Exploring Numerical Solution Methods of Electric Systems Using Circuit Principles (Supplementary Information)

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1 Derivation of multiple-node Gaussian Elimination method using circuit principles

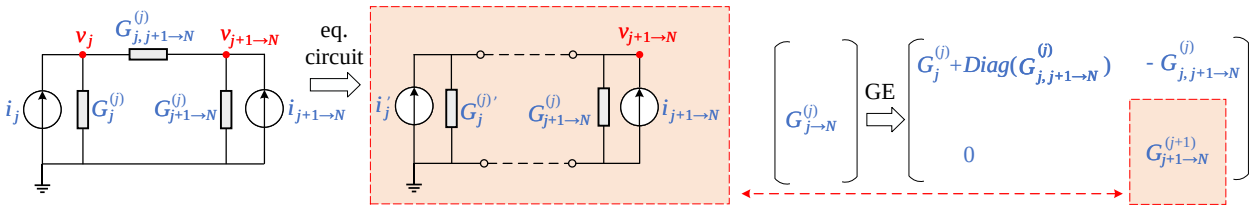

Figure 1: Extending the two-node case into multiple-node networks, in step j of FE, only considering the network containing node j to N. The circuit at the left-side of the network containing node $j+1$ to N is equivalent to a current source $i_j^{'}$ in parallel with a conductance $G_j^{(j)'}$, and then the obtained network for node $j+1$ to N is equivalent to the matrix equation after the j^{th} FE step.

The equivalence of GE and circuit principles in the two-node circuit described in Section II(A) of the paper can be easily extended to multiple-node networks. The basic idea is that, in step $j \in \{1, 2, ..., N-1\}$, only considering the network containing node $j \to N$ after the $(j-1)^{\text{th}}$ FE operation, and the corresponding matrix to be operated is $G_{j\to N}^{(j)}$, where the superscript (j) denotes the j^{th} FE operation. Using the same equivalence techniques as used in the two-node case, the circuit at the left-side of the network containing node $j+1$ to N is equivalent to a current source $i_j^{'}$ in parallel with a conductance $G_j^{(j)'}$, and then the obtained network for node $j+1$ to N is equivalent to the matrix equation with $G_{j+1\to N}^{(j+1)}$ after the j^{th} GE step, as illustrated in Fig. 1. Here, the coupling conductance $G_{j,j+1\to N}^{(j)}$ is a vector containing the branch conductance values between node j and node $j+1$ to N:

$$G_{j,j+1\to N}^{(j)} = [G_{j,j+1}^{(j)}, G_{j,j+2}^{(j)}, ..., G_{j,N}^{(j)}], \tag{1}$$

and $Diag(G_{j,j+1\to N}^{(j)})$ is the sum of values in the vector. The resulting equivalent circuit after eliminating node j is shown in Fig. 1, where the equivalent conductance $G_j^{(j)'}$ is a matrix with the size of $(N-j)\times(N-j)$. Similar to the two-node case, the equivalent conductance should be:

$$G_j^{(j)'} = ([G_j^{(j)}]^{-1} + [G_{j,j+1\to N}^{(j)}]^{-1})^{-1}, \tag{2}$$

where all the elements in $[G_j^{(j)}]$ have the same value:

$$[G_j^{(j)}] = \begin{bmatrix} G_j^{(j)} & \dots & G_j^{(j)} \\ \vdots & \ddots & \vdots \\ G_j^{(j)} & \dots & G_j^{(j)} \end{bmatrix}, \tag{3}$$

and $[G_{j,j+1\to N}^{(j)}]$ is diagonal matrix with branch conductance values in diagonal:

$$[G_{j,j+1\to N}^{(j)}] = \begin{bmatrix} G_{j,j+1}^{(j)} & 0 & \dots & 0 \\ 0 & G_{j,j+2}^{(j)} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & G_{j,N}^{(j)} \end{bmatrix}. \tag{4}$$

However, there may not exist branch between node j and some node $j+p \in \{j+1, ..., N\}$, thus $G_{j,j+p}^{(j)} = 0$ and the matrix $[G_{j,j+1\to N}^{(j)}]$ is irreversible. In practical computation, the corresponding $[G_j^{(j)}]$ and $[G_{j,j+1\to N}^{(j)}]$ should be modified according to the branches between node j and other nodes. To achieve this purpose, two operations are involved: deleting and padding. (1) **Deleting**. Assuming there are m branches, then the size of the two matrices should be $m \times m$, by deleting the rows and columns where the diagonal elements with values of zero are located. For example, if $G_{j,j+p}^{(j)} = 0$, then the p^{th} row and column should be deleted from the matrix $[G_j^{(j)}]$ and $[G_{j,j+1\to N}^{(j)}]$. (2) **Padding**. After modifying the matrices, the computed $G_j^{(j)'}$ from (2) also has the size of $m \times m$, then it should be extended to $(N-j) \times (N-j)$ by padding rows and columns with zero-elements according to the deleting locations in $[G_j^{(j)}]$ and $[G_{j,j+1\to N}^{(j)}]$. Combining the two operations together, (2) is rewritten as:

$$G_j^{(j)'} = \text{padding}\{([G_j^{(j)}]_{\text{del}}^{-1} + [G_{j,j+1\to N}^{(j)}]_{\text{del}}^{-1})^{-1}\}. \tag{5}$$

2 Derivation of pivoting method for sparsity of GE/LU

In electric power systems, the conductance matrices are generally sparse, with majority elements equal to zero. Therefore, sparse techniques are usually utilized to solve large scale systems to speed up circuit simulation. In Gauss's algorithm for LU factorization (equivalent to GE, denoted as GE/LU), pivoting methods achieve sparsity by reordering the matrix up-front via row and column exchanges. We use a four-node circuit topology to simply demonstrate the principles. The conductance matrix is expressed using the structural representation to avoid showing specific numerical calculations, as shown in Fig. 2(a), where $\times$ denotes a non-zero element. Using the original node order, the matrix changes to a dense matrix after the first FE step, which is not preferred to keep sparsity. And if reorder the nodes by exchanging the node number of 1 and 4, the resulting matrix could keep sparsity after the first FE step, as shown in Fig. 2(c).

