10.018 Modelling Space and Systems
Term 2, 2021

Homework Week 4

Due Date: 6:30pm, Feb. 23, 2021

BASIC problems TO BE SUBMITTED.

The BASIC set of problems is designed to be a very easy and straightforward application of the definitions from lectures and cohorts (you might have to do some calculations, but not much). It's a good way to start your homework. If you have trouble starting any of the questions do consult your cohort instructors (in office hours, via email or via Piazza).

I. (a) The following diagram (Figure 1a) shows some level curves of f centred at $(0,0)$ and the curve $g = 0$ (the red curve). Find by inspection the global maximum value and the global minimum value of f subject to the constraint $g = 0$, and briefly explain your reasoning.

(b) The constraint condition is changed to $g \leq 0$, as defined by both the boundary and interior of the orange circle (see Figure 1b). Find by inspection the global maximum value and the global minimum value of f subject to this new constraint, and briefly explain your reasoning.

Solution:

(a) Global maximum value: 10; global minimum value: 4

Since the curve $g = 0$ is closed and bounded and f is continous, by Extreme Value Theorem, the global extrema exist. By the method of Lagrange multipliers, the critical points are the points where the tangent of the curve $g = 0$ is parallel to the tangent of the level curve of f. From the diagram, we see that the critical points are at the level curves $f = 10$ and $f = 4$. Hence the global maximum is 10 and the global minimum is 4.

(b) Global maximum value: 10; global minimum value: 0

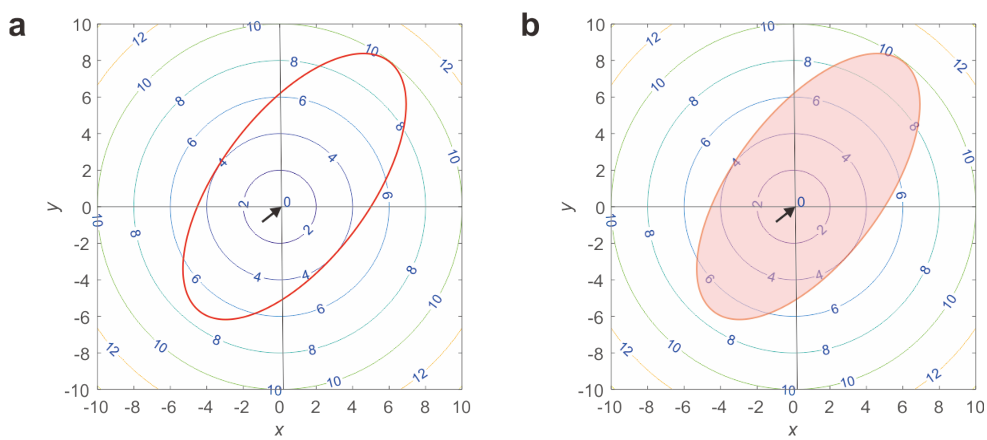

Figure 1: (a) level contours of f and the level contour $g = 0$. (b) level curves of f and the region $g \leq 0$.

Since the region $g \leq 0$ is closed and bounded and f is continous, by Extreme Value Theorem, the global extrema exist. It is easy to see that the global maximum is 10 and the global minimum is 0.

II. *(Evaluating double integrals I)*

For these double integrals

- **Draw** the region of integration in the xy-axes (label your axes and equation of lines properly).
- **Identify** if the region is horizontally simple, vertically simple, both, or neither.
- **Evaluate the integrals.**

(a) $\int_1^3 \int_2^4 9x^3y^2 \, dy \, dx$

(b) $\int_1^2 \int_1^x xy \, dy \, dx$

Solution:

(a) Draw the lines $y = 2, y = 4, x = 1, x = 3$ and find the region bounded by these lines. Figure 2a shows us the region of integration, which is both horizontally and vertically simple.

