10.018 Modelling Space and Systems

Lecture 1: Vectors and the Dot Product

Term 2, 2021

SINGAPORE UNIVERSITY OF TECHNOLOGY AND DESIGN

Syllabus and Course Structure

- Two course leads:
  - CHEONG Kang Hao (Multivariable Calculus: W1-W6)
  - Keegan KANG (Linear Algebra: W8-W14)
- All of you are signed up on Piazza already...just like in previous Math module
- Syllabus and assessment format on e-Dimension ; living document that will be updated continually
- Read the syllabus! :)

Course Communication / Material

- Course material downloadable from **e-Dimension**
- **Urgent communication** via email
  - Student to faculty: submit **Medical Certificate**, reasons for not submitting homework, missing an exam, etc
  - Faculty to student: follow up if you have not submitted homework / low attendance / etc (you do not want to receive an urgent email from us)
  - Faculty to math reps: follow up on cohorts (math reps will get a separate email with details)
- **All other communication:** via Piazza (configure settings if needed to see Piazza posts from faculty)

Why Piazza?

- Avenue for non-homework / non-exam questions (discuss how material is useful? potential real world applications using the content? UROPs? etc
- Can seek clarification for any ambiguity (notes, lectures, cohorts, etc) and get an answer
- Can refer to **previous semester's** clarifications. (some concepts taught this term could be very useful for your future module. can check back easily if required)

Progressive Learning Sessions

Some students will receive an email inviting them to the **Progressive Learning Sessions**. Venue and timings are indicated in the email. Progressive Learning Sessions start running from Week 1.

You **should not** take it as a sign that you are in a difficult situation. We are trying to make sure that you can catch up with the pace of the class. Attendance for this group of students is compulsory.

Consulting faculty during office hours / after class

- Feel free to look for any of your 2 instructors if you need help. There should be a fixed office hours.

Bootcamp: A Friendly Reminder :-)

You will be issued a grade at the end of the module (unlike previous Math module). Students with a D or D+ grade will have to attend the Bootcamp (from 30 Aug to 10 Sep).The attendance of the Bootcamp is **compulsory**, attending less than 80% of the Bootcamp will result in an immediate failure of the Bootcamp. **Do not plan for any activities** during the Bootcamp period if you are in the danger zone of failing the course based on your performance during the term (consult your instructors if you are not sure).

Students with an F grade can take the bootcamp exam as well, but the final passing grade will consist of the non-exam part and the bootcamp exam part.

Those who failed the bootcamp exam **cannot proceed into their Pillar year!** They would need to retake the class next year.

The Difference Between Math I and Math II

$$\underset{\text{(point on line)}}{x} \longrightarrow \boxed{f(x)} \longrightarrow \underset{\text{(scalar)}}{y}$$

$$\underset{\text{(point in space)}}{\vec{x}} \longrightarrow \boxed{f(\vec{x})} \longrightarrow \underset{\text{(scalar)}}{y}$$

Want To Find

- Graph
- Gradient at a point
- Concavity and convexity
- Limit at a point
- Continuity at a point
- (Global) min / max
- Linear approximations
- Taylor approximations
- Exact Integral
- Approximate Integral
- Line Integrals

A Multivariable World

Comments

- Understanding Math 1 **concepts** makes the jump to Math 2 easier
- But cannot just use / modify Math 1 “formulae” without due consideration - must ensure the “one dimensional” change to “two dimensional” change makes sense
- Good news: Many **concepts** learnt for “two dimensions” in Math 2 are exactly the same needed for “n dimensions” (more realistic scenario)
- Every cohort and subsequent lecture will have a “Before we start” slide that will mention the corresponding 1D analogue. **Please review the corresponding concepts from Math I before attending lecture / cohort**.
- No time to cover all 2D analogues (Math I is 12 weeks, multivariable calculus is 6 weeks, but will learn tools that allow you to derive 2D analogues on your own / understand derivation in textbooks)

