Week 4 - Day 2

Capacitance

Concept 1: Work Done to Charge Up a Capacitor and Energy Stored in the Electric Field

Concept 2: Effect of Dielectric in Capacitor

↓

Supercapacitor vs. Battery, Capacitive Touch Screen/Switch

Reading:

University Physics with Modern Physics – Chapter 24

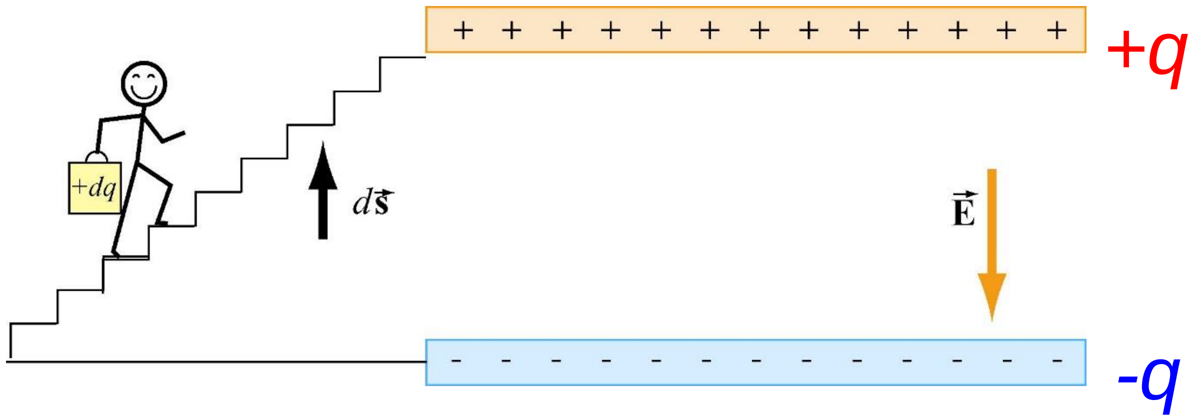
Introduction to Electricity and Magnetism – Chapter 5

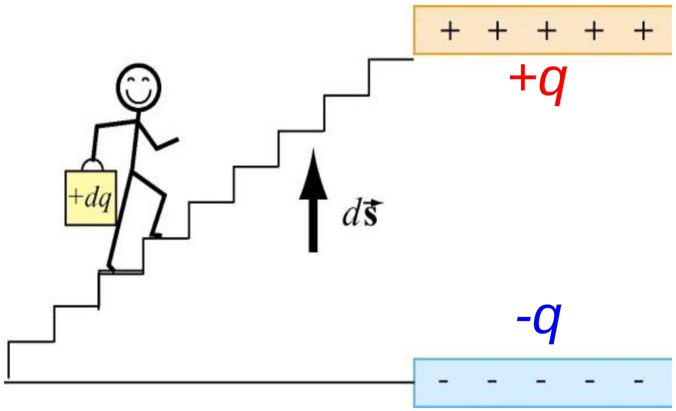
Concept 1: Work Done to Charge Up a Capacitor and Energy Stored in the Electric Field

Work Done/Energy Needed to Charge a Capacitor

- Steps to charge an empty capacitor (uncharged):

1. Carry small amount of charge packet $+dq$ from bottom to the top one at a time.
2. At some point the top plate has charge $+q$ and bottom has charge $-q$.
3. Carry $+dq$ from the bottom to the top. Now $q_{top} = q + dq$, $q_{bottom} = -q - dq$.
4. Continue until the top has charge *+Q* and the bottom *–Q*
5. At some point, when the top plate has charge *+q* and bottom has charge *–q*, *t*he potential difference is $\Delta V = q/C$.
6. The work done to lift another *dq* is $dW = (dq)\Delta V = (dq)q/C$.





- So the work done to move *dq* is:
- $dW = dq\,\Delta V = dq\frac{q}{C} = \frac{1}{C}qdq$
- The total energy to charge to *q* = *Q* is given by:

$$W = \int dW = \frac{1}{C}\int_0^Q q\,dq = \frac{1}{C}\frac{Q^2}{2}$$

- It also implies the energy stored in the capacitor, $U = \frac{1}{2}\frac{Q^2}{C}$
- Since $C = \frac{Q}{|\Delta V|}$

$$U = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}Q|\Delta V| = \frac{1}{2}C|\Delta V|^2$$

Energy Stored in a Capacitor and Energy Density of $\vec{E}$

- Since $C = \frac{Q}{|\Delta V|}$
- $U = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}Q|\Delta V| = \frac{1}{2}C|\Delta V|^2$
- But where is the energy stored???

- For Parallel-plate capacitor:
- $C = \frac{\varepsilon_o A}{d}$ and $V = Ed$
- $U = \frac{1}{2}CV^2 == \frac{1}{2}\frac{\varepsilon_o A}{d}(Ed)^2 == \frac{\varepsilon_o E^2}{2} \times (Ad) = u_E \times (volume)$
- The energy is stored in the **electric field** that permeates the space between the conductors where charges are stored!
- $u_E \equiv E$ field energy density $= \frac{\varepsilon_o E^2}{2}$

SINGAPORE UNIVERSITY OF TECHNOLOGY AND DESIGN

Concept Question 1.1

- A parallel-plate capacitor, disconnected from a battery, has plates with equal and opposite charges, separated by a distance d. Suppose the plates are pulled apart until separated by a distance $D > d$.
- How does the final electrostatic energy stored in the capacitor compare to the initial energy?

1. The final stored energy is smaller
2. The final stored energy is larger
3. Stored energy does not change.

Concept Question 1.1: Solution

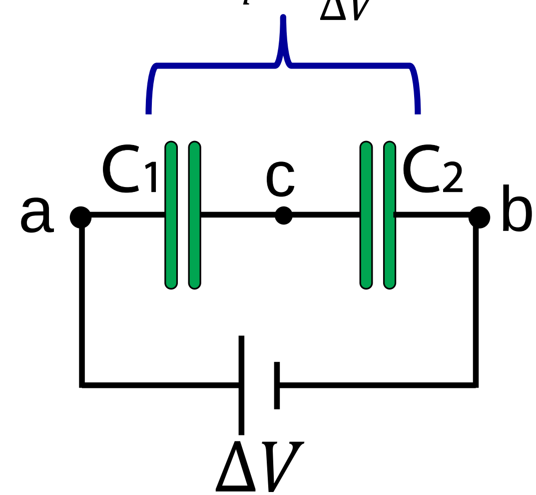
- Answer: 2. The stored energy increases
- Q is fix, C becomes smaller, U increases

As you pull apart the capacitor plates you increase the amount of space in which the E field is non-zero and hence increase the stored energy. Where does the extra energy come from? From the work you do pulling the plates apart.

