10.018 Modelling Space and Systems

Cohort 5.1

Understanding region of integration, polar coordinates

Term 2, 2021

SINGAPORE UNIVERSITY OF TECHNOLOGY AND DESIGN

Before we start....

To get the most out of this cohort, you should already be familiar with

1. Double Integration (last week, lectures)

as we will be going through

1. (more) Double integration
2. Polar Coordinates

Warm Up Activity 1 (10 minutes)

Let's play a game! Your goal is **to confuse your neighbour**!

(1) Sketch individually four regions in the xy-plane in **random order** which are

- vertically but not horizontally simple.
- horizontally but not vertically simple.
- both vertically and horizontally simple.
- neither vertically nor horizontally simple.

(they could be all of one type, or all of different types. Up to you!)

(2) Write and hide your answer key.

(3) Pass your sketches to your neighbour and see if they can identify correctly your regions.

Activity 1 (solution)

There is no solution to this activity.

Activity 2 (10 minutes)

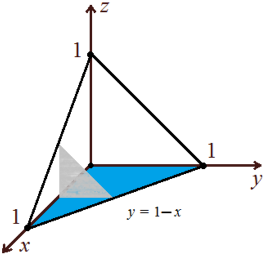
Set up a double integral to find the volume of the tetrahedron bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $x + y + z = 1$. You don't have to evaluate it.

Hint 1: We want to find $\iint\limits_R f(x, y)\ \mathrm{d}x\ \mathrm{d}y$.

Hint 2: Sketch R to figure out what $\iint\limits_R$ are.

Hint 3: How should $f(x, y)$ look like?

Activity 2 (solution)



The region of integration is a triangle in the xy-plane, described by
$0 \leq x \leq 1, \quad 0 \leq y \leq 1 - x.$

The surface we are integrating under is given by $z = 1 - x - y$.

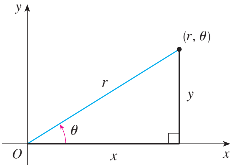
Hence the volume is given by

$$\int_0^1 \int_0^{1-x} (1 - x - y)\,\mathrm{d}y\,\mathrm{d}x$$
$$= \int_0^1 \left[y - xy - \frac{y^2}{2} \right]_0^{1-x} \mathrm{d}x$$
$$= \left[\frac{x}{2} - \frac{x^2}{2} + \frac{x^3}{6} \right]_0^1 = \frac{1}{6}.$$

This agrees with the geometric result that the volume of a pyramid is $\frac{1}{3} \times$ base $\times$ height.

In fact, the rigorous way to prove geometric formulas like this is via integrals.

Polar coordinates



Recall that we can convert between Cartesian and polar coordinates:

$$x = r\cos(\theta), \qquad y = r\sin(\theta);$$
$$r = \sqrt{x^2 + y^2}, \qquad \theta = \arctan\left(\frac{y}{x}\right) \text{ (if } x > 0\text{)}.$$

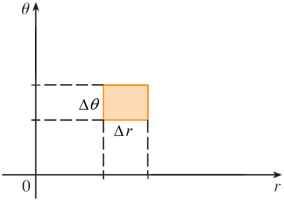
To evaluate $\iint_R f(x, y)\,\mathrm{d}A$, it is sometimes better to make a change of variables to polar coordinates, for instance when

- The region R is a disk or a sector,
- The function f contains terms like $x^2 + y^2$.

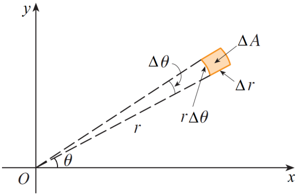
Polar coordinates – change of variables

In the limit definition of a double integral $\iint_R f(x,y)\,\mathrm{d}A$, we sum over many small rectangles of area ΔA. In Cartesian coordinates, we often make the sides of the rectangles parallel to the axes, namely $\Delta A = \Delta x \Delta y$. In the limit, this becomes the area element $\mathrm{d}A = \mathrm{d}x\,\mathrm{d}y$.

However, when working with polar coordinates, we need to sum over different rectangles to better reflect the geometry involved. Consider a Δr-by-$\Delta\theta$ rectangle in the $r\theta$-plane, where Δr and $\Delta\theta$ are small:



$\Longrightarrow$



In the xy-plane, this becomes an approximate rectangle with side lengths Δr and $r\Delta\theta$ (remember that θ is in radians).

