Method of Manufactured Solutions (MMS) Benchmarks for the First Order Stokes' Ice Sheet Equations

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July 29, 2013

1 Introduction

This document gives some method of manufactured solutions (MMS) test cases for the first order Stokes equations with Glen's Law viscosity derived by I. Kalashnikova.

2 3D First Order Stokes Equations with Glen's Law Viscosity

The first order Stokes equation are given by:

$$\begin{cases} -\nabla \cdot (2\mu \dot{\boldsymbol{\epsilon}}_1) & = & -\rho g \frac{\partial s}{\partial x}, \\ -\nabla \cdot (2\mu \dot{\boldsymbol{\epsilon}}_2) & = & -\rho g \frac{\partial s}{\partial y} \end{cases} \tag{1}$$

Here, in the most general (3D) case,

$$\dot{\boldsymbol{\epsilon}}_1^T = \begin{pmatrix} 2\dot{\epsilon}_{xx} + \dot{\epsilon}_{yy}, & \dot{\epsilon}_{xy}, & \dot{\epsilon}_{xz} \end{pmatrix} \tag{2}$$

$$\dot{\boldsymbol{\epsilon}}_2^T = \begin{pmatrix} \dot{\epsilon}_{xy}, & \dot{\epsilon}_{xx} + 2\dot{\epsilon}_{yy}, & \dot{\epsilon}_{yz} \end{pmatrix} \tag{3}$$

$$\dot{\epsilon}_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{4}$$

for $i, j \in \{x, y, z\}$. The viscosity μ is given by Glen's law:

$$\mu = \frac{1}{2} A^{-\frac{1}{n}} \left(\dot{\epsilon}_{xx}^2 + \dot{\epsilon}_{yy}^2 + \dot{\epsilon}_{xx}\dot{\epsilon}_{yy} + \dot{\epsilon}_{xy}^2 + \dot{\epsilon}_{xz}^2 + \dot{\epsilon}_{yz}^2\right)^{\left(\frac{1}{2n} - \frac{1}{2}\right)} \tag{5}$$

Here, A is the flow rate factor, and n is the Glen's law exponent ($n = 3$ for ice sheets).

3 2D First Order Stokes Equations with Glen's Law Viscosity

The MMS problems given in this document are for a simplified version of the first order Stokes equations, namely (1) in two spatial dimensions (2D) and on a rectangular geometry aligned with the $x-$ and y–axes: $\Omega = (0, 1) \times (0, 1)$. For this geometry, $s(x, y) = const$, so the gravity-based source terms in (1) vanish.

Some additional sources f_1 and f_2 are added to the equations, however. Toward this effect, the governing equations for the MMS problems presented here are the following:

$$\left\{ \begin{array}{rcl} -\nabla \cdot (2\mu \dot{\boldsymbol{\epsilon}}_1) + f_1 & = & 0, \\ -\nabla \cdot (2\mu \dot{\boldsymbol{\epsilon}}_2) + f_2 & = & 0 \end{array} \right. \tag{6}$$

where

$$\dot{\boldsymbol{\epsilon}}_1^T = \left(\begin{array}{cc} 2\dot{\epsilon}_{xx} + \dot{\epsilon}_{yy}, & \dot{\epsilon}_{xy} \end{array} \right) \tag{7}$$

$$\dot{\boldsymbol{\epsilon}}_2^T = \left(\begin{array}{cc} \dot{\epsilon}_{xy}, & \dot{\epsilon}_{xx} + 2\dot{\epsilon}_{yy} \end{array} \right) \tag{8}$$

and

$$\mu = \frac{1}{2} A^{-\frac{1}{n}} \left(\dot{\epsilon}_{xx}^2 + \dot{\epsilon}_{yy}^2 + \dot{\epsilon}_{xx}\dot{\epsilon}_{yy} + \dot{\epsilon}_{xy}^2 \right)^{\left(\frac{1}{2n} - \frac{1}{2}\right)} \tag{9}$$

with $\dot{\epsilon}_{ij}$ given by (4). Written out, (6) has the form:

$$\left\{ \begin{array}{rcl} -\frac{\partial}{\partial x}\left(4\mu\frac{\partial u}{\partial x} + 2\mu\frac{\partial v}{\partial y}\right) - \frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y} + \mu\frac{\partial v}{\partial x}\right) + f_1 & = & 0, \\ -\frac{\partial}{\partial x}\left(\mu\frac{\partial u}{\partial x} + \mu\frac{\partial v}{\partial y}\right) - \frac{\partial}{\partial y}\left(2\mu\frac{\partial u}{\partial x} + 4\mu\frac{\partial v}{\partial y}\right) + f_2 & = & 0 \end{array} \right. \tag{10}$$

