2005 American Control Conference
June 8-10, 2005. Portland, OR, USA

Optimal Velocity Profile Generation for Given Acceleration Limits: Theoretical Analysis

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***Abstract*— A semi-analytical method is proposed to generate minimum-time optimal velocity profiles for a vehicle with given acceleration limits driving along a specified path. The method is formally proven to provide optimal results using optimal control theory. In addition, several undesirable cases, where loss of controllability occurs, and which have been neglected in the literature, are being dealt with in this work.**

I. Introduction

Several results for trajectory planning of high-speed land vehicles using numerical methods have been recently published [1], [2], [3], [4], [5]. These results incorporate accurate, high order dynamical models in the optimization process and thus produce quite realistic results. However, these numerical optimization approaches are computationally costly and may not be appropriate for real-time implementation.

In order to develop a scheme for fast autonomous vehicle operation that can be applied in real time, we need a method that produces optimal or near-optimal solutions with low computational cost. Such methods have been proposed, for instance, in [6] and [7]. The path in these references is designed using geometric principles, and an intuitively "optimal" velocity profile is generated using a semi-analytical approach, by taking into consideration the maximum acceleration available to the vehicle at each point on the path.

In this work we concentrate on the generation of the optimal velocity profile along a specified path, given the acceleration limits of the vehicle. A point mass model of the vehicle is used, and the problem is formulated using an optimal control framework. A formal proof of the optimality of the semi-analytical method proposed in [6] and [7] to generate the optimal velocity profile is provided. Several problematic cases, in which loss of controllability occurs, and which were neglected in [6] and [7], are also discussed. The methodology is of minimal computational cost, compared to common numerical optimization approaches, and is suitable for on-line implementation.

II. Problem Statement

Consider a vehicle modelled by a point mass m travelling through a prescribed path, with given acceleration limits and fixed boundary conditions, that is, fixed initial and final position and velocity. We seek the velocity profile along the path for minimum travel time. The path is described by the radius at each point of the path as a function of the path length coordinate s, $R(s)$ (Fig. 1), or equivalently by the curvature $k(s)$ along the path. The cartesian coordinates at any point on the path may be calculated using a standard transformation [3]. The equations of motion are given by

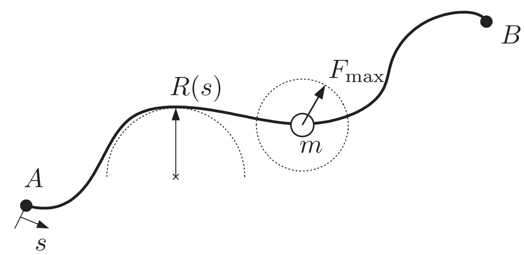

Fig. 1. The vehicle of mass m travels along the prescribed path $R(s)$ in minimum time, given the maximum acceleration limit $F_{\max}/m$.

$$m\frac{\mathrm{d}^2 s}{\mathrm{d}t^2} = f_t \ , \quad \frac{m\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2}{R(s)} = f_n \ , \tag{1}$$

where, f_t is the tangential component of the force along the path, and f_n is the normal (centripetal) force such that the vehicle tracks the prescribed path. Consider now the following state assignment and change of time scale: $\tau = \beta t$, $z_1 = \alpha\beta s$, $z_2 = \alpha\frac{\mathrm{d}s}{\mathrm{d}t}$, with $\alpha = \sqrt{m/F_{\max}}$ and $\beta = \alpha F_{\max}/m$. The control input in this formulation is f_t, and the maximum overall acceleration limit $F_{\max}/m$ translates to a state-dependent control constraint. Introducing the control variable u, the control constraint may be written as

$$f_t/F_{\max} = u\sqrt{1-\left(z_2^2/R(z_1)\right)^2}, \quad u \in [-1,+1] \ . \tag{2}$$

The dynamics of the system may then be written as

$$\dot{z}_1 = z_2 \ , \quad \dot{z}_2 = u\sqrt{1-\left(z_2^2/R(z_1)\right)^2}, \quad u \in [-1,+1] \ . \tag{3}$$

Note that for the dynamics to be well defined and controllability to be maintained the trajectories have to remain inside the region $\mathcal{S}$ of the state space defined by

$$\mathcal{S} = \{(z_1,z_2) : |R(z_1)| > z_2^2\}. \tag{4}$$

In the sequel we assume that $(z_1(\tau), z_2(\tau)) \in \mathcal{S}$, for all $\tau \in [0, \tau_f]$, unless stated otherwise.

