Week 3 – Day 2

Electric Potential

Concept 1: Electric Potential Field of a Continuous Charge Body

Concept 2: Electric Field and Gradient of Electric Potential Field

Lighting rod

Reading:

1. University Physics with Modern Physics – Chapter 23
2. Introduction to Electricity and Magnetism – Chapter 4

Concept 1: Electric Potential Field of a Continuous Charge Body

- Method 1: From continuous charge distribution
- Method 2: Using Gauss's Law to Find Electric Potential from Electric Field

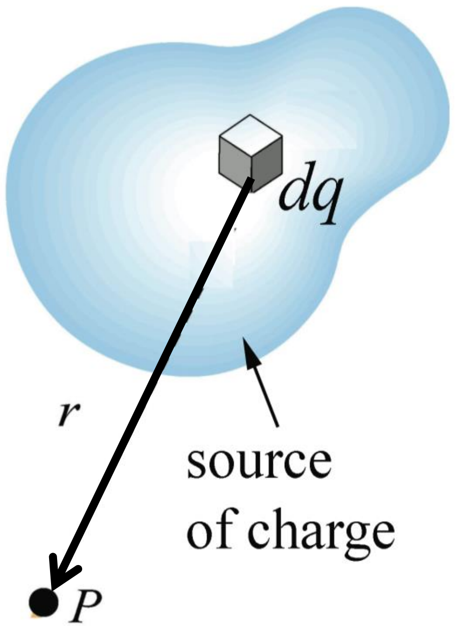
Method 1: Continuous Charge Distributions

- Break the continuous charge distribution into infinitesimal charged elements, each of charge dq. Electric potential difference between infinity and P due to dq.
- $dV \equiv V_{dq}(P) - V_{dq}(\infty) = k_e \frac{dq}{r}$
- Superposition Principle:

$$V(P) - V(\infty) = k_e \int_{source} \frac{dq}{r}$$

- Reference Point:

$$V(\infty) = 0 \Rightarrow V(P) = k_e \int_{source} \frac{dq}{r}$$



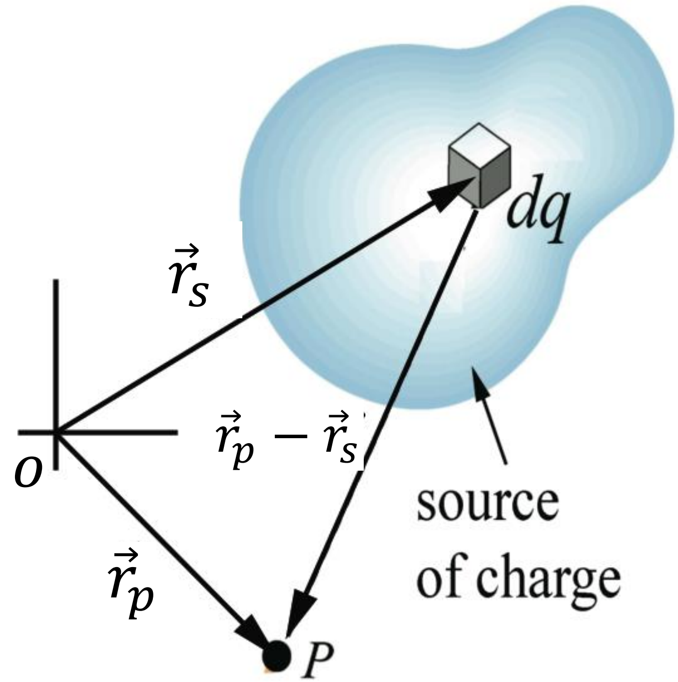
Calculating Electric Potential Difference for Continuous Distributions

1. Choose $V(\infty) = 0$
2. From the continuous charge, choose a small dq. Identify the dq position from the origin, $\vec{r}_s$.
3. Identify

$$dq = \begin{cases} \lambda ds \\ \sigma dA \\ \rho dV \end{cases}$$

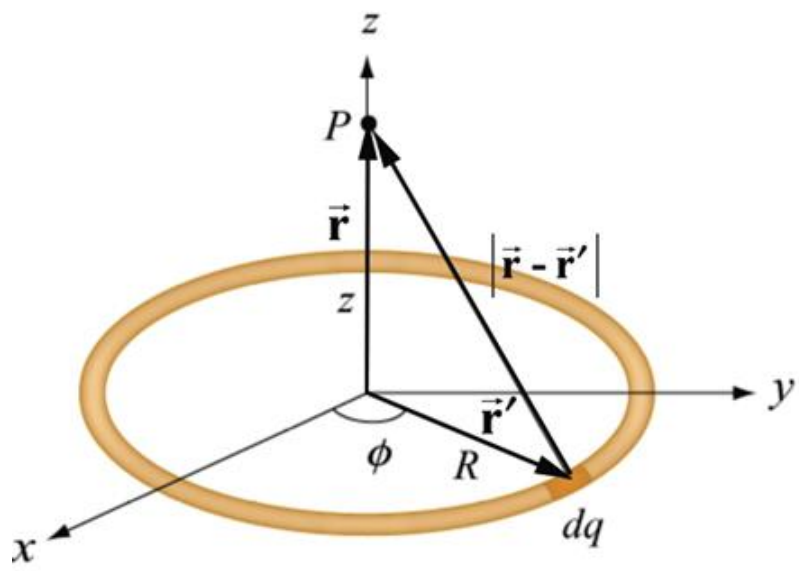
4. Identify the position of point of interest P, $\vec{r}_p$.
5. Calculate dq to point P distance, $\left|\vec{r}_p - \vec{r}_s\right|$
6. Electric potential due to dq is $dV = k_e \frac{dq}{|\vec{r}_p - \vec{r}_s|}$
7. Define limits of integral and integrate

$$V(\vec{\mathrm{r}}) = k_e \int_{\text{source}} \frac{dq}{\left|\vec{r}_p - \vec{r}_s\right|}$$



