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## **Abkürzungsverzeichnis**

O.B.d.A                    ...

## Symbolverzeichnis

$G(s)$	Übertragungsfunktion eines LTI Systems
$(A, B, C, D)$	Realization des Systems im Zustandsraum



# Chapter 1

## Recap

Topics to be considered:

- (a) Vector norms
- (b) GP
- (c) direct sum

Add Monotonic convergence!

By theorem ??, the reight inverse of  $G = G(A, B, C, D)$  exists if and only if the matrix  $D$  has full row rank.

The choice of  $\beta$  is the only decision to make in this algorithm. For  $\beta$  close to 0 or 2, we get slower convergence and better robustness. For  $\beta = 1$  we get convergence in one iteration, but as discussed above, this choice might be not robust at all. **evtl kommen Robustheit Beispiele.**

If there exists (only) a left inverse of  $G$ , we can also calculate a better choice of input  $u$ . But in this algorithm we do not have guarantee of zero-convergence.



# Chapter 2

## Basics

### <Einfuehrung>

We consider a discrete time iteration

$$\begin{aligned}x(t+1) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t),\end{aligned}\tag{2.1}$$

$$x(0) = x_0, \quad t \geq 0,\tag{2.2}$$

with error

$$e(t) = r(t) - y(t)\tag{2.3}$$

at time  $t = 0, 1, 2, \dots$ .

Here  $A \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^{n \times l}$ ,  $C \in \mathbb{R}^{m \times n}$ , and  $D \in \mathbb{R}^{m \times l}$  are real matrices,  $r(t) \in \mathbb{R}^m$  is a reference signal at time  $t$ . For notation simplification, we write  $(A, B, C, D)$  and mean it to be the discrete-time control system. in (2.1).

We aim to find a "good" input signal  $u(t)$ , such that the reference  $r(t)$  is tracked possibly well for all  $t \geq 0$ . With other words, it means, that the error norm  $\|e(\cdot)\|$  must stay small in some  $\|\cdot\|$ , e.g. the  $L_2$ -norm. We can achieve it by finding of a stabilizing controller. To be able to construct it, we must first clearly define what stability means. Since let us first formulate the common definitions and entertain ideas of a stabilizing controller construction.

## 2.1 Definitions

**Definition 1** *A discrete time difference system*

$$x(t+1) = Ax(t)\tag{2.4}$$

---

for  $t \geq 0$  is said to be asymptotically stable, if for all initial condition  $x(0) \in \mathbb{R}^n$

$$\lim_{t \rightarrow \infty} x(t) = 0 \quad (2.5)$$

is satisfied. It is said to be stable, if

$$\limsup_{t \rightarrow \infty} \|x(t)\| < \infty \quad (2.6)$$

for all initial condition  $x(0) \in \mathbb{R}^n$ . We call the matrix  $A$  (asymptotically) stable, if the dynamical system (2.4) is (asymptotically) stable.

**Theoreme habe ich rausgemacht, da ich sie eh nicht benutze.**

In the same way we want to define (asymptotic) stabilizability and detectability.

First, a discrete time system (2.1) or the pair  $(A, B)$  is (asymptotically) stabilizable, if there exists a matrix  $F$ , such that  $A - BF$  is (asymptotically) stable.

The system or the pair  $(A, C)$  is detectable, if  $(A^T, C^T)$  is asymptotically stabilizable.

## 2.2 Stabilizing Controller based on Separation Principle

We can find a stabilizing controller for (2.1) using feedback control. First let us consider the case  $y(t) = x(t)$ .

Then the system (2.1) becomes stable, if we choose  $u(t) = -Fx(t)$  with some matrix  $F$ , such that  $(A - BF)$  is stable.

Clearly, if the system is (asymptotically) stabilizable, we can find such matrix  $F$ .

But what is a "good" choice for  $F$ ? One option is the so-called Linear-Quadratic-Regulator (LQR), which minimizes the cost function

$$V(u(\cdot)) = \sum_{t=0}^{\infty} ((x(t))^T Q(x(t)) + u(t)^T R u(t)), \quad (2.7)$$

overall stabilizing piecewise constant input signals  $u(\cdot) \in \mathbb{R}^m$ , and with  $x(\cdot)$  satisfying (2.1) (with  $C = I$  and  $D = 0$ ).  $Q \succ 0$  and  $R \succ 0$  are here some weighting matrices.

The solution of this optimal control problem can be found by choosing

$$F = (R + B^T P B)^{-1} B^T P A, \quad (2.8)$$

where  $P$  is a unique symmetric positive definite matrix, the stabilizing solution of the discrete time algebraic Riccati equation

$$A^T A - A^T P B (R + B^T P B)^{-1} B^T P A + Q - P = 0. \quad (2.9)$$

For the case  $y(t) \neq x(t)$ , we can additionally construct an observer to approximate the signal  $x(\cdot)$ . The controller we get is said to be based on *separation principle*. It requires the pair  $(A, C)$  to be asymptotically detectable which allows us to find a matrix  $J$ , such that  $A - JC$  is asymptotically stable.

We set

$$u = -F\hat{x}, \text{ with } \hat{x} \text{ given via dynamical system} \quad (2.10)$$

$$\hat{x}(t+1) = A\hat{x}(t) + Bu(t) + J(y - \hat{y}(t)), \quad (2.11)$$

$$\hat{y}(t) = C\hat{x}(t) + Du(t), \quad (2.12)$$

$$\hat{x}(0) = x_0, \quad t = 0, 1, 2, \dots, N. \quad (2.13)$$

With this observer system the estimation error can be calculated as

$$\tilde{x}(t+1) = x(t+1) - \hat{x}(t+1) = (A - JC)\tilde{x}(t) \quad (2.14)$$

for  $t \geq 0$  and hence  $\hat{x}(\cdot)$  converges to  $x(\cdot)$  at least asymptotically.

We set  $M$  as a right inverse of the matrix

$$D + (C - DF)(I - A + BF)^{-1}B \quad (2.15)$$

Then the choice

$$u = -F\hat{x} + Mr, \quad (2.16)$$

solves tracking issue for constant  $r(\cdot)$ . It still can be used for piecewise constant  $r(\cdot)$ , if the signal does not change too fast.



# Chapter 3

## ILC Algorithms

### <Einführung>

Let us imagine that we need to process the same action multiple times. For example, a robot manipulator must put some objects in a box with high accuracy, while the objects to put are always located in the same place. If we already have some input sequence, which solves this issue, we can use the same data for the next iteration and get the same precision. The other possibility is to "learn" from the previous iteration and try to enhance the exactness, while the tracking objective  $r(\cdot)$  stays the same over all iterations.

More formally, we consider the system (2.1) over a finite time horizon  $t = 0, 1, 2, \dots, N$ ,  $N \in \mathbb{N}_0$ . As we discussed in the previous chapter, we are interested in good tracking of the signal  $r(\cdot)$ . At first glance, the stability concept seems to be not necessary for  $N < \infty$ . Still, in this thesis, it is the requirement we presuppose on the system (2.1). The reason is that also the discrete system over finite horizon can produce vast signals for large  $N$ , if the system is not stable. Especially, it means, that  $A$  has at least one eigenvalue larger than one, which implies very large norm of the matrices  $A^t$  for  $t = 0, 1, \dots, N$ . This makes difficult the numerical calculations, since the solution of the difference equation (2.1) includes it.

Moreover, for our purposes, for stabilizable systems we can assume stability without loss of generality.

If the system is stabilizable, we can find such a matrix  $F$ , such that  $(A + BF)$  is stable. We define  $u(t) = -Fx(t) + \tilde{u}(t)$ ,  $t = 0, 1, 2, \dots, N$  with a new input signal  $\tilde{u}(t) \in \mathbb{R}^l$ . Then the system (2.1) has the form

$$\begin{aligned} x(t+1) &= (A - BF)x(t) + B\tilde{u}(t), \\ y(t) &= (C - DF)x(t) + D\tilde{u}(t), \quad t = 0, 1, 2, \dots, N. \end{aligned} \tag{3.1}$$

---

We can set

$$\begin{aligned}\tilde{A} &= (A - BF), & \tilde{B} &= B \\ \tilde{C} &= (C - DF), & \tilde{D} &= D.\end{aligned}\tag{3.2}$$

The shifted system  $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$  is stable and we still can improve the input signal by considering  $\tilde{u}(\cdot)$ .

**Example 1** Let us consider the system  $(A, B, C, D)$  with

$$A = \begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix}, B = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, C = \begin{pmatrix} 0 & 1 \end{pmatrix}, D = 2.\tag{3.3}$$

This system is controllable and observable, and hence we can calculate the matrices  $F$  and  $J$ , such that

$$(A - BF) \text{ and } (A^T - C^T J) \text{ are stable.}\tag{3.4}$$

We calculate with MATLAB

$$F = \begin{pmatrix} 1.9282 & 1.2679 \end{pmatrix} \text{ and } J = \begin{pmatrix} 1.8304 & 4.6385 \end{pmatrix}.\tag{3.5}$$

This matrices are calculated using discrete algebraic Riccati equation, while we set  $Q = I_m$ ,  $R = I_l$ .

Then we re-describe our system as

$$\begin{aligned}x(t+1) &= \begin{pmatrix} 0.0718 & -0.2679 \\ 0.1436 & 0.4641 \end{pmatrix} x(t) + \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tilde{u}(t) \\ y(t) &= \begin{pmatrix} -3.8564 & -1.5359 \end{pmatrix} x(t) + 2\tilde{u}(t), \quad t = 0, 1, 2, \dots, N.\end{aligned}\tag{3.6}$$

We take as a reference  $r(\cdot)$  the pulse signal and set  $N = 20$ .

With controller based on separation principle we get the tracking, illustrated in Figure ??.

As one can see, the tracking in Example 1 is not perfect at all. However, we can extract a bunch of information from this controlled system. For example, we get the input and output sequences

$$\{u(0), u(1), u(2) \dots, u(N)\} \subset \mathbb{R}^l\tag{3.7}$$

$$\{y(0), y(1), y(2), \dots, y(N)\} \subset \mathbb{R}^m.\tag{3.8}$$

This sequences are limited due to the finite time horizon. It seems to be an advantage. Indeed, if we put the signals we got from simulation together, we get a big, but finite



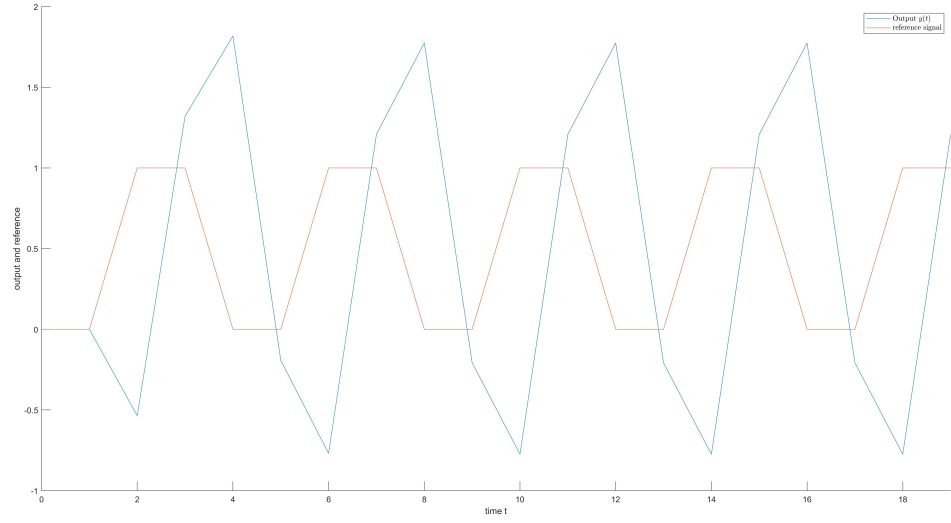


Figure 3.1: Tracking with controller based on separation principle for system (3.6)

dimensional vector. That means, the standard tools from linear algebra can be applied. As a matter of fact, we can express our whole simulated system as a linear equation. We call such vectors *Supervectors*, and the whole system is said to be "lifted".

