Assessing the relevance of web pages using PageRank

Model behind PageRank: Random walk

- Imagine a web surfer doing a random walk on the web
  - Start at a random page
  - At each step, go out of the current page along one of the links on that page, equiprobably
- In the steady state, each page has a long-term visit rate.
- This long-term visit rate is the page's PageRank.
- PageRank = long-term visit rate = steady state probability □

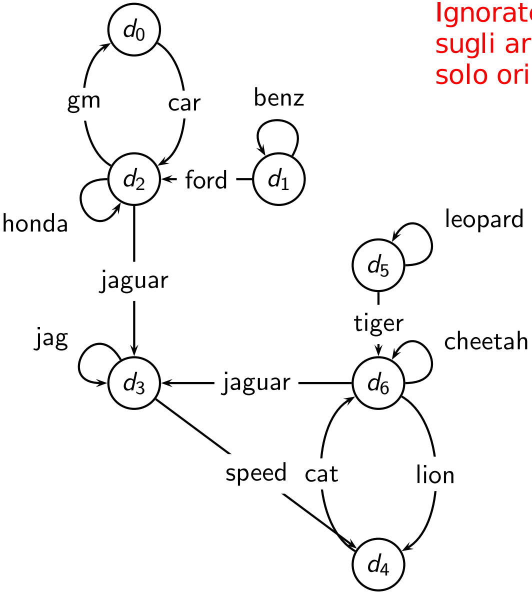
Quanto spesso un surfer
che si muove sul web
capita su una certa pagina

Formalization of random walk: Markov chains

- A Markov chain consists of N states, plus an $N \times N$ transition probability matrix P.
- state = page
- At each step, we are on exactly one of the pages.
- For $1 \leq i, j \leq N$, the matrix entry P_{ij} tells us the probability of j being the next page, given we are currently on page i.
- Clearly, for all i, $\sum_{j=1}^{N} P_{ij} = 1$

$$d_i \xrightarrow{P_{ij}} d_j$$

Example web graph



Ignorate le etichette sugli archi: interessano solo origine e destinazione

Link matrix for example

	d_0	d_1	d_2	d_3	d_4	d_5	d_6
d_0	0	0	1	0	0	0	0
d_1	0	1	1	0	0	0	0
d_2	1	0	1	1	0	0	0
d_3	0	0	0	1	1	0	0
d_4	0	0	0	0	0	0	1
d_5	0	0	0	0	0	1	1
d_6	0	0	0	1	1	0	1

Matrice di adiacenza
del grafo: ad esempio
d2 ha archi diretti a
d0, d2, d3

Transition probability matrix P for example

	d_0	d_1	d_2	d_3	d_4	d_5	d_6
d_0	0.00	0.00	1.00	0.00	0.00	0.00	0.00
d_1	0.00	0.50	0.50	0.00	0.00	0.00	0.00
d_2	0.33	0.00	0.33	0.33	0.00	0.00	0.00
d_3	0.00	0.00	0.00	0.50	0.50	0.00	0.00
d_4	0.00	0.00	0.00	0.00	0.00	0.00	1.00
d_5	0.00	0.00	0.00	0.00	0.00	0.50	0.50
d_6	0.00	0.00	0.00	0.33	0.33	0.00	0.33

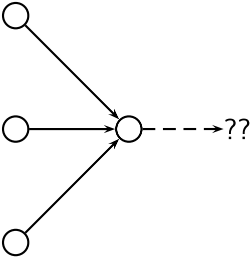
Assumiamo che il random surfer segua a caso uno dei link della pagina

Probabilità del surfer di muoversi da una pagina all'altra

Long-term visit rate

- Recall: PageRank = long-term visit rate
- Long-term visit rate of page d is the probability that a web surfer is at page d at a given point in time.
- Next: what properties must hold of the web graph for the long-term visit rate to be well defined?
- The web graph must correspond to an ergodic Markov chain.
- First a special case: The web graph must not contain dead ends.

Dead ends



- The web is full of dead ends.
- Random walk can get stuck in dead ends.
- If there are dead ends, long-term visit rates are not well-defined (or non-sensical).

Teleporting – to get us out of dead ends

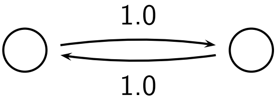
- At a dead end, jump to a random web page with prob. $1/N$.
- At a non-dead end, with probability 10%, jump to a random web page (to each with a probability of $0.1/N$).
- With remaining probability (90%), go out on a random hyperlink.
  - For example, if the page has 4 outgoing links: randomly choose one with probability (1-0.10)/4=0.225
- 10% is a parameter, the teleportation rate.
- Note: "jumping" from dead end is independent of teleportation rate. vuol dire che il teleportation rate (90% nell'esmpio) ha effetto solo sui nodi che non sono dead end

Result of teleporting

- With teleporting, we cannot get stuck in a dead end.
- But even without dead ends, a graph may not have well-defined long-term visit rates.
- More generally, we require that the Markov chain be ergodic.

