PFL

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1 Introduction

xxx (Collins et al., 2021)

Algorithm 1

Input: Participation rate r, step size η, number of local updates for the head τ_w, for the shortcut τ_s and for the representation τ_b, number of communication rounds T.

Initialize $\mathbf{B}^0, \mathbf{w}_1^0, ..., \mathbf{w}_n^0, \mathbf{s}_1^0, ..., \mathbf{s}_n^0$
for $t = 0, 1, 2, ..., T-1$ **do**
  Server receives a batch of clients $\mathcal{I}^t$ of size rn
  Server sends current representation ϕ^t to clients in $\mathcal{I}^t$
  for each client i in $\mathcal{I}^t$ **do**
    Client i initializes $\mathbf{w}_i^{t,0} \leftarrow \mathbf{w}_i^{t-1,\tau_h}$
    Client updates its head for τ_h steps:
    for $\tau = 1$ to τ_w **do**
      $\mathbf{w}_i^{t,\tau} \leftarrow \mathrm{GRD}\left(f_i\left(\mathbf{w}_i^{t,\tau-1}, \mathbf{B}^{t-1}, \mathbf{s}_i^{t-1,\tau_s}\right), \mathbf{w}_i^{t,\tau-1}, \eta\right)$
    end for
    Client i initializes $\mathbf{B}_i^{t,0} \leftarrow \mathbf{B}^{t-1}$
    Client i updates its representation for τ_b steps:
    for $\tau = 1$ to τ_b **do**
      $\mathbf{B}_i^{t,\tau} \leftarrow \mathrm{GRD}\left(f_i\left(\mathbf{w}_i^{t,\tau_w}, \mathbf{B}_i^{t,\tau-1}, \mathbf{s}_i^{t-1,\tau_s}\right), \mathbf{B}_i^{t,\tau-1}, \eta\right)$
    end for
    Client i sends updated representation $\mathbf{B}_i^{t,\tau_b}$ to server
    Client i initializes $\mathbf{s}_i^{t,0} \leftarrow \mathbf{s}_i^{t-1,\tau_s}$
    Client i updates its shortcut for τ_s steps:
    for $\tau = 1$ to τ_s **do**
      $\mathbf{s}_i^{t,\tau} \leftarrow \mathrm{GRD}\left(f_i\left(\mathbf{w}_i^{t,\tau_w}, \mathbf{B}^{t-1}, \mathbf{s}_i^{t,\tau-1}\right), \mathbf{s}_i^{t,\tau-1}, \eta\right)$
    end for
  end for
  for each client j not in $\mathcal{I}^t$ **do**
    Set $\mathbf{w}_i^{t,\tau_w} \leftarrow \mathbf{w}_i^{t-1,\tau_w}$ and $\mathbf{s}_i^{t,\tau_s} \leftarrow \mathbf{s}_i^{t-1,\tau_s}$
  end for
  Server computes new representation: $\mathbf{B}^t = \frac{1}{rn}\sum_{i\in\mathcal{I}^t} \mathbf{B}_i^{t,\tau_b}$
end for

1.1 Preliminaries

First, we establish the notations that will be used throughout our proof. Let $\mathbf{S} := [\mathbf{s}_1, ..., \mathbf{s}_{rn}] \in \mathbb{R}^{d\times rn}$ represent the personalized layers, and let $\mathbf{W} := [\mathbf{w}_1, ..., \mathbf{w}_{rn}] \in \mathbb{R}^{k\times rn}$ denote the personalized heads, which follow the global representation $\mathbf{B}$. Since our algorithm updates $\mathbf{w}_i$ and $\mathbf{s}_i$ for each client i simultaneously, we define $\mathbf{h}_i^\top := [\mathbf{w}_i^\top, \mathbf{s}_i^\top]$ and $\mathbf{H} := [\mathbf{h}_1, ..., \mathbf{h}_{rn}] \in \mathbb{R}^{(k+d)\times rn}$.

