Notes from *Foundations of Game Engine Development Volume 1: Mathematics*

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March 25, 2025

1 Vectors and Matrices

1.2 Basic Vector Operations

"It's important to realize that a vector by itself does not have any specific location in space. The information it posseses is merely an oriented magnitude and nothing more."

1.4.2 Matrix Multiplication

- When an $n \times p$ matrix $\mathbf{A}$ is multiplied by $p \times m$ matrix $\mathbf{B}$ the (i, j) entry of the matrix product $\mathbf{AB}$ is given by:

$$(\mathbf{AB})_{ij} = \sum_{k=0}^{p-1} A_{ik} B_{kj} \tag{1}$$

- The (i, j) entry of $(\mathbf{AB})^T$ is the (j, i) entry of $\mathbf{AB}$:

$$(\mathbf{AB})_{ij}^{\mathrm{T}} = (\mathbf{AB})_{ji} = \sum_{k=0}^{p-1} A_{jk} B_{ki} \tag{2}$$

- Multiplying a 3×3 matrix $\mathbf{M}$ multiplied by a 3×1 column vector $\mathbf{v}$:

$$\mathbf{Mv} = \begin{bmatrix} M_{00} & M_{01} & M_{02} \\ M_{10} & M_{11} & M_{12} \\ M_{20} & M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = \begin{bmatrix} M_{00}v_x & + & M_{01}v_y & + & M_{02}v_z \\ M_{10}v_x & + & M_{11}v_y & + & M_{12}v_z \\ M_{20}v_x & + & M_{21}v_y & + & M_{22}v_z \end{bmatrix}$$

- If we write $\mathbf{M} = [\mathbf{a}, \mathbf{b}, \mathbf{c}]$, then we have:

$$\mathbf{Mv} = v_x \mathbf{a} + v_x \mathbf{b} + v_x \mathbf{c} \tag{3}$$

- Multiplication of transposed matrices:

$$(\mathbf{AB})^T = \mathbf{B}^T \mathbf{A}^T \tag{4}$$

1.5.1 Dot Product

$$\mathbf{a} \cdot \mathbf{b} = \sum_{i=0}^{n-1} a_i b_i = \mathbf{a}^{\mathrm{T}} \mathbf{b} = \|\mathbf{a}\| \, \|\mathbf{b}\| \cos \theta \tag{5}$$

- Provides a computationally cheap way to determine how much two vectors are parallel or perpendicular to each other.
- If $\mathbf{a}$ and $\mathbf{b}$ are unit vectors, then $\mathbf{a} \cdot \mathbf{b} = \cos \theta$ and the range of the cosine function is $[-1, 1]$
- Assuming $\|\mathbf{a}\| = \|\mathbf{b}\|$, $\mathbf{a} \cdot \mathbf{b}$ is:
  - Maximally positive when $\mathbf{a}$ and $\mathbf{b}$ point in the same direction.
  - Maximally negative when $\mathbf{a}$ and $\mathbf{b}$ point in opposite directions.
  - Zero when $\mathbf{a}$ and $\mathbf{b}$ are perpendicular, regardless of magnitude
- Squared magnitude of a vector:

$$v^2 = \mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2 \tag{6}$$

1.5.2 Cross Product

$$\mathbf{a} \times \mathbf{b} = (a_y b_z - a_z b_y, a_z b_x - a_x b_z, a_x b_y - a_y b_x) \tag{7}$$

- Can also be expressed as a matrix product by forming a special 3×3 antisymmetric matrix denoted by

$$[\mathbf{a}]_\times = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \tag{8}$$

$$\mathbf{a} \times \mathbf{b} = [\mathbf{a}]_\times \mathbf{b} = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} = \begin{bmatrix} -a_z b_y + a_y b_z \\ a_z b_x - a_x b_z \\ -a_y b_x + a_x b_y \end{bmatrix}$$

- Only defined for 3-dimensions, whereas dot product is defined for all numbers of dimensions
- Actually a subtle misinterpretation of a more general and more algebraically sound operation called the *wedge product*.
- Zero when **a** and **b** are parallel
- When **a** and **b** are not parallel, $\mathbf{a} \times \mathbf{b}$ is a new vector that is perpendicular to both **a** and **b**
- Has a magnitude equal to the area of the parallelogram having sides **a** and **b**:

$$\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\| \, \|\mathbf{b}\| \sin\theta \tag{9}$$

