Week 12 - Day 2

Secure Quantum Communication

Concept 1: State vector

Concept 2: Superposition

↓

Quantum computation (Week 13)

Reference:
Six Quantum Pieces: A First Course in Quantum Physics
Chapter 1 except 1.3

Is quantum mechanics important/relevant?

- Microelectronics functions thanks to our understanding of basic quantum mechanics -> phones, computers …
- GPS functions thanks to atomic clocks … again basic quantum mechanics
- Nuclear power is based on quantum mechanics … and photovoltaics and thermoelectric too … quantum mechanics can help solve energy problems
- New materials function following quantum mechanics: graphene, superconductors, semiconductors … with vast current and potential applications

- More recently, people have been developing technologies based on quantum mechanics for quantum telecommunication, quantum cryptography and quantum computing!

In short, the answer is a resounding **YES**!
And more so in the near future … so be ready!!

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The Decade of Quantum Computing Is Upon Us, IBM Exec Says

With business uses poised to accelerate, CIOs need to start tinkering with platforms, forming work groups and looking for problems the technology could address, says IBM Research's Dario Gil

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Technology

Israel Allocates $60 Million to Build First Quantum Computer

By Yaacov Benmeleh
March 3, 2021, 8:07 PM GMT+8

Friday, 5th March 2021
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4 March, 2021 - 15:40 By Tony Quested

Cambridge quantum computing pioneer breaks new ground

Forbes

Mar 3, 2021, 02:46pm EST | 1,740 views

The Bitcoin Boom And The Quantum Threat

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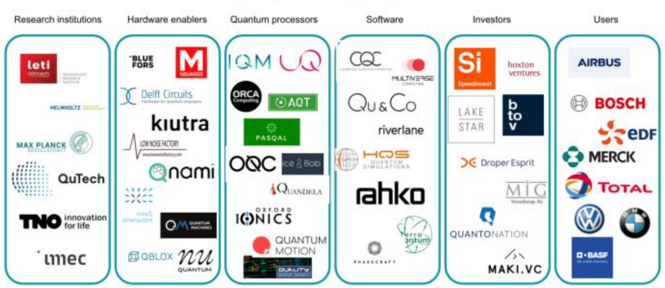
Here's how quantum computing could transform the future

European quantum computing stack



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Inside the race to keep secrets safe from the quantum computing revolution

Start-ups are taking part in a global competition to upgrade encryption to fend off the quantum computing threat

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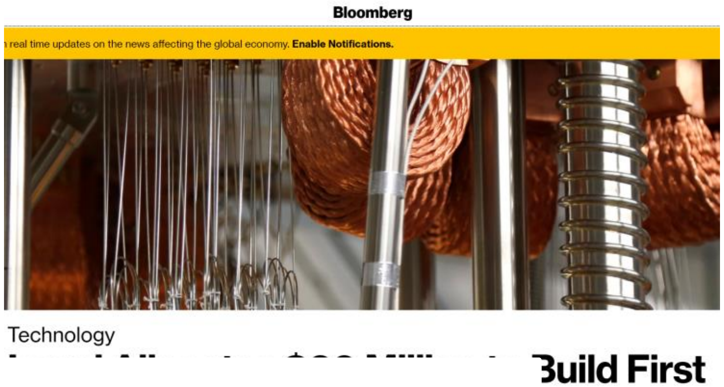
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Build First


Friday, 5th March 2021

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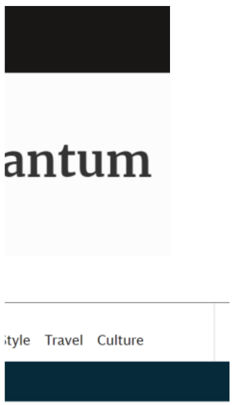
4 March, 2021 - 15:40 By Tony Quest

Cambridge quantun new ground

A huge industry is opening in front of our eyes.

Even if you do not work directly on quantum systems, there is a great need of **enabling technologies**: better optics, electronics, materials, algorithms, …

Be prepared to have an opportunity to seize them!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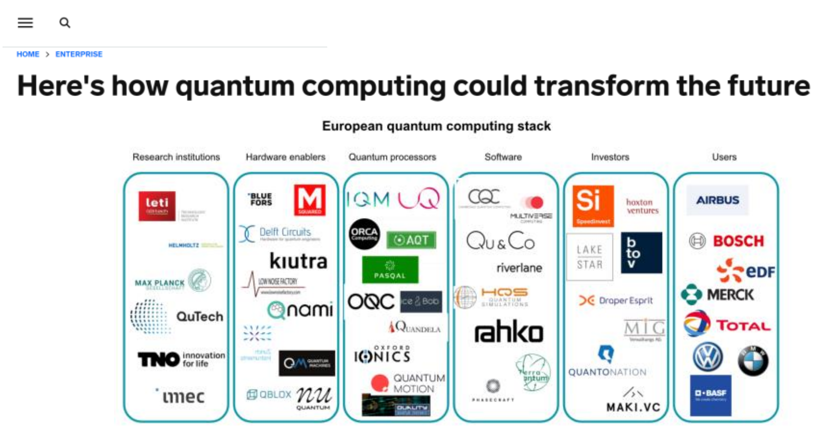
afe from the quantum computing revolution

Start-ups are taking part in a global competition to upgrade encryption to fend off the quantum computing threat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Here's how quantum computing could transform the future


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China realizes 4,600-km secure, stable quantum communication network

Xinhua

Secure quantum communication

All around the world, and in Singapore too, many companies are working on secure quantum telecommunication.

Companies in Europe

Spooky-1, Singapore-made prototypical satellite for quantum communication

https://www.hardwarezone.com.sg/tech-news-singapore-s-nanosatellite-spooqy-1-orbit-now-test-ultra-secure-quantum-communication-techn

Within the next few lessons we will learn why so much money, time and effort is dedicated to going beyond classical telecommunication.

FYI: for the scared and/or interested student

For a no-math introduction to quantum mechanics, to get a “feeling” about some concepts, see this wonderful series of 7 articles.

https://arstechnica.com/series/exploring-the-quantum-world/

EXPLORING THE QUANTUM WORLD / Ars Series

A curious observer’s guide to quantum mechanics, pt 7: The quantum century

Manipulating quantum devices has been like getting an intoxicating new superpower for society.

MIGUEL F. MORALES – 2/21/2021, 10:00 PM

A curious observer’s guide to quantum mechanics, Pt. 6: Two quantum spooks

Proof that the world can be much stranger than we expect.

MIGUEL MORALES – 2/14/2021, 10:00 PM

A curious observer’s guide to quantum mechanics, pt. 5: Catching a wave

When it comes to quantum mechanics, confinement is not necessarily bad.

MIGUEL MORALES – 2/7/2021, 10:00 PM

A curious observer’s guide to quantum mechanics, pt. 4: Looking at the stars

How do photons travel across light years? (Their quantum waviness enables modern telescopes.)

