10.018: Modelling Space and Systems

Cohort 3.2
Optimisation

Term 2, 2021

SINGAPORE UNIVERSITY OF TECHNOLOGY AND DESIGN

Before we start....

You should already be familiar with

1. Sets and Intervals
2. Extreme Value Theorem
3. Level curves and directional derivatives

We will be going through

1. Types of regions in Math II: The 2D analogue of line segments
2. Prelude to constrained optimization

General approach

In general, apart from finding maxima and minima when $\nabla f = \vec{0}$, we also need to consider the points where the *partial derivatives do not exist*.

We also need to inspect the boundary of the domain of the function, and in the interior of the domain as well. We need to check these points separately.

In summary, to find maxima and minima of a function:

- Find all points where $\nabla f = \vec{0}$.
- Classify these points using the second derivative test.
- Study all points where the ∇f is not defined.
- Study all points on the boundary of the domain.

Complete the squares, use the AM-GM inequality or the Extreme Value Theorem when investigating global maxima or global minima.

Activity 1 (30 minutes)

Find all local and global maxima and minima of the following functions:

1. $f(x, y) = \sqrt{(x-2)^2 + (y-1)^2}$.
2. $g(x, y) = -(x+5)^2$.

Activity 1 (solution)

1. Set the partial derivatives to zero:

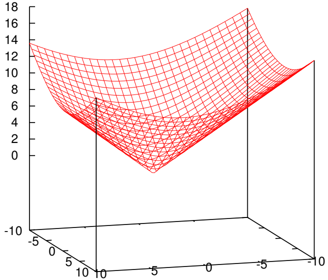
$$\frac{\partial f}{\partial x} = \frac{(x-2)}{\sqrt{(x-2)^2+(y-1)^2}} = 0, \quad \frac{\partial f}{\partial y} = \frac{(y-1)}{\sqrt{(x-2)^2+(y-1)^2}} = 0.$$

The partial derivatives are *not defined* at $(2,1)$, so the above equations have no solutions.

However, let us study the function at $(2,1)$.

$f(2,1)=0$, while in general $f(x,y)\geq 0$.

Therefore f has a (global) minimum at $(2,1)$.



Activity 1 (solution, continued)

2. g is differentiable everywhere, and

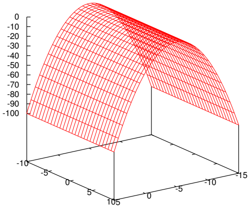
$$\frac{\partial g}{\partial x} = -2(x+5), \qquad \frac{\partial g}{\partial y} = 0.$$

The entire line $x = -5$ consists of critical points.

$g_{xx} = -2, g_{xy} = g_{yx} = g_{yy} = 0$, so $D = 0$ and the second derivative test is inconclusive.

However, we can *sketch* the surface: for each y, the graph is a downward parabola, with maximum at $x = -5$.

So the entire line $x = -5$ consists of global maxima.

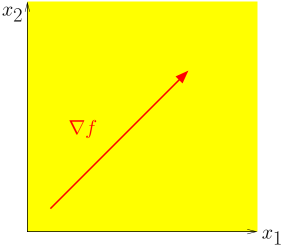


Alternative solution: The entire line $x = -5$ consists of critical points. On this line, g takes the value of $-(-5+5)^2 = 0$. We know that $g(x, y) = -(x+5)^2 \leq 0$ for all x, y since squares are nonnegative. So the entire line $x = -5$ consists of global maxima.

Unboundedness

What happens if region R is unbounded and closed: Nothing can be concluded a priori without examining the problem instance.

Consider the problem of maximizing $f(x_1, x_2) = x_1 + x_2$ over the nonnegative quadrant.

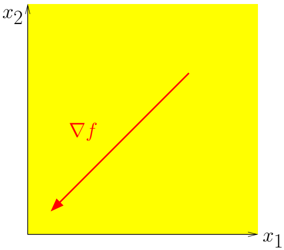


Question: Does this problem have an optimal solution? Why or why not?

Answer: No. The objective function goes to ∞.

More on unboundedness

Consider the problem of maximizing $f(x_1, x_2) = -x_1 - x_2$ over the nonnegative quadrant.

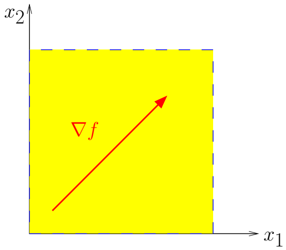


Question: Does this problem have an optimal solution? Why or why not?