III. Optimal Control Formulation

Given fixed boundary conditions $z_{10} = z_1(\tau = 0)$ (point A), $z_{20} = z_2(\tau = 0)$, $z_{1f} = z_1(\tau = \tau_f)$ (point B) and

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$z_{2f} = z_2(\tau = \tau_f)$, we desire the optimal control u that drives the system (3) from point A to point B in minimum time τ_f. At this point we make the natural assumption that the boundary conditions are chosen in such a way that the optimal velocity does not change sign, that is, $z_2(\tau) \geq 0,\ \forall\, \tau \geq 0$. The Hamiltonian for the corresponding minimum time problem is

$$H = 1 + \lambda_1 z_2 + \lambda_2 u\sqrt{1 - \left(z_2^2/R(z_1)\right)^2}\ . \tag{5}$$

The system of adjoint equations is

$$\dot{\lambda}_1 = -\frac{\partial H}{\partial z_1} = -\lambda_2 u \frac{z_2^4}{\sqrt{1-\left(z_2^2/R(z_1)\right)^2}}\frac{R'}{R^3}\ , \tag{6}$$

$$\dot{\lambda}_2 = -\frac{\partial H}{\partial z_2} = -\lambda_1 + 2\lambda_2 u \frac{z_2^3}{R^2\sqrt{1-\left(z_2^2/R(z_1)\right)^2}}, \tag{7}$$

where R' denotes the partial derivative of the path radius with respect to z_1, that is, $R' = \frac{\partial R}{\partial z_1}$. From the transversality condition we get $H(\tau_f) = 0$. Since the Hamiltonian does not depend explicitly on time, it follows that $H(\tau) = 0$, for all $\tau \in [0, \tau_f]$.

The optimal control is found using Pontryagin's Maximum Principle

$$u^* = \mathrm{argmin}_{u\in[-1,+1]} H(z,\lambda,u) = \begin{cases} -1 \text{ for } \lambda_2 > 0, \\ +1 \text{ for } \lambda_2 < 0, \end{cases} \tag{8}$$

which implies $u^* = -\mathrm{sgn}\,\lambda_2$. Therefore λ_2 is the switching function, which determines the value of u^*.
The occurrence of singular controls, i.e. intervals for which $\lambda_2 \equiv 0$, can be easily ruled out.

Proposition 1: Assuming that the optimal trajectory remains in $\mathcal{S}$, there can be no singular sub-arc. Subsequently, the optimal trajectory is composed only of bang-bang subarcs ($u = +1$ or $u = -1$).

IV. Solution for Special Cases of $R(s)$

In the following subsections we investigate the solution of the optimal time problem introduced above for several special cases of $R(s)$. We consider the two simple cases of paths of non-increasing and non-decreasing curvature. Using the solutions of these simple cases along with *Bellman's Principle of Optimality* we will show how to construct the solution for the general case of $R(s)$.

A. Case 1: Path of Non-Increasing Curvature

A path of non-increasing curvature is defined by $RR' \geq 0$. Equations (6), (7) with $RR' \geq 0$ give that $\dot{\lambda}_1 \geq 0$.

Suppose now that there exist a switching time τ_1. It follows that $\lambda_2(\tau_1) = 0$. The transversality condition implies

$$\lambda_1(\tau_1) = -\frac{1}{z_2(\tau_1)}\ . \tag{9}$$

For any time $\tau < \tau_1$ we have that

$$\begin{aligned} -\frac{1}{z_2(\tau)} &\leq \frac{-1 + |\lambda_2|\sqrt{1-\left(\frac{z_2^2}{R(z_1)}\right)^2}}{z_2(\tau)} \\ &= \lambda_1(\tau) \leq \lambda_1(\tau_1) = -\frac{1}{z_2(\tau_1)} \end{aligned} \tag{10}$$

since λ_1 is non-decreasing. We conclude that τ_1 is a switching point from acceleration ($u = +1$) to deceleration ($u = -1$).

