10.018 Modelling Space and Systems

Lecture 3: Optimization

Term 2, 2021

SINGAPORE UNIVERSITY OF TECHNOLOGY AND DESIGN

Before we start....

To get the most out of this lecture, you should already be familiar with

1. Minima & Maxima (Math I)
2. Finding global optima (Math I)
3. Extreme Value Theorem (Math I)

as we will be going through

1. Optimization in Math II: The 2D analogue of optimization

Math Modeling

You have finished (or almost finished) your FIRST math modeling problem!

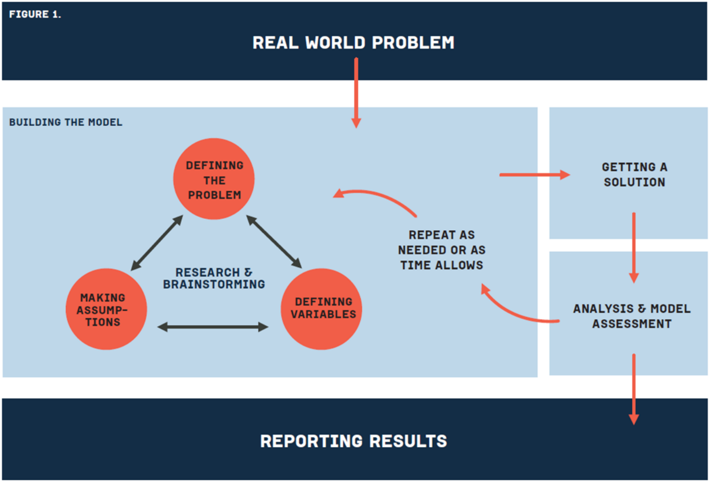
Math Modeling

You have finished (or almost finished) your FIRST math modeling problem!

Pat yourself on the back for your accomplishments!

Math Modeling

You have done just one cycle of a Math Modeling process. The full picture looks something like this:

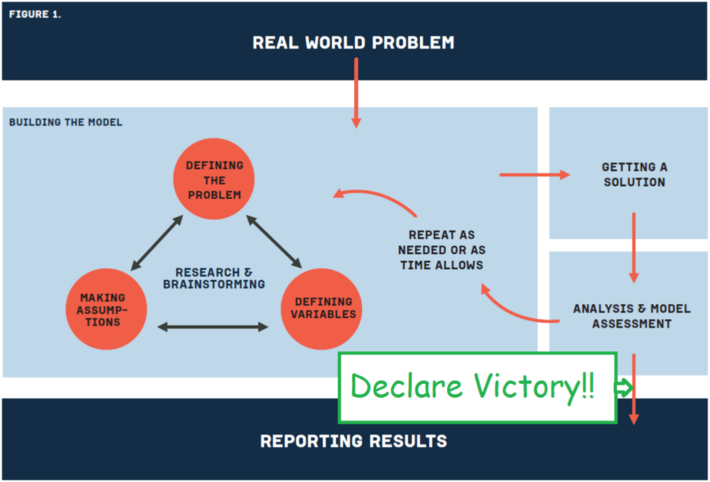


Math Modeling

So when do you stop iterating and cycling through the process making your model better and better after each cycle?

Math Modeling

So when do you stop iterating and cycling through the process making your model better and better after each cycle?



Go global!

Solving an optimization problem typically requires finding a global optimum (either global min or global max):

- Maximize profit.
- Minimize costs.
- Maximize production etc.

However, the sufficient conditions for min/max from previous week give us only local min/max.
We will learn a few tools to determine if the found min/max local or global.

Example

Consider the problem:

$$\min\ x^2 + 3x + y^2 - 2y, \qquad x, y \in \mathbb{R}.$$

Let $f(x, y)$ be the objective function, and set $\nabla f = \vec{0}$:

$$\begin{aligned} \frac{\partial f}{\partial x} &= 2x + 3 = 0 \\ \frac{\partial f}{\partial y} &= 2y - 2 = 0. \end{aligned}$$

Critical point: $x_0 = -3/2, y_0 = 1$.
When we compute D at (x_0, y_0), as well as f_{xx}, we can conclude that it is a local minimum.

Example

Consider the problem:

$$\min\ x^2 + 3x + y^2 - 2y, \qquad x, y \in \mathbb{R}.$$

Let $f(x, y)$ be the objective function, and set $\nabla f = \vec{0}$:

$$\begin{aligned} \frac{\partial f}{\partial x} &= 2x + 3 = 0 \\ \frac{\partial f}{\partial y} &= 2y - 2 = 0. \end{aligned}$$

Critical point: $x_0 = -3/2, y_0 = 1$.
When we compute D at (x_0, y_0), as well as f_{xx}, we can conclude that it is a local minimum.