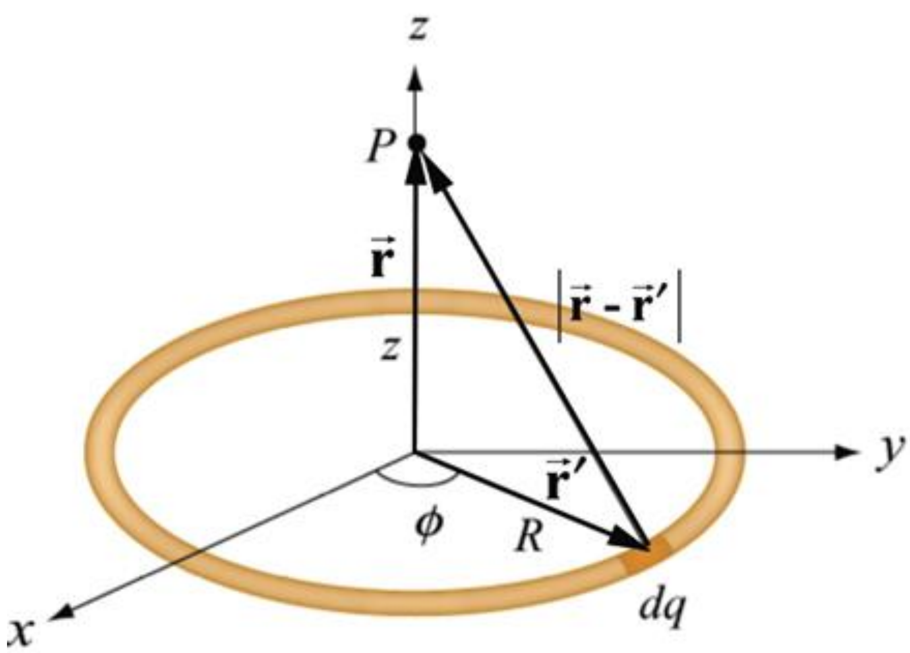
Worked Example: Uniform Charged Ring

- Consider a uniformly charged ring with total charge Q. Find the electric potential difference between infinity and a point P along the symmetric axis a distance z from the center of the ring.



Worked Example: Solution

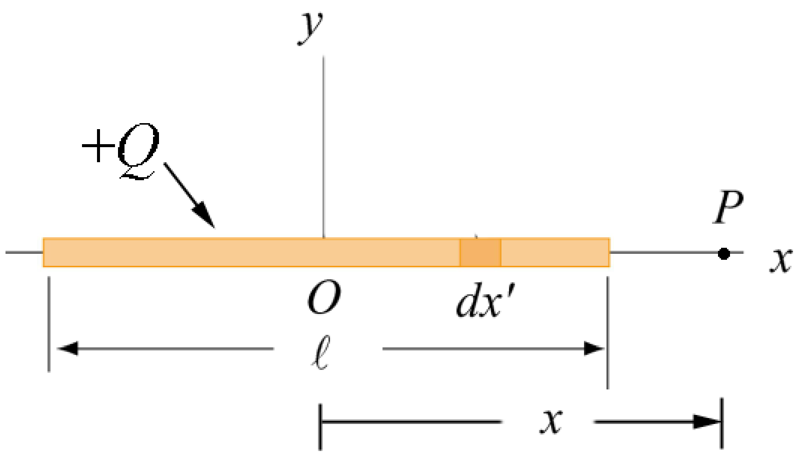
- $\lambda = \frac{Q}{2\pi R}; dq = \lambda dA = \lambda R d\varphi$
- Choose $V(\infty) = 0$
- $\vec{r}' = R\hat{r}$
- $\vec{r} = z\hat{\mathrm{k}}$
- $|\vec{r} - \vec{r}'| = \sqrt{R^2 + z^2}$
- $dV = \frac{1}{4\pi\varepsilon_0}\frac{dq}{|\vec{\mathrm{r}} - \vec{\mathrm{r}}\prime|} = \frac{1}{4\pi\varepsilon_0}\frac{\lambda R d\varphi}{\sqrt{R^2+z^2}}$



$$V(z) = \frac{\lambda R}{4\pi\varepsilon_0\sqrt{R^2 + z^2}}\int_0^{2\pi} d\varphi = \frac{k_e \lambda R(2\pi)}{\sqrt{R^2 + z^2}} = \frac{k_e Q}{\sqrt{R^2 + z^2}}$$

Case Problem 1.1

- A thin rod extends along the *x*-axis from $x = -l/2$ to $x = l/2$. The rod carries a uniformly distributed positive charge $+Q$. Calculate the electric potential difference between infinity and at a point P along the *x*-axis.

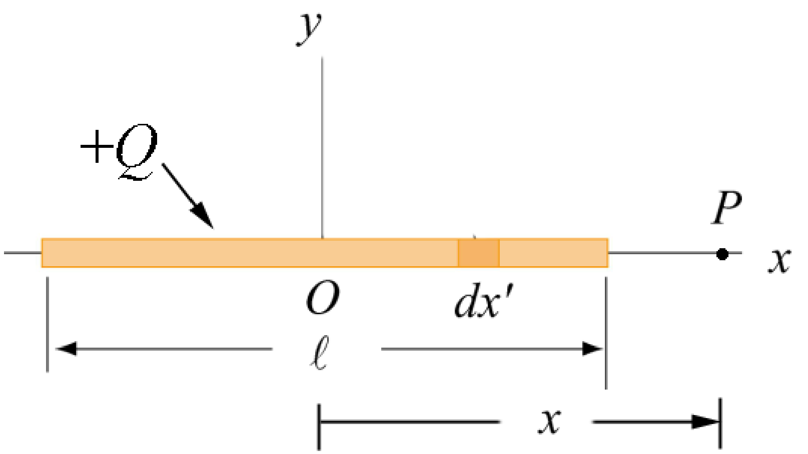


- You may find this info useful: $\int \frac{dx\prime}{x-x\prime} = -\ln|x - x'|$

