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An Obstacle-Avoiding Minimum Variation B-spline Problem

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Abstract

We study the problem of computing a planar curve, restricted to lie between two given polygonal chains, such that the integral of the square of arc-length derivative of curvature along the curve is minimized. We introduce the Minimum Variation B-spline problem *which is a linearly constrained optimization problem over curves defined by B-spline functions only.*

An empirical investigation indicates that this problem has one unique solution among all uniform quartic B-spline functions. Furthermore, we prove that, for any B-spline function, the convexity properties of the problem are preserved subject to a scaling and translation of the knot sequence defining the B-spline.

1 Introduction

There are at least two areas dealing with the construction of smooth curves. One area concerns path-planning for autonomous vehicles and robots, see e.g. [9, 5, 13, 2]. A smooth path is easy for a vehicle to follow and yield low wear on for example the vehicle steering gear [9]. The other area deals with curve and surface design and reconstruction [12]. In this case, the smoothness is, for example, used to make surfaces resemble real objects having a smooth shape or to create an appealing form.

The literature contains different ways to define smoothness. One of the more natural definitions comes from minimizing the square of the arc-length derivative of curvature along the curve. This results in a curve that is known as a *minimum variation curve* (MVC) [12].

Consider any curve in the plane and let $K(s)$ denote the curvature at arc-length s along it. Then the curve is an MVC if it minimizes the cost function

$$\int \left(\frac{d}{ds}K(s)\right)^2 ds, \tag{1}$$

subject to constraints on curve position, and optionally, curve tangent and curvature, at certain points along the curve. We refer to the problem of computing an MVC as the *MVC-problem* (P_{MVC}).

Even though an MVC has pleasing properties it also possesses some negative ones. Solely imposing constraints on the position, direction, and curvature of the curve at its endpoints, the MVC is described by an *intrinsic spline* of degree 2 [5]. The position of an intrinsic spline can not be written as a closed form expression and it is costly to compute. More importantly, to our knowledge, it is not known how to effectively bound it in order for the curve to avoid, i.e. not intersect, obstacles such as other curves.

Another class of curves that are used for describing smooth curves are the so called *B-splines* [4, 6], which are defined in terms of *B-spline bases*. One thing that makes B-splines attractive is the ease by which the shape of the resulting curve can be controlled. For this reason, B-splines are widely used in a variety of different contexts such as data fitting, computer aided design (CAD), automated manufacturing (CAM), and computer graphics [7, 15]. Another advantage of B-splines compared to intrinsic splines is that the former can be given in closed form. It is also possible to bound them in the plane by piecewise linear envelopes in terms of the parameters describing the splines, as shown by Lutterkort and Peters [10].

Contribution

In contrast to the general MVC-problem, mentioned above, we introduce the *minimum variation B-spline problem* and some variants of it. These problems are linearly constrained optimization problems minimizing the same cost function as in (1) but for curves restricted to being B-spline functions only.

We study the problem of how to compute a uniform B-spline of degree 4 (a quartic B-spline) in the plane restricted to lie between two given polygonal chains such that the integral of the square of arc-length derivative of curvature along the curve is minimized. In particular, we are interested in the convexity properties of the problem since we want to solve it in practice [2]. If the problem is convex, there is only one local minimum and a solution computed by a numerical solver is expected to be the global minimum [1].

We investigate if there is a unique solution to this problem by studying the convexity properties of the related problems. We undertake an empirical investigation that indicates that it has one unique solution in cases involving 1 up to 20 B-spline bases. Furthermore, we prove that the convexity properties of all the problems are preserved subject to a scaling and translation of the knot sequence defining the B-spline.

2 The problem of computing an obstacle-avoiding uniform quartic B-spline

In this section we define the main problem that we study. We formulate it as an optimization problem with a cost function $f_{\tilde{B}}(b)$ being minimized over the vector b subject to a set of linear constraints (the complete formulation can be found on page 3).

2.1 B-splines and their minimum variation cost function

The cost function involves a measure of smoothness, which is based on the arc-length derivative of curvature along the curve. The curve is defined as a B-spline function. Let $B(b,x) = \sum_{i=1}^{m} b_i N_{i,d}(x) \in R$ be a *B-spline function* (B-spline) of *degree* d. It is defined by the *B-spline basis functions* $N_{i,d}(x)$ [4, 6], the *B-spline coefficient vector* $b = [b_1, \ldots, b_m]^T \in R^m$, and a non-decreasing *knot sequence* $\tau = \{\tau_i\}_{i=1,\ldots,m+d+1}$. We refer to the number of B-spline bases m as being the *dimension* of the problems that we study.