Is it a global minimum?

Example

We just need to complete the square

$$
\begin{aligned}
x^2 + 3x + y^2 - 2y &= x^2 + 2\,\frac{3}{2}x + \left(\frac{3}{2}\right)^2 + y^2 - 2y + 1 - \left(\frac{3}{2}\right)^2 - 1 \\
&= (x + \frac{3}{2})^2 + (y-1)^2 - \frac{13}{4} \\
&\geq -\frac{13}{4} = f(-\frac{3}{2}, 1).
\end{aligned}
$$

In other words, $f(x, y)$ is always greater or equal than $f(x_0, y_0)$. Hence, it is a **global minimum**.

AM-GM inequality

Another example of an inequality that could be very useful is an inequality of arithmetic and geometric means. It states

AM-GM inequality

For nonnegative real numbers $x_1, \ldots, x_n$ arithmetic mean is always at least their geometric mean:

$$\frac{x_1 + \ldots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \ldots x_n}.$$

with equality if and only if $x_1 = x_2 = \ldots = x_n$.

We will illustrate an application of this inequality in determining whether min/max is global or not.

AM-GM inequality intuition

Application of AM-GM inequality

Assume we have found that $(10, 10)$ is a local minimum of $f(a, b) = 0.25ab + \frac{250}{b} + \frac{250}{a}$. But is $(10, 10)$ a global minimum?

Recall the AM-GM inequality:

$$\frac{x_1 + \ldots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \ldots x_n}.$$

Applying it to this case:

$$\begin{aligned} \frac{0.25ab + \frac{250}{b} + \frac{250}{a}}{3} &\geq \sqrt[3]{0.25ab \cdot \frac{250}{b} \cdot \frac{250}{a}} \\ &= \sqrt[3]{\frac{250 \cdot 250}{4}} = 25 = \frac{f(10, 10)}{3} \end{aligned}$$

Re-arranging, we see that $f(a, b) \geq f(10, 10)$ for all a, b! Therefore, $(10, 10)$ is a global minimum.

Disproving global min/max

Recall $\vec{x}^*$ a global minimum if $f(\vec{x}^*) \leq f(\vec{x})$ for all $\vec{x}$.

In cohort 2.2 we found a local minimum at $(18, 6)$ of a function $f(x, y) = x^2 + y^3 - 6xy$.

Disproving global min/max

Recall $\vec{x}^*$ a global minimum if $f(\vec{x}^*) \leq f(\vec{x})$ for all $\vec{x}$.

In cohort 2.2 we found a local minimum at $(18, 6)$ of a function $f(x, y) = x^2 + y^3 - 6xy$.

Is it a global minimum? How to disprove it?

Disproving global min/max

Recall $\vec{x}^*$ a global minimum if $f(\vec{x}^*) \leq f(\vec{x})$ for all $\vec{x}$.

In cohort 2.2 we found a local minimum at $(18, 6)$ of a function $f(x, y) = x^2 + y^3 - 6xy$.

Is it a global minimum? How to disprove it?

If we want to *disprove* it, we need to demonstrate just one point with the function value *lower* than the value $f(18, 6) = -108$.

Disproving global min/max

Recall $\vec{x}^*$ a global minimum if $f(\vec{x}^*) \leq f(\vec{x})$ for all $\vec{x}$.

In cohort 2.2 we found a local minimum at $(18, 6)$ of a function $f(x, y) = x^2 + y^3 - 6xy$.

Is it a global minimum? How to disprove it?

If we want to *disprove* it, we need to demonstrate just one point with the function value *lower* than the value $f(18, 6) = -108$.

Just consider $x = 0, y = -100$, then
$f(0, -100) = -10^6 < f(18, 6)$, so $(18, 6)$ **can't** be a global min!

Constrained Optimization

Suppose we have a constrained maximization problem (P1):

$$\max_{(x_1,\ldots,x_n)\in\mathbb{R}^n} f(x_1,\ldots,x_n), \quad \text{subject to} \quad \begin{aligned} g_1(x_1,\ldots,x_n) &= 0,\\ &\vdots\\ g_s(x_1,\ldots,x_n) &= 0,\\ h_1(x_1,\ldots,x_n) &\geq 0,\\ &\vdots\\ \text{and } h_t(x_1,\ldots,x_n) &\geq 0 \end{aligned}$$

Here $g_1(x_1,\ldots,x_n)=0,\ \ldots,\ g_s(x_1,\ldots,x_n)=0$ are the equality constraints and $h_1(x_1,\ldots,x_n)\geq 0,\ \ldots,\ h_t(x_1,\ldots,x_n)\geq 0$ are the inequality constraints that define the feasible region.