Case problem 1.1: Solution

- Info: $\int \frac{dx'}{x-x'} = -\ln|x-x'|$

- $V(\infty) = 0$
- $dV = \frac{dq}{4\pi\varepsilon_0 r} = \frac{1}{4\pi\varepsilon_0}\frac{\lambda dx'}{x-x'}$



$$V = \frac{\lambda}{4\pi\varepsilon_0}\int_{-l/2}^{l/2}\frac{dx'}{x-x'} = -\frac{\lambda}{4\pi\varepsilon_0}\ln|x-x'|\Big|_{-l/2}^{l/2}$$

$$= -\frac{\lambda}{4\pi\varepsilon_0}\ln\left[\frac{x-l/2}{x+l/2}\right] > 0$$

- In the limit $x >> l$

$$\ln\left[1-\frac{l}{2x}\right] \approx -\frac{l}{2x} + \cdots$$

$$\ln\left[1+\frac{l}{2x}\right] \approx \frac{l}{2x} + \cdots$$

- $\ln\left[\frac{x-l/2}{x+l/2}\right] = \ln\left[\frac{1-\frac{l}{2x}}{1+\frac{l}{2x}}\right] = \ln\left[1-\frac{l}{2x}\right] - \ln\left[1+\frac{l}{2x}\right]$

$$\approx -\frac{l}{2x} - \frac{l}{2x} = -\frac{l}{x}$$

- $V = \frac{\lambda}{4\pi\varepsilon_0}\frac{l}{x} = \frac{Q}{4\pi\varepsilon_0 x}$

Summary: Electric Field and Potential

- A point charge q creates an electric field and potential around it

$$\vec{E}(\vec{r}) = k_e \frac{q}{r^2}\hat{\mathbf{r}} \quad ; \quad V(r) = k_e \frac{q}{r}$$

- Use superposition for systems of charges
  - Set of discrete charges:

$$\vec{E}(\vec{r}) = k_e \sum_i \frac{q_i}{\left|\vec{r}_p - \vec{r}_s\right|^3}(\vec{r}_p - \vec{r}_s)$$

$$V(\vec{r}) - V(\infty) = k_e \sum_i \frac{q_i}{\left|\vec{r}_p - \vec{r}_s\right|}$$

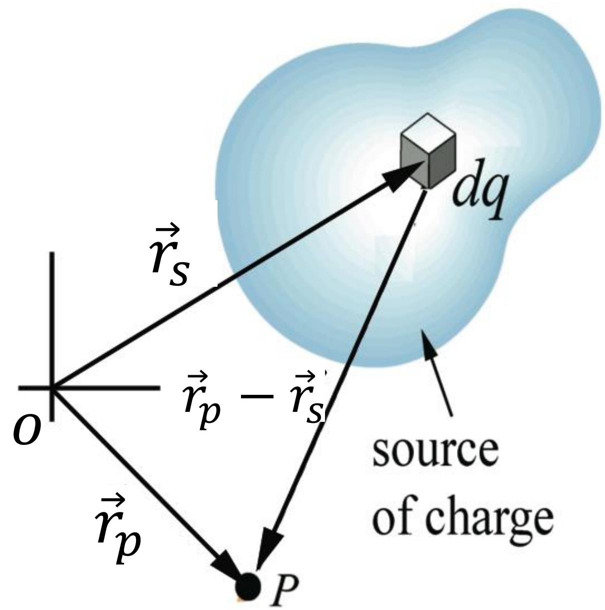
  - Continuous charges:

$$\vec{E}(\vec{r}) = k_e \int \frac{dq}{\left|\vec{\mathrm{r}}_p - \vec{r}_s\right|^3}(\vec{\mathrm{r}}_p - \vec{r}_s)$$

$$V(\vec{\boldsymbol{r}}) - V(\infty) = k_e \int \frac{dq}{\left|\vec{r}_p - \vec{r}_s\right|}$$

- If you already know electric field (usually from Gauss's Law), we can compute the electric potential difference using

$$V(\vec{\mathrm{r}}) - V(\infty) = -\int_{\infty}^{\vec{\mathrm{r}}} \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}}$$



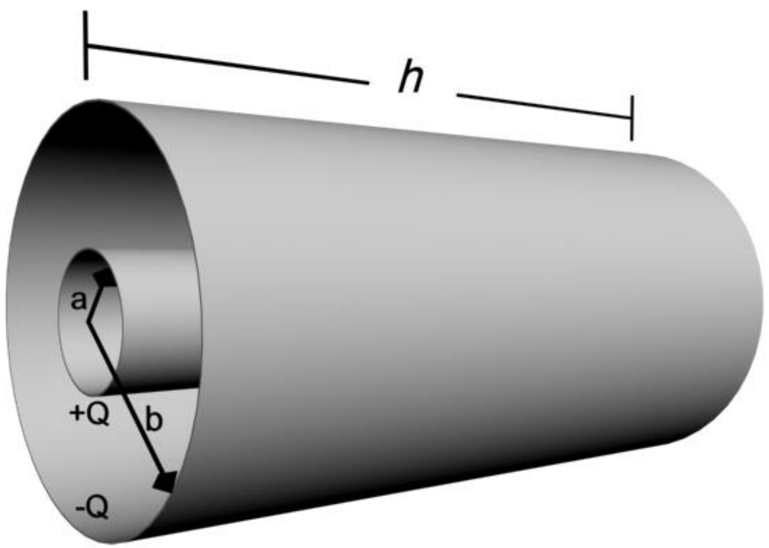
Method 2: Find Electric Potential from Electric Field (Gauss's Law)