Combining Capacitors (Series)

- When capacitors are connected in parallel or series, how do their capacitance values combine?
- Consider the capacitors in series as shown below.
- The left plate of C_1 carries a charge $+Q$. Since the charges on opposite plates of C_1 are the same in magnitude, the right plate of C_1 carries a charge $-Q$. This $-Q$ can only have come from the left plate of C_2, which must have charge $+Q$. Thus, the right plate of C_2 has a charge of $-Q$. Hence, different capacitors in series carry the same charge.
- Let us now consider the relation
- $\Delta V_{ac} = \frac{Q}{C_1}$ and $\Delta V_{cb} = \frac{Q}{C_2}$ and $\Delta V_{ac} + \Delta V_{cb} = \Delta V_{ab} = \Delta V$
- $\Delta V = \frac{Q}{C_{1_1}} + \frac{Q}{C_2} = \frac{Q}{C_{eq}}$
- Thus, $C_{eq} = \left(\frac{1}{C_1} + \frac{1}{C_2}\right)^{-1}$
- C_{eq} is the equivalent capacitance of the network

$$C_{eq} = \frac{Q}{\Delta V}$$



SINGAPORE UNIVERSITY OF TECHNOLOGY AND DESIGN

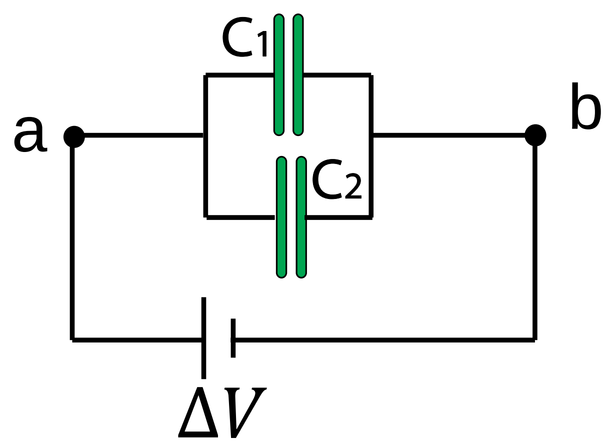
Combining Capacitors (Parallel)

- Since the potential difference across them is the same value ΔV
- $Q_1 = C_1 \Delta V$ and $Q_2 = C_2 \Delta V$
- The total charge is the combination of both capacitors

$$Q = Q_1 + Q_2 = (C_1 + C_2)\Delta V = C_{eq}\Delta V$$

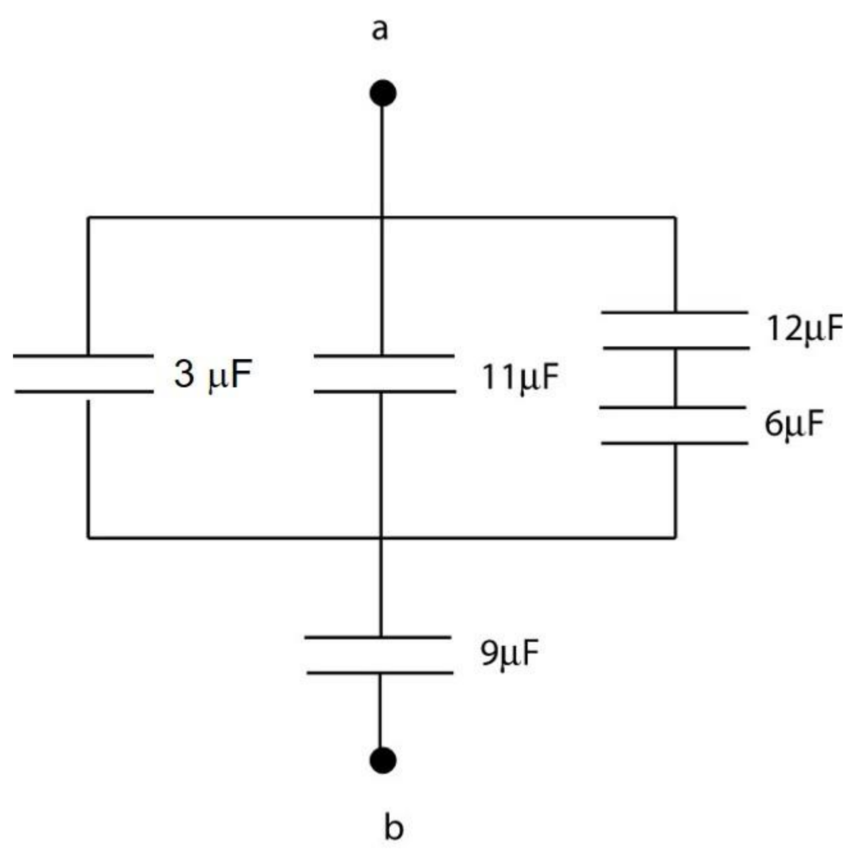
- which shows that

$$C_{eq} = C_1 + C_2$$

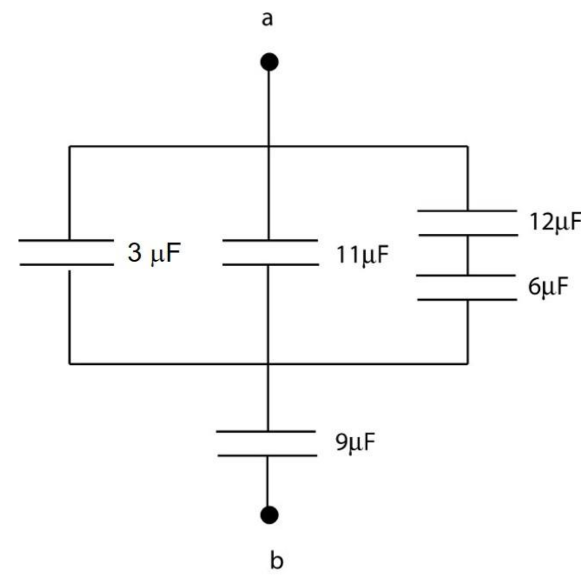


Case Problem 1.1

Find the equivalent capacitance of the combination shown in the figure below



Case Problem 1.1: Solution



The five capacitors in the figure are neither all in series nor all in parallel. We can, however, identify portions of the arrangement that are either in series or parallel, which we combine to find the net equivalent capacitance.