Answer: Yes, the origin. The region is unbounded, but the optimal value is zero and an optimum is attained.

Open regions

Consider the problem of maximizing $f(x_1, x_2) = x_1 + x_2$ over the box $\{(x_1, x_2) \in \mathbb{R}^2 : 0 < x_1 < 1, 0 < x_2 < 1\}$. Note: the region is open.



Question: Does this problem have an optimal solution? Why or why not?

Answer: No. While the objective function is bounded above, there is no optimum: moving closer to the upper righthand corner (1, 1) always increases the objective function value towards the upper bound.

Activity 2 (30 minutes)

Let a, b be constants ($a \neq b$), and let $y < x$. Suppose we have the function

$$f(x,y) = \begin{cases} \frac{1}{(x-y)^2} & \text{if } y \leq a \leq x \text{ and } y \leq b \leq x \\ 0 & \text{otherwise} \end{cases}$$

Find the values of x, y which **maximizes** this function and explain your answer when $a = 3$ and $b = 5$.

How does your answer change if a, b can take on any real number?

Activity 2 (hint)

Note that if any of the following conditions hold: $\{3 > x\}$ or $\{5 > x\}$, or $\{3 < y\}$ or $\{5 < y\}$, then $f(x, y) = 0$.

If you cannot see this, draw a graph!

Activity 2 (hint)

The partial derivatives of this function are

$$\frac{\partial f}{\partial x} = \begin{cases} -\frac{2}{(x-y)^3} & \text{if } y \leq 3 \text{ and } x > 5 \\ \text{undefined} & \text{if } y \leq 3 \text{ and } x = 5 \\ 0 & \text{otherwise} \end{cases}$$

$$\frac{\partial f}{\partial y} = \begin{cases} \frac{2}{(x-y)^3} & \text{if } y < 3 \text{ and } x \geq 5 \\ \text{undefined} & \text{if } y = 3 \text{ and } x \geq 5 \\ 0 & \text{otherwise} \end{cases}$$

Note that the case when the partial derivatives are equal to zero is where $f(x, y)$ is **minimized**, and not maximized.

Let's start by finding what values x and y have to be for the function to be non-zero, and then trying to find a maximum within these values.

Activity 2 (solution)

$f(x, y)$ is non-zero when $\{y \leq 3\}$ **and** $\{x \geq 5\}$.

We set $\tilde{y} = 3$, $\tilde{x} = 5$, and consider

$$f(\tilde{x}, \tilde{y}) = \frac{1}{(\tilde{x} - \tilde{y})^2} = \frac{1}{4}$$

Suppose we set $L = \tilde{x} - \tilde{y}$.

We can see that for any $x' > \tilde{x}$, and for any $y' < \tilde{y}$, we must have

$$L' = x' - y' > L$$

which implies $\frac{1}{L^2} > \frac{1}{L'^2}$, and hence $f(\tilde{x}, \tilde{y}) > f(x', y')$ for $x' > \tilde{x}, y' < \tilde{y}$.

hence we see that the optimal solution is $x^* = 5$, and $y^* = 3$.

Activity 2 (solution)

What happens if a and b can take on any real value?

We can repeat the following steps above, with some changes.

$f(x, y)$ is non-zero when $\{y \leq \min\{a, b\}\}$ **and** $\{x \geq \max\{a, b\}\}$.

We set $\tilde{y} = \min\{a, b\}$, $\tilde{x} = \max\{a, b\}$, and consider

$$f(\tilde{x}, \tilde{y}) = \frac{1}{(\tilde{x} - \tilde{y})^2}$$

Suppose we set $L = \tilde{x} - \tilde{y}$.

We can see that for any $x' > \tilde{x}$, and for any $y' < \tilde{y}$, we must have

$$L' = x' - y' > L$$

which implies $\frac{1}{L^2} > \frac{1}{L'^2}$, and hence $f(\tilde{x}, \tilde{y}) > f(x', y')$ for $x' > \tilde{x}, y' < \tilde{y}$.

hence we see that the optimal solution is $x^* = \max\{a, b\}$, and $y^* = \min\{a, b\}$.

Activity 2 Takeaways

- *Important Problem Solving Heuristic:* Most problems will start with "Show the general case". Try specific cases related to the problem e.g. $a = 3, b = 5$ in this question, which may give ideas / directions on how to solve the general case.
- *Applications:* This is a maximum likelihood estimation problem in statistics to find the parameters of a uniform distribution $U[a, b]$ given 2 observations (but can generalize to N observations).