Constrained optimization

Consider

$$\min\{f(\vec{x}) : \vec{x} \in \mathbb{R}^n\} \qquad \text{(P2)}.$$

(P1) is called an constrained optimization problem, (P2) is called a unconstrained optimization problem.

So far we only dealt with problems of the type (P2). How do we approach problems of the type (P1), constrained optimization problems?

An example

Find the global minimum of the following problem:

$$\left.\begin{array}{lrcl}
\min & x^2+3x+y^2-2y & & \\
\text{s.t.:} & & & \\
 & x^2+y^2 & \leq & 4 \\
 & x-y & \leq & 0
\end{array}\right\}$$

An example

Find the global minimum of the following problem:

$$\left.\begin{array}{lrcl} \min & x^2 + 3x + y^2 - 2y & & \\ \text{s.t.:} & & & \\ & x^2 + y^2 & \leq & 4 \\ & x - y & \leq & 0 \end{array}\right\}$$

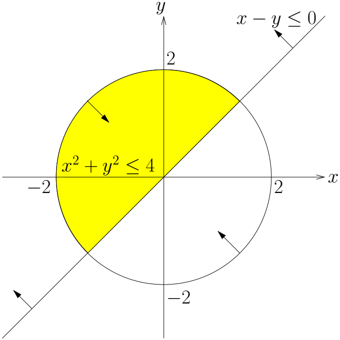
How do we proceed?

An example

Find the global minimum of the following problem:

$$\left.\begin{array}{lrcl} \min & x^2 + 3x + y^2 - 2y & & \\ \text{s.t.:} & & & \\ & x^2 + y^2 & \leq & 4 \\ & x - y & \leq & 0 \end{array}\right\}$$

How do we proceed?



Relaxation

One problem solving strategy is: **ignore them!**

Relaxation

One problem solving strategy is: **ignore them!**
This is called a relaxation of the original problem.

Relaxation

One problem solving strategy is: **ignore them!**
This is called a relaxation of the original problem.

BEFORE:

$$\min\ x^2 + 3x + y^2 - 2y$$

subject to:

$$x^2 + y^2 \leq 4$$
$$x - y \leq 0$$

Relaxation

One problem solving strategy is: **ignore them!**
This is called a relaxation of the original problem.

BEFORE:

$$\min\ x^2 + 3x + y^2 - 2y$$

subject to:

$$x^2 + y^2 \leq 4$$
$$x - y \leq 0$$

AFTER:

$$\min\ x^2 + 3x + y^2 - 2y$$

This is a **relaxation** of the original problem.

How does this help? We know how to solve the relaxed problem (without constraints). How are the solutions to the two problems related?

Relaxation

Consider the following situation. Supposed you need to find the coldest room at SUTD, but you have to look only in Bldg 1.

$$\min \text{ (Temperature of the Room on Campus)} \qquad (P1)$$

subject to: Room in Bldg 1

Relaxation

Consider the following situation. Supposed you need to find the coldest room at SUTD, but you have to look only in Bldg 1.

$$\min \text{ (Temperature of the Room on Campus)} \qquad (P1)$$
$$\text{subject to: Room in Bldg 1}$$

A relaxed problem:

$$\min \text{ (Temperature of the Room on Campus)} \qquad (P2)$$

You can ask the Office of Campus Infrastructure, but they can't look exactly in Bldg 1. They can only give you info about the whole campus, i.e. they are solving the *relaxed* problem (P2).

How are the solutions of (P1) and (P2) related? There are two possible cases:

Relaxation

Consider the following situation. Supposed you need to find the coldest room at SUTD, but you have to look only in Bldg 1.

$$\min \text{ (Temperature of the Room on Campus)} \qquad (P1)$$

$$\text{subject to: Room in Bldg 1}$$

A relaxed problem:

$$\min \text{ (Temperature of the Room on Campus)} \qquad (P2)$$

You can ask the Office of Campus Infrastructure, but they can't look exactly in Bldg 1. They can only give you info about the whole campus, i.e. they are solving the *relaxed* problem (P2).

How are the solutions of (P1) and (P2) related? There are two possible cases:

Case 1: You found the coolest room is LT2 (which is in Bldg 1). Is this the solution to your original problem?