## 3.1 Supervector description

For the sequences (3.7) and (3.8) we define the supervectors

$$u = \begin{pmatrix} u(0) \\ u(1) \\ u(2) \\ \vdots \\ u(N) \end{pmatrix} \in \mathbb{R}^{l(N+1)}, \quad y = \begin{pmatrix} y(0) \\ y(1) \\ y(2) \\ \vdots \\ y(N) \end{pmatrix} \in \mathbb{R}^{m(N+1)}. \quad (3.9)$$

Respectively, we determine supervectors  $r \in \mathbb{R}^{m(N+1)}$  and  $e \in \mathbb{R}^{m(N+1)}$ .

For system (2.1) the relation between  $u$  and  $y$ , and the error  $e$  can be written in a linear matrix form

$$y = Gu + d, \quad (3.10)$$

$$e = r - y. \quad (3.11)$$

---

The matrix  $G$  represents here the state-space model and is given via

$$G = G(A, B, C, D) = \begin{pmatrix} D & 0 & \cdots & 0 & 0 & 0 \\ CB & D & \cdots & 0 & 0 & 0 \\ CAB & CB & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ CA^{N-2}B & CA^{N-3}B & \cdots & CB & D & 0 \\ CA^{N-1}B & CA^{N-2}B & \cdots & CAB & CB & D \end{pmatrix} \in \mathbb{R}^{m(N+1) \times l(N+1)}. \quad (3.12)$$

The vector  $d$  depends on the initial condition  $x_0$ :

$$d = d(C, A, x_0) = \begin{pmatrix} Cx_0 \\ CAx_0 \\ CA^2x_0 \\ \vdots \\ CA^Nx_0 \end{pmatrix} \in \mathbb{R}^{m(N+1)}. \quad (3.13)$$

It looks pretty simple!

If we know the signal  $r$  we want to track, we can find a perfect solution by choosing

$$u_\infty = G^{-1}r - d. \quad (3.14)$$

However, this choice does not look greatly credible.

For example, if our system is uncertain and is described by  $\tilde{G}$ , but modeled by  $G$ , we get

$$e = r - y = (I - \tilde{G}G^{-1})r. \quad (3.15)$$

This error might be non-zero, even for a little model perturbations. Besides, the matrix  $G$  might be singular and do not have an inverse.

Still it can have a right or left inverse. If we can reformulate this idea in a more robust way, we may get a good and simple applicable algorithm, which we call the inverse model algorithm.

## 3.2 Inverse Model Algorithms

**Algorithm 1** *Let the matrix  $D$  in (2.1) have full row rank. Then the matrix  $G = G(A, B, C, D)$  has a right inverse  $G_R$ :*

$$GG_R = I.$$

The Right Inverse Model Algorithm (RIA) is given via input update law

$$\begin{aligned} u_{k+1} &= u_k + \beta G_R e_k, \\ u_0 &\in \mathbb{R}^{l(N+1)}, \end{aligned} \tag{3.16}$$

with error evolution

$$\begin{aligned} e_{k+1} &= (1 - \beta)e_k, \quad k \geq 0, \\ e_0 &= r - Gu_0 - d. \end{aligned} \tag{3.17}$$

In particular,  $(e_k)_{k \geq 0}$  converges to zero for  $k \rightarrow \infty$  for any initial error  $e_0 \in \mathbb{R}^{m(N+1)}$  if and only if

$$0 < \beta < 2.$$

**Proof:**

Firstly, the matrix  $G$  has lower triangular structure, with the matrix  $D$  on its diagonal. Hence,  $G$  has full row rank if and only if  $D$  has the full row rank. Taking the limit over relation (3.17) results in

$$\lim_{k \rightarrow \infty} e_{k+1} = \lim_{k \rightarrow \infty} (1 - \beta)e_k = \lim_{k \rightarrow \infty} (1 - \beta)^k e_0. \tag{*1}$$

Choosing  $0 < \beta < 2$  yields the proof. ■

For  $\beta$  close to 0 or 2, we get slower convergence and better robustness. For  $\beta = 1$  we get convergence in one iteration, but this choice might be highly non-robust [1], pp 149, 152-155.

If there exists (only) a left inverse of  $G$ , we can also calculate a better choice of input  $u$ . But in this algorithm, we do not have a guarantee of zero-convergence.

**Algorithm 2** Let the matrix  $D$  in (2.1) have full column rank. Then the matrix  $G = G(A, B, C, D)$  has a left inverse  $G_L$ :

$$G_L G = I.$$

The Left Inverse Model Algorithm (LIA) is given via input update law

$$\begin{aligned} u_{k+1} &= u_k + \beta G_L e_k, \quad k \geq 0 \\ u_0 &\in \mathbb{R}^{l(N+1)}. \end{aligned} \tag{3.18}$$

---

with error evolution

$$\begin{aligned} e_k &= (I - \beta GG_L)e_{k-1} = (I + [(1 - \beta)^{k-1} - 1] GG_L)e_0, \quad k \geq 1, \\ e_0 &= r - Gu_0 - d. \end{aligned} \tag{3.19}$$

Monotonic convergence to

$$e_\infty = \lim_{k \rightarrow \infty} e_k = (I - GG_L)e_0 \tag{3.20}$$

for  $k \rightarrow \infty$  is guaranteed if and only if

$$0 < \beta < 2.$$

**Proof:**

Firstly, the matrix  $G$  has lower triangular structure, with the matrix  $D$  on its diagonal. Hence,  $G$  has full column rank if and only if  $D$  has the full column rank.

Using the relation  $(GG_L)^2 = GG_L GG_L = GG_L$  the error evolution formula (3.19) can be proven by mathematical induction:

for  $k = 1$  it follows:

$$e_1 = (I - \beta GG_L)e_0 = (I + GG_L - GG_L - \beta GG_L)e_0 = (I + [(1 - \beta) + 1] GG_L)e_0.$$

For  $k \in \mathbb{N}$

$$\begin{aligned} e_{k+1} &= (I - \beta GG_L)e_k = (I - \beta GG_L)(I + [(1 - \beta)^{k-1} - 1] GG_L)e_0 = \\ &= (I - \beta GG_L + (1 - \beta)^{k-1} GG_L - GG_L - \beta GG_L [(1 - \beta)^{k-1} - 1] GG_L)e_0 = \\ &= (I - (\beta - (1 - \beta)^{k-1} + \beta [(1 - \beta)^{k-1} - 1]) GG_L)e_0 \\ &= (I - (\beta - (1 - \beta)^{k-1} + 1 + \beta(1 - \beta)^{k-1} - \beta) GG_L)e_0 \\ &= (I + [(1 - \beta)^k - 1] GG_L)e_0 \end{aligned}$$

To prove (3.20) recall that

$$\mathbb{R}^{m(N+1)} = \text{im}(G) \oplus \ker(G_L).$$

Hence  $e_0$  can be written as

$$e_0 = Gw_0 + v_0,$$

where  $v_0 = (I - GG_L)e_0 \in \ker(G_L)$  is uniquely defined and  $w_0 = G_L e_0 \in \mathbb{R}^{l(N+1)}$ . It follows:

$$e_k = (1 + [(1 - \beta)^{k-1} - 1] GG_L)e_0 = (1 - \beta)^{k-1} Gw_0 + v_0.$$

Then for  $\beta \in (0, 2)$  the error  $e_k$  converges to  $v_0$  for  $k \rightarrow \infty$ . ■

The choice of  $\beta$  is identical to this in Algorithm 1.

**Example 2** For the system (3.6) we can apply the Right Inverse Model Algorithm. Result for  $N = 20$  and  $\beta = .1$  is illustrated in Figure

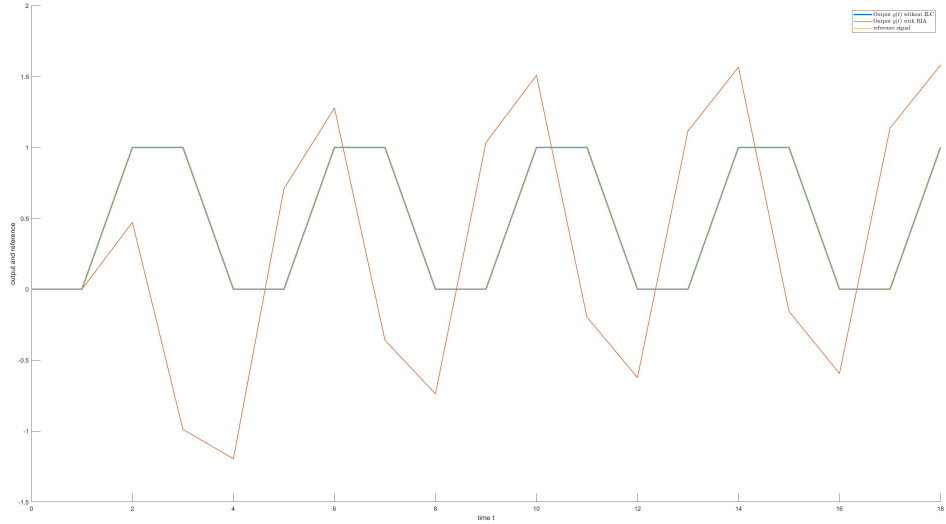


Figure 3.2: Tracking with LQR Controller (without ILC Algorithm), and tracking with Right Inverse Model Algorithm for the system (3.6), with pulse reference signal

Right and Left Inverse Model Algorithms belong to the class of so-called Unit Memory Algorithms.

Indeed, we can formulate such algorithms more general.

### 3.3 Unit Memory Algorithm

We are interested in the processes, which are executed repetitively. Let  $k = 0, 1, 2, \dots$  be the number of completed iterations. Then we get a system

$$\begin{aligned} y_k &= Gu_k + d, \\ e_k &= r - y_k, \end{aligned} \tag{3.21}$$

$u_0 \in \mathbb{R}^{l(N+1)}$ ,  $k = 0, 1, 2, \dots$ . This is again a discrete-time system, with time increment  $k$ .

To achieve a good tracking we need to ensure the norm  $\|e_k\|$  to be small for large  $k$ . A better tracking can be achieved, if the convergence is monotone, i.e.

$$\|e_{k+1}\| < \|e_k\| \text{ for all } k \geq 0. \tag{3.22}$$

---

The asymptotically perfect tracking means

$$\lim_{k \rightarrow \infty} \|e_k\| = 0. \quad (3.23)$$

We have already seen, that it is not always possible to achieve zero convergence. In Algorithm 2 the perfect tracking is possible, if the reference signal  $r$  is feasible. In common the convergence properties can also depend on the choice  $u_0$ .

We assume linear dependency of  $u_{k+1}$  upon  $u_k$  and  $e_k$  for  $k \geq 0$ , and define the following algorithm using feedback control.

**Algorithm 3** *The Unit Memory Algorithm is given via input update law*

$$\begin{aligned} u_{k+1} &= u_k + K e_k, \\ u_0 &\in \mathbb{R}^{l(N+1)}. \end{aligned} \quad (3.24)$$

*The error dynamic is then given via*

$$e_{k+1} = (I - GK)e_k, \quad k \geq 0, \quad (3.25)$$

$$e_0 = r - Gu_0 - d. \quad (3.26)$$

$K \in \mathbb{R}^{l(N+1) \times m(N+1)}$  is called **learning matrix**.