...

The global objective can be rewritten as

$$\min_{\mathbf{B}\in\mathbb{R}^{d\times k},\mathbf{H}\in\mathbb{R}^{rn\times(k+d)}} \left\{ F(\hat{\mathbf{B}}, \mathbf{H}) := \frac{1}{2rnm}\mathbb{E}_{\mathcal{A},\mathcal{I}} \left\| \mathbf{Y} - \mathcal{A}(\mathbf{H}_{\mathcal{I}}^\top \widetilde{\mathbf{B}}^\top) \right\|_2^2 \right\}, \tag{1}$$

where $\mathbf{Y} = \mathcal{A}(\mathbf{H}_{\mathcal{I}}^* \widetilde{\mathbf{B}}^{*\top}) \in \mathbb{R}^{rnm}$. Then we give the update rules of our algorithm:

$$\mathbf{H}_{\mathcal{I}^t}^{t+1} = \operatorname*{arg\,min}_{\mathbf{H}_{\mathcal{I}^t}\in\mathbb{R}^{(k+d)\times rn}} \frac{1}{2rnm} \left\| \mathcal{A}^t \left(\mathbf{H}_{\mathcal{I}^t}^{*\top} \widetilde{\mathbf{B}}^{*\top} - \mathbf{H}_{\mathcal{I}^t}^\top \widetilde{\mathbf{B}}^{t\top} \right) \right\|_2^2, \tag{2}$$

$$\bar{\mathbf{B}} = \hat{\mathbf{B}}^t - \frac{\eta}{rnm} \left((\mathcal{A}^t)^\top \mathcal{A}^t (\mathbf{H}_{\mathcal{I}^t}^{t+1\top} \widetilde{\mathbf{B}}^{t\top} - \mathbf{H}_{\mathcal{I}^t}^{*\top} \widetilde{\mathbf{B}}^{*\top}) \right)^\top \mathbf{W}_{\mathcal{I}^t}^{t+1\top}, \tag{3}$$

$$\hat{\mathbf{B}}^{t+1}, \mathbf{R}^{t+1} = \mathrm{QR}(\bar{\mathbf{B}}^t), \tag{4}$$

$$\widetilde{\mathbf{B}}^{t+1} = [(1-\alpha)\hat{\mathbf{B}}^{t+1}, \alpha \mathbf{I}_d]. \tag{5}$$

As for separated update on $\mathbf{W}$ and $\mathbf{S}$:

$$\mathbf{W}^{t+1} = \operatorname*{arg\,min}_{\mathbf{W}\in\mathbb{R}^{k\times rn}} \frac{1}{2rnm} \left\| \mathcal{A}^t \left(\mathbf{W}^{*\top} \hat{\mathbf{B}}^{*\top} - \mathbf{W}^\top \hat{\mathbf{B}}^{t\top} + \mathbf{S}^{*\top} - \mathbf{S}^{t\top} \right) \right\|_2^2, \tag{6}$$

$$\bar{\mathbf{B}} = \hat{\mathbf{B}}^t - \frac{\eta}{rnm} \left((\mathcal{A}^t)^\dagger \mathcal{A}^t (\mathbf{W}^{t+1\top} \hat{\mathbf{B}}^{t\top} - \mathbf{W}^{*\top} \hat{\mathbf{B}}^{*\top} + \mathbf{S}^{t\top} - \mathbf{S}^{*\top}) \right)^\top \mathbf{W}_{\mathcal{I}^t}^{t+1\top}, \tag{7}$$

$$\hat{\mathbf{B}}^{t+1}, \mathbf{R}^{t+1} = \mathrm{QR}(\bar{\mathbf{B}}^t), \tag{8}$$

$$\mathbf{S}^{t+1} = \operatorname*{arg\,min}_{\mathbf{S}\in\mathbb{R}^{d\times rn}} \frac{1}{2rnm} \left\| \mathcal{A}^t \left(\mathbf{W}^{*\top} \hat{\mathbf{B}}^{*\top} - \mathbf{W}^{t+1\top} \hat{\mathbf{B}}^{t\top} + \mathbf{S}^{*\top} - \mathbf{S}^\top \right) \right\|_2^2 \tag{9}$$

