

School of Electronics Engineering and Computer Science
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Mathematics in Olympiad in Informatics

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Introduction

Number Theory

Division, Prime and Coprime
Congruence Modulo

Introduction to Calculus

Differential of a Function
Calculus

Introduction



English is **very helpful** to our future study and work. Please try to use English as much as possible, from now on.

Therefore, the slides are written in English.

If there's no specific instruction, the complexity is always time complexity.

For the convenience of view, I recommend you to open it with Adobe Reader.



Fundamental theorem of arithmetic

Every integer greater than 1 either is prime itself or is the product of prime numbers, and that this product is unique, up to the order of the factors. i.e.

$$\forall n > 1, n \in \mathbb{N}, n = p_1^{a_1} p_2^{a_2} \dots p_n^{a_n}$$

where $p_k (1 \leq k \leq n), p_1 < p_2 < \dots < p_n$ is prime.



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Prime Number Theorem

$$\pi(N) \sim \frac{N}{\log(N)}$$

where $\pi(N)$ is the prime counting function, and $\log(N)$ is the natural logarithm of N . The equation means that for large enough N , the probability that a random natural number not greater than N is prime is very close to $\frac{1}{\log(N)}$.



How to check whether a number N is prime?

Pseudo-code for prime number checking.

```
1: for  $i = 2$  to  $\sqrt{N}$  do  
2:   if  $i$  is divisor of  $N$  then  
3:     return False  
4:   end if  
5: end for
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The time complexity of such algorithm is $O(\sqrt{N})$.
The space complexity of such algorithm is $O(1)$.



How to generate a prime number list from 1 to N ?

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1. Check whether a number is prime one by one. $O(N\sqrt{N})$
2. Sieve of Eratosthenes $O(N \log(N))$
3. Linear Sieve



Pseudo-code for linear sieve.

