Project Euler: The Fibonacci Golden Nuggets

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1 Problem 137: Ordinary Fibonacci Golden Nuggets

Consider the infinite polynomial series $A_F(x) = xF_1 + x^2F_2 + x^3F_3 + \cdots$, where F_n is the nth term in the Fibonacci sequence: $1, 1, 2, 3, 5, 8, \cdots$; that is, $F_n = F_{n-1} + F_{n-2}$, $F_1 = 1$ and $F_2 = 1$. For this problem we shall be interested in values of x for which $A_F(x)$ is a positive integer. Surprisingly,

$$A_F(1/2) = (1/2)+(1/2)^2+2(1/2)^3+3(1/2)^4+5(1/2)^5+\cdots = 1/2+1/4+2/8+3/16+5/32+\cdots = 2.$$

The corresponding values of x for the first five natural numbers are shown below.

x	$A_F(x)$
$\sqrt{2}-1$	1
$\frac{1}{2}$	2
$\frac{\sqrt{13}-2}{3}$	3
$\frac{\sqrt{89}-5}{8}$	4
$\frac{\sqrt{34}-3}{5}$	5

Table 1: Values of x where $A_F(x)$ is a positive integer.

We shall call $A_F(x)$ a golden nugget if x is rational, because they become increasingly rarer; for example, the 10th golden nugget is 74049690. Find the 15th golden nugget.

2 Preliminaries

Before we discuss the solution of Problem 137, we visit some well-known results related to the Fibonacci sequence in this section. The Fibonacci sequence, $F_n = F_{n-1} + F_{n-2}$, $F_1 = 1$ and $F_2 = 1$, $n \geq 3$, is undoubtedly one of the most famous number sequence in mathematics. Here, we pay attention to several well-known results that will be needed in the solutions later.

Lemma 2.1 (Cassini's Identity)**.** *For any nth Fibonacci number,*

$$F_{n-1}F_{n+1} - F_n^2 = (-1)^n. \tag{1}$$

Proof. Note that it is not difficult to prove the following result by mathematical induction:

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix} \quad \text{(Exercise!)}. \tag{2}$$

Here, note that

$$F_{n-1}F_{n+1} - F_n^2 = \det\left(\begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix} \right) = \det\left(\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \right)^n = (-1)^n. \tag{3}$$

□

Definition 2.2 (Lucas Numbers)**.** The Lucas numbers is defined as follows:

$$L_n := \begin{cases} 2 & \text{if } n = 0; \\ 1 & \text{if } n = 1; \\ L_{n-1} + L_{n-2} & \text{if } n > 1. \end{cases} \tag{4}$$

In other words, the first few terms of the L_n are 2, 1, 3, 4, 7, 11, 18, 29, 47, $\cdots$. There are some interesting properties in L_n. For instance, we can rewrite L_n as

$$L_n = F_{n-1} + F_{n+1}, \tag{5}$$

and many more can also be found in [2]. More importantly, we also note the generating function of Lucas numbers.

$$\begin{aligned} \Phi(x) &= L_0 + L_1 x + \sum_{n=2}^{\infty} L_n x^n && (6) \\ &= 2 + x + \sum_{n=2}^{\infty} (L_{n-1} + L_{n-2}) x^n && (7) \\ &= 2 + x + \sum_{n=1}^{\infty} L_n x^{n+1} + \sum_{n=0}^{\infty} L_n x^{n+2} && (8) \\ &= 2 + x + x(\Phi(x) - 2) + x^2 \Phi(x) && (9) \end{aligned}$$

which can be rearranged as

$$\Phi(x) = \frac{2 - x}{1 - x - x^2}.$$

Our last preliminary in this section is to introduce the unique sequence of numbers H_n generated by the sum of $F_{n+1} + L_n$, $n \geq 1$. The sequence of H_n thus looks like the following:

$$2, 5, 7, 12, 19, 31, 50, 81, 131, \cdots \tag{10}$$

This sequence will be needed for Problem 140.

3 Solution to Problem 137

Now, we are ready to discuss the solution to Problem 137. First, we require the generating function of the Fibonacci numbers. First, notice that

$$\begin{aligned}
A_F(x) = \sum_{n=1}^{\infty} F_n x^n &= x + x^2 + \sum_{n=1}^{\infty} F_{n+2} x^{n+2} \\
&= x + x^2 + \sum_{n=1}^{\infty} F_{n+1} x^{n+2} + \sum_{n=1}^{\infty} F_n x^{n+2} \\
&= x + x^2 + x \sum_{n=1}^{\infty} F_{n+1} x^{n+1} + x^2 \sum_{n=1}^{\infty} F_n x^n \\
&= x + x^2 + x(A_F(x) - x) + x^2 A_F(x) \\
&\implies (1 - x - x^2) A_F(x) = x
\end{aligned}$$