In particular, we consider two different *quartic* B-splines, i.e. B-splines of degree 4. These are $\hat{B}(b,x)$ and $\tilde{B}(b,x)$ defined by the two *uniform* knot sequences $\tau = \hat{t} = \{\hat{t}_i\}_{i=1,\ldots,m+5}$ and $\tau = \tilde{t} = \{\tilde{t}_i\}_{i=1,\ldots m+5}$ respectively. These sequences are defined by

$$\hat{t}_i = \hat{t}_1 + (i-1)\Delta, \; i = 2, \ldots, m+5. \tag{2}$$

$$\tilde{t}_i = \begin{cases} \hat{t}_1, & i < 5, \\ \hat{t}_i, & i = 5, \ldots, m+1 \\ \hat{t}_{m+1}, & i > m+1. \end{cases} \tag{3}$$

where $\hat{t}_1, \Delta \in R$. Note that $\tilde{t}$ has 5 *multiple* knots at each end.

The curvature $K_B(b,x)$ (see O'Neill [14]) of $B(b,x)$ is given by

$$K(s) = K_B(b,x) = \frac{\frac{\partial^2}{\partial x^2} B(b,x)}{(1 + (\frac{\partial}{\partial x} B(b,x))^2)^{\frac{3}{2}}}. \tag{4}$$

Our measure of smoothness of a curve is based on the arc-length derivative of curvature $d/ds(K(s))$. Using $ds = \sqrt{1 + (\partial/\partial x(B(b,x)))^2}\, dx$ we find that

$$\frac{d}{ds} K(s) = \frac{\frac{\partial}{\partial x} K_B(b,x)}{\sqrt{1 + (\frac{\partial}{\partial x} B(b,x))^2}}. \tag{5}$$

It can be derived, using the recursive definition of $B(b,x)$ [4], that 4 is the least degree of $B(b,x)$ for which $\partial^3/\partial x^3(B(b,x))$, and thereby the derivative of curvature $d/ds(K(s(x))$, is continuous with respect to x. Thus, for a curve defined by a general B-spline $B(b,x)$ with knot sequence τ, the *minimum variation B-spline cost function* $f_B(b)$ is given by

$$f_B(b) = \int_{\tau_1}^{\tau_{m+d+1}} \frac{(\frac{\partial}{\partial x} K_B(b,x))^2}{\sqrt{1 + (\frac{\partial}{\partial x} B(b,x))^2}}\, dx. \tag{6}$$

Note that this is the same cost function as (1) but for a B-spline.

2.2 Linear constraints

We now consider a uniform quartic B-spline $\tilde{B}(b,x)$, defined by b and $\tilde{t}$ from (3). The set of constraints consists of two parts. The first part stipulates boundary conditions by requiring that the value of $\tilde{B}(b,x)$ and its first three derivatives are equal to given constants at the end points, $\tilde{t}_1$ and $\tilde{t}_{m+5}$, of the curve. These constraints are:

$$\begin{cases} \frac{\partial^k}{\partial x^k} \tilde{B}(b, \tilde{t}_1) = \alpha_k, & k = 0, \ldots, 3, \\ \frac{\partial^k}{\partial x^k} \tilde{B}(b, \tilde{t}_{m+5}) = \beta_k, & k = 0, \ldots, 3, \end{cases} \tag{7}$$

where α_k and β_k, $k = 0, \ldots, 3$, are given constants.

The second part makes sure the curve does not intersect two given obstacles defined as non-intersecting polygonal chains which in this case are piecewise linear functions of x with domain $[\tilde{t}_1, \tilde{t}_{m+5}]$. These chains are denoted $\underline{c}$ and $\overline{c}$, where $\underline{c}(x) \leq \overline{c}(x)$, $x \in [\tilde{t}_1, \tilde{t}_{m+5}]$. The region between the chains constitutes the region in the plane which contains the curve. To ensure that the curve does not intersect the obstacles we use an *envelope* to bound it and require that the envelope does not intersect the obstacles. The envelope also consists of two piecewise linear functions $\underline{e}(b,x)$ and $\overline{e}(b,x)$ that are uniquely defined by $\tilde{B}(b,x)$ [10]. For every

