Week 10 - Day 2

Tuning circuit in all wireless communication, Oscillator

Circuit Analysis

Concept 1: RL circuit (cont')

Concept 2: LC Circuit Analysis

Concept 3: Introduction of Undriven RLC Circuit

Reading:

University Physics with Modern Physics – Chapter 30

Introduction to Electricity and Magnetism – Chapter 11

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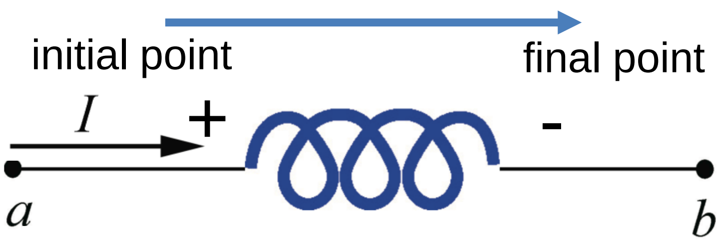
Concept 1: Circuits with Inductors (RL circuit)

Applying Modified Kirchhoff's Law (Really Just Faraday's Law)

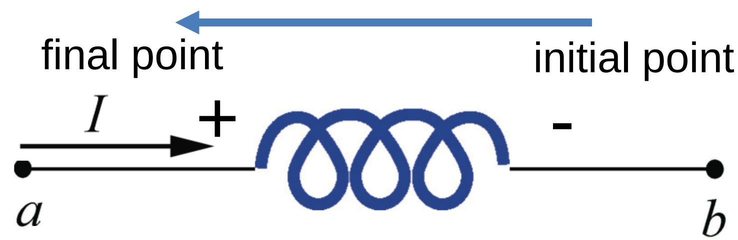
Sign Conventions - Inductor

Convention: Similar to resistor, we take the entrance of current across an inductor to have higher potential, as shown.

- When moving across an inductor in the direction of current, the emf across an inductor decreases (negative sign).
- When moving across an inductor opposite to the direction of current, the emf across an inductor increases (positive sign).



$$V_b - V_a = \varepsilon = -L\frac{dI}{dt}$$



$$V_a - V_b = \varepsilon = +L\frac{dI}{dt}$$

Note: The sign depends on the direction of I (that you assign) and the direction of the loop (that you choose), when you apply the sign convention in KVL.

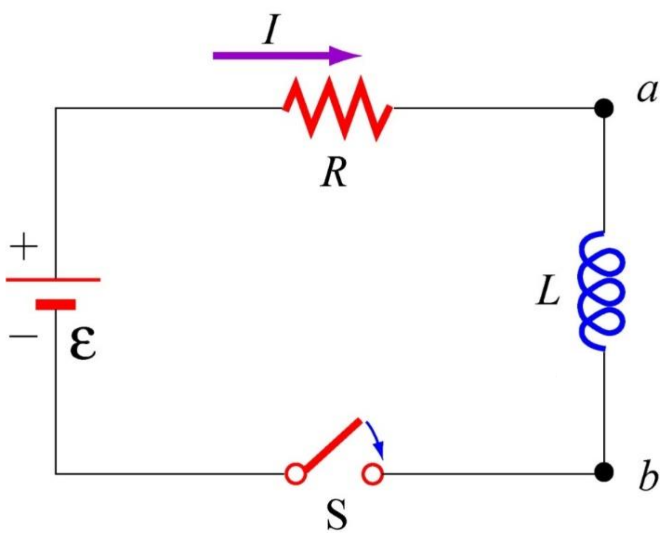
RL Circuit

- Applying KVL to this simple RL circuit (remember the sign convention), you will get the following circuit equation:

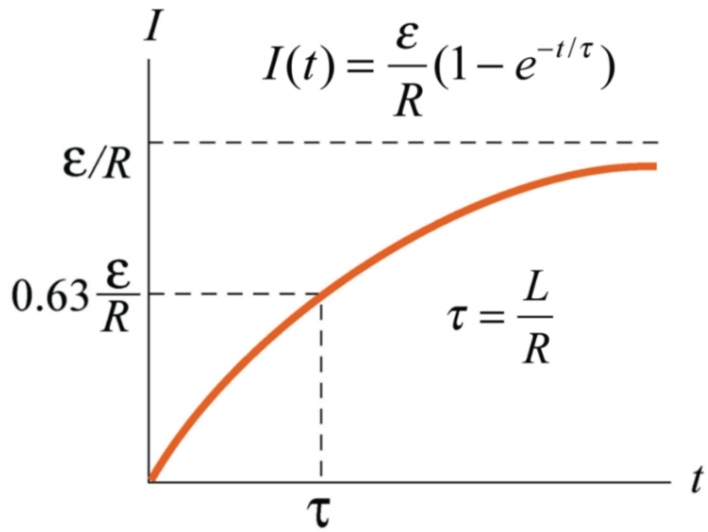
$$\varepsilon - IR - L\frac{dI}{dt} = 0 \Rightarrow \frac{dI}{dt} = -\frac{R}{L}\left(I - \frac{\varepsilon}{R}\right)$$

First Order Linear Nonhomogeneous Differential Equation

- Note: This equation is similar to the C charging equation.

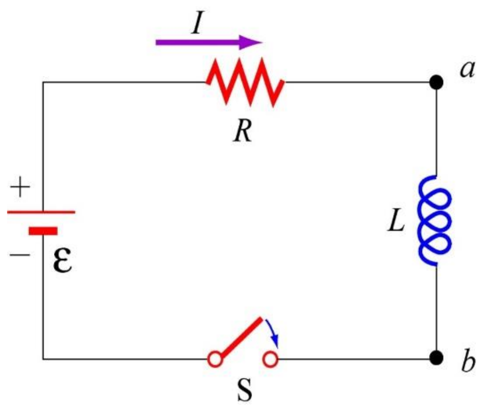


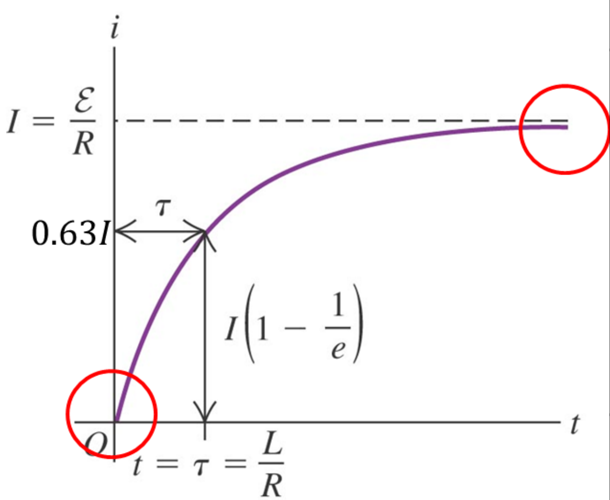
RL Circuit



- Solving this 1st order differential equation when the switch is closed at t = 0:
- $\frac{dI}{dt} = -\frac{R}{L}\left(I - \frac{\varepsilon}{R}\right) \Rightarrow I(t) = \frac{\varepsilon}{R}\left(1 - e^{-t/\left(\frac{L}{R}\right)}\right)$
- We can set $\tau = \frac{L}{R}$ is called time constant (units: seconds)
- $I(t) = \frac{\varepsilon}{R}\left(1 - e^{-t/\tau}\right)$
- Recall: Time constant is defined to be the time needed for a current to rise up to approximately 63% of its max, i.e. $I(\tau) = I(0)(1 - e^{-1})$
- It gives us an idea how fast the current changes with time in a RL circuit