1.2 Auxiliary Lemmas

We first consider the update for $\mathbf{H}$. According to the update rule of (2), $\mathbf{H}^{t+1}$ minimizes the function of $\widetilde{F}\left(\mathbf{H}, \hat{\mathbf{B}}^t\right) := \frac{1}{2rnm} \left\| \mathcal{A}\left(\mathbf{H}^{*\top} \widetilde{\mathbf{B}}^{*\top} - \mathbf{H}^\top \widetilde{\mathbf{B}}^{t\top} \right) \right\|_2^2$

Let $\mathcal{H}_p$ be the p-th column of $\mathbf{H}^{t+1\top}$, $\mathcal{H}_p^*$ denote the p-th column of $\mathbf{H}^{*\top}$ and $\mathbf{b}_p^t$ be the p-th column of $\widetilde{\mathbf{B}}^t$, then for any $p \in [k+d]$, we have

$$\begin{aligned}
\mathbf{0} &= \nabla_{\mathcal{H}_p} \widetilde{F}\left(\mathbf{H}, \hat{\mathbf{B}}^t\right) \\
&= \frac{1}{rnm} \sum_{i=1}^{rn} \sum_{j=1}^{m} \left(\left\langle \mathbf{A}_{i,j}, \mathbf{H}^{t+1\top} \widetilde{\mathbf{B}}^{t\top} - \mathbf{H}^{*\top} \widetilde{\mathbf{B}}^{*\top} \right\rangle \right) \mathbf{A}_{i,j} \mathbf{b}_p^t \\
&= \frac{1}{rnm} \sum_{i=1}^{rn} \sum_{j=1}^{m} \left(\sum_{q=1}^{k+d} \mathbf{b}_q^{t\top} \mathbf{A}_{i,j}^\top \mathcal{H}_q^{t+1} - \sum_{q=1}^{k+d} \mathbf{b}_q^{*\top} \mathbf{A}_{i,j}^\top \mathcal{H}_q^* \right) \mathbf{A}_{i,j} \mathbf{b}_p^t,
\end{aligned} \tag{10}$$

which means

$$\frac{1}{m}\sum_{q=1}^{k+d}\left(\sum_{i=1}^{rn}\sum_{j=1}^{m}\mathbf{A}_{i,j}\mathbf{b}_p^t\mathbf{b}_q^t\mathbf{A}_{i,j}^\top\right)\mathcal{H}_q^{t+1}=\frac{1}{m}\sum_{q=1}^{k+d}\left(\sum_{i=1}^{rn}\sum_{j=1}^{m}\mathbf{A}_{i,j}\mathbf{b}_p^t\mathbf{b}_q^*\mathbf{A}_{i,j}^\top\right)\mathcal{H}_q^*. \tag{11}$$