To ensure the stability of the algorithm, it is enough to consider the iteration process

$$e_{k+1} = (I - GK)e_k = (I - GK)^k e_0 := L^k e_0, \quad k \geq 0. \quad (3.27)$$

That it is a discrete dynamical system over  $k$ . We can reformulate our goal as follows: find some matrix  $K \in \mathbb{R}^{l(N+1) \times m(N+1)}$ , which renders (3.27) stable.

We can also rewrite our system as

$$\begin{pmatrix} e_{k+1} \\ e_k \end{pmatrix} = \left( \begin{array}{c|c} I & -G \\ \hline K & 0 \end{array} \right) \begin{pmatrix} e_k \\ v_k \end{pmatrix} \quad (3.28)$$

Then our problem is reduced to finding a stabilizing controller  $K$  with

$$v_k = K e_k. \quad (3.29)$$

We know how to work with such systems – for example, LQR controller provides a good solution. This is a reliable method, but since the matrix  $G$  can be large, it also can be too costly. Often we can achieve good results with much more simple methods. For example, we can reduce the number of parameters to be chosen.

In the Algorithms 1 and 2: we choose the fixed matrix  $K_0$  and as right or left inverse of the matrix  $G$ , and leave the parameter  $\beta$  to decide on.

However, these algorithms have a big disadvantage: the numerical calculation might be inaccurate even for right and left inverse, – or quite impossible if the matrix  $G$  does not have full row/column rank. We illustrate it in the following example.

**Example 3** *We consider again our system (3.6).*

*We have seen, that for  $N = 20$  we can achieve good and robust results. But how does it look for larger  $N$ ?*

*Let us consider the case  $N = 40$ . The error evolution does not depend on the inverse model, and hence converges as expected to 0 (Figure 3.3).*

*But once we try to apply calculated result to our system, we do not see good tracking (Figure 3.4). Quite the same we get a large discrepancy between the reference signal and the system output.*

*If we write our error evolution as*

$$e_k = r - Gu_k - d, \text{ for } k \geq 0, \quad (*)$$

*we can observe a quite different picture compared to the error evolution*

$$e_k = (1 - \beta)^k e_0, \quad k \geq 0. \quad (**)$$

*We illustrate it in the Figure 3.5.*

*The reason might be the condition number of the matrix  $G$ , which equals  $1.0137e+18$ .*

An algorithm, which does not require the model inversion, might be here of use.

## 3.4 Gradient Algorithms

We can begin to define a new algorithm with the requirements on it. We want this algorithm to be monotonic convergent, so we can easily decide on termination criterion. The direct calculation provides

$$\|e_{k+1}\| < \|e_k\| \text{ for all } k > 0, \quad (3.30)$$

in some norm  $\|\cdot\|$  in  $\mathbb{R}^{m(N+1)}$ .

We chose some positive definite weighting matrices  $Q(t)$  and  $R(t)$ ,  $t = 0, 1, 2, \dots, N$ , and define the scalar products

$$\langle y, z \rangle_Q = \sum_{t=0}^N y(t)^T Q(t) z(t), \quad \langle u, v \rangle_R = \sum_{t=0}^N u(t)^T R(t) v(t). \quad (3.31)$$

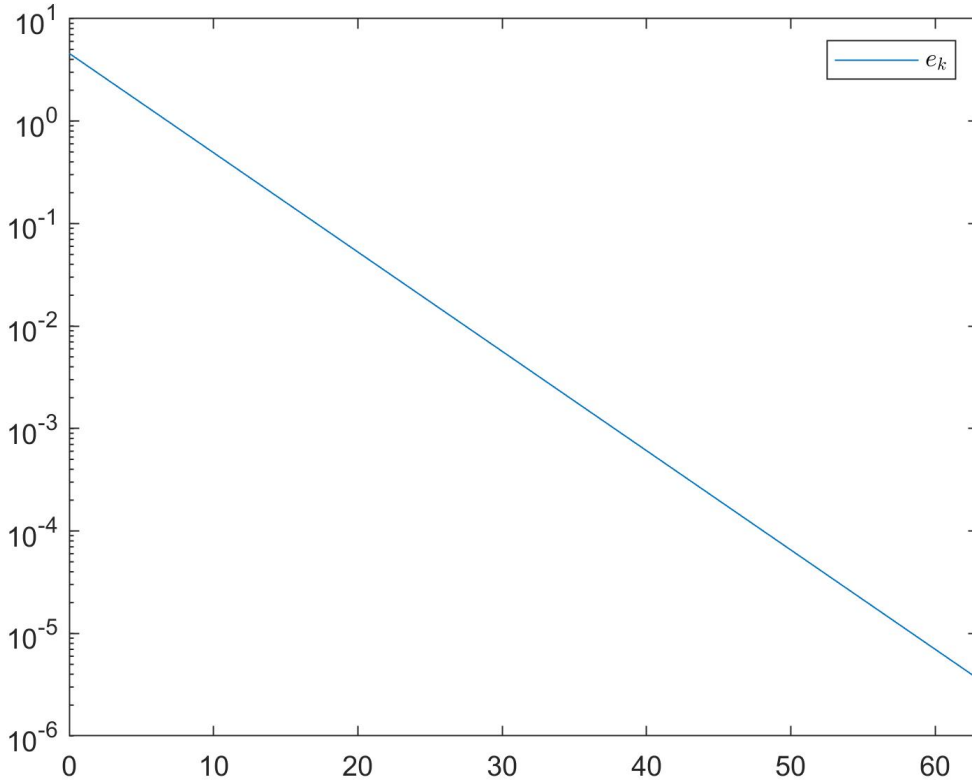


Figure 3.3: Error evolution calculated with (\*) in Right Inverse Model Algorithm for ??

We denote with  $\|\cdot\|_Q$  and  $\|\cdot\|_R$  the associated norms, and set

$$\begin{aligned} G^* &= R^{-1}G^T Q \text{ for all } G \in \mathbb{R}^{m(N+1) \times l(N+1)}, \\ K^* &= Q^{-1}K^T R \text{ for all } K \in \mathbb{R}^{l(N+1) \times m(N+1)}, \end{aligned}$$

where

$$Q := \text{diag}(Q(0), Q(1), Q(2), \dots, Q(N)) \text{ and } R := \text{diag}(R(0), R(1), R(2), \dots, R(N)). \quad (3.32)$$

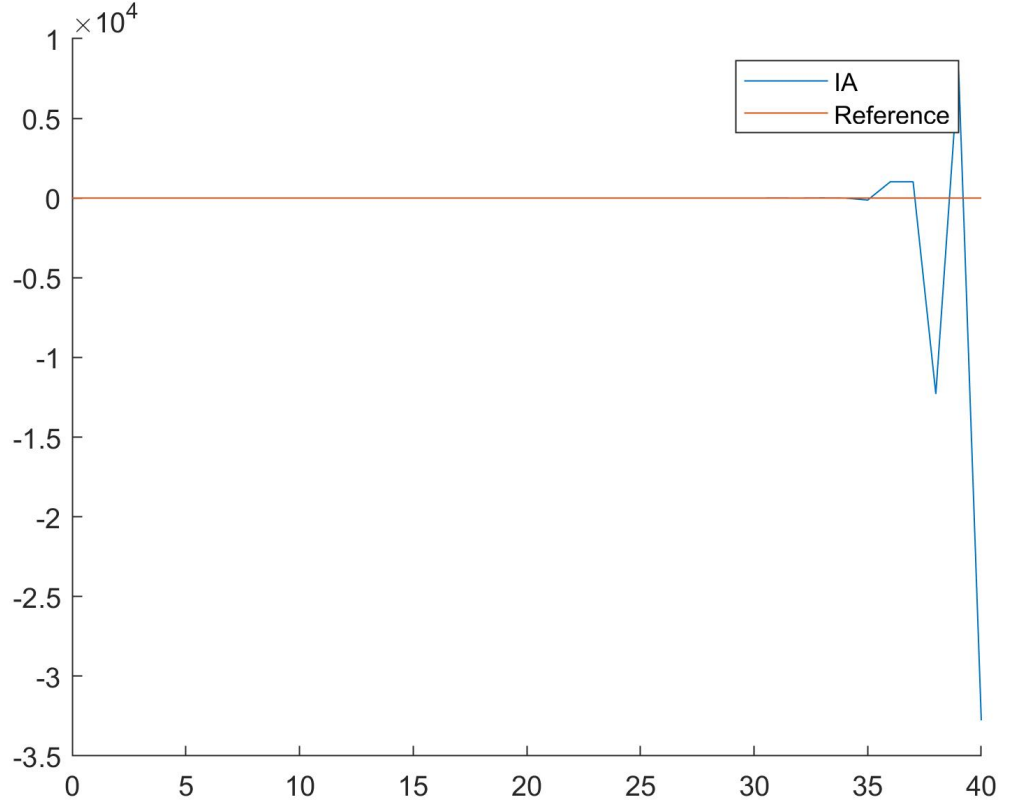
In the same way, we can define

$$L^* = Q^{-1}L^T Q \text{ for all } L \in \mathbb{R}^{m(N+1) \times m(N+1)} \text{ and} \quad (3.33)$$

$$M^* = R^{-1}M^T R \text{ for all } M \in \mathbb{R}^{l(N+1) \times l(N+1)}. \quad (3.34)$$

Recall, that for  $R = I$  and  $Q = I$  the matrices  $G^*$  and  $K^*$  are just the conjugate transpose. The notation from above can be seen as conjugate transpose in the vector space with scalar products (3.31).




 Figure 3.4: Application of the result of RIA for 3.6 for  $N = 40$ 

Applying the weighted norms on (3.30) provides

$$\begin{aligned}
 \|e_{k+1}\|_Q^2 &= \|e_k + (e_{k+1} - e_k)\|_Q^2 = \\
 &= \|e_k\|_Q^2 + 2\langle e_k, e_{k+1} - e_k \rangle_Q + \|e_{k+1} - e_k\|_Q^2 \\
 &= \|e_k\|_Q^2 + 2\langle e_k, (I - GK)e_k - e_k \rangle_Q + \|(I - GK)e_k - e_k\|_Q^2 \\
 &= \|e_k\|_Q^2 - 2\langle e_k, GK e_k \rangle_Q + \|GK e_k\|_Q^2 \\
 &< \|e_k\|_Q^2, \text{ if } \|GK e_k\|_Q^2 < 2\langle e_k, GK e_k \rangle_Q.
 \end{aligned} \tag{3.35}$$

