We write the coefficient 3 in the matrix L at row 2 and column 1. The same can be done for the coefficient 2 for row 3 and column 1.

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \qquad A = \begin{pmatrix} 1 & -3 & -1 \\ 0 & 16 & 8 \\ 0 & 4 & 6 \end{pmatrix}$$

We perform the following operation on matrix A

$$R_3 = -\frac{1}{4}R_2 + R_3$$

We write the coefficient $\frac{1}{4}$ in matrix L at row 3 and column 2.

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & \frac{1}{4} & 1 \end{pmatrix} \qquad A = \begin{pmatrix} 1 & -3 & -1 \\ 0 & 16 & 8 \\ 0 & 0 & 4 \end{pmatrix} = U$$

1.2 Solution for 1b

LU decomposition is a technique for factorizing a square matrix A as a product of two other matrices.

$$A = LU$$

where L is a lower triangular matrix and U is upper triangular matrix.
To have LU decomposition the matrix A must be square matrix ($m \times m$ dimensions) and invertible (A^{-1} exists).
The algorithm used to solve this particular matrix is LU without pivoting. In this algorithm we initialize L to be identity matrix and we modify A to be an upper triangular matrix like U by performing row operations to create zeroes below the diagonal for each column. Then each multiplier/coefficient is recorded in L for the coresponding row and column.
The most common use for LU decomposition is to solve system of linear equations.

$$Ax = b$$

For the system we can substitute $A = LU$ and we can let $y = Ux$. Then the system becomes $Ly = b$ and after solving it we can solve the other system $Ux = y$

2 Problem 2

$$A = \begin{bmatrix} 5 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & 2 & 1 \\ 7 & 4 & 2 \end{bmatrix}$$

2.1 Solution for 2a

First we perform the Grand-Schmidt process for each column of the matrix

$$v_1 = \begin{bmatrix} 5 \\ 2 \\ 3 \\ 7 \end{bmatrix} \quad v_2 = \begin{bmatrix} 1 \\ -3 \\ 2 \\ 4 \end{bmatrix} \quad v_3 = \begin{bmatrix} -1 \\ 4 \\ 1 \\ 2 \end{bmatrix}$$

We determine the orthogonal vectors e_k by the following formula

$$e_k = \frac{u_k}{||u_k||} \text{ where } \quad u_k = v_k - \sum_{j=1}^{k} proj_{u_j} v_k \quad ||u_k|| = \sqrt{\sum_{j=1}^{n} |u_i|^2} \quad proj_a(b) = \frac{a \cdot b}{a \cdot a} a$$

We solve for each column

$$u_1 = v_1 = \begin{bmatrix} 5 \\ 2 \\ 3 \\ 7 \end{bmatrix} \implies e_1 = \frac{u_1}{||u_1||} = \frac{u_1}{\sqrt{5^2 + 2^2 + 3^2 + 7^2}} = \frac{u_1}{\sqrt{87}} = \begin{bmatrix} \frac{5\sqrt{87}}{87} \\ \frac{2\sqrt{87}}{87} \\ \frac{3\sqrt{87}}{87} \\ \frac{7\sqrt{87}}{87} \end{bmatrix} \approx \begin{bmatrix} 0.54 \\ 0.21 \\ 0.32 \\ 0.75 \end{bmatrix}$$

$$u_2 = v_2 - proj_{u_1}(v_2) = v_2 - \frac{u_1 \cdot v_2}{u_1 \cdot u_1} u_1 = v_2 - \frac{(1 \cdot 5) + (-3) \cdot 2 + (2 \cdot 3) + (4 \cdot 7)}{\sqrt{87}^2} u_1 =$$

$$u_2 = v_2 - \frac{33}{87} u_1 = v_2 - \frac{11}{29} u_1 = \begin{bmatrix} 1 \\ -3 \\ 2 \\ 4 \end{bmatrix} - \frac{11}{29} \begin{bmatrix} 5 \\ 2 \\ 3 \\ 7 \end{bmatrix} = \begin{bmatrix} 1 - \frac{55}{29} \\ -3 - \frac{22}{29} \\ 2 - \frac{33}{29} \\ 4 - \frac{77}{29} \end{bmatrix} = \begin{bmatrix} -\frac{26}{29} \\ -\frac{109}{29} \\ \frac{25}{29} \\ \frac{39}{29} \end{bmatrix} \approx \begin{bmatrix} -0.90 \\ -3.76 \\ 0.86 \\ 0.75 \end{bmatrix}$$