Suppose now that there exist a second switching instant τ_2. Using the same reasoning as above we conclude that τ_2 has to be a switching point from acceleration to deceleration. It follows that there are no switchings from deceleration to acceleration. Finally, for the case of non-increasing curvature there may be at most one switching of the control from acceleration to deceleration.

B. Case 2: Path of Non-Decreasing Curvature

A path of non-decreasing curvature is defined by $RR' \leq 0$. Using a similar approach as before we may conclude that in the case of a non-decreasing curvature there may be at most one switching of the control from acceleration to deceleration.

V. Loss of Controllability

In the previous formulation of the optimal control problem we have assumed that the vehicle tracks the given path exactly, and thus the centripetal force is given by $f_n = F_{\max} z_2^2/R(z_1)$. When the speed of the vehicle takes the value of

$$z_{2-\mathrm{critical}} = \sqrt{|R(z_1)|}, \tag{11}$$

we have that $(z_1, z_2) \in \partial\mathcal{S}$. In this case $f_n = F_{\max}$ and $\dot{z}_2 = 0$ from (3). The control input u cannot affect the value of the velocity and loss of controllability ensues. This may also be interpreted as loss of the ability to generate tangential force f_t, since the whole force capacity $F_{\max}$ is used to produce the centripetal force f_n. In order to analyze this pathological case, we introduce the state constraint

$$|R(z_1)| - z_2^2 = \epsilon \tag{12}$$

and we let $\epsilon \to 0^+$. Taking the derivative with respect to time of (12) we may solve for the control input that ensures that the vehicle stays on the characteristic path (12). It is easily shown that this control input is given by

$$u_{\mathrm{sc}} = \frac{RR'}{2z_2^2\sqrt{1 - z_2^4/R^2}}. \tag{13}$$

Next we investigate the case of loss of controllability at paths of decreasing, constant and increasing curvature.

A. Path of Monotonically Decreasing Curvature

Consider a path of monotonically decreasing curvature. Assume that at some point in time τ_c, $z_2(\tau_c) = z_{2-\text{critical}}$. The tangential component of the acceleration becomes zero and $\dot{z}_2(\tau_c) = 0$. Since the vehicle travels on a path of decreasing curvature (increasing radius), $R(z_1(\tau_c^+)) > R(z_1(\tau_c))$, while $z_2(\tau_c^+) = z_2(\tau_c)$. It follows that the square root in the rhs of equation (3) will take a positive, non-zero value at $\tau = \tau_c^+$ and the system will regain controllability. Thus, in a path of monotonically decreasing curvature controllability may be lost only instantaneously.

B. Path of Constant Curvature

Consider now a path of constant curvature, and assume that at some point τ_c we have that $z_2(\tau_c) = z_{2-\text{critical}}$. Since the square root in equation (3) becomes zero, the tangential acceleration becomes zero and hence $\dot{z}_2(\tau_c) = 0$. Since we are on a path of constant curvature, $z_2(\tau) = z_{2-\text{critical}}$ for all $\tau \geq \tau_c$ and thus once controllability is lost, it cannot be regained.

In Fig. 2 the curve (i) is the characteristic path of maximum acceleration $(u = +1)$ from the starting point A, (ii-a) is the characteristic path $z_2 = z_{2-\text{critical}}$, and (iii) is the characteristic path of maximum deceleration $(u = -1)$ towards the end point F. Once the point C is reached accelerating from A, controllability is lost and the vehicle travels with velocity $z_{2-\text{critical}}$ on the path (ii-a). At the point D the control would have to change to braking $(u = -1)$ in order for the vehicle to reach the end point D. This is not possible however due to loss of controllability, and the vehicle continues to travel with velocity $z_{2-\text{critical}}$ even after the point D.