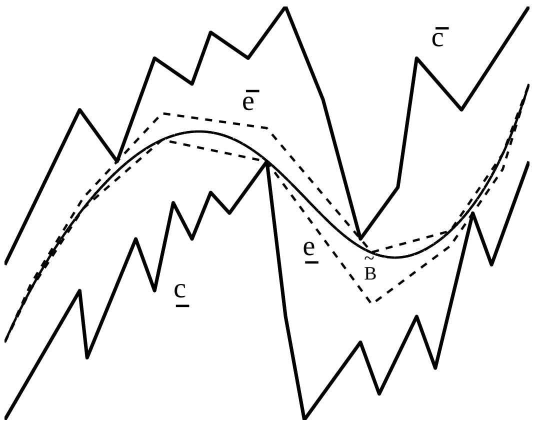

Figure 1. A B-spline function that is computed to solve an instance of P$_O$. The B-spline is built from 10 B-spline bases and its envelope and the obstacles of the problem are shown in the figure.

$x \in [\tilde{t}_1, \tilde{t}_{m+5}]$, they satisfy $\underline{e}(b,x) \leq \tilde{B}(b,x) \leq \overline{e}(b,x)$. This is used for the second part of the constraints which enforces

$$\begin{cases} \underline{c}(x) & \leq & \underline{e}(b,x), & x \in [\tilde{t}_1, \tilde{t}_{m+5}], \\ \overline{e}(b,x) & \leq & \overline{c}(x), & x \in [\tilde{t}_1, \tilde{t}_{m+5}]. \end{cases} \tag{8}$$

2.3 The optimization problem

We summarize the outline in the preceding sections as follows:

Problem P$_O$ (*The obstacle-avoiding uniform quartic minimum variation B-spline problem*):

Given a knot sequence $\tilde{t}$ (according to (3)), two polygonal chains $\underline{c}(x) \leq \overline{c}(x)$, and constants α_k and β_k, $k = 0, \ldots, 3$, compute

$$\min_{b \in R^m} \int_{\tilde{t}_1}^{\tilde{t}_{m+5}} \frac{(\frac{\partial}{\partial x} K_{\tilde{B}}(b,x))^2}{\sqrt{1 + (\frac{\partial}{\partial x}\tilde{B}(b,x))^2}}\, dx$$

such that

$$\begin{array}{lcl} \frac{\partial^k}{\partial x^k}\tilde{B}(b,\tilde{t}_1) & = & \alpha_k, k = 0, \ldots, 3, \\ \frac{\partial^k}{\partial x^k}\tilde{B}(b,\tilde{t}_{m+5}) & = & \beta_k, k = 0, \ldots, 3, \\ \underline{c}(x) & \leq & \underline{e}(b,x), x \in [\tilde{t}_1, \tilde{t}_{m+5}], \\ \overline{e}(b,x) & \leq & \overline{c}(x), x \in [\tilde{t}_1, \tilde{t}_{m+5}], \end{array}$$

where $\tilde{B}(b,x)$ is the uniform quartic B-spline defined by b and $\tilde{t}$, $\underline{e}(b,x) \leq \overline{e}(b,x)$ is the envelope and $K_{\tilde{B}}(b,x)$ is the curvature of $\tilde{B}(b,x)$.

The last two constraints can be expressed by constraints linear in b [10]. The number of such constraints is no more than $6m - 2$ plus the number of vertices in the obstacles. These constraints express the fact that since the computed curve is contained in its envelope it is enough to require that the envelope lies between the obstacles in order to ensure that the computed curve does not intersect the obstacles. Note that, as the derivatives of $\tilde{B}(b,x)$ are also linear in b, all constraints are linear in b. Figure 1 shows a computed solution to an instance of P$_O$, in which $\tilde{B}(b,x)$ is built from 10 B-spline bases. It also visualizes how $\tilde{B}(b,x)$ is constrained to lie between the obstacles, i.e. the two polygonal chains, by means of its envelope.

3 Convexity properties of P$_O$

We are interested in if there is a unique solution to P$_O$. The problem has a unique solution if it is convex. In this section we show that a convexity investigation of P$_O$ can be reduced to a convexity investigation of a simpler problem defined using one particular knot sequence.

3.1 Related problems

Below we define three problems that relate to P$_O$. Except from the first problem they are used to investigate if there is a unique solution to P$_O$.