We first replace the 12-μF and 6-μF series combination by its equivalent capacitance; calling that *C'* we use the fact that the reciprocals of the capacitance adds up

$$\frac{1}{C'} = \frac{1}{12\mu F} + \frac{1}{6\mu F} \;\rightarrow C' = 4\mu F$$

Next, we find the equivalent capacitance of the three capacitors in parallel. Calling their equivalent capacitance *C"*, we have

$$C'' = 3\mu F + 11\mu F + 4\mu F = 18\mu F$$

Finally, we find the equivalent capacitance C_{eq} of these two capacitors in series

$$\frac{1}{C_{eq}} = \frac{1}{18\mu F} + \frac{1}{9\mu F} \;\rightarrow\; C_{eq} = 6\mu F$$

The equivalent capacitance of the network is 6μF; that is, if a potential difference V_{ab} is applied across the network, the net charge on the network is 6μF times V_{ab}.

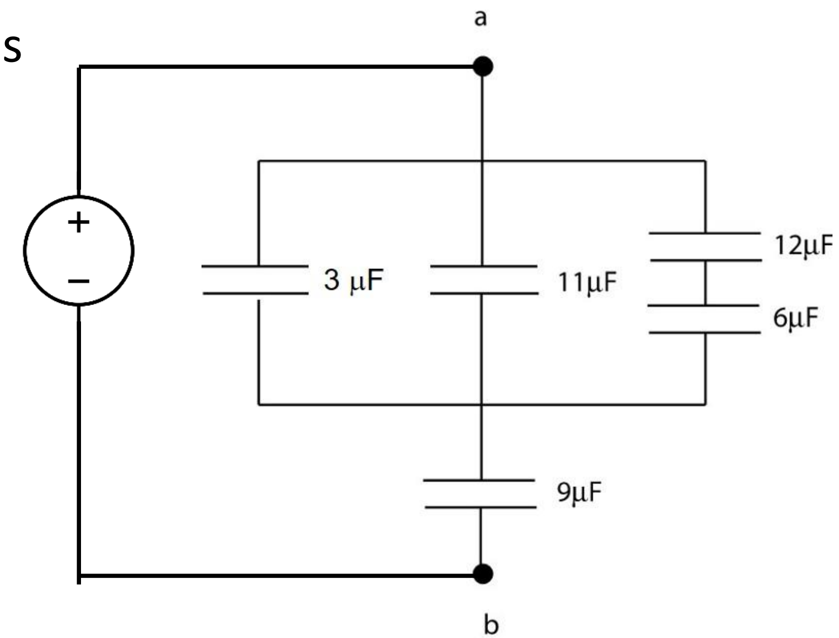
Case Problem 1.2

A 9V battery is connected to the same circuit as Case Problem 1.1. Find

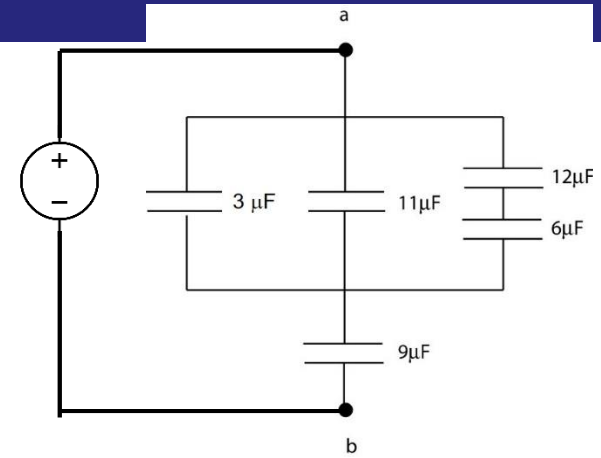
a) What is the total energy that stored in the circuit.

b) The voltage across the $9\mu F$ capacitor.

c) The charge stored in the $11\mu F$ capacitor.



Case Problem 1.2 Solution



a) Energy in C, $U = \frac{1}{2}C\Delta V^2$; $C = C_{eq} = 6\mu F$

Thus, $U_{total} = \frac{1}{2}(6\mu F)(9V)^2 = 243\mu J$

Note: You can check yourself as an exercise that the summation of energy stored in each capacitor is equal to $243\mu J$.

b) In a series of capacitor circuit, the charge stored in both capacitors is the same.

$$9V = \Delta V_{18\mu F} + \Delta V_{9\mu F} = \frac{Q}{18\mu} + \frac{Q}{9\mu}$$
$$3Q = 9(18\mu) \Rightarrow Q = 54\mu C$$

Thus, $\Delta V_{9\mu F} = \frac{54\mu C}{9\mu V} = 6V$ and $\Delta V_{18\mu F} = 3\text{V}$

c) Voltage across $11\mu F$ is 3V. Thus, $Q = C\Delta V = 11\mu F(3V) = 33\mu C$

For checking: $Q_{3\mu F} = 3\mu(3) = 9\mu C$; $Q_{4\mu F} = 4\mu(3) = 12\mu C$.

For parallel capacitors, the total charge, $Q = 33\mu + 9\mu + 12\mu = 54\mu C$, which is consistent in part b).

