School of Electronics Engineering and Computer Science
Peking University

Mathematics in Olympiad in Informatics Part 2

Frederica Haoyue Shi
hyshi@pku.edu.cn

January 22, 2017

Content

Gaussian Elimination

It's an intuitive idea to find the solution for a linear equation set. Let us look at examples:

System of equations	Row operations	Augmented matrix
$\begin{aligned} 2x + y - z &= 8 \\ -3x - y + 2z &= -11 \\ -2x + y + 2z &= -3 \end{aligned}$		$\left[\begin{array}{ccc\|c} 2 & 1 & -1 & 8 \\ -3 & -1 & 2 & -11 \\ -2 & 1 & 2 & -3 \end{array}\right]$
$\begin{aligned} 2x + y - z &= 8 \\ \tfrac{1}{2}y + \tfrac{1}{2}z &= 1 \\ 2y + z &= 5 \end{aligned}$	$\begin{aligned} L_2 + \tfrac{3}{2}L_1 &\to L_2 \\ L_3 + L_1 &\to L_3 \end{aligned}$	$\left[\begin{array}{ccc\|c} 2 & 1 & -1 & 8 \\ 0 & 1/2 & 1/2 & 1 \\ 0 & 2 & 1 & 5 \end{array}\right]$
$\begin{aligned} 2x + y - z &= 8 \\ \tfrac{1}{2}y + \tfrac{1}{2}z &= 1 \\ -z &= 1 \end{aligned}$	$L_3 + -4L_2 \to L_3$	$\left[\begin{array}{ccc\|c} 2 & 1 & -1 & 8 \\ 0 & 1/2 & 1/2 & 1 \\ 0 & 0 & -1 & 1 \end{array}\right]$
The matrix is now in echelon form (also called triangular form)		

The matrix is now in echelon form (also called triangular form)		
$\begin{aligned} 2x + y &= 7 \\ \tfrac{1}{2}y &= \tfrac{3}{2} \\ -z &= 1 \end{aligned}$	$\begin{aligned} L_2 + \tfrac{1}{2}L_3 &\to L_2 \\ L_1 - L_3 &\to L_1 \end{aligned}$	$\left[\begin{array}{ccc\|c} 2 & 1 & 0 & 7 \\ 0 & 1/2 & 0 & 3/2 \\ 0 & 0 & -1 & 1 \end{array}\right]$
$\begin{aligned} 2x + y &= 7 \\ y &= 3 \\ z &= -1 \end{aligned}$	$\begin{aligned} 2L_2 &\to L_2 \\ -L_3 &\to L_3 \end{aligned}$	$\left[\begin{array}{ccc\|c} 2 & 1 & 0 & 7 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right]$
$\begin{aligned} x &= 2 \\ y &= 3 \\ z &= -1 \end{aligned}$	$\begin{aligned} L_1 - L_2 &\to L_1 \\ \tfrac{1}{2}L_1 &\to L_1 \end{aligned}$	$\left[\begin{array}{ccc\|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right]$

Gaussian Elimination

It's an intuitive idea to find the solution for a linear equation set. Let us look at examples:

System of equations	Row operations	Augmented matrix
$\begin{aligned} 2x + y - z &= 8 \\ -3x - y + 2z &= -11 \\ -2x + y + 2z &= -3 \end{aligned}$		$\left[\begin{array}{ccc\|c} 2 & 1 & -1 & 8 \\ -3 & -1 & 2 & -11 \\ -2 & 1 & 2 & -3 \end{array}\right]$
$\begin{aligned} 2x + y - z &= 8 \\ \frac{1}{2}y + \frac{1}{2}z &= 1 \\ 2y + z &= 5 \end{aligned}$	$\begin{aligned} L_2 + \frac{3}{2}L_1 &\to L_2 \\ L_3 + L_1 &\to L_3 \end{aligned}$	$\left[\begin{array}{ccc\|c} 2 & 1 & -1 & 8 \\ 0 & 1/2 & 1/2 & 1 \\ 0 & 2 & 1 & 5 \end{array}\right]$
$\begin{aligned} 2x + y - z &= 8 \\ \frac{1}{2}y + \frac{1}{2}z &= 1 \\ -z &= 1 \end{aligned}$	$L_3 + -4L_2 \to L_3$	$\left[\begin{array}{ccc\|c} 2 & 1 & -1 & 8 \\ 0 & 1/2 & 1/2 & 1 \\ 0 & 0 & -1 & 1 \end{array}\right]$
The matrix is now in echelon form (also called triangular form)		

