Week 13 - Day 1

Secure Quantum Communication

Concept 1: Measurement

Quantum computation (Week 13)

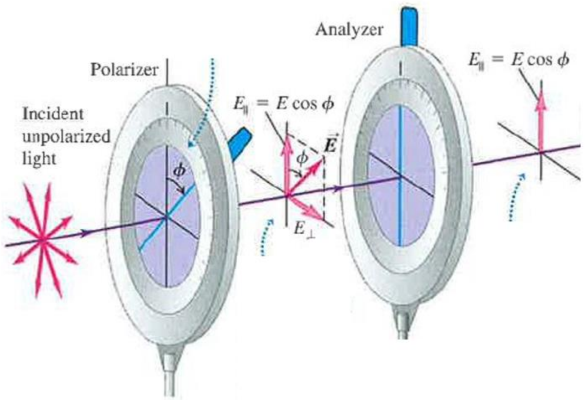
Reference:
Six Quantum Pieces: A First Course in Quantum Physics
Chapter 1 section 1.3 and Chapter 2 sections 2.1, 2.2 and 2.3

Concept 1: Measurement

Measurement

Let us consider again the setup of the two polarizers. The analyzer can be thought of as a measurement device. It measures whether the photon is vertically polarized or not.

If the photon goes through you measure 1 (presence of a signal), if the photon does not go through you measure a 0 (absence of a signal). polarizer.



Let us consider a single photon prepared in the state

$$|\varphi_+\rangle = \begin{pmatrix} \sin(\varphi) \\ \cos(\varphi) \end{pmatrix} = \sin(\varphi)\,|H\rangle + \cos(\varphi)\,|V\rangle$$ by the first polarizer.

With probability $|\sin(\varphi)|^2$ the photon is measured in state $|H\rangle$ and with probability $|\cos(\varphi)|^2$ it is measured in state $|V\rangle$.

Measurement

The average output, if we prepare a photon in the same state, and we repeat the measurement many times, we would get as an average output denoted as

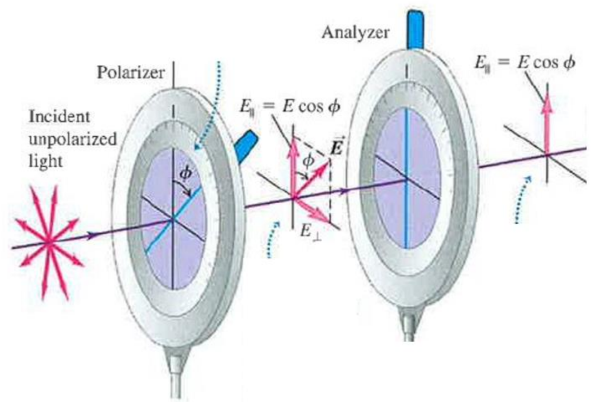
$$\langle\hat{O}\rangle = O_V P_V + O_H P_H$$

Average output of come quantity $\hat{O}$ (in our case polarization in V direction). This notation will become clearer later.

Value of the measurement for that outcome

Probability of the outcome

where $P_{V/H}$ is the probability of the state to be in vertically/horizontally polarized and $O_{V/H}$ is the value of the outcome.



This is a general result which can readily expanded to systems with more than 2 outcomes.

In our case $O_V = 1$, $O_H = 0$, $P_V = |\cos(\varphi)|^2$ and $P_H = |\sin(\varphi)|^2$, so we get

$$\langle\hat{O}\rangle = O_V P_V + O_H P_H = 1 \cdot |\cos(\varphi)|^2 + 0 \cdot |\sin(\varphi)|^2 = |\cos(\varphi)|^2$$

Measurement

With a bit of linear algebra, we can now gain a deeper insight into the measurement.

We can write

$$\langle \hat{O} \rangle = O_V |\langle V|\psi\rangle|^2 + O_H |\langle H|\psi\rangle|^2$$

because $P_V = |\langle V|\psi\rangle|^2$, and $P_H = |\langle H|\psi\rangle|^2$. We also notice that

$$|\langle H|\psi\rangle|^2 = \langle\psi|H\rangle\langle H|\psi\rangle = (\alpha^* \ \beta^*) \left[\begin{pmatrix}1\\0\end{pmatrix}(1\ 0)\right]\begin{pmatrix}\alpha\\\beta\end{pmatrix}$$

where the part $|H\rangle\langle H| = \left[\begin{pmatrix}1\\0\end{pmatrix}(1\ 0)\right]$ can be written as a matrix

$$|H\rangle\langle H| = \left[\begin{pmatrix}1\\0\end{pmatrix}(1\ 0)\right] = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}$$

Scalar quantities like O_V can be freely moved around

and so you can think of $O_H |\langle H|\psi\rangle|^2 = O_H \langle\psi|H\rangle\langle H|\psi\rangle = O_H (\alpha^* \ \beta^*) \begin{pmatrix}1\\0\end{pmatrix}(1\ 0)\begin{pmatrix}\alpha\\\beta\end{pmatrix} =$

$(\alpha^* \ \beta^*) \left[O_H \begin{pmatrix}1\\0\end{pmatrix}(1,0)\right]\begin{pmatrix}\alpha\\\beta\end{pmatrix} = \langle\psi| \left[O_H |H\rangle\langle H|\right] |\psi\rangle$ where the part in green is a matrix.