In this section, we show that the pivoting method of GE/LU could also be derived based on the equivalent circuit described in Section II(A) of the paper. Using the equivalent circuit shown in Fig. 2(b), the objective of the first FE step (eliminating node 1) to keeping sparsity is equivalent to find a proper reordering to get a sparse $G_1^{(1)'}$. As described in the above section:

$$G_1^{(1)'} = \text{padding}\{([G_1^{(1)}]_{\text{del}}^{-1} + [G_{1,2\to 4}^{(1)}]_{\text{del}}^{-1})^{-1}\}. \tag{6}$$

Since $[G_1^{(1)}]$ is a full matrix with all elements equal to $G_1^{(1)}$, the size of $[G_1^{(1)}]_{\text{del}}$ actually determines the number of non-zeros in $G_1^{(1)'}$; i.e., if $[G_1^{(1)}]_{\text{del}}$ is a $m \times m$ matrix, then $G_1^{(1)'}$ has m^2 non-zeros. According to the deleting operation, the less branches between node 1 and node 2, 3, 4, the smaller size of $[G_1^{(1)}]_{\text{del}}$ and $[G_{1,2\to 4}^{(1)}]_{\text{del}}$. As shown in Fig. 2(d), if using the original node order, node 1 has coupling conductance with node 2, 3, 4, thus $[G_{1,2\to 4}^{(1)}]_{\text{del}} = [G_{1,2\to 4}^{(1)}] = diag(G_{1,2}^{(1)}, G_{1,3}^{(1)}, G_{1,4}^{(4)})$ with size of 3×3, and $G_1^{(1)'}$ is a full matrix with size of 3×3. If exchange the indices of node 1 with node 4, then there is no coupling conductance between node 1 and node 2, 3, which results in the deleting operations over $[G_{1,2\to 4}^{(1)}]$, and the size of $[G_{1,2\to 4}^{(1)}]_{\text{del}}$ changes to 1×1, thus leading to a sparse $G_1^{(1)'}$.

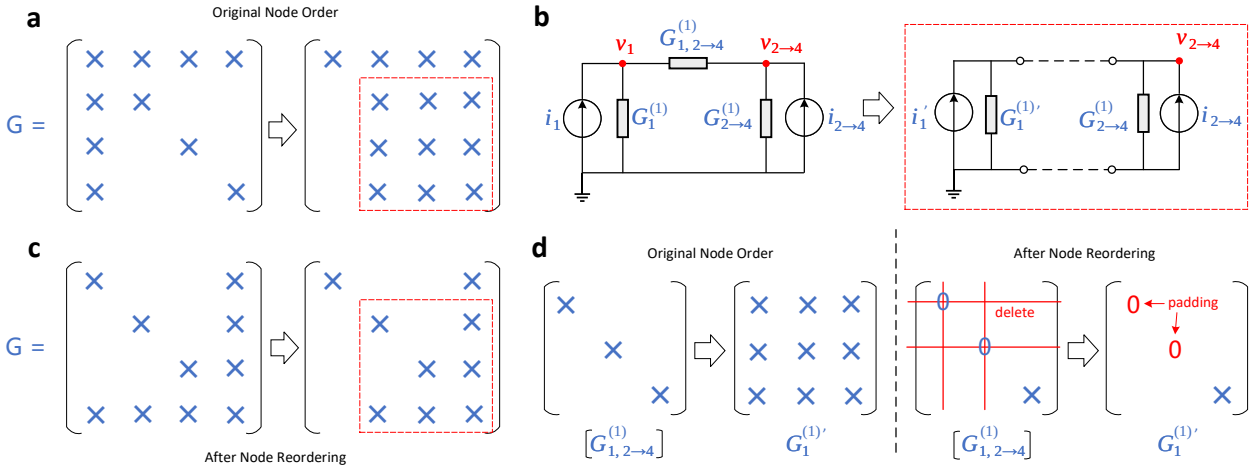

Figure 2: Explaining the pivoting methods for sparsity using circuit principles. (a) Conductance matrix of the example circuit with four nodes, where $\times$ denotes a non-zero element in the structural representation; after the first FE step, the remaining matrix to be eliminated becomes a full matrix with several fill-ins. (b) In the view of equivalent circuit, the example circuit changes to a three-node circuit with node 1 being eliminated. (c) After node reordering, the first FE step does not introduce extra fill-ins to the matrix. (d) In the view of equivalent circuit, the equivalent conductance $G_1^{(1)'}$ under the original node order is a full matrix, while after the node reordering $G_1^{(1)'}$ becomes a sparse matrix.

Therefore, to obtain a sparse $G_1^{(1)'}$, the principle is that the number of branches between node 1 and other nodes should be minimized. This principle is just the same as the minimum degree (MD) algorithm proposed by Tinney and Walker [1], which aims to find the minimum degree vertex in the matrix graph. The MD strategy has been very successful in practice and becomes the base of modern sparse techniques such as the multiple minimum degree (MMD) and average minimum degree (AMD).

References

[1] W. F. Tinney and J. W. Walker, "Direct solutions of sparse network equations by optimally ordered triangular factorization," *Proc. IEEE,* vol. 55, no. 11, pp. 1801-1809, Nov. 1967.