$$e_2 = \frac{u_2}{||u_2||} = \frac{u_2}{\sqrt{-0.9^2 + -3.76^2 + 0.86^2 + 1.34^2}} = \frac{u_2}{\sqrt{17.48}} \approx \begin{bmatrix} -0.21 \\ -0.90 \\ 0.21 \\ 0.32 \end{bmatrix}$$

$$u_3 = v_3 - proj_{u_1}(v_3) - proj_{u_2}(v_3) = v_3 - \frac{u_1 \cdot v_3}{u_1 \cdot u_1} u_1 - \frac{u_2 \cdot v_3}{u_2 \cdot u_2} u_2$$

$$\frac{u_1 \cdot v_3}{u_1 \cdot u_1} u_1 = \frac{-5 + (4 \cdot 2) + (1 \cdot 3) + (2 \cdot 7)}{\sqrt{87}^2} u_1 = \frac{20}{87} \begin{bmatrix} 5 \\ 2 \\ 3 \\ 7 \end{bmatrix} = \begin{bmatrix} 1.14 \\ 0.46 \\ 0.69 \\ 1.61 \end{bmatrix}$$

$$\frac{u_2 \cdot v_3}{u_2 \cdot u_2} u_2 = \frac{0.9 + (-3.76 \cdot 4) + 0.86 + (0.75 \cdot 2)}{17.48} u_2 = \frac{-11.78}{17.48} u_2 = \begin{bmatrix} 0.65 \\ 2.72 \\ -0.62 \\ -0.54 \end{bmatrix}$$

$$u_3 = \begin{bmatrix} -1 \\ 4 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 1.14 \\ 0.46 \\ 0.69 \\ 1.61 \end{bmatrix} - \begin{bmatrix} 0.65 \\ 2.72 \\ -0.62 \\ -0.54 \end{bmatrix} = \begin{bmatrix} -2.69 \\ 1.26 \\ 0.83 \\ 1.21 \end{bmatrix}$$

$$e_3 = \frac{u_3}{||u_3||} = \frac{u_3}{\sqrt{-2.69^2 + 1.26^2 + 0.83^2 + 1.21^2}} = \frac{u_3}{\sqrt{10.99}} = \begin{bmatrix} -0.81 \\ 0.38 \\ 0.25 \\ 0.36 \end{bmatrix}$$

After we create the orthogonal basis we can create the Q matrix.

$$Q = \begin{pmatrix} 0.54 & -0.21 & -0.81 \\ 0.21 & -0.90 & 0.38 \\ 0.32 & 0.21 & 0.25 \\ 0.75 & 0.32 & 0.36 \end{pmatrix}$$

We can find R by the following formula

$$R = Q^T A$$

$$Q^T = \begin{pmatrix} 0.54 & 0.21 & 0.32 & 0.75 \\ -0.21 & -0.9 & 0.21 & 0.32 \\ -0.81 & 0.38 & 0.25 & 0.36 \end{pmatrix}$$

$$Q^T A = \begin{pmatrix} 0.54 & 0.21 & 0.32 & 0.75 \\ -0.21 & -0.9 & 0.21 & 0.32 \\ -0.81 & 0.38 & 0.25 & 0.36 \end{pmatrix} \begin{bmatrix} 5 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & 2 & 1 \\ 7 & 4 & 2 \end{bmatrix} = \begin{pmatrix} 9.33 & 3.54 & 2.14 \\ 0 & 4.18 & -2.53 \\ 0 & 0 & 3.31 \end{pmatrix} = R$$

2.2 Solution for 2b

3 Problem 3

$$A = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 2 \end{pmatrix}$$

3.1 Solution for 3a

To prove that A is positive defined we need to check if the leading principal minors of the matrix are all positive numbers. The leading principal minors correspond to the determinants of the top-left submatrices

$$A_{1\times1} = \begin{pmatrix} 2 \end{pmatrix} \qquad A_{2\times2} = \begin{pmatrix} 2 & -1 \\ -1 & 5 \end{pmatrix} \qquad A_{3\times3} = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 2 \end{pmatrix}$$

$$\begin{aligned}
&\det A_{1\times1} = 2 > 0 \\
&\det A_{2\times2} = (2 \cdot 5) - (-1 \cdot (-1)) = 10 - 1 = 9 > 0 \\
&\det A_{3\times3} = (2 \cdot 5 \cdot 2) + ((-1) \cdot (-1) \cdot 1) + (1 \cdot (-1) \cdot (-1)) \\
&-(1 \cdot 5 \cdot 1) - ((-1) \cdot (-1) \cdot 2) - (2 \cdot (-1) \cdot (-1)) = 20 + 1 + 1 - 5 - 2 - 2 = 13 > 0
\end{aligned}$$