We can evaluate this integral to get

$$\begin{aligned}\int_1^3 \int_2^4 9x^3y^2 \, dy \, dx &= \int_1^3 \left[3x^3y^3\right]_2^4 \, dx \\ &= \int_1^3 168x^3 \, dx \\ &= \left[42x^4\right]_1^3 \\ &= 3402 - 42 \\ &= 3360\end{aligned}$$

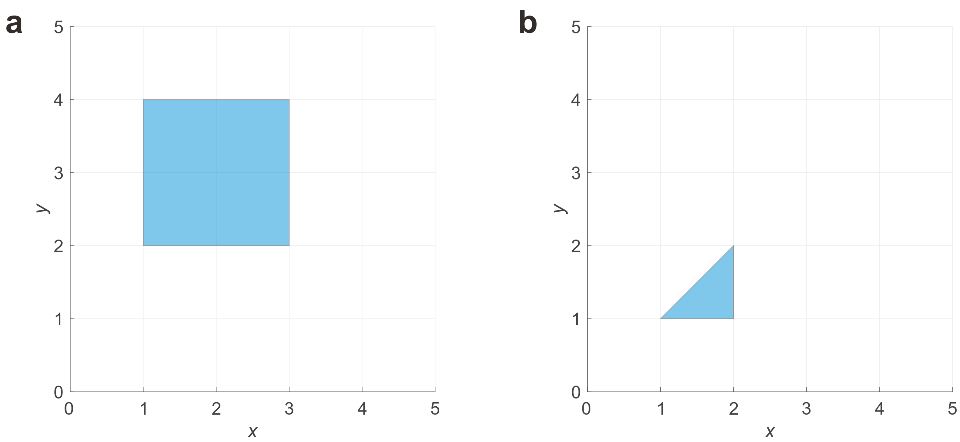

Figure 2: Region of integration for II

(b) Draw the lines $y = 1, y = x, x = 1, x = 2$ and find the region bounded by these lines. Figure 2b shows us the region of integration, which is both horizontally and vertically simple.

We can evaluate this integral to get

$$\begin{aligned}
\int_1^2 \int_1^x xy \, \mathrm{d}y \, \mathrm{d}x &= \int_1^2 \left[x\frac{y^2}{2} \right]_1^x \mathrm{d}x \\
&= \int_1^2 \frac{x^3}{2} - \frac{x}{2} \, \mathrm{d}x \\
&= \left[\frac{x^4}{8} - \frac{x^2}{4} \right]_1^2 \\
&= \frac{9}{8}
\end{aligned}$$

INTERMEDIATE problems TO BE SUBMITTED.

The INTERMEDIATE set of problems is a *little* harder (but not by much) than the BASIC one. If you have trouble starting any of the questions do consult your cohort instructors (in office hours, via email or via Piazza).

1. In Problems (a)-(f), decide (**without calculation**) whether the integrals are positive, negative, or zero. Let D be the region inside the unit circle centred at the origin, let R be the right half of D and let B be the bottom of D. We have drawn the region of integration for you in Figure 3.
Hint: it will help to visualize the surfaces you are integrating over. If you can't visualize them, use some plotting tools (Wolframalpha, Matlab, Python, Geogebra etc).

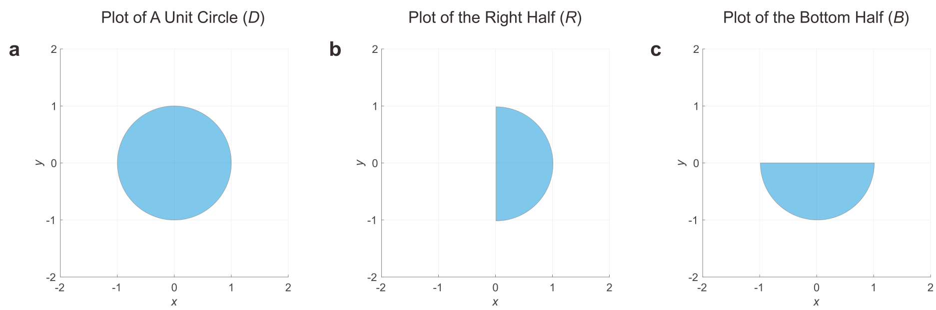

Figure 3: Regions D, R, B of integration.