Matrix $O_H \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}$

Measurement

In this way we can write

$$\langle\hat{O}\rangle = O_V|\langle V|\psi\rangle|^2 + O_H|\langle H|\psi\rangle|^2 = O_V\langle\psi|V\rangle\langle V|\psi\rangle + O_H\langle\psi|H\rangle\langle H|\psi\rangle$$
$$= \langle\psi|\ \boxed{(O_V|V\rangle\langle V|\ +\ O_H|H\rangle\langle H|)}\ |\psi\rangle = \langle\psi|\boxed{\hat{O}}|\psi\rangle$$

Or in short $\langle\hat{O}\rangle = O_V P_V + O_H P_H = \langle\psi|\hat{O}|\psi\rangle$ where $\hat{O}$ is the matrix constructed with the possible measured outcomes $O_{V/H}$ and the corresponding output states $|V\rangle$ or $|H\rangle$.

$$\hat{O} = O_V|V\rangle\langle V|\ +\ O_H|H\rangle\langle H|$$

Possible outcomes of measuring energy

Corresponding output states

This is true for whichever basis states one wants to use, or measures in. For example, the polarizer is at an angle φ, then one would consider the two orthonormal basis states $|\varphi_+\rangle$ and $|\varphi_-\rangle$ and two possible outcomes O_+, and O_-, and the corresponding observable

$$\hat{O} = O_+|\varphi_+\rangle\langle\varphi_+|\ +\ O_-|\varphi_-\rangle\langle\varphi_-|$$

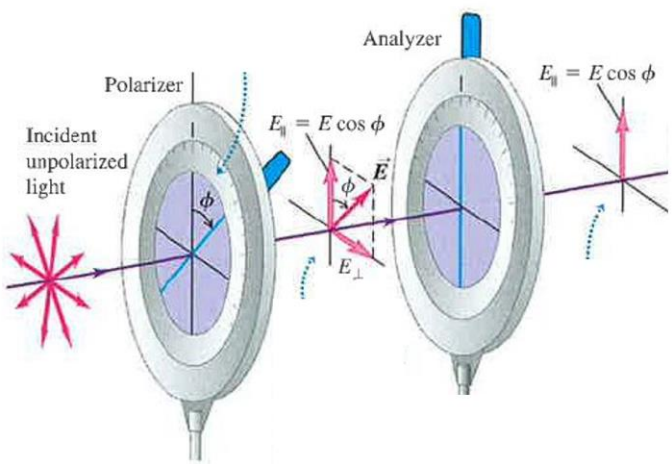
Case problem 1.1

Marie is using a polarizer given to her by her assistant Enrico.

She does not know how Enrico set up the polarizer, but she knows how to prepare single photons either vertically $|V\rangle$ or horizontally $|H\rangle$ polarized.

Marie launches and equal number of $|V\rangle$ and $|H\rangle$ photons and she gets that the number of $|V\rangle$ photons that goes through is 3 times that of the $|H\rangle$ photons.

What can the angle of the polarizer set by her assistant Enrico be?



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Case problem 1.1: solution

The polarizer is set to let a state proportional to $|\varphi\rangle = \begin{pmatrix} \sin(\varphi) \\ \cos(\varphi) \end{pmatrix}$ to go through.

The probability that state $|H\rangle$ goes through is $|\langle H|\varphi\rangle|^2 = \sin(\varphi)^2$, while the probability that $|V\rangle$ goes through is $|\langle V|\varphi\rangle|^2 = \cos(\varphi)^2$.

So we get

$$\frac{\sin(\varphi)^2}{\cos(\varphi)^2} = \tan(\varphi)^2 = 1/3 \quad \rightarrow \quad \tan(\varphi) = 1/\sqrt{3} \quad \rightarrow \varphi = \arctan\left(1/\sqrt{3}\right) = \frac{\pi}{6} = 30^\circ$$

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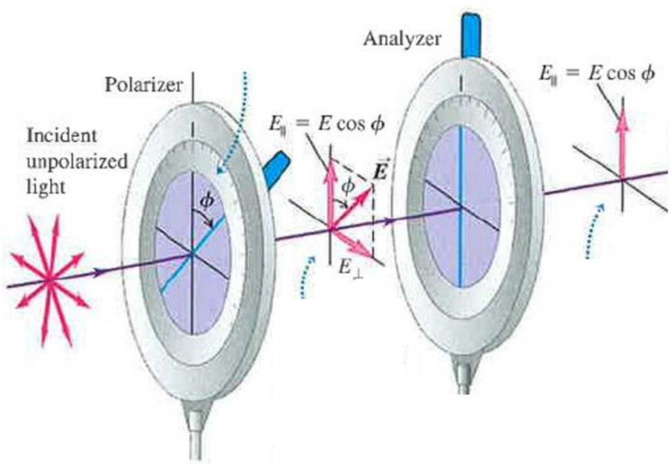
Case problem 1.2

Consider a polarizer set in a direction with $\varphi = 45^{\circ}$, i.e. if a photon goes through then it would be measured to be in a state proportional to $|+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$ with probability 1.