Hence, we have

$$A_F(x) = \frac{x}{1 - x - x^2}. \tag{11}$$

Next, we try several values of integers k such that

$$A_F(x) = \frac{x}{1 - x - x^2} = k \tag{12}$$

and see if there is any pattern for the values of x when x is rational.

	k	x
1	2	1/2
2	15	3/5
3	104	8/13
4	714	21/34

Table 2: Fibonacci Golden Nuggets

We refer to a research journal in 2015 (see [1]) where it discusses the technique to generate these golden nuggets. The technique used actually depends on the pattern discovered by the values of x of golden nuggets. Here, we cannot immediately find any intuitive pattern in the values of k but we can do so for the values of x. It is not difficult to check that the values of x above satisfy the sequence $x = \frac{F_n}{F_{n+1}}$ for $n \geq 1$. Hence, we substitute this expression into $A_F(x)$ to get a closed form formula for

k. This yields the following result.

$$\begin{aligned} A_F\left(\frac{F_n}{F_{n+1}}\right) &= \frac{\frac{F_n}{F_{n+1}}}{1-(\frac{F_n}{F_{n+1}})-(\frac{F_n}{F_{n+1}})^2} \\ &= \frac{F_nF_{n+1}}{F_{n+1}^2 - F_nF_{n+1} - F_n^2} \\ &= \frac{F_nF_{n+1}}{F_{n+1}(F_{n+1}-F_n) - F_n^2} \\ &= \frac{F_nF_{n+1}}{F_{n+1}(F_{n-1}) - F_n^2} \\ &= (-1)^n F_nF_{n+1} \quad \text{(By Cassini's Identity).} \end{aligned}$$

Since we are only considering values where k is a positive integer, then the possible values of n which satisfy this expression is when n is even. Therefore, this simplifies to

$$k = F_nF_{n+1}.$$

Remark 3.1. Note that F_nF_{n+1} can be found under OEIS A081018, hence the answer to this problem can actually be cheated from OEIS :). In OEIS, it is also indicated that an alternative formula for this sequence is $(L_{4n+1} - 1)/5$. The derivation of this alternative formula can also be found in [1]. Nevertheless, as this is a Project Euler Problem, there exists another problem 140 which is slightly more difficult than this problem and cannot be cheated from OEIS. XD

4 Problem 140: Modified Fibonacci Golden Nuggets

Consider the infinite polynomial series $A_G(x) = xG_1 + x^2G_2 + x^3G_3 + \cdots$, where G_k is the kth term of the second order recurrence relation $G_k = G_{k-1} + G_{k-2}$, $G_1 = 1$ and $G_2 = 4$; that is, $1, 4, 5, 9, 14, 23, \cdots$. For this problem we shall be concerned with values of x for which $A_G(x)$ is a positive integer. The corresponding values of x for the first five natural numbers are shown below.

x	$A_G(x)$
$\frac{\sqrt{5}-1}{4}$	1
$\frac{2}{5}$	2
$\frac{\sqrt{22}-2}{6}$	3
$\frac{\sqrt{137}-5}{14}$	4
$\frac{1}{2}$	5

Table 3: Values of x where $A_G(x)$ is a positive integer.

We shall call $A_G(x)$ a golden nugget if x is rational, because they become increasingly rarer; for example, the 20th golden nugget is 211345365. Find the sum of the first thirty golden nuggets.

5 Solution to Problem 140

Similar to Problem 137, now we want to find integers k such that

$$A_G(x) = x + 4x^2 + 5x^3 + 9x^4 + 14x^5 + \cdots = k, \tag{13}$$

where the coefficients of x^n are defined by the sequence $G_n = G_{n-1} + G_{n-2}$ for $n \geq 3$, $G_1 = 1$ and $G_2 = 4$.

Lemma 5.1 (Generating Function of $A_G(x)$)**.**

$$A_G(x) = \frac{x(1+3x)}{1-x-x^2} = \frac{3-2x}{1-x-x^2} - 3. \tag{14}$$

Proof. First, notice that

$$\begin{aligned} A_G(x) = \sum_{n=1}^{\infty} G_n x^n &= x + 4x^2 + \sum_{n=1}^{\infty} G_{n+2} x^{n+2} \\ &= x + 4x^2 + \sum_{n=1}^{\infty} G_{n+1} x^{n+2} + \sum_{n=1}^{\infty} G_n x^{n+2} \\ &= x + 4x^2 + x\sum_{n=1}^{\infty} G_{n+1} x^{n+1} + x^2 \sum_{n=1}^{\infty} G_n x^n \\ &= x + 4x^2 + x(A_G(x) - x) + x^2 A_G(x) \\ &\implies (1-x-x^2)A_G(x) = x + 3x^2 = x(1+3x) \end{aligned}$$