Notice that the term of $\frac{1}{m}\sum_{i=1}^{rn}\sum_{j=1}^{m}\mathbf{A}_{i,j}\mathbf{b}_p^t\mathbf{b}_q^t\mathbf{A}_{i,j}^\top$ is a matrix with dimension of $rn\times rn$, and so is the term of $\frac{1}{m}\sum_{i=1}^{rn}\sum_{j=1}^{m}\mathbf{A}_{i,j}\mathbf{b}_p^t\mathbf{b}_q^*\mathbf{A}_{i,j}^\top$. To solve the function in (11), we define $\mathbf{G}_{pq}:=\frac{1}{m}\sum_{i=1}^{rn}\sum_{j=1}^{m}\mathbf{A}_{i,j}\mathbf{b}_p^t\mathbf{b}_q^t\mathbf{A}_{i,j}^\top$, $\mathbf{C}_{pq}:=\frac{1}{m}\sum_{i=1}^{rn}\sum_{j=1}^{m}\mathbf{A}_{i,j}\mathbf{b}_p^t\mathbf{b}_q^*\mathbf{A}_{i,j}^\top$ and $\mathbf{D}_{pq}:=\frac{1}{m}\sum_{i=1}^{rn}\sum_{j=1}^{m}\left\langle\mathbf{b}_p^t,\mathbf{b}_q^*\right\rangle\mathbf{I}_{rn}$, for all $p,q\in[k+d]$. Further, we define block matrices $\mathbf{G},\mathbf{C},\mathbf{D}\in\mathbb{R}^{rn(k+d)\times rn(k+d)}$, which are formed by $\mathbf{G}_{pq},\mathbf{C}_{pq},\mathbf{D}_{pq}$ respectively. In detail, take $\mathbf{G}$ for example,

$$\mathbf{G}:=\begin{bmatrix}\mathbf{G}_{11} & \cdots & \mathbf{G}_{1(k+d)}\\ \vdots & \ddots & \vdots\\ \mathbf{G}_{(k+d)1} & \cdots & \mathbf{G}_{(k+d)(k+d)}\end{bmatrix}. \tag{12}$$

Then we define $\widetilde{\mathcal{H}}^{t+1}:=\mathrm{vec}(\mathbf{H}^{t+1\top})\in\mathbb{R}^{rn(k+d)}$ and $\widetilde{\mathcal{H}}^*:=\mathrm{vec}(\mathbf{H}^{*\top})\in\mathbb{R}^{rn(k+d)}$. From (11) we reach,

$$\begin{aligned}\widetilde{\mathcal{H}}^{t+1}&=\mathbf{G}^{-1}\mathbf{C}\widetilde{\mathcal{H}}^*\\ &=\mathbf{D}\widetilde{\mathcal{H}}^*-\mathbf{G}^{-1}\left(\mathbf{G}\mathbf{D}-\mathbf{C}\right)\widetilde{\mathcal{H}}^*,\end{aligned} \tag{13}$$

where $\mathbf{G}$ is invertible will be proved in the following lemma. Here, we consider $\mathbf{G}_{pq}$,

$$\begin{aligned}\mathbf{G}_{pq}&=\frac{1}{m}\sum_{i=1}^{rn}\sum_{j=1}^{m}\mathbf{A}_{i,j}\mathbf{b}_p\mathbf{b}_q^\top\mathbf{A}_{i,j}^\top\\ &=\frac{1}{m}\sum_{i=1}^{rn}\sum_{j=1}^{m}\mathbf{e}_i\left(\mathbf{x}_i^j\right)^\top\mathbf{b}_p\mathbf{b}_q^\top\mathbf{x}_i^j\mathbf{e}_i^\top,\end{aligned} \tag{14}$$

meaning that $\mathbf{G}_{pq}$ is diagonal with diagonal entries

$$\left(\mathbf{G}_{pq}\right)_{ii}=\frac{1}{m}\sum_{j=1}^{m}\left(\mathbf{x}_i^j\right)^\top\mathbf{b}_p\mathbf{b}_q^\top\mathbf{x}_i^j=\mathbf{b}_p^\top\left(\frac{1}{m}\sum_{j=1}^{m}\mathbf{x}_i^j\left(\mathbf{x}_i^j\right)^\top\right)\mathbf{b}_q. \tag{15}$$