Important Concepts: RL Circuit





	$t = 0^+$ Right after the switch is closed	$t = \infty$ After the switch is closed for a long time
Current, $I(t)$	Current is zero, $I(0^+) = 0$	Current is steady, $I(\infty) = const.$
Rate of change of current	$\frac{dI(t)}{dt} > 0$	$\frac{dI(t)}{dt} = 0$
Behaviour of the inductor	Inductor induces a back emf (like a battery in opposite direction of the current) to oppose the current	Inductor does nothing and looks like a wire
Behaviour between a and b	Looks like an open circuit momentarily between a and b	Looks like a short circuit between a and b

Concept Question 1.1:

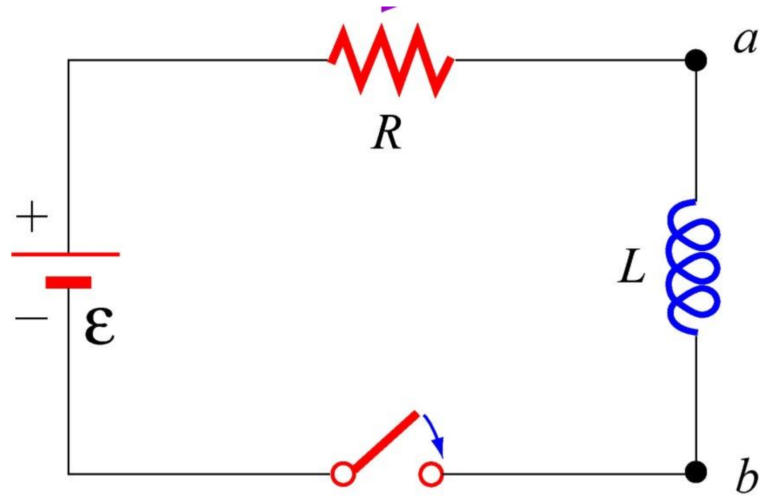
In the RL circuit below, what is the maximum emf that can be self-induced by the L?

A. ε

B. $\frac{\varepsilon}{2}$

C. 2ε

D. $> \varepsilon$



Concept Question 1.1 Solution:

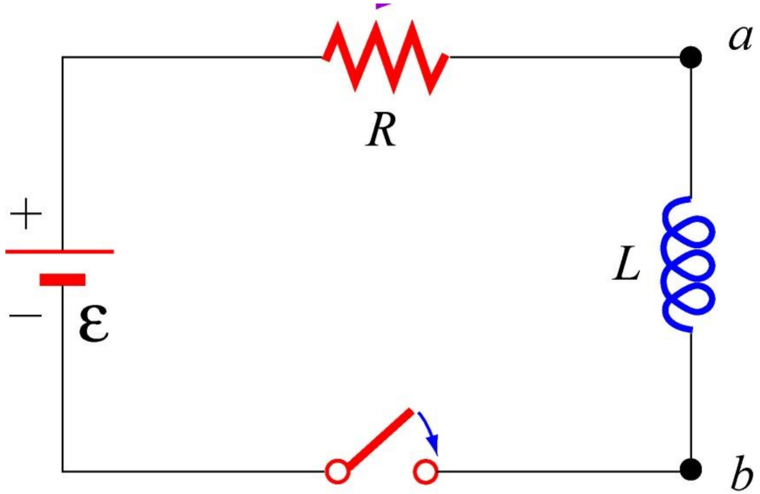
- In the RL circuit below, what is the maximum emf that can be self-induced by the L?

  A. ε

  B. $\frac{\varepsilon}{2}$

  C. 2ε

  D. $> \varepsilon$



- Answer: A
- The self-induced emf by the inductor can never be greater than the emf supplied by the battery.

Concept Question 1.2: "Voltage" Across Inductor

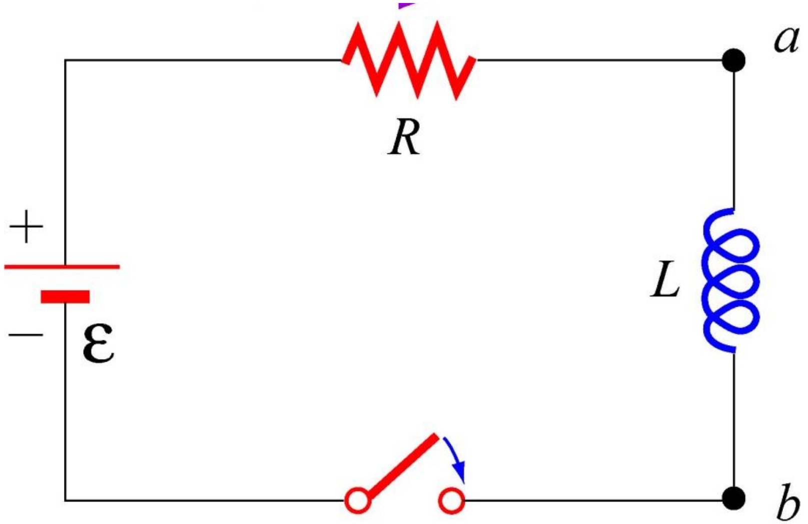
- In the circuit at right the switch is closed at t = 0. A voltmeter hooked across the inductor will read:

A. $V_L = \varepsilon e^{-\frac{t}{\tau}}$

B. $V_L = \varepsilon\left(1 - e^{-\frac{t}{\tau}}\right)$

C. $V_L = 0$

D. $V_L = \varepsilon$, which is a constant.



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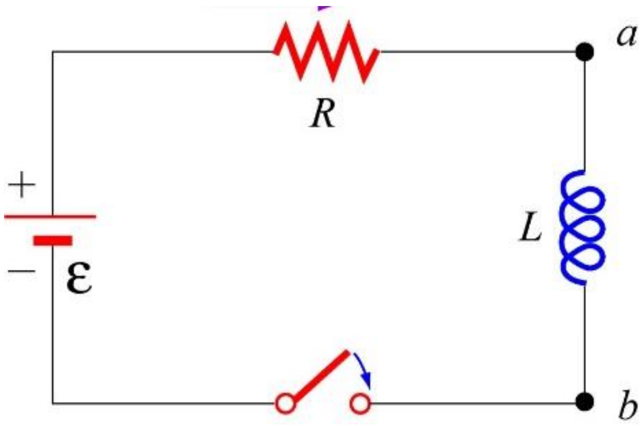
Concept Question 1.2: Solution

- Answer: A. $V_L = \varepsilon e^{-\frac{t}{\tau}}$

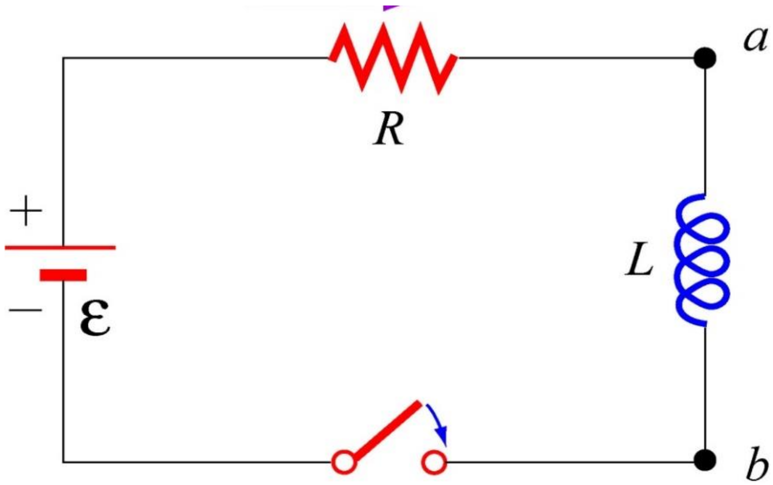
- The inductor "*works hard*" at first, preventing current flow, then "*relaxes*" as the current becomes constant in time.
- Although "voltage differences" between two points isn't completely meaningful for the inductor, we certainly can hook a voltmeter across an inductor and measure the EMF it generates.



