SINGAPORE UNIVERSITY OF TECHNOLOGY AND DESIGN

10.018: Modelling Space and Systems
Term 2, 2021

Homework Week 2

Due Date: 6:30pm, Feb. 09, 2021

Reminder: Part 3 and Part 4 of Math Modeling are TO BE SUBMITTED on PIAZZA, 2% of your grade

You need to submit **at most 2 pages** (readable pics/screenshots) by uploading on piazza in an appropriate thread, by Mon 6pm, Feb. 08 (meaning the MM posting on piazza is due one day before your homework).

Format of the submissions:

- Cohort name, list of team members with their official full names and student IDs.
- Restatement of the problem from Part 1 (incorporate any comments you have received thus far).
- List of VALID assumptions and variables from Part 2 (incorporate any comments you have received thus far, make sure variables are named in a logical manner and units are listed) *You may have more assumptions appearing here as the result of Solving Math modeling problem.*
- A solution to your model (Part 3).
- An analysis and model assessment. *Include at least 1 weakness and at least 1 strength of the model* (Part 4).
- You may skip details in your computations/derivations, keep the main steps and main results.

P.S.: Do not spend more than 1-2 hours on this part, we don't want you to burn out at this point. Just give it your best shot!

BASIC problems TO BE SUBMITTED.

The BASIC set of problems is designed to be a very easy and straightforward application of the definitions from lectures and cohorts (you might have to do some calculations, but not much). If you have trouble starting any of the questions do consult your cohort instructors (in office hours, via email or via Piazza).

1. *Find $\frac{dz}{dt}$ using the chain rule, where $z = x\sin y + ye^x$ and $x = t^2$, $y = \ln t$.*

   Solution: By the Chain Rule,

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}\\ &= (\sin y + y\,e^x)2t + (x\cos y + e^x)\,\frac{1}{t}\\ &= 2t\left(\sin(\ln t) + (\ln t)e^{t^2}\right) + t\cos(\ln t) + \frac{e^{t^2}}{t}.\end{aligned}$$

   This agrees with the derivative of $z = x\sin y + ye^x = (t^2)\sin(\ln t) + (\ln t)e^{t^2}$, as obtained by substitution, namely

$$\begin{aligned}\frac{\mathrm{d}z}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}((t^2)\sin(\ln t) + (\ln t)e^{t^2})\\ &= (t^2\cos(\ln t)\frac{1}{t} + 2t\sin(\ln t)) + ((\ln t)e^{t^2}2t + \frac{1}{t}e^{t^2})\\ &= 2t\left(\sin(\ln t) + (\ln t)e^{t^2}\right) + t\cos(\ln t) + \frac{e^{t^2}}{t}.\end{aligned}$$

2. *Find $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$, where $z = \ln(xy)$, and $x = u^2 + \sinh v$, $y = \tanh u + v^3$.*

   Solution: By the Chain Rule,

$$\begin{aligned}\frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}\\ &= \frac{2u}{x} + \frac{\operatorname{sech}^2 u}{y}\\ &= \frac{2u}{u^2 + \sinh v} + \frac{\operatorname{sech}^2 u}{\tanh u + v^3}\end{aligned}$$

   and

$$\begin{aligned}\frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}\\ &= \frac{\cosh v}{x} + \frac{3v^2}{y}\\ &= \frac{\cosh v}{u^2 + \sinh v} + \frac{3v^2}{\tanh u + v^3}\end{aligned}$$

   Alternate solution: Substituting $x = u^2 + \sinh v$, $y = \tanh u + v^3$ into $z = \ln(x, y)$,

$$z = \ln(xy) = \ln((u^2 + \sinh v)(\tanh u + v^3)) = \ln(u^2 + \sinh v) + \ln(\tanh u + v^3).$$

   Therefore

$$\frac{\partial z}{\partial u} = \frac{2u}{u^2 + \sinh v} + \frac{\operatorname{sech}^2 u}{\tanh u + v^3}$$

and

$$\frac{\partial z}{\partial v} = \frac{\cosh v}{u^2 + \sinh v} + \frac{3v^2}{\tanh u + v^3}.$$

3. *Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by*

$$f(x, y) = x^3 - 27x + 2y^2. \tag{1}$$

   (a) Find the 2 critical points of $f(x, y)$.

   (b) Use the second derivative test to classify these critical points.

   Solution:

   (a) Taking the first partial derivatives, we have the following equations which we set to zero

$$f_x = 3x^2 - 27 = 0 \tag{2}$$
$$f_y = 4y = 0 \tag{3}$$

   in order to find the critical points.

   From (3), we get $y = 0$. Now, if we look at (2), we have

$$x^2 - 9 = 0 \tag{4}$$
$$\Rightarrow x = 3 \quad \text{or} \quad x = -3. \tag{5}$$

   Therefore, the critical points are $(3, 0)$ and $(-3, 0)$.