Define $\mathbf{\Pi}^i:=\frac{1}{m}\sum_{j=1}^{m}\mathbf{x}_i^j\left(\mathbf{x}_i^j\right)^\top$ for all $i\in[rn]$, then $\mathbf{C}_{pq}$ is also diagonal with entries $\left(\mathbf{C}_{pq}\right)_{ii}=\mathbf{b}_p^\top\mathbf{\Pi}^i\mathbf{b}_q^*$. Note that $\mathbf{D}_{pq}=\left\langle\mathbf{b}_p,\mathbf{b}_q^*\right\rangle\mathbf{I}_{rn}$ is also diagonal. Then we define

$$\mathbf{G}^i:=\left[\mathbf{b}_p^\top\mathbf{\Pi}^i\mathbf{b}_q\right]_{1\le p,q\le k+d}=\widetilde{\mathbf{B}}^\top\mathbf{\Pi}^i\widetilde{\mathbf{B}},\qquad \mathbf{C}^i:=\left[\mathbf{b}_p^\top\mathbf{\Pi}^i\mathbf{b}_q^*\right]_{1\le p,q\le k+d}=\widetilde{\mathbf{B}}^\top\mathbf{\Pi}^i\widetilde{\mathbf{B}}^*, \tag{16}$$

note that $\mathbf{G}^i$ and $\mathbf{C}^i$ are the $(k+d)\times(k+d)$ matrices that formed by taking the i-th diagonal entry of each block $\mathbf{G}_{pq},\mathbf{C}_{pq}$, respectively. Similarly, we define $\mathbf{D}^i:=\left[\left\langle\mathbf{b}_p,\mathbf{b}_q^*\right\rangle\right]_{1\le p,q\le k+d}=\widetilde{\mathbf{B}}^\top\widetilde{\mathbf{B}}^*$. Then we can decouple the term of $\mathbf{G}^{-1}\left(\mathbf{G}\mathbf{D}-\mathbf{C}\right)\widetilde{\mathcal{H}}^*$ in (13) into i vectors, defined as

$$\mathbf{f}_i:=\left(\mathbf{G}^i\right)^{-1}\left(\mathbf{G}^i\mathbf{D}^i-\mathbf{C}^i\right)\widetilde{\mathcal{H}}_i^*, \tag{17}$$

where $\widetilde{\mathcal{H}}_i^* \in \mathbb{R}^{k+d}$ is the vector formed by taking the $((p-1)rn+i)$-th elements of $\widetilde{\mathcal{H}}^*$ for $p = 1, ..., k+d$. Next, we consider the vector $\widetilde{\mathcal{H}}_i^{t+1}$ formed by taking the $((p-1)rn+i)$-th elements of $\widetilde{\mathcal{H}}^{t+1}$ for $p = 1, ..., k+d$, from (13) we have

$$\begin{aligned}\widetilde{\mathcal{H}}_i^{t+1} &= \mathbf{D}^i\widetilde{\mathcal{H}}_i^* - \left(\mathbf{G}^i\right)^{-1}\left(\mathbf{G}^i\mathbf{D}^i - \mathbf{C}^i\right)\widetilde{\mathcal{H}}_i^* \\ &= \widetilde{\mathbf{B}}^\top\widetilde{\mathbf{B}}^*\widetilde{\mathcal{H}}_i^* - \mathbf{f}_i.\end{aligned} \tag{18}$$

Finally, we reach to the update of $\mathbf{H}^{t+1}$ as

$$\mathbf{H}^{t+1} = \widetilde{\mathbf{B}}^\top\widetilde{\mathbf{B}}^*\mathbf{H}^* - \mathbf{F}, \tag{19}$$

where $\mathbf{F} := [\mathbf{f}_1, \mathbf{f}_2, ..., \mathbf{f}_{rn}]$. Note that for each $\mathbf{f}_i$, the first k elements form a vector defined as $\mathbf{f}_{i1}$, which is for updating $\mathbf{w}_i$ and elements from $k+1$ to $k+d$ can form a vector defined as $\mathbf{f}_{i2}$, which is for updating $\mathbf{s}_i$. Further, we can obtain the update for $\mathbf{W}^{t+1}$ and $\mathbf{S}^{t+1}$ from (19),