MIGUEL F. MORALES – 1/31/2021, 10:00 PM

A curious observer’s guide to quantum mechanics, pt. 3: Rose colored glasses

“How big is a particle?” Well, that’s a subtle (and, unsurprisingly, complex) question.

MIGUEL F. MORALES – 1/24/2021, 10:00 PM

A curious observer’s guide to quantum mechanics, pt. 2: The particle melting pot

In which lasers do things that make absolutely no sense but give us great clocks.

MIGUEL F. MORALES – 1/17/2021, 10:00 PM

A “no math” (but seven-part) guide to modern quantum mechanics

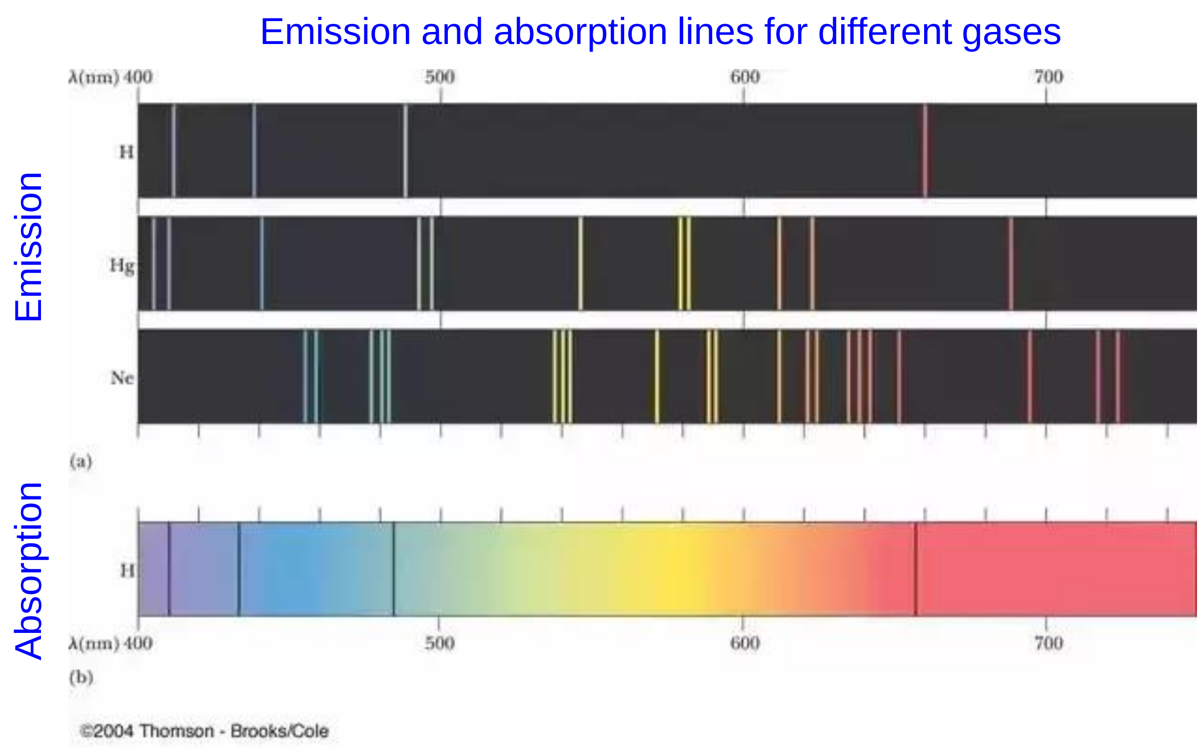
Welcome to "The curious observer’s guide to quantum mechanics"–featuring particle/wave duality.

MIGUEL F. MORALES – 1/10/2021, 10:00 PM

Concept 1: State vector

Preamble

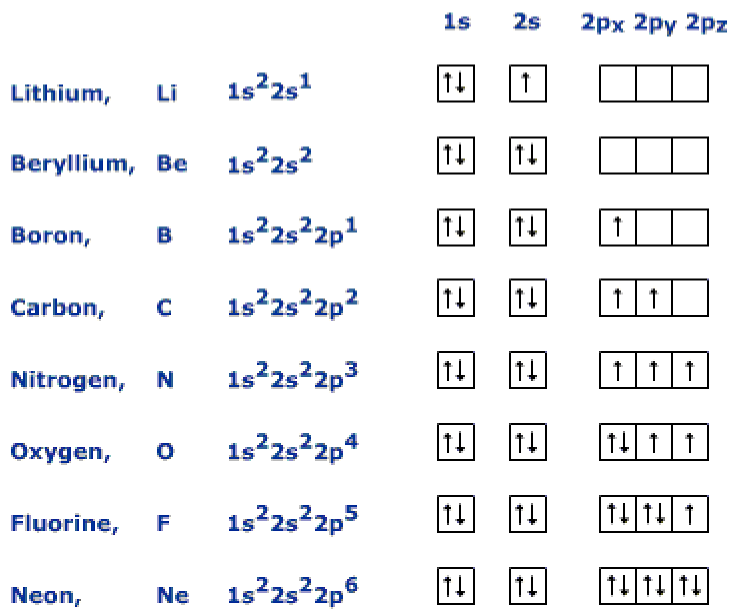
There are some aspects of quantum mechanics you may have encountered already which can give you big clues on how to work with it.



Atoms emit or absorb only certain frequencies of light.

Preamble

From chemistry you may also know that not only do electrons occupy different energy orbitals, but they appear in two different flavors called spins.



socratic.org

An electron is always measured either in a spin up or spin down configuration.

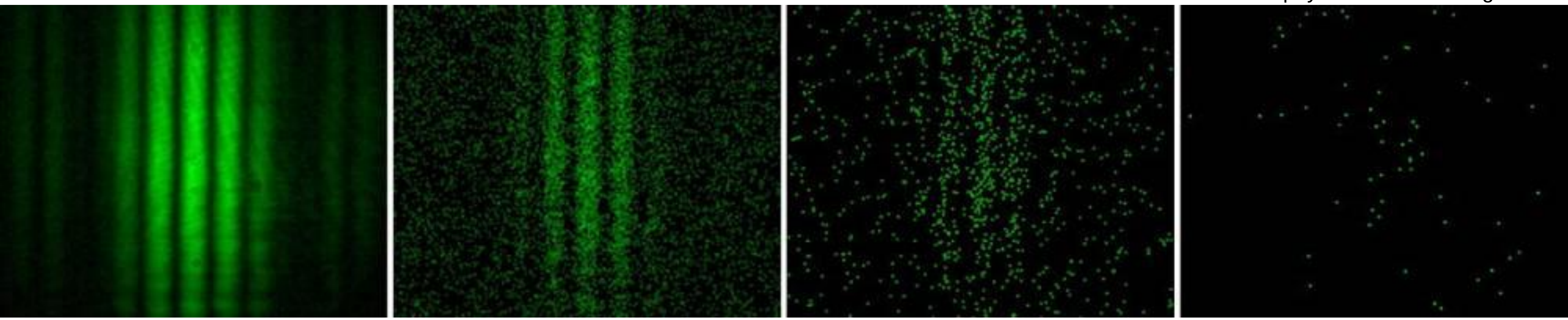
Preamble

Let us go back light which we encountered last week.