Problem P$_{MVB}$: The *minimum variation B-spline problem* is defined as P$_{MVC}$ but the curve is restricted to being a B-spline function $B(b,x)$ with knot sequence τ.

Problem P: The *uniform quartic minimum variation B-spline problem* is a subproblem to P$_{MVB}$. It is defined in the same way as P$_O$ (see Section 2.3), but the constraints on the envelope of $\tilde{B}(b,x)$ are omitted. Therefore, P$_O$ itself is in fact a subproblem of P. Since P$_O$ is defined for $\tilde{B}(b,x)$ on the knot sequence $\tilde{t}$, so is also P.

Problem P$_T$: The *trivial uniform quartic minimum variation B-spline problem* is stated in the same way as P$_{MVB}$ with the restriction that it only considers quartic B-splines $\hat{B}(b,x)$ on the knot sequence $\hat{t}$ according to (2). This knot sequence implies that the value of the B-spline and its three first derivatives vanish at the endpoints $\hat{t}_1$ and $\hat{t}_{m+5}$. In turn, there is always the trivial solution to P$_T$, namely $b = [0, \ldots, 0]^T$ yielding $\hat{B}(b,x) \equiv 0$.

The relations between the problems of this paper are outlined in Figure 2. Each subproblem is presented with its additional restrictions or constraints compared to the problem on its left hand side.

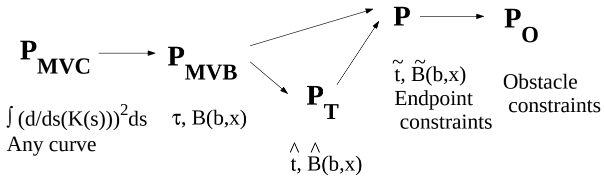

Figure 2. Sketch of how P$_{MVC}$ and its subproblems discussed in this paper are related. A subproblem of a problem stands to the right of the problem and its additional restrictions are outlined in below of it.

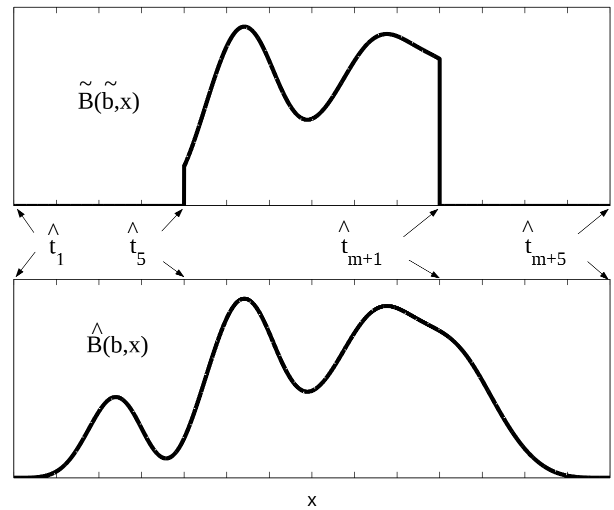

Figure 3. An example of when $\tilde{B}(\tilde{b},x) \equiv \hat{B}(b,x)$ over $x \in [\hat{t}_5, \hat{t}_{m+1})$.

3.2 Problem P$_O$ is convex if P$_T$ is convex

We show that investigating the convexity of P$_O$ can be done by investigating the convexity of the simpler problem P$_T$, for which the global minimum (the zero function) is known.

Lemma 1 *Assume that P$_T$ is a convex problem. Then P$_O$ is convex.*

Proof: First we introduce a new problem, P$_{\breve{T}}$, which we get from P$_T$ by adding the same constraints as those in P at the points $\hat{t}_5 = \tilde{t}_5$ and $\hat{t}_{m+1} = \tilde{t}_{m+1}$ (see (7)). This implies that the values of the B-spline and its first three derivatives are specified at those points. Imposing linear constraints on a convex problem yields a problem that is convex. Hence, by the assumption that P$_T$ is convex, it follows that P$_{\breve{T}}$ is convex.