$$\mathbf{W}^{t+1} = (1-\alpha)^2\hat{\mathbf{B}}^{t\top}\hat{\mathbf{B}}^*\mathbf{W}^* + \alpha(1-\alpha)\hat{\mathbf{B}}^{t\top}\mathbf{S}^* - \mathbf{F}_1, \tag{20}$$

$$\mathbf{S}^{t+1} = \alpha(1-\alpha)\hat{\mathbf{B}}^*\mathbf{W}^* + \alpha^2\mathbf{S}^* - \mathbf{F}_2, \tag{21}$$

where $\mathbf{F}_1$ and $\mathbf{F}_1$ are matrices formed by $\mathbf{f}_{i1}$ and $\mathbf{f}_{i2}$, respectively.

Lemma 1 *Bounding* $\left\|\mathbf{G}^{-1}\right\|_2$

In order to give bounding on $\left\|\mathbf{G}^{-1}\right\|_2$, we need to lower bound $\sigma_{min}(\mathbf{G})$. For some vector $\mathbf{z} \in \mathbb{R}^{rn(k+d)}$, let $\mathbf{z}^i \in \mathbb{R}^{k+d}$ be the vector formed by taking the $((p-1)rn+i)$-th elements of $\mathbf{z}$ for $p = 1, ..., k+d$, then we have

$$\begin{aligned}\sigma_{min}(\mathbf{G}) &= \min_{\mathbf{z}:\|\mathbf{z}\|_2=1}\mathbf{z}^\top\mathbf{G}\mathbf{z} \\ &= \min_{\mathbf{z}:\|\mathbf{z}\|_2=1}\sum_{i=1}^{rn}\left(\mathbf{z}^i\right)^\top\mathbf{G}^i\mathbf{z}^i \\ &\geq \min_{i\in[rn]}\sigma_{min}\left(\mathbf{G}^i\right),\end{aligned}$$

where

$$\mathbf{G}^i = \widetilde{\mathbf{B}}^\top\mathbf{\Pi}^i\widetilde{\mathbf{B}} = \begin{bmatrix}(1-\alpha)\hat{\mathbf{B}}^\top \\ \alpha\mathbf{I}_d\end{bmatrix}\mathbf{\Pi}^i\begin{bmatrix}(1-\alpha)\hat{\mathbf{B}} & \alpha\mathbf{I}_d\end{bmatrix} = \begin{bmatrix}(1-\alpha)^2\hat{\mathbf{B}}^\top\mathbf{\Pi}^i\hat{\mathbf{B}} & \alpha(1-\alpha)\hat{\mathbf{B}}^\top\mathbf{\Pi}^i \\ \alpha(1-\alpha)\mathbf{\Pi}^i\hat{\mathbf{B}} & \alpha^2\mathbf{\Pi}^i\end{bmatrix}. \tag{22}$$

Lemma 2 ...

Let $\mathbf{Q}^{t\top} = \mathbf{H}^{t+1\top}\widetilde{\mathbf{B}}^{t\top} - \mathbf{H}^{*\top}\widetilde{\mathbf{B}}^{*\top}$. To bound $\frac{1}{rn}\left\|\left(\frac{1}{m}\mathcal{A}^\top\mathcal{A}(\mathbf{Q}^{t\top}) - \mathbf{Q}^{t\top}\right)^\top\mathbf{W}^{t+1\top}\right\|_2$, we first consider the bound of the columns of $\mathbf{Q}$. Let $\mathbf{q}_i \in \mathbb{R}^d$ be the i-th column of $\mathbf{Q}$, for all $i \in [rn]$ we have