We have seen that light is a wave, and like sea waves, it can show ripples of high and low intensity.

First thing that we notice is that light, although it follows the wave equation we have recently discovered, is that light is made of particles called photons.

physics.stackexchange.com



Higher intensity light

(one can observe its wave-like properties)

Lower intensity light

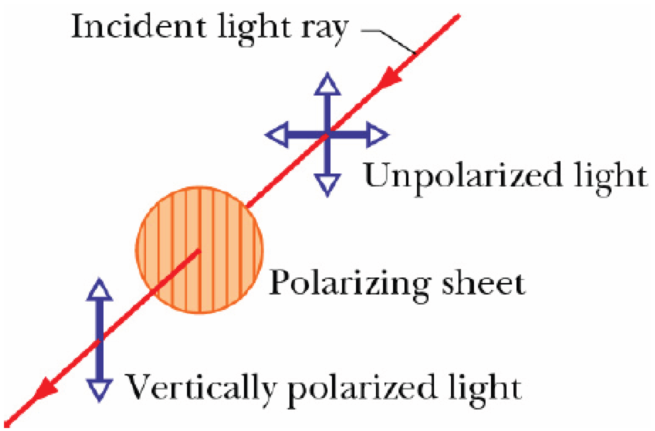
(one can observe that it is made of particles)

Fun fact: one of the way to build light sources which emit one single photon at the time was developed by our own SUTD Prof Joel Yang!

Preamble

Let us take this in, and let us go back also to light polarization.

If we take light from a typical source, and we shine it through a polarizing sheet then we get that about 50% of the light goes through the sheet, and what comes out is linearly polarized light.



Now think of this same experiment, but consider the light going through the sheet one photon at the time.

There are only 2 possible outcomes: either (i) the photon goes through and will be linearly polarized in the direction that is not absorbed, or (ii) it is absorbed.

Preamble

Whether we consider light, atoms or electrons, experiments tell us that anything can be described by a number of discrete different options

atoms emit light at certain frequencies (many, yet discrete, options)

electrons are measured to have spins up or down (2 options)

photons can be either absorbed or they go through (2 options)

A great deal of the main concepts of quantum mechanics can be well understood simply studying systems with just 2 options, which are referred to as **two-level systems**.

In the following we will mainly focus on these systems (the required mathematics for them is also the simplest) and we leave the extension to other systems to the more interested students☺

State vectors

While what we are going to do is valid for any two-level system, to make things more complete, in the following we will focus on single photons and their polarization.

We have learned that a photon can be polarized in the direction of absorption of the polarizing sheet, in the direction that let it go through, or in something in between … in the latter case, experiments show that the photon has a probability of going through.

It is thus natural to use a vector notation to describe the two level system, and the linear algebra that you are learning now will save your day☺

We thus describe the photon using the ket notation $|\psi\rangle$

- If the photon is horizontally polarized we write $|\psi\rangle = |H\rangle$
- If the photon is vertically polarized we write $|\psi\rangle = |V\rangle$
- If the photon is partially horizontally and partially vertically polarized, then we can use $|H\rangle$ and $|V\rangle$ as orthonormal basis and write $|\psi\rangle = \alpha|H\rangle + \beta|V\rangle$

where α and β are two scalar which we will discuss in more detail later.

State vectors

Relying on linear algebra, we can write that

$$|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

and the two basis vectors are

$$|V\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and

$$|H\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Indeed, $|\psi\rangle = \alpha|H\rangle + \beta|V\rangle = \alpha\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Given the vectorial nature of $|\psi\rangle$, it is often referred to as state vector.

Since $|\psi\rangle$ is use to described wave-like objects, for example as light, it is also referred to as wave-function.

State vectors

What is the meaning of the state vector $|\psi\rangle$? How can we interpret it?

Think back at the EM waves … they are described by the $\vec{E}$ and $\vec{B}$ fields, but their energy, carried by each of their photons, is given by the square of these fields $\left|\vec{E}\right|^2$ and $\left|\vec{B}\right|^2$.

It is thus fairly natural to think that the modulus square of the elements of $|\psi\rangle$ are related to some "intensity" of the object described.

State vectors

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It is thus fairly natural to think that the modulus square of the elements of $|\psi\rangle$ are related to some "intensity" of the object described.

Indeed, the state vector $|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ is a vector that represent all the required information to describe the physical system you consider, and it is such that

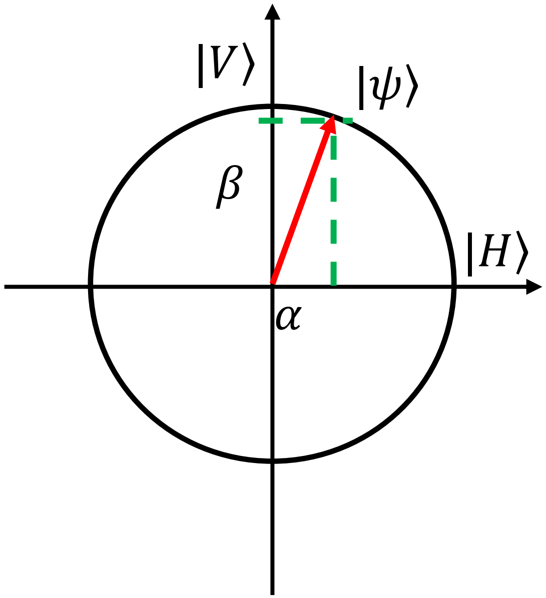
$|\alpha|^2$ →probability that the photon is horizontally polarized, e.g. in the state $|H\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$|\beta|^2$ →probability that the photon is vertically polarized, e.g. in the state $|V\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

It results that $|\alpha|^2 + |\beta|^2 = 1$ (the photon has a total probability 1) which is the only required condition for $|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ to be physically meaningful.

State vectors

If α and β are real, one can think of the state $|\psi\rangle$ as being a point on the circle of radius 1.



Actually α and β are generically complex, and hence you would need another axis to describe a generic two-level system. But we will not worry about that in this course.

Concept question 1.1

Which of these vectors can represent a quantum physical state?

A. $\begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$ B. $\begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$ C. $\begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix}$ D. $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ E. $\begin{pmatrix} \sqrt{2/3} \\ \sqrt{1/3} \end{pmatrix}$

A. Only A, C and D
B. Only B, C and D
C. Only C, D and E
D. Only B, D and E
E. Only A and B

Multiple Choice

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Concept question 1.1: solution

Which of these vectors can represent a quantum physical state?