As already discussed, the solution $A \to C \to D \to F$ using the characteristic paths (i), (ii-a) and (iii) is not possible. In fact, it is not allowable for the vehicle to reach $z_{2-\text{critical}}$ (unless at the end point $z_{2f} = z_{2-\text{critical}}$). Consider now the solution $z_2^*(z_1)$, from $A \to B$ on (i) with $u = +1$, then from $B \to E$ on (ii-b) with $u = 0$, and finally from $E \to F$ on (iii) with $u = -1$. The characteristic path (ii-b) has constant velocity, which is less than the critical velocity but it is as close to it as possible (let $\epsilon \to 0^+$ in (12)). Notice that this solution provides a smaller travel time than any other solution satisfying $R(z_1) - z_2^2 \geq \epsilon$, since it has pointwise the largest possible velocity. The necessary optimality conditions are also satisfied since for $R' = 0$ (13) gives $u_{\text{sc}} = 0$, which is the control applied on the $B \to E$ subarc. Note however that an optimal solution does not exist in this case since the optimal trajectory for $\epsilon \to 0^+$ does not lie in $\mathcal{S}$.

We point out that the characteristic path (ii-a) corresponds to the solution with free boundary conditions, and thus it represents the fastest possible way to travel through a path of constant curvature.

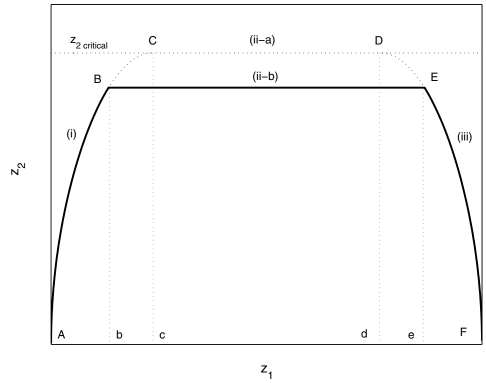


Fig. 2. Constant radius path; uncontrollable case.

C. Path of Monotonically Increasing Curvature

Consider finally a path of monotonically increasing curvature, and a time τ_c such that $z_2(\tau_c) = z_{2-\text{critical}}$. Once again, $\dot{z}_2(\tau_c) = 0$ and $z_2(\tau_c^+) = z_2(\tau_c)$, and since $R(z_1(\tau_c^+)) < R(z_1(\tau_c))$, it follows that a $z_2(\tau_c^+)^2 \geq R(z_1(\tau_c^+))$. The quantity inside the square root in the rhs of (3) becomes negative at τ_c^+. The equations are infeasible and a larger centripetal force than the available $F_{\max}$ is needed for the vehicle to negotiate the path. In other words the vehicle cannot follow the prescribed path.

In the sequel, an approach to generate a suboptimal solution that is guaranteed to maintain $z_2 < z_{2-\text{critical}}$ is proposed. Although the proposed path is not necessarily optimal, it is feasible, controllability is maintained at all times, and recovers the optimal solution when $R'(z_1) \to 0$.

In order to construct this feasible path, notice that the smallest possible slope of the z_1-z_2 plot of any feasible solution is achieved at maximum deceleration ($u = -1$). In particular, we have

$$\dot{z}_2 = u\sqrt{1 - z_2^4/R^2} \Rightarrow z_2' = \frac{u}{z_2}\sqrt{1 - z_2^4/R^2} \tag{14}$$

which for $u = -1$ yields

$$z_{2\min}' = -\frac{1}{z_2}\sqrt{1 - z_2^4/R^2}. \tag{15}$$

On the other hand, the slope of the $z_{2-\text{critical}}$ characteristic path of (11) is

$$z_{2-\text{critical}}' = \frac{\text{sgn}(R)R'}{2\sqrt{|R|}} = \rho \tag{16}$$

We enforce the inequality $z_{2\min}' \leq z_{2-\text{critical}}'$, which implies, by taking into consideration equations (15) and (16), that the following polynomial inequality should hold

$$P(z_2) = z_2^4 + R^2\rho^2 z_2^2 - R^2 \leq 0. \tag{17}$$

Solving for z_2^2, the roots of $P(z_2)$ are $r_{1,2} = (-R^2\rho^2 \mp \sqrt{R^4\rho^4 + 4R^2})/2$ and for (17) to hold, given $z_2 > 0$ we must have $z_2 \leq \sqrt{r_2}$. The following characteristic path is then generated

$$z_{2-\text{safe}} = \sqrt{r_2}. \tag{18}$$

The explicit relationship between $z_{2-\text{safe}}$ and $z_{2-\text{critical}}$ is given by the following equation

$$z_{2-\text{safe}}^4 - z_{2-\text{critical}}^4 + \left(R'^2/4\right) z_{2-\text{safe}}^2 z_{2-\text{critical}}^2 = 0 \ , \quad (19)$$

from which we conclude that $z_{2-\text{safe}} \to z_{2-\text{critical}}$ when $R' \to 0$.