Polar coordinates – formula

We sum over these approximate rectangles of area $\Delta A \approx (\Delta r)\,(r\Delta\theta)$. In the limit as $\Delta r, \Delta\theta \to 0$,

$$\mathrm{d}A = r\,\mathrm{d}r\,\mathrm{d}\theta,$$

so we obtain:

Integration in polar coordinates

$$\iint\limits_R f(x,y)\,\mathrm{d}A = \iint\limits_R f(r\cos(\theta),\ r\sin(\theta))\,r\,\mathrm{d}r\,\mathrm{d}\theta.$$

Pay attention: the region of integration R is the **same** for LHS and RHS (geometrically). But when you write it as iterated integral, the limits of integration will be **different.**

Polar coordinates – example

In this toy example, we verify the formula for the area of a circle using double integration.

Let R be the circular region defined by $x^2 + y^2 \leq a^2$. $f = 1$ gives the area.

Hence, the area is $\iint_R 1 \, \mathrm{d}A$, which equals

$$\iint\limits_R 1 \, r \, \mathrm{d}r \, \mathrm{d}\theta$$

$$= \int_0^{2\pi} \int_0^{a} r \, \mathrm{d}r \, \mathrm{d}\theta$$

$$= \int_0^{2\pi} \frac{1}{2} a^2 \, \mathrm{d}\theta = \pi a^2.$$

Remember to express the limits of integration in polar coordinates.

Activity 3 (15 minutes)

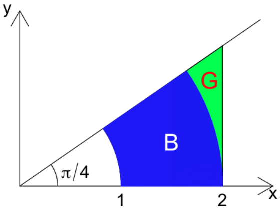
(1) Calculate the area of the region B shown on the right, using an iterated integral in polar coordinates.

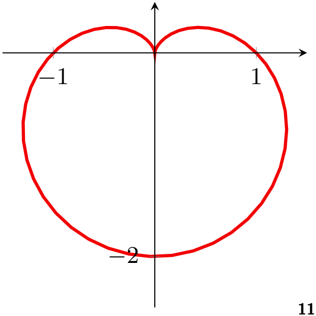
(2) Set up the integral to compute the area of the regions $B + G$ shown on the right, using an iterated integral in polar coordinates.

(3) Set up the integral (**do not evaluate it**) to calculate the area enclosed by the cardioid, described by the formula. .

$$r = 1 - \sin(\theta).$$

It's called a *cardioid* since it vaguely resembles a heart.





Activity 3 (solution)

(1) The area of the sector is given by

$$
\begin{aligned}
\iint\limits_R 1 \, \mathrm{d}A &= \iint\limits_R 1 \, r \, \mathrm{d}r \, \mathrm{d}\theta \\
&= \int_0^{\pi/4} \int_1^2 r \, \mathrm{d}r \, \mathrm{d}\theta \\
&= \int_0^{\pi/4} \left. \frac{r^2}{2} \right|_{r=1}^{r=2} \mathrm{d}\theta \\
&= \frac{3}{2} \int_0^{\pi/4} \mathrm{d}\theta \\
&= \frac{3\pi}{8}.
\end{aligned}
$$

Activity 3 (solution)

(2) Let us try to re-express the points on the vertical line $x = 2$ in polar coordinates, i.e. try an expression for $(2, y)$ in terms of (r, θ). Using trigonometry, we must have that $y = 2\tan\theta$ and $r = \dfrac{2}{\cos\theta}$. Hence the integral is given by:

$$\iint\limits_R 1\,\mathrm{d}A = \iint\limits_R 1\,r\,\mathrm{d}r\,\mathrm{d}\theta = \int_0^{\pi/4}\int_1^{2/\cos\theta} r\,\mathrm{d}r\,\mathrm{d}\theta$$

Activity 3 (solution)

(3) Let R denote the interior of the cardioid. The enclosed area is given by

$$\iint\limits_R 1\,\mathrm{d}A = \iint\limits_R 1\,r\,\mathrm{d}r\,\mathrm{d}\theta$$

$$= \int_0^{2\pi} \int_0^{1-\sin\theta} r\,\mathrm{d}r\,\mathrm{d}\theta$$

$$= \int_0^{2\pi} \left[\frac{1}{2}r^2\right]_0^{1-\sin\theta} \mathrm{d}\theta$$

$$= \int_0^{2\pi} \frac{1}{2}(1-\sin\theta)^2\,\mathrm{d}\theta$$

$$= \left[\frac{3}{4}\theta + \cos(\theta) - \frac{1}{8}\sin(2\theta)\right]_0^{2\pi} = \frac{3}{2}\pi.$$

To integrate $\sin^2(\theta)$, use the identity $\sin^2(\theta) = \dfrac{1}{2}[1-\cos(2\theta)]$.

