10.018 Modelling Space and Systems

Cohort 6.2

Line Integrals along Vector Fields

Term 2, 2021

SINGAPORE UNIVERSITY OF TECHNOLOGY AND DESIGN

Before we start....

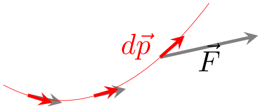
To get the most out of this lecture, you should already be familiar with

1. Line integrals in Physics, and Modeling Space and Systems lecture 6 and cohort this week
2. Vector fields in Physics, and Modeling Space and Systems lecture 6 and cohort this week

as we will be going through

1. Line integrals along vector fields

Lecture: Line integral along a vector field



With the notation $\mathrm{d}\vec{p} = [\mathrm{d}x_1, \mathrm{d}x_2, \ldots, \mathrm{d}x_n]$, the line integral along a vector field $\vec{F}$ is denoted by $\int_\gamma \vec{F}(\vec{p}) \cdot \mathrm{d}\vec{p}$, and is computed using

$$\int_\gamma \vec{F}(\vec{p}) \cdot \mathrm{d}\vec{p} = \int_a^b \vec{F}\big(\vec{p}(t)\big) \cdot \vec{p}\,'(t)\,\mathrm{d}t.$$

Example from physics: if $\vec{F}$ denotes an electric or gravitational force field, then **the work done on a particle, traveling along a curve** γ (parametrized by $\vec{p}$) is given by

$$W = \int_\gamma \vec{F}(\vec{p}) \cdot \mathrm{d}\vec{p}.$$

Lecture: Line integral along a vector field – example

In $\mathbb{R}^2$, if we write $\vec{F}(x,y) = \begin{bmatrix} F_1(x,y) \\ F_2(x,y) \end{bmatrix}$ and $\mathrm{d}\vec{p} = [\mathrm{d}x,\ \mathrm{d}y]$, then it is customary to write

$$\int_\gamma \vec{F}(\vec{p}) \cdot \mathrm{d}\vec{p} = \int_\gamma F_1\,\mathrm{d}x + F_2\,\mathrm{d}y.$$

Example: evaluate $\int_\gamma xy\,\mathrm{d}x + (x-y)\,\mathrm{d}y$, where γ is a segment of $y = x^2$ from $(0,0)$ to $(2,4)$.

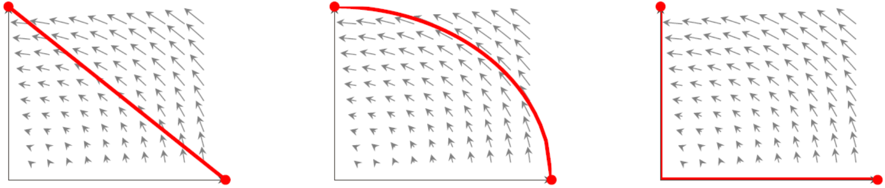
Solution: let $\vec{p}(t) = [x(t),\ y(t)] = [t,\ t^2]$, $t \in [0,2]$, then

$$\begin{aligned}\int_\gamma xy\,\mathrm{d}x + (x-y)\,\mathrm{d}y &= \int_0^2 xy\,\frac{\mathrm{d}x}{\mathrm{d}t}\mathrm{d}t + (x-y)\,\frac{\mathrm{d}y}{\mathrm{d}t}\mathrm{d}t \\ &= \int_0^2 t\,t^2\,1\,\mathrm{d}t + (t-t^2)\,2t\,\mathrm{d}t \\ &= \int_0^2 (2t^2 - t^3)\,\mathrm{d}t = \frac{4}{3}.\end{aligned}$$

Activity 1 (15 minutes)

Estimate the value of the line integral $\int_\gamma \vec{F}(\vec{p}) \cdot \mathrm{d}\vec{p}$, where $\vec{F}(x, y) = \begin{bmatrix} -y \\ x \end{bmatrix}$, and γ is a curve from $(1, 0)$ to $(0, 1)$, along

(1) a line segment (2) a quarter of a circle (3) two line segments. Is it POSITIVE, NEGATIVE, approximately ZERO, **without computing it**?



(4) Find the actual values of the integral.
(5) What if we swap the endpoints of the curve in (1)?

Activity 1 (solution)

From physical interpretation we can see that the line integral along the vector field will be positive in cases (1) and (2), and approximately zero in case (3).