Which is the corresponding observable matrix $\hat{O}$?



Case problem 1.2: solution

Using

$$\hat{O} = O_+|\varphi_+\rangle\langle\varphi_+| + O_-|\varphi_-\rangle\langle\varphi_-|$$

we know that the result is $O_+ = 1$ if the state is $|+\rangle$ and $O_- = 0$ for $|-\rangle$.

We can thus write

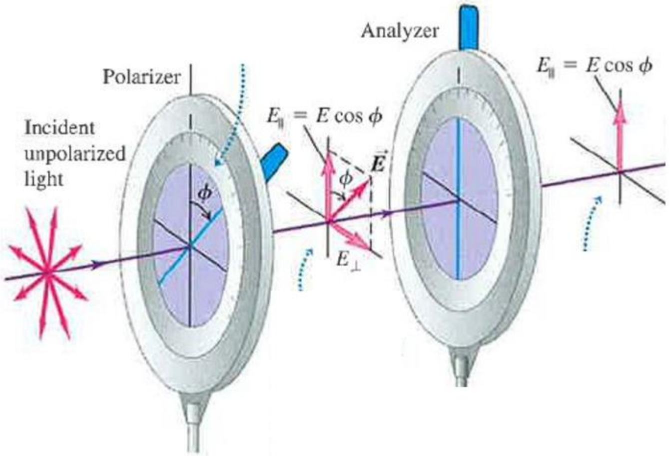
$$\hat{O} = 1|+\rangle\langle+| + 0\,|-\rangle\langle-| = \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}$$

Case problem 1.3

Consider the observable matrix $\hat{O} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ corresponding to a particular choice of angle of the polarizer.

(1) What is outcome for measurement of single photon in the state $|\psi\rangle = \frac{1}{\sqrt{3}}|H\rangle + \sqrt{\frac{2}{3}}|V\rangle$, a superposition of vertical and horizontal linear polarization?

(2) What is the average outcome of many measurements of this polarizer with many single photons prepared in the same state as at point (1) $|\psi\rangle = \frac{1}{\sqrt{3}}|H\rangle + \sqrt{\frac{2}{3}}|V\rangle$?



Case problem 1.3: solution

(1) For the observable matrix $\hat{O} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ corresponding to a particular choice of angle of the polarizer, at a measurement of each single photon one can measure either 0 or 1 (the photon has gone through or not).

(2) For this point we can use the expression for the expectation value given that each time one could measure 0 or 1.

$$\langle \hat{O} \rangle = \langle \psi | \hat{O} | \psi \rangle = \frac{1}{2}\begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{\sqrt{2}}{\sqrt{3}} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1/\sqrt{3} \\ \sqrt{2}/\sqrt{3} \end{pmatrix} = \frac{1}{6}\begin{pmatrix} 1 & \sqrt{2} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}$$

$$= \frac{1}{6}\begin{pmatrix} 1+\sqrt{2} & 1+\sqrt{2} \end{pmatrix}\begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} = \frac{1}{6}(3 + 2\sqrt{2}) \approx 0.97$$

Measurement

Interestingly, given that observables can be written as

$$\hat{O} = O_A|A\rangle\langle A| \; + \; O_B|B\rangle\langle B|$$

with $|A\rangle$ and $|B\rangle$ orthonormal vector states and $O_{A/B}$ two real number, then the resulting matrix $\hat{O}$ is Hermitian, i.e. $\hat{O} = \hat{O}^\dagger = (\hat{O}^T)^*$ where T means transposition of the matrix and * is complex conjugation of all its elements.

Example:

$$\left(\begin{pmatrix} 1 & 1+2i \\ 1-2i & 3 \end{pmatrix}^T\right)^* = \left(\begin{pmatrix} 1 & 1-2i \\ 1+2i & 3 \end{pmatrix}\right)^* = \begin{pmatrix} 1 & 1+2i \\ 1-2i & 3 \end{pmatrix}$$

Try it at home: Prove that $\hat{O} = (\hat{O}^T)^*$ for any $\hat{O}$ written as in the equation in red above.

Concept question 1.1

What is the average outcome for the following observable and vector state?

$$|\psi\rangle = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ -1 \end{pmatrix} \qquad \hat{A} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

Hint: $\langle \hat{A} \rangle = \langle \psi | \hat{A} | \psi \rangle$

A. wait, this does not make sense
B. 1.2
C. 0.4
D. π

Multiple Choice

Concept question 1.1: solution

What is the average outcome for the following observable and vector state?

$$|\psi\rangle = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ -1 \end{pmatrix} \qquad \hat{A} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

Hint: $\langle\hat{A}\rangle = \langle\psi|\hat{A}|\psi\rangle$

1. wait, this does not make sense
2. **1.2**
3. 0.4
4. π

$$\langle\hat{A}\rangle = \langle\psi|\hat{A}|\psi\rangle = \frac{1}{5}\begin{pmatrix} 2 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 2 & -2 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \frac{6}{5} = 1.2$$