The last inequality is equivalent to

$$\frac{1}{2}e_k^T K^T G^T Q G K e_k < e_k^T G K e_k, \tag{3.36}$$

which is satisfied if

$$(GK)^* GK \prec GK + (GK)^*. \tag{3.37}$$

If we set  $K = \beta G^*$  for some scalar  $\beta > 0$ , the last matrix inequality becomes

$$\beta(GG^*)^2 - 2GG^* \prec 0. \tag{3.38}$$

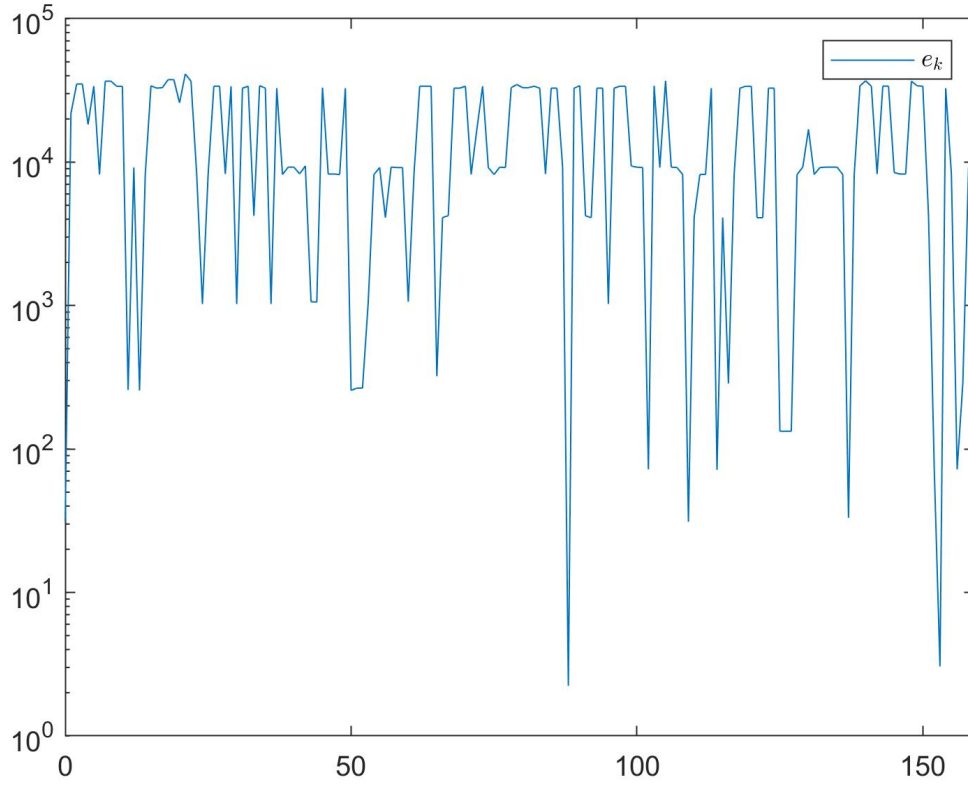


Figure 3.5: Error evolution calculated with (\*\*) in Right Inverse Model Algorithm for 3.6

We choose  $\beta \in (0, \tilde{\beta})$ , with  $\tilde{\beta} = \frac{2}{\sigma_{\max}^2}$ . The  $\sigma_{\max} := \sigma_{\max}(G) = \sigma_{\max}(G^*) := \sqrt{\lambda_{\max}(GG^*)}$  denotes here the square root taken over the maximum eigenvalue of the matrix  $GG^*$ .

Then the last inequality is fulfilled, since

$$\lambda_{\max} \left( \frac{1}{\sigma_{\max}^2} (GG^*)^2 \right) = \frac{1}{\sigma_{\max}^2} \sigma_{\max}^4 = \sigma_{\max}^2 = \lambda_{\max}(GG^*). \quad (3.39)$$

and hence

$$\frac{1}{2} \beta (GG^*)^2 - G^*G \prec \frac{1}{\sigma_{\max}^2} (G^*G)^2 - GG^* \preceq 0. \quad (3.40)$$

This choice of  $K$  determines the *Steepest Descent Algorithm*

### 3.4.1 Steepest Descent Algorithm

**Algorithm 4** *The Steepest Descent Algorithm is characterized by choosing  $K = \beta G^*$ , where  $\beta > 0$  is a real scalar gain. The iterative law is given via*

$$\begin{aligned} u_{k+1} &= u_k + \beta G^* e_k, \\ e_{k+1} &= (I - \beta G G^*) e_k, \quad k \geq 0, \\ u_0 &\in \mathbb{R}^{l(N+1)}. \end{aligned} \tag{3.41}$$

*Monotonic convergence to*

$$e_\infty = \lim_{k \rightarrow \infty} e_k = P_{\ker[G^*]} e_0, \tag{3.42}$$

*is guaranteed if*

$$0 < \beta < \frac{2}{\sigma_{\max}^2}.$$

*Here  $P_{\ker[G^*]}$  denotes the positive orthogonal projection operator onto  $\ker[G^*]$ . In particular, the zero convergence is attainable, if  $e_0 \in \text{im}[GG^*]$ .*

**Proof:**

As next, since  $L^* = L$ , for any  $p \in \mathbb{N}$  it holds:

$$\begin{aligned} \sum_{k=0}^p \|e_k\|^2 &= \sum_{k=0}^p \langle L^k e_0, L^k e_0 \rangle_Q = \sum_{k=0}^p \langle e_0, L^{2k} e_0 \rangle_Q = \sum_{k=0}^p \langle e_0, L^{2k} (I - L^2) w_1 \rangle_Q \\ &= \langle e_0, \left[ \left( \sum_{k=0}^p L^{2k} \right) (I - L^2) \right] w_1 \rangle_Q = \langle e_0, (I - L^{2(p+1)}) w_1 \rangle_Q \\ &\leq \|e_0\|_Q \|w_1\|_Q (1 + \|L\|^{2(p+1)}) \leq 2 \|e_0\|_Q \|w_1\|_Q. \end{aligned} \tag{*1}$$

This yields the convergence of the series, if we let  $p \rightarrow \infty$ . Hence  $(\|e_k\|^2)_k$  is a null sequence, what proofs

$$\lim_{k \rightarrow \infty} e_k = 0. \tag{*2}$$

Now, let  $e_0 \in \mathbb{R}^{m(N+1)}$  be arbitrary. Since  $\mathbb{R}^{m(N+1)} = \ker[I - L] \oplus \text{im}[I - L]$ , we can write  $e_0$  as

$$e_0 = e_{\ker} + e_{\text{im}}, \tag{*3}$$

where  $e_{\ker} \in \ker[I - L]$  and  $e_{\text{im}} \in \text{im}[I - L]$  are uniquely defined.

Hence

$$e_k = L^k e_{\text{im}} + e_{\text{ker}}, \quad (*4)$$

and as  $L^k e_{\text{im}}$  converges to 0 for  $k \rightarrow \infty$ . This shows (3.42), as

$$\lim_{k \rightarrow \infty} e_k = e_{\text{ker}} = P_{\text{ker}[GG^*]} e_0 = P_{\text{ker}[G^*]} e_0. \quad (*5)$$

■

An interesting observation is, that via this algorithms got input signal  $u_\infty$  is the unique solution of the (minimum norm) optimization problem

$$u_\infty = \arg \min_{u \in \mathbb{R}^{l(N+1)}} \{ \|u - u_0\|_R^2 : \text{subject to } r = Gu + d \}. \quad (3.43)$$

One consequence is that the choice of  $u_0$  has more significance than the simple intuition. A good choice will influence convergence rates beneficially. More generally, the limit  $u_\infty$  is the closest input to  $u_0$  in the norm, and since choosing  $u_0 = 0$  leads to minimum energy solution  $u_\infty$  [1] p. 168.

### Choice of the Weighting Matrices

With weighting matrices  $Q$  and  $R$  we can impact the behavior of the algorithm. For example, with the choice of the matrix  $Q$  we can weight some elements of inputs and outputs signals, depending on their importance in measuring accuracies and convergence rates, or accent the relative importance of different time intervals in measuring tracking accuracy and required convergence rates. For rapidly convergence in the initial parts of the time interval, we can set  $Q(t) = \varepsilon^{2t} \tilde{Q}$  with some time independent  $\tilde{Q}$  and  $\varepsilon \in (0, 1)$ .

The choice of  $R(\cdot)$  may arise out of the real need to converge to a minimum input energy solution. Again, using the  $\varepsilon$ -weighed  $R(t) = \varepsilon^{2t} \tilde{R}$  with some fixed  $\tilde{R}$ , we can accent the input signals at some times  $0 \leq t_1 \leq t \leq t_2 \leq N$ . This can be used if we want to limit the control action at some time intervals, or if we need to reflect the physical units used.

**Example 4** For the system (3.6) we choose  $N = 40$ ,  $\beta = 0.0026$  and  $R = I$ ,  $Q = I$ . The error evolution shows much more anticipated behavior (Figure 3.6). The reference tracking is illustrated in Figure 3.6. This time we have indeed the perfect following for the signal  $r$ .

### 3.4.2 Suppression of eigenvalues

In the considered algorithms we assumed the learning matrix  $K$  to be constant for all iterations. We can presume another controller structure and set variable  $K_k$  for  $k \geq 0$ .

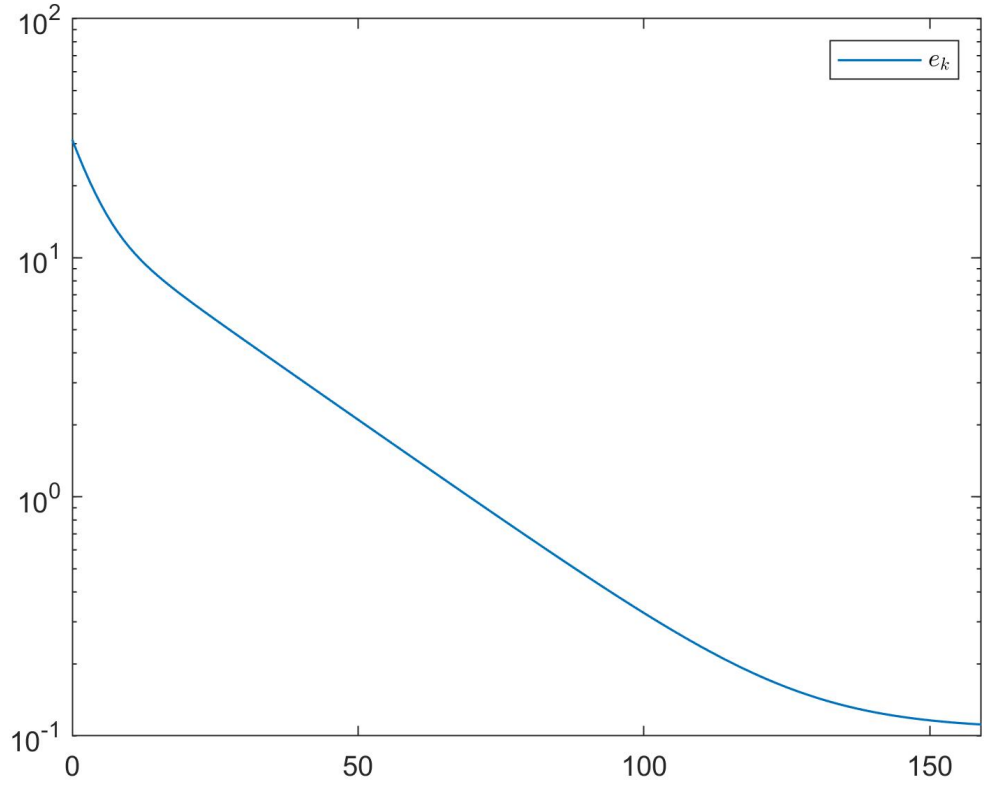


Figure 3.6: Error evolution with Steepest Descent Algorithm for the system (3.6)

The simplest modification way is to set an iteration varying gain  $\beta$  for each step:  $K_k = \beta_k K_0$ ,  $k \geq 0$  ( $K_0$  is here a fixed matrix).

Let us consider the eigenvalues of the matrix  $GG^*$  and choose the gains  $\beta_k$ ,  $k \geq 0$  according to them.