Concept Question 1.3: Inserting a Core

From the simple RL circuit below, when you insert the iron core into the solenoid (inductor), what happens to B in the solenoid, the inductance L and the time constant, τ of the RL circuit?

A. B increases, L increases, τ increases.
B. B increases, L decreases, τ increases.
C. B increases, L decreases, τ decreases.
D. B decreases, L increases, τ increases.
E. B decreases, L decreases, τ increases.
F. B decreases, L decreases, τ decreases.
G. B, L and τ stay the same.



Concept Question 1.3: Solution

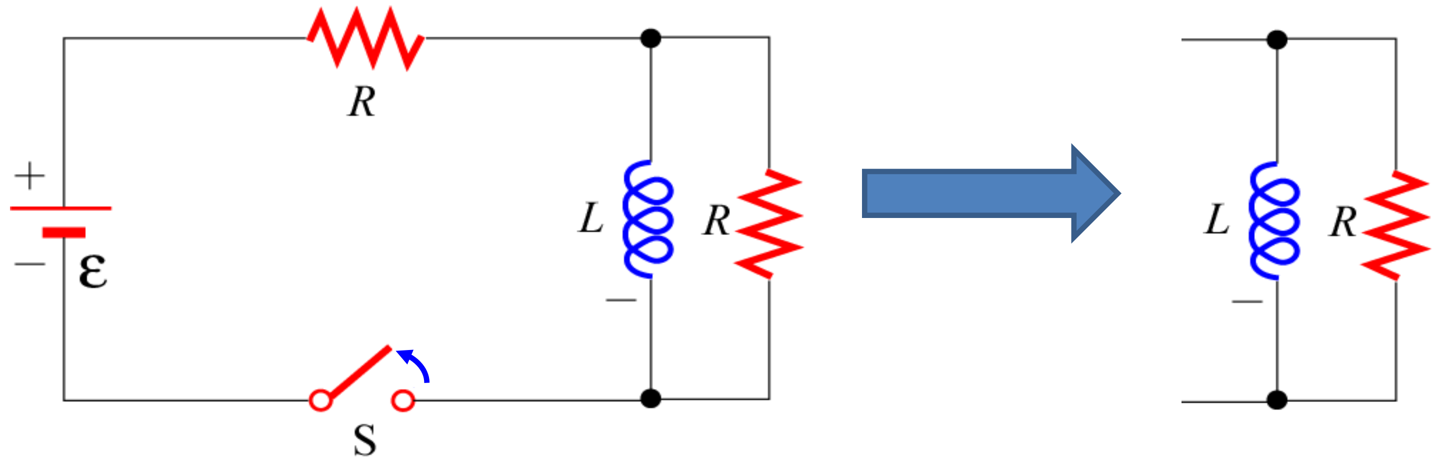
- Answer: A. B increases, L increases, τ increases.

- When you insert the iron core into the coil, the magnetic field increases in the core. It is because the dipole moments of the iron core align with the external field, increase the overall B field in space. Hence, the flux through the coil increases and thus its inductance too.
- The time constant, $\tau = \frac{L}{R}$. The L increases, the τ increases.

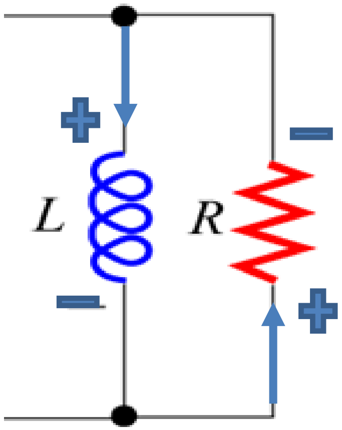
Case Problem 1.1: Current Decay in RL Circuit

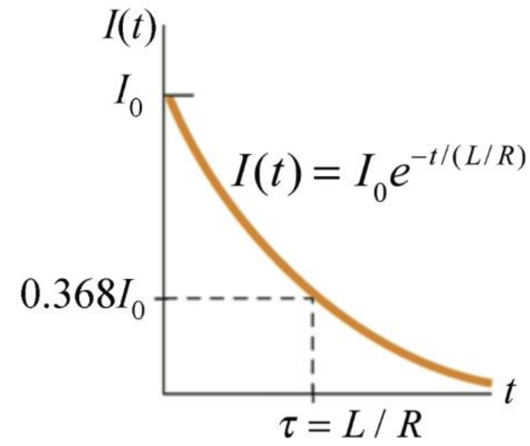
- Following the concept question, after the switch is closed for a long time, at t=0, the switch is open.

1. Form the differential equation for the current I in this RL circuit, using Kirchhoff's Voltage Law, after the switch is open.
2. With the initial condition, at $t = 0, I(t = 0) = I_o = \frac{\varepsilon}{R}$, solve the equation to get the current as a function of time, $I(t)$.



Case Problem 1.1: Solution





- Based on KVL,
- $-L\dfrac{dI}{dt} - IR = 0 \;\rightarrow\; L\dfrac{dI}{dt} = -IR$
- The solution: $I = I_o e^{-\left(\frac{R}{L}\right)t} = I_o e^{-\frac{t}{\tau}}$
- Where $\tau = \frac{L}{R}$ (time constant)

Note:

- At $t = 0^+$,
  - The current maintains the same value as before though the switch is open.
  - The inductor acts like a battery (induces a back emf) to maintain the current while the current is trying to reduce.
- At $t = \infty$,
  - No more change of the current, no more emf induced. Nothing happens anymore.

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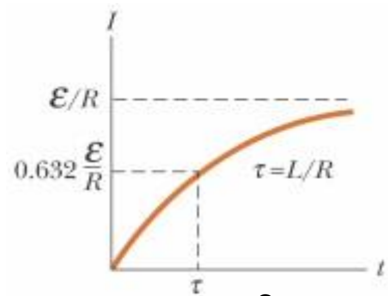
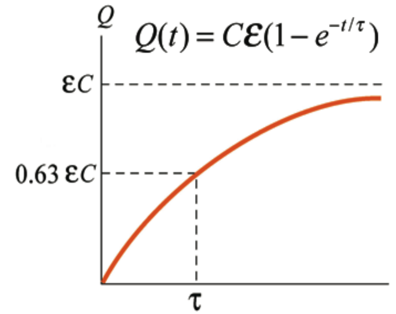
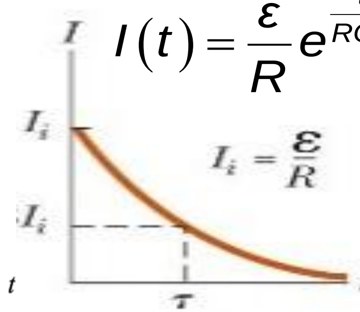
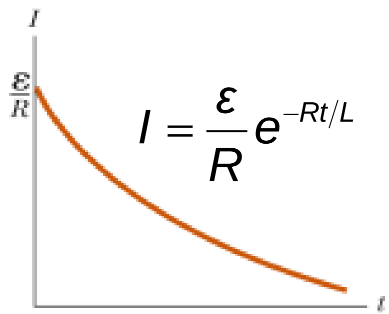
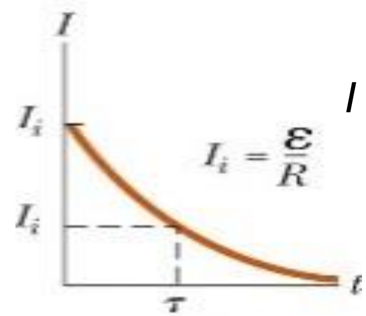
Summary: RL and RC Circuits – A Comparison

