10.018: Modelling Space and Systems

Cohort 3.1

Application of the Extreme Value Theorem to optimisation

Term 2, 2021

SINGAPORE UNIVERSITY OF TECHNOLOGY AND DESIGN

Before we start....

To get the most out of this cohort, you should already be familiar with

1. Minima & Maxima (Math I)
2. Finding global optima (Math I)

as we will be going through

1. Optimization in Math II: The 2D analogue of optimization

Go global!

Solving an optimization problem typically requires finding a global optimum (either global min or global max):

- Maximize profit.
- Minimize costs.
- Maximize production etc.

However, the sufficient conditions for min/max from previous week give us only local min/max.

We will learn a few tools to determine if any of the local min/max found is global.

Constrained Optimization

Suppose we have a constrained maximization problem:

$$\max_{(x_1,\ldots,x_n)\in\mathbb{R}^n} f(x_1,\ldots,x_n), \quad \text{subject to} \quad \begin{aligned} g_1(x_1,\ldots,x_n) &= 0, \\ &\vdots \\ g_s(x_1,\ldots,x_n) &= 0, \\ h_1(x_1,\ldots,x_n) &\geq 0, \\ &\vdots \\ \text{and } h_t(x_1,\ldots,x_n) &\geq 0 \end{aligned}$$

Here $g_1(x_1,\ldots,x_n) = 0, \ldots, g_s(x_1,\ldots,x_n) = 0$ are the equality constraints and $h_1(x_1,\ldots,x_n) \geq 0, \ldots, h_t(x_1,\ldots,x_n) \geq 0$ are the inequality constraints that define the feasible region.

Bounded region

We will need the following definitions to describe the region:

Bounded region

A region in $\mathbb{R}^n$ is *bounded* if it is contained in some ball.

Examples:

- The square $-1 \leq x, y \leq 1$ in $\mathbb{R}^2$ is bounded.
- The first quadrant $x, y \geq 0$ of $\mathbb{R}^2$ is unbounded.
- The disk $x^2 + y^2 < 1$ in $\mathbb{R}^2$ is bounded.
- Suppose one of a, b, c is nonzero. The plane $ax + by + cz = 1$ in $\mathbb{R}^3$ is unbounded.

Open and Closed

Open and Closed regions

- A region is **closed** if it contains its boundary.
- An region is **open** if it does not contain any part of its boundary.
- $\mathbb{R}^n$ and empty set $\emptyset$ are both open and closed **by definition**.

Remarks:

- If constraints have only $\leq$, $\geq$, $=$ then it is a **closed** region.
- If constraints have only $<$ or $>$ it is open.

Open and Closed: Examples

Examples:

- The square $-1 \leq x \leq 1, -1 \leq y \leq 1$ in $\mathbb{R}^2$ is closed.
- The first quadrant $x \geq 0, y \geq 0$ of $\mathbb{R}^2$ is closed.
- The disk $x^2 + y^2 < 1$ in $\mathbb{R}^2$ is open.
- $(0, 1)$ and $(0, \infty)$ in $\mathbb{R}$ are open.
- $[0, 1]$ and $[0, \infty)$ in $\mathbb{R}$ are closed,
- $(0, 1]$ and $[0, 1)$ in $\mathbb{R}$ are neither open nor closed.

Language Warning! Note, 'bounded' and 'containing the boundary' (i.e. closed) are completely different concepts!

Remark:

- There are closed regions that is not bounded, for example $[0, \infty)$ in $\mathbb{R}$.
- There are bounded regions that are not closed, for example $(0, 1)$ in $\mathbb{R}$.

Activity 1 (15 minutes)

For each of these sets, discuss with your neighbors what the set looks like (from a geometric perspective). Tell if the set is **open, closed, or neither**; if it is **bounded or unbounded**; and justify your choices. Draw the regions whenever you can.

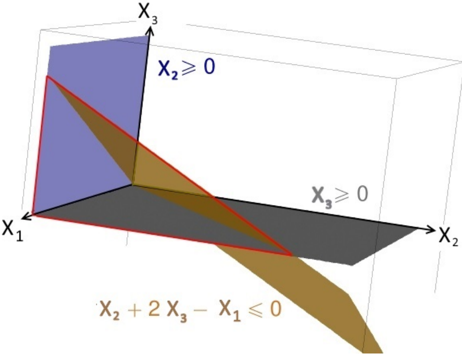
1. $\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$.
2. $\{x \in \mathbb{R} : x \geq 0\}$.
3. $\{(x,y) \in \mathbb{R}^2 : x + y \leq 1, x \geq 0, y \geq 0\}$.
4. $\{(x_1, \ldots, x_n) \in \mathbb{R}^n : 0 \leq x_i < 1 \text{ for all } i = 1, \ldots, n\}$.
5. $\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_2 + 2x_3 - x_1 \leq 0, x_1 \geq 0, x_2 \geq 0, x_3 \geq 0\}$.