It is easy to show that this characteristic path may be followed by the vehicle using the control law

$$u_{\text{safe}} = \frac{z_2 z'_{2-\text{safe}}}{\sqrt{1 - z_2^4/R^2}}. \quad (20)$$

This is shown in Fig. 3, where the trajectory starts at point A with maximum acceleration $u = +1$ and moves along the characteristic (i). Afterwards the control switches to u_{safe} at the point B, where the path (i) intersects the characteristic path (ii-b) where $z_2 = z_{2-\text{safe}}$. Observe that the path (ii-b) and the $z_{2-\text{critical}}$ path denoted by (ii-a) never meet. Finally, at the point C the control switches to $u = -1$ and the trajectory follows the path (iii) in order to reach the desirable final point D.

Proposition 2: The control law u_{safe} approaches the optimal control law u_{sc} as $\epsilon \to 0^+$.

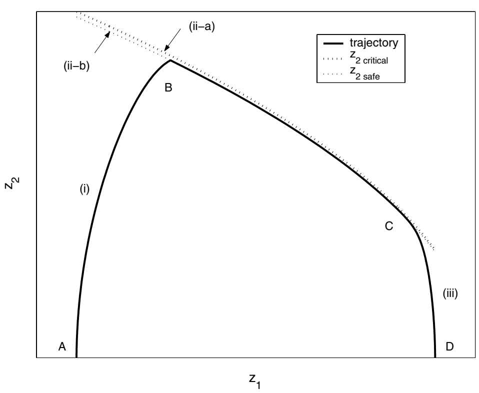


Fig. 3. Path of increasing curvature; uncontrollable case.

VI. PATH WITH $\min R(s)$

Consider a path that has a unique point of minimum radius at C, as in Fig. 4. Let z_{2-C}^* denote the velocity at point C of this path. According to Bellman's Principle of Optimality if the solution $A \to B$ is optimal then the first part of this solution, $A \to C$, solves the minimum time problem from $A \to C$ with the final condition $z_2(C) = z_{2-C}^*$. Similarly, the second part, $C \to B$, solves the minimum-time problem $C \to B$ with initial condition $z_2(C) = z_{2-C}^*$.

On the part $A \to C$ we have a path of decreasing radius, and according to Section IV-B the possible optimal velocity profiles, summarized in Fig. 5(a), are: pure acceleration (Case 1), pure deceleration (Case 2) or pure acceleration that switches once to pure deceleration (Case 3). Similarly, on the part $C \to B$ we have a path of increasing radius and according to Section IV-A the possible optimal velocity profiles, summarized in Fig. 5(b), are: pure acceleration (Case a), pure deceleration (Case b) or pure acceleration that switches once to pure deceleration (Case c).

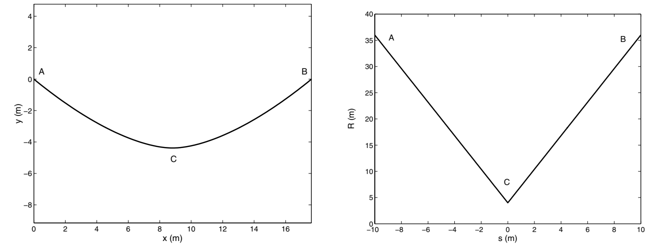
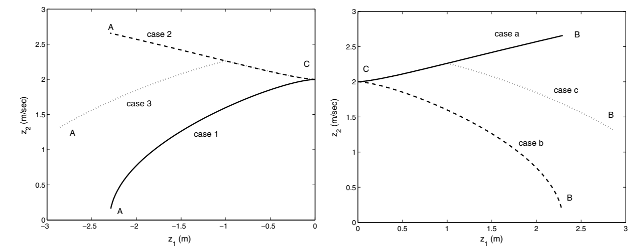


Fig. 4. Path with minimum radius at point C, in cartesian coordinates (left), radius as a function of path length (right).