	RL	***RC***
Charging	 $I = \frac{\varepsilon}{R}\left(1 - e^{-Rt/L}\right)$	  $I(t) = \frac{\varepsilon}{R} e^{\frac{-t}{RC}}$
Discharging	 $I = \frac{\varepsilon}{R} e^{-Rt/L}$	 $I(t) = \frac{Q}{RC} e^{\frac{-t}{RC}}$
Energy	$U_L = \frac{1}{2} L I^2$	$U_C = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2} C(\Delta V)^2 = \frac{1}{2} Q(\Delta V)$

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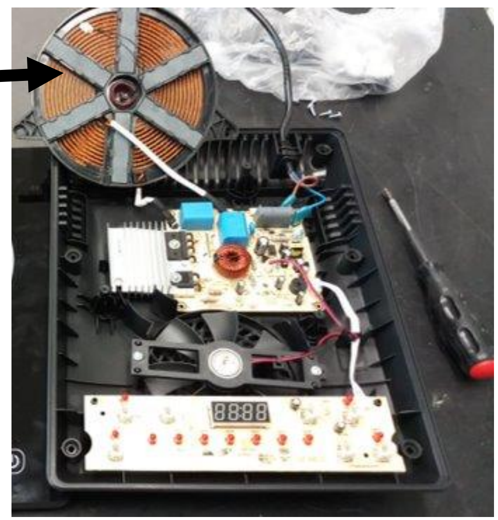
Concept 2: LC Circuit Analysis

Recall:

- We learnt that induction cookers use changing magnetic flux to induce eddy current to heat up the bottom of a ferromagnetic pot.
- An alternating current is passing through a wire coil to generate a changing magnetic flux.
- We are going to learn how to generate this alternating current from the circuit point of view.

Primary coil



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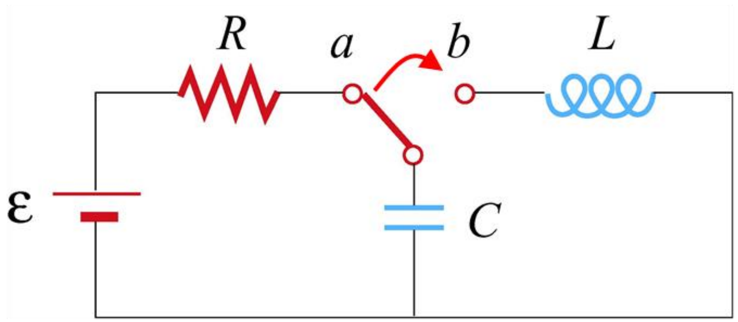
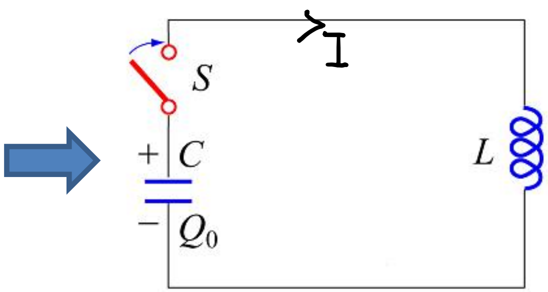
Application: LC circuit

- All commercial wireless communication systems (phone, GPS, Bluetooth, RFID, etc.) have an oscillation (LC) circuit at specific frequency (MHz to GHz) to transmit and receive signals.
- Tuning circuit in TV, radio, phone, etc.
- Metal Detector https://youtu.be/FWMhk6x785Q (42:32 - 50:09)
- Many other electronic devices to generate resonant oscillating signal: amplifier, filter, tuner, oscillator, mixer, etc.

LC Circuit and Equation

- Consider the circuit below with capacitor, inductor, resistor, and battery. Let the capacitor become fully charged. Throw the switch from ***a*** to ***b***. What happens?
- Firstly, it forms a simple LC circuit.



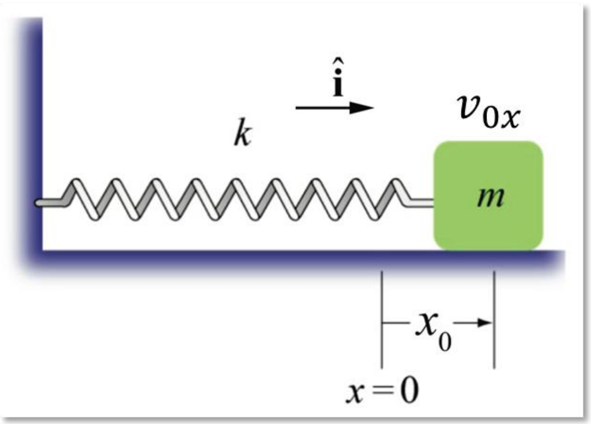


- By KVL: $\frac{Q}{C} - L\frac{dI}{dt} = 0$
- Note that it is a discharging process: $I = -\frac{dQ}{dt} \rightarrow \frac{dI}{dt} = -\frac{d^2Q}{dt^2}$

$$\frac{d^2Q}{dt^2} + \frac{1}{LC}Q = 0 \quad \text{OR} \quad \frac{d^2Q}{dt^2} = -\frac{1}{LC}Q$$

This is a **Simple Harmonic Motion (SHM)** differential equation. We saw this equation in Physical World!

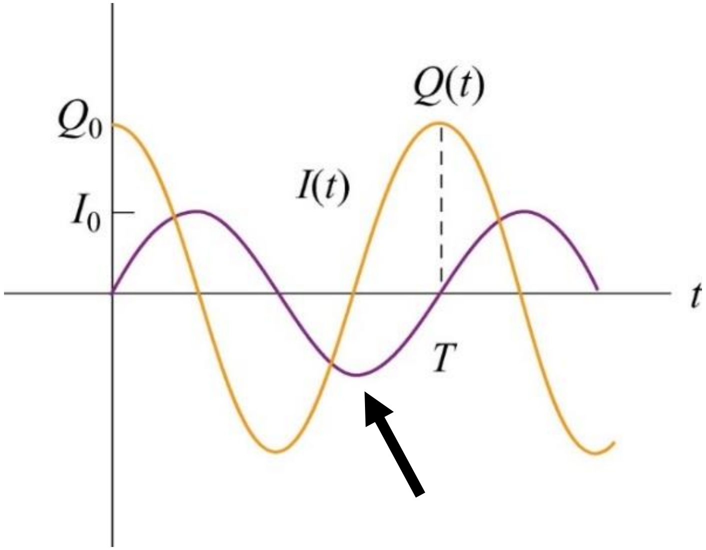
LC Circuit Solution



- Recall SHM differential eq.: $\frac{d^2x}{dt} = -\omega_o x$
- The general solution: $\mathrm{x}(t) = A\cos(\omega_o t + \phi)$

- In LC circuit: $\frac{d^2Q}{dt^2} = -\frac{1}{LC}Q$
- General Solution: $Q(t) = Q_o \cos(\omega_o t + \phi)$
- Q_O: Amplitude of charge oscillation
- ϕ: Phase (time offset)
- The oscillation angular frequency, $\omega_o = \frac{1}{\sqrt{LC}}$
- Note 1: Q_O and ϕ can be determined from the initial conditions given.

