Techincal University - Sofia
Faculty of Applied Mathematics and Informatics

# Project 1 - Topics of Algebra

## Solution for version 4

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# Technical University of Sofia

**Informatics and Software Science - Optimization theory and big data analytics (2024/2025)**

## Project 1 - Topics of Algebra

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**Problem 1.** Let be given the following real matrix:

$$A = \begin{pmatrix} 1 & -3 & -1 \\ 3 & 7 & 5 \\ 2 & -2 & 4 \end{pmatrix}$$

Then

a) find its **LU** decomposition;

b) explain rigorously what is a **LU** decomposition.

**Problem 2.** Let be given the following real matrix:

$$A = \begin{bmatrix} 5 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & 2 & 1 \\ 7 & 4 & 2 \end{bmatrix}$$

Then

a) find its **QR** decomposition and **LQ** decomposition;

b) explain rigorously what is a **QR** decomposition and a **LQ** decomposition.

**Problem 3.** Let be given the following real matrix:

$$A = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 2 \end{pmatrix}$$

Then

a) prove by pivoting that $A$ is positive defined;

b) find its **Cholesky-Banachiewicz** and $\mathbf{LDL}^T$ decompositions.

**Problem 4.** Let be given the following real matrix:

$$A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix}$$

Then

a) find its **SVD** decomposition;

b) explain rigorously what is a **SVD** decomposition.

# 1 Problem 1

$$A = \begin{pmatrix} 1 & -3 & -1 \\ 3 & 7 & 5 \\ 2 & -2 & 4 \end{pmatrix}$$

## 1.1 Solution for 1a

Lets start with L being the identity matrix

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We perform the following operations on matrix A:

$$\left| \begin{array}{l} R_2 = R_2 - 3R_1 \\ R_3 = R_3 - 2R_1 \end{array} \right.$$

Where $R_i$ is the $i$th row of the matrix.
We write the coefficient 3 in the matrix L at row 2 and column 1. The same can be done for the coefficient 2 for row 3 and column 1.

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \qquad A = \begin{pmatrix} 1 & -3 & -1 \\ 0 & 16 & 8 \\ 0 & 4 & 6 \end{pmatrix}$$

We perform the following operation on matrix A

$$R_3 = -\frac{1}{4}R_2 + R_3$$

We write the coefficient $\frac{1}{4}$ in matrix L at row 3 and column 2.

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & \frac{1}{4} & 1 \end{pmatrix} \qquad A = \begin{pmatrix} 1 & -3 & -1 \\ 0 & 16 & 8 \\ 0 & 0 & 4 \end{pmatrix} = U$$

## 1.2 Solution for 1b

LU decomposition is a technique for factorizing a square matrix A as a product of two other matrices.

$$A = LU$$

where L is a lower triangular matrix and U is upper triangular matrix.
To have LU decomposition the matrix A must be square matrix ($m \times m$ dimensions) and invertible ($A^{-1}$ exists).
The algorithm used to solve this particular matrix is LU without pivoting. In this algorithm we initialize L to be identity matrix and we modify A to be an upper triangular matrix like U by performing row operations to create zeroes below the diagonal for each column. Then each multiplier/coefficient is recorded in L for the coresponding row and column.
The most common use for LU decomposition is to solve system of linear equations.

$$Ax = b$$

For the system we can substitute $A = LU$ and we can let $y = Ux$. Then the system becomes $Ly = b$ and after solving it we can solve the other system $Ux = y$

# 2 Problem 2

$$A = \begin{bmatrix} 5 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & 2 & 1 \\ 7 & 4 & 2 \end{bmatrix}$$

## 2.1 Solution for 2a

First we perform the Grand-Schmidt process for each column of the matrix

$$v_1 = \begin{bmatrix} 5 \\ 2 \\ 3 \\ 7 \end{bmatrix} \quad v_2 = \begin{bmatrix} 1 \\ -3 \\ 2 \\ 4 \end{bmatrix} \quad v_3 = \begin{bmatrix} -1 \\ 4 \\ 1 \\ 2 \end{bmatrix}$$

