Project Euler Problem 516: 5-Smooth Totients

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1 Introduction

It is assumed that the reader knows the following:

$$\text{For any 5-smooth number } n, \text{then } n \text{ is of the form } 2^a3^b5^c.$$

If not, it is recommended that the reader should work on Project Euler Problem 204 first before returning to this problem.

This problem wants us to find the numbers n such that

$$\varphi(n) \text{ is a 5-smooth number.}$$

$$\text{i.e. } \varphi(n) = n\prod_{p|n}\left(1-\frac{1}{p}\right) = 2^a3^b5^c.$$

2 Mathematical Part of the Problem

By using the fact that every number n can be expressed as a product of primes $p_1^{a_1}p_2^{a_2}\cdots$, we arrive at

$$\begin{aligned}
\varphi(n) &= n\prod_{p|n}\left(1-\frac{1}{p}\right)\\
&= (p_1^{a_1}p_2^{a_2}\cdots)\prod_{p|n}\left(1-\frac{1}{p}\right)\\
&= (p_1^{a_1}p_2^{a_2}\cdots)\left[\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\cdots\right]\\
&= (p_1^{a_1}p_2^{a_2}\cdots)\left[\left(\frac{p_1-1}{p_1}\right)\left(\frac{p_2-1}{p_2}\right)\cdots\right]\\
&= (p_1^{a_1-1}(p_1-1))(p_2^{a_2-1}(p_2-1))\cdots\\
&= 2^a3^b5^c
\end{aligned}$$

This shows that in order for $\varphi(n)$ to be a 5-smooth number, we need to have the following conditions:

- Every $p_i - 1$ is either in the form of 2^a, 3^b or 5^c. (i.e. $p_i - 1$ is a 5-smooth number)
- Every $p_i^{a_i - 1}$ is also either in the form of 2^a, 3^b or 5^c. This can be satisfied if (1) $p_i = 2, 3, 5$ with any $a_i \geq 0$ or (2) $p_i = 7, 11, 13, \cdots$ but with $a_i = 1$ (Very Important Point!)

Hence, your implementation should perform the following:

- It is not difficult to realise that generating 5-smooth numbers $< 10^{12}$ to get all the necessary primes is much more efficient than sieving primes and checking smoothness up to 10^{12}
- After obtaining all the primes p where $p - 1$ is 5-smooth, you can then use these specific set of primes to generate all the possible values of n. Hint: The number of $p \leq 10^{12}$ where $p - 1$ is 5-smooth is 545. Have Fun!!! :)