$$\begin{aligned}\mathbf{q}_i &= \widetilde{\mathbf{B}}^t\widetilde{\mathbf{B}}^{t\top}\widetilde{\mathbf{B}}^*\mathbf{h}_i^* - \widetilde{\mathbf{B}}^t\mathbf{f}_i - \widetilde{\mathbf{B}}^*\mathbf{h}_i^* \\ &= \left((1-\alpha)^2\hat{\mathbf{B}}^t\hat{\mathbf{B}}^{t\top} + \alpha^2\mathbf{I}_d\right)\widetilde{\mathbf{B}}^*\mathbf{h}_i^* - \widetilde{\mathbf{B}}^t\mathbf{f}_i - \widetilde{\mathbf{B}}^*\mathbf{h}_i^* \\ &= \left((1-\alpha)^2\hat{\mathbf{B}}^t\hat{\mathbf{B}}^{t\top} + \alpha^2\mathbf{I}_d\right)\left((1-\alpha)\hat{\mathbf{B}}^*\mathbf{w}_i^* + \alpha\mathbf{s}_i^*\right) - (1-\alpha)\hat{\mathbf{B}}^t\mathbf{f}_{i1} - \alpha\mathbf{f}_{i2} - (1-\alpha)\hat{\mathbf{B}}^*\mathbf{w}_i^* - \alpha\mathbf{s}_i^*\end{aligned}$$

Thus,

1.3 Main Result

Recall that $\mathbf{Q}^{t\top} = \mathbf{H}^{t+1\top}\widetilde{\mathbf{B}}^{t\top} - \mathbf{H}^{*\top}\widetilde{\mathbf{B}}^{*\top}$, plugging this into (3), and without losing generality, we drop the subscripts of $\mathcal{I}^t$ and obtain

$$\begin{aligned}\bar{\mathbf{B}}^{t+1} &= \hat{\mathbf{B}}^t - \frac{\eta}{rnm}\left((\mathcal{A}^t)^\top \mathcal{A}^t(\mathbf{Q}^{t\top})\right)^\top \mathbf{W}^{t+1\top} \\ &= \hat{\mathbf{B}}^t - \frac{\eta}{rn}\mathbf{Q}^t\mathbf{W}^{t+1\top} - \frac{\eta}{rn}\left(\frac{1}{m}(\mathcal{A}^t)^\top \mathcal{A}^t(\mathbf{Q}^{t\top}) - \mathbf{Q}^{t\top}\right)^\top \mathbf{W}^{t+1\top}. \end{aligned} \tag{23}$$

Since $\bar{\mathbf{B}}^{t+1} = \hat{\mathbf{B}}^{t+1}\mathbf{R}^{t+1}$, we right multiply $(\mathbf{R}^{t+1})^{-1}$ and left multiply $\hat{\mathbf{B}}_\perp^{*\top}$ on both sides to get

$$\hat{\mathbf{B}}_\perp^{*\top}\hat{\mathbf{B}}^{t+1} = \left(\hat{\mathbf{B}}_\perp^{*\top}\hat{\mathbf{B}}^t - \frac{\eta}{rn}\hat{\mathbf{B}}_\perp^{*\top}\mathbf{Q}^t\mathbf{W}^{t+1\top} - \frac{\eta}{rn}\hat{\mathbf{B}}_\perp^{*\top}\left(\frac{1}{m}(\mathcal{A}^t)^\top \mathcal{A}^t(\mathbf{Q}^{t\top}) - \mathbf{Q}^{t\top}\right)^\top \mathbf{W}^{t+1\top}\right)(\mathbf{R}^{t+1})^{-1}. \tag{24}$$