A B-spline function $\hat{B}(b,x)$ defined on knot sequence $\hat{t}$ (as in P$_{\breve{T}}$) is closely related to a B-spline function $\tilde{B}(\tilde{b},x)$ defined on the knot sequence $\tilde{t}$ (as in P). It can be verified, by applying the recursive definition of the B-spline, that, $\tilde{B}(\tilde{b},x) \equiv \hat{B}(b,x),\ x \in [\hat{t}_5, \hat{t}_{m+1})$, if and only if

$$\begin{bmatrix} \tilde{b}_1 \\ \tilde{b}_2 \\ \tilde{b}_3 \\ \tilde{b}_4 \end{bmatrix} = \begin{bmatrix} \frac{1}{24} & \frac{11}{24} & \frac{11}{24} & \frac{1}{24} \\ 0 & \frac{1}{3} & \frac{7}{12} & \frac{1}{12} \\ 0 & 0 & \frac{3}{4} & \frac{1}{4} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix},$$

$$\tilde{b}_j = b_j,\ j = 5, \ldots, m-4,$$

$$\begin{bmatrix} \tilde{b}_{m-3} \\ \tilde{b}_{m-2} \\ \tilde{b}_{m-1} \\ \tilde{b}_m \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{4} & \frac{3}{4} & 0 & 0 \\ \frac{1}{12} & \frac{7}{12} & \frac{1}{3} & 0 \\ \frac{1}{24} & \frac{11}{24} & \frac{11}{24} & \frac{1}{24} \end{bmatrix} \begin{bmatrix} b_{m-3} \\ b_{m-2} \\ b_{m-1} \\ b_m \end{bmatrix}.$$

Figure 3 shows an example of $\tilde{B}(\tilde{b},x)$ and $\hat{B}(b,x)$ together with parts of their knot sequences when $\tilde{B}(\tilde{b},x) \equiv \hat{B}(b,x)$ over $x \in [\hat{t}_5, \hat{t}_{m+1})$.

The cost function of P$_{\breve{T}}$ can be split into three terms

$$\int_{\hat{t}_1}^{\hat{t}_5} g(b,x)dx + \int_{\hat{t}_5}^{\hat{t}_{m+1}} g(b,x)dx + \int_{\hat{t}_{m+1}}^{\hat{t}_{m+5}} g(b,x)dx$$

where

$$g(b,x) = \frac{(\frac{\partial}{\partial x} K_{\hat{B}}(b,x))^2}{\sqrt{1 + (\frac{\partial}{\partial x}\hat{B}(b,x))^2}}.$$

Since the constraints completely determine the coefficients $b_1, \ldots, b_4$ and $b_{m-3}, \ldots, b_m$, the first and third integral is fixed and only the second integral is affected by the minimization. Hence, the convex problem P$_{\breve{T}}$ is the same as finding $b_5, \ldots, b_{m-4}$ such that this second integral is minimized. The linear one-to-one relationship between b and $\tilde{b}$ for $x \in [\hat{t}_5, \hat{t}_{m+1})$ implies that this is problem equivalent to P. Hence, P$_{\breve{T}}$ and P have the same convexity properties. Since P$_O$ is constructed from P by adding linear constraints, i.e. the ones considering the envelope, it follows that P$_O$ is convex. □

3.3 The convexity of P$_{MVB}$ is preserved due to scaling and translation.

We show that the convexity of P$_{MVB}$ is preserved due to a scaling and translation of the B-spline $B(b,x)$ and its defining knot sequence τ. The result is valid for an arbitrary knot sequence τ, dimension m, and B-spline degree

d. In turn, as we are only dealing with linear constraints, this property holds for any of the subproblems of P_{MVB}. In particular, it holds for P_T and implies that it is sufficient to investigate the convexity of P_T for a specific uniform knot sequence (for example $\hat{t} = \{0, 1, \ldots, m + d\}$) in order to investigate the convexity of P_T for a general uniform knot sequence $\hat{t}$.

Problem P_{MVB} is convex on $\Omega \subset R^m$ if the cost function $f_B(b)$ defined by (6) is convex with respect to $b \in \Omega$. Therefore, we turn our attention to the convexity of $f_B(b)$.

Lemma 2 *Let the cost function $f_B(b)$ be defined by (6) using a B-spline function $B(b, x)$ of degree d defined on a knot sequence $\tau = \{\tau_1, \tau_2, \ldots, \tau_{m+d+1}\}$. Let the cost function $f_{\check{B}}(b)$ be defined by (6) using the B-spline function $\check{B}(b, x) = y_0 + wB(b, (x - x_0)/v)$ defined on the knot sequence $\check{\tau} = \{x_0 + v\tau_1, x_0 + v\tau_2, \ldots, x_0 + v\tau_{m+d+1}\}$, where $v, w > 0$ and $x_0, y_0 \in R$.*

If $f_B(b)$ is convex on $\Omega \subset R^m$, then $f_{\check{B}}(b)$ is convex on $\check{\Omega} = \{\check{b} \,|\, \check{b} = vb/w,\ b \in \Omega\}$.