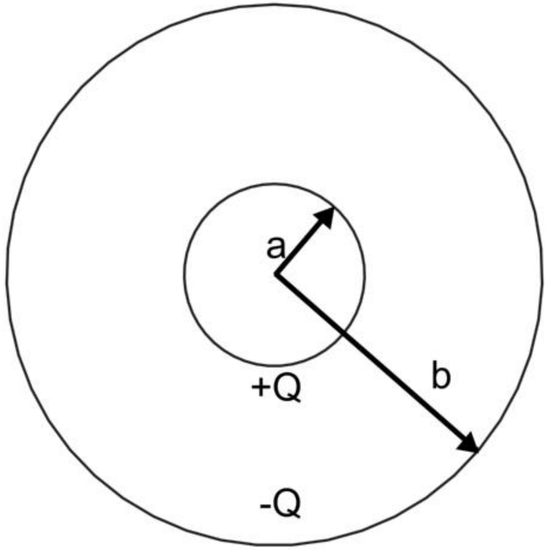
- If the charge distribution has a lot of symmetry, we use Gauss's Law to calculate the electric field first, then calculate the electric potential V using

$$V_B - V_A = -\int_A^B \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}}$$

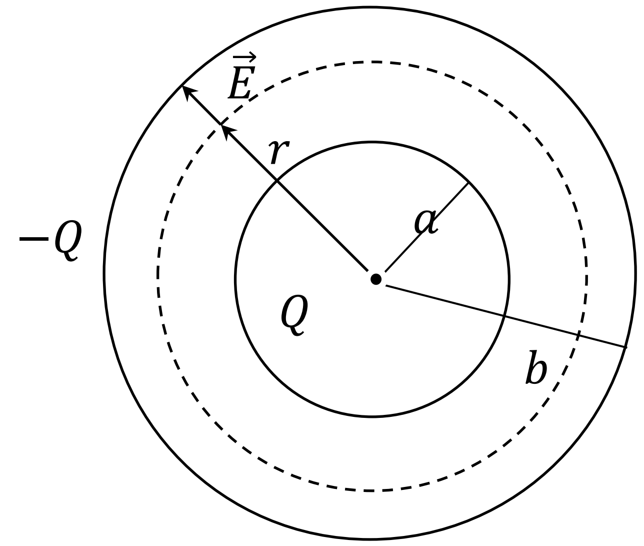
Worked Example: Coaxial Cylinders

A very long thin uniformly charged cylindrical shell (length h and radius a) carrying a positive charge +Q is surrounded by a thin uniformly charged cylindrical shell (length h and radius b) with negative charge -Q , as shown in the figure. You may ignore edge effects. Find V(b) – V(a).





Worked Example: Solution



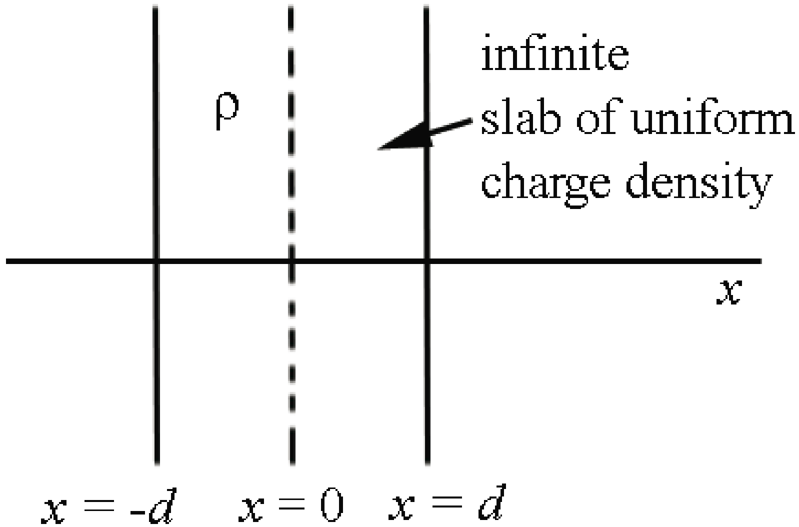
- Consider symmetry, we have
- $\oiint \vec{E} \cdot d\vec{A} = 2\pi r h' E = \dfrac{q_{enc}}{\varepsilon_0} = \dfrac{\sigma 2\pi a h\prime}{\varepsilon_0}$
- $\Rightarrow E = \dfrac{\sigma a}{\varepsilon_0 r}; \sigma = \dfrac{Q}{2\pi a h}$
- $\Delta V = V(b) - V(a) = -\int_a^b \vec{E} \cdot d\vec{s}$

$$= -\int_a^b E dr = -\int_a^b \frac{\sigma a}{\varepsilon_0} \frac{dr}{r} = \frac{\sigma a}{\varepsilon_0} \ln\left(\frac{a}{b}\right)$$

- which is <0. The result is negative because the electric field points in the direction of decreasing electric potential.

Case Problem 1.3: Charge Slab

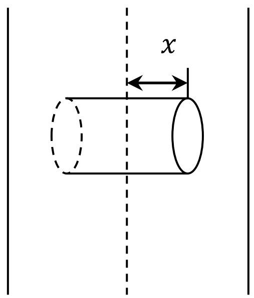
- Infinite slab of thickness $2d$, centered at $x = 0$ with uniform charge density, ρ. Find $V(x_A) - V(0); x_A > d$



Case Problem 1.3: Solution

- First, find the electric field both inside and outside the slab.
- The symmetry is planar so we use Gaussian surface (cylinders of cross-section A and height 2x and place symmetrically w.r.t x=0)
- There are two distinct regions of space, inside and outside of the slab. By symmetry the magnitude of the field will be the same on the left as on the right of the slab but will point in the opposite direction.