The matrix is now in echelon form (also called triangular form)		
$\begin{aligned} 2x + y &= 7 \\ \frac{1}{2}y &= \frac{3}{2} \\ -z &= 1 \end{aligned}$	$\begin{aligned} L_2 + \frac{1}{2}L_3 &\to L_2 \\ L_1 - L_3 &\to L_1 \end{aligned}$	$\left[\begin{array}{ccc\|c} 2 & 1 & 0 & 7 \\ 0 & 1/2 & 0 & 3/2 \\ 0 & 0 & -1 & 1 \end{array}\right]$
$\begin{aligned} 2x + y &= 7 \\ y &= 3 \\ z &= -1 \end{aligned}$	$\begin{aligned} 2L_2 &\to L_2 \\ -L_3 &\to L_3 \end{aligned}$	$\left[\begin{array}{ccc\|c} 2 & 1 & 0 & 7 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right]$
$\begin{aligned} x &= 2 \\ y &= 3 \\ z &= -1 \end{aligned}$	$\begin{aligned} L_1 - L_2 &\to L_1 \\ \frac{1}{2}L_1 &\to L_1 \end{aligned}$	$\left[\begin{array}{ccc\|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right]$

Practice: JSOI2008 sphere

Task: Given n integers $\{a_1, a_2, ..., a_n\}$, select some s.t. the xor sum is maximum.

Linear Algebra

Task: Given n integers $\{a_1, a_2, ..., a_n\}$, select some s.t. the xor sum is maximum.

$n \leq 20$ Brute-force search.

Task: Given n integers $\{a_1, a_2, ..., a_n\}$, select some s.t. the xor sum is maximum.

$n \leq 20$ Brute-force search.
That's definitely a terrible choice for $n \leq 100,000$.

Hint: apply Gaussian elimination to the xor function.

Task: Given n integers $\{a_1, a_2, ..., a_n\}$, select some s.t. the xor sum is maximum.

$n \leq 20$ Brute-force search.
That's definitely a terrible choice for $n \leq 100,000$.

Hint: apply Gaussian elimination to the xor function.
The time complexity is $O(n \log(MAX))$

Practice Time

Given n integers $\{a_1, a_2, ..., a_n\}, n \leq 100,000$.

Practice Time

Given n integers $\{a_1, a_2, ..., a_n\}, n \leq 100,000$.

- Select some, s.t. they have the minimum xor sum.

Practice Time

Given n integers $\{a_1, a_2, ..., a_n\}, n \leq 100,000$.

- Select some, s.t. they have the minimum xor sum.
- Select some, s.t. they have the maximum/minimum xor sum after appending another given integer a.

Practice Time

Given n integers $\{a_1, a_2, ..., a_n\}, n \leq 100,000$.

- Select some, s.t. they have the minimum xor sum.
- Select some, s.t. they have the maximum/minimum xor sum after appending another given integer a.
- Select some, s.t. their xor sum is another given integer a.

Practice Time

Given n integers $\{a_1, a_2, ..., a_n\}, n \leq 100,000$.

- Select some, s.t. they have the minimum xor sum.
- Select some, s.t. they have the maximum/minimum xor sum after appending another given integer a.
- Select some, s.t. their xor sum is another given integer a.
- Select one, s.t. it has maximum xor result with another given integer a, multiple queries (up to $100,000$).

Probability

Probability is the measure of the likelihood that an event will occur.

Probability and Expectation

Probability

Probability is the measure of the likelihood that an event will occur.

Expected value

Expected value of a random variable, intuitively, is the long-run average value of repetitions of the experiment it represents.

$$E[X] = \sum_{k \in K} x_k P(k)$$

where K is the set consists of all the events, $P(k)$ is the probability of event k.

Example

Let X represent the outcome of a roll of a fair six-sided die. More specifically, X will be the number of pips showing on the top face of the die after the toss. The possible values for X are $1, 2, 3, 4, 5$ and 6, all equally likely (each having the probability of $\frac{1}{6}$). The expectation of X is

$$E[X] = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6} = 3.5$$

Practice: POJ2096

A system has s sub-systems, and it may produce n kinds of bug. Ivan finds out 1 bug everyday, and it belongs to one kind and one sub-system. The probability bug belongs to each sub-system and each kind are uniform. Output the expected days that Ivan finds bugs from every sub-system and every kind.

Practice: POJ2096

A system has s sub-systems, and it may produce n kinds of bug. Ivan finds out 1 bug everyday, and it belongs to one kind and one sub-system. The probability bug belongs to each sub-system and each kind are uniform. Output the expected days that Ivan finds bugs from every sub-system and every kind.

Hint: $f[i][j]$ represents the expected remaining days when Ivan has found i kinds of bugs from j sub-system.

Part 2 ends here. Thank you for attention!
Powereed by $\LaTeX$.