(a) $A \to C$ (b) $C \to B$

Fig. 5. Possible optimal velocity profiles before (left) and after (right) point C. Similarly for a path with maximum radius at point C.

All the possible optimal velocity profiles for the overall problem from A to B are shown in Fig 6. In the following, we discuss each case separately in order to compute the optimal velocity at point C.

Case 1a corresponds to pure acceleration in both sub-arcs, $A \to C$ and $C \to B$. The velocity at point C has to be less or equal to $z_{2-\text{critical}}$. In this case the optimal velocity z_{2-C}^* is determined exclusively by the boundary conditions at A and B.

Case 1b corresponds to pure acceleration from A to C and pure deceleration from C to D. Consider a different solution using the sequence of characteristic paths (I),(III),(IV),(II) shown in Fig. 6, Case 1b, which also satisfies the IC's at points A and B. However, it is obvious that the solution using one switching from (I) to (II) uses the highest possible velocity point-wise between A and B (for the given boundary conditions), and thus this is the optimal solution. Again, the velocity at point C has to be less than or equal to $z_{2-\text{critical}}$.

Case 1c corresponds to pure acceleration from A to C and switching of the control from acceleration to deceleration along the sub-arc $C \to B$. This case is similar to the Case 1b.

Case 2a corresponds to pure deceleration from A to C and pure acceleration from C to B. Assume, as shown in Case 2a of Fig 6, that there are other solutions that consist of admissible switchings, and which satisfy the

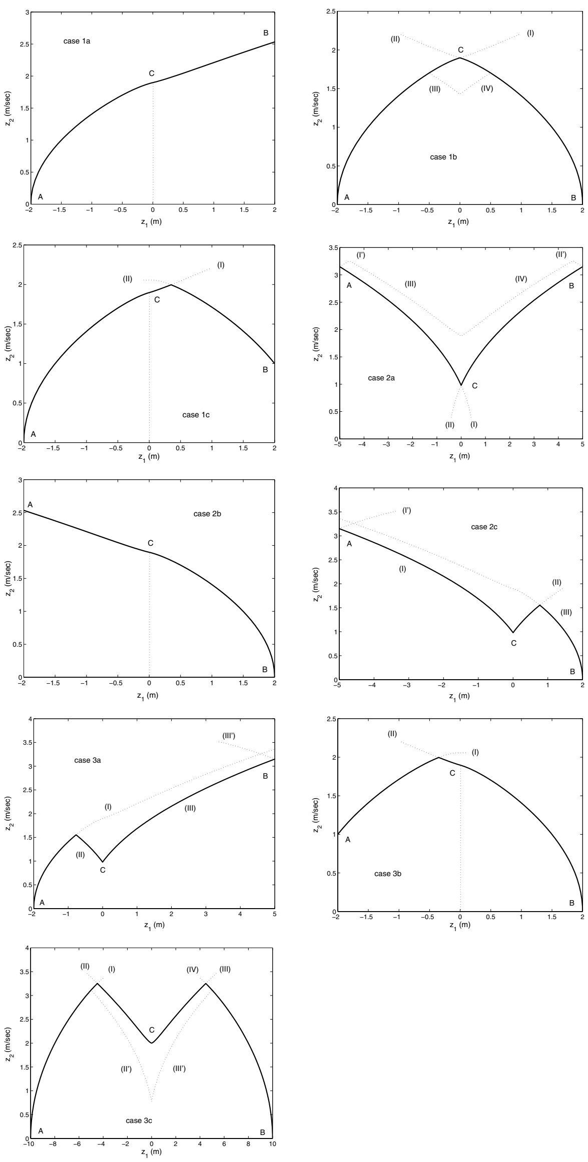


Fig. 6. All possible optimal velocity profiles from A to B.

same boundary conditions at points A and B. In fact, the solution that corresponds to Case 2a, constructed by the characteristic paths (I) and (II), is the one with the lowest velocity point-wise between A and B. We conclude that Case 2a will be optimal only if $z_{2-C}^{*} = z_{2-C-\mathrm{critical}}$. In that case any other solution would require the vehicle to pass through C with velocity greater than the critical one, which is not possible.