Case Problem 1.3: Solution

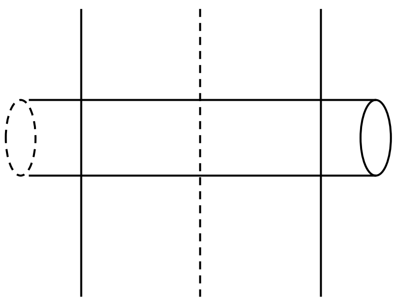


Inside

$$\oiint \vec{E}\cdot d\vec{A} = \iint\limits_{\text{left cap}} \vec{E}\cdot d\vec{A} + \iint\limits_{\text{sides}} \vec{E}\cdot d\vec{A} + \iint\limits_{\text{right cap}} \vec{E}\cdot d\vec{A} = 2EA$$

0

$$2EA = \frac{q_{enc}}{\varepsilon_0} = \frac{\rho A(2x)}{\varepsilon_0} \;\Rightarrow\; E_{in} = \frac{\rho x}{\varepsilon_0}$$



Outside

$$\oiint \vec{E}\cdot d\vec{A} = \iint\limits_{\text{left cap}} \vec{E}\cdot d\vec{A} + \iint\limits_{\text{sides}} \vec{E}\cdot d\vec{A} + \iint\limits_{\text{right cap}} \vec{E}\cdot d\vec{A} = 2EA$$

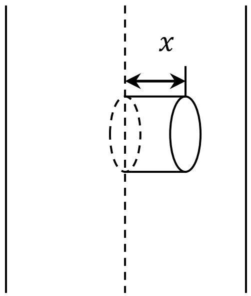
0

$$2EA = \frac{q_{enc}}{\varepsilon_0} = \frac{\rho A(2d)}{\varepsilon_0} \;\Rightarrow\; E_{out} = \frac{\rho d}{\varepsilon_0}$$

Case Problem 1.3 Alternative Solution to find $\vec{E}$

- First, find the electric field both inside and outside the slab.
- Consider symmetry, on the plane $x = 0$
- The symmetry is planar so we use Gaussian surface (cylinders of cross-section A and height x and place one end of the pillbox at $x = 0$ to take advantage of the fact that $\vec{E} = 0$ there)
- There are two distinct regions of space, inside and outside of the slab. By symmetry the magnitude of the field will be the same on the left as on the right of the slab, but will point in the opposite direction. We only consider explicitly for $x > 0$

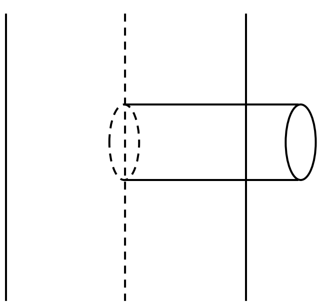
Case Problem 1.3 Alternative Solution to find $\vec{E}$



Inside

$$\oiint \vec{E}\cdot d\vec{A} = \iint_{\text{left cap}} \vec{E}\cdot d\vec{A} + \underbrace{\iint_{\text{sides}} \vec{E}\cdot d\vec{A}}_{0} + \underbrace{\iint_{\text{right cap}} \vec{E}\cdot d\vec{A}}_{0} = EA$$

$$EA = \frac{q_{enc}}{\varepsilon_0} = \frac{\rho A x}{\varepsilon_0} \Rightarrow E_{in} = \frac{\rho x}{\varepsilon_0}$$



Outside

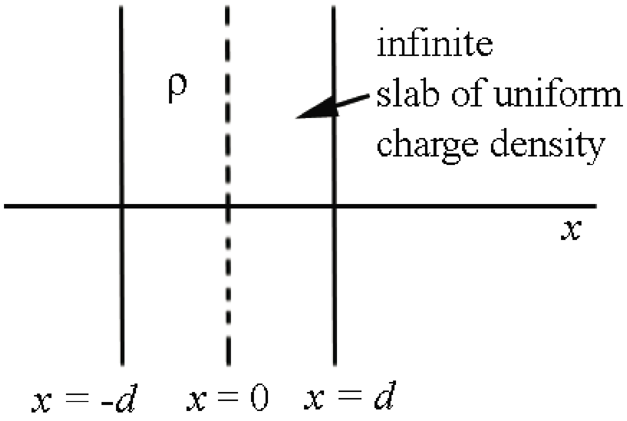
$$\oiint \vec{E}\cdot d\vec{A} = \iint_{\text{left cap}} \vec{E}\cdot d\vec{A} + \underbrace{\iint_{\text{sides}} \vec{E}\cdot d\vec{A}}_{0} + \underbrace{\iint_{\text{right cap}} \vec{E}\cdot d\vec{A}}_{0} = EA$$

$$EA = \frac{q_{enc}}{\varepsilon_0} = \frac{\rho A d}{\varepsilon_0} \Rightarrow E_{out} = \frac{\rho d}{\varepsilon_0}$$

Note: more than one choice of Gaussian surface!

Case Problem 1.3 Solution

- $$\vec{E} = \begin{cases} \dfrac{\rho d}{\varepsilon_0}\hat{\imath} & \text{for } x \geq d \\ \dfrac{\rho x}{\varepsilon_0}\hat{\imath} & \text{for } -d < x < d \\ -\dfrac{\rho d}{\varepsilon_0}\hat{\imath} & \text{for } x \leq -d \end{cases}$$



- $V(x_A) - V(0) = -\int_0^{x_A} E(x)dx$

$$= -\int_0^d \frac{\rho x}{\varepsilon_0} dx - \int_d^{x_A} \frac{\rho d}{\varepsilon_0} dx$$

$$= -\frac{\rho}{\varepsilon_0}\frac{d^2}{2} - \frac{\rho}{\varepsilon_0} d(x_A - d)$$

$$= -\frac{\rho d}{\varepsilon_0}\left(x_A - \frac{d}{2}\right) \text{ for } x_A \geq d$$