Proof: The cost function for $B(b, x)$ is

$$f_B(b) = \int_{\tau_1}^{\tau_{m+d+1}} \frac{(\frac{\partial}{\partial x} K_B(b, x))^2}{\sqrt{1 + (\frac{\partial}{\partial x} B(b, x))^2}}\, dx$$

and the cost function for $\check{B}(x)$ is

$$f_{\check{B}}(b) = \int_{x_0 + v\tau_1}^{x_0 + v\tau_{m+d+1}} \frac{(\frac{\partial}{\partial x} K_{\check{B}}(b, x))^2}{\sqrt{1 + (\frac{\partial}{\partial x} \check{B}(b, x))^2}}\, dx.$$

Letting $z = (x - x_0)/v$ implies that $dx = vdz$ and for $k = 1, 2 \ldots$, $\partial^k/\partial x^k(\check{B}(b, x)) = w(\partial^k/\partial z^k(B(b, z)))/v^k$. We use the relation $w(\partial^k/\partial z^k(B(b, z)))/v = \partial^k/\partial z^k(B(wb/v, z))$, $k = 0, 1, \ldots$, which comes from the linearity of B-splines, and derive

$$f_{\check{B}}(b) = \frac{1}{v^3} f_B\left(\frac{w}{v} b\right). \tag{9}$$

If $f_B(b)$ is (strictly) convex on $\Omega \subset R^m$, then, for every $b^{(1)}, b^{(2)} \in \Omega$ and $\lambda \in [0, 1]$, $f_B(\lambda b^{(1)} + (1 - \lambda)b^{(2)}) < \lambda f_B(b^{(1)}) + (1 - \lambda) f_B(b^{(2)})$. Relation (9) states that $f_{\check{B}}(vb/w) = f_B(b)/v^3$. This in turn yields that

$$\begin{aligned}
& f_{\check{B}}\left(\lambda\left(\frac{v}{w} b^{(1)}\right) + (1 - \lambda)\left(\frac{v}{w} b^{(2)}\right)\right) \\
&= \frac{1}{v^3} f_B\left(\lambda b^{(1)} + (1 - \lambda) b^{(2)}\right) \\
&< \frac{1}{v^3}\left(\lambda f_B\left(b^{(1)}\right) + (1 - \lambda) f_B\left(b^{(2)}\right)\right) \\
&= \lambda f_{\check{B}}\left(\frac{v}{w} b^{(1)}\right) + (1 - \lambda) f_{\check{B}}\left(\frac{v}{w} b^{(2)}\right).
\end{aligned}$$

Thus, if $f_B(b)$ is convex on $\Omega \subset R^m$, then $f_{\check{B}}(b)$ is convex on $\check{\Omega} = \{\check{b} \,|\, \check{b} = vb/w,\ b \in \Omega\}$. □

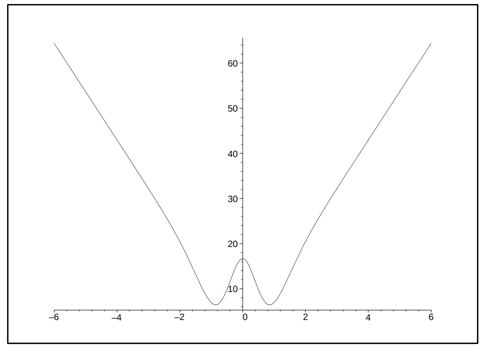

Figure 4. Plot of the second derivative of the cost function $f_{\hat{B}}(b)$ for $b = b_1 \in \Omega = [-6, 6]$.

4 An empirical investigation of the convexity of P_O

In order to investigate the convexity of P_O we investigate the convexity of P_T (see Lemma 1). We have made some attempts to investigate the convexity of the cost function $f_{\hat{B}}(b)$ analytically by for example studying if the Hessian of $f_{\hat{B}}(b)$ is positive definite. But expressions involved in the analysis grow rapidly in size, and yet we have no significant analytical results. Therefore, we rely on an empirical investigation.

