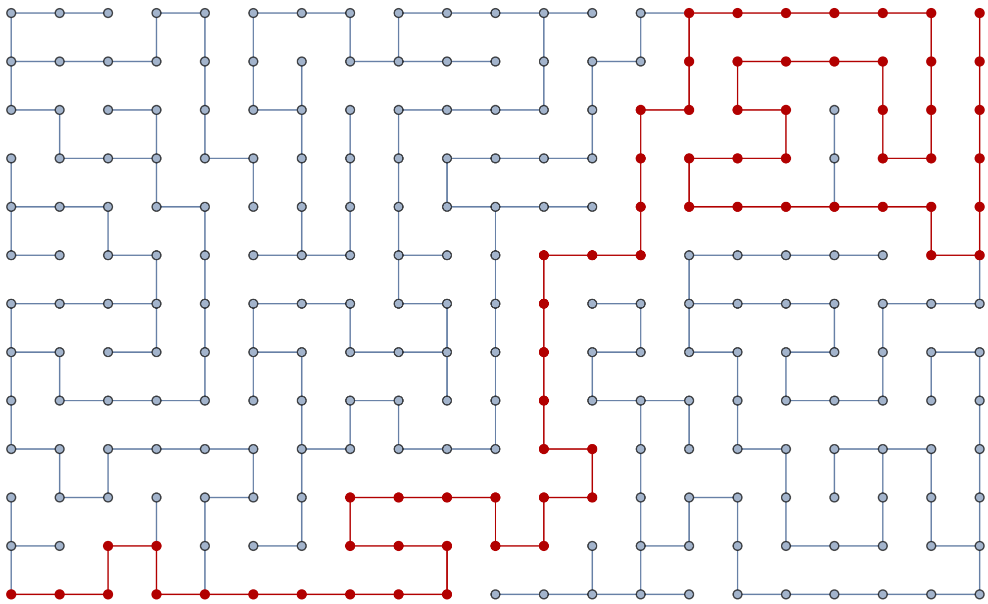
Project Euler Problem 380: Amazing Mazes

Wee JunJie

December 3, 2019

1 Introduction



An $m \times n$ maze is an $m \times n$ rectangular grid with walls placed between grid cells such that there is exactly one path from the top-left square to any other square. Let $C(m, n)$ be the number of distinct mazes with dimensions m and n. Mazes which can be formed by rotation and reflection from another maze are considered distinct. It can be verified that C(1,1) = 1, C(2,2) = 4, C(3,4) = 2415, and C(9,12) = 2.5720e46 (in scientific notation rounded to 5 significant digits).

2 Solution

The first idea of this solution is to realise that such $m \times n$ mazes can be visualized in the form of spanning trees in an $m \times n$ grid as shown above. Hence, this problem translates into the following:

"How many spanning trees are there in an $m \times n$ grid-graph?"

We require a few results from graph theory in order to compute $C(m, n)$.

Definition 2.1. The Cartesian product $G\Box H$ of graphs G and H is a graph such that

- the vertex set of $G\Box H$ is the Cartesian product $V(G) \times V(H)$.
- two vertices (u, u') and (v, v') are adjacent in $G\Box H$ if and only if either $u = v$ and u' is adjacent to v' in H, or $u' = v'$ and u is adjacent to v in G.

Definition 2.2. (Laplacian Matrix) Given a simple graph G with n vertices, the Laplacian matrix L of G is

$$L = D - A \tag{1}$$

where D is the degree matrix and A is the adjacency matrix of G.
In other words, L can also be defined as

$$L_{i,j} := \begin{cases} \deg(v_i) & \text{if } i = j \\ -1 & \text{if } i \neq j \text{ and } v_i \text{ is adjacent to } v_j \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

Theorem 2.3. *(Kirchoff's Matrix Tree Theorem) For a given connected graph G with n labelled vertices, let $\lambda_1,\ \lambda_2,\ \cdots,\ \lambda_{n-1}$ be the non-zero eigenvalues of its Laplacian matrix. Then the number of spanning trees of G is*

$$t(G) = \frac{1}{n}\prod_{k=1}^{n-1} \lambda_k$$

Equivalently the number of spanning trees is equal to any cofactor of the Laplacian matrix of G.