A. $\begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$ B. $\begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$ C. $\begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix}$ D. $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ E. $\begin{pmatrix} \sqrt{2/3} \\ \sqrt{1/3} \end{pmatrix}$

A. Only A, C and D
B. Only B, C and D
C. Only C, D and E
D. Only B, D and E
E. Only A and B

Only for these states $|\alpha|^2 + |\beta|^2 = 1$

A. $|1/2|^2 + |1/2|^2 = 1/2$

B. $\left|1/\sqrt{2}\right|^2 + \left|-1/\sqrt{2}\right|^2 = 1$

C. $|2/3|^2 + |1/3|^2 = 5/9$

D. $|1|^2 + |0|^2 = 1$

E. $\left|\sqrt{2/3}\right|^2 + \left|\sqrt{1/3}\right|^2 = 1$

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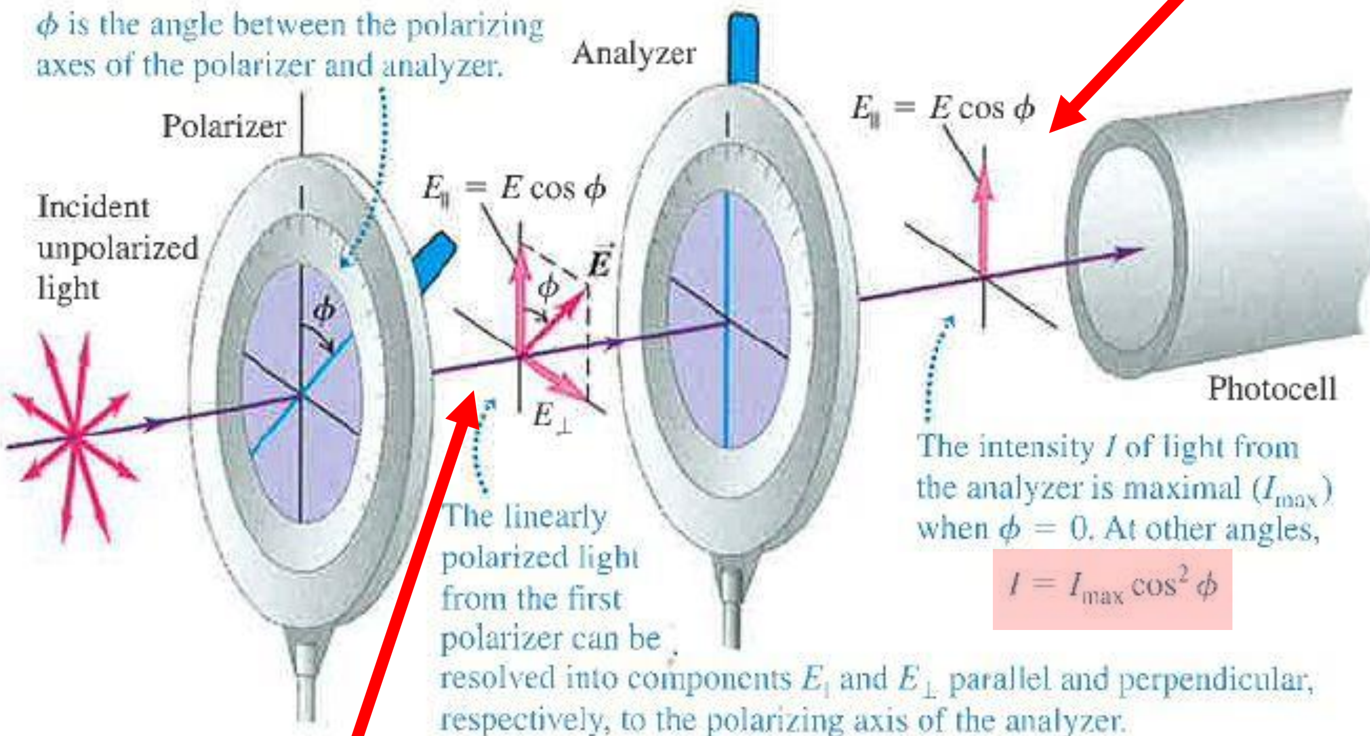
Concept 2: Superposition

Last week we have seen Malus law.

The intensity drops as $[\cos(\varphi)]^2$. It means that a photon goes through the analyzer with a probability $[\cos(\varphi)]^2$



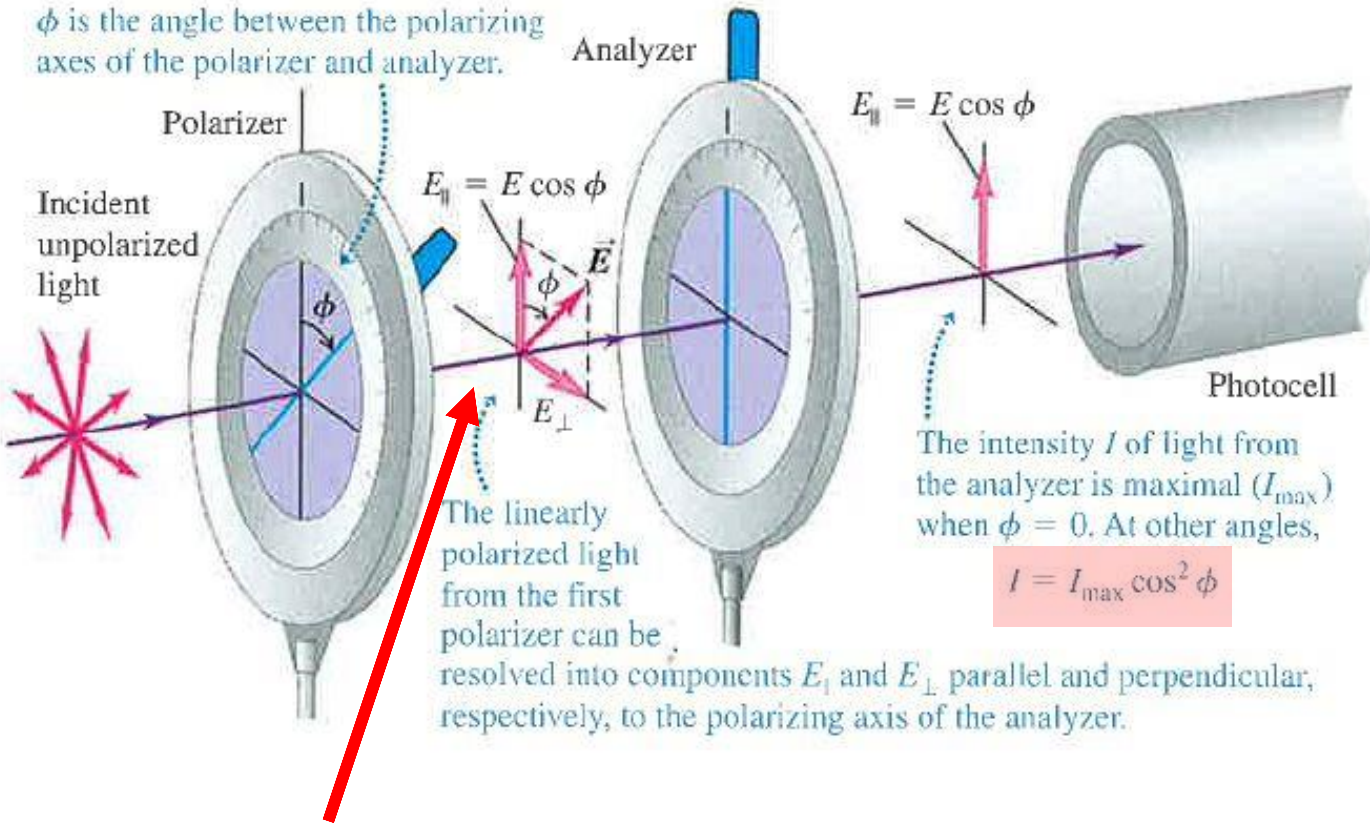
Photons are all polarized in a particular direction here.