Activity 3 (30 minutes)

Design a milk container that has a capacity of 1250 cm^3.

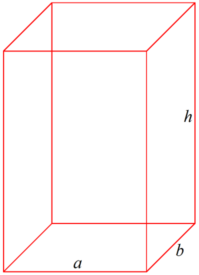
You may only use:

- Cardboard, which costs 0.1 cent/cm^2 for the sides or the top of the container, and 0.2 cent/cm^2 for the bottom, and
- Aluminium, which costs 0.15 cent/cm^2 for any face of the container.

You can mix and match the two materials. The goal is to minimize the cost of the container.

This is an open-ended problem with multiple solutions. You are free to experiment with ideas and discuss within your group.

Activity 3 (solution)



One solution is to use a rectangular prism with base $a \times b$ and height h.

Volume $V = abh = 1250$, so $h = \dfrac{1250}{ab}$.

We choose cardboard sides and top, and aluminium bottom.

The total cost is:

$$f(a,b) = 0.15 \cdot ab + 0.1 \cdot ab + 0.1 \cdot 2ah + 0.1 \cdot 2bh = 0.25ab + \frac{250}{b} + \frac{250}{a}.$$

Set the first-order partial derivatives to zero:

$$\frac{\partial f}{\partial a} = 0.25b - \frac{250}{a^2} = 0, \qquad \frac{\partial f}{\partial b} = 0.25a - \frac{250}{b^2} = 0.$$

Activity 3 (solution, continued)

Hence $a^2b = 1000$, $ab^2 = 1000$. *Dividing* the two equations, we get $a/b = 1$, so $(a, b) = (10, 10)$.

Second derivatives are

$$f_{aa}(10,10) = \frac{500}{1000}, f_{ba}(10,10) = f_{ab}(10,10) = 0.25, f_{bb}(10,10) = \frac{500}{1000}$$

so $D = 0.5^2 - 0.25^2 > 0$; also, $f_{aa}(10,10) > 0$, thus $(10,10)$ is a minimum.

Therefore, the optimal container has dimensions $10 \times 10 \times 12.5$ cm, and the total cost is 75 cents.

Another plausible solution is to use a cylinder, with base radius $r = 5(4/\pi)^{1/3} \approx 5.42$ cm. The total cost is $75(\pi/4)^{1/3} \approx 69.2$ cents. (Why might this solution be impractical?)

Activity 3 (solution, continued)

How to show that $(10, 10)$ is a global minimum of $f(a,b) = 0.25ab + \frac{250}{b} + \frac{250}{a}$?

Recall the AM-GM inequality from the Lecture:

$$\frac{x_1 + \ldots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \ldots x_n}.$$

Applying it to this case:

$$\begin{aligned}\frac{0.25ab + \frac{250}{b} + \frac{250}{a}}{3} &\geq \sqrt[3]{0.25ab \cdot \frac{250}{b} \cdot \frac{250}{a}} \\ &= \sqrt[3]{\frac{250 \cdot 250}{4}} = 25 = \frac{f(10,10)}{3}\end{aligned}$$

In other words, $f(a,b) \geq f(10,10)$ for all a, b! Therefore, $(10, 10)$ is a global minimum.

Summary

- General approach to local optimisation and global optimisation.
- Possible outcomes of optimization problems.
- Arithmetic Mean-Geometric Mean Inequality

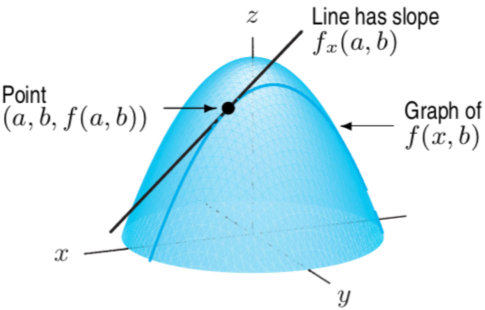
Self-reading: Visualizing partial derivatives

We have computed partial derivatives in Week 1, but how do they look like?

We will use diagrams from the textbook *Calculus Single and Multivariable*.