(1) We can use the parametrization $\vec{p}(t) = [1-t,\, t],\ t \in [0,1]$.

$$\begin{aligned}
\int_\gamma \vec{F}(\vec{p}) \cdot \mathrm{d}\vec{p} &= \int_0^1 \vec{F}(\vec{p}(t)) \cdot \vec{p}\,'(t)\,\mathrm{d}t \\
&= \int_0^1 \begin{bmatrix} -t \\ 1-t \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} \mathrm{d}t \\
&= \int_0^1 (t + 1 - t)\,\mathrm{d}t = 1.
\end{aligned}$$

Activity 1 (solution)

(2) Let $\vec{p}(t) = [\cos(t), \sin(t)],\ t \in [0, \pi/2]$.

$$\begin{aligned}
\int_\gamma \vec{F}(\vec{p}) \cdot \mathrm{d}\vec{p} &= \int_0^{\pi/2} \vec{F}(\vec{p}(t)) \cdot \vec{p}\,'(t)\,\mathrm{d}t \\
&= \int_0^{\pi/2} \begin{bmatrix} -\sin(t) \\ \cos(t) \end{bmatrix} \cdot \begin{bmatrix} -\sin(t) \\ \cos(t) \end{bmatrix} \mathrm{d}t \\
&= \int_0^{\pi/2} \left(\sin^2(t) + \cos^2(t)\right) \mathrm{d}t = \frac{\pi}{2}.
\end{aligned}$$

Activity 1 (solution, continued)

(3) We can pick the parametrization

$$\vec{p}(t) = \begin{cases} [1-t,\, 0], & t \in [0,1] \\ [0,\, t-1], & t \in [1,2] \end{cases}.$$

$$\begin{aligned}
\int_\gamma \vec{F}(\vec{p}) \cdot \mathrm{d}\vec{p} &= \int_0^1 \vec{F}(\vec{p}(t)) \cdot \vec{p}\,'(t)\,\mathrm{d}t + \int_1^2 \vec{F}(\vec{p}(t)) \cdot \vec{p}\,'(t)\,\mathrm{d}t \\
&= \int_0^1 \begin{bmatrix} 0 \\ 1-t \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 0 \end{bmatrix} \mathrm{d}t + \int_1^2 \begin{bmatrix} 1-t \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mathrm{d}t \\
&= \int_0^1 0\,\mathrm{d}t + \int_1^2 0\,\mathrm{d}t = 0.
\end{aligned}$$

This answer is expected, since at every point on the curve, the vector field is perpendicular to the curve.

Activity 1 (solution, continued)

(5) Let $\vec{p}(t) = [t,\ 1-t],\ t \in [0,1]$.

$$\begin{aligned}\int_0^1 \vec{F}(\vec{p}(t)) \cdot \vec{p}\,'(t)\,\mathrm{d}t &= \int_0^1 \begin{bmatrix} t-1 \\ t \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} \mathrm{d}t \\ &= \int_0^1 (t-1-t)\,\mathrm{d}t = -1.\end{aligned}$$

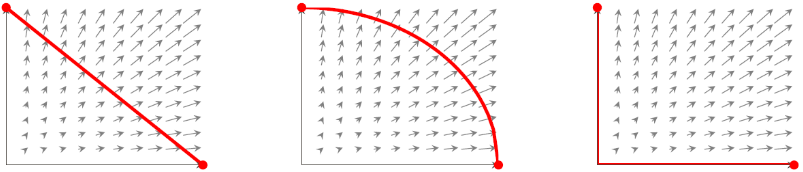
Activity 2 (10 minutes)

Compute the line integral

$$\int_\gamma \vec{F}(\vec{p}) \cdot \mathrm{d}\vec{p},$$

where $\vec{F}(x,y) = \begin{bmatrix} x \\ y \end{bmatrix}$, and γ is a curve from $(1,0)$ to $(0,1)$, along

(1) a line segment (2) a quarter of a circle (3) two line segments.