Ergodic Markov chains

- A Markov chain is ergodic iff it is irreducible and aperiodic.
- Irreducibility. Roughly: there is a path from any page to any other page.
- Aperiodicity. Roughly: The pages cannot be partitioned such that the random walker visits the partitions sequentially.



- A non-ergodic Markov chain:

Ergodic Markov chains

- Theorem: For any ergodic Markov chain, there is a unique long-term visit rate for each state.
- This is the steady-state probability distribution.
- Over a long time period, we visit each state in proportion to this rate.
- It doesn't matter where we start.
- Teleporting makes the web graph ergodic.
- $\Rightarrow$ Web-graph+teleporting has a steady-state probability distribution.
- $\Rightarrow$ Each page in the web-graph+teleporting has a PageRank. □

Where we are

- We now know what to do to make sure we have a well-defined PageRank for each page.
- Next: how to compute PageRank

In teoria potremmo emulare il random surfer e segnarci quanto spesso capita su ogni pagina, ma questo sarebbe un metodo molto inefficiente

Formalization of "visit": Probability vector

- A probability (row) vector $\vec{x} = (x_1, \ldots, x_N)$ tells us where the random walk is at any point.
- Example: $\begin{array}{cccccccccc} (& 0 & 0 & 0 & \ldots & 1 & \ldots & 0 & 0 & 0 &) \\ & 1 & 2 & 3 & \ldots & i & \ldots & \text{N-2} & \text{N-1} & \text{N} & \end{array}$
- More generally: the random walk is on page i with probability x_i.
- Example:
  $\begin{array}{cccccccccc} (& 0.05 & 0.01 & 0.0 & \ldots & 0.2 & \ldots & 0.01 & 0.05 & 0.03 &) \\ & 1 & 2 & 3 & \ldots & i & \ldots & \text{N-2} & \text{N-1} & \text{N} & \end{array}$
- $\sum x_i = 1$

Change in probability vector

- If the probability vector is $\vec{x} = (x_1, \ldots, x_N)$ at this step, what is it at the next step?
- Recall that row i of the transition probability matrix P tells us where we go next from state i.
- So from $\vec{x}$, our next state is distributed as $\vec{x}P$.

Nel nostro esempio la matrice con il teleporting diventa:

$$P = \begin{array}{c|ccccccc} & d_0 & d_1 & d_2 & d_3 & d_4 & d_5 & d_6 \\ \hline d_0 & 0.02 & 0.02 & 0.88 & 0.02 & 0.02 & 0.02 & 0.02 \\ d_1 & 0.02 & 0.45 & 0.45 & 0.02 & 0.02 & 0.02 & 0.02 \\ d_2 & 0.31 & 0.02 & 0.31 & 0.31 & 0.02 & 0.02 & 0.02 \\ d_3 & 0.02 & 0.02 & 0.02 & 0.45 & 0.45 & 0.02 & 0.02 \\ d_4 & 0.02 & 0.02 & 0.02 & 0.02 & 0.02 & 0.02 & 0.88 \\ d_5 & 0.02 & 0.02 & 0.02 & 0.02 & 0.02 & 0.45 & 0.45 \\ d_6 & 0.02 & 0.02 & 0.02 & 0.31 & 0.31 & 0.02 & 0.31 \end{array}$$

Calcolo Page Rank:

$$x_0 = \left(\frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}\right)$$

$$x_1 = x_0 P, \qquad x_2 = x_1 P$$

$$\ldots \; x_n = x_{n-1} P = x_0 P^n$$

Page Rank = $\lim_{n \to \infty} x_n$

Steady state in vector notation

- The steady state in vector notation is simply a vector $\vec{\pi} = (\pi_1, \pi_2, \ldots, \pi_N)$ of probabilities.
- (We use $\vec{\pi}$ to distinguish it from the notation for the probability vector $\vec{x}$.)
- π_i is the long-term visit rate (or PageRank) of page i.
- So we can think of PageRank as a very long vector – one entry per page.

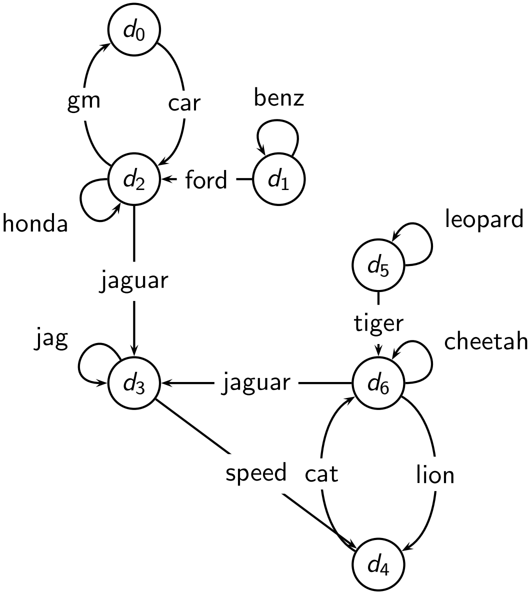
One way of computing the PageRank $\vec{\pi}$