3.1 `sincos2D` MMS test case

The first MMS test case derived will be referred to as the `sincos2D` test case. It is an Dirichlet/homogeneous Neumann BVP on the unit square: $\Omega = (0,1) \times (0,1)$. The force terms in (6) and boundary conditions are derived such that the exact solution to the problem is given by:

$$\begin{array}{rcl} u & = & \sin(2\pi x + \phi)\cos(2\pi y + \psi) + 3\pi x \\ v & = & -\cos(2\pi x + \phi)\sin(2\pi y + \psi) - 3\pi y \end{array} \tag{11}$$

The parameters $\phi, \psi \in [0, 2\pi)$ in (22) are phase shifts that can be used to generate a family of solutions. In order to obtain this exact solution, the source terms in (6) are given by:

$$f_1 = -8\mu\pi^2 \sin(2\pi x + \phi)\cos(2\pi y + \psi) + A^{-\frac{1}{n}} \left(\frac{1}{n} - 1\right) \tilde{\mu}^{\frac{1}{n}-2} \left(\frac{\partial \tilde{\mu}}{\partial x}(2\epsilon_{xx} + \epsilon_{yy}) + \frac{\partial \tilde{\mu}}{\partial y}\epsilon_{xy} \right) \tag{12}$$

$$f_2 = 8\mu\pi^2 \cos(2\pi x + \phi)\sin(2\pi y + \psi) + A^{-\frac{1}{n}} \left(\frac{1}{n} - 1\right) \tilde{\mu}^{\frac{1}{n}-2} \left(\frac{\partial \tilde{\mu}}{\partial x}\epsilon_{xy} + \frac{\partial \tilde{\mu}}{\partial y}(\epsilon_{xx} + 2\epsilon_{yy}) \right) \tag{13}$$

where

$$\tilde{\mu} = 2\pi\cos(2\pi x + \phi)\cos(2\pi y + \psi) + 3\pi x \tag{14}$$

$$\frac{\partial \tilde{\mu}}{\partial x} = -4\pi^2 \sin(2\pi x + \phi)\cos(2\pi y + \psi) \tag{15}$$

$$\frac{\partial \tilde{\mu}}{\partial y} = -4\pi^2 \cos(2\pi x + \phi)\sin(2\pi y + \psi) \tag{16}$$