(a)	$\iint_D x \, \mathrm{d}A$	(b)	$\iint_R x \, \mathrm{d}A$	(c)	$\iint_B x \, \mathrm{d}A$
(d)	$\iint_D y \, \mathrm{d}A$	(e)	$\iint_R y \, \mathrm{d}A$	(f)	$\iint_B y \, \mathrm{d}A$

Problem (a) has been solved as an example for you: to do this question, it helps to draw level curves to decide how the shape of the object described by our **region of integration (base)** and the **surface (height)** looks like.

Figure 4 shows the level curves of the surface $z = x$. After drawing the level curves, it is straightforward to see that $\iint_D x \, \mathrm{d}A$ has zero volume (equal volume above zero to the right, and equal volume below zero to the left). **IMPORTANT**: you could use your own methods/words to briefly justify your answer.

Solution:

(b) $\iint_R x \, \mathrm{d}A$ must have positive volume

(c) $\iint_B x \, \mathrm{d}A$ must have zero volume

(d) $\iint_D y \, \mathrm{d}A$ has zero volume

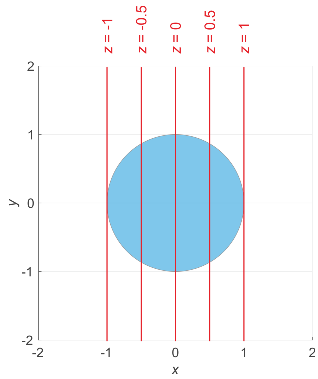

Figure 4: Level curves of $f(x, y) = x$

(e) $\iint_R y \, \mathrm{d}A$ must have zero volume

(f) $\iint_B y \, \mathrm{d}A$ must have negative volume

2. Ah Meng plans to open a factory in Wonderland to produce plastic toys for kids. Ah Meng hired an undergraduate from SUTD to model his cost and profit in this planned factory. The student conducted extensive market research and noticed that the revenue R of such a factory is approximately $R(l, p) = 100l^{\frac{2}{3}}p^{\frac{1}{3}}$, where l represents the hours of labour and p represents the tons of plastic materials. In Wonderland, all workers received an hourly pay of \$20. The cost of plastic materials is \$100/ton. Ah Meng has raised a fund of \$200,000 from Alice Venture Capital to set up this factory. What is the maximum possible revenue that he could produce? Solve this problem using the method of Lagrange Multipliers. Justify that the point you have found is a global maximum.

Solution:

The revenue of the planned factory is $R(l, p) = 100l^{\frac{2}{3}}p^{\frac{1}{3}}$. This total cost of the factory, including both labour and material cost is $20l + 100p$. We want to find the maximum revenue R, subject to constraints $C(l, p) = 20l + 100p - 200,000 = 0$, and $l \geq 0$ and $p \geq 0$.

We can ignore the nonnegativity constraints for now and proceed to solve this optimization problem.

By the Lagrange multipliers, we have: $\nabla R + \lambda \nabla C = \vec{0}$ and $C(l, p) = 0$

$$\nabla R = \begin{bmatrix} \frac{\partial R}{\partial l} \\ \frac{\partial R}{\partial p} \end{bmatrix} = \begin{bmatrix} 100 \times \frac{2}{3} \times l^{-\frac{1}{3}}p^{\frac{1}{3}} \\ 100 \times \frac{1}{3} \times l^{\frac{2}{3}}p^{-\frac{2}{3}} \end{bmatrix}, \quad \nabla C = \begin{bmatrix} \frac{\partial C}{\partial l} \\ \frac{\partial C}{\partial p} \end{bmatrix} = \begin{bmatrix} 20 \\ 100 \end{bmatrix}$$

$$