- Other identities:

$$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$$
$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}\,(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}\,(\mathbf{a} \cdot \mathbf{b})$$

1.5.3 Scalar Triple Product

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = (\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a} = (\mathbf{c} \times \mathbf{a}) \cdot \mathbf{b} \tag{10}$$

1.6 Vector Projection

- Projection of **a** onto **b**:

$$\mathbf{a}_{\|\mathbf{b}} = \frac{\mathbf{a} \cdot \mathbf{b}}{b^2} \mathbf{b} \tag{11}$$

- $\mathbf{a}_{||b}$ indicates the component of the vector **a** that is parallel to the vector **b** (alternatively $\mathrm{proj}_{\mathbf{b}}\mathbf{a}$)

$$\mathbf{a}_{\|\mathbf{b}} = \frac{1}{b^2} \mathbf{b}\mathbf{b}^{\mathrm{T}} \mathbf{a} = \frac{1}{b^2} \begin{bmatrix} b_x^2 & b_x b_y & b_x b_z \\ b_x b_y & b_y^2 & b_y b_z \\ b_x b_z & b_y b_z & b_z^2 \end{bmatrix} \tag{12}$$

- Where $\mathbf{b}\mathbf{b}^{\mathrm{T}}$ is a symmetric matrix and an example of an *outer product*

$$\mathbf{u} \otimes \mathbf{v} = \mathbf{u}\mathbf{v}^{\mathrm{T}} = \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} \begin{bmatrix} v_x & v_y & v_z \end{bmatrix} = \begin{bmatrix} u_x v_x & u_x v_y & u_x v_z \\ u_y v_x & u_y v_y & u_y v_z \\ u_z v_x & u_z v_y & u_z v_z \end{bmatrix} \tag{13}$$

- If we subtract the projection $\mathbf{a}_{||b}$ from the original vector $\mathbf{a}$, the we get the part that is perpendicular to the vector $\mathbf{b}$, called the rejection of $\mathbf{a}$ from $\mathbf{b}$.

$$\mathbf{a}_{\perp\mathbf{b}} = \mathbf{a} - \mathbf{a}_{\|\mathbf{b}} = \mathbf{a} - \frac{\mathbf{a}\cdot\mathbf{b}}{b^2}\mathbf{b} \tag{14}$$

- From a set of n linearly independent vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$, the *Gram-Schmidt process* can be used to produce a set of mutually orthogonal vectors $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$. For example, a set of 3 vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is orthogonalized using the calculations:

$$\begin{aligned}
\mathbf{u}_1 &= \mathbf{v}_1 \\
\mathbf{u}_2 &= \mathbf{v}_2 - (\mathbf{v}_2)_{||\mathbf{u}_1} \\
\mathbf{u}_3 &= \mathbf{v}_3 - (\mathbf{v}_3)_{||\mathbf{u}_1} - (\mathbf{v}_3)_{||\mathbf{u}_2}
\end{aligned}$$

  It is common that the vectors $\mathbf{u}_i$ be renormalized to unit length after the orthogonalization process

1.7 Matrix Inversion

The identity matrix:

$$\mathbf{I}_n = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \end{bmatrix}$$

which allows us to define the inverse of a matrix:

$$\mathbf{M}^{-1}\mathbf{M} = \mathbf{M}\mathbf{M}^{-1} = \mathbf{I} \tag{15}$$

1.7.2 Determinants

- A matrix has an inverse if, and only if, its determinant is not zero
- The determinant of any $n \times n$ matrix $\mathbf{M}$ can be expressed using the Leibniz formula for determinants:

$$\det(\mathbf{M}) = \sum_{\sigma\in S_n}\left(\operatorname{sgn}(\sigma)\prod_{k=0}^{n-1} M_{k,\sigma(k)}\right) \tag{16}$$

- Using expansion by minors, , the determinant of an $n \times n$ matrix $\mathbf{M}$ is given by:

$$\det(\mathbf{M}) = \sum_{j=0}^{n-1} M_{kj}(-1)^{k+j}\left|\mathbf{M}_{\overline{kj}}\right| \tag{17}$$

  where $\mathbf{M}_{\overline{ij}}$ is the submatrix that excludes row i and column j, and k can be chosen to be any fixed row in the matrix.