```
1: for  $i = 2$  to  $N$  do
2:   set  $IsPrime[i] = True$ 
3: end for
4: for  $i = 2$  to  $N$  do
5:   if  $IsPrime[i]$  then
6:     push  $i$  into PrimeNumberList
7:   end if
8:    $j = 0$ 
9:   while  $j < \text{CurrentPrimeNumber}$  and  $i \cdot \text{PrimeNumberList}[j] < N$  do
10:     $IsPrime[i \cdot \text{PrimeNumberList}[j]] = False$ 
11:    if  $\text{PrimeNumberList}[j]$  is divisor of  $i$  then
12:      break
13:    end if
14:    increase  $j$ 
15:   end while
16: end for
```



Linear Sieve

The time complexity of such algorithm is $O(N)$.

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Practice: POJ 2689

Given $L, U (1 \leq L < U < 2^{31}, U - L \leq 1,000,000)$, you are to find the two adjacent primes C_1 and $C_2 (L \leq C_1 < C_2 \leq U)$ that are closest (i.e. $C_2 - C_1$ is the minimum), and the two adjacent primes D_1 and $D_2 (L \leq D_1 < D_2 \leq U)$ that are the most distant (i.e. $D_2 - D_1$ is the maximum).



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Hint: You are definitely required to generate a prime number list in the range of $[L, U]$. But how?

Optional: Mathematical Induction



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It contains two steps:

1. **base case**, to prove the given statement for the least element in the well-ordered set.

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It contains two steps:

1. **base case**, to prove the given statement for the least element in the well-ordered set.
2. **inductive step**, to prove that, if the statement is assumed to be true for any element, then it must be true for the next element as well.



An Example

Prove $\sum_{i=0}^n = \frac{n(n+1)}{2}, \forall n \in \mathbb{N}$



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Assume that $\sum_{i=0}^n = \frac{n(n+1)}{2}$ when $n = k$,

then $\sum_{i=0}^{k+1} = \frac{k(k+1)}{2} + k + 1 = \frac{k(k+1)+2(k+1)}{2} = \frac{(k+1)(k+2)}{2}$.

Hence, the statement is true when $n = k + 1$.



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Therefore, $\sum_{i=0}^n = \frac{n(n+1)}{2}$, $\forall n \in \mathbb{N}$. #



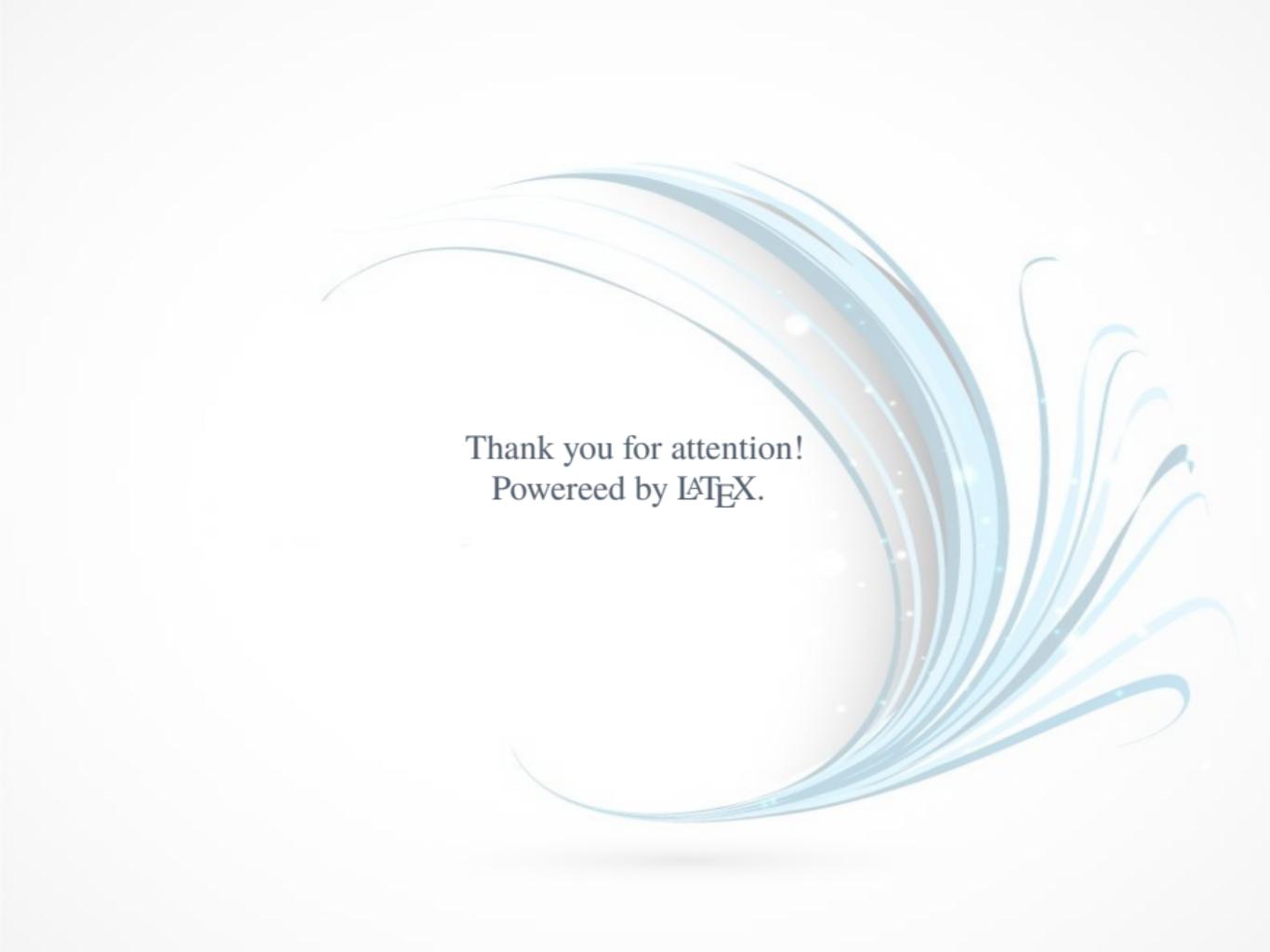
Euclid Algorithm for Greatest Common Divisor(GCD)

Pseudo-code for linear sieve.

```
1: function GCD(integer a,b)
2: if b == 0 then
3:   return a
4: else
5:   return GCD(b, a mod b)
6: end if
```



Euler's totient function

The background features a central white circle with a thin black outline. From the right side, several light blue, wavy lines radiate outwards, resembling a fan or a stylized flower. The lines are thin and have a slight gradient, with some having small white dots along their length. The overall effect is elegant and modern.

Thank you for attention!
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