Example 2.4. *(Complete Graph) For any complete graph K_n of $n \geq 3$,*

$$L = \begin{bmatrix} n-1 & -1 & \cdots & -1 \\ -1 & n-1 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & n-1 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 0 \end{bmatrix}, \quad D = \begin{bmatrix} n-1 & 0 & \cdots & 0 \\ 0 & n-1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & n-1 \end{bmatrix}$$

Note that

$$A = J - I \quad \textit{where } J \textit{ is the all-ones matrix and } I \textit{ is identity matrix,}$$

and J has n linearly dependent columns hence it has rank 1. Hence, the all-ones vector is an eigenvector of J with eigenvalue n. Since rank(J)=1, the remaining eigenvalues of J are zero. Therefore the eigenvalues x of L are as follows:

$$Lx = (D - A)x = (D - J + I)x$$

Hence, $x = 0$ with multiplicity 1 and $x = n$ with multiplicity $n - 1$. By the Matrix Tree Theorem, we have

$$t(K_n) = \frac{1}{n}n^{n-1} = n^{n-2} \tag{3}$$

This is also known as the Cayley's Formula which also represents the number of trees on n labelled vertices.

Example 2.5. *(Path Graph) For any path graph P_n of $n \geq 1$,*

$$L = \begin{bmatrix} 1 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & \ddots & \ddots & \\ & & \ddots & 2 & -1 \\ & & & -1 & 1 \end{bmatrix} \tag{4}$$

which is a tridiagonal Toeplitz matrix with the eigenvalues given by

$$\lambda_j = 2 - 2\cos\left(\frac{\pi j}{n}\right), \quad 1 \leq j \leq n. \tag{5}$$

Assuming this fact, we apply the Matrix Tree Theorem and get

$$t(P_n) = \frac{1}{n}\prod_{k=1}^{n-1}\lambda_k = \frac{1}{n}\prod_{k=1}^{n-1}\left(2 - 2\cos\left(\frac{\pi k}{n}\right)\right) \tag{6}$$

Lastly, we require a corollary for a special type of graphs.

Corollary 2.6. *(Spanning Trees of Cartesian Products of Graphs) Let G and H be graphs with m and n vertices respectively. If the eigenvalues of Laplacian of G are λ_k, $1 \leq k \leq m-1$ and the eigenvalues of Laplacian of H are μ_h, $1 \leq h \leq n-1$ Then the number of spanning trees of $G\square H$ is given by*

$$t(G\square H) = \frac{1}{mn}\prod_{k=1}^{m-1}\prod_{h=1}^{n-1}(\lambda_k + \mu_h) \tag{7}$$

Finally, as a solution to this problem, we find that the number of spanning trees in any $m \times n$ grid-graph $(P_m\square P_n)$ is

$$\begin{aligned} t(P_m\square P_n) &= \frac{1}{mn}\prod_{k=1}^{m-1}\prod_{h=1}^{n-1}\left[4 - 2\cos\left(\frac{\pi h}{n}\right) - 2\cos\left(\frac{\pi k}{m}\right)\right] && (8) \\ &= \frac{1}{mn}\prod_{k=1}^{m-1}\prod_{h=1}^{n-1}\left[4\sin^2\left(\frac{\pi h}{2n}\right) + 4\sin^2\left(\frac{\pi k}{2m}\right)\right] \quad \text{(By Double-Angle Formula)} && (9) \end{aligned}$$

```python
import numpy as np
import sympy as sp

prod = sp.N(1)
m, n = 100, 500
for k in range(1, m):
    sub_prod = sp.N(1)
    for h in range(1, n):
        sub_prod *= 4*(math.sin((k*math.pi)/(2*m))**2) +
        4*(math.sin((h*math.pi)/(2*n))**2)
    prod *= sub_prod
```