We determine the orthogonal vectors $e_k$ by the following formula

$$e_k = \frac{u_k}{||u_k||} \text{ where } \quad u_k = v_k - \sum_{j=1}^{k} proj_{u_j} v_k \quad ||u_k|| = \sqrt{\sum_{j=1}^{n} |u_i|^2} \quad proj_a(b) = \frac{a \cdot b}{a \cdot a} a$$

We solve for each column

$$u_1 = v_1 = \begin{bmatrix} 5 \\ 2 \\ 3 \\ 7 \end{bmatrix} \implies e_1 = \frac{u_1}{||u_1||} = \frac{u_1}{\sqrt{5^2 + 2^2 + 3^2 + 7^2}} = \frac{u_1}{\sqrt{87}} = \begin{bmatrix} \frac{5\sqrt{87}}{87} \\ \frac{2\sqrt{87}}{87} \\ \frac{3\sqrt{87}}{87} \\ \frac{7\sqrt{87}}{87} \end{bmatrix} \approx \begin{bmatrix} 0.54 \\ 0.21 \\ 0.32 \\ 0.75 \end{bmatrix}$$

$$u_2 = v_2 - proj_{u_1}(v_2) = v_2 - \frac{u_1 \cdot v_2}{u_1 \cdot u_1} u_1 = v_2 - \frac{(1 \cdot 5) + (-3) \cdot 2 + (2 \cdot 3) + (4 \cdot 7)}{\sqrt{87}^2} u_1 =$$

$$u_2 = v_2 - \frac{33}{87} u_1 = v_2 - \frac{11}{29} u_1 = \begin{bmatrix} 1 \\ -3 \\ 2 \\ 4 \end{bmatrix} - \frac{11}{29} \begin{bmatrix} 5 \\ 2 \\ 3 \\ 7 \end{bmatrix} = \begin{bmatrix} 1 - \frac{55}{29} \\ -3 - \frac{22}{29} \\ 2 - \frac{33}{29} \\ 4 - \frac{77}{29} \end{bmatrix} = \begin{bmatrix} -\frac{26}{29} \\ -\frac{109}{29} \\ \frac{25}{29} \\ \frac{39}{29} \end{bmatrix} \approx \begin{bmatrix} -0.90 \\ -3.76 \\ 0.86 \\ 0.75 \end{bmatrix}$$

$$e_2 = \frac{u_2}{||u_2||} = \frac{u_2}{\sqrt{-0.9^2 + -3.76^2 + 0.86^2 + 1.34^2}} = \frac{u_2}{\sqrt{17.48}} \approx \begin{bmatrix} -0.21 \\ -0.90 \\ 0.21 \\ 0.32 \end{bmatrix}$$

$$u_3 = v_3 - proj_{u_1}(v_3) - proj_{u_2}(v_3) = v_3 - \frac{u_1 \cdot v_3}{u_1 \cdot u_1}u_1 - \frac{u_2 \cdot v_3}{u_2 \cdot u_2}u_2$$

$$\frac{u_1 \cdot v_3}{u_1 \cdot u_1}u_1 = \frac{-5 + (4 \cdot 2) + (1 \cdot 3) + (2 \cdot 7)}{\sqrt{87}^2}u_1 = \frac{20}{87}\begin{bmatrix} 5 \\ 2 \\ 3 \\ 7 \end{bmatrix} = \begin{bmatrix} 1.14 \\ 0.46 \\ 0.69 \\ 1.61 \end{bmatrix}$$

$$\frac{u_2 \cdot v_3}{u_2 \cdot u_2}u_2 = \frac{0.9 + (-3.76 \cdot 4) + 0.86 + (0.75 \cdot 2)}{17.48}u_2 = \frac{-11.78}{17.48}u_2 = \begin{bmatrix} 0.65 \\ 2.72 \\ -0.62 \\ -0.54 \end{bmatrix}$$

$$u_3 = \begin{bmatrix} -1 \\ 4 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 1.14 \\ 0.46 \\ 0.69 \\ 1.61 \end{bmatrix} - \begin{bmatrix} 0.65 \\ 2.72 \\ -0.62 \\ -0.54 \end{bmatrix} = \begin{bmatrix} -2.69 \\ 1.26 \\ 0.83 \\ 1.21 \end{bmatrix}$$

$$e_3 = \frac{u_3}{||u_3||} = \frac{u_3}{\sqrt{-2.69^2 + 1.26^2 + 0.83^2 + 1.21^2}} = \frac{u_3}{\sqrt{10.99}} = \begin{bmatrix} -0.81 \\ 0.38 \\ 0.25 \\ 0.36 \end{bmatrix}$$