Vectors

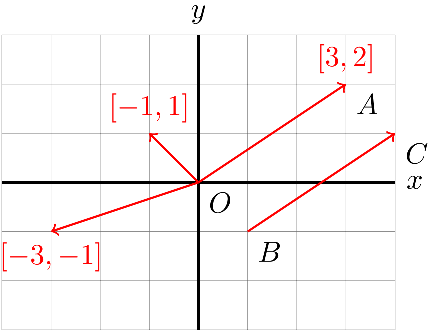
In $\mathbb{R}^2$ and $\mathbb{R}^3$, a *vector* $\overrightarrow{AB}$ is a directed line segment that corresponds to a displacement from point A to point B.

You have seen vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$ in physics already; graphically, you can think of them as arrows.

It is natural to represent vectors using coordinates. For example, if O denotes the origin and $A = (3, 2)$, then **the position vector** $\overrightarrow{OA} = [3, 2]$.

We will encounter more abstract vectors soon. Although they are not physical arrows with magnitude and direction, yet they can still be represented by coordinates.

Vectors in $\mathbb{R}^2$



The **position vector** from the origin to $A = (3, 2)$ is denoted by $\vec{a} = \overrightarrow{OA} = [3, 2]$. Also, $\overrightarrow{OA} \cong \overrightarrow{BC}$.

Notation: *Coordinates* are given in round brackets, *vectors* are given in square brackets. Denote vectors by an arrow on top, e.g. $\vec{a}$.

Vector operations

Vectors can be multiplied by *scalars* coordinate-wise. In this course we generally take scalars to be the *real* numbers, though occasionally we also work with complex scalars (we will let you know such occasions).

Vectors can be added or subtracted coordinate-wise, that is, by adding/subtracting the coordinates.

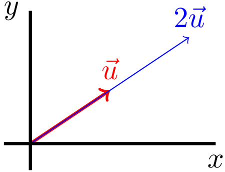
Vector subtraction: $\vec{w} - \vec{v} = \vec{w} + (-\vec{v})$.

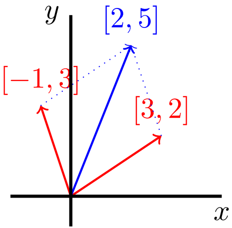
It is convenient to use *column vectors* as well as *row vectors*, as different representations of the same vector.

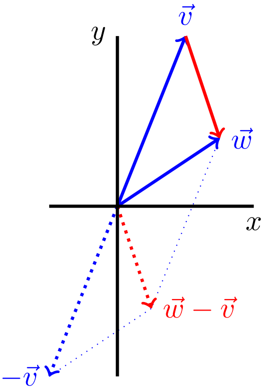
For example, another representation of $[3, 2]$ is $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

In future modules, depending on the context, there will be a difference in row and column representation of vectors.

Vector operations – examples







Test Yourself

Let $\vec{u} = [2, 1]$ and $\vec{v} = [4, -5]$.

Calculate

- $\vec{u} + \vec{v} =$ ________________
- $\vec{v} - \vec{u} =$ ________________
- $3\vec{u} + 2\vec{v} =$ ________________

Vectors in $\mathbb{R}^n$

Definition: we can represent a vector in $\mathbb{R}^n$ as an ordered set of n real numbers, written as a row or a column. Thus, a vector $\vec{v} \in \mathbb{R}^n$ is of the form

$$[v_1, v_2, \ldots, v_n] \quad \text{or} \quad \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

v_i is called the ith *component* of $\vec{v}$.

Vectors in $\mathbb{R}^n$

Why do we want vectors in n dimensions? Each point of 3-dimensional space may be regarded as a position vector relative to a chosen origin. But we can use vectors in a variety of other ways:

- (*Physics*) Hermann Minkowski used vectors with four components when he introduced spacetime coordinates (three for its position in space and one for time): $\vec{v} = [x, y, z, t]$.
- (*Economics*) A consumption vector, $\vec{q} = [q_1, q_2, \dots, q_n]$ shows the quantities $q_1, q_2, \dots, q_n$ consumed of each of n different goods.
- (*Computer Science*) A document vector, $\vec{w} = [1, 0, 0, 1, \dots, 1] \in \mathbb{R}^N$, where N is the number of words in the language, and i^{th} entry denotes whether the word appears in the document or not.