Then we consider the term of $\hat{\mathbf{B}}_\perp^{*\top}\mathbf{Q}^t\mathbf{W}^{t+1\top}$:

$$\begin{aligned}\hat{\mathbf{B}}_\perp^{*\top}\mathbf{Q}^t\mathbf{W}^{t+1\top} &= \hat{\mathbf{B}}_\perp^{*\top}\left(\widetilde{\mathbf{B}}^t\mathbf{H}^{t+1} - \widetilde{\mathbf{B}}^*\mathbf{H}^*\right)\mathbf{W}^{t+1\top} \\ &= \left((1-\alpha)\,\hat{\mathbf{B}}_\perp^{*\top}\hat{\mathbf{B}}^t\mathbf{W}^{t+1} + \alpha\hat{\mathbf{B}}_\perp^{*\top}\mathbf{S}^{t+1} - \alpha\hat{\mathbf{B}}_\perp^{*\top}\mathbf{S}^*\right)\mathbf{W}^{t+1\top} \\ &= (1-\alpha)\,\hat{\mathbf{B}}_\perp^{*\top}\hat{\mathbf{B}}^t\mathbf{W}^{t+1}\mathbf{W}^{t+1\top} - \alpha\hat{\mathbf{B}}_\perp^{*\top}\left(\mathbf{S}^* - \mathbf{S}^{t+1}\right)\mathbf{W}^{t+1\top}, \end{aligned} \tag{25}$$

plugging this into (24) then we reach to

$$\begin{aligned}\hat{\mathbf{B}}_\perp^{*\top}\hat{\mathbf{B}}^{t+1} = \Bigg(&\hat{\mathbf{B}}_\perp^{*\top}\hat{\mathbf{B}}^t\left(\mathbf{I}_k - \frac{(1-\alpha)\,\eta}{rn}\mathbf{W}^{t+1}\mathbf{W}^{t+1\top}\right) + \frac{\alpha\eta}{rn}\hat{\mathbf{B}}_\perp^{*\top}\left(\mathbf{S}^* - \mathbf{S}^{t+1}\right)\mathbf{W}^{t+1\top} \\ &- \frac{\eta}{rn}\hat{\mathbf{B}}_\perp^{*\top}\left(\frac{1}{m}(\mathcal{A}^t)^\top \mathcal{A}^t(\mathbf{Q}^{t\top}) - \mathbf{Q}^{t\top}\right)^\top \mathbf{W}^{t+1\top}\Bigg)(\mathbf{R}^{t+1})^{-1}. \end{aligned} \tag{26}$$

Therefore,

$$\begin{aligned}\mathrm{dist}(\hat{\mathbf{B}}^{t+1}, \hat{\mathbf{B}}^*) &= \left\|\hat{\mathbf{B}}_\perp^{*\top}\hat{\mathbf{B}}^{t+1}\right\|_2 \\ &\le \left\|\hat{\mathbf{B}}_\perp^{*\top}\hat{\mathbf{B}}^t\left(\mathbf{I}_k - \frac{(1-\alpha)\,\eta}{rn}\mathbf{W}^{t+1}\mathbf{W}^{t+1\top}\right)\right\|_2 \left\|(\mathbf{R}^{t+1})^{-1}\right\|_2 \\ &\quad + \frac{\eta}{rn}\left\|\hat{\mathbf{B}}_\perp^{*\top}\left(\frac{1}{m}(\mathcal{A}^t)^\top \mathcal{A}^t(\mathbf{Q}^{t\top}) - \mathbf{Q}^{t\top}\right)^\top \mathbf{W}^{t+1\top}\right\|_2 \left\|(\mathbf{R}^{t+1})^{-1}\right\|_2 \\ &\quad + \frac{\alpha\eta}{rn}\left\|\hat{\mathbf{B}}_\perp^{*\top}\left(\mathbf{S}^* - \mathbf{S}^{t+1}\right)\mathbf{W}^{t+1\top}\right\|_2 \left\|(\mathbf{R}^{t+1})^{-1}\right\|_2. \end{aligned} \tag{27}$$

References

Liam Collins, Hamed Hassani, Aryan Mokhtari, and Sanjay Shakkottai. Exploiting shared representations for personalized federated learning. In *International Conference on Machine Learning*, pages 2089–2099. PMLR, 2021.

A Proofs