1.7.3 Elementary Matrices

There are 3 elementary row operations:

1. Multiply one row of matrix $\mathbf{M}$ by a nonzero scalar value t

$$\mathbf{E} = \begin{array}{cl} \begin{bmatrix} 1 & \dots & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & & \vdots \\ 0 & \dots & t & \dots & 0 \\ \vdots & & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \dots & 1 \end{bmatrix} & \begin{array}{l} \\ \\ \leftarrow \text{row } r \\ \\ \\ \end{array} \end{array} \tag{18}$$

   This causes the determinant of $\mathbf{M}$ to be multiplied by t.

2. Exchange 2 rows of $\mathbf{M}$

$$\mathbf{E} = \begin{array}{cl} \begin{bmatrix} 1 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & & \vdots & & \vdots \\ 0 & \dots & 0 & \dots & 1 & \dots & 0 \\ \vdots & & \vdots & \ddots & \vdots & & \vdots \\ 0 & \dots & 1 & \dots & 0 & \dots & 0 \\ \vdots & & \vdots & & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \dots & 0 & \dots & 1 \end{bmatrix} & \begin{array}{l} \\ \\ \leftarrow \text{row } r \\ \\ \leftarrow \text{row } s \\ \\ \\ \end{array} \end{array} \tag{19}$$

   This causes the determinant of $\mathbf{M}$ to be negated. Because of this, the determinant of $\mathbf{M}$ is zero if any 2 rows are the same.

3. Add a scalar multiple of one row of $\mathbf{M}$ to another row of $\mathbf{M}$

$$\mathbf{E} = \begin{array}{cl} \begin{array}{c} \text{column } s \\ \downarrow \end{array} & \\ \begin{bmatrix} 1 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & & \vdots & & \vdots \\ 0 & \dots & 0 & \dots & 1 & \dots & 0 \\ \vdots & & \vdots & \ddots & \vdots & & \vdots \\ 0 & \dots & 1 & \dots & 0 & \dots & 0 \\ \vdots & & \vdots & & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \dots & 0 & \dots & 1 \end{bmatrix} & \begin{array}{l} \\ \\ \leftarrow \text{row } r \\ \\ \leftarrow \text{row } s \\ \\ \\ \end{array} \end{array} \tag{20}$$

   This does not change the determinant of $\mathbf{M}$.

1.7.4 Inverse Calculation

If it exists, the inverse of a matrix can be found using *Gauss-Jordan elimination*, where elementary row operations are successively applied to the matrix until it is transformed into the identity matrix.

- General method for square matrices of any size
- For matrices of smaller size, faster methods exist

1.7.5 Inverses of Small Matrices

This approach uses the minors with alternating signs that appear in the formula for the determinant given by (17).

- The *cofactor* of the (i, j) entry of $\mathbf{M}$:

$$C_{ij}(\mathbf{M}) = (-1)^{i+j} \left|\mathbf{M}_{\overline{ij}}\right| \tag{21}$$

- The *cofactor matrix* $\mathbf{C}(\mathbf{M})$ of an $n \times n$ matrix $\mathbf{M}$ is the matrix in which every entry of $\mathbf{M}$ has been replaced by the corresponding cofactor.

- The formula for the inverse of a matrix $\mathbf{M}$ using its cofactor matrix:

$$\mathbf{M}^{-1} = \frac{\mathbf{C}^{\mathrm{T}}(\mathbf{M})}{\det(\mathbf{M})} \tag{22}$$

- The matrix $\mathbf{C}^{\mathrm{T}}$ is called the *adjugate* of the matrix $\mathbf{M}$, and it is denoted by $\operatorname{adj}(\mathbf{M})$

- For a 2×2 matrix $\mathbf{A}$, the explicit inverse formula is

$$\mathbf{A}^{-1} = \frac{1}{A_{00}A_{11} - A_{01}A_{10}} \begin{bmatrix} A_{11} & -A_{01} \\ -A_{10} & -A_{00} \end{bmatrix} \tag{23}$$