The dot product

Vectors in $\mathbb{R}^n$ can be added/subtracted coordinate-wise, but it is not obvious how to 'multiply' them (what does multiplication even mean here?).
One way of "multiplication" is the dot product.

Definition (algebraic): if $\vec{u} = [u_1, u_2, u_3]$ and $\vec{v} = [v_1, v_2, v_3]$, then the dot product $\vec{u} \cdot \vec{v}$ is given by

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3.$$

Definition (geometric):

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos\theta.$$

Where θ is the angle between *non-zero* vectors $\vec{u}$, $\vec{v}$, and $\|\vec{u}\|$, $\|\vec{v}\|$ denotes the *lengths*, or *norms* of vectors $\vec{u}, \vec{v}$ respectively.

Note that the **dot product is a scalar**, not a vector.

Length and angle

Definition: the length, or *norm* of a vector $\vec{v} = [v_1, v_2, v_3] \in \mathbb{R}^3$ is denoted by $\|\vec{v}\|$, and

$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2} = \sqrt{\vec{v} \cdot \vec{v}}.$$

In 2D and 3D, the first equality follows from Pythagoras' Theorem.

An important property of the dot product follows from this:
$\vec{u}$ and $\vec{v}$ are perpendicular iff (if and only if) $\vec{u} \cdot \vec{v} = 0$.

The dot product in $\mathbb{R}^n$

Definition: if $\vec{u} = [u_1, u_2, \ldots, u_n]$ and $\vec{v} = [v_1, v_2, \ldots, v_n]$, then the dot product $\vec{u} \cdot \vec{v}$ is given by

$$\vec{u} \cdot \vec{v} = \sum_{i=1}^{n} u_i v_i = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n.$$

Note that the **dot product is a scalar**, not a vector.

Reminder:

$$\sum_{i=a}^{b} f_i = f_a + f_{a+1} + \cdots + f_{b-1} + f_b.$$

The *length*, or *norm* (or a *magnitude*), of a vector $\vec{v} \in \mathbb{R}^n$ is denoted by $\|\vec{v}\|$, and

$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2} = \sqrt{\vec{v} \cdot \vec{v}}.$$

Test Yourself

Let $\vec{u} = [2, 1]$ and $\vec{v} = [4, -5]$.

Calculate

- $\vec{u} \cdot \vec{v} =$ ____________
- $\vec{v} \cdot \vec{u} =$ ____________
- $\|\vec{u}\| =$ ____________ and $\|\vec{v}\| =$ ____________.
- The angle between $\vec{u}$ and $\vec{v} =$ ____________.
- Let $\vec{w} = [-2, 1, 0]$. $\vec{u} + \vec{w} =$ ____________ and $\vec{u} \cdot \vec{w} =$ ____________.

Properties of the Dot Product

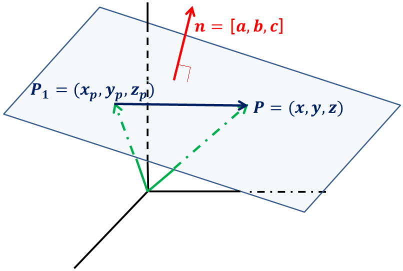
For any vectors $\vec{u}, \vec{v}, \vec{w}$ and any scalar c:

1. $\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$.
2. $(c\,\vec{v}) \cdot \vec{w} = c\,(\vec{v} \cdot \vec{w}) = \vec{v} \cdot (c\,\vec{w})$.
3. $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$.

Equation for a plane

A plane in $\mathbb{R}^3$ can be determined using 2 pieces of information: a point on the plane, and a vector perpendicular to the plane (called a *normal* vector).