**Theorem 5** Assume, that the matrix  $GG^*$  has  $q$  non-zero ordered eigenvalues  $\lambda_0, \lambda_1, \dots, \lambda_q$ . Set  $\beta_k = \frac{1}{\lambda_k}$  for  $k = 0, 1, \dots, q$  and consider the update law

$$\begin{aligned} u_{k+1} &= u_k + \beta_k G^* e_k, \\ e_{k+1} &= (I - \beta_{k+1} GG^*) e_k = \prod_{l=0}^{k+1} (I - \beta_l GG^*) e_0, \quad k \geq 0, \\ u_0 &\in \mathbb{R}^{l(N+1)}. \end{aligned} \tag{3.44}$$

Then the error sequence  $(e_k)_{k \geq 0}$  converges in a finite number of iterations.

**Proof:**

Set  $\tilde{N} := m(N+1) - 1$ . With spectral theorem, there exists a basis of the eigenvectors  $\{v_0, v_1, \dots, v_{\tilde{N}}\}$  with corresponding eigenvalues  $\lambda_0, \lambda_1, \dots, \lambda_q, 0, \dots, 0$ . Then we can

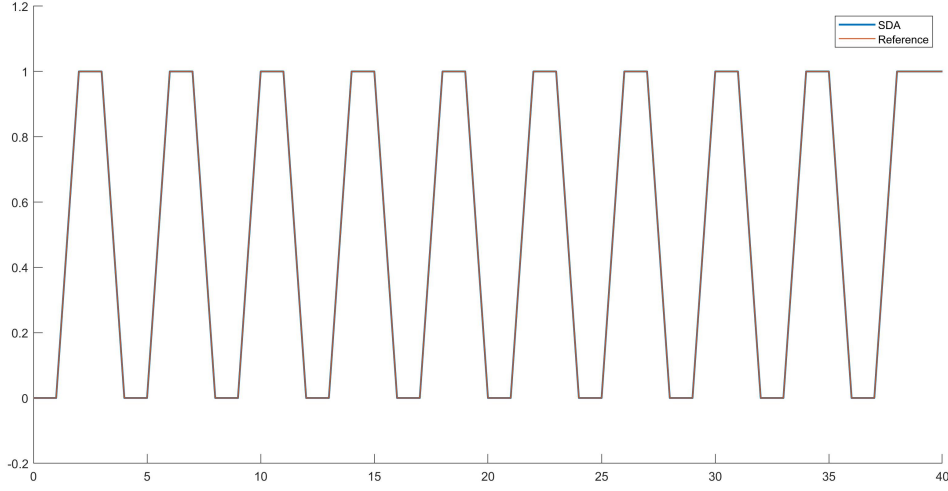


Figure 3.7: Reference signal tracking with Steepest Descent Algorithm for the system (3.6)

write  $e_0$  as

$$e_0 = \sum_{p=0}^{\tilde{N}} \gamma_p v_p \quad (*1)$$

for some uniquely defined  $\gamma_p \in \mathbb{R}$ ,  $p = 0, 1, \dots, \tilde{N}$ .

For  $k \in \mathbb{N}$  it follows

$$e_k = \left( \prod_{j=0}^k (I - \beta_j G G^*) \right) e_0 = \sum_{p=0}^{\tilde{N}} \gamma_p \left( \prod_{j=0}^k (I - \beta_j \lambda_p I) \right) v_p. \quad (*2)$$

For the first iteration the error becomes

$$e_1 = \sum_{p=0}^{\tilde{N}} \gamma_p (I - \beta_1 \lambda_p I) v_p = \sum_{p=1}^{\tilde{N}} \gamma_p (I - \beta_1 \lambda_p I) v_p. \quad (*3)$$

The component  $v_0$  is eliminated from the error. Assuming, that for  $k \geq 1$  the first  $k$  components are eliminated, we get

$$e_{k+1} = \sum_{p=0}^{\tilde{N}} \gamma_p \left( \prod_{j=0}^k (I - \beta_j \lambda_p I) \right) v_p = \sum_{p=k+1}^{\tilde{N}} \gamma_p \left( \prod_{j=0}^k (I - \beta_j \lambda_p I) \right) v_p, \quad (*4)$$

and hence the first  $k + 1$  components  $v_0, v_1, \dots, v_k$  are eliminated. By induction, the iteration process terminates after, at most,  $m(N + 1) - q - 1$  iterations, as all non-zero eigenvalues have been covered and hence all corresponding eigenvectors are eliminated.

■

Although conceptually interesting this algorithm is not well applicable for the real

problems. One can see, that the small non-zero eigenvalues of  $GG^*$  will lead to very large values of  $\beta_k$  and hence we get extremely large transient variations in error norm. Also computing the eigenvalues can be numerically questionable. Moreover, to achieve the monotonic convergence we need to consider only the eigenvalues  $\lambda > \frac{1}{\sigma_{\max}^2}$ . If our model is inaccurate, and some eigenvalues are uncertain, it has directly impact on the iteration result.

That all makes this algorithm not solid. Still, we can use the idea, to apply the eigenvalues of  $GG^*$  – but not directly. We can try to "pick" the compatible eigenvalues.

**Algorithm 5** Choose a finite number  $N_p + 1$  of points  $p_0, p_1, \dots$  spread over the half-open interval  $(\frac{1}{2}\sigma_{\max}^2, \sigma_{\max}^2]$ . The Gradient Algorithm with Suppression of Eigenvalues is defined via choosing of the iteration-depending control law  $K_k = \beta_k G^* e_k$ , where

$$\begin{aligned} \beta_k &= \frac{1}{p_k} \text{ for } k = 0, 1, \dots, N_p, \\ \beta_k &= \beta \text{ for } k > N_p, \end{aligned} \quad (3.45)$$

with  $\beta \in (0, \frac{2}{\sigma_{\max}^2})$ . The iteration law is given via

$$\begin{aligned} u_{k+1} &= u_k + \beta_k G^* e_k, \quad k \geq 0 \\ e_k &= \left[ \prod_{l=0}^k (I - \beta_l GG^*) \right] e_0, \quad k = 0, 1, \dots, N_p, \\ e_k &= (I - \beta GG^*)^{k-N_p} e_{N_p}, \quad k > N_p, \\ e_0 &= r - Gu_0 - d, \quad u_0 \in \mathbb{R}^{l(N+1)}. \end{aligned} \quad (3.46)$$

This algorithm has the same convergence properties as Algorithm 4, but potentially better convergence rates due to the eigenvalue suppression. Intuitively, the approach will increase the convergence speed in the first  $N_p + 1$  iterations, if  $N_p$  is large enough for good approximation of the interval  $(\frac{1}{2}\sigma_{\max}^2, \sigma_{\max}^2]$ .

**Example 6** We apply the SoE Algorithm on the system (3.6). The error evolution we get is illustrated in Figure 3.8. The new algorithm needs only 148 iteration, while classical SDA terminates by 382 iterations. We also get a much better improvement  $\|e_0/e_\infty\|$ , while  $e_\infty$  is meant to be the error in the last iteration.

In fact, for each eigenvalue  $\lambda$  of  $GG^*$  holds

$$\lambda > 2/\sigma_{\max} = .1444. \quad (3.47)$$

This explained the better performance of the algorithm, and better convergence rate.

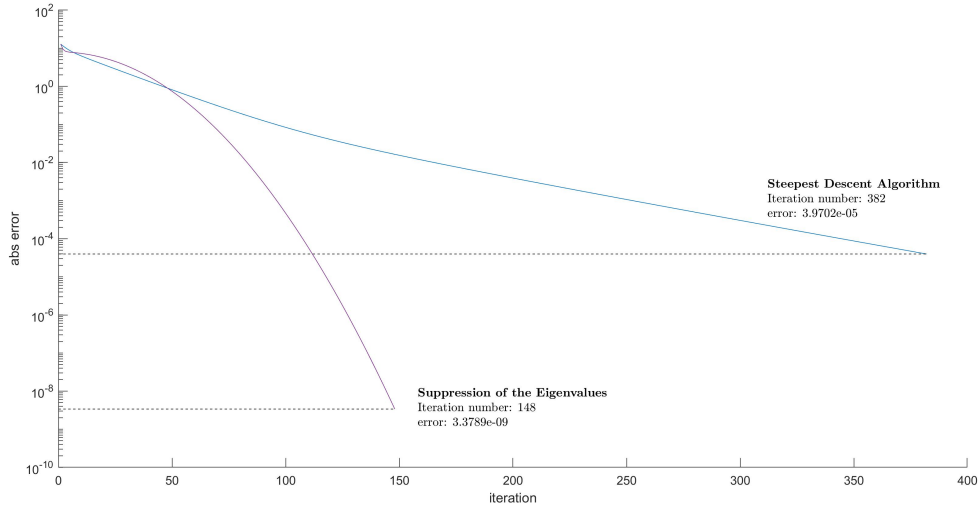


Figure 3.8: Error evolution for classical SDA and SoE for system (3.6)

### 3.5 Feedback Design

With the previous algorithms we can achieve a better tracking for repetitively executed processes. But they do not accent the special features of the model. For example, we can not manage the non-minimum-phase zeros, which can impact the convergence rate [1]. If we consider the plant structure and construct a feedback controller, and then converts it into a steepest descent-like algorithm, we possibly can achieve better performance and robustness properties.

More precisely, we consider a forward path compensator  $K_c$  in a unity negative feedback system for (2.1). The design criteria for  $K_c$  include the closed-loop stability and the ability to track, albeit approximately, the reference signal  $r$ . With this compensator, depending on the model, we can remedy the plant properties such as oscillation, loop interaction, or the effects of non-minimum-phase zeros.

We denote the closed-loop matrix  $T$  and sensitivity matrix  $S$  of the resultant closed-loop system by

$$T = (I + GK_c)^{-1}GK_c \text{ and } S = I - T = (I + GK_c)^{-1}. \quad (3.48)$$

Then the compensated Steepest Descent Algorithm is given as follows.

**Algorithm 6** *The compensated Steepest Descent Algorithm is characterized by choosing  $K = \beta K_c S$ ,  $\beta \in \mathbb{R}$ . The iterative law is given via*

$$\begin{aligned} u_{k+1} &= u_k + \beta K_c S T^* e_k, \\ e_{k+1} &= (I - \beta T T^*) e_k, \quad k \geq 0, \\ u_0 &\in \mathbb{R}^{l(N+1)}. \end{aligned} \quad (3.49)$$



*Monotonic convergence to*

$$e_\infty = \lim_{k \rightarrow \infty} e_k = P_{\ker[T^*]} e_0, \quad (3.50)$$

*is guaranteed if*

$$0 < \beta < \frac{2}{\sigma_{\max}(T)^2}.$$

*$T^*$  stays here for conjugate transpose of the matrix  $T$ .*

One can see, that the error evolution corresponds to this of the model  $y = T\tilde{u} + d$  with input  $\tilde{u} \in \mathbb{R}^{m(N+1)}$ , if we apply on it the Steepest Descent Algorithm.

The effect of the operator  $TT^*$  can be seen by considering the closed-loop relation

$$y = Tr. \quad (3.51)$$

We get a perfect tracking, if  $T = I$ , which is not achievable by feedback control. Still, we can assume a good tracking if  $\|T\| \approx 1$ .

Therefore if  $K_c$  provides excellent feedback control of  $G$ , then rapid convergence could be attained with a choice of gain  $\beta \approx 1$ .



# Chapter 4

## Robustness of the ILC Algorithms

If our model is not precisely known a chosen ILC algorithm might not provide the desired result. Therefore it is useful to consider how robust the algorithms are. In this thesis, we are going to consider the parametric uncertainty. That means, we do not know the true value of a real parameter (or the parameters) but know some range in which this parameter must lie.