Relaxation

Consider the following situation. Supposed you need to find the coldest room at SUTD, but you have to look only in Bldg 1.

$$\min \text{ (Temperature of the Room on Campus)} \qquad (P1)$$
$$\text{subject to: Room in Bldg 1}$$

A relaxed problem:

$$\min \text{ (Temperature of the Room on Campus)} \qquad (P2)$$

You can ask the Office of Campus Infrastructure, but they can't look exactly in Bldg 1. They can only give you info about the whole campus, i.e. they are solving the *relaxed* problem (P2).

How are the solutions of (P1) and (P2) related? There are two possible cases:

Case 1: You found the coolest room is LT2 (which is in Bldg 1). Is this the solution to your original problem? Yes it is! It satisfies the constraint. Therefore it is the coldest room overall, and in Bldg 1 in particular!

Relaxation

Consider the following situation. Supposed you need to find the coldest room at SUTD, but you have to look only in Bldg 1.

$$\min \text{ (Temperature of the Room on Campus)} \qquad (P1)$$

$$\text{subject to: Room in Bldg 1}$$

A relaxed problem:

$$\min \text{ (Temperature of the Room on Campus)} \qquad (P2)$$

You can ask the Office of Campus Infrastructure, but they can't look exactly in Bldg 1. They can only give you info about the whole campus, i.e. they are solving the *relaxed* problem (P2).

How are the solutions of (P1) and (P2) related? There are two possible cases:

Case 1: You found the coolest room is LT2 (which is in Bldg 1). Is this the solution to your original problem? Yes it is! It satisfies the constraint. Therefore it is the coldest room overall, and in Bldg 1 in particular!

Case 2: You found the coolest room outside of Bldg 1.

Relaxation

Consider the following situation. Supposed you need to find the coldest room at SUTD, but you have to look only in Bldg 1.

$$\min \text{ (Temperature of the Room on Campus)} \qquad (P1)$$

subject to: Room in Bldg 1

A relaxed problem:

$$\min \text{ (Temperature of the Room on Campus)} \qquad (P2)$$

You can ask the Office of Campus Infrastructure, but they can't look exactly in Bldg 1. They can only give you info about the whole campus, i.e. they are solving the *relaxed* problem (P2).

How are the solutions of (P1) and (P2) related? There are two possible cases:

Case 1: You found the coolest room is LT2 (which is in Bldg 1). Is this the solution to your original problem? Yes it is! It satisfies the constraint. Therefore it is the coldest room overall, and in Bldg 1 in particular!

Case 2: You found the coolest room outside of Bldg 1. Too bad!! It does not satisfy the constraint. Back to the drawing board...

Second Approach

Note: if you use the relaxation method you always have to check if the solution of the relaxed problem satisfies the constraints of the original problem!

Second Approach

Note: if you use the relaxation method you always have to check if the solution of the relaxed problem satisfies the constraints of the original problem!

If we can't ignore the constraints, we have to deal with them.

This is the topic of Week 4: constrained optimization (Lagrange Multipliers).

Extreme Value Theorem

It turns out that for special regions we are always guaranteed to find a global max and a global min. In particular,

Extreme Value Theorem for Multivariable Functions

If $f(x, y)$ is a continuous function on a **closed and bounded** region R, then $f(x, y)$ has a global maximum at some point (x_0, y_0) in R and a global minimum at some point (x_1, y_1) in R.

Note, EVT doesn't tell us where exactly are the global optima or how to find them. It only tells us that they exist.

Closed Set

Just as a closed interval contains its endpoints, a **closed set** in $\mathbb{R}^2$ is one that contains all its boundary points.

(a) Closed sets

(b) Sets that are not closed

For example, the disk

$$D = \{(x, y)\,|x^2 + y^2 \leq 1\}$$

which consists of all points on and inside the circle $x^2 + y^2 = 1$, is a closed set because it contains all of its boundary points (which are the points on the circle $x^2 + y^2 = 1$). But if **even** one point on the boundary curve were omitted, the set would not be closed. (See figure.)

Bounded Set

A **bounded set** in $\mathbb{R}^2$ is one that is contained in some ball.
In other words, it is finite in extent.
In some textbooks, they say "contained within some disk".

Summary

We have covered:

- Global optimality.
- Relaxation of optimization problem.
- Recall of EVT.

Summary

We have covered:

- Global optimality.
- Relaxation of optimization problem.
- Recall of EVT.

Textbook (Hughes-Hallett, *Calculus Single and Multivariable*): read Section 15.2, then try some of Exercises 15.2.1–15.2.12. You may discuss them on Piazza.