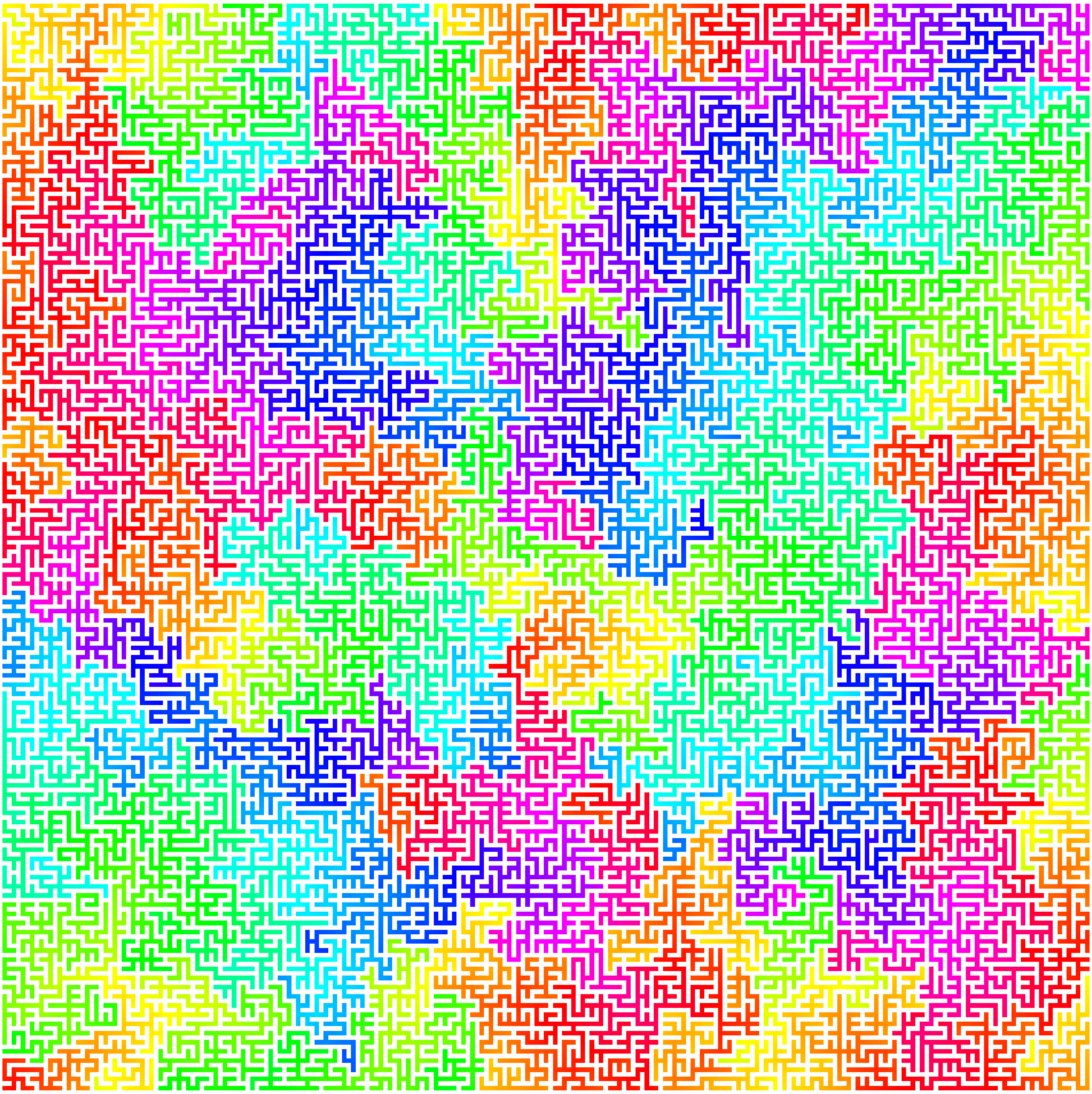


Figure 1: Maze Transversal via Wilson's Algorithm II