   (b) To classify the critical points, we look at the second partial derivatives, where we have

$$f_{xx} = 6x \tag{6}$$
$$f_{yy} = 4 \tag{7}$$
$$f_{xy} = f_{yx} = 0. \tag{8}$$

   Now we can classify the critical points using the second derivative test.

   At $(3, 0)$, we have $D = 72 > 0$ with $f_{xx} > 0$. Therefore $(3, 0)$ is a local minimum.

   At $(-3, 0)$, we have $D = -72 < 0$, so $(-3, 0)$ is a saddle point.

INTERMEDIATE problems TO BE SUBMITTED.

The INTERMEDIATE set of problems is a *little* harder (but not by much) than the BASIC one. If you have trouble starting any of the questions do consult your cohort instructors (in office hours, via email or via Piazza).

4. *Find and classify the critical points of the following function*

$$f(x, y) = x^3 + y^2 - 6xy + 6x + 3y. \tag{9}$$

   (Hint: In the equations for f_x and f_y, express y in terms of x and solve.)

   Solution:

   (a) Taking the first partial derivatives, we have the following equations which we set to zero

$$f_x = 3x^2 - 6y + 6 = 0 \tag{10}$$
$$f_y = 2y - 6x + 3 = 0 \tag{11}$$

in order to find the critical points. As f_y can be re-written as $2y = 6x - 3$, then we can substitute $2y$ into (10) to get

$$\begin{aligned} & x^2 - 2y + 2 = 0 && (12) \\ \Rightarrow & x^2 - (6x - 3) + 2 = 0 && (13) \\ \Rightarrow & x^2 - 6x + 5 = 0 && (14) \\ \Rightarrow & (x - 5)(x - 1) = 0 && (15) \end{aligned}$$

which leads to the x-coordinates of the critical points being $x = 5$, $x = 1$. Substituting these values of x into $f(x, y)$, we get the critical points to be $\left(1, \frac{3}{2}\right), \left(5, \frac{27}{2}\right)$.

Taking the second partial derivatives give

$$\begin{aligned} f_{xx} &= 6x && (16) \\ f_{yy} &= 2 && (17) \\ f_{xy} = f_{yx} &= -6 && (18) \end{aligned}$$

and therefore we can classify the critical points using the second derivative test. At $\left(1, \frac{3}{2}\right)$, the second derivative test gives $D = -24 < 0$. Therefore $\left(1, \frac{3}{2}\right)$ is a saddle point.

At $\left(5, \frac{27}{2}\right)$, the second derivative test gives $D = 24 > 0$ with $f_{xx} > 0$. Therefore $\left(5, \frac{27}{2}\right)$ is a local minimum.

5. *The following figure is the topographic map of a geograhic region (i.e. the contour map of elevation as a function of location). As with topographic maps, all distances regardless of direction are drawn to the same scale. At which base station, P, Q, R or S, does the gradient vector point closest to the north?*

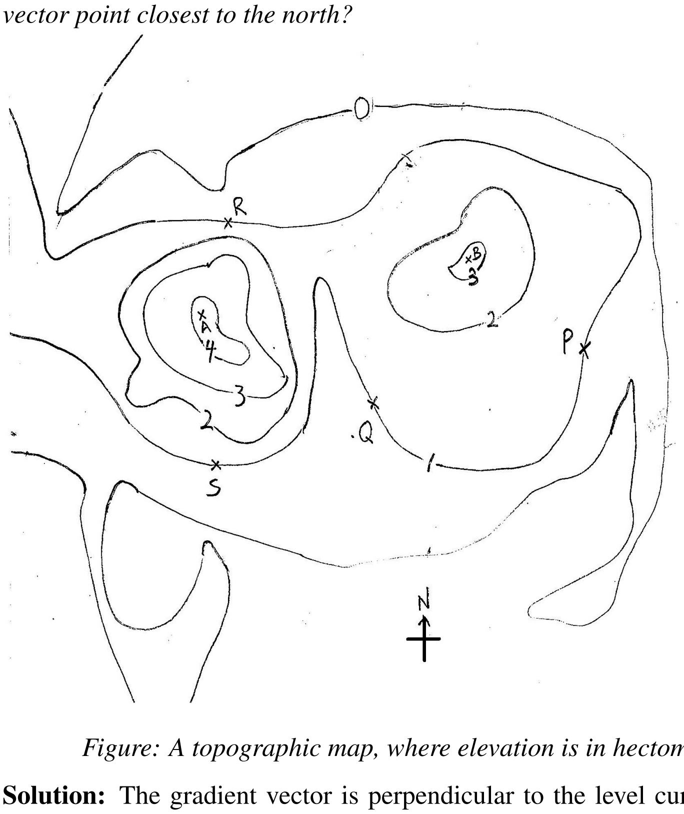
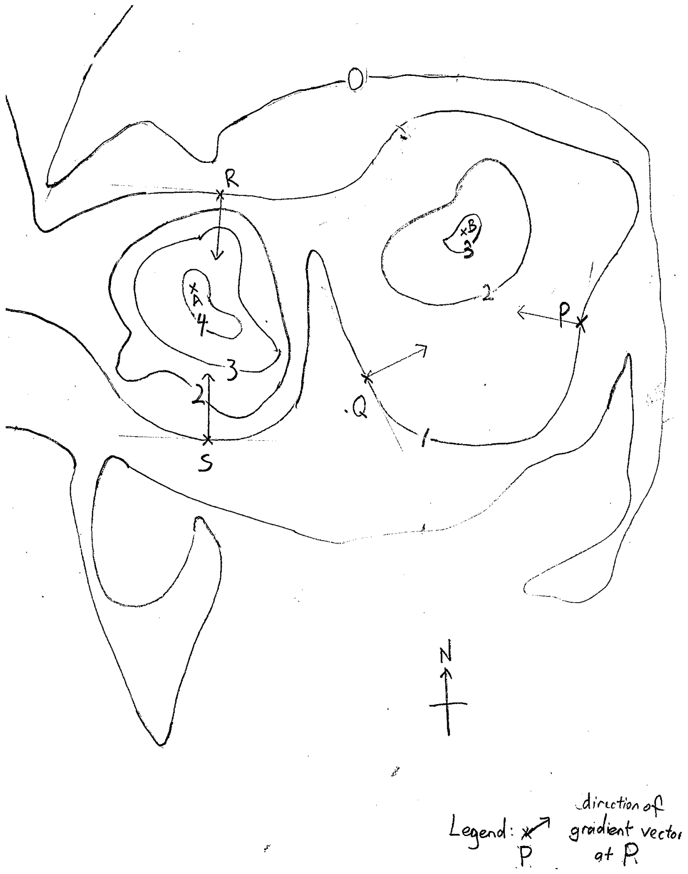