Activity 2 (solution)

(1) $\vec{p}(t) = [1-t,\ t],\ \ t \in [0,1]$.

$$\begin{aligned}\int_\gamma \vec{F}(\vec{p}) \cdot \mathrm{d}\vec{p} &= \int_0^1 \vec{F}(\vec{p}(t)) \cdot \vec{p}\,'(t)\,\mathrm{d}t \\ &= \int_0^1 \begin{bmatrix} 1-t \\ t \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} \mathrm{d}t \\ &= \int_0^1 (2t-1)\,\mathrm{d}t = 0.\end{aligned}$$

(2) $\vec{p}(t) = [\cos(t),\ \sin(t)],\ \ t \in [0, \pi/2]$.

$$\begin{aligned}\int_\gamma \vec{F}(\vec{p}) \cdot \mathrm{d}\vec{p} &= \int_0^{\pi/2} \vec{F}(\vec{p}(t)) \cdot \vec{p}\,'(t)\,\mathrm{d}t \\ &= \int_0^{\pi/2} \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix} \cdot \begin{bmatrix} -\sin(t) \\ \cos(t) \end{bmatrix} \mathrm{d}t \\ &= \int_0^{\pi/2} \big(-\cos(t)\sin(t) + \sin(t)\cos(t) \big)\,\mathrm{d}t = 0.\end{aligned}$$

Activity 2 (solution, continued)

(3) $\vec{p}(t) = \begin{cases} [1-t,\, 0], & t \in [0,1] \\ [0,\, t-1], & t \in [1,2] \end{cases}.$

$$\begin{aligned}
\int_\gamma \vec{F}(\vec{p}) \cdot \mathrm{d}\vec{p} &= \int_0^1 \vec{F}(\vec{p}(t)) \cdot \vec{p}\,'(t)\,\mathrm{d}t + \int_1^2 \vec{F}(\vec{p}(t)) \cdot \vec{p}\,'(t)\,\mathrm{d}t \\
&= \int_0^1 \begin{bmatrix} 1-t \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 0 \end{bmatrix} \mathrm{d}t + \int_1^2 \begin{bmatrix} 0 \\ t-1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mathrm{d}t \\
&= \int_0^1 (t-1)\,\mathrm{d}t + \int_1^2 (t-1)\,\mathrm{d}t \\
&= -\frac{1}{2} + \frac{1}{2} = 0.
\end{aligned}$$

Unlike in Activity 1, the line integrals along the vector field here **do not seem to depend on the path** taken.

Break

5 min break

Don't be late

Conservative vector fields

Definition: a vector field $\vec{F} : \mathbb{R}^n \to \mathbb{R}^n$ is called conservative if there exists an $f : \mathbb{R}^n \to \mathbb{R}$ such that

$$\vec{F} = \nabla f,$$

that is, if $\vec{F}$ is the *gradient* of a scalar field.

Example from Physics: if f is a potential function (e. g. electric or gravitational potential), then $\vec{F}$ is the corresponding field. So, if $f(x, y) = \dfrac{c}{\sqrt{x^2 + y^2}}$, then

$$\vec{F}(x, y) = \frac{-c}{(x^2 + y^2)^{3/2}} \begin{bmatrix} x \\ y \end{bmatrix}, \qquad \|\vec{F}\| = \frac{|c|}{x^2 + y^2} = \frac{|c|}{r^2},$$

(c. f. inverse square law for the corresponding forces).

Gradient theorem

Gradient theorem

Let $\vec{F} = \nabla f$ be a conservative vector field from $\mathbb{R}^n$ to $\mathbb{R}^n$. Let γ be *any* (piecewise) differentiable curve in $\mathbb{R}^n$ with initial point $\vec{a}$ and final point $\vec{b}$. Then

$$\int_\gamma \vec{F}(\vec{p}) \cdot \mathrm{d}\vec{p} = f(\vec{b}) - f(\vec{a}).$$

- In particular, if γ is a closed loop, then $\displaystyle\int_\gamma \vec{F}(\vec{p}) \cdot \mathrm{d}\vec{p} = 0$.
- Note how the gradient theorem is similar to the fundamental theorem of calculus (for single integrals): if $g = G'$, then

  $$\int_a^b g(x)\,\mathrm{d}x = G(b) - G(a).$$

  Only the end points matter.
- In Physics: the theorem implies that work done against gravity is path independent, i. e. gravity is a *conservative* force.

Gradient theorem – example

Let us revisit example 2 from Lecture 6, where
$\vec{F}(x,y) = \begin{bmatrix} 3+2xy \\ x^2-3y^2 \end{bmatrix}$ and γ is a curve from $(0,0)$ to $(1,1)$.