Let us try to understand this using state vector representation of photons.

What happens if you send a single photon through a polarizer?

It cannot lose intensity depending on its angle, because it is just a single photon.

So what happens?

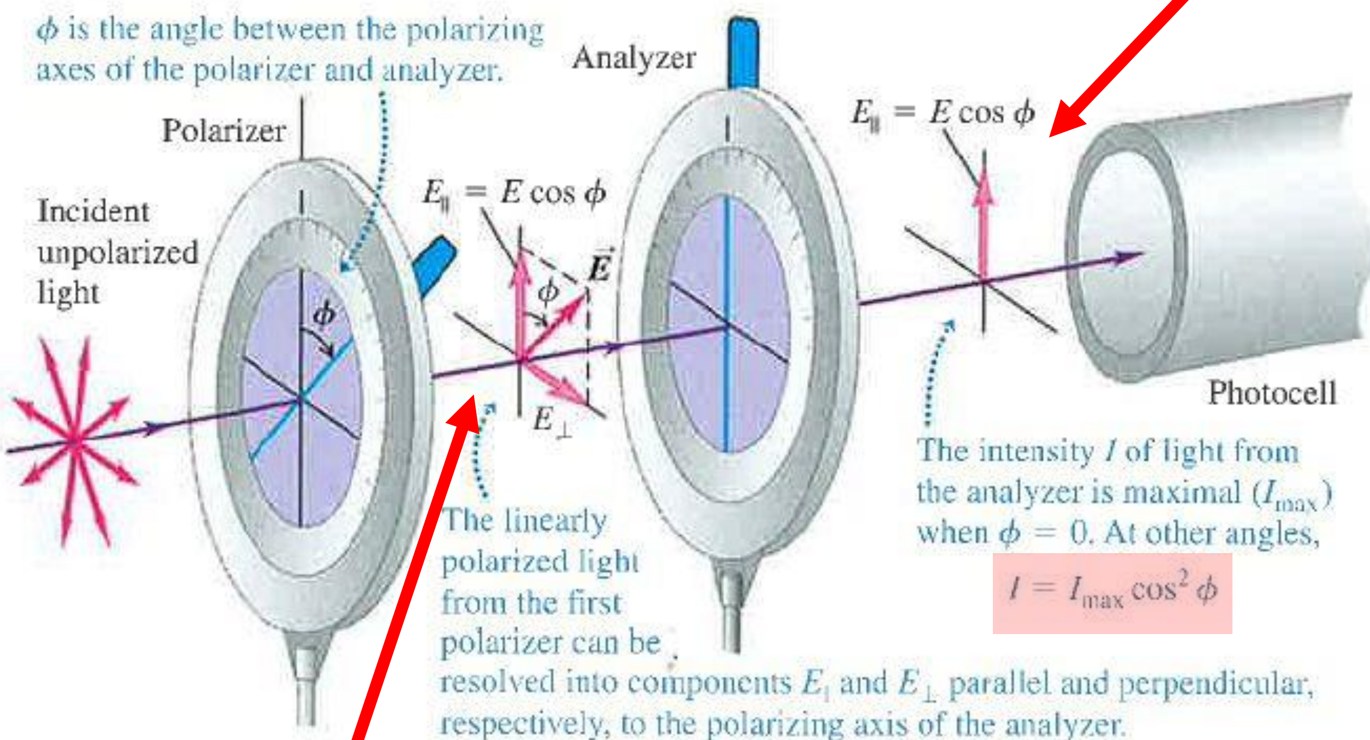


What happens if you send a **single photon** through a polarizer?

It cannot lose intensity depending on its angle, because it is just a single photon.

So what happens?

Here the emerging photon would be polarized in a direction rotated with respect to $|V\rangle$ and $|H\rangle$



Here a photon is described by the state vector $|\psi\rangle = |H\rangle = \begin{pmatrix}1\\0\end{pmatrix}$.

We need to learn to write state vectors in different, rotated, basis.

Superposition

Linear algebra would turn in use here again. Example:

The state $|H\rangle = \begin{pmatrix}1\\0\end{pmatrix}$ can be written as the sum of two vectors

$$\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1/2\\1/2\end{pmatrix} + \begin{pmatrix}1/2\\-1/2\end{pmatrix}$$

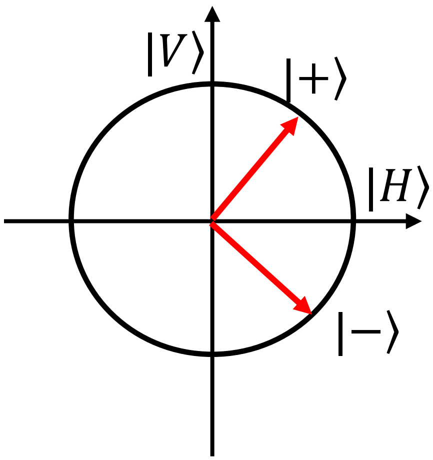
but the vectors $\begin{pmatrix}1/2\\1/2\end{pmatrix}$ and $\begin{pmatrix}1/2\\-1/2\end{pmatrix}$ are not state vectors because for them $|\alpha|^2 + |\beta|^2 \neq 1$.

But we can use the orthonormal basis $|+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$ and $|-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$ and write

$$\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}\right]$$

or in other words

$$|H\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$$



We refer to this procedure as to write $|H\rangle$ as a superposition of $|+\rangle$ and $|-\rangle$, and the fact that you can do this as the superposition principle.

Case Problem 1.1

Since $|H\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$ with $|+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$ and $|-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$ how would one write $|V\rangle$ as a superposition of state vectors $|+\rangle$ and $|-\rangle$?