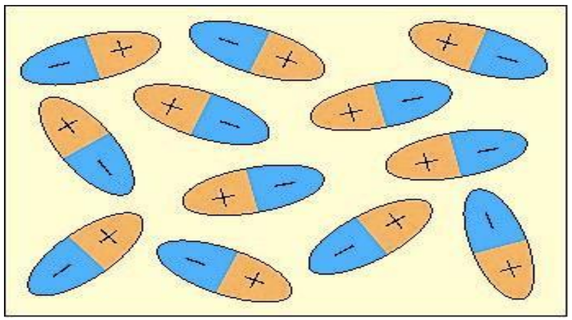
Concept 2: Effect of Dielectric in Capacitor

- A dielectric is a non-conductor or insulator.
- Examples: rubber, glass, waxed paper
- When placed in an isolated charged capacitor, **the dielectric reduces the potential difference between the two plates.**

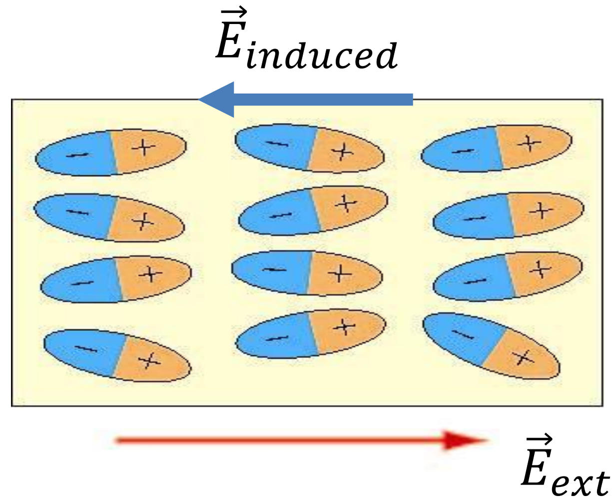
But why is that so? What is the physical mechanism?

Molecular View of Dielectrics

- **Polar Dielectrics :** Dielectrics with permanent electric dipole moments
- Example: Water

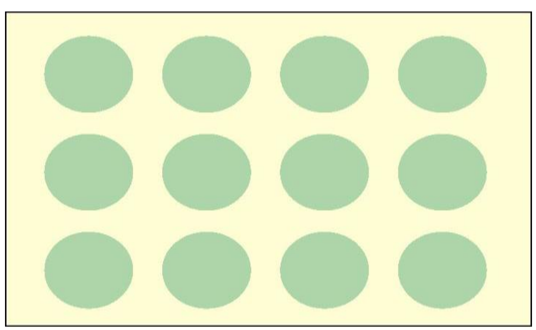


No external E field $\vec{E}_{ext} = 0$. The dipole molecules are randomly oriented.

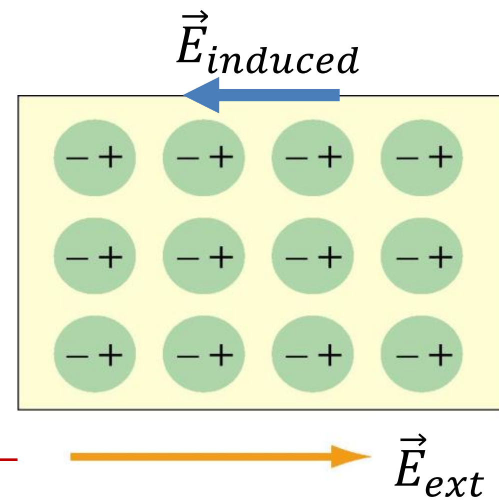


The dipole molecules are aligned along $\vec{E}_{ext}$.

- **Non-Polar Dielectrics :** Dielectrics with induced electric dipole moments
- Example: CH_4



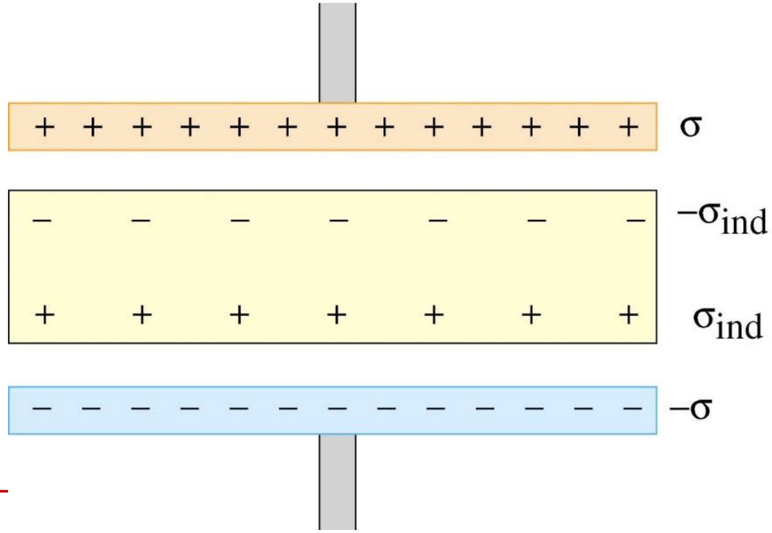
No external E field $\vec{E}_{ext} = 0$



Induced dipoles due to $\vec{E}_{ext}$.

Dielectric in Capacitor

- We can see that external electric field induces an internal field within the dielectric material. Thus, the overall field, $\vec{E} = \vec{E}_{ext} + \vec{E}_{induced}$
- Unlike conductor, the induced field in dielectric does not compensate all the external field, but only decreases the electric field!
- Likewise, the potential difference across the 2 parallel plate capacitor decreases because dielectric polarization decreases the Electric Field (within the dielectric material)!
- Next question: How much does the electric field decrease? What is the new electric field between the plates?



Dielectric Constant, κ

- We characterize the weaken electric field due to dielectric material by dielectric constant, κ. E_o is the original field.

$$E = \frac{E_o}{\kappa}$$

- Generally,

$$\varepsilon = \kappa\varepsilon_o$$

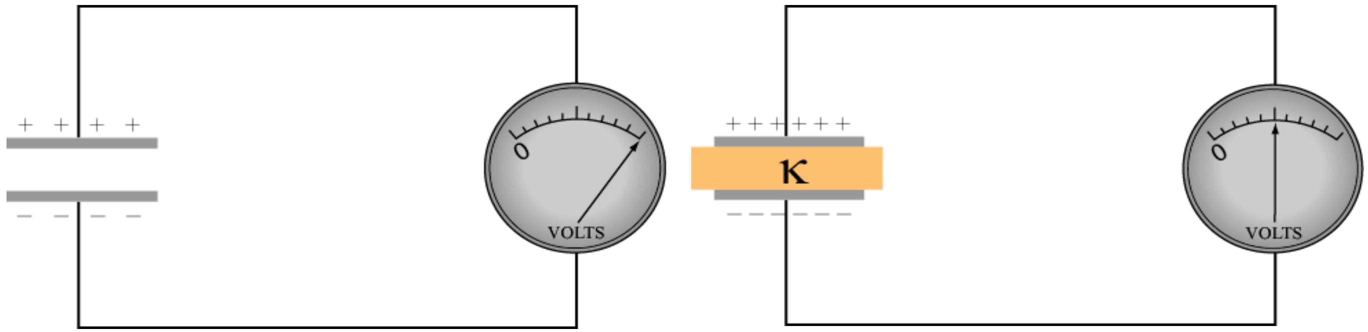
- κ is called dielectric constant, also called relative permittivity, ε_r.
- ε is permittivity of material; ε_o is permittivity of free space.
- Note: $\kappa = \varepsilon_r = \frac{\varepsilon}{\varepsilon_o}$
- Example:

Dielectric constants	
Vacuum	1.0
Paper	3.7
Pyrex Glass	5.6
Water	80

Can you guess a general relation between κ of polar and that of non-polar dielectrics?