We can normalize such uncertainties without loss of generality, as any uncertainty  $\delta \in [a, b] \subset \mathbb{R}$  can be expressed as

$$\delta = \delta_0 + w\hat{\delta} \quad (4.1)$$

with  $\hat{\delta} \in [-1, 1]$ , nominal value  $\delta_0 \in \mathbb{R}$  and weight  $w \in \mathbb{R}$ .

In order to guarantee the convergence of the algorithms despite such parametric uncertainties it is required that (3.27) stays stable for all possible uncertainties.

Let us consider, how the uncertainty looks in the lifted system like and deliberate over the method to solve the robustness issue.

<Einfuehrung>

### 4.1 Uncertain Lifted System

Let us consider a system (2.1) with uncertainty  $\hat{\Delta}$  from the set

$$\hat{\Delta} = \left\{ \begin{pmatrix} \delta_1 I_{p_1} & & \\ & \ddots & \\ & & \delta_\tau I_{p_\tau} \end{pmatrix} \mid \delta_l \in [-1, 1], p_l \in \mathbb{N} \text{ for } l = 1, 2, \dots, \tau. \right\}. \quad (4.2)$$

$\tau \in \mathbb{N}$  is the number of the uncertain parameter.

Each uncertain system  $\left[ \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] (\hat{\Delta})$  we can rewrite as a feedback interconnection

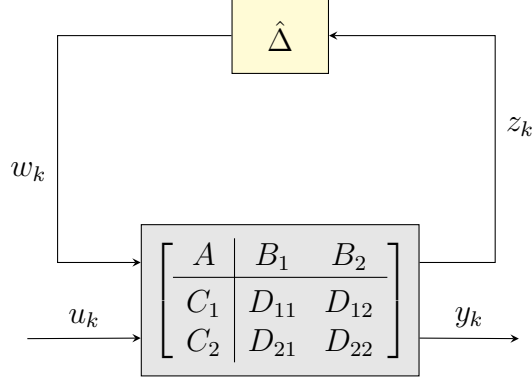


Figure 4.1: Uncertain system as feedback interconnection of LFR and the uncertainty  $\hat{\Delta}$

of the system  $\left[ \begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array} \right]$  and  $\hat{\Delta}$ . Then the original system can be find as star product (or linear fractional transformation, LFT)

$$\left[ \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] = \hat{\Delta} \star \left[ \begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array} \right] \quad (4.3)$$

We illustrate it in the Figure 4.1. We speak here about the "pulling out" the uncertainty, or the *linear fractional representation* (LFR).

With LFR we rewrite the system (2.1) as

$$\begin{aligned} x(t+1) &= Ax(t) + B_1u(t) + B_2w(t), \\ y(t) &= C_1x(t) + D_{11}u(t) + D_{12}w(t), \\ z(t) &= C_2x(t) + D_{21}u(t) + D_{22}w(t), \\ w(t) &= \hat{\Delta}z(t), \\ x(0) &\in \mathbb{R}^n, t = 0, 1, 2, \dots, N, \end{aligned} \quad (4.4)$$

with error  $e(\cdot) = r(\cdot) - y(\cdot)$ .

The corresponding lifted system has the form

$$y = G_1u + G_2w + d_1, \quad (4.5)$$

$$z = G_3u + G_4w + d_2, \quad (4.6)$$

$$e = r - y, \quad (4.7)$$

$$w = \Delta z, \quad (4.8)$$

with the matrices

$$\begin{aligned}
 G_1 &= \\
 &= \begin{pmatrix} D_{11} & 0 & \cdots & 0 & 0 & 0 \\ C_1 B_1 & D_{11} & \cdots & 0 & 0 & 0 \\ C_1 A B_1 & C_1 B_1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ C_1 A^{N-2} B_1 & C_1 A^{N-3} B_1 & \cdots & C_1 B_1 & D_{11} & 0 \\ C_1 A^{N-1} B_1 & C_1 A^{N-2} B_1 & \cdots & C_1 A B_1 & C_1 B_1 & D_{11} \end{pmatrix}, \quad (4.9)
 \end{aligned}$$

$$\begin{aligned}
 G_2 &= \\
 &= \begin{pmatrix} D_{12} & 0 & \cdots & 0 & 0 & 0 \\ C_1 B_2 & D_{12} & \cdots & 0 & 0 & 0 \\ C_1 A B_2 & C_1 B_2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ C_1 A^{N-2} B_2 & C_1 A^{N-3} B_2 & \cdots & C_1 B_2 & D_{12} & 0 \\ C_1 A^{N-1} B_2 & C_1 A^{N-2} B_2 & \cdots & C_1 A B_2 & C_1 B_2 & D_{12} \end{pmatrix}, \quad (4.10)
 \end{aligned}$$

$$\begin{aligned}
 G_3 &= \\
 &= \begin{pmatrix} D_{21} & 0 & \cdots & 0 & 0 & 0 \\ C_2 B_1 & D_{21} & \cdots & 0 & 0 & 0 \\ C_2 A B_1 & C_2 B_1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ C_2 A^{N-2} B_1 & C_2 A^{N-3} B_1 & \cdots & C_2 B_1 & D_{21} & 0 \\ C_2 A^{N-1} B_1 & C_2 A^{N-2} B_1 & \cdots & C_2 A B_1 & C_2 B_1 & D_{21} \end{pmatrix}, \quad (4.11)
 \end{aligned}$$

$$\begin{aligned}
 G_4 &= \\
 &= \begin{pmatrix} D_{22} & 0 & \cdots & 0 & 0 & 0 \\ C_2 B_2 & D_{22} & \cdots & 0 & 0 & 0 \\ C_2 A B_2 & C_2 B_2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ C_2 A^{N-2} B_2 & C_2 A^{N-3} B_2 & \cdots & C_2 B_2 & D_{22} & 0 \\ C_2 A^{N-1} B_2 & C_2 A^{N-2} B_2 & \cdots & C_2 A B_2 & C_2 B_2 & D_{22} \end{pmatrix}, \quad (4.12)
 \end{aligned}$$

---

and vectors

$$d_1 = \begin{pmatrix} C_1 x_0 \\ C_1 A x_0 \\ C_1 A^2 x_0 \\ \vdots \\ C_1 A^N x_0 \end{pmatrix}, \quad (4.13)$$

$$d_2 = \begin{pmatrix} C_2 x_0 \\ C_2 A x_0 \\ C_2 A^2 x_0 \\ \vdots \\ C_2 A^N x_0 \end{pmatrix}. \quad (4.14)$$

$\Delta$  is here a block diagonal matrix

$$\Delta = \text{diag}(\underbrace{\hat{\Delta}, \hat{\Delta}, \dots, \hat{\Delta}}_{(N+1) \text{ times}}) \quad (4.15)$$

and we define

$$\mathbf{\Delta} = \{\text{diag}(\hat{\Delta}, \hat{\Delta}, \dots, \hat{\Delta}) \mid \hat{\Delta} \in \hat{\mathbf{\Delta}}\}. \quad (4.16)$$

Using LFT, we get

$$y = G(\Delta)u + d(\Delta) := [G_1 + G_2\Delta(I - G_4\Delta)^{-1}G_3] u + [G_2\Delta(I - G_4\Delta)^{-1}d_2 + d_1], \quad (4.17)$$

while the inverse of the matrix  $(I - G_4\Delta)$  is assumed to exist. It is always the case if the norm of the matrix  $G_4$  is (strictly) bounded by 1 [2].

With ILC algorithms we get the input update rule

$$u_{k+1} = u_k + K e_k, \quad (4.18)$$

and hence

$$e_{k+1} = L(\Delta)e_k = (I - G(\Delta)K)e_k. \quad (4.19)$$

If we pull out the uncertainty from  $L(\Delta)$ , we get

$$\begin{pmatrix} e_{k+1} \\ z_k \end{pmatrix} = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix} \begin{pmatrix} e_k \\ w_k \end{pmatrix} := \begin{pmatrix} I - G_1 & -G_2 \\ G_3 K & G_4 \end{pmatrix} \begin{pmatrix} e_k \\ w_k \end{pmatrix}, \quad (4.20)$$

$$w_k = \Delta z_k,$$

with the uncertain system defined through the uncertain matrix

$$L(\Delta) = \Delta \star \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix} = \mathcal{A} + \mathcal{B}\Delta(I - \mathcal{D}\Delta)^{-1}\mathcal{C}. \quad (4.21)$$

We are interested in ensuring of the stability for the system (4.19).

The most obviously way is to ensure  $\|L(\Delta)\| < 1$  for all uncertainties  $\Delta \in \mathbf{\Delta}$ . However, this criteria might be too restrictive. We illustrate it on the following example.

**Example 7** *Let us consider the system*

$$e_{k+1} = L(\alpha)e_k = \begin{pmatrix} .1 & \alpha \\ 0 & .1 \end{pmatrix} e_k \quad (4.22)$$

*with uncertain parameter  $\alpha \in \mathbb{R}$ . Then the norm  $\|L(\alpha)\|$  is proportional to  $|\alpha|$ .*

*For example, with the spectral norm, we can ensure stability only for  $|\alpha| < 1$ .*

*On the other hand, one can directly see, that the system is stable for all  $\alpha \in \mathbb{R}$ , since the eigenvalues of the matrix equal .1.*

This example motivates to devise another method for ensuring the stability of the uncertain systems. Much more elegant results we can reach using Full-Block S-Procedure.

## 4.2 Full-Block S-Procedure

First, let us consider the uncertainty  $\Delta$  without taking into account its structure. Using the Lyapunov LMI we formulate the following Theorem.

**Theorem 8** *Suppose that there exists some positive definite matrix  $X$  satisfying*

$$\begin{pmatrix} I \\ L(\Delta) \end{pmatrix}^T \begin{pmatrix} -X & 0 \\ 0 & X \end{pmatrix} \begin{pmatrix} I \\ L(\Delta) \end{pmatrix} = L(\Delta)XL(\Delta) - X \preceq 0 \text{ for all } \Delta \in \mathbf{\Delta}. \quad (4.23)$$

*Then the discrete dynamical system (4.19) is stable for all  $\Delta \in \mathbf{\Delta}$  and for all  $e_0 \in \mathbb{R}^{m(N+1)}$ . It is asymptotically stable if the inequality is strict.*

**Proof:**

First, let us prove the Theorem for the strict LMI. Then we can find some  $\gamma \in (0, 1)$ ,

such that

$$\begin{pmatrix} I \\ L(\Delta) \end{pmatrix} \begin{pmatrix} -\gamma X & 0 \\ 0 & X \end{pmatrix} \begin{pmatrix} I \\ L(\Delta) \end{pmatrix} = L(\Delta)^T X L(\Delta) - \gamma X \prec 0 \text{ for all } \Delta \in \mathbf{\Delta}. \quad (*1)$$

Moreover, since  $X \succ 0$ , there exist some  $\alpha, \beta > 0$ , such that

$$\alpha I \prec X \prec \beta I. \quad (*2)$$

Set  $\eta_k := e_k^T X e_k$ . Then we have

$$\eta_{k+1} = e_{k+1}^T X e_{k+1} = e_k^T L(\Delta)^T X L(\Delta) e_k \leq \gamma e_k^T X e_k = \gamma \eta_k \text{ for all } k \geq 0. \quad (*3)$$