Concept question 1.2

What is the average outcome for the following observable and vector state?

$$|\psi\rangle = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad \hat{A} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

Hint: $\langle \hat{A} \rangle = \langle \psi | \hat{A} | \psi \rangle$

1. wait, this does not make sense
2. -0.2
3. 0.6
4. π

Multiple Choice

Concept question 1.2: solution

What is the average outcome for the following observable and vector state?

$$|\psi\rangle = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad \hat{A} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

Hint: $\langle \hat{A} \rangle = \langle \psi | \hat{A} | \psi \rangle$

1. **wait, this does not make sense**
2. -0.2
3. 0.6
4. π

The state vector is not normalized

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Concept question 1.3

What is the average outcome for the following observable and vector state?

$$|\psi\rangle = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ -1 \end{pmatrix} \qquad \hat{A} = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$$

Hint: $\langle \hat{A} \rangle = \langle \psi | \hat{A} | \psi \rangle$

1. wait, this does not make sense
2. -0.2
3. 0.6
4. π

Multiple Choice

Concept question 1.3: solution

What is the average outcome for the following observable and vector state?

$$|\psi\rangle = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ -1 \end{pmatrix} \qquad \hat{A} = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$$

Hint: $\langle \hat{A} \rangle = \langle \psi | \hat{A} | \psi \rangle$

1. **wait, this does not make sense**
2. -0.2
3. 0.6
4. π

The observable is not Hermitian $\hat{A} \neq \hat{A}^\dagger$

Measurement

Note that until now we have only considered photons going through polarizers, for which not only there are just 2 outcomes, but one of them is 0.

In general, the outcome of a measurement can be any real number (we never measure a complex quantity).

For example, you can measure the spin of an electron, the energy of a particle, the momentum, the angular momentum etc….

Suppose one sets up an experiment and informs you that the matrix representing the observation that you can do is given by $\hat{O}$.

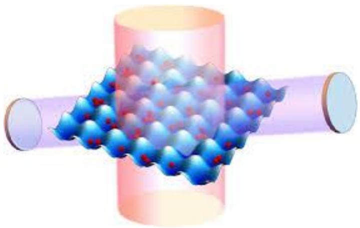
Can you know figure out what are the possible outcomes of the measurement and what would be the state of the system after the measurement?

Measurement

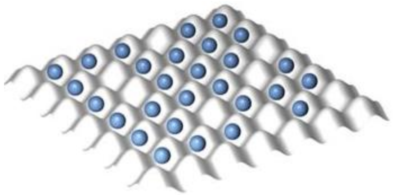
To make things more concrete, since energy is something one can measure, we consider energy as the observable we are interested into.

To visualize things a bit more, let us consider atoms trapped by laser light.

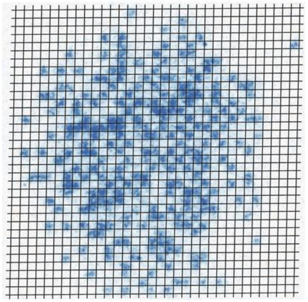
Indeed, if atoms are cold enough and you do things properly, you can trap atoms in magnetic fields and lasers (ask our Max Colla who has been doing this over and over again).



Optical lattice made by counterpropagating lasers



Depiction of atoms in a potential made by the lasers



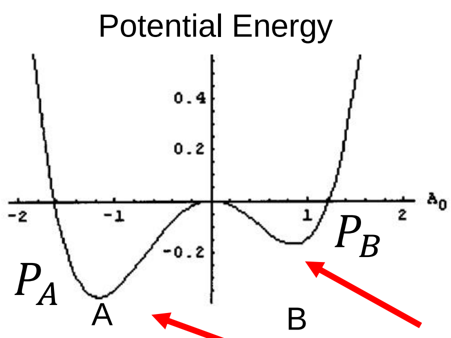
Actual image from an experiment. Each blue dot is a single atom.

Gross and Bloch, Science, Vol. 357, pp. 995-1001 (2017)

Such systems are a bit advanced to study, but what we can do is to study the behavior of a single atom in a trap with just two deeps (called a double well).

Measurement

A double well can be visualized by the following potential



double well potential

Ultracold atoms will only be found at position A or B, because you would need a larger temperature for them to move around more freely.

Since the atoms can be just at position A or B, we can write the potential energy operator $\hat{P}$ very easily.

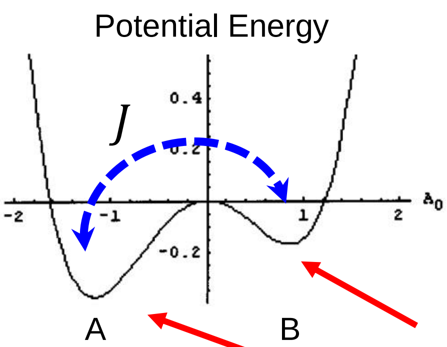
The potential energy $\hat{P}$ in B is higher than in A, so we can write the potential energy as the matrix

$$\hat{P} = \begin{pmatrix} P_A & 0 \\ 0 & P_B \end{pmatrix}$$

where P_A is the potential energy at point A and P_B is the potential energy at point B.