-> Polar dielectrics have a high κ, compared to that of non-polar dielectrics.

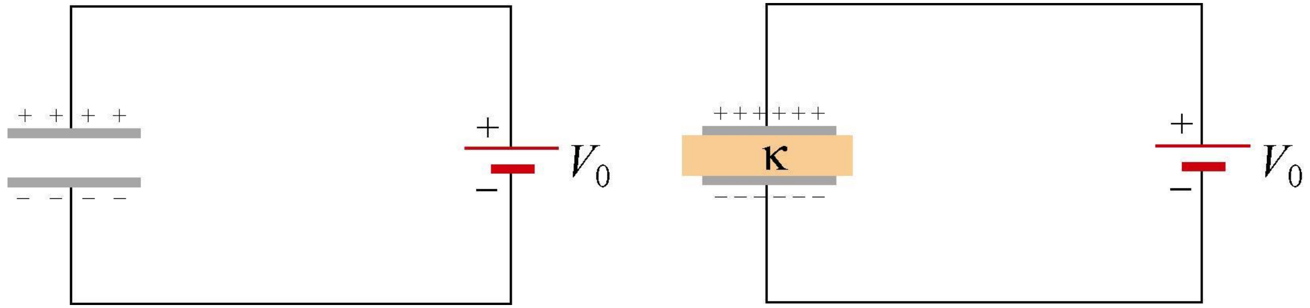
Dielectric in a Capacitor



Scenario 1: A capacitor is charged up to V and it is disconnected to the battery. A dielectric is then inserted.

- Note: Q_o= constant
- Upon inserting a dielectric: $\Delta V = \frac{\Delta V_o}{\kappa}$
- $C = \frac{Q}{\Delta V} = \frac{Q_o}{\frac{\Delta V_o}{\kappa}} = \kappa \frac{Q_o}{V_o} = \kappa C_o$
- Capacitance of new configuration, C increases by κ, compared to that of the initial configuration, C_o.

Dielectric in a Capacitor



Scenario 2: A battery remains connected to a capacitor, while a dielectric is inserted into it.

- Note: ΔV_o = constant throughout the process while battery remains connected.
- Upon inserting a dielectric:

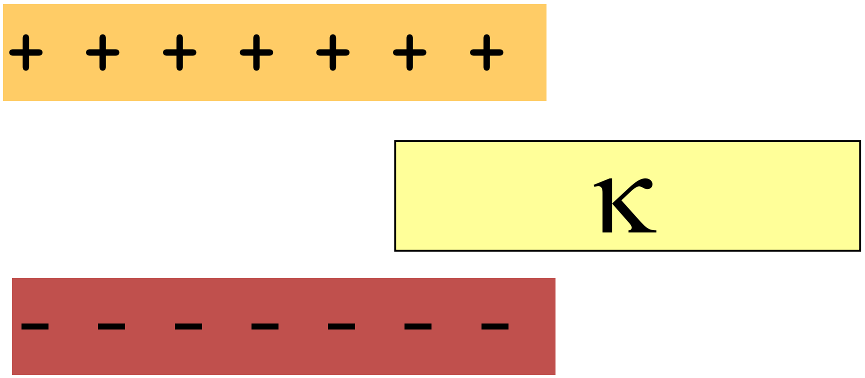
$$Q = C\Delta V_o = \kappa C_o V_o$$
$$Q = \kappa Q_o$$

- Charge stored in the new configuration, Q increased by κ, compared to that of the initial configuration, Q_o.
- Additional charge is supplied by the battery.

Concept Question 2.1

A parallel plate capacitor is charged to a total charge Q and the battery removed. A slab of material with dielectric constant k in inserted between the plates. The **charge** stored in the capacitor

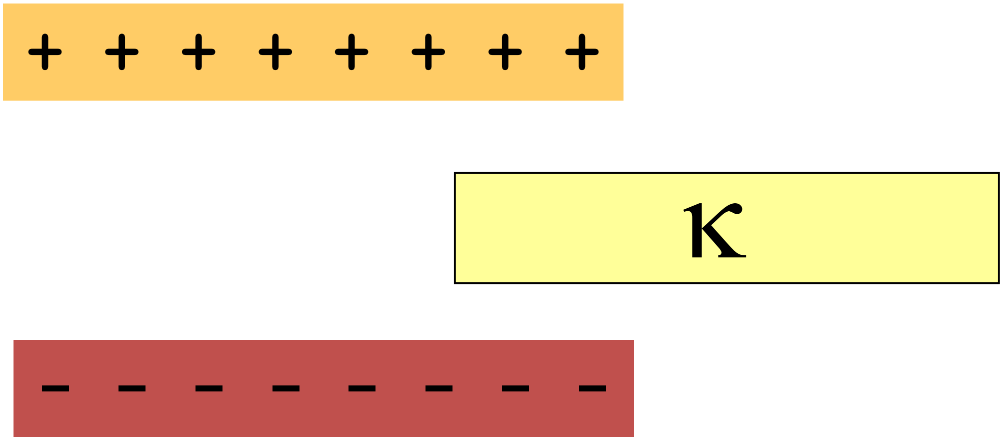
1. Increases
2. Decreases
3. Stays the same



Concept Question 2.1: Solution

Answer: 3. Charge stays the same

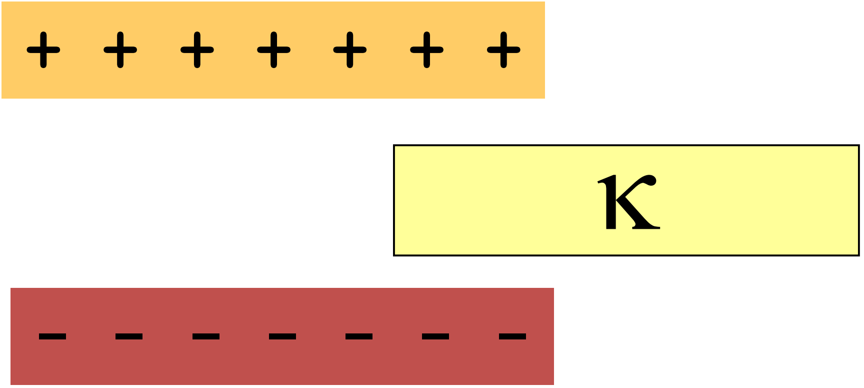
Since the capacitor is disconnected from a battery there is no way for the amount of charge on it to change.