Case 2b corresponds to pure deceleration in both sub-arcs, $A \to C$ and $C \to B$. It is completely equivalent to Case 1a if we reverse the boundary conditions at the points A and B.

Case 2c corresponds to pure deceleration from A to C and one switching from acceleration to deceleration inside the sub-arc from C to B. As in Case 2a, unless $z_{2-C}^{*} = z_{2-C-\mathrm{critical}}$ there are other solutions that satisfy the boundary conditions at points A and B consisting of higher velocities at all points between A and B.

Case 3a is completely equivalent to Case 2c.

Case 3b is completely equivalent to Case 1c.

Case 3c corresponds to one switching from acceleration to deceleration along the sub-arc $A \to C$ and another switching from acceleration to deceleration along the sub-arc $C \to B$. Obviously, there are other solutions that satisfy the boundary conditions at the points A and B, consisting of smaller velocities at all points (using the sequence (I), (II'), (III'), (IV), for instance). Unless $z_{2-C}^{*} = z_{2-C-\mathrm{critical}}$ there are other solutions that satisfy the boundary conditions at points A and B, which consist of higher velocity at all points. Thus Case 3c is optimal if $z_{2-C}^{*} = z_{2-C-\mathrm{critical}}$.

From the previous analysis we conclude that there are only two possible scenarios for the value of z_{2-C}^{*}. In Cases 1a, 2a, 3a, 2b and 3b we have $z_{2-C}^{*} \leq z_{2-C-\mathrm{critical}}$. In all these cases there exists at most one control switching. In Cases 2a, 2c, 3a and 3c we have $z_{2-C}^{*} = z_{2-C-\mathrm{critical}}$. In such cases we may have up to three control switchings.

We now propose a methodology to construct the overall optimal solution. Starting from point A we calculate the characteristic path using full acceleration ($u = +1$), say $z_2^{\mathrm{i}}(z_1)$. Starting at point B we construct the characteristic path of full deceleration ($u = -1$) using backward integration, say $z_2^{\mathrm{ii}}(z_1)$. Finally, we construct the characteristic $z_2^{\mathrm{iii}}(z_1)$ of full deceleration up to point C and full acceleration starting from point C. In fact, the path $z_2^{\mathrm{iii}}(z_1)$ is exactly the characteristic path of Case 2a, and the solution to the free boundary condition problem. The optimal velocity profile is then given by

$$z_2^{*}(z_1) \quad = \quad \min\left\{z_2^{\mathrm{i}}(z_1), z_2^{\mathrm{ii}}(z_1), z_2^{\mathrm{iii}}(z_1)\right\} \qquad (21)$$

It is easy to show that (21) reproduces all the cases of Fig. 6.

VII. Path with $\max R(s)$

Consider now a path that has a unique point of maximum radius at C. As before, we may assume that the total path consists of two parts, one of increasing radius, from A to C, and one of decreasing radius, from C to the final point B.

All possible scenarios that may appear along the sub-arcs $A \to C$ and $C \to B$ according to the solutions presented in Sections IV-A and IV-B may be summarized as in Fig. 5. The same arguments as in Section VI hold for each one of the Cases 1a-3c. Thus, we conclude that the optimal velocity at point C is determined solely by the boundary conditions at points A and B for each of the Cases 1a, 1b, 1c, 2b and 3b. In all these cases the optimal solution involves at most one switching, from acceleration to deceleration.