- For a 3×3 matrix $\mathbf{B}$, the explicit inverse formula is

$$\mathbf{B}^{-1} = \frac{1}{\det(\mathbf{B})} \begin{bmatrix} B_{11}B_{22} - B_{12}B_{21} & B_{02}B_{21} - B_{01}B_{22} & B_{01}B_{12} - B_{02}B_{11} \\ B_{12}B_{20} - B_{10}B_{22} & B_{00}B_{22} - B_{02}B_{20} & B_{02}B_{10} - B_{00}B_{12} \\ B_{10}B_{21} - B_{11}B_{20} & B_{01}B_{20} - B_{00}B_{21} & B_{00}B_{11} - B_{01}B_{10} \end{bmatrix} \tag{24}$$

  Note that each row in this formula is a cross product of two columns of the matrix $\mathbf{B}$, and the determinant is equal to the triple product (10) of the three columns of the matrix.

- The inverse of a matrix $\mathbf{M} = [\mathbf{a}, \mathbf{b}, \mathbf{c}]$ whose column are the 3D vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ can be written as

$$\mathbf{M}^{-1} = \frac{1}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]} \begin{bmatrix} \mathbf{b} \times \mathbf{c} \\ \mathbf{c} \times \mathbf{a} \\ \mathbf{a} \times \mathbf{b} \end{bmatrix} \tag{25}$$

  where the cross products are treated as row vectors.

- Let $\mathbf{M}$ be a 4×4 matrix whose first 3 rows are filled by the four 3D column vectors $\mathbf{a}$,$\mathbf{b}$,$\mathbf{c}$, and $\mathbf{d}$, and whose fourth row contains the entries $\begin{bmatrix} x & y & z & w \end{bmatrix}$

$$\mathbf{M} = \begin{bmatrix} \uparrow & \uparrow & \uparrow & \uparrow \\ \mathbf{a} & \mathbf{b} & \mathbf{c} & \mathbf{d} \\ \downarrow & \downarrow & \downarrow & \downarrow \\ - & - & - & - \\ x & y & z & w \end{bmatrix} \tag{26}$$

Then we define the four vectors $\mathbf{s}$, $\mathbf{t}$, $\mathbf{u}$, and $\mathbf{v}$, as

$$\mathbf{s} = \mathbf{a} \times \mathbf{b} \tag{27}$$
$$\mathbf{t} = \mathbf{c} \times \mathbf{d} \tag{28}$$
$$\mathbf{u} = y\mathbf{a} - x\mathbf{b} \tag{29}$$
$$\mathbf{v} = w\mathbf{c} - z\mathbf{d} \tag{30}$$

The determinant takes the form

$$\det(\mathbf{M}) = \mathbf{s} \cdot \mathbf{v} + \mathbf{t} \cdot \mathbf{u} \tag{31}$$

and the inverse of $\mathbf{M}$ is given by

$$\mathbf{M}^{-1} = \frac{1}{\mathbf{s} \cdot \mathbf{v} + \mathbf{t} \cdot \mathbf{u}} \left[\begin{array}{c|c} \mathbf{b} \times \mathbf{v} + y\mathbf{t} & -\mathbf{b} \cdot \mathbf{t} \\ \mathbf{v} \times \mathbf{a} - x\mathbf{t} & \mathbf{a} \cdot \mathbf{t} \\ \mathbf{d} \times \mathbf{u} + w\mathbf{s} & -\mathbf{d} \cdot \mathbf{s} \\ \mathbf{u} \times \mathbf{c} + z\mathbf{s} & \mathbf{c} \cdot \mathbf{s} \end{array}\right] \tag{32}$$

2 Transformations

2.1 Coordinate Spaces

2.1.1 Transformation Matrices

- The transformationfrom a position $\mathbf{p}_A$ in coordinate system A to the position $\mathbf{p}_B$ in coordinate system B can be expressed as

$$\mathbf{p}_B = \mathbf{M}\mathbf{p}_A + \mathbf{t} \tag{33}$$

where $\mathbf{M}$ is a 3×3 matrix that reorients the coordinate axes, and $\mathbf{t}$ is a 3D translation vector that moves the origin of the coordinate system.

