

Assignment 1: Kleinberg and Tardos - Chapter 01 - Some Representative Problems
Exercises 1.2, 1.3, 1.5, 1.8

For full credit, please adhere to the following:

- Unsupported answers receive no credit.
- All answers can be typed or handwritten, and should be readable.
- Submit the assignment in one pdf file

1. (25 points) **Exercise 1.1:** Decide whether you think the following statement is true or false. If it is true, give a short explanation. If it is false, give a counterexample.

True or false? Consider an instance of the Stable Matching Problem in which there exists a man m and a woman w such that m is ranked first on the preference list of w and w is ranked first on the preference list of m . Then, in every stable matching S for this instance, the pair (m, w) belongs to S .

2. (25 points) **Exercise 1.3:** There are many other settings in which we can ask questions related to some type of “stability” principle. Here’s one, involving competition.

Suppose we have two television networks, whom we’ll call A and B . There are n prime-time programming slots, and each network has n TV shows. Each network wants to devise a *schedule* — an assignment of each show to a distinct slot — so as to attract as much share as possible.

Here is the way we determine how well the two networks perform relative to each other, given their schedules: Each show has a fixed *rating*, which is based on the number of people who watched it last year; we’ll assume that no two shows have exactly the same rating. A network *wins* a given time slot if the show that it schedules for the time slot has a larger rating than the show the other network schedules for that time slot. The goal of each network is to win as many time slots as possible.

Suppose in the opening week of the fall season, Network A reveals a schedule S and Network B reveals a schedule T . On the basis of this pair of schedules, each network wins certain time slots, according to the rule above. We’ll say that the pair of schedules (S, T) is *stable* if neither network can unilaterally change its own schedule and win more time slots. That is, there is no schedule S' such that Network A wins more slots with the pair (S', T) than it did with the pair (S, T) ; and symmetrically, there is no schedule T' such that Network B wins more slots with the pair (S, T') than it did with the pair (S, T) .

The analogue of Gale Shapley's question for this kind of stability is the following: For every set of TV shows and ratings, is there always a stable pair of schedules? Resolve this questions by doing one of the following two things:

- (a) Give an algorithm that, for any set of TV shows and associated ratings, produces a stable pair of schedules; or
 - (b) Give an example of a set of TV shows and associated ratings for which there is no stable pair of schedules.
3. (25 points) **Exercise 1.5:** The Stable Matching Problem assumes that all men and women have a fully ordered list of preferences. In this problem we will consider a version of the problem in which men and women can be *indifferent* between certain options. As before we have a set, M , of n men and a set, W , of n women. Assume each man and each woman ranks the members of the opposite gender, but now we allow ties in the ranking. For example (with $n = 4$), a woman could say that m_1 is ranked in first place; second place is a tie between m_2 and m_3 (she has no preference between them); and m_4 is in last place. We will say that w *prefers* m to m' if m is ranked higher than m' on her preference list (they are not tied).

With indifferences in the rankings, there could be two natural notions for stability. And for each, we can ask about the existence of stable matchings, as follows:

- (a) A *strong instability* in a perfect matching S consists of a man m and a woman w , such that each of m and w prefers the other to their partner in S . Does there always exist a perfect matching with no strong instability? Either give an example of a set of men and women with pereference lists for which every perfect matching has a strong instability; or give an algorithm that is guaranteed to find a perfect matching with no strong instability.
- (b) A *weak instability* in a perfect matching S consists of a man m and a woman w such that their partners in S are w' and m' , respectively, and one of the following holds:
 - m prefers w to w' , and w either prefers m to m' or is indifferent between these two choices; or
 - w prefers m to m' , and m either prefers w to w' or is indifferent between these two choices.

In other words, the pairing between m and w is either preferred by both, or preferred by one while the other is indifferent. Does there always exist a perfect matching with no weak instability? Either give an example of a set of men and women with preference lists for which every perfect matching has a weak instability; or give an algorithm that is guaranteed to find a perfect matching with no weak instability.

4. (25 points) **Exercise 1.8:** For this problem, we will explore the issue of *truthfulness* in the Stable Matching Problem and specifically, in the Gale-Shapley algorithm. The basic question is: Can a man or a woman end up better off by lying about his or her preferences? More concretely, we suppose each participant has a true preference order. Now consider a woman w . Suppose w prefers man m to m' , but both m and m' are low on her list of preferences. Can it be the case that by switching the order of m and m' on her list of preferences (i.e., by falsely claiming that she prefers m' to m) and running the algorithm with this false preference list, w will end up with a man m'' that she truly prefers to both m and m' ? (We can ask the same question for men, but will focus on the case for women for purposes of this question.)

Resolve this question by doing one of the following two things:

- (a) Give a proof that, for any set of preference lists, switching the order of a pair on the list cannot improve a woman's partner in the Gale-Shapley algorithm; or
- (b) Give an example of a set of preference lists for which there is a switch that would improve the partner of a woman who switched preferences.