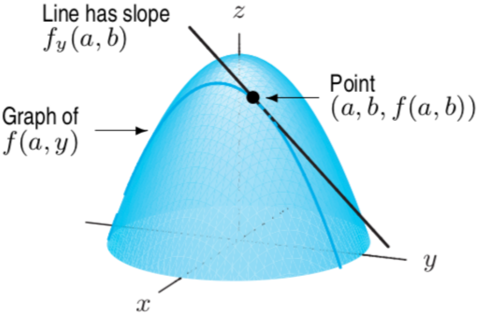
Self-reading: Visualizing $f_x(a, b)$

Suppose we have a surface given by $z = f(x, y)$, and we want to look at $f_x(a, b)$.
We will look at the graph of $f(x, b)$ (b is kept fixed), which is the curve where the vertical plane $y = b$ cuts the graph of $z = f(x, y)$.
Hence $f_x(a, b)$ is the slope of the tangent line to this curve at $x = a$.



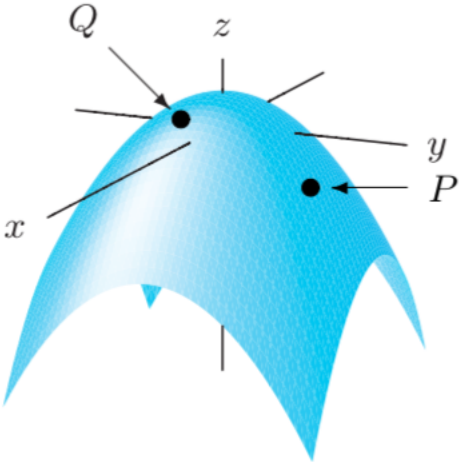
Self-reading: Visualizing $f_y(a, b)$

Suppose we have a surface given by $z = f(x, y)$, and we want to look at $f_y(a, b)$.
We will look at the graph of $f(a, y)$ (a is kept fixed), which is the curve where the vertical plane $x = a$ cuts the graph of $z = f(x, y)$.
Hence $f_y(a, b)$ is the slope of the tangent line to this curve at $y = b$.



Self-reading Question 1

What is the sign of the partial derivatives f_x, f_y at P and Q?



Solution to Self-reading Question 1

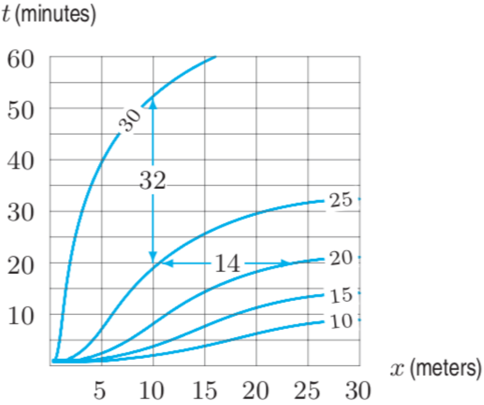
The positive x-axis points out of the page. Imaging heading off in this direction from the point marked P; we descend steeply. So the partial derivative with respect to x is negative at P, with quite a large absolute value. The same is true for the partial derivative with respect to y at P, since there is also a steep descent in the positive y-direction.

At the point marked Q, heading in the positive x-direction results in a gentle descent, whereas heading in the positive y-direction results in a gentle ascent. Thus, the partial derivative f_x at Q is negative but small (that is, near zero), and the partial derivative f_y is positive but small.

Self-reading Question 2

Here is a contour diagram for the temperature $H(x,t)$ in a room as a function of distance x from a heater and time t in minutes after the heater has been turned on.

What are the signs of $H_x(10,20)$ and $H_t(10,20)$?



Solution to Self-reading Question 2

The point $(10, 20)$ is nearly on the $H = 25$ contour. As x increases, we move toward the $H = 20$ contour, so H is decreasing and $H_x(10, 20)$ is negative. This makes sense because the $H = 30$ contour is to the left: As we move further from the heater, the temperature drops.

On the other hand, as t increases, we move toward the $H = 30$ contour, so H is increasing; as t decreases H decreases. Thus, $H_t(10, 20)$ is positive. This says that as time passes, the room warms up.

Individual Quiz (10 minutes)

The MCQs will be released later, together with a website link to enter your answers. You will have 10 minutes to do this quiz on your own.

It will not be graded. But please do not discuss the questions with your peers, instructors or on piazza. We will be providing VR lab session for you to confirm your answer. Solutions will be provided later.

Note: the VR lab session is **optional**, but students who go for the session need to stay within their same grouping.