Denote a normal vector by $\vec{n} = [a, b, c]$, a particular (**fixed**) point on the plane by $P_1 = (x_p, y_p, z_p)$, and an arbitrary (**variable**) point on the plane by $P = (x, y, z)$.



Equation for a plane – continued

We have the vector relation:

$$\begin{aligned} \vec{n} \cdot \overrightarrow{P_1 P} &= 0 \\ a(x - x_p) + b(y - y_p) + c(z - z_p) &= 0 \\ ax + by + cz &= d \end{aligned}$$

where $d = ax_p + by_p + cz_p$ is a constant. (Note, d is equal to the dot product of the normal vector and a vector $\overrightarrow{OP_1}$.)

Hence a general plane can be described by the equation

$ax + by + cz = d$, or $\vec{n} \cdot \vec{x} = d$ where $\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

Equation for a plane – continued

There is another way of understanding the equation for a plane. Consider equation of the line $y = m\,x + a$. The coefficient m corresponds to a slope of the line, i.e. $m = \Delta y / \Delta x$.

We can do similarly for the plane. If a plane has slope m in the x-direction (i.e. $\frac{\Delta z}{\Delta x} = m$), has slope n in the y-direction (i.e. $\frac{\Delta z}{\Delta y} = n$), and passes through the point (x_0, y_0, z_0), then its equation is

$$z = z_0 + m\,(x - x_0) + n\,(y - y_0).$$

Cross product

In $\mathbb{R}^3$, there is another useful vector product, the cross product.

Definition: the *cross product* of $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ is

$$\vec{u} \times \vec{v} = \begin{bmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{bmatrix}$$

Note that the cross product is a *vector*.

Important: cross product is defined only for vectors in $\mathbb{R}^3$!

Cross product, properties

You should check that $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$, and

$$(\vec{u} \times \vec{v}) \cdot \vec{u} = (\vec{u} \times \vec{v}) \cdot \vec{v} = 0.$$

Hence, $\vec{u} \times \vec{v}$ is perpendicular to both $\vec{u}$ and $\vec{v}$.

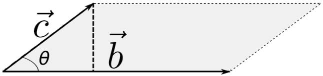
It can be shown that the cross product points in the direction given by the *right hand rule*: as your right hand wraps around from $\vec{u}$ to $\vec{v}$, your thumb points in the direction of $\vec{u} \times \vec{v}$.

You can also check the identity $\|\vec{u} \times \vec{v}\|^2 = \|\vec{u}\|^2\|\vec{v}\|^2 - (\vec{u} \cdot \vec{v})^2$. Since $\vec{u} \cdot \vec{v} = \|\vec{u}\|\|\vec{v}\| \cos\theta$, it follows that the norm of the cross product

$$\|\vec{u} \times \vec{v}\| = \|\vec{u}\|\|\vec{v}\| \sin\theta.$$

Cross product and parallelogram

Consider a parallelogram with edges given by the vectors $\vec{b}$ and $\vec{c}$.



Its area is given by

$$\text{Area} = \text{(length of base)}\ \text{(height)} = \|\vec{b}\|\|\vec{c}\|\sin\theta$$

Observe that's exactly the norm of the cross product of $\vec{b}$ and $\vec{c}$.

Properties of the Cross product

For vectors $\vec{u}, \vec{v}, \vec{w}$ and scalar c:

1. $\vec{w} \times \vec{v} = -\vec{v} \times \vec{w}$.
2. $(c\,\vec{v}) \times \vec{w} = c\,(\vec{v} \times \vec{w}) = \vec{v} \times (c\,\vec{w})$.
3. $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$.

Summary

We have covered:

- Vectors in $\mathbb{R}^n$, their geometric interpretations, addition and scalar multiplication.
- The dot product and its geometric interpretation in terms of length and angle.
- The equation for a plane in 3D.
- Cross product of vectors in 3D.