Activity 1 (solution)

1. $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$. This is a disk without its boundary. Open, bounded.
2. $\{x \in \mathbb{R} : x \geq 0\}$. This is a half-line, including its starting point. Closed, unbounded.
3. $\{(x, y) \in \mathbb{R}^2 : x + y \leq 1, x \geq 0, y \geq 0\}$. This is a triangle including the boundary. Closed, bounded.
4. $\{(x_1, \ldots, x_n) \in \mathbb{R}^n : 0 \leq x_i < 1$ for all $i = 1, \ldots, n\}$. This is a hypercube that includes only part of its faces. Neither open nor closed, bounded.

Activity 1 (solution)

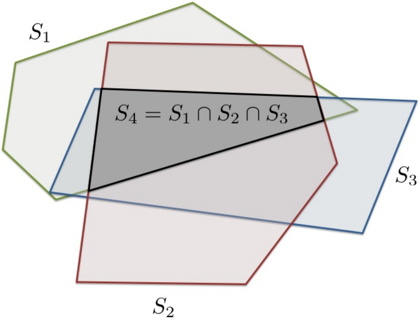
5. Closed, unbounded.



Intersection of closed sets

Theorem

The intersection of any number of closed sets is closed.



Extreme Value Theorem

It turns out that for special regions we are always guaranteed to find a global max and a global min. In particular,

Extreme Value Theorem for Multivariable Functions

If $f(x, y)$ is a continuous function on a **closed and bounded** region R, then $f(x, y)$ has a global maximum at some point (x_0, y_0) in R and a global minimum at some point (x_1, y_1) in R.

Note, EVT doesn't tell us where exactly are the global optima or how to find them. It only tells us that they exist.

Activity 2 (20 minutes)

Find all local and global maxima and minima of the following function:

$h(x, y) = x^2 + xy$, where the domain is $[-10, 10] \times [-10, 10]$.

Activity 2 (solution, continued)

$\dfrac{\partial h}{\partial x} = 2x + y$, $\dfrac{\partial h}{\partial y} = x$, so $\nabla h = \vec{0}$ at $(0,0)$.

$h_{xx}(0,0) = 2, h_{xy}(0,0) = h_{yx}(0,0) = 1, h_{yy}(0,0) = 0$,
$D = -1 < 0$, so $(0,0)$ is a saddle point.

To find the maxima and minima of h, we need to check the *boundary* of the domain $[-10,10] \times [-10,10]$.

The boundary is made up of four line segments:

$$\begin{aligned}
x = -10: &\quad h(-10,y) = 100 - 10y. \\
x = 10: &\quad h(10,y) = 100 + 10y. \\
y = -10: &\quad h(x,-10) = x^2 - 10x, \quad \text{local minimum at } x = 5. \\
y = 10: &\quad h(x,10) = x^2 + 10x, \quad \text{local minimum at } x = -5.
\end{aligned}$$

Activity 2 (solution, continued)

For the functions $h(\pm 10, y)$ and $h(x, \pm 10)$, we also need to check the *boundary* of their domain (y or $x \in [-10, 10]$), which are the corners of the square $[-10, 10] \times [-10, 10]$:

$$h(-10,-10) = 200, \quad h(-10,10) = 0, \quad h(10,-10) = 0, \quad h(10,10) = 200.$$

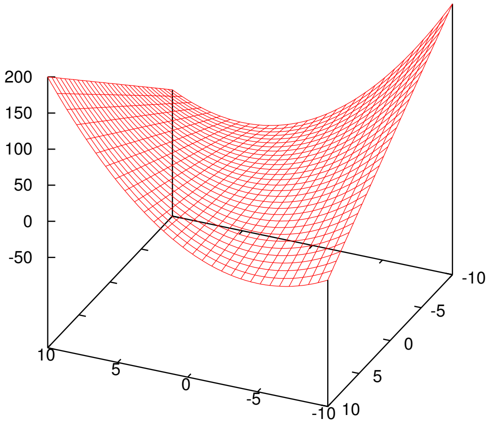
Also, for $h(x, \pm 10)$, the values at the critical points are

$$h(5,-10) = -25, \quad h(-5,10) = -25.$$

Since h is continuous and the feasible region is closed and bounded, by the EVT, the maxima and minima must be among the above values.

Comparing the values, we see that h has global maxima at $(-10,-10)$ and $(10,10)$, and global minima at $(5,-10)$ and $(-5,10)$.

Activity 2 (solution, continued)



What happens at $(-10, 10)$ and $(10, -10)$?