We use two different empirical methods for studying the convexity of $f_{\hat{B}}(b)$. Both methods use the fact that $f_{\hat{B}}(b)$, where $b \in R^m$, has at least one minimum, namely the origin $b = [0, \ldots, 0]^T$. In each method, we study a set $\Omega \subset R^m$, of B-spline coefficient vectors b, that contains the origin. We perform our investigations using integer knot sequences only since, by Lemma 2, the results carry over to investigations performed using any uniform knot sequence. First, we plot $f_{\hat{B}}(b)$ in one and two dimensions, i.e. for $b \in \Omega \subset R$ and $b \in \Omega \subset R^2$, and rely on a visual inspection of the plots to draw conclusions about the convexity of $f_{\hat{B}}(b)$. Second, we search for other minima of $f_{\hat{B}}(b)$ than the one at the origin by solving instances of P_T with a numerical solver.

The plots are based on numerical computations performed in MAPLE. In one dimension we have successfully plotted a positive second derivative – indicating convexity – for sizes of Ω up to $[-20000, 20000]$ before having numerical problems. Figure 4 shows the second derivative of $f_{\hat{B}}(b)$ for $b = b_1 \in \Omega = [-6, 6]$. In the two-dimensional case we plotted $f_{\hat{B}}(b)$ for $[b_1, b_2]^T \in \Omega = [-k, k] \times [-k, k] \subset R^2$ for $k \leq 5000$ without problems. For $k > 5000$ we get numerical difficulties as the value of $f_{\hat{B}}(b)$ grows rapidly with the size of b. Plots for the two-dimensional case also indicate convexity but are omitted here due to page limitations. They are included in the full report [3].

Our search for other minima is performed using a numerical solver with many randomly chosen initial values. Given an initial value $b^{(0)} \in R^m$, the solver provides a local minimum by iteratively converging to a solution b^* [8, 11]. For dimensions 1 to 20 ($m = 1, \ldots, 20$), we try to find out if $f_{\hat{B}}(b)$ has any other local minima than the global minimum $b = [0, \ldots, 0]^T$. The space over which the search is performed is $\Omega = [-10^q, 10^q] \times \cdots \times [-10^q, 10^q] \subset R^m$, where $q = 0, 1, 2$. If the cost function $f_{\hat{B}}(b)$ is convex over $b \in \Omega$ the *only* minimum of $f_{\hat{B}}(b)$ is at the origin. Thus, if the search is successful, $f_{\hat{B}}(b)$ is not convex and if the search is unsuccessful, $f_{\hat{B}}(b)$ might be convex. To minimize $f_{\hat{B}}(b)$ starting at $b = b^{(0)}$, we use `fminunc`, which is a solver for unconstrained optimization problems provided by MATLAB. Test results show that, for all $m = 1, \ldots, 20$ and $q = 0, 1, 2$, the solver converges to the origin. Thus, we have not found any local minima other than the origin. Even if this does not necessarily mean that $f_{\hat{B}}(b)$ is convex, the test results together with the plots, indicate that $f_{\hat{B}}(b)$ (and hence P_T and thereby P_O) is convex.

5 Conclusions and future work

In this paper we have studied the problem of computing a smooth planar curve where the smoothness of the curve was defined as the integral of the square of arc-length derivative of curvature along the curve. We introduced the minimum variation B-spline problem which is a linearly constrained optimization problem over curves defined by B-spline functions only. Our focus lay on properties of the variant of the problem asking for a curve restricted to lie between two given polygonal chains. This problem finds application in path planning among obstacles [2].

An empirical investigation, based on plots and numerous tests using numerical methods, indicates that each instance of this problem has one unique solution among all uniform quartic B-spline functions. We conjecture that there is but one unique solution of the problem. The problem might in fact be convex. The practical implication of this is that a solution computed by a numerical solver can be trusted to be the global minimum. Furthermore, we prove that, for any B-spline function, the convexity properties of the problem are preserved subject to a scaling and translation of the knot sequence defining the B-spline.

Our use of envelopes makes it possible to compute curves that go free of obstacles but it also has a drawback. There is, by construction, some space to accommodate an envelope between the obstacles and a curve [10]. Therefore, the curve is probably not the smoothest possible. It would be interesting to determine how much worse than optimal our curves actually are. It is likely that the difference eventually vanishes as the number of knots is increased. However, this is at the expense of having to spend more time computing the curve. What is the relation between time and improved smoothness?

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