Case Problem 1.1: solution

Since $|H\rangle = \frac{1}{\sqrt{2}}(|+\rangle \ + \ |-\rangle)$ with $|+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$ and $|-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$ how would one write $|V\rangle$ as a superposition of state vectors $|+\rangle$ and $|-\rangle$?

We can consider $|+\rangle - |-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} - \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\2\end{pmatrix} = \begin{pmatrix}0\\\sqrt{2}\end{pmatrix} = \sqrt{2}\begin{pmatrix}0\\1\end{pmatrix} = \sqrt{2}|V\rangle$

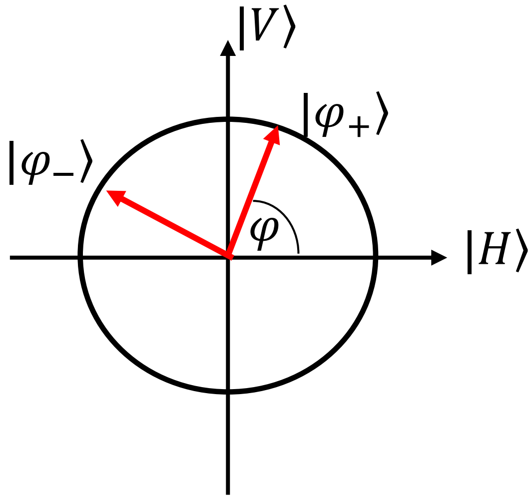
Hence $|V\rangle = \frac{1}{\sqrt{2}}(|+\rangle \ - \ |-\rangle)$

Superposition

Describing the polarization of photons at a different angle can be simply seen as a change of basis!

One can choose the basis states $|V\rangle$ and $|H\rangle$, or the basis states $|+\rangle$ and $|-\rangle$, or any other orthonormal basis, for instance

$$|\varphi_+\rangle = \begin{pmatrix}\cos(\varphi)\\ \sin(\varphi)\end{pmatrix}, \qquad |\varphi_-\rangle = \begin{pmatrix}-\sin(\varphi)\\ \cos(\varphi)\end{pmatrix}$$



The state $|\varphi_+\rangle = \cos(\varphi)|H\rangle + \sin(\varphi)\,|V\rangle$ and $|\varphi_-\rangle = -\sin(\varphi)|H\rangle + \cos(\varphi)\,|V\rangle$, are indeed physical because $[\cos(\varphi)]^2 + [\sin(\varphi)]^2 = 1$.

Furthermore they are orthogonal, in fact the scalar product

$$\begin{pmatrix}\cos(\varphi)\\ \sin(\varphi)\end{pmatrix} \cdot \begin{pmatrix}-\sin(\varphi)\\ \cos(\varphi)\end{pmatrix} = -\cos(\varphi)\sin(\varphi) + \sin(\varphi)\cos(\varphi) = 0,$$

so they form a good orthonormal basis.

Superposition

With this in mind we can go back to our first state vector for a two-level system $|\psi\rangle = \alpha|H\rangle + \beta|V\rangle$.

The probability of being in state $|H\rangle$ is $|\alpha|^2$.
Using linear algebra this can be written as the modulus square of the scalar product of the two state vectors $|\psi\rangle$ and $|H\rangle$.

Since the elements of the state vectors can be complex numbers, we alight here that the scalar product between two complex vectors $\vec{v}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ is given by

$$\vec{v}_1^{\dagger}\ \vec{v}_2 = (\vec{v}_1^{*})^T\ \vec{v}_2 = (x_1^*\ \ y_1^*)\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$$

where T stands for transposition, a^* means the complex conjugate of a, and $a^\dagger$ (which you read as "a dagger") means that you do both transposition and complex conjugation.

Superposition

We can thus introduce the bra notation $\langle\psi| = (\alpha^* \quad \beta^*)$ which is the transpose and complex conjugate of $|\psi\rangle = \begin{pmatrix}\alpha \\ \beta\end{pmatrix}$.

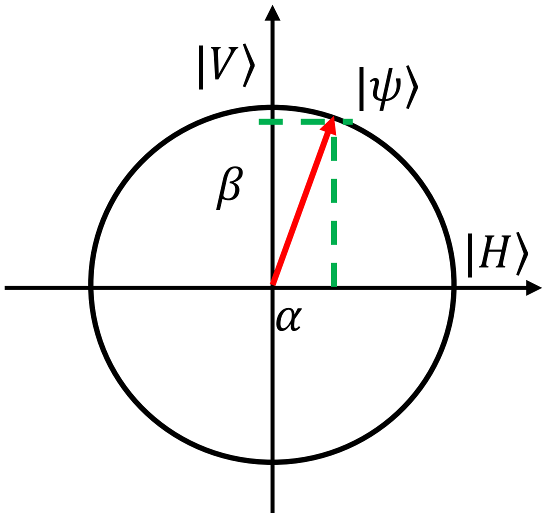
The scalar product between two state vectors $|\psi_1\rangle = \begin{pmatrix}\alpha_1 \\ \beta_1\end{pmatrix}$ and $|\psi_2\rangle = \begin{pmatrix}\alpha_2 \\ \beta_2\end{pmatrix}$ can be written as

$$\langle\psi_1|\psi_2\rangle = ({\alpha_1}^* \quad {\beta_1}^*)\begin{pmatrix}\alpha_2 \\ \beta_2\end{pmatrix}$$

The probability of state $|\psi_1\rangle$ to be in state $|\psi_2\rangle$ is given by the modulus square of the scalar product of the two state vectors $|\langle\psi_1|\psi_2\rangle|^2$

Let us do a concrete example:

Consider the $|\psi\rangle = \alpha|H\rangle + \beta|V\rangle$ for which we already know that the probability of being in the vertical polarization state $|V\rangle$ is $|\beta|^2$.



Let us use what we have just learned. The probability of $|\psi\rangle$ being in state $|H\rangle$ is given by

$$|\langle H|\psi\rangle|^2 = |(1 \quad 0)\begin{pmatrix}\alpha \\ \beta\end{pmatrix}|^2 = |\alpha|^2$$

Concept question 2.1

For the following state vector,

$$|\psi\rangle = \begin{pmatrix} 1/\sqrt{3} \\ \sqrt{2/3} \end{pmatrix}$$

what is the probability to be in state horizontally polarized $|H\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$?

A. $1/\sqrt{3}$ B. $1/3$

C. $\sqrt{2/3}$

D. $2/3$

Multiple Choice

Concept question 2.1: solution

For the following state vector,

$$|\psi\rangle = \begin{pmatrix} 1/\sqrt{3} \\ \sqrt{2/3} \end{pmatrix}$$

what is the probability to be in state horizontally polarized $|H\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$?