Week 3 – Day 2

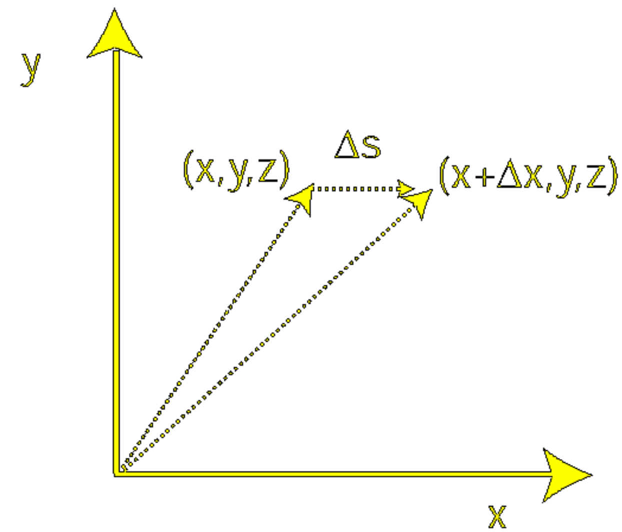
Concept 2: Electric Field and Gradient of Electric Potential

Deriving E from V

$$\Delta V = -\int_A^B \vec{E} \cdot d\vec{s}$$

$$A = (x, y, z), B = (x + \Delta x, y, z)$$

$$\Delta\vec{s} = \Delta x\hat{\imath}$$



$$\Delta V = -\int_{(x,y,z)}^{(x+\Delta x,y,z)} \vec{E} \cdot d\vec{s} \cong -\vec{E} \cdot \Delta\vec{s} = -\vec{E} \cdot (\Delta x\hat{\imath})$$

$$= -E_x \Delta x$$

$$E_x \cong -\frac{\Delta V}{\Delta x} \rightarrow -\frac{\partial V}{\partial x}$$

E_x = Rate of change in V wrt. x while y and z held constant.

Deriving E from V

- If we combine all directions together,

$$\vec{E} = -\left(\frac{\partial V}{\partial x}\hat{\imath} + \frac{\partial V}{\partial y}\hat{\jmath} + \frac{\partial V}{\partial z}\hat{k}\right)$$

$$= -\underbrace{\left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right)}_{}V$$

Gradient (del) operator:

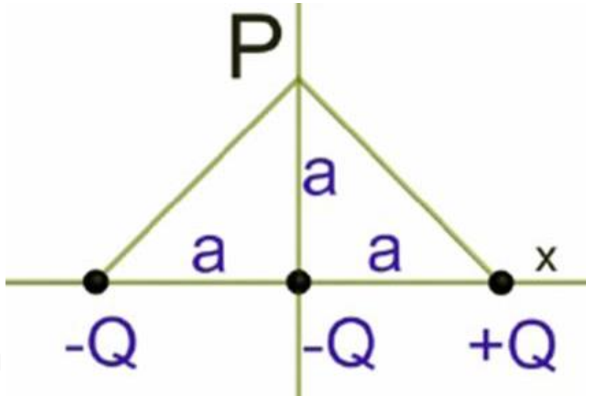
$$\vec{\nabla} \equiv \frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}$$

$$\vec{E} = -\vec{\nabla}V$$

Note: The equation simply says that electric field is the negative gradient of electric potential.

Concept Question 2.1: E from V

- Consider the point-like charged objects arranged in the figure below. The electric potential difference between the point P and infinity and is
- $V(P) = -\frac{kQ}{a}$



- From that can you derive E(P)?

A. Yes, its kQ/a^2 (up)

B. Yes, its kQ/a^2 (down)

C. Yes in theory, but I don't know how to take a gra

D. No, you can't get E(P) from V(P)

Multiple Choice

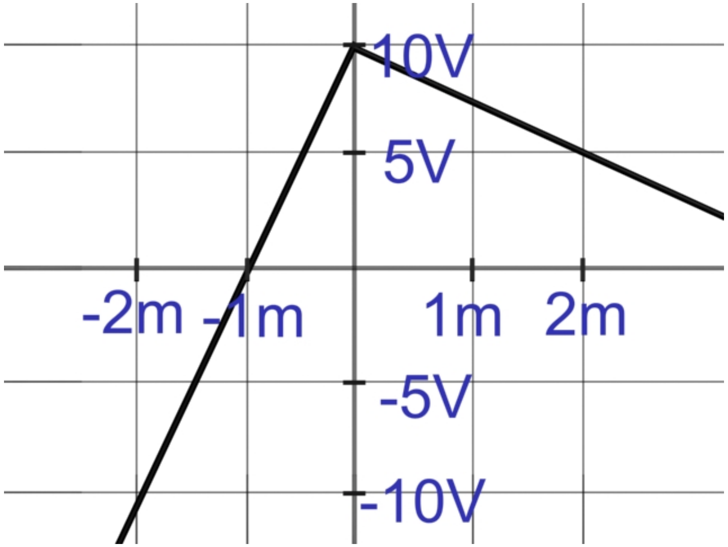
Concept Question 2.1: Solution

Answer : D. No, you can’t get E(P) from V(P)

The electric field is the gradient (spatial derivative) of the potential. Knowing the potential at a single point tells you nothing about its derivative.

People commonly make the mistake of trying to do this. Don’t!