2.1.2 Orthogonal Transforms

- The inverse of an orthogonal matrix is equal to its transpose. Assuming that $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ all have unit length and are mutually perpendicular

$$\mathbf{M}^{\mathrm{T}}\mathbf{M} = \begin{bmatrix} \leftarrow & \mathbf{a}^{\mathrm{T}} & \rightarrow \\ \leftarrow & \mathbf{b}^{\mathrm{T}} & \rightarrow \\ \leftarrow & \mathbf{c}^{\mathrm{T}} & \rightarrow \end{bmatrix} \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \mathbf{a} & \mathbf{b} & \mathbf{c} \\ \downarrow & \downarrow & \downarrow \end{bmatrix} = \begin{bmatrix} a^2 & \mathbf{a} \cdot \mathbf{b} & \mathbf{a} \cdot \mathbf{c} \\ \mathbf{b} \cdot \mathbf{a} & b^2 & \mathbf{b} \cdot \mathbf{c} \\ \mathbf{c} \cdot \mathbf{a} & \mathbf{c} \cdot \mathbf{b} & c^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{34}$$

Since $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ each have unit length and are perpendicular, all diagonal entries are ones and all other entries are zero. This means $\mathbf{M}^{\mathrm{T}}\mathbf{M} = \mathbf{I}$ and, therefore, $\mathbf{M}^{\mathrm{T}} = \mathbf{M}^{-1}$.

- Orthogonal matrices preserve the dot product between any two vectors $\mathbf{a}$ and $\mathbf{b}$. Given the vectors after they are transformed by orthogonal matrix $\mathbf{M}$

$$(\mathbf{M}\mathbf{a}) \cdot (\mathbf{M}\mathbf{b}) = (\mathbf{M}\mathbf{a})^{\mathrm{T}} \cdot (\mathbf{M}\mathbf{b}) = \mathbf{a}^{\mathrm{T}}\mathbf{M}^{\mathrm{T}} \cdot \mathbf{M}\mathbf{b} = \mathbf{a}^{\mathrm{T}}\mathbf{b} = \mathbf{a} \cdot \mathbf{b} \tag{35}$$

Since $\mathbf{a} \cdot \mathbf{a}$ is the squared magnitude of $\mathbf{a}$, (35) also proves that magnitude is not changed by an orthogonal matrix. It must therefore also be true that the angle θ between $\mathbf{a}$ and $\mathbf{b}$ is unchanged.

- The transform performed by an orthogonal matrix is always a rotation, a reflection, or a combination of the two.

- The determinant of an orthogonal matrix is always ± 1, positive for a pure rotation and negative for a rotation with a reflection.

2.1.3 Transform Composition

- Whenever a vector $\mathbf{v}$ is transformed by a matrix $\mathbf{M}_1$, then by a matrix $\mathbf{M}_2$, the result $\mathbf{v}'$ is calculated by:

$$\mathbf{v}' = \mathbf{M}_2\left(\mathbf{M}_2\mathbf{v}\right) = \left(\mathbf{M}_2\mathbf{M}_1\right)\mathbf{v} \tag{36}$$

- To perform a transform $\mathbf{A}$ in coordinate system A in coordinate system B, where matrix $\mathbf{M}$ transforms vectors from A to B, the equivalient transform $\mathbf{B}$ in coordinate system B is

$$\mathbf{B} = \mathbf{MAM}^{-1} \tag{37}$$

2.2 Rotations

2.2.1 Rotation About a Coordinate Axis

$$\mathbf{M}_{\text{rot}\,x}\left(\theta\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \tag{38}$$

$$\mathbf{M}_{\text{rot}\,y}\left(\theta\right) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \tag{39}$$

$$\mathbf{M}_{\text{rot}\,z}\left(\theta\right) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{40}$$