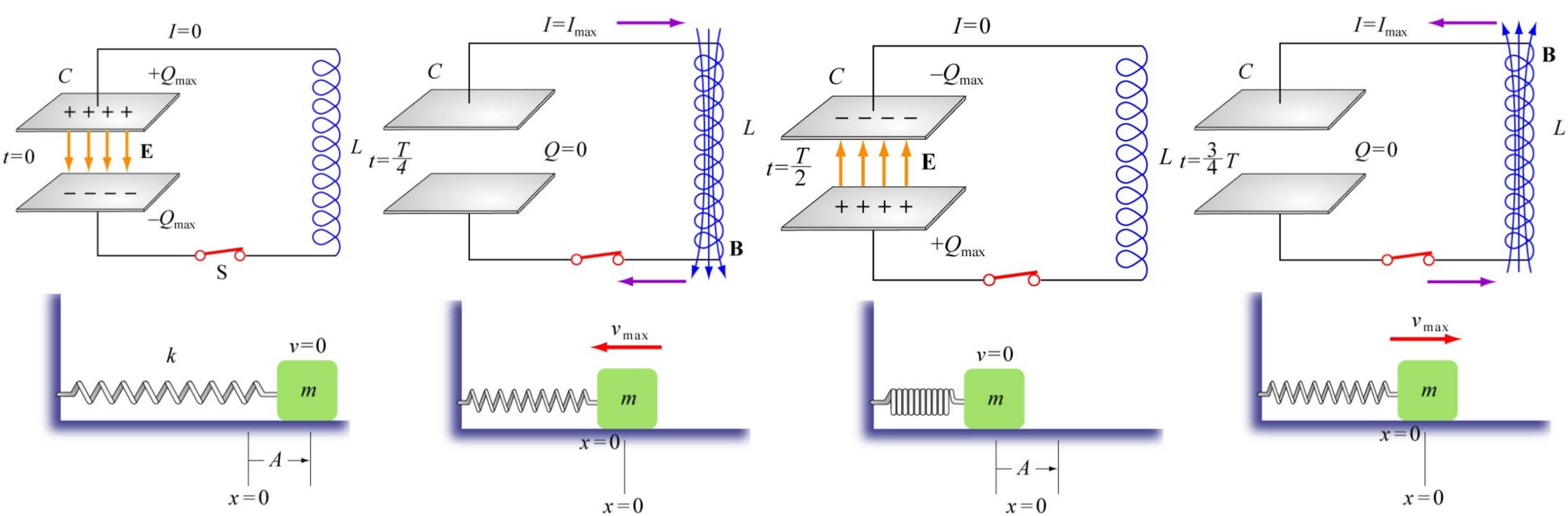
  Note 2: $I = -\frac{dQ}{dt} = Q_o\omega_o \sin(\omega_o t + \phi)$



Notice relative phases between Q and I

LC Circuit and Spring-Mass Analogy

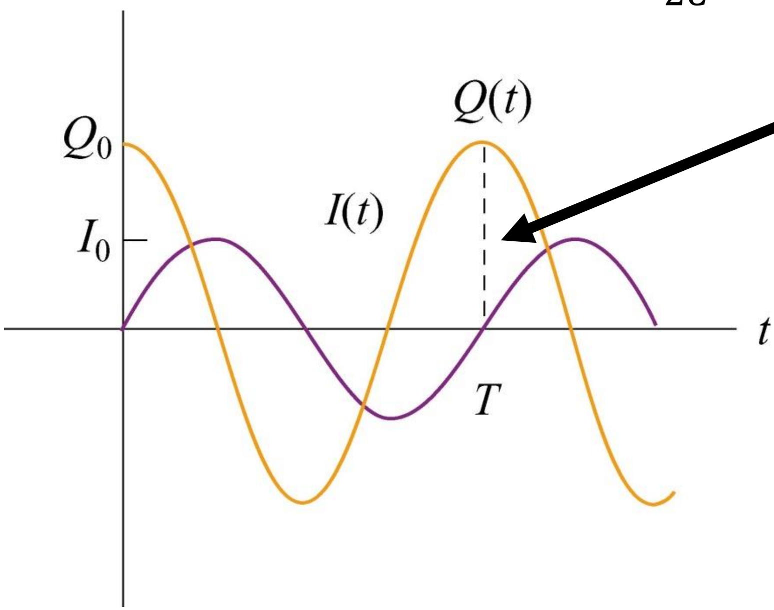
- Inductor L and capacitor C cause the charge and current oscillating in the circuit. Thus, LC circuit is also known as oscillating circuit or oscillator.
- The effect is analogous to mechanical mass on a spring system - **Simple Harmonic Motion**, with trade-off between charge on capacitor (Spring) and current in inductor (Mass).



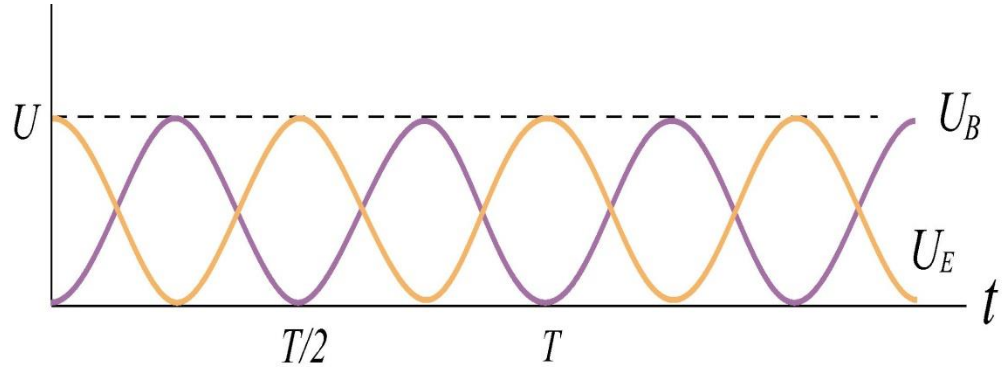
LC Oscillations: Energy point of view

- Energy stored in C, $U_E = \frac{Q^2}{2C} = \left(\frac{Q_o^2}{2C}\right)\cos^2 \omega_o t$
- Energy stored in L, $U_B = \frac{1}{2}LI^2 = \frac{1}{2}LI_O^2 \sin^2 \omega_o t = \left(\frac{Q_o^2}{2C}\right)\sin^2 \omega_o t$
- Note: $\frac{1}{2}LI_O^2 = \frac{1}{2}\frac{Q_o^2}{C}$ (U_{max} in L = U_{max} in C)
- Thus, $U_{total} = U_E + U_B = \frac{Q_o^2}{2C} = \text{constant}$

Total energy is conserved !!