Measurement

A double well can be visualized by the following potential

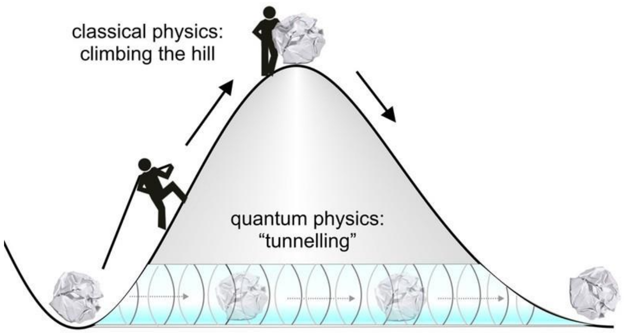


double well potential

Ultracold atoms will only be found at position A or B

Even at extremely cold temperatures, so that atoms do not fidget around, atoms are still allowed to tunnel from one side to the other.

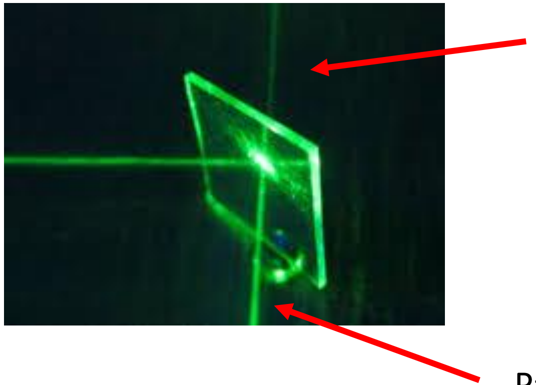
This is due to the kinetic energy part of the total energy.



Roughly speaking, this is due to the fact that atoms have a non-zero probability to be on both sides of the hill.

(parenthesis on quantum tunneling)

This “quantum tunneling” should not come too much as a shock. Consider a semi-reflective mirror as shown here



Some light is reflected, and some is transmitter.

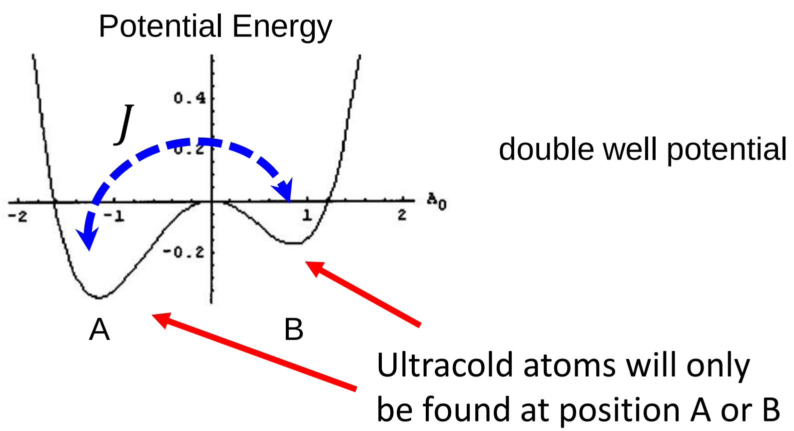
This occurs for each single photon!

After the photon “hits” the mirror, it is partially reflected and partially transmitted, thus partially “tunneling” through the mirror and exiting on the other side.

This can occur for any type of particle against any type of potential!

Measurement

A double well can be visualized by the following potential



The kinetic energy $\hat{K}$ allows the atom to go from position A to position B and can be described by the matrix

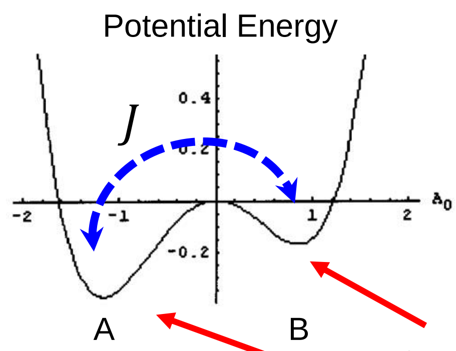
$$\hat{K} = \begin{pmatrix} 0 & J \\ J & 0 \end{pmatrix}$$

where J is the magnitude of the kinetic energy.

A larger J means that the motion occurs very rapidly, while a smaller J implies the opposite.

Measurement

A double well can be visualized by the following potential



double well potential

Ultracold atoms will only be found at position A or B

Does this operator really moves a state from position A to position B?

Let us try, $\hat{K}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0 & J\\ J & 0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = J\begin{pmatrix}0\\1\end{pmatrix}$, which means that a state $\begin{pmatrix}1\\0\end{pmatrix}$ goes to $\begin{pmatrix}0\\1\end{pmatrix}$ after action of the kinetic energy operator.

At the same time, $\hat{K}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0 & J\\ J & 0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = J\begin{pmatrix}1\\0\end{pmatrix}$, which means that the matrix $\hat{K}$ moves the state from being in state $\begin{pmatrix}0\\1\end{pmatrix}$ to state $\begin{pmatrix}1\\0\end{pmatrix}$.