Concept Question 2.2: E from V



The graph above shows a potential V as a function of x. The *magnitude* of the electric field for x > 0 is

A. larger than that for x < 0

B. smaller than that for x < 0

C. equal to that for x < 0

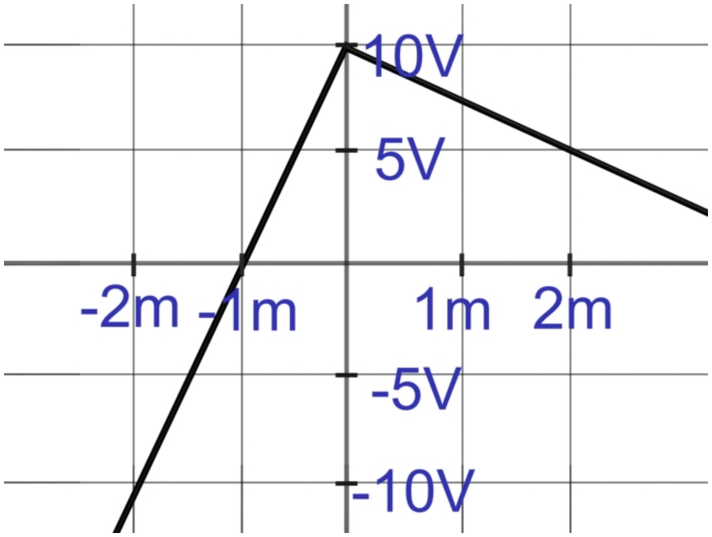
Multiple Choice

Concept Question 2.2: Solution

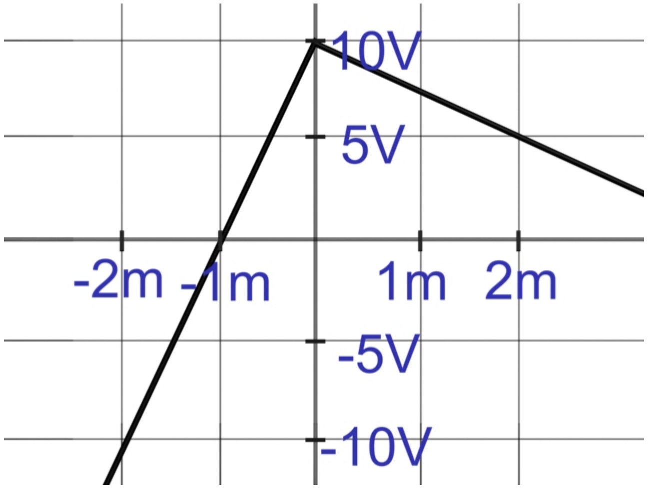
Answer : B. The *magnitude* of the electric field for x > 0 is *smaller* than that for x < 0

The slope is smaller for x > 0 than x < 0

Translation: The hill is steeper on the left than on the right.



Concept Question 2.3: E from V



The above shows potential V(x). Which is true?

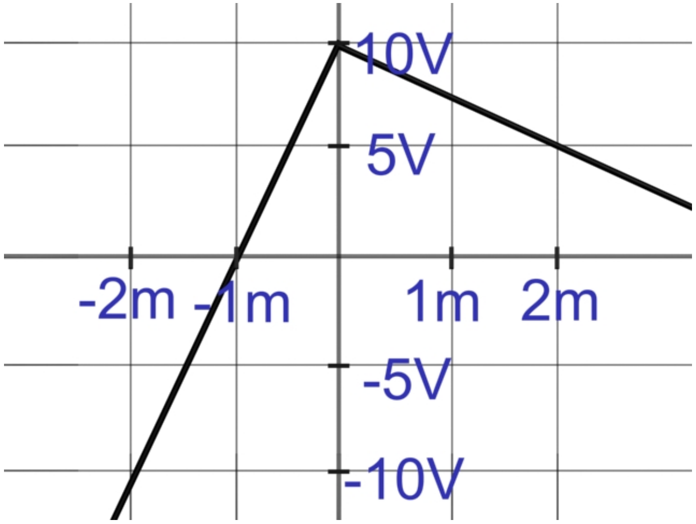
A. $E_{x>0}$ is positive and $E_{x<0}$ is positive
B. $E_{x>0}$ is positive and $E_{x<0}$ is negative
C. $E_{x>0}$ is negative and $E_{x<0}$ is negative
D. $E_{x>0}$ is negative and $E_{x<0}$ is positive

Multiple Choice

Concept Question 2.3: Solution

Answer : B. $E_{x>0}$ is positive and $E_{x<0}$ is negative

E is the negative slope of the potential, positive on the right, negative on the left,
Translation: "Downhill" is to the left on the left and to the right on the right.



Concept Question 2.4:

- The electric potential V(x,y,z) = A + B xy + C z where A, B and C are non-zero constants. Which of the following statements is correct?

A. *E* is 0 at the origin

B. *E* has a component in all directions

C. Diminishing A will lower the value of *E* everywhere

D. Diminishing A will increase the value of *E* everywhere

Multiple Choice

Concept Question 2.4: Solution

- Answer is B. *E* has a component in all directions.
- Due to the gradient of V:

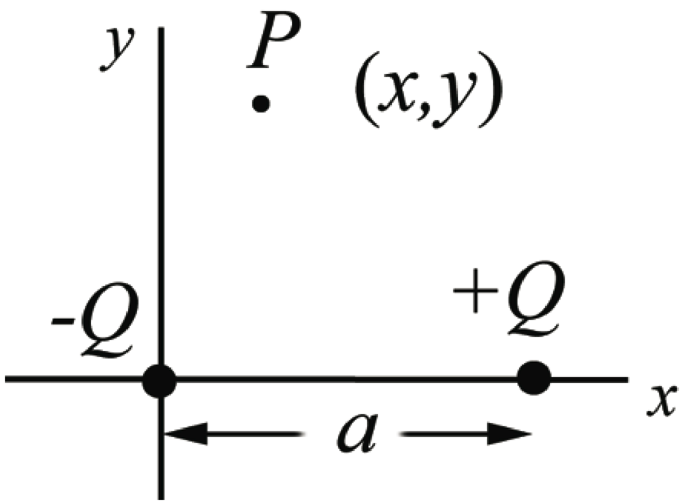
$$E_x = -\frac{\partial V}{\partial x} = -By,$$
$$E_y = -\frac{\partial V}{\partial y} = -Bx,$$
$$E_z = -\frac{\partial V}{\partial z} = -C$$

Case Problem 2.1: E from V

- Consider two point like charged objects with charge –Q located at the origin and +Q located at the point (0,a).