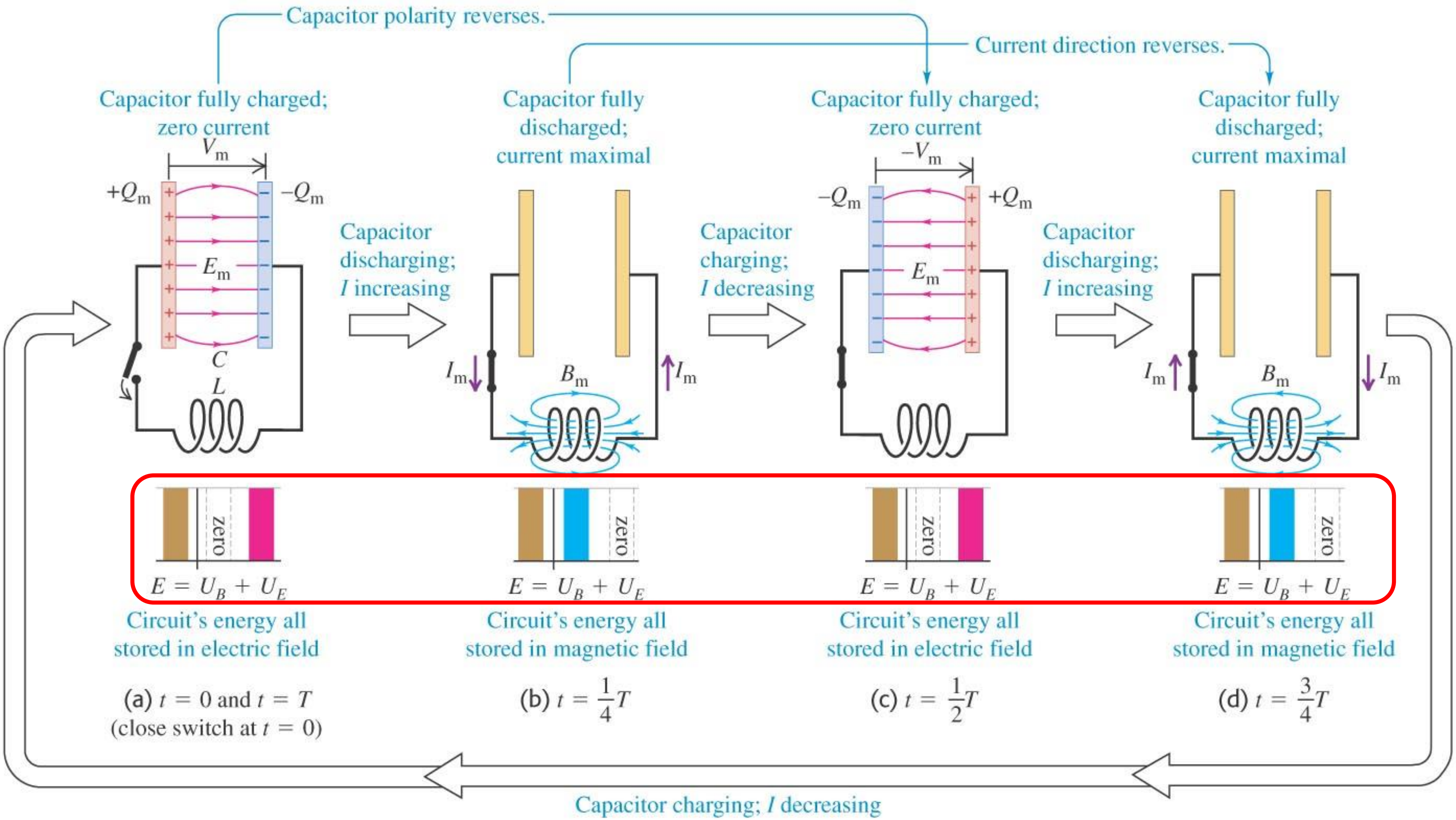


Notice relative phases



FYI: LC circuit – Energy point of view

- An *L-C circuit* contains an inductor and a capacitor and is an *oscillating* circuit.



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LC Oscillations: From Energy Eq. to SHM Differential Eq.

$$U_t = U_E + U_B = \frac{Q^2}{2C} + \frac{1}{2}LI^2 = \frac{Q_o}{2C} = \text{constant}$$

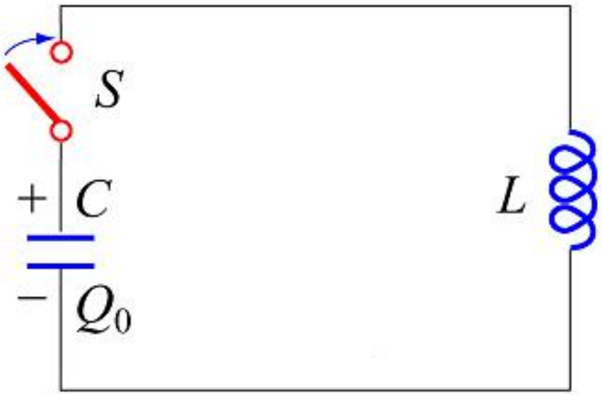
Total energy is conserved !! It also implies $\frac{dU_t}{dt} = 0$ (Recall Physical World!).

Let's differentiate U_t wrt. time

$$\frac{dU_t}{dt} = 0 = \frac{d}{dt}\left(\frac{Q^2}{2C} + \frac{1}{2}LI^2\right) = \frac{Q}{C}\frac{dQ}{dt} + LI\frac{dI}{dt}$$

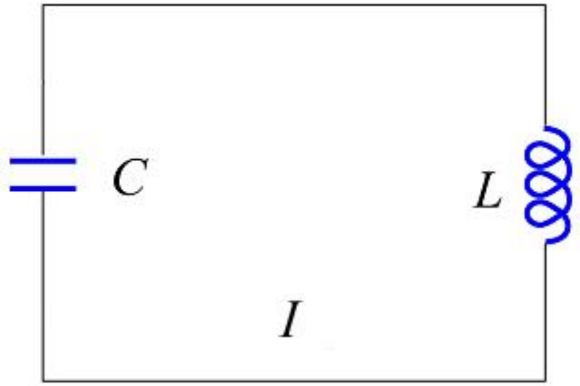
Since $I = -\frac{dQ}{dt}$, and $\frac{dI}{dt} = -\frac{d^2Q}{dt^2}$, we get $0 = \frac{Q}{C}(-I) + LI\left(-\frac{d^2Q}{dt^2}\right) = 0$

Thus, $\frac{d^2Q}{dt^2} + \frac{Q}{LC} = 0$ (leads back to the differential eq. of SHM)



Concept Question 2.1: LC Circuit

Consider the LC circuit on the right. At the time shown, the current has its maximum value. At this time:



A. The charge on the capacitor has its maximum value.

B. The magnetic field is zero.

C. The electric field has its maximum value.

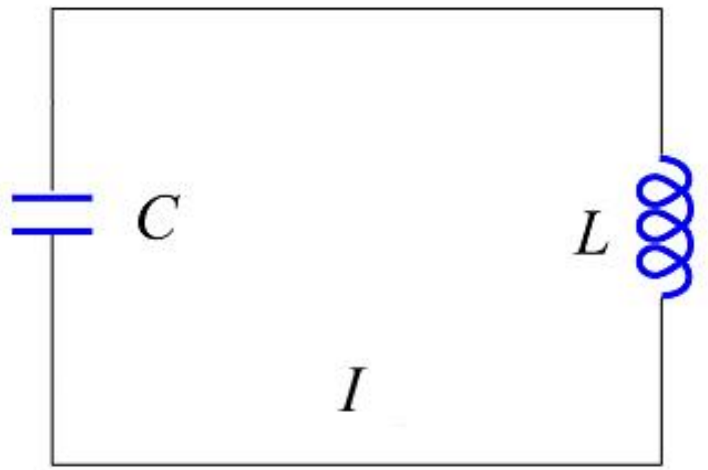
D. The charge on the capacitor is zero.

E. Don't have a clue. ☹

Concept Question 2.1: Solution

Answer: D. The current is maximum when the charge on the capacitor is zero

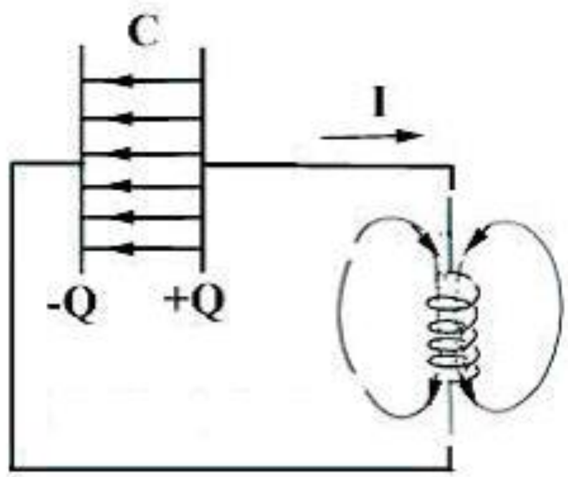
Current and charge are exactly 90 degrees out of phase in an ideal LC circuit (no resistance), so when the current is maximum, the charge must be identically zero.