After we create the orthogonal basis we can create the Q matrix.

$$Q = \begin{pmatrix} 0.54 & -0.21 & -0.81 \\ 0.21 & -0.90 & 0.38 \\ 0.32 & 0.21 & 0.25 \\ 0.75 & 0.32 & 0.36 \end{pmatrix}$$

We can find R by the following formula

$$R = Q^T A$$

$$Q^T = \begin{pmatrix} 0.54 & 0.21 & 0.32 & 0.75 \\ -0.21 & -0.9 & 0.21 & 0.32 \\ -0.81 & 0.38 & 0.25 & 0.36 \end{pmatrix}$$

$$Q^T A = \begin{pmatrix} 0.54 & 0.21 & 0.32 & 0.75 \\ -0.21 & -0.9 & 0.21 & 0.32 \\ -0.81 & 0.38 & 0.25 & 0.36 \end{pmatrix} \begin{bmatrix} 5 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & 2 & 1 \\ 7 & 4 & 2 \end{bmatrix} = \begin{pmatrix} 9.33 & 3.54 & 2.14 \\ 0 & 4.18 & -2.53 \\ 0 & 0 & 3.31 \end{pmatrix} = R$$

We can find $L$ by transposing $R$.

$$L = R^T = \begin{pmatrix} 9.33 & 0 & 0 \\ 3.54 & 4.18 & 0 \\ 2.14 & -2.53 & 3.31 \end{pmatrix}$$

### 2.2 Solution for 2b

# 3 Problem 3

$$A = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 2 \end{pmatrix}$$

## 3.1 Solution for 3a

To prove that A is positive defined by pivoting we will transform the matrix into an upper triangular matrix and check if every pivot is positive. The first pivot is $a_{11} = 2$. Then we perform the following operations on matrix A:

$$\left| \begin{array}{l} R_2 = R_2 + \frac{R_1}{2} \\ R_3 = R_3 - \frac{R_1}{2} \end{array} \right.$$

$$A = \begin{pmatrix} 2 & -1 & 1 \\ 0 & \frac{9}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{3}{2} \end{pmatrix}$$

The next pivot is $a_{22} = \frac{9}{2}$. We perform the following operation

$$R_3 = R3 - \left( \frac{-\frac{1}{2}}{\frac{9}{2}} \right) R2 = R_3 + \frac{1}{9} R_2$$

$$A = \begin{pmatrix} 2 & -1 & 1 \\ 0 & \frac{9}{2} & -\frac{1}{2} \\ 0 & 0 & \frac{13}{9} \end{pmatrix}$$

The pivots are $\left(2, \frac{9}{2}, \frac{13}{9}\right)$ which are all positive numbers.

## 3.2 Solution for 3b

The Cholesky decomposition is decomposing the matrix in the form

$$A = L \cdot L^T$$

where L is lower triangular matrix, where each element is calculated by the following formulas

$$
\begin{aligned}
l_{11} &= \sqrt{a_{11}} \\
l_{j1} &= \frac{a_{j1}}{l_{11}} \quad j \in [2, n] \\
l_{ii} &= \sqrt{a_{ii} - \sum_{k=1}^{i-1} l_{ik}} \quad i \in [2, n] \\
l_{ji} &= \frac{\left(a_{ij} - \sum_{k=1}^{i-1} l_{ik} l_{jk}\right)}{l_{ii}} \quad i \in [2, n-1], j \in [i+1, n]
\end{aligned}
$$