From (\*2) we get

$$\alpha \|e_k\|^2 \leq \eta_k \leq \beta \|e_k\|^2 \text{ for all } k \geq 0, \quad (*4)$$

and with (\*3) it follows

$$\|e_k\|^2 \leq \frac{1}{\alpha} \eta_k \leq \frac{\gamma^k}{\alpha} \eta(0) \leq \frac{\beta}{\alpha} \gamma^k \|e_0\|^2. \quad (*5)$$

Since  $\gamma \in (0, 1)$ ,  $\|e_k\|^2 \rightarrow 0$  for  $k \rightarrow \infty$ , and taking the square root yields the claim. For not strict case, we set in (\*3) and (\*5)  $\gamma = 1$ , and replace in (\*1) " $\prec$ " with " $\preceq$ ". Then  $\|e_k\|^2$  (and hence  $\|e_k\|$ ) is bounded over all  $k \geq 0$  and hence the system (4.19) is stable. ■

In this Theorem, we still have the uncertainty in our LMI problem. To get rid of it, we fix the matrix  $X \succ 0$  and try to find a multiplier  $P = P^T$ , which fulfills

$$\begin{pmatrix} I \\ \Delta \end{pmatrix}^T P \begin{pmatrix} I \\ \Delta \end{pmatrix} \succeq 0, \quad (4.24)$$

and, additionally,

$$\begin{pmatrix} I & 0 \\ \mathcal{A} & \mathcal{B} \end{pmatrix}^T \begin{pmatrix} -X & 0 \\ 0 & X \end{pmatrix} \begin{pmatrix} I & 0 \\ \mathcal{A} & \mathcal{B} \end{pmatrix} + \begin{pmatrix} \mathcal{C} & \mathcal{D} \\ 0 & I \end{pmatrix}^T P \begin{pmatrix} \mathcal{C} & \mathcal{D} \\ 0 & I \end{pmatrix} \preceq 0. \quad (4.25)$$

If we can find such a matrix  $P$ , the stability of the system (4.19) can be proven.

We formulate it as the following Theorem.

**Theorem 9** *Let  $\mathbb{P}$  be a subset of the symmetric matrices, and  $X \succ 0$  be fixed. Then the system (4.19) is stable if there exists a matrix  $P = P^T \in \mathbb{P}$ , such that (4.24) and (4.25) are fulfilled. If the inequality (4.25) is strict, system becomes asymptotic stable.*



**Proof:**

Let  $P = P^T \in \mathbb{P}$  be some matrix, which fulfills the relations (4.24) and (4.25).

We multiply (4.25) with

$$H = \begin{pmatrix} I \\ \Delta(I - \mathcal{D}\Delta)^{-1}C \end{pmatrix} \quad (*1)$$

right and its transpose left. Then the first term results in

$$H^T \begin{pmatrix} I & 0 \\ \mathcal{A} & \mathcal{B} \end{pmatrix}^T \begin{pmatrix} -X & 0 \\ 0 & X \end{pmatrix} \begin{pmatrix} I & 0 \\ \mathcal{A} & \mathcal{B} \end{pmatrix} H = L(\Delta)^T X L(\Delta) - X. \quad (*2)$$

Hence, if we can prove that the second term multiplied with  $H$  right and its transpose left is positive semi-definite, this matrix  $P$  will ensure the monotonic convergence of the algorithms. Multiplying  $H$  yields to

$$H^T \begin{pmatrix} \mathcal{C} & \mathcal{D} \\ 0 & I \end{pmatrix}^T P \begin{pmatrix} \mathcal{C} & \mathcal{D} \\ 0 & I \end{pmatrix} H = (I - \Delta\mathcal{D})^{-T} \underbrace{\begin{pmatrix} I \\ \Delta \end{pmatrix}^T P \begin{pmatrix} I \\ \Delta \end{pmatrix}}_{\succeq 0} (I - \Delta\mathcal{D}) \succeq 0,$$

what proves the first statement. If the inequality (4.25) is strict, the LMI (4.23) is also strict and we achieve the asymptotic stability.  $\blacksquare$

Hence, the robustness issue can be reduced to a certain LMI over a multiplier set  $\mathbb{P}$ . The choosing of the set  $\mathbb{P}$  is here the cornerstone. It must be large enough, to be able to find such multiplier  $P$ , if the system is stable, but also adequate small, such that the searching action is not too costly.

Let us begin with a single uncertain parameter  $\delta$ . Then the uncertainty  $\Delta = \delta I$  with  $\delta \in [-1, 1]$ . A good choice for multiplier set is

$$\mathbb{P} = \left\{ \begin{pmatrix} S & R \\ R^T & -S \end{pmatrix} \mid S \succeq 0, R + R^T = 0 \right\}. \quad (4.26)$$

There is

$$\begin{pmatrix} I \\ \delta I \end{pmatrix}^T \begin{pmatrix} S & R \\ R^T & -S \end{pmatrix} \begin{pmatrix} I \\ \delta I \end{pmatrix} = \underbrace{(1 - \delta^2)S}_{\succeq 0} + \delta \underbrace{(R + R^T)}_{=0} \succeq 0, \quad (4.27)$$

and hence the requirements on the set  $\mathbb{P}$  for the Theorem (9) are fulfilled.

For  $\tau \in \mathbb{N}$  uncertain real parameters the uncertainty has the form

$$\Delta = \text{diag}(\underbrace{\hat{\Delta}_\tau, \hat{\Delta}_\tau, \dots, \hat{\Delta}_\tau}_{(N+1) \text{ times}}) \quad (4.28)$$

with

$$\hat{\Delta}_\tau = \begin{pmatrix} \delta_1 I_{p_1} & & & \\ & \delta_2 I_{p_2} & & \\ & & \ddots & \\ & & & \delta_\tau I_{p_\tau} \end{pmatrix}. \quad (4.29)$$

If we choose a permutation matrix  $\mathcal{P}$ , such that

$$\Delta_{\text{perm}} = \mathcal{P}\Delta = \begin{pmatrix} \delta_1 I_{(p_1 \cdot N)} & & & \\ & \delta_2 I_{(p_2 \cdot N)} & & \\ & & \ddots & \\ & & & \delta_\tau I_{(p_\tau \cdot N)} \end{pmatrix}. \quad (4.30)$$

the convergence properties of the algorithm are not impacted:  $G(\Delta)$  in (4.17) can be rewritten as

$$G(\Delta) = G_1 + G_2 \mathcal{P}^T \mathcal{P} \Delta (I - G_4 \mathcal{P}^T \mathcal{P} \Delta)^{-1} G_3 \quad (4.31)$$

$$= (\mathcal{P}\Delta) \star \left[ \left( \begin{array}{c|c} G_1 & G_2 \\ \hline G_3 & G_4 \end{array} \right) \begin{pmatrix} I & \\ & \mathcal{P}^T \end{pmatrix} \right], \quad (4.32)$$

and the uncertain vector  $d(\Delta)$  as

$$d(\Delta) = G_2 \mathcal{P}^T \mathcal{P} \Delta (I - G_4 \mathcal{P}^T \mathcal{P} \Delta)^{-1} d_2 + d_1 \quad (4.33)$$

$$= (\mathcal{P}\Delta) \star \left[ \left( \begin{array}{c|c} d_1 & G_2 \\ \hline d_2 & G_4 \end{array} \right) \begin{pmatrix} I & \\ & \mathcal{P}^T \end{pmatrix} \right]. \quad (4.34)$$

Hence we can use the structure as in (4.30) without loss of generality. The multiplier set can be chosen as

$$\mathbb{P}_\tau = \left\{ \begin{pmatrix} S & R \\ R^T & -S \end{pmatrix} \middle| \begin{array}{l} S = \text{diag}(S_1, S_2, \dots, S_\tau) \succeq 0, \\ R = \text{diag}(R_1, R_2, \dots, R_\tau), R + R^T = 0 \end{array} \right\}. \quad (4.35)$$

For  $P_\tau \in \mathbb{P}_\tau$

$$\begin{pmatrix} I \\ \Delta_{\text{perm}} \end{pmatrix}^T P_\tau \begin{pmatrix} I \\ \Delta_{\text{perm}} \end{pmatrix} = \begin{pmatrix} (1 - \delta_1^2) S_1 & & & \\ & (1 - \delta_2^2) S_2 & & \\ & & \ddots & \\ & & & (1 - \delta_\tau^2) S_\tau \end{pmatrix}. \quad (4.36)$$

Go back to the example 7. Let  $\alpha \in [0, 2]$  be an uncertainty with the nominal value

$\alpha_0 = 1$ . With linear fractional transformation the system

$$e_{k+1} = \begin{pmatrix} .1 & \alpha \\ 0 & .1 \end{pmatrix} e_k \quad (4.37)$$

results in

$$\begin{pmatrix} e_{k+1} \\ z_k \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & .1 & 1 \\ 0 & 0 & .1 \end{pmatrix} \begin{pmatrix} e_k \\ w \end{pmatrix}. \quad (4.38)$$

With the choice

$$X = \text{diag}(2, .5) \quad (4.39)$$

we can compute with MATLAB the solution

$$P = \text{diag}(4.61, -4.61), \quad (4.40)$$

which renders the eigenvalues of the left side term in the relation (4.25) to

$$\lambda_1 = -2.35, \lambda_2 = -1.21 \text{ and } \lambda_3 = -.488. \quad (4.41)$$

Hence, according to the Theorem 9, the system is stable for all  $\alpha \in [0, 2]$ .

**Example 10** *In the chapter 3 we said, that the choice of  $\beta$  might impact the robustness properties of the Inverse Model Algorithms. Let us illustrate it on the system (3.3), but with uncertain parameter  $a \in [\underline{a}, \bar{a}]$  and nominal value  $a_0 = 4$ :*

$$A = \begin{pmatrix} 2 & 1 \\ a & 3 \end{pmatrix} \quad (4.42)$$

*For  $N = 23$  and  $\beta = .1$  our system still stays robust for  $\bar{a} = -\underline{a} = 2$ . If we set  $\beta = .4$ , the system is not robust even for  $\bar{a} = -\underline{a} = .1$ .*

*However, the ability to find the adequate matrices  $X$  and  $P$  depends on the time horizon  $N$ . For  $N = 20$ , we get a much more robust system, for which the stability can be proven for  $\bar{a} = -\underline{a} = 2$ .*



# Chapter 5

## Examples

Ein paar Beispiele, die ich spaeter in die Kapitel reinstelle.

### Example 11

$$A = \begin{pmatrix} -.5 & 0 \\ 0 & -.3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, N = 200. \quad (5.1)$$

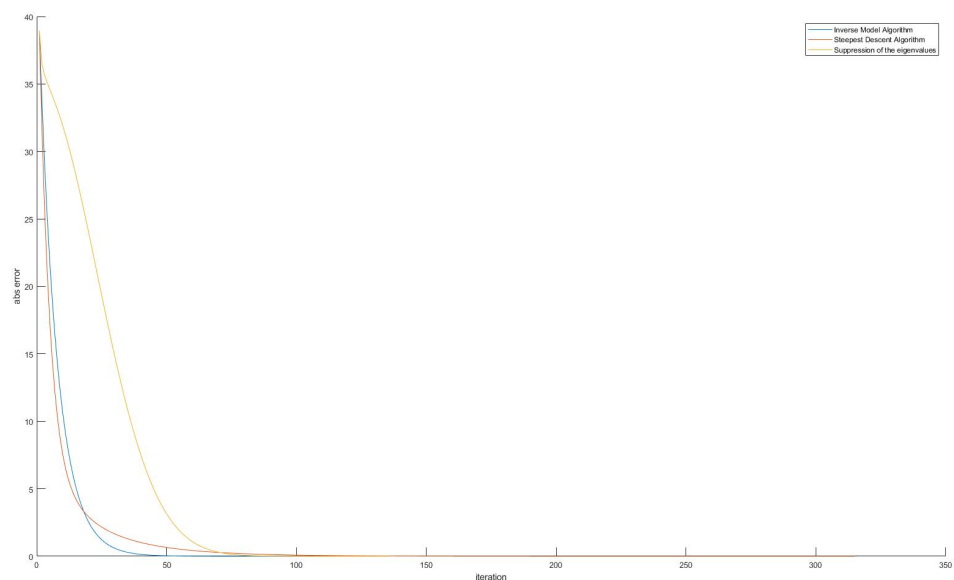


Figure 5.1: Example 11.  $u_0 = .1$ ,  $r(t)$  is random for  $t = 0, 1, 2, \dots, N$ .