Measurement

The total energy observable $\hat{H}$ (typically referred to as the Hamiltonian) is given by the sum of kinetic and potential energy (as usual!) giving

$$\hat{H} = \hat{K} + \hat{P} = \begin{pmatrix} P_A & J \\ J & P_B \end{pmatrix}.$$

But as everything that we can measure, this can be written as

$$\hat{H} = E_1 |E_1\rangle\langle E_1| + E_2 |E_2\rangle\langle E_2|$$

Possible outcomes of measuring energy

Corresponding output states

with two possible energy outcomes E_1, and E_2, and two possible states after the measurement $|E_1\rangle$ and $|E_2\rangle$.

How do we find the values E_1, and E_2, and two states $|E_1\rangle$ and $|E_2\rangle$?

Measurement

We note that $\widehat{H}\,|E_1\rangle = E_1\,|E_1\rangle$ and $\widehat{H}\,|E_2\rangle = E_2\,|E_2\rangle$ because the two states form an orthonormal basis, i.e. $\langle E_1|E_2\rangle = 0$ and $\langle E_1|E_1\rangle = \langle E_2|E_2\rangle = 1$.

Since $\widehat{H}$ is a matrix and $|E_1\rangle$ and $|E_2\rangle$ are vectors, this means that we have an expression of the type

Matrix

Scalar

$$\begin{pmatrix} * & * \\ * & * \end{pmatrix}\begin{pmatrix} * \\ * \end{pmatrix} = \lambda \begin{pmatrix} * \\ * \end{pmatrix}$$

Same vector

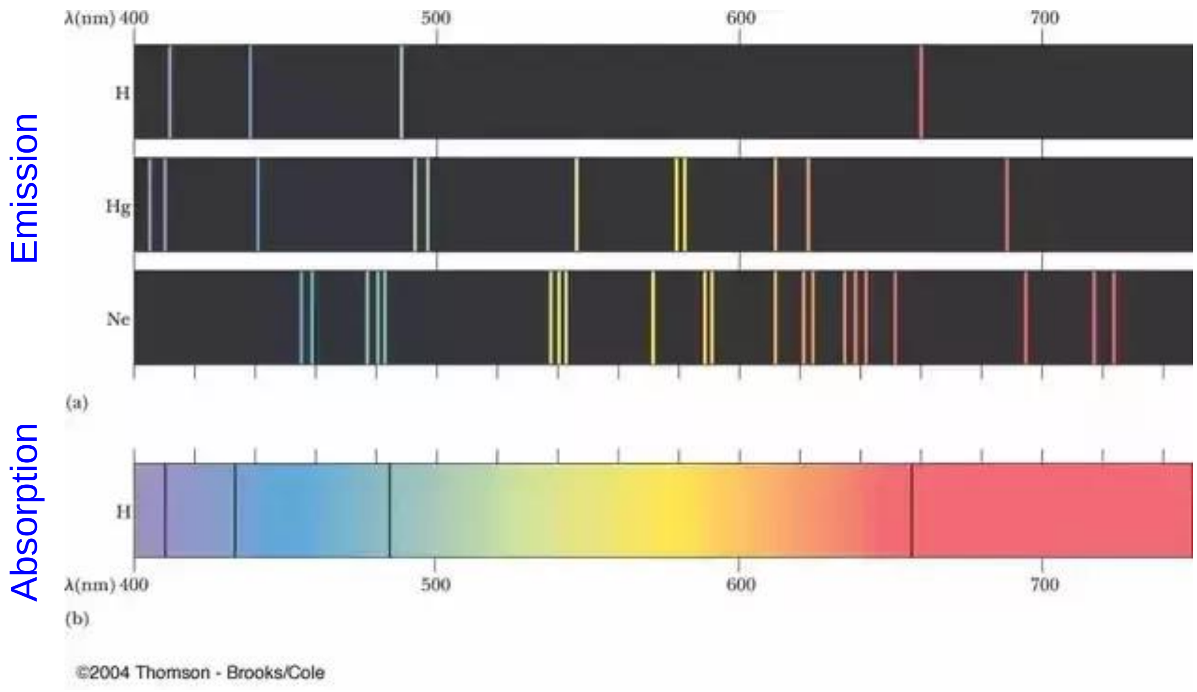
where the matrix only stretches or compresses the vector, but it does not change its direction.

This means that the outcomes of the measurement are the eigenvalues of the matrix , while the state after the measurement is the corresponding eigenvector.

Measurement

The fact that the observable energy is a matrix and the possible energy outcomes are the (discrete) eigenvalues of this matrix is fundamentally the reason why we observe a discrete spectrum of light from atoms (see Week 12, day 2 material)

Emission and absorption lines for different gases



Measurement: reminder (see also Modeling Space and Systems)

How to compute the eigenvalues of a matrix?

$$M\vec{v} = \lambda\vec{v}$$

We can rewrite the above as

$$M\vec{v} - \lambda\vec{v} = 0$$
$$(M - \lambda\, I)\vec{v} = 0$$

Where I is the identity matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

For this linear equation not to have only the trivial solution $\vec{v} = \vec{0}$, one needs that the determinant of $(M - \lambda\, I)$ is 0, i.e.

$$\det[M - \lambda\, I] = 0$$

This is one way to find the eigenvalues λ of the matrix M.