A. $1/\sqrt{3}$

B. $1/3$

C. $\sqrt{2/3}$

D. $2/3$

$$|\langle H|\psi\rangle|^2 = \left|\begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 1/\sqrt{3} \\ \sqrt{2/3} \end{pmatrix}\right|^2 = |1/\sqrt{3}|^2 = 1/3$$

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Concept question 2.2

For the following state vector,

$$|\psi\rangle = \begin{pmatrix} 1/\sqrt{3} \\ \sqrt{2/3} \end{pmatrix}$$

what is the probability to be in state $|+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$?

A. $1/2$
B. $1/3$
C. ≈ 0.97
D. ≈ 0.03

Multiple Choice

Concept question 2.2: solution

For the following state vector,

$$|\psi\rangle = \begin{pmatrix} 1/\sqrt{3} \\ \sqrt{2/3} \end{pmatrix}$$

what is the probability to be in state $|+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$?

A. $1/2$

B. $1/3$

C. $\approx \mathbf{0.97}$

D. ≈ 0.03

$$|\langle + | \psi \rangle|^2 = \left| \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{3} \\ \sqrt{2/3} \end{pmatrix} \right|^2 = \frac{1}{2} \left| \frac{1}{\sqrt{3}} + \sqrt{2/3} \right|^2 \approx 0.97$$

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Concept question 2.3

For the following state vector,

$$|\psi\rangle = \begin{pmatrix} 1/\sqrt{3} \\ \sqrt{2/3} \end{pmatrix}$$

what is the probability to be in state $|-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$?

A. $1/2$
B. $1/3$
C. ≈ 0.97
D. ≈ 0.03

Multiple Choice

Concept question 2.3: solution

For the following state vector,

$$|\psi\rangle = \begin{pmatrix} 1/\sqrt{3} \\ \sqrt{2/3} \end{pmatrix}$$

what is the probability to be in state $|-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$?

A. $1/2$

B. $1/3$

C. ≈ 0.97

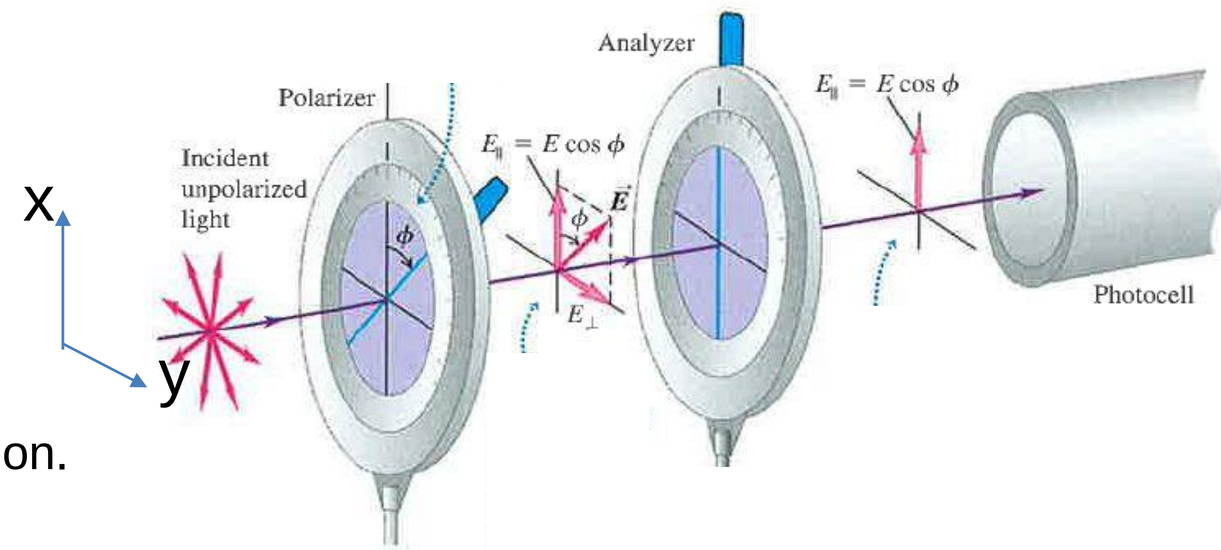
D. $\approx \mathbf{0.03}$

$$|\langle -|\psi\rangle|^2 = \left|\frac{1}{\sqrt{2}}(1 \quad -1)\begin{pmatrix} 1/\sqrt{3} \\ \sqrt{2/3} \end{pmatrix}\right|^2 = \frac{1}{2}\left|\frac{1}{\sqrt{3}} - \sqrt{2/3}\right|^2 \approx 0.03$$

Case Problem 2.1: Malus law

Considering a photon after the polarizer to be in the state $|\varphi_+\rangle = \begin{pmatrix} \cos(\varphi) \\ \sin(\varphi) \end{pmatrix}$ and after the analyser in the state $|H\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

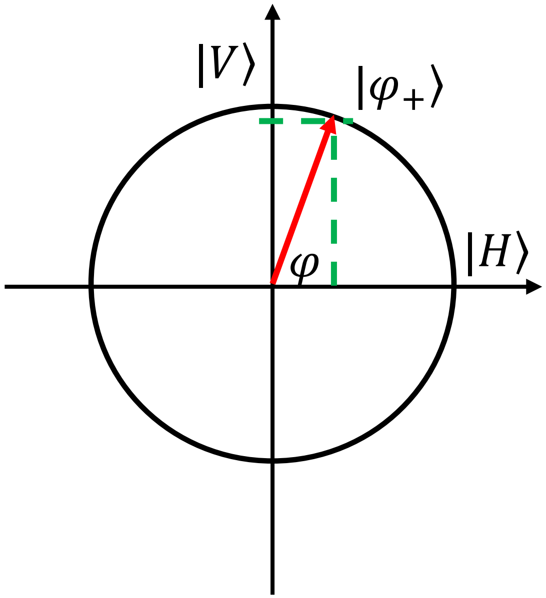
Take note of the coordinate that we define in this question.



1) Represent the states $|\varphi_+\rangle$ and $|H\rangle$ on a circle of radius 1.
2) What is the probability that a single photon goes through the analyzer?
3) Derive Malus law.

Case Problem 2.1: Malus law: Solution

1) Represent the states $|\varphi_+\rangle$ and $|H\rangle$ on a circle of radius 1.



2) The probability that a single photon goes through the analyzer is given by

$$|\langle H|\varphi_+\rangle|^2 = \left|\begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}\cos(\varphi)\\ \sin(\varphi)\end{pmatrix}\right|^2 = \cos^2(\varphi)$$

Case Problem 2.1: Malus law: Solution

3) Malus law's states that the intensity of light outside of the polarizer would be given by $I = I_0 \cos^2(\varphi)$ where I_0 is the initial intensity and I the final one. The intensity of light is proportional to the number of photons.