### Example 12 *Satellite Example*

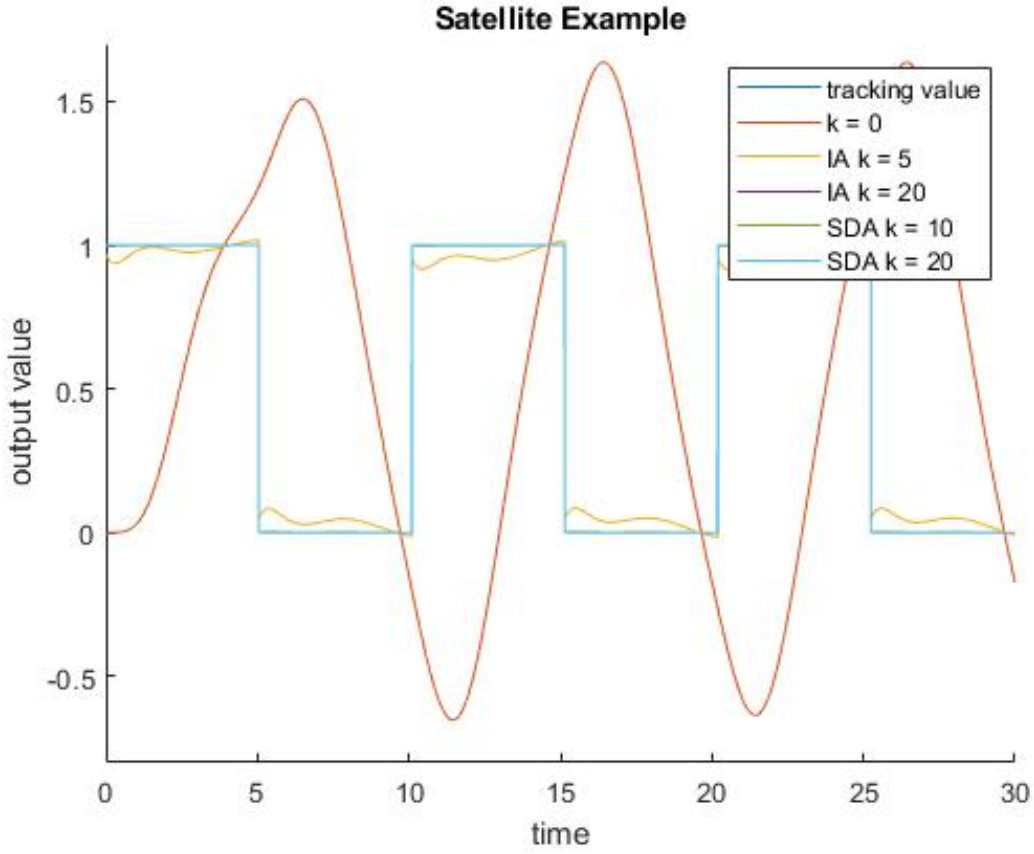


Figure 5.2

**Example 13** *Uncertain Example*

$$A = \begin{pmatrix} -.5 + a & 0 \\ 0 & -.4 - a \end{pmatrix}, B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, C = \begin{pmatrix} .1 & .2 \end{pmatrix}, D = 1, x_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, N = 200. \quad (5.2)$$

$a$  is an uncertain parameter with nominal value  $\hat{a} = 1.5$  and variability  $a \in [1.3, 1.8]$ .

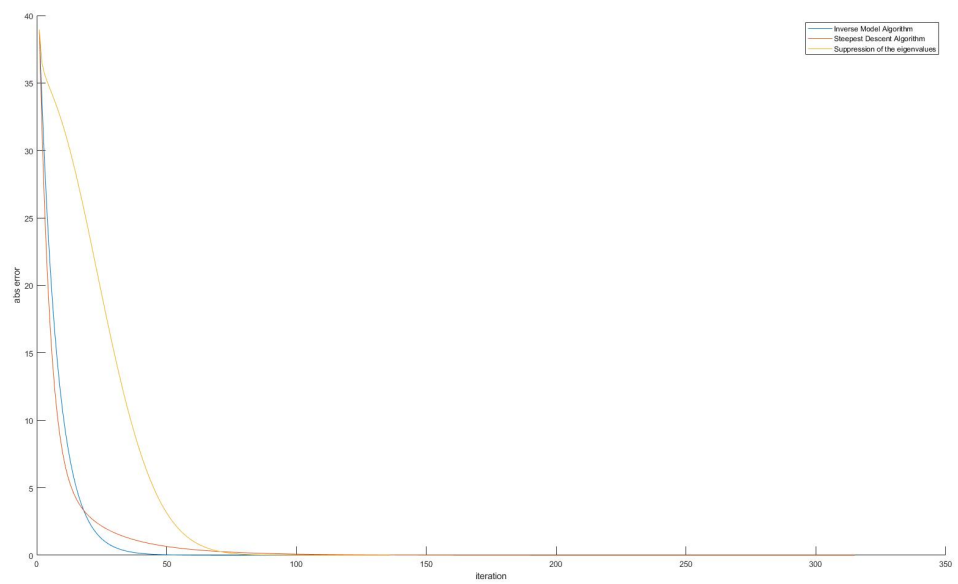


Figure 5.3: Example 13.  $u(0) = .1$ ,  $r(t)$  is random for  $t = 0, 1, 2, \dots, N$ .

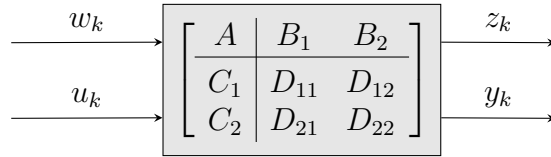
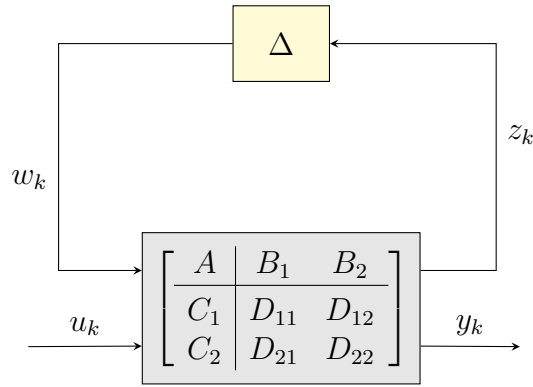
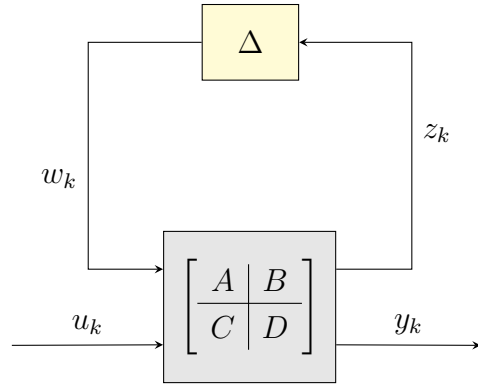
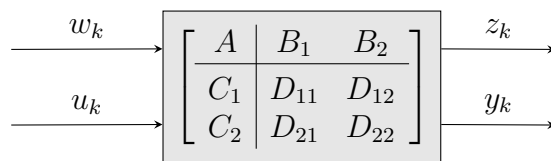


Figure 5.4: Linear Fractional Representation: open loop



(a) Closed loop



(b) Open loop

Figure 5.5: Linear Fractional Representation



# Chapter 6

## Questions

### 6.1 Session 30.02.2020

System we consider

$$e_{k+1} = L(\Delta)e_k = \Delta \star Le_k. \quad (6.1)$$

$L$  is already a closed loop, i.e.  $L = N$  from RC. Thm 6.19: If  $\mu_{V_e}(L) \leq 1$ , then RS and RP are guaranteed for all  $\Delta \in \mathbf{\Delta}$  with  $\|\Delta\| < 1$  (if well-defined).

**Exercise:**  $(I - D_{22}\Delta)^{-1}$  exists and is **stable** if there exists a matrix  $P = P^T$ , such that

$$\begin{bmatrix} I \\ \Delta(\delta) \end{bmatrix}^T P \begin{bmatrix} I \\ \Delta(\delta) \end{bmatrix} \succeq 0, \text{ and } \begin{bmatrix} D_{22} \\ I \end{bmatrix}^T P \begin{bmatrix} D_{22} \\ I \end{bmatrix} \prec 0. \quad (*)$$

For real parametric uncertainty with one parameter  $\Delta(\delta) = \delta I$ :  $(*) \Rightarrow (I - D_{22}\delta I)^{-1}$  exists and is **stable**.

**Proof:**

$M = D_{22}$ . Consider  $\{\delta I : \delta \in [-1, 1]\} \subset \mathbf{\Delta} = \{\delta I : \delta \in \mathbb{C}, |\delta| < 1\}$ . Suppose there exists some  $\delta_0 \in \mathbb{C}$ ,  $|\delta_0| < 1$ , such that there exists some  $\lambda \in \mathbb{C}$ ,  $|\lambda| > 1$ ,  $\lambda \in \text{eig}(I - M\delta_0)$ .

$$\Rightarrow \exists x \neq 0 : (I - M\delta_0)x = \lambda x \Rightarrow M\delta_0 x = (1 - \lambda)x \Rightarrow \frac{1 - \lambda}{\delta_0} \in \text{eig}(M).$$

$$\text{If } \left| \frac{1 - \lambda}{\delta_0} \right| \geq 1 \Rightarrow M \text{ is unstable } \nexists.$$

$$\text{If } \left| \frac{1 - \lambda}{\delta_0} \right| < 1 \Rightarrow \frac{1 - \lambda}{\delta_0} I \in \mathbf{\Delta} \Rightarrow \frac{-1 + \lambda}{\delta_0} I \in \mathbf{\Delta} \Rightarrow \det \left( I + M \frac{-1 + \lambda}{\delta_0} \right) \neq 0 \Rightarrow \frac{1 - \lambda}{\delta_0} \notin \text{eig}(M)$$

■



# Chapter 7

## Zusammenfassung und Ausblick

Hier können auch Schlußfolgerungen präsentiert und ein Ausblick gegeben werden.



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## .1 Beweise

### .1.1 Beweis 1

$$1 + 1 = 2 \tag{1}$$

## Eidesstattliche Erklärung

Ich erkläre, dass ich meine Bachelor-Arbeit [*Titel der Arbeit*] selbstständig und ohne Benutzung anderer als der angegebenen Hilfsmittel angefertigt habe und dass ich alle Stellen, die ich wörtlich oder sinngemäß aus Veröffentlichungen entnommen habe, als solche kenntlich gemacht habe. Die Arbeit hat bisher in gleicher oder ähnlicher Form oder auszugsweise noch keiner Prüfungsbehörde vorgelegen.

Stuttgart, den XX.XX.2013

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(Name des Kandidaten)