$$
\begin{aligned}
l_{11} &= \sqrt{2} = 1.41 \\
l_{21} &= \frac{-1}{1.41} = -0.70 \\
l_{31} &= \frac{1}{1.41} = 0.70 \\
l_{22} &= \sqrt{a_{22} - l_{21}^2} = \sqrt{5 - (-0.70)^2} = \sqrt{4.5} = 2.12 \\
l_{32} &= \frac{a_{32} - l_{32} l_{31}}{l_{22}} = \frac{-1 - (0.7)(-0.7)}{2.12} = -0.24 \\
l_{33} &= \sqrt{a_{33} - (l_{31}^2 + l_{32}^2)} = \sqrt{2 - (0.7^2 + (-0.24^2))} = 1.20
\end{aligned}
$$

$$L = \begin{pmatrix} 1.41 & 0 & 0 \\ -0.70 & 2.12 & 0 \\ 0.70 & -0.24 & 1.20 \end{pmatrix} \implies L^T = \begin{pmatrix} 1.41 & -0.70 & 0.70 \\ 0 & 2.12 & -0.23 \\ 0 & 0 & 1.20 \end{pmatrix}$$

To find the $LDL^T$ decomposition we need to find the two matrices: $L$ and $D$. L is a lower triangular matrix and D is a diagonal matrix. We can find each element with the following formulas

$$d_{ii} = a_{ii} - \sum_{j=1}^{i-1} l_{ij}^2 \cdot d_{jj} \qquad l_{ii} = \frac{1}{d_{ii}} \left( a_{ij} - \sum_{k=1}^{i-1} l_{ik} \cdot l_{jk} \cdot d_{kk} \right)$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix} \qquad D = \begin{pmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{pmatrix}$$

$$d_{11} = a_{11} = 2$$

$$l_{21} = \frac{1}{d_{11}} \left( a_{21} - \sum_{k=1}^{0} l_{2k} l_{1k} d_{kk} \right) = \frac{1}{2}(-1) = -0.5$$

$$l_{31} = \frac{1}{d_{11}} \left( a_{31} - \sum_{k=1}^{0} l_{2k} l_{1k} d_{kk} \right) = \frac{1}{2}(1) = 0.5$$

$$d_{22} = a_{22} - \sum_{k=1}^{1} l_{2k}^2 d_{kk} = 5 - (l_{21}^2 \cdot d_{11}) = 5 - (0.5^2 \cdot 2) = 5 - 0.5 = 4.5$$

$$l_{32} = \frac{1}{d_{22}} \left( a_{32} - \sum_{k=1}^{1} l_{3k} l_{2k} d_{kk} \right) = \frac{1}{4.5} (-1 - (l_{31} \cdot l_{21} \cdot d_{11}) =$$
$$\frac{1}{4.5}(-1 - (0.5 \cdot (-0.5) \cdot 2)) = \frac{1}{4.5}(-1 + 0.5) = \frac{1}{4.5} \cdot (-0.5) \approx -0.111$$

$$d_{33} = a_{33} - \sum_{k=1}^{2} l_{3k}^2 d_{kk} = 2 - (l_{31}^2 d_{11} + l_{32}^2 d_{22}) = 2 - (2 \cdot 0.5^2 + 4.5 \cdot (-0.111)^2) \approx 1.444$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ -0.5 & 1 & 0 \\ 0.5 & -0.111 & 1 \end{pmatrix} \quad D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4.5 & 0 \\ 0 & 0 & 1.444 \end{pmatrix} \quad L^T = \begin{pmatrix} 1 & -0.5 & 0.5 \\ 0 & 1 & -0.111 \\ 0 & 0 & 1 \end{pmatrix}$$

# 4 Problem 4

$$A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix}$$

## 4.1 Solution for 4a

To find the Singular Value Decomposition or SVD we need to construct the matrises $U, S, V$. To do that we need to create $A^T$.

$$A^T = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}$$

We calculate $W = A^T A$

$$W = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} \cdot \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 17 & 8 \\ 8 & 17 \end{bmatrix}$$