Concept Question 2.2

A parallel plate capacitor is charged to a total charge Q and the battery removed. A slab of material with dielectric constant k in inserted between the plates. The **energy** stored in the capacitor

1. Increases
2. Decreases
3. Stays the same



Concept Question 2.2: Answer

Answer: 2. Energy stored decreases

The dielectric reduces the electric field and hence reduces the amount of energy stored in the field.

The easiest way to think about this is that the capacitance is increased while the charge remains the same so $U = Q^2/2C$.

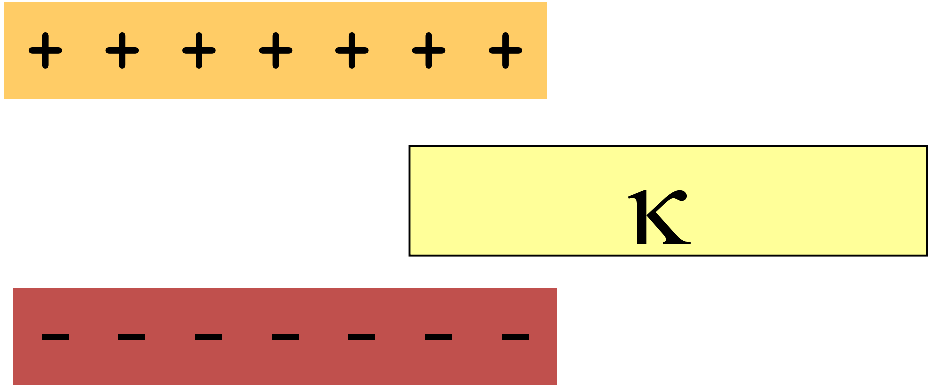
Also from energy density:

$$u_{E,0} = \frac{1}{2}\varepsilon_o E^2 \Rightarrow \frac{1}{2}\kappa\varepsilon_o\left(\frac{E}{\kappa}\right)^2 < u_{E,0}$$

Concept Question 2.3

A parallel plate capacitor is charged to a total charge Q and the battery removed. A slab of material with dielectric constant k in inserted between the plates. The **force on the dielectric**

1. pulls in the dielectric
2. pushes out the dielectric
3. is zero



Concept Question 2.3: Solution

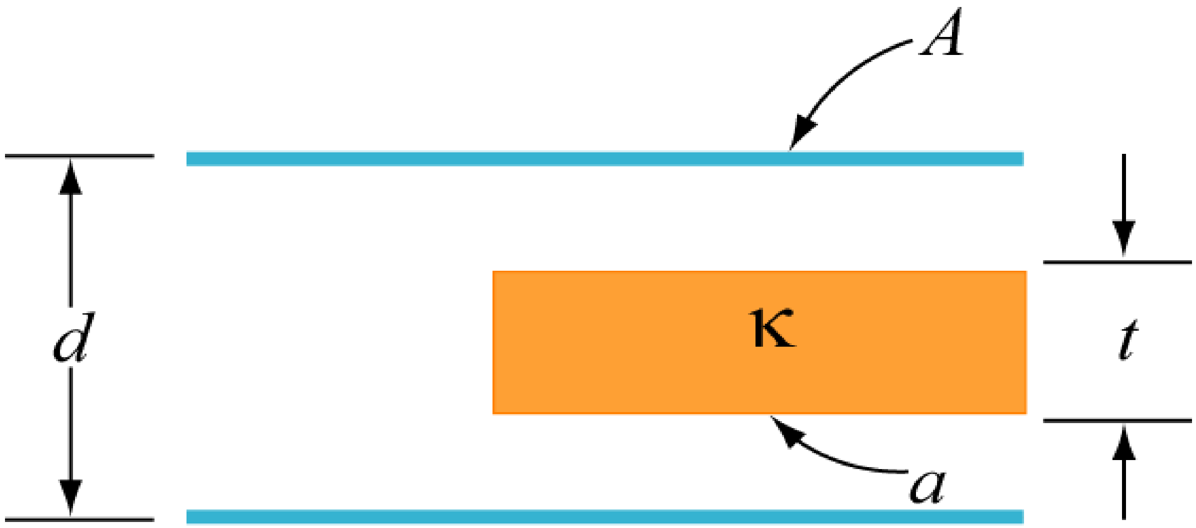
Answer: 1. The dielectric is pulled in

We just saw that the energy is reduced by the introduction of a dielectric. Since systems want to reduce their energy, the dielectric will be sucked into the capacitor.

Alternatively, since opposing charges are induced on the dielectric surfaces close to the plates, the attraction between these will lead to the attractive force.

Case Problem 2.1: Partially Filled Capacitor

- What is the capacitance of this capacitor?



Case Problem 2.1: Solution

- The total capacitance could be treated as the parallel connection of capacitor area *a* with capacitor of area (*A*-*a*).
- For capacitor of area *a,* let the charge density on the upper electrode be *σ*. Then, the electric field in the space between the dielectric and the electrode is given by
- $E = \frac{\sigma}{\varepsilon_o}$ pointing downward from upper electrode to lower electrode.
- By the continuity of the electric flux, assuming no free charge on the surface of the dielectric, then the electric field in the dielectric is given by
- $E_d = \frac{E}{\kappa} = \frac{\sigma}{\kappa\varepsilon_o}$ pointing downward from upper electrode to lower electrode.
- The potential difference of upper electrode with respect to the lower electrode is
- $V = \int_0^t E_d dy + \int_t^d E dy = \frac{\sigma t}{\kappa\varepsilon_o} + \frac{\sigma(d-t)}{\varepsilon_o} = \frac{\sigma[\kappa d+(1-\kappa)t]}{\kappa\varepsilon_o}$