Concept Question 2.2: LC Circuit

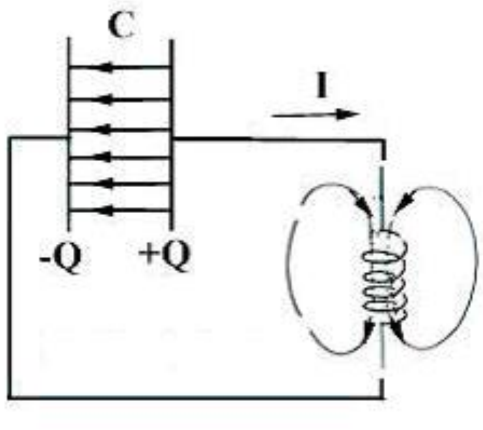
In the LC circuit on the right, the current is in the direction shown and the charges on the capacitor have the signs shown. At this time,

A. I is increasing and Q is increasing.
B. I is increasing and Q is decreasing.
C. I is decreasing and Q is increasing.
D. I is decreasing and Q is decreasing.
E. Don't have a clue.



Concept Question 2.2: Solution

Answer: B. I is increasing; Q is decreasing.



With current in the direction shown, the capacitor is discharging (Q is decreasing).

But since ***Q*** on the right plate is positive, ***I*** must be increasing. The positive charge *wants* to flow, and the current will increase until the charge on the capacitor changes sign. That is, we are in the first quarter period of the discharge of the capacitor, when ***Q*** is decreasing and positive and ***I*** is increasing and positive.

Case Problem 2.1: LC Circuit

Consider the circuit shown in the figure below. Suppose the switch that has been connected to point a for a long time is suddenly thrown to b at t=0. Find the following quantities:

A. The frequency of oscillation of the LC circuit.

B. The maximum charge that appears on the capacitor.

C. The total energy the circuit possesses at any time t.

D. The maximum current in the inductor.

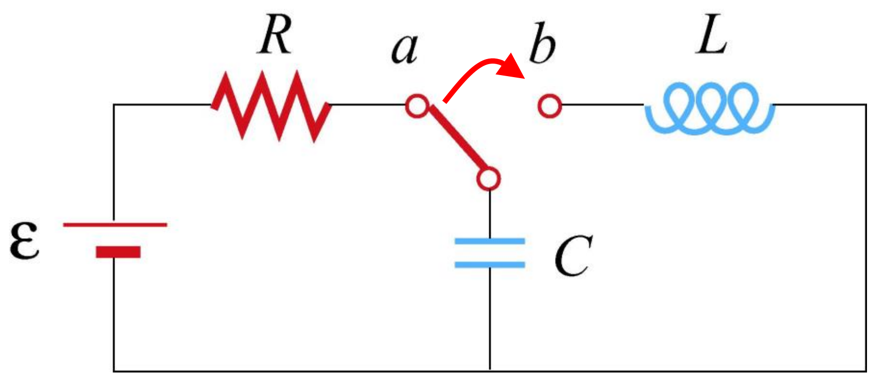


Figure *LC* circuit

Case Problem 2.1: Solution

A. The angular frequency of oscillation of the LC circuit is given by $\omega = \frac{1}{\sqrt{LC}} = 2\pi f$. Therefore, the frequency is $f = \frac{1}{2\pi\sqrt{LC}}$

B. From $Q = C\,|\Delta V_c|$, the maximum charge stored in the capacitor before the switch is thrown to b is $Q_o = C\varepsilon$.

C. At any time, the total energy in the LC circuit is equal to the initial energy that stored in the capacitor , $U_c = \frac{1}{2}C\varepsilon^2$.

D. In LC oscillation, the energy in the capacitor would be transferred to the inductor. When the energy is maximum in L, the current through it is also maximum. Thus,

$$\frac{1}{2}C\varepsilon^2 = \frac{1}{2}LI_o^2 \;\Rightarrow I_o = \sqrt{\frac{C}{L}}\,\varepsilon$$

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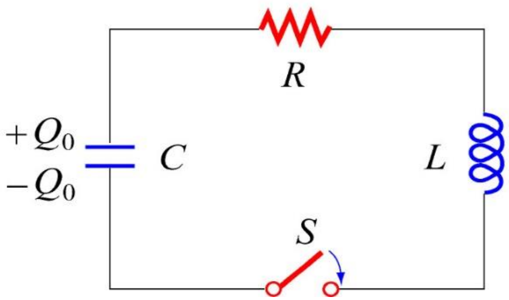
Concept 3: Undriven RLC Circuits

What happens when you add a R to the LC circuit?

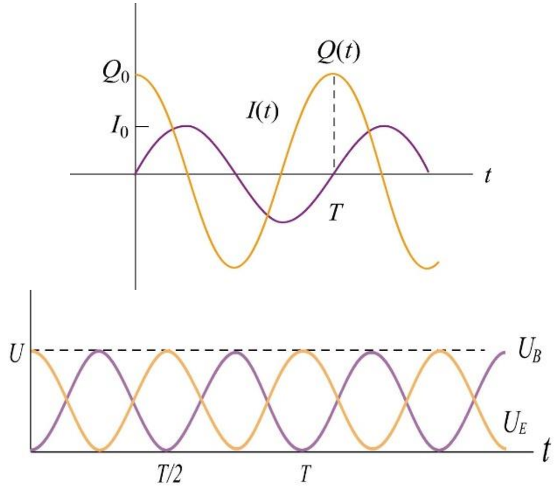
For the LC circuit we saw that energy is conserved and it oscillates between being mostly stored in the inductor (strong magnetic field there) or mostly stored in the capacitor (strong electric field there).

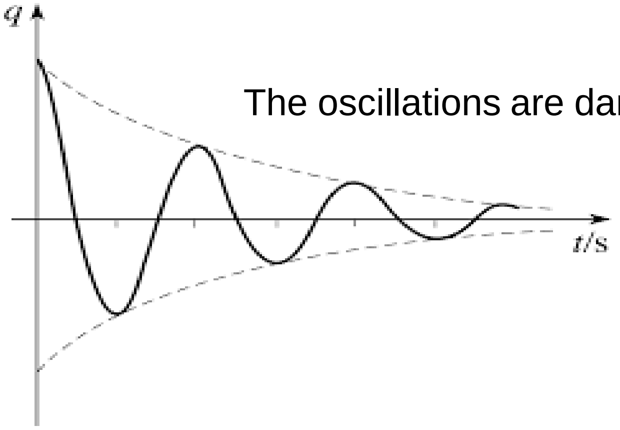


Suppose we now add a resistance to the circuit.

Each time the current goes through the resistor, some energy is lost.

This implies that the oscillations cannot last forever, but the charge and current will eventually be completely damped.