Next we need to find the eigenvalues and eigenvectors. To find the eigenvalues we need to calculate

$$\det(W - \lambda I) = 0$$

$$\det(W - \lambda I) = 0 \implies \begin{vmatrix} 17-\lambda & 8 \\ 8 & 17-\lambda \end{vmatrix} = 0$$
$$(17-\lambda)^2 - 8^2 = 0$$
$$(17-\lambda+8)(17-\lambda-8) = 0$$
$$(25-\lambda)(9-\lambda) = 0$$
$$\lambda_1 = 25 \qquad \lambda_2 = 9$$

To find the eigenvectors we need to calculate for each eigenvalue the following expression

$$(W - \lambda I)X = 0$$

where $X = (x_1, x_2)^T$

$$\lambda_1 = 25 \implies \begin{pmatrix} 17-25 & 8 \\ 8 & 17-25 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -8 & 8 \\ 8 & -8 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$
$$-8x_1 + 8x_2 = 0 \implies x_1 = x_2 \implies v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$||v|| = \sqrt{1^2 + 1^2} = \sqrt{2} \implies v_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\lambda_2 = 9 \implies \begin{pmatrix} 17-9 & 8 \\ 8 & 17-9 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 8 & 8 \\ 8 & 8 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$8x_1 + 8x_2 = 0 \implies x_1 = -x_2 \implies v = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$||v|| = \sqrt{1^2 + (-1)^2} = \sqrt{2} \implies v_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Using $v_1, v_2$ we can create the matrix $V$

$$V = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

To get the $S$ matrix we need to calculate the singular values, which are the square root of each nonzero eigenvalue

$$\sigma_1 = 5 \qquad \sigma_2 = 3 \implies S = \begin{bmatrix} 5 & 0 \\ 0 & 3 \\ 0 & 0 \end{bmatrix}$$

To construct $U$ we need to calculate $u_1 = \frac{Av_1}{\sigma_1}$ and $u_2 = \frac{Av_2}{\sigma_2}$

$$Av_1 = \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 5 \\ 5 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{5}{\sqrt{2}} \\ \frac{5}{\sqrt{2}} \\ 0 \end{bmatrix}$$

$$u_1 = \frac{Av_1}{\sigma_1} = \frac{1}{5} \begin{bmatrix} \frac{5}{\sqrt{2}} \\ \frac{5}{\sqrt{2}} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{bmatrix}$$

$$Av_2 = \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ \frac{4}{\sqrt{2}} \end{bmatrix}$$

$$u_2 = \frac{Av_2}{\sigma_2} = \frac{1}{3} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ \frac{4}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{6} \\ -\frac{\sqrt{2}}{6} \\ \frac{2\sqrt{2}}{3} \end{bmatrix}$$

To calculate $u_3$ we need to find a vector which makes an orthonormal basis for $\mathbb{R}^3$ with $u_1, u_2$ and is a unit vector. To do that we need to solve the following system

$$\left| \begin{array}{ll} \frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y + 0z = 0 & | \cdot \frac{2}{\sqrt{2}} \\ \frac{\sqrt{2}}{6}x - \frac{\sqrt{2}}{6}y + \frac{2\sqrt{2}}{3}z = 0 & | \cdot \frac{6}{\sqrt{2}} \end{array} \right.$$

$$\left| \begin{array}{ll} x + y + 0z = 0 & |R_2 = R_2 + -(R_1) \\ x - y + 4z = 0 & \end{array} \right.$$

$$-2y + 4z = 0 \implies y = 2z$$

$$x = -y = -2z$$

$$u_3 = \begin{bmatrix} -2z \\ 2z \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$$

We need to find the unit vector

$$||u_3|| = \sqrt{(-2)^2 + 2^2 + 1^2} = \sqrt{9} = 3 \implies u_3 = \begin{bmatrix} -\frac{2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}$$

We can build U using $u_1, u_2, u_3$

$$U = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{6} & -\frac{2}{3} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & \frac{2}{3} \\ 0 & \frac{2\sqrt{2}}{3} & \frac{1}{3} \end{pmatrix}$$

### 4.2 Solution for 4b