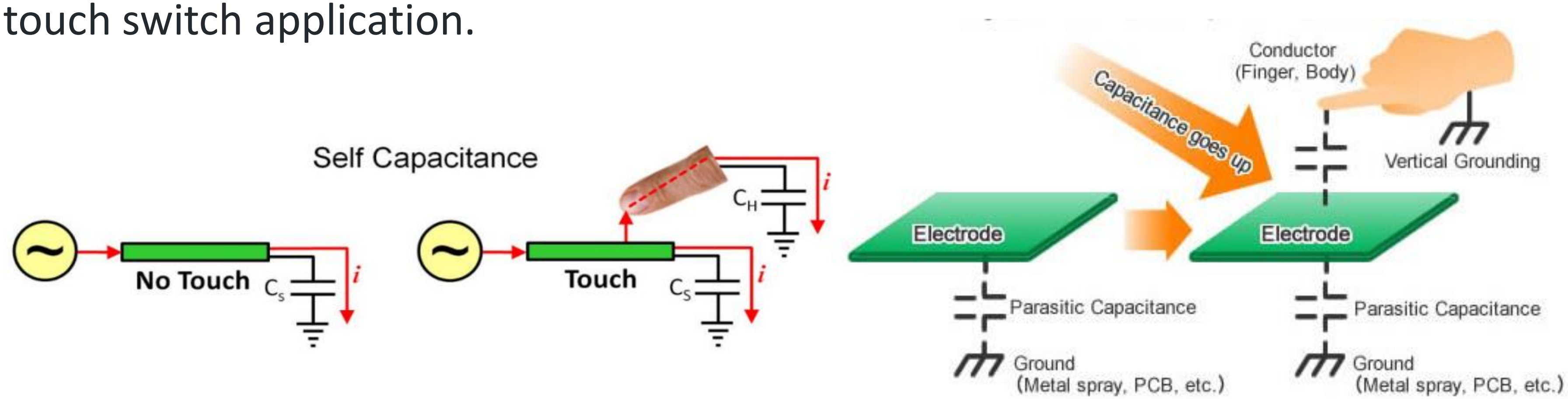
Case Problem 2.1: Solution

- The capacitance of the capacitor of area *a* is given by
- $C = \frac{Q}{\Delta V} = \frac{\sigma a}{\frac{\sigma[\kappa d+(1-\kappa)t]}{\kappa \varepsilon_o}} = \frac{a\kappa\varepsilon_o}{\kappa d+(1-\kappa)t}$
- The total capacitance is given by
- $C = C_a + C_{A-a} = \frac{a\kappa\varepsilon_o}{\kappa d+(1-\kappa)t} + \frac{(A-a)\varepsilon_o}{d}$
- This problem can also be solved using series and parallel combination!
- Note: the position of the dielectric within the parallel plates does not affect the total capacitance.

Application: Projected Capacitance Touchscreen/ Switch

The operating principle for a projected capacitance (PCAP) touchscreen is based on the introduction of a finger (or stylus, etc.) resulting in a localized change in capacitance. It is commonly done in 2 methods:

Method 1: Self-capacitance
When a finger is near an electrode, the human-body capacitance forms a parallel capacitor circuit with the electrode, causing an increase in capacitance. A circuit measures the current change due to the change in the time constant of an RC circuit (we shall learn RC circuit in Week 10). Self capacitance is usually used for touch switch application.



Self capacitance sensing response – forming a parallel C circuit (increase in capacitance)

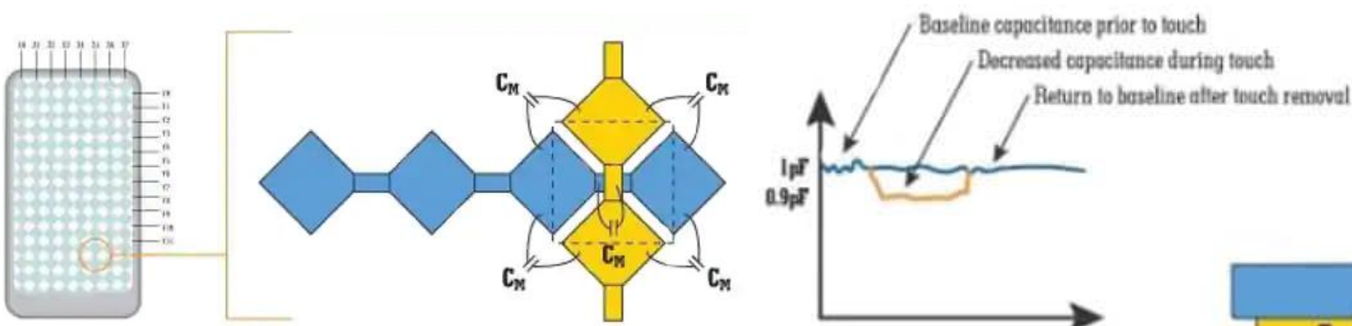
Image credit: Geoff Walker's *Fundamentals of Projected-Capacitive Touch Technology (2014)*

Application: Projected Capacitance Touchscreen/ Switch

Method 2: Mutual-capacitance

2 near-by electrodes form a capacitor. When a finger comes closer, a conductive human body (or the human-body capacitance) perturbs the electric field between the electrodes, which causes a decrease in capacitance between two electrodes. It is like forming capacitors in series. Most smart phones sense mutual capacitance sensing. It supports multi-touch sensing at every position on the screen.

Sensing mutual capacitance involves measuring the interaction between an X and a Y sensor. A signal driven on each Y line and each X line is sensed to detect the level of coupling between the sensors.