Hence, we have

$$A_G(x) = \frac{x(1+3x)}{1-x-x^2} = \frac{3-2x}{1-x-x^2} - 3. \tag{15}$$

Note that $\frac{3-2x}{1-x-x^2}$ would be a generating function for $A_G(x)$ with the sequence starting from 0 with $G_0 = 3$. $\square$

With the generating functions, we now refer back to (13) and get the following equation to solve for integers k:

$$\frac{x(1+3x)}{1-x-x^2} = k \iff (k+3)x^2 + (k+1)x - k = 0, \quad k \in \mathbb{N}. \tag{16}$$

Again, we attempt to search for rational values of x for integers k. We try to find some pattern in the numbers for the first few rational terms.

	k	x
1	2	2/5
2	5	1/2
3	21	7/12
4	42	3/5
5	152	19/31
6	296	8/13

Table 4: Modified Fibonacci Golden Nuggets

Unfortunately, one will find that the sequence of values of k is not intuitive and it cannot be found in OEIS. However, by re-applying the technique in [1], we can derive a new set of formulas to compute these special values of k efficiently. This is done similar to the Ordinary Fibonacci Golden Nuggets where we find some pattern in the values of x. Comparing these new set of values of x with those in Problem 137, notice that the even index entries of x follow the Fibonacci ratio as well. i.e. $1/2, 3/5, 8/13, \cdots$. Hence, for even entries of x, we simply have

$$x = \frac{F_n}{F_{n+1}}. \tag{17}$$

By substituting the expression to the generating function of $A_G(x)$, we obtain a closed formula to compute the even entries of k. For $x = \frac{F_n}{F_{n+1}}$, we have

$$\begin{aligned} k = A_G(x) = \frac{x(1+3x)}{1-x-x^2} &= \frac{\frac{F_n}{F_{n+1}}(1+3\frac{F_n}{F_{n+1}})}{1-\frac{F_n}{F_{n+1}}-(\frac{F_n}{F_{n+1}})^2} \\ &= \frac{F_nF_{n+1}+3F_n}{F_{n+1}^2-F_nF_{n+1}-F_n^2} \\ &= (-1)^n(F_nF_{n+1}+3F_n) \quad \text{(By Cassini's Identity)} \\ &= F_nF_{n+1}+3F_n \quad \because n \text{ is even.} \end{aligned}$$

As for the odd entries, this is slightly more complicated. Notice closely that the odd entries are actually ratios of Fibonacci sequence but with starting terms of 2 and 5. i.e. $2/5, 7/12, 19/31, \cdots$. This is connected to an earlier sequence we introduce which is $H_n = F_{n+1} + L_n$, $n \in \mathbb{N}$. Hence, for the odd entries of x, we have

$$x = \frac{H_n}{H_{n+1}} = \frac{F_{n+1}+L_n}{F_{n+2}+L_{n+1}}. \tag{18}$$

Therefore, for odd entries of k, we have

$$k = \frac{(\frac{H_n}{H_{n+1}})(1+3(\frac{H_n}{H_{n+1}}))}{1-(\frac{H_n}{H_{n+1}})-(\frac{H_n}{H_{n+1}})^2}. \tag{19}$$

In order to compute the 30 golden nuggets, we just need to compute this

$$k = \begin{cases} \dfrac{(\frac{H_n}{H_{n+1}})(1+3(\frac{H_n}{H_{n+1}}))}{1-(\frac{H_n}{H_{n+1}})-(\frac{H_n}{H_{n+1}})^2} & \text{if } n \text{ is odd;} \\ F_nF_{n+1}+3F_n & \text{if } n \text{ is even;} \end{cases} \tag{20}$$

Finally, based on two subsequences of k, we generate the following table below.

	k
1	2
2	5
3	21
4	42
5	152
6	296
7	1050
8	2037
9	7205
10	13970
11	49392
12	95760
13	338546
14	656357
15	2320437
16	4498746
17	15904520
18	30834872
19	109011210
20	211345365
21	747173957
22	1448582690
23	5121206496
24	9928733472
25	35101271522
26	68052551621
27	240587694165
28	466439127882
29	1649012587640
30	3197021343560

Table 5: First 30 Modified Fibonacci Golden Nuggets

Lastly, notice that problem 137 and 140 is posted many years before 2015, before even [1] was being published. There are other solutions lying around used by other Project Euler solvers. However, I find the simple technique used in [1] pretty elegant.

References

[1] Hong, D. S. (2015). When is the generating function of the Fibonacci numbers an integer?. The College Mathematics Journal, 46(2), 110-112.

[2] https://en.wikipedia.org/wiki/Lucas_number