Another way to compute the line integral is to check that

$$F = \nabla f, \quad \text{where } f(x,y) = 3x + x^2y - y^3,$$

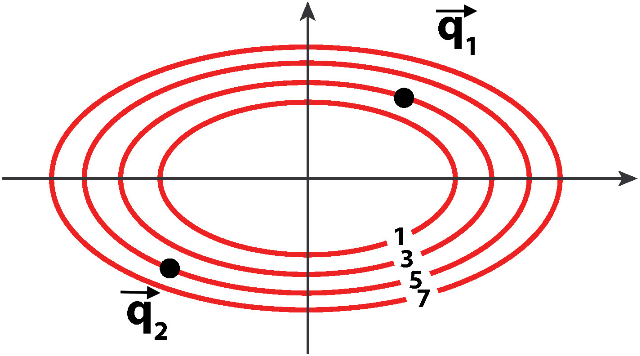
therefore by the gradient theorem,

$$\int_\gamma \vec{F}(\vec{p}) \cdot \mathrm{d}\vec{p} = f(1,1) - f(0,0) = 3.$$

Activity 3 (10 minutes)

(1) Use the gradient theorem to solve Activity 2.

(2) A contour map of g is given below. Sketch the conservative vector field $\vec{G} = \nabla g$.



(3) If γ is a curve from $\vec{q}_1$ to $\vec{q}_2$, evaluate $\displaystyle\int_\gamma \vec{G}(\vec{p}) \cdot \mathrm{d}\vec{p}$.

Activity 3 (solution)

(1) We can check that $\vec{F}(x,y) = \begin{bmatrix} x \\ y \end{bmatrix} = \nabla f$ where $f(x,y) = \frac{1}{2}(x^2 + y^2)$. Therefore, by the gradient theorem,

$$\int_\gamma \vec{F}(\vec{p}) \cdot \mathrm{d}\vec{p} = f(0,1) - f(1,0) = 0,$$

for any curve γ from $(1,0)$ to $(0,1)$.

(2) $\vec{G} = \nabla g$. Using properties of the gradient, which we proved in Class 16 Activity 5, the vectors in $\vec{G}$ are *perpendicular* to the level curves in the contour map, and point from the smaller function values to the larger function values.

(3) By the gradient theorem,

$$\int_\gamma \vec{G}(\vec{p}) \cdot \mathrm{d}\vec{p} = g(\vec{q}_2) - g(\vec{q}_1) = 5 - 3 = 2.$$

Activity 4 (10 minutes)

Prove the gradient theorem, that is, if $\vec{F} = \nabla f$, then

$$\int_\gamma \vec{F}(\vec{p}) \cdot \mathrm{d}\vec{p} = f(\vec{b}) - f(\vec{a}),$$

the path γ is parametrized by $\vec{p}(t) = [x_1(t), \ldots, x_n(t)]$; $p(a) = \vec{a}$ is the initial point of γ; $p(b) = \vec{b}$ is the final point of γ.

Hint: use definition of the gradient, the multivariable chain rule, and the fundamental theorem of calculus.

Activity 4 (solution)

$$\begin{aligned}
\int_\gamma \vec{F}(\vec{p}) \cdot \mathrm{d}\vec{p} &= \int_a^b (\nabla f)\big(\vec{p}(t)\big) \cdot \vec{p}\,'(t)\,\mathrm{d}t \\
&= \int_a^b \left[\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n}\right] \cdot \left[\frac{\mathrm{d}x_1}{\mathrm{d}t}, \frac{\mathrm{d}x_2}{\mathrm{d}t}, \dots, \frac{\mathrm{d}x_n}{\mathrm{d}t}\right] \mathrm{d}t \\
&= \int_a^b \left(\frac{\partial f}{\partial x_1}\frac{\mathrm{d}x_1}{\mathrm{d}t} + \frac{\partial f}{\partial x_2}\frac{\mathrm{d}x_2}{\mathrm{d}t} + \cdots + \frac{\partial f}{\partial x_n}\frac{\mathrm{d}x_n}{\mathrm{d}t}\right) \mathrm{d}t \\
&= \int_a^b \frac{\mathrm{d}f\big(\vec{p}(t)\big)}{\mathrm{d}t}\,\mathrm{d}t \qquad \text{(by the chain rule)} \\
&= f\big(\vec{p}(b)\big) - f\big(\vec{p}(a)\big) \qquad \text{(by the FTOC)} \\
&= f(\vec{b}) - f(\vec{a}).
\end{aligned}$$

Hence the line integral only depends on the endpoints of the curve, and not the curve itself.

Summary

We have covered:

- Line integrals along vector fields.
- The gradient theorem.

Textbook: read Sections 21.1 and 21.2, then try some of Exercises 21.1.2–21.1.8 and Exercises 21.2.1–21.2.10.