Activity 2 (solution, continued)

Remark: Proof that $(-10, 10)$ is not a local extremum. We want to show that in any neighbourhood of $(-10, 10)$: $h(-10, 10)$ is neither the smallest nor the largest. That is, if we can show that $h(a, b) < h(-10, 10) < h(c, d)$, for some (a, b) and (c, d) in any neighbourhood of $(-10, 10)$, then we are done.

For any $0 < \varepsilon \ll 1$, and rewriting $h(x, y) = x(x + y)$.

$$h(-10+\varepsilon, 10) = (-10+\varepsilon)\varepsilon \quad \text{[this is negative]} < 0 = h(-10, 10),$$
$$h(-10, 10-\varepsilon) = (-10)(-\varepsilon) \quad \text{[this is positive]} > 0 = h(-10, 10).$$

Hence, $h(-10, 10)$ is neither the smallest nor the largest. The same can be done for $(10, -10)$.

Application of EVT: Activity 3 (10 min)

How is EVT useful in solving optimization problems?

Assume the region R is **closed** and **bounded**.
Discuss with your classmates what does EVT tell you about the location of **global minima and maxima** in the following situations:

a. A continuous function f doesn't have any critical points in the interior of region R.
b. A continuous function f has only one critical point in the interior of R and it's a local minimum.
c. Function f is not continuous in the region R.
d. A continuous function f has only two critical points in the interior of R and it's a local minimum and a local maximum.

Illustrate each situation with a sketch of a function of one variable.

Application of EVT: Activity 3 (solution)

(a) *There are no critical points of f in the interior of region R.* Since region R is closed and bounded, global maximum and global minimum are achieved **on the boundary**. The highest/lowest value on the boundary will be automatically global max/global min on R by EVT.

Application of EVT: Activity 3 (solution)

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(c) We cannot use EVT in this case. We may still have min/max in the region R.

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b) *Function f has only one critical point in the interior of R and it's a local minimum.* It means that global max will be on the boundary. Global min could be at the interior point that you found or on the boundary as well.

c) We cannot use EVT in this case. We may still have min/max in the region R.

d) *Function f has only two critical points in the interior of R and it's a local minimum and a local maximum.* We may still have higher values on the boundary. Look for maximum on the boundary, if the values on the boundary are lower than at the critical point in the interior of R, then the local max you have found will be a global max. Similarly for min.

Math Modeling

- Defining the Problem Statement.
- Making Assumptions. Defining Variables.
- Getting a Solution.
- Analysis and Model Assessment.
- **Reporting the Results.**

Reporting the Results

A technical report typically starts with a summary page, also called an **Executive Summary**. It is at most one page long.
It is a place to *summarize* the problem solution and to provide a brief description of the results.

The Executive Summary should **restate the problem, state the assumptions made, briefly describe the chosen solution methods, provide the final results and conclusions, and should discuss briefly strengths and weaknesses**.

Describe your results in complete sentences that can stand on its own. E.S. lets the reader know what to expect in the report but does not overwhelm them with too much detail.

Think of it as a summary page you give to your boss who gave you this task. He wants to know what you were solving, how you did it and what are the results. But he has the time to read only one page.

Activity 4 (15 mins)

Take a look at the provided executive summaries (see eDim) and order them from best to worst.

Read them carefully and see whether they satisfy the conditions in bold from the previous page. Which parts are missing? How would you change the summaries to make them better?

Activity 4 (Solution)

SAMPLE A: too much on the final result and not enough on the process, no assumptions.

SAMPLE B: too much technical detail, info-dump, too much on the process.

SAMPLE C: almost “ideal” content but in a mess, three large chunks, better paragraphs and organization is required.

Part 5 of Math Modeling TO BE SUBMITTED on PIAZZA. This will be the last 1% of your MM grade

You need to submit **1 page** (readable pics/screenshots) by uploading on Piazza in your own thread, by Mon 6pm, Feb 15th.

Format of the submission:

1. List of team members with their official full names and student IDs.
2. Executive Summary of your math modeling program solution (please type out...imagine this is for your boss to read). *It should restate the problem, state the assumptions made, briefly describe the chosen solution methods, and provide the final results and conclusions, should discuss briefly strengths and weaknesses.*

Self-reflection TO BE SUBMITTED via Google Form

This is an **individual work** to be submitted at
`https://forms.gle/D8QJk4F3BkJ6uyuAA`

As mathematical modelling is your newly found tool, we would like you to carry out a reflection and share your responses on the following questions.

1. Reflect on your learning process on the mathematical modelling. What is it about the mathematical modelling that you enjoy the most and why?
2. What is it about mathematical modelling that you are still unsure of?
3. Can you think of a few scenarios that you personally will like to apply mathematical modelling to?

Summary

We have covered:

- Finding local and global optima
- Math Modeling: Reporting the Results

Textbook (Hughes-Hallett, *Calculus Single and Multivariable*): read Section 15.2, then try some of Exercises 15.2.1–15.2.12. You may discuss them on Piazza.