a. Find the electric potential V(x,y)at the point P located at (x,y).
b. Find the x-and y-components of the electric field at the point P using

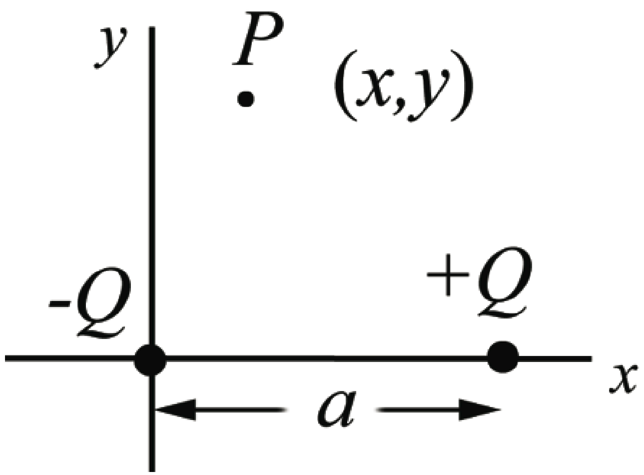
$$E_x(x,y) = -\frac{\partial V}{\partial x}, E_y(x,y) = -\frac{\partial V}{\partial y}$$



Case Problem 2.1: Solution

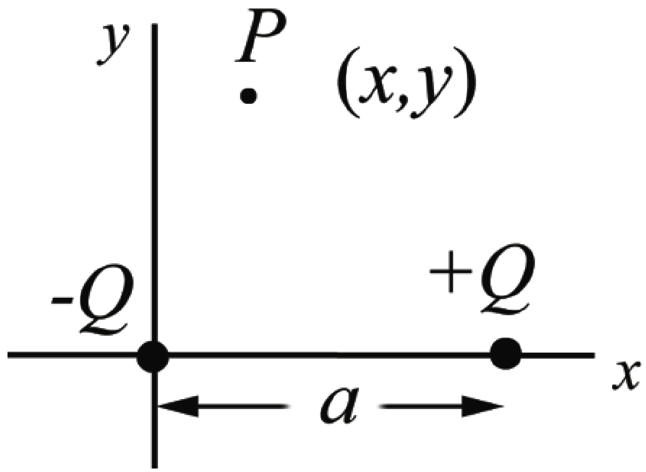
- $V = \sum_{i=1,2} V_i = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1}{r_1} + \frac{q_2}{r_2}\right)$

- $q_1 = -Q, \quad r_1 = \sqrt{x^2 + y^2}$
- $q_2 = Q, \quad r_2 = \sqrt{(a-x)^2 + y^2}$



$$V = \frac{1}{4\pi\varepsilon_0}\left(\frac{-Q}{\sqrt{x^2+y^2}} + \frac{Q}{\sqrt{(a-x)^2+y^2}}\right)$$
$$= \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{\sqrt{(a-x)^2+y^2}} - \frac{1}{\sqrt{x^2+y^2}}\right)$$

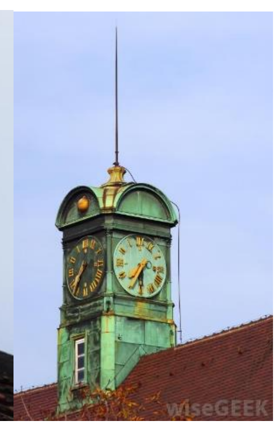
- $V = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{\sqrt{(a-x)^2+y^2}} - \frac{1}{\sqrt{x^2+y^2}}\right)$
- $E_x = -\frac{\partial V}{\partial x}, E_y = -\frac{\partial V}{\partial y}, E_z = -\frac{\partial V}{\partial z}$

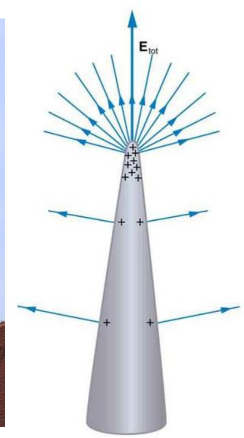


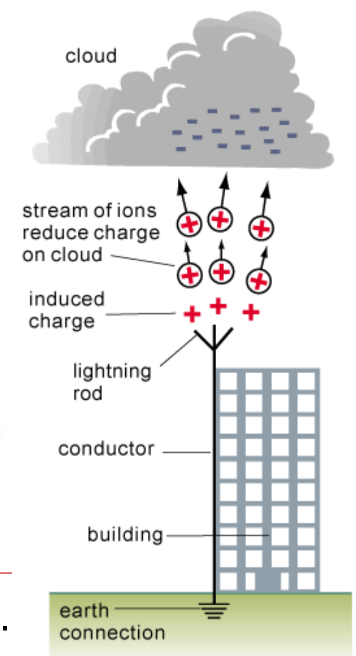
- $E_x = \frac{Q}{4\pi\varepsilon_0}\left(\frac{(x-a)}{[(a-x)^2+y^2]^{3/2}} - \frac{x}{[x^2+y^2]^{3/2}}\right)$
- $E_y = \frac{Q}{4\pi\varepsilon_0}\left(\frac{y}{[(a-x)^2+y^2]^{3/2}} - \frac{y}{[x^2+y^2]^{3/2}}\right)$
- $E_z = 0$

Application: Lightning Rod

- Why do lightning rods have a sharp point at the top?
- From the In-Class Worksheet, we shall see a small radius curve of a conductor can setup a higher local electric field. A very pointed conductor has a large charge concentration at the point. It can exert a force large enough to transfer charge on or off the conductor.
- The large negative charges created in storm clouds induce an opposite charge on a building that can result in a lightning bolt hitting the building.
- The induced (positive) charge is discharged away continually by a lightning rod, reducing the chance of more dramatic lightning strike.
- But if it does strike, charges flow to earth.







A lightning rod is pointed to facilitate the transfer of charge.

In Class Worksheet