- Start with any distribution $\vec{x}$, e.g., uniform distribution
- After one step, we're at $\vec{x}P$.
- After two steps, we're at $\vec{x}P^2$.
- After k steps, we're at $\vec{x}P^k$.
- Algorithm: multiply $\vec{x}$ by increasing powers of P until convergence.
- This is called the power method.
- Recall: regardless of where we start, we eventually reach the steady state $\vec{\pi}$.
- Thus: we will eventually (in asymptotia) reach the steady state. □

PageRank summary

- Preprocessing
  - Given graph of links, build matrix P
  - Apply teleportation
  - From modified matrix, compute $\vec{\pi}$
  - $\vec{\pi}_i$ is the PageRank of page i.

Example web graph



Transition (probability) matrix

	d_0	d_1	d_2	d_3	d_4	d_5	d_6
d_0	0.00	0.00	1.00	0.00	0.00	0.00	0.00
d_1	0.00	0.50	0.50	0.00	0.00	0.00	0.00
d_2	0.33	0.00	0.33	0.33	0.00	0.00	0.00
d_3	0.00	0.00	0.00	0.50	0.50	0.00	0.00
d_4	0.00	0.00	0.00	0.00	0.00	0.00	1.00
d_5	0.00	0.00	0.00	0.00	0.00	0.50	0.50
d_6	0.00	0.00	0.00	0.33	0.33	0.00	0.33

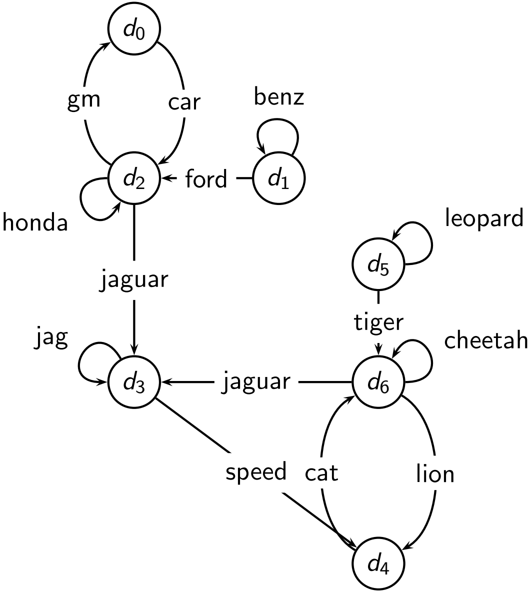
Transition matrix with teleporting

	d_0	d_1	d_2	d_3	d_4	d_5	d_6
d_0	0.02	0.02	0.88	0.02	0.02	0.02	0.02
d_1	0.02	0.45	0.45	0.02	0.02	0.02	0.02
d_2	0.31	0.02	0.31	0.31	0.02	0.02	0.02
d_3	0.02	0.02	0.02	0.45	0.45	0.02	0.02
d_4	0.02	0.02	0.02	0.02	0.02	0.02	0.88
d_5	0.02	0.02	0.02	0.02	0.02	0.45	0.45
d_6	0.02	0.02	0.02	0.31	0.31	0.02	0.31

Power method vectors $\vec{x}P^k$

	$\vec{x}$	$\vec{x}P^1$	$\vec{x}P^2$	$\vec{x}P^3$	$\vec{x}P^4$	$\vec{x}P^5$	$\vec{x}P^6$	$\vec{x}P^7$	$\vec{x}P^8$	$\vec{x}P^9$	$\vec{x}P^{10}$	$\vec{x}P^{11}$	$\vec{x}P^{12}$	$\vec{x}P^{13}$
d_0	0.14	0.06	0.09	0.07	0.07	0.06	0.06	0.06	0.06	0.05	0.05	0.05	0.05	0.05
d_1	0.14	0.08	0.06	0.04	0.04	0.04	0.04	0.04	0.04	0.04	0.04	0.04	0.04	0.04
d_2	0.14	0.25	0.18	0.17	0.15	0.14	0.13	0.12	0.12	0.12	0.12	0.11	0.11	0.11
d_3	0.14	0.16	0.23	0.24	0.24	0.24	0.24	0.25	0.25	0.25	0.25	0.25	0.25	0.25
d_4	0.14	0.12	0.16	0.19	0.19	0.20	0.21	0.21	0.21	0.21	0.21	0.21	0.21	0.21
d_5	0.14	0.08	0.06	0.04	0.04	0.04	0.04	0.04	0.04	0.04	0.04	0.04	0.04	0.04
d_6	0.14	0.25	0.23	0.25	0.27	0.28	0.29	0.29	0.30	0.30	0.30	0.30	0.31	0.31

Example web graph



	PageRank
d_0	0.05
d_1	0.04
d_2	0.11
d_3	0.25
d_4	0.21
d_5	0.04
d_6	0.31

PageRank(d2)$<$ PageRank(d6): why?