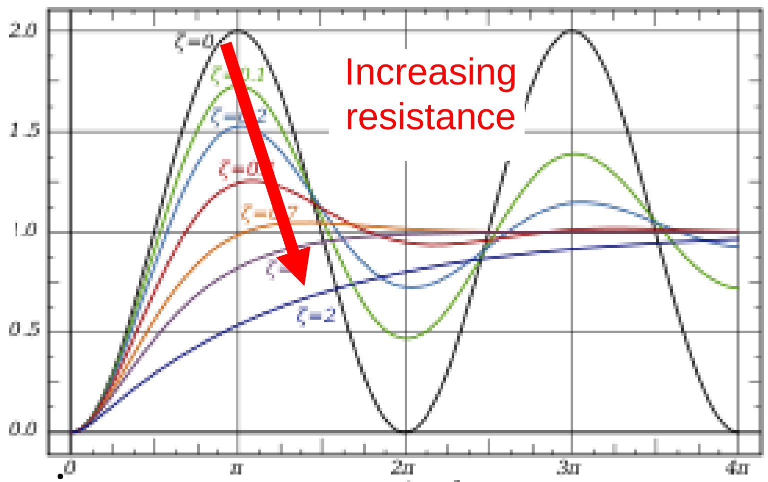




What happens when you add a R to the LC circuit?

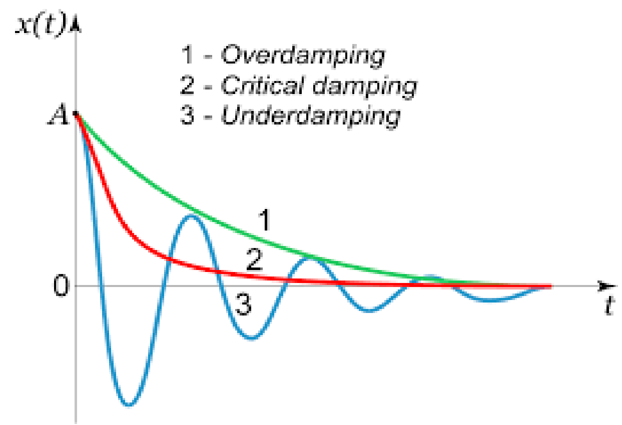
If the resistance is small, the oscillations will decay slowly, but if the resistance is very large, the energy can be lost even before a single oscillation is done.

Current or charge



Time

or



3 possible scenarios:

- Overdamping -> R is large enough and no oscillations occur (exponential decay only)
- Critical damping -> R is just nice that the behavior is neither oscillatory not exponential decay
- Underdamping -> R is small enough and oscillations are still present (exponential decay envelope of oscillations)

Appendix

Quick Comparison: Resistance, Capacitance and Inductance

Resistors	**Capacitors**	**Inductors**
Ohm's law defines resistance. $$R \equiv \frac{\Delta V}{I}$$	Capacitance, the ability to hold charge.	Inductance, the ability to "hold" current (moving charge).
Do not store energy, but dissipate energy.	Capacitors store electric energy once they have been connected across a potential difference	Inductors store magnetic energy once current flows through it.
They transform electrical energy into thermal energy at a rate of $$P = \Delta V \cdot I = \frac{\Delta V^2}{R} = I^2 R$$	Capacitors store electric energy once charged: $$U_E = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}C(\Delta V)^2$$	Magnetic energy stored in the inductor. $$U_B = \frac{1}{2}LI^2$$

Extra Case Problem 1

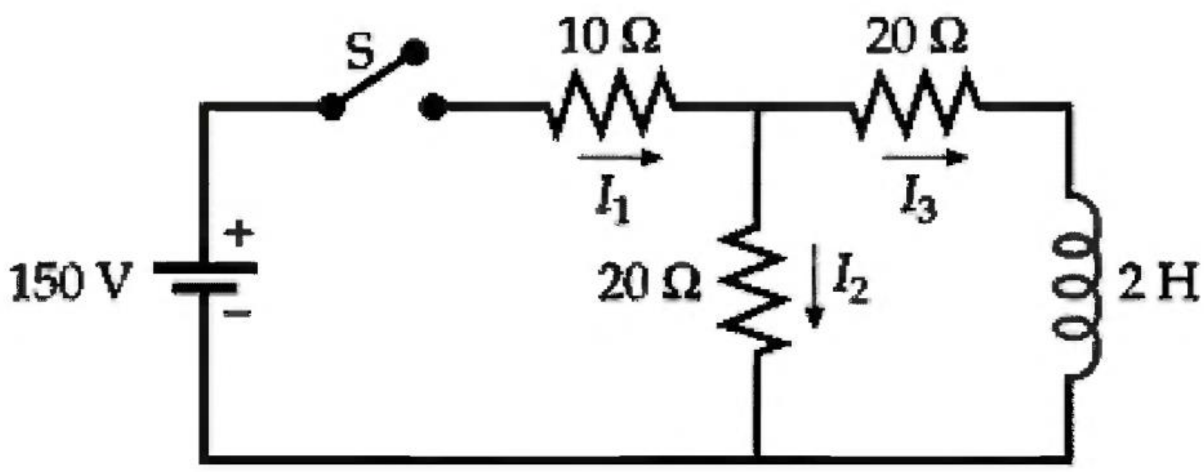
For the circuit shown in the figure. Find currents I_1, I_2, I_3

A. immediately after switch S is closed, $t = 0^+$.

B. a long time after switch S has been closed, $t = T$

C. immediately after switch S has been opened, $t = T^+$.

D. a long time after switch S has been opened, $t \to \infty$.



Extra Case Problem 1 Solution

A. The current in an inductor cannot change instantaneously from 0 A to the steady state current, therefore, the current in the inductor must be zero after the switch is closed. $I_3(t=0)=0$.

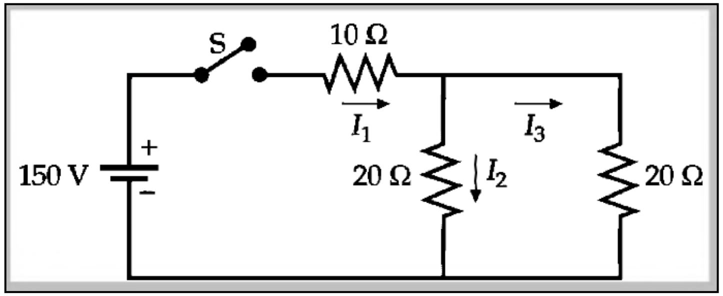
The current in the left loop equals the EMF divided by the equivalent resistance of the two resistors in series. $R_{eq}=10\Omega+20\Omega=30\Omega$

$$I_1=I_2=\frac{150V}{30\Omega}=5A$$

B. A long time after switch S has been closed, the current reaches its steady-state value, dI/dt = 0, so there is no potential drop across the inductor. The inductor acts like a short circuit (a wire with zero resistance).

$R_{eq}=[(20\Omega)^{-1}+(20\Omega)^{-1}]^{-1}+10\Omega=20\Omega$. Thus, $I_1=\frac{150V}{20\Omega}=7.5\text{A}$.

With a current divider, $I_2=I_3=3.75A$.



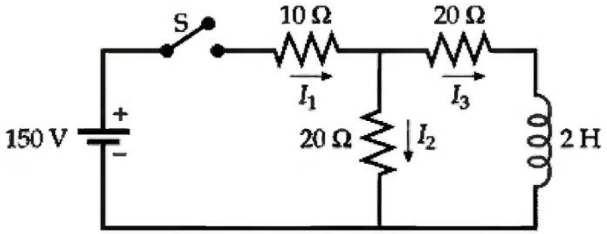
Extra Case Problem 1 Solution

C. immediately after switch S has been opened.

The current in an inductor cannot change instantaneously from the steady state current to 0 A, therefore, the current in the inductor must be the same after the switch is opened as it was just before the switch is opened.

Thus, $I_3 = 3.75\ A$.

- When the switch is opened, I_1 drops to 0 A.
- To oppose the change in the direction of the magnetic flux through the inductor, the direction of the induced current in the loop changes direction; $I_2 = -I_3 = -3.75\ A$.



D. all currents must be 0A, a long time after the switch is opened. $I_1 = I_2 = I_3$