Break

5 min break

Don't be late.

Activity 4 (15 minutes)

The indefinite integral $\int e^{-x^2}\,\mathrm{d}x$ cannot be expressed as a finite combination of elementary functions. However, it is possible to find the value of the definite integral $I = \int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x$.

(1) Write the double integral

$$\iint_{\mathbb{R}^2} e^{-x^2-y^2}\,\mathrm{d}A$$

as an iterated integral (in Cartesian coordinates), and simplify it in terms of I.

(2) Make a change of variables to polar coordinates, and then evaluate the integral. Hence, find the value of I.

Activity 4 (solution)

(1)

$$
\begin{aligned}
\iint\limits_{\mathbb{R}^2} e^{-x^2-y^2}\,\mathrm{d}A &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2}e^{-y^2}\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_{-\infty}^{\infty} e^{-y^2}\left(\int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x\right)\mathrm{d}y \\
&= \int_{-\infty}^{\infty} e^{-y^2}\, I\,\mathrm{d}y \\
&= I\int_{-\infty}^{\infty} e^{-y^2}\;\mathrm{d}y \quad \text{(since } I \text{ is a constant)} \\
&= I^2.
\end{aligned}
$$

Activity 4 (solution, continued)

(2) In polar coordinates, remember that $x^2 + y^2 = r^2$, hence:

$$\begin{aligned}\iint_{\mathbb{R}^2} e^{-x^2-y^2}\,\mathrm{d}A &= \int_0^{2\pi}\int_0^{\infty} e^{-r^2}\,r\,\mathrm{d}r\,\mathrm{d}\theta \\ &= \int_0^{2\pi}\left[-\frac{1}{2}e^{-r^2}\right]_0^{\infty}\mathrm{d}\theta \\ &= \int_0^{2\pi}\frac{1}{2}\,\mathrm{d}\theta \\ &= \pi.\end{aligned}$$

Combined with the answer to part (1), we have $I^2 = \pi$, so

$$I = \int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x = \sqrt{\pi}.$$

The value of I is of interest in probability and statistics, since e^{-x^2} describes the shape of a bell curve.

Activity 5 (10 minutes)

Consider a region R given by $x^2 + y^2 \leq 9$. Is the following statements correct? If no, what should be the correct version?

(1) $$\iint\limits_R (x^2 + y^2)\,\mathrm{d}A = \int_0^{2\pi} \int_0^3 r^2\,\mathrm{d}r\,\mathrm{d}\theta.$$

(2) $$\iint\limits_R 1\,\mathrm{d}A = \int_0^3 \int_0^{2\pi} r\,\mathrm{d}\theta\,\mathrm{d}r.$$

(3) $$\iint\limits_R \frac{y}{x}\,\mathrm{d}A = \int_0^{2\pi} \int_0^3 r\tan\theta\,\mathrm{d}r\,\mathrm{d}\theta.$$

Activity 5 (solution)

(1) We know $x^2 + y^2 = r^2$ and $\mathrm{d}A = r\mathrm{d}r\,\mathrm{d}\theta$. A factor of r is missing. The correct statement is

$$\iint\limits_R (x^2 + y^2)\,\mathrm{d}A = \int_0^{2\pi} \int_0^3 r^3\,\mathrm{d}r\,\mathrm{d}\theta.$$

(2) and (3) are correct.

Summary

We have covered:

- Vertically and horizontally simple regions.
- Finding region of integration to solve problems
- Polar coordinates

Textbook: read Section 20.3 to 20.5 (for this entire week at least), then try some of Exercises 20.3.1–20.3.14, Exercises 20.4.1–20.4.41, and Exercises 20.5.1–20.5.30. You may discuss them on Piazza.