3.2 Solution for 3b

The Cholesky decomposition is decomposing the matrix in the form

$$A = L \cdot L^T$$

where L is lower triangular matrix, where each element is calculated by the following formulas

$$\begin{aligned}
&l_{11} = \sqrt{a_{11}} \\
&l_{j1} = \frac{a_{j1}}{l_{11}} \quad j \in [2, n] \\
&l_{ii} = \sqrt{a_{ii} - \sum_{k=1}^{i-1} l_{ik}} \quad i \in [2, n] \\
&l_{ji} = \frac{\left(a_{ij} - \sum_{k=1}^{i-1} l_{ik} l_{jk}\right)}{l_{ii}} \quad i \in [2, n-1], j \in [i+1, n]
\end{aligned}$$

$$
\begin{aligned}
l_{11} &= \sqrt{2} = 1.41 \\
l_{21} &= \frac{-1}{1.41} = -0.70 \\
l_{31} &= \frac{1}{1.41} = 0.70 \\
l_{22} &= \sqrt{a_{22} - l_{21}^2} = \sqrt{5 - (-0.70)^2} = \sqrt{4.5} = 2.12 \\
l_{32} &= \frac{a_{32} - l_{32} l_{31}}{l_{22}} = \frac{-1 - (0.7)(-0.7)}{2.12} = -0.24 \\
l_{33} &= \sqrt{a_{33} - (l_{31}^2 + l_{32}^2)} = \sqrt{2 - (0.7^2 + (-0.24^2))} = 1.20
\end{aligned}
$$

$$
L = \begin{pmatrix} 1.41 & 0 & 0 \\ -0.70 & 2.12 & 0 \\ 0.70 & -0.24 & 1.20 \end{pmatrix} \implies L^T = \begin{pmatrix} 1.41 & -0.70 & 0.70 \\ 0 & 2.12 & -0.23 \\ 0 & 0 & 1.20 \end{pmatrix}
$$

To find the LDL^T decomposition we need to find D which is the

4 Problem 4

$$A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix}$$

4.1 Solution for 4a

To find the Singular Value Decomposition or SVD we need to construct the matrises S, U, V. To do that we need to create A^T.

$$A^T = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}$$

We calculate AA^T

$$AA^T = \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix} \cdot \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} = \begin{bmatrix} 13 & 12 & 2 \\ 12 & 13 & -2 \\ 2 & -2 & 8 \end{bmatrix}$$

Next we need to find the eigenvalues and eigenvectors. To do that we need $\det(AA^T - \lambda I) = 0$

$$\det(AA^T - \lambda I) = 0 \implies \begin{vmatrix} 13-\lambda & 12 & 2 \\ 12 & 13-\lambda & -2 \\ 2 & -2 & 8-\lambda \end{vmatrix} = 0 \implies -\lambda(\lambda - 25)(\lambda - 9) = 0$$

$$\lambda_1 = 25 \quad \lambda_2 = 9 \quad \lambda_3 = 0$$

To find the eigenvectors we need to calculate for each eigenvalue the following expression

$$(AA^T - \lambda I)X = 0$$

$$\lambda_1 = 25 \implies \text{eigenvector} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\lambda_2 = 9 \implies \text{eigenvector} \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{4} \\ 1 \end{bmatrix}$$

$$\lambda_3 = 0 \implies \text{eigenvector} \begin{bmatrix} -2 \\ 2 \\ 2 \end{bmatrix}$$

To get the S matrix we calculate the square root of the nonzero eigenvalues

$$\sigma_1 = 5 \qquad \sigma_2 = 3$$

The S matrix is a zero matrix with the eigenvalues on its diagonal

$$S = \begin{bmatrix} 5 & 0 \\ 0 & 3 \\ 0 & 0 \end{bmatrix}$$

4.2 Solution for 4b