Figure: A topographic map, where elevation is in hectometres above sea level.

Solution: The gradient vector is perpendicular to the level curve and is in the direction of increasing elevation. The gradient vector at S points closest to the north, see the following figure which indicates the direction (but not the norm) of the gradient vector at P, Q, R and S respectively.



The same topographic map revealing extra land to the south, with the direction (but not the norm) of gradient vectors indicated.

Challenging problem [UNGRADED].

6. For a function of one variable, the second-order Taylor expansion about 0

$$F(h) \approx F(0) + F'(0)h + \frac{F''(0)}{2}h^2, \tag{19}$$

gives a reasonably good approximation to $F(h)$ in the neighbourhood of 0. Now let

$$f(x, y) = \sin x \, e^{2y-1}.$$

Use (19) to compute the value of $f(0.1, 0.6)$ up to 2 decimal places. (Hint: Define

$$F(h) := f(x + hu, y + hv),$$

and choose x, y, u, v appropriately.)

Solution: Computing the terms in the Taylor expansion (19), we have

$$F'(h) = \frac{\partial f}{\partial x}(x+hu, y+hv)u + \frac{\partial f}{\partial y}(x+hu, y+hv)v,$$
$$F''(h) = \frac{\partial^2 f}{\partial x^2}(x+hu, y+hv)u^2 + 2\frac{\partial^2 f}{\partial x \partial y}(x+hu, y+hv)uv + \frac{\partial^2 f}{\partial y^2}(x+hu, y+hv)v^2,$$

where

$$\frac{\partial f}{\partial x} = \cos x\, e^{2y-1}, \quad \frac{\partial f}{\partial y} = 2\sin x\, e^{2y-1},$$

and

$$\frac{\partial^2 f}{\partial x^2} = -\sin x\, e^{2y-1}, \quad \frac{\partial^2 f}{\partial y^2} = 4\sin x\, e^{2y-1},$$
$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = 2\cos x\, e^{2y-1}.$$

Now let $x = 0, y = 0.5, u = 0.1$ and $v = 0.1$. Then substituting everything back into (19) and setting $h = 1$, we get

$$\begin{aligned}
f(0.1, 0.6) &\approx f(0, 0.5) + (0.1)\left[\frac{\partial f}{\partial x}(0, 0.5) + \frac{\partial f}{\partial x}(0, 0.5)\right] \\
&\quad + \frac{1}{2}(0.1)^2\left[\frac{\partial^2 f}{\partial x^2}(0, 0.5) + 2\frac{\partial^2 f}{\partial x \partial y}(0, 0.5) + \frac{\partial^2 f}{\partial y^2}(0, 0.5)\right] \\
&= \sin 0\, e^0 + (0.1)\left[\cos 0\, e^0 + 2\sin 0\, e^0\right] \\
&\quad + \frac{1}{2}(0.1)^2\left[-\sin 0\, e^0 + 2(2\cos 0\, e^0) + 4\sin 0\, e^0\right] \\
&= 0.1 + 0.01(2) = 0.12.
\end{aligned}$$