Worked example

What are the possible outcomes of the following observable?

$$\hat{A} = \begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix}$$

And what can be the updated vector states after the measurement?

Worked example: solution

We need to find the eigenvalues and eigenvectors of $\hat{A} = \begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix}$

For the eigenvalues we get

$$\det[\hat{A} - \lambda\, I] = \det\left[\begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] = \det\left[\begin{pmatrix} 5-\lambda & 1 \\ 1 & 5-\lambda \end{pmatrix}\right] = \lambda^2 - 10\lambda + 24$$

which gives $\lambda_1 = 4$ and $\lambda_2 = 6$.

For the eigenvectors we can simply write

$$\hat{A}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \lambda \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

and check which vector $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ satisfies such equation (since we already know λ).

For $\lambda_1 = 4$ we get

$$\begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 4\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

which gives $5\alpha + \beta = 4\,\alpha$, i.e. $\beta = -\alpha$

$$|\psi_1\rangle = \alpha \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

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Worked example: solution

$|\psi_1\rangle = \alpha \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is not normalized. So we need to find a α such that $\langle \psi_1 | \psi_1 \rangle = 1$.

This results in $\alpha^2\left(1^2 + (-1)^2\right) = 2\alpha^2 = 1$ which gives as a solution

$$\alpha = \frac{1}{\sqrt{2}}$$

which then results in

$$|\psi_1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

For $\lambda_2 = 6$ we get

$$\begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 6 \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

which gives $5\alpha + \beta = 6\,\alpha$, i.e. $\beta = \alpha$

$$|\psi_2\rangle = \alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

and thus

$$|\psi_2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Case Problem 1.4

What are the possible outcomes of the following observable?

$$\hat{B} = \begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix}$$

And what can the updated vector state be after the measurement?

Case Problem 1.4: solution

In a similar manner to the example, one can work for the observable $\hat{B} = \begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix}$.

For the eigenvalues we get

$$\det[\hat{B} - \lambda\, I] = \det\left[\begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] = \det\left[\begin{pmatrix} 4-\lambda & -2 \\ -2 & 1-\lambda \end{pmatrix}\right] = \lambda^2 - 5\lambda$$

which gives $\lambda_1 = 0$ and $\lambda_2 = 5$.

For the eigenvectors we can simply write

$$\hat{A}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \lambda \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

and check which vector $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ satisfies such equation (since we already know λ).

For $\lambda_1 = 0$ we get

$$\begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

which gives $4\alpha - 2\beta = 0$, i.e. $\beta = 2\alpha$

$$|\psi_1\rangle = \alpha \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Case Problem 1.4: solution

$|\psi_1\rangle = \alpha\begin{pmatrix}1\\2\end{pmatrix}$ is not normalized. So we need to find a α such that $\langle\psi_1|\psi_1\rangle = 1$.

This results in $\alpha^2(1^2+2^2) = 5\alpha^2 = 1$ which gives as a solution

$$\alpha = \frac{1}{\sqrt{5}}$$

which then results in

$$|\psi_1\rangle = \frac{1}{\sqrt{5}}\begin{pmatrix}1\\2\end{pmatrix}$$

For $\lambda_2 = 5$ we get

$$\begin{pmatrix}4 & -2\\-2 & 1\end{pmatrix}\begin{pmatrix}\alpha\\\beta\end{pmatrix} = 5\begin{pmatrix}\alpha\\\beta\end{pmatrix}$$

which gives $4\alpha - 2\beta = 5\,\alpha$, i.e. $\alpha = -2\beta$

$$|\psi_2\rangle = \beta\begin{pmatrix}-2\\1\end{pmatrix}$$

and thus

$$|\psi_2\rangle = \frac{1}{\sqrt{5}}\begin{pmatrix}-2\\1\end{pmatrix}$$

Case Problem 1.5

Enrico has figured out that an observable is described by the following matrix

$$\hat{A} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

and he observed that the two possible outcomes occur with equal probability.

1) What are the possible outcomes?
2) What is the average outcome that he observes after many measurements?

Case Problem 1.5: solution

1) We can first figure out which are the possible outcomes, i.e. the eigenvalues of $\hat{A}$.
We can thus use $\det[\hat{A} - \lambda\, I] = 0$.