In Section VI we concluded that Cases 2a, 2c, 3a and 3c may appear as the optimal solutions only if the velocity at C is $z_{2-C}^{*} = z_{2-C-\mathrm{critical}}$. Since point C is a point of maximum radius the critical velocity at point C is larger compared to any other point from A to B

$$z_{2-\text{critical}} < z_{2-C-\text{critical}} \ . \tag{22}$$

On the other hand, in Cases 2a, 2c, 3a and 3c the velocity at point C is a local minimum. That is, in an area around point C

$$z^*_{2-C} \leq z_2 \ . \tag{23}$$

For $z^*_{2-C} = z_{2-C-\text{critical}}$ equations (22), (23) imply that $z_2 > z_{2-\text{critical}}$ and the vehicle cannot follow the prescribed path. We conclude that Cases 2a, 2c, 3a and 3c cannot appear as optimal solutions in the case of a path with a point C of maximum radius. The only possible scenarios are Cases 1a, 1b, 1c, 2b and 3b, where the optimal velocity at C is determined uniquely by the initial and final boundary conditions.

VIII. General Solution

In this section we show how to construct the optimal solution for the general case path, that is, a path that includes any combination of constant, decreasing and increasing curvature. Consider the general case path having the radius profile shown in Fig. 7(a). There are several points of minimum radius (points P_1, P_2 and P_5) and a sub-arc of constant radius $P_3 \rightarrow P_4$. In Fig. 7(b) we show the solutions of the of the problems with free boundary conditions, for the sub-arcs containing a minimum radius and the sub-arcs of constant curvature. In particular, (i) corresponds to the fastest way to cross the sub-arc with minimum radius at point P_1 regardless of the boundary conditions; likewise, (ii) is the fastest way to cross the sub-arc containing P_2, (iii) is the fastest way to cross $P_3 \rightarrow P_4$, and (iv) is the fastest way to cross the sub-arc containing P_5.

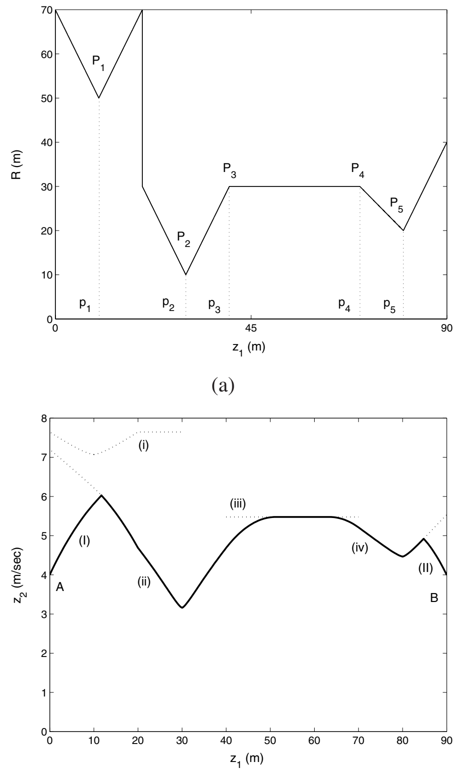
Next, consider fixed boundary conditions at the initial and final points A and B as in Fig. 7(b). Also consider the characteristic paths (I) and (II) of maximum acceleration from A and maximum deceleration towards B, respectively. The solid line in Fig. 7(b) is constructed as in [7] by taking

$$z^*_2(z_1) = \min\left\{z^k_2(z_1)\right\}, \quad k = \mathrm{I}, \mathrm{i}, \mathrm{ii}, \mathrm{iii}, \mathrm{iv}, \mathrm{II} \tag{24}$$

where $z^k_2(z_1)$ is the trajectory $z_2(z_1)$ on the kth characteristic path. It is easy to verify the optimality of the proposed solution from point-wise maximality of the velocity.

IX. Conclusions

Formal proof of optimality for a method to determine the optimal velocity profile for the minimum time travel of a vehicle through a specified path, with given acceleration limits has been presented. Several problematic cases that lead to loss of controllability have been identified, and for these cases alternative suboptimal solutions have been proposed. The optimal path is synthesized by combining together paths of non-increasing and non-decreasing curvature. Bellman's Principle of Optimality allows the elimination of non-optimal combination of such paths. The method is computationally very fast and can be easily implemented on-line, unlike other numerical optimization approaches.



(a)

(b)

Fig. 7. (a) A general case radius profile path (top); (b) Optimal velocity profile for the general case path.

Acknowledgment: This work has been supported in part by the US Army Research Office, award no. DAAD19-00-1-0473.

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