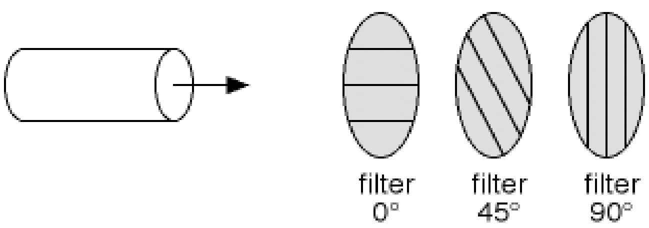
Since the probability that the photon goes through the analyzer is affected by a factor $\cos(\varphi)^2$
then the light intensity will be affected by the same factor, thus giving Malus's law

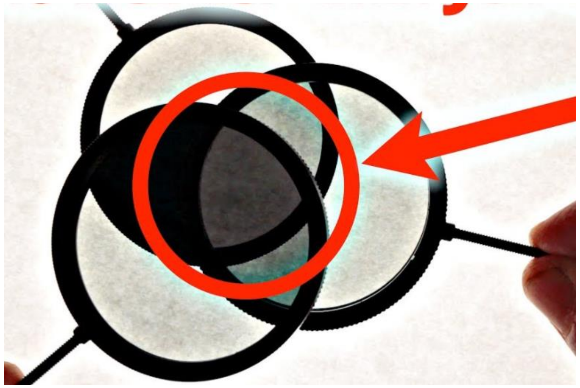
$$I = I_0 \cos^2(\varphi)$$

Case Problem 2.2: Two and three polarizers

Considering a photon after the polarizer to be in the state $|H\rangle = \begin{pmatrix}1\\0\end{pmatrix}$. The analyser is oriented with a relative angle $\varphi = 90^{\circ}$ with respect to the polarizer.

1) Show that the probability of a photon exiting the polarizer and then going through the analyzer is 0.





We now introduce an intermediate polarizer between the polarizer and the analyser, with an orientation at 45° from each of them.

After the intermediate polarizer the photon can only be in a state at 45°, and after the analyser it can only be in a state at 90° from the beginning.

2) Write what the state of the photon can be after each polarizer and what is the probability that the photon goes across all of them.

3) Compare the probability of a photon to go through the 3 polarizers with the case studied at point 1 with one less filter → isn't this amazing?

Case Problem 2.2: Two and three polarizing lenses: solution

1) At 90° with $|H\rangle = \begin{pmatrix}1\\0\end{pmatrix}$, the analyzer would let through photons oriented as $|V\rangle = \begin{pmatrix}0\\1\end{pmatrix}$. One thus gets $|\langle H|V\rangle|^2 = \left|\begin{pmatrix}0 & 1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\right|^2 = 0$

2) After the first polarizer the state would be proportional to $|H\rangle = \begin{pmatrix}1\\0\end{pmatrix}$, after the second proportional to $|+\rangle = \begin{pmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{pmatrix}$, and after the third it would be proportional to $|V\rangle = \begin{pmatrix}0\\1\end{pmatrix}$ (we write proportional because we could multiply any state $|\psi\rangle$ by a "global phase" $e^{i\varphi}|\psi\rangle$ … more on this on the last slide of the second day session).

3) The events of photon going through one polarizer or going though the other are independent. So the probability of going through them is the product of each individual probability. So one gets

$$P = |\langle H|+\rangle|^2 \; |\langle +|V\rangle|^2 = \left|\begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{pmatrix}\right|^2 \left|\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}\right|^2 = \left|1/\sqrt{2}\right|^2 \left|1/\sqrt{2}\right|^2$$
$$= 1/4$$

This is larger than what examined in case 1. Cool!

Case Problem 2.3: A large number of polarizers (Extra Problem)

Consider now the case of inserting $n-1$ polarizers between the first polarizer and the analyser, each equally tilted with respect to each other with an angle of $\pi/2n$.

1) What is the general expression, as a function of n, for the probability of the photon to go through all the n polarizers?
2) What is the limit for n going to infinity of this probability?

Isn't the result super cool?

Case Problem 2.3: A large number of polarizing lenses (Extra Problem): solution

Let us call the state coming out of a polarizer with angle $\frac{\pi}{2}\frac{m}{n}$ as

$$|\varphi_{m,n}\rangle = \begin{pmatrix} \cos\left(\frac{\pi}{2}\frac{m}{n}\right) \\ \sin\left(\frac{\pi}{2}\frac{m}{n}\right) \end{pmatrix}$$

This symbol means product for m going from 1 to n-1.

It is a more compact way to write what is written in blue

where $|\varphi_{0,n}\rangle = |H\rangle$ and $|\varphi_{n,n}\rangle = |V\rangle$. The overall probability is given by

$$P = \left|\langle\varphi_{0,n}|\varphi_{1,n}\rangle\right|^2 \left|\langle\varphi_{2,n}|\varphi_{3,n}\rangle\right|^2 \cdots \left|\langle\varphi_{n-1,n}|\varphi_{n,n}\rangle\right|^2 = \prod_{m=1}^{n} \left|\langle\varphi_{m-1,n}|\varphi_{m,n}\rangle\right|^2$$

$$= \prod_{m=1}^{n} |\cos(\pi(m-1)/2n)\cos(\pi\, m/2n) + \sin(\pi\,(m-1)/2n)\sin(\pi\, m/2n)|^2$$

Now we can use the identity $\cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta) = \cos(\alpha-\beta)$ so we get

$$P = \prod_{m=1}^{n} \left|\langle\varphi_{m-1,n}|\varphi_{m,n}\rangle\right|^2 = \prod_{m=1}^{n} |\cos(\pi/2n)|^2 = |\cos(\pi/2n)|^{2n}$$

In the limit of n going to infinity one gets $\cos(\pi/2n) \approx 1 - (\pi/2n)^2$ so

$$P_\infty = \lim_{n\to\infty} \left|1 - (\pi/2n)^2\right|^{2n} = 1$$

Superposition: FYI

Let us consider again the two orthogonal states given by the two superposition states of $|H\rangle$ and $|V\rangle$ we have encountered earlier, $|+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$ and $|-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$.

The probability that $|+\rangle$ is vertically polarized is $|\langle V|+\rangle|^2 = 1/2$,
the probability that $|-\rangle$ is vertically polarized is $|\langle V|-\rangle|^2 = 1/2$,
and also $|\langle H|+\rangle|^2 = |\langle H|-\rangle|^2 = 1/2$.

We cannot distinguish the two photons $|+\rangle$ and $|-\rangle$ simply measuring the probability of these photons to pass through a polarizing lens at 0° or 90° degrees.

However $|+\rangle$ and $|-\rangle$ are very different as they form a basis (we have seen that they are orthogonal).

More generally, given $|\psi\rangle = \begin{pmatrix}\alpha\\\beta\end{pmatrix}$, note that as of now we have only set a constraint on $|\alpha|^2 + |\beta|^2 = 1$, but α and β carry more information than that. They are complex numbers $\alpha = |\alpha|e^{i\varphi_\alpha}$, and $\beta = |\beta|e^{i\varphi_\beta}$, where φ_α and φ_β are real numbers and represent the phase of the complex number.