We thus get $\det\left[\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] = 0$ which we can rewrite as

$$\det\left[\begin{pmatrix} 1-\lambda & 1 \\ 1 & 2-\lambda \end{pmatrix}\right] = (1-\lambda)(2-\lambda) - 1 = 1 - 3\lambda + \lambda^2 = 0$$

which is satisfied for $\lambda_{1,2} = (3 \pm \sqrt{5})/2$

2) Since each possible outcome appears with 50% chances, the expectation value is

$$\langle \hat{A} \rangle = P_1\lambda_1 + P_2\lambda_2 = \frac{1}{2}\frac{(3+\sqrt{5})}{2} + \frac{1}{2}\frac{(3-\sqrt{5})}{2} = \frac{3}{2}$$

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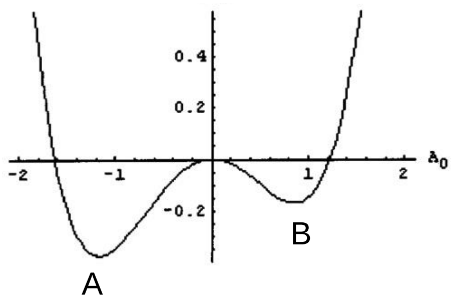
Case Problem 1.6

In the past 20 year researchers have learned to trap single atoms potentials made of laser light. The atoms can feel a potential like the one drawn here

Potential Energy



double well potential

Given the difference in the two wells, there is an energy difference 2Δ between the two wells, and the atom may also be able to tunnel from one well to the other more easily if the barrier between the two wells is smaller (ability to tunnel is parametrized by J).

In these conditions the energy of the atom can be given by

$$\hat{H} = \begin{pmatrix} -\Delta & J \\ J & \Delta \end{pmatrix}$$

1) Compute the value of the lowest energy possible of the atom as a function of J.

2) One can also measure whether the atom is in the left or in the right well, which is determined by the observable $\hat{Z} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

What is the average position of the lowest energy state in the limit $\Delta/J \to \infty$?

Case Problem 1.6: solution

1) The possible energies are given by the eigenvalues of $\widehat{H} = \begin{pmatrix} -\Delta & J \\ J & \Delta \end{pmatrix}$

Hence we get $\det\left[\begin{pmatrix} -\Delta & J \\ J & \Delta \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] = \det\left[\begin{pmatrix} -\Delta - \lambda & J \\ J & \Delta - \lambda \end{pmatrix}\right] = -J^2 - \Delta^2 + \lambda^2 = 0$ which gives

$$\lambda_{1,2} = \pm\sqrt{\Delta^2 + J^2}$$

which means that the lowest energy state has energy $\lambda_1 = -\sqrt{\Delta^2 + J^2}$

2) In the limit $\Delta/J \to \infty$ the energy operator is approximated by $\widehat{H} \approx \begin{pmatrix} -\Delta & 0 \\ 0 & \Delta \end{pmatrix} = \Delta\,\hat{Z}$

which means that the lowest energy state is $|\psi_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ which has average position

$$\langle\psi_1|\hat{A}|\psi_1\rangle = \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = -1$$

Measurement (SUMMARY)

Take home message:

1) Measurement outcomes take different discrete values O_i
2) The average outcome of a measurement, once repeated many times, is given by the expected outcome times its probability $\langle \hat{O} \rangle = O_V P_V + O_H P_H$
3) All observables can be represented by Hermitian matrices $\hat{O}$
4) The possible outcomes O_i are the eigenvalue of the matrix $\hat{O}$
5) The state of the system after the measurement $\hat{O}$ is the eigenvector corresponding to the measured outcome

So far, we have looked at polarization and atoms in double well potentials, but this analysis applies to all two-level systems and with simple extension also beyond them.

Demystifying notations

Some of you may feel a bit uneasy with the “new” notations. Here we would like to remind you that they are not that shocking:

$|\psi\rangle$ represent your information of the physical state and it is just a column vector $\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$

$\langle\psi|$ is the adjoint of $|\psi\rangle$, very useful to compute expectation values and build operators. It is simply $(\alpha^* \quad \beta^*)$

$\hat{O}$ is what you can observe and it stands for an Hermitian matrix $\begin{pmatrix}a & b\\ b^* & c\end{pmatrix}$

$\langle\hat{O}\rangle$ is the expectation value after many measurements and it just means $\langle\psi|\hat{O}|\psi\rangle$

And this is all the “new” notation☺

Easy peasy

FYI: Phases

We have written the quantum states as and we have mostly considered only cases in which they are real.

Two comments are long due

1) the states $|\psi_1\rangle = \begin{pmatrix}\alpha\\ \beta\end{pmatrix}$ and $|\psi_2\rangle = e^{i\varphi_g}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$ are indistinguishable from each other.

Any measurement would give the same result. This is because they are only different is a "so-called" global phase $\varphi_g$in the term $e^{i\varphi_g}$.

Example: $\langle\psi_2|\hat{A}|\psi_2\rangle = e^{-i\varphi_g}\cdot e^{i\varphi_g}\langle\psi_1|\hat{A}|\psi_1\rangle = \langle\psi_1|\hat{A}|\psi_1\rangle$

2) the states $|\psi_1\rangle = \begin{pmatrix}\alpha\\ \beta\end{pmatrix}$ and $|\psi_2\rangle = \begin{pmatrix}\alpha\\ e^{i\varphi_r}\beta\end{pmatrix}$ can be extremely different from each other, even orthogonal, depending on the values of α, β and φ_r. In this case φ is a relative phase between α and β.

Example: $|+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix}$ and $|-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -1\end{pmatrix}$ only differ because of a relative phase $\varphi_r = \pi$, but they are orthogonal.