Mutual capacitance sensing response

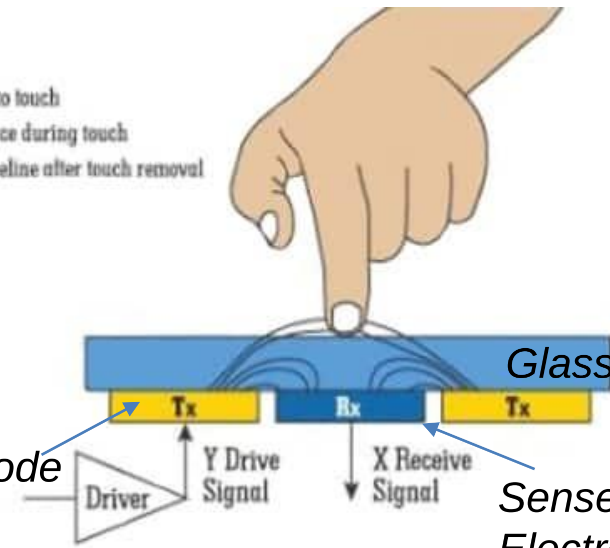


Image: https://www.digikey.com/en/articles/from-touch-to-call-tracing-the-path-of-a-touch-gesture

Application: Projected Capacitance Touchscreen/ Switch

- Sensing mutual capacitance is fundamentally different from sensing self capacitance.
- In method 1 (self-capacitance), a finger touch increases the overall capacitance value of the circuit.
- In method 2 (mutual-capacitance), a finger touch (or a human-body capacitance) decreases the mutual-capacitance coupling.
- In both cases, no charge is transferred through the finger/body.
- The system must react to changes in capacitance, not raw capacitance.
- A microcontroller system is employed to detect the change of capacitance and position.

More Information:

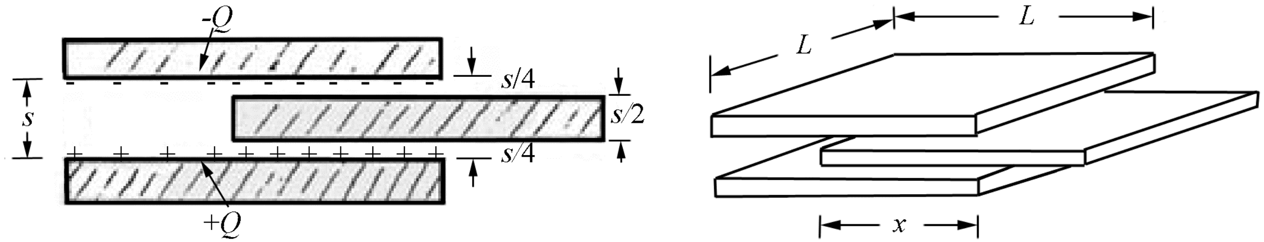
https://charliemicou.com/docs/touchscreens

https://www.digikey.com/en/articles/from-touch-to-call-tracing-the-path-of-a-touch-gesture

https://www.dmccoltd.com/english/museum/touchscreens/

Extra Case Problem: Capacitance, Stored Energy, and Electrostatic Work

Two flat, square metal plates have sides of length L, and thickness s/2, are arranged parallel to each other with a separation of s, where s<<L so you may ignore fringing fields. A charge Q is moved from the upper plate to the lower plate. A third uncharged conducting plate of the same thickness s/2 is placed so that it lies between the other two plates to a depth x, maintaining the same spacing s/4 between its surface and the surfaces of the other two.



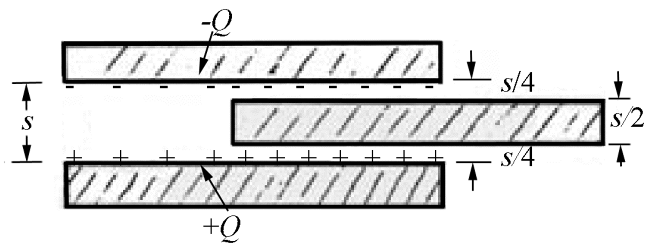
a) What is the capacitance of this system?

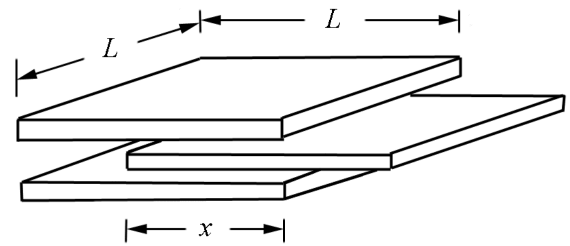
b) How much energy is stored in the electric field?

c) If the middle plate is released, it starts to move. Will it move to the right or left?

d) Find the horizontal force exerted on the middle plate by the charge distribution on the outer plates that cause it to move.

Extra Case Problem Solution





a) What is the capacitance of this system?

- The three plates can be thought as being two capacitors in parallel of which the right one is the series of two other capacitors Unlike the example with a dielectric inserted, there is no electric field inside the slab of metal.
- $C_1 = \frac{\varepsilon_0 L(L-x)}{s}$; $\frac{1}{C_2} = \frac{1}{C} + \frac{1}{C} \Rightarrow C_2 = \frac{C}{2} = \frac{1}{2}\frac{\varepsilon_0 Lx}{\frac{s}{4}} = \frac{2\varepsilon_0 Lx}{s}$
- $C_{total} = C_1 + C_2 = \frac{\varepsilon_0 L(L-x)}{s} + \frac{2\varepsilon_0 Lx}{s} = \frac{\varepsilon_0 L(L+x)}{s}$
- Note: the change distribution on the outer plates between region 1 and 2 are different. By conservation of charge, $\mathrm{Q} = \mathrm{Q}_1 + \mathrm{Q}_2$; For conductors are equipotential object, $\Delta V_1 = \Delta \mathrm{V}_2 \rightarrow \frac{Q_1}{C_1} = \frac{Q_2}{C_2}$.
- Thus, $\mathrm{Q} = Q_1 + \frac{C_2}{C_1} Q_1 \rightarrow Q_1 = \frac{C_1}{C_1 + C_2} Q = \frac{L-x}{L+x} Q$; $Q_2 = \frac{2x}{L+x} Q$

SINGAPORE UNIVERSITY OF TECHNOLOGY AND DESIGN

Extra Case Problem Solution

b) The energy stored is equal to $U = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}\frac{(Q^2 s)}{\varepsilon_o L(L+x)}$

c) The middle plate will move to the left once it is released. The middle plate will be polarized by the outer plates. It creates an attractive force to the left side. The result is similar to the force acting on a dielectric in between 2 parallel plates.

From energy point of view, a system always tends to move towards lower potential energy. From part b), increasing x causes a reduction in U.

d) Recall the general relation: $\vec{F} = -\frac{dU}{dx}\hat{\imath}$ from mechanics. The magnitude of $F = \frac{dU}{dx}$. From $U = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}\frac{(Q^2 s)}{\varepsilon_o L(L+x)}$

- $|\vec{F}| = \frac{1}{2}\frac{Q^2 s}{\varepsilon_o L(L+x)^2}$. The force is pointing to the left.