2.2.2 Rotation About an Arbitrary Axis

For a vector $\mathbf{v}$ rotated about a vector$\mathbf{a}$ by an angle θ where the angle between $\mathbf{v}$ and $\mathbf{a}$ is α

$$\mathbf{v}' = \mathbf{v}_{\parallel\mathbf{a}} + \mathbf{v}_{\perp\mathbf{a}}\cos\theta + \left(\mathbf{a}\times\mathbf{v}\right)\sin\theta \tag{41}$$

which is expressed in matrix format as

$$\mathbf{M}_{\text{rot}}\left(\theta,\mathbf{a}\right) = \begin{bmatrix} \cos\theta + (1-\cos\theta)a_x^2 & (1-\cos\theta)a_xa_y - (\sin\theta)a_z & (1-\cos\theta)a_xa_z + (\sin\theta)a_y \\ (1-\cos\theta)a_xa_y + (\sin\theta)a_z & \cos\theta + (1-\cos\theta)a_y^2 & (1-\cos\theta)a_ya_z - (\sin\theta)a_x \\ (1-\cos\theta)a_xa_z - (\sin\theta)a_y & (1-\cos\theta)a_ya_z + (\sin\theta)a_x & \cos\theta + (1-\cos\theta)a_z^2 \end{bmatrix} \tag{42}$$

2.3 Reflections

Vector $\mathbf{v}$ can be reflected through a plane perpendicular to vector $\mathbf{a}$ (assuming $\mathbf{a}$ has unit length)

$$\mathbf{v}' = \mathbf{v}_{\perp\mathbf{a}} - \mathbf{v}_{\parallel\mathbf{a}} \tag{43}$$

with matrix representation

$$\mathbf{v}' = \begin{bmatrix} 1-a_x^2 & -a_xa_y & -a_xa_z \\ -a_xa_y & 1-a_y^2 & -a_ya_z \\ -a_xa_z & -a_ya_z & 1-a_z^2 \end{bmatrix}\mathbf{v} - \begin{bmatrix} a_x^2 & a_xa_y & a_xa_z \\ a_xa_y & a_y^2 & a_ya_z \\ a_xa_z & a_ya_z & a_z^2 \end{bmatrix} \tag{44}$$

Combining the matrix terms into a single matrix, we arrive at the formula

$$\mathbf{M}_{\text{reflect}}\left(\mathbf{a}\right) = \begin{bmatrix} 1-2a_x^2 & -2a_xa_y & -2a_xa_z \\ -2a_xa_y & 1-2a_y^2 & -2a_ya_z \\ -2a_xa_z & -2a_ya_z & 1-2a_z^2 \end{bmatrix} \tag{45}$$

We can also construct a transform that negates the the perpendicular component instead of the parallel component

$$\mathbf{v}' = \mathbf{v}_{\parallel \mathbf{a}} - \mathbf{v}_{\perp \mathbf{a}} \tag{46}$$

with the matrix form

$$\mathbf{M}_{\text{invol}}\left(\mathbf{a}\right) = \begin{bmatrix} 2a_x^2 - 1 & 2a_x a_y & 2a_x a_z \\ 2a_x a_y & 2a_y^2 - 1 & 2a_y a_z \\ 2a_x a_z & 2a_y a_z & 2a_z^2 - 1 \end{bmatrix} \tag{47}$$

where $\mathbf{M}_{\text{invol}}\left(\mathbf{a}\right)$ denotes $\mathbf{M}$ as an *involution*, which is a matrix that, when multiplied by itself, produces the identity matrix.

2.4 Scales

A scale transformation aligned to the coordinate axes

$$\mathbf{M}_{\text{scale}}\left(s_x, s_y, s_z\right) = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & s_z \end{bmatrix} \tag{48}$$

To scale a vector $\mathbf{v}$ along a single arbitrary direction $\mathbf{a}$ while preserving the vector's size in every direction orthogonal to $\mathbf{a}$

$$\mathbf{v}' = s\mathbf{v}_{\parallel \mathbf{a}} + \mathbf{v}_{\perp \mathbf{a}} \tag{49}$$

where the transformation matrix is

$$\mathbf{M}_{\text{scale}}\left(s, \mathbf{a}\right) = \begin{bmatrix} (s-1)a_x^2 + 1 & (s-1)a_x a_y & (s-1)a_x a_z \\ (s-1)a_x a_y & (s-1)a_y^2 + 1 & (s-1)a_y a_z \\ (s-1)a_x a_z & (s-1)a_y a_z & (s-1)a_z^2 + 1 \end{bmatrix} \tag{50}$$