$$\epsilon_{xx} = 2\pi\cos(2\pi x + \phi)\cos(2\pi y + \psi) + 3\pi \tag{17}$$

$$\epsilon_{yy} = -2\pi\cos(2\pi x + \phi)\cos(2\pi y + \psi) - 3\pi \tag{18}$$

$$\epsilon_{xy} = 0 \tag{19}$$

$$\mu = \frac{1}{2} A^{-\frac{1}{n}} \tilde{\mu}^{\frac{1}{n}-1} \tag{20}$$

The following piece of code specifies f_1 and f_2 (taken from the Albany code base):

```
double r = 3.0*pi;
MeshScalarT x2pi = 2.0*pi*coordVec(cell,qp,0);
MeshScalarT y2pi = 2.0*pi*coordVec(cell,qp,1);
MeshScalarT muargt = 2.0*pi*cos(x2pi + xphase)*cos(y2pi + yphase) + r;
MeshScalarT muqp = 0.5*pow(A, -1.0/n)*pow(muargt, 1.0/n - 1.0);
MeshScalarT dmuargtdx = -4.0*pi*pi*sin(x2pi + xphase)*cos(y2pi + yphase);
MeshScalarT dmuargtdy = -4.0*pi*pi*cos(x2pi + xphase)*sin(y2pi + yphase);
MeshScalarT exx = 2.0*pi*cos(x2pi + xphase)*cos(y2pi + yphase) + r;
MeshScalarT eyy = -2.0*pi*cos(x2pi + xphase)*cos(y2pi + yphase) - r;
MeshScalarT exy = 0.0;
f[0] = 2.0*muqp*(-4.0*pi*pi*sin(x2pi + xphase)*cos(y2pi + yphase))
     + 2.0*0.5*pow(A, -1.0/n)*(1.0/n - 1.0)*pow(muargt, 1.0/n
     - 2.0)*(dmuargtdx*(2.0*exx + eyy) + dmuargtdy*exy);
f[1] = 2.0*muqp*(4.0*pi*pi*cos(x2pi + xphase)*sin(y2pi + yphase))
     + 2.0*0.5*pow(A, -1.0/n)*(1.0/n - 1.0)*pow(muargt, 1.0/n
     - 2.0)*(dmuargtdx*exy + dmuargtdy*(exx + 2.0*eyy));
```

To finish the problem specification, we still need to give the boundary conditions. They are:

$$\begin{array}{rcll} u &=& 0, & \text{at } x=0 \\ u &=& 3\pi, & \text{at } x=1 \\ 2\mu\dot{\boldsymbol{\epsilon}}_2\cdot\mathbf{n} &=& 0, & \text{at } y=0, y=1 \\ 2\mu\dot{\boldsymbol{\epsilon}}_1\cdot\mathbf{n} &=& 0, & \text{at } x=0, x=1 \\ v &=& 0, & \text{at } y=0 \\ v &=& -3\pi, & \text{at } y=1 \end{array} \tag{21}$$

Note that the `sincos2D` test case can be turned into a constant viscosity test case by setting $n = 1$.

3.2 `cosexp2D` MMS test case

The second MMS test case derived will be referred to as the `cosexp2D` test case. The problem specification includes Robin boundary conditions resembling the basal sliding boundary condition. For this problem, the force terms in (6) and boundary conditions are derived such that the exact solution to the problem is given by:

$$\begin{array}{rcl} u &=& e^x\sin(2\pi y) \\ v &=& e^x\cos(2\pi y) \end{array} \tag{22}$$

In order to obtain this exact solution, the source terms in (6) are given by:

$$f_1 = 2\mu\exp(x)\sin(2\pi y)\left[2-3\pi-2\pi^2\right] + A^{-\frac{1}{n}}\left(\frac{1}{n}-1\right)\tilde{\mu}^{\frac{1}{n}-2}\left(\frac{\partial\tilde{\mu}}{\partial x}(2\epsilon_{xx}+\epsilon_{yy})+\frac{\partial\tilde{\mu}}{\partial y}\epsilon_{xy}\right) \tag{23}$$

$$f_2 = 2\mu\exp(x)\cos(2\pi y)\left[3\pi+\frac{1}{2}-8\pi^2\right] + A^{-\frac{1}{n}}\left(\frac{1}{n}-1\right)\tilde{\mu}^{\frac{1}{n}-2}\left(\frac{\partial\tilde{\mu}}{\partial x}\epsilon_{xy}+\frac{\partial\tilde{\mu}}{\partial y}(\epsilon_{xx}+2\epsilon_{yy})\right) \tag{24}$$

where

$$\tilde{\mu} = \exp(x)\sqrt{(1+4\pi^2-2\pi)\sin^2(2\pi y)+\frac{1}{4}(2\pi+1)^2\cos^2(2\pi y)} \tag{25}$$

$$\frac{\partial \tilde{\mu}}{\partial x} = \tilde{\mu} \tag{26}$$

$$\frac{\partial \tilde{\mu}}{\partial y} = \frac{3}{2} \frac{\pi(1+4\pi^2-4\pi)\cos(2\pi y)\sin(2\pi y)\exp(x)}{\sqrt{(1+4\pi^2-2\pi)\sin^2(2\pi y)+\frac{1}{4}(2\pi+1)^2\cos^2(2\pi y)}} \tag{27}$$

$$\epsilon_{xx} = \exp(x)\sin(2\pi y) \tag{28}$$

$$\epsilon_{yy} = -2\pi\exp(x)\sin(2\pi y) \tag{29}$$

$$\epsilon_{xy} = \frac{1}{2}(2\pi+1)\exp(x)\cos(2\pi y) \tag{30}$$

$$\mu = \frac{1}{2}A^{-\frac{1}{n}}\tilde{\mu}^{\frac{1}{n}-1} \tag{31}$$

The following piece of code specifies f_1 and f_2 (taken from the Albany code base):

```
double a = 1.0;
MeshScalarT x2pi = 2.0*pi*coordVec(cell,qp,0);
MeshScalarT x = coordVec(cell,qp,0);
MeshScalarT y2pi = 2.0*pi*coordVec(cell,qp,1);
MeshScalarT muargt = (a*a + 4.0*pi*pi - 2.0*pi*a)*sin(y2pi)*sin(y2pi)
                   + 1.0/4.0*(2.0*pi+a)*(2.0*pi+a)*cos(y2pi)*cos(y2pi);
muargt = sqrt(muargt)*exp(a*x);
MeshScalarT muqp = 1.0/2.0*pow(A, -1.0/n)*pow(muargt, 1.0/n - 1.0);
MeshScalarT dmuargtdx = a*muargt;
MeshScalarT dmuargtdy = 3.0/2.0*pi*(a*a+4.0*pi*pi
                      -4.0*pi*a)*cos(y2pi)*sin(y2pi)*exp(a*x)/sqrt((a*a
                      + 4.0*pi*pi - 2.0*pi*a)*sin(y2pi)*sin(y2pi)
                      + 1.0/4.0*(2.0*pi+a)*(2.0*pi+a)*cos(y2pi)*cos(y2pi));
MeshScalarT exx = a*exp(a*x)*sin(y2pi);
MeshScalarT eyy = -2.0*pi*exp(a*x)*sin(y2pi);
MeshScalarT exy = 1.0/2.0*(2.0*pi+a)*exp(a*x)*cos(y2pi);
f[0] = 2.0*muqp*(2.0*a*a*exp(a*x)*sin(y2pi) - 3.0*pi*a*exp(a*x)*sin(y2pi)
     - 2.0*pi*pi*exp(a*x)*sin(y2pi))
     + 2.0*0.5*pow(A, -1.0/n)*(1.0/n-1.0)*pow(muargt, 1.0/n-2.0)*(dmuargtdx*(2.0*exx
     + eyy) + dmuargtdy*exy);
f[1] = 2.0*muqp*(3.0*a*pi*exp(a*x)*cos(y2pi) + 1.0/2.0*a*a*exp(a*x)*cos(y2pi)
     - 8.0*pi*pi*exp(a*x)*cos(y2pi)) + 2.0*0.5*pow(A, -1.0/n)*(1.0/n-1.0)*pow(muargt,
     1.0/n-2.0)*(dmuargtdx*exy + dmuargtdy*(exx + 2.0*eyy));
```

To finish the problem specification, we still need to give the boundary conditions. They are:

$$\begin{array}{rcll} u & = & 0, & \text{at } y=0,1 \\ v & = & 0, & \text{at } (x,y)=(0,0) \\ 2\mu\dot{\boldsymbol{\epsilon}}_1\cdot\mathbf{n} & = & 4(\pi-1)\mu u, & \text{at } x=0 \\ 2\mu\dot{\boldsymbol{\epsilon}}_1\cdot\mathbf{n} & = & -4(\pi-1)\mu u, & \text{at } x=1 \\ 2\mu\dot{\boldsymbol{\epsilon}}_2\cdot\mathbf{n} & = & -(2\pi+1)v, & \text{at } x=0 \\ 2\mu\dot{\boldsymbol{\epsilon}}_2\cdot\mathbf{n} & = & (2\pi+1)v, & \text{at } x=1 \\ 2\mu\dot{\boldsymbol{\epsilon}}_2\cdot\mathbf{n} & = & 0, & \text{at } y=0,1 \end{array} \tag{32}$$

The boundary condition at the single point $(x, y) = (0, 0)$ for v is specified to make v unique (otherwise, v could be non-unique up to a constant).

Note that the `cosexp2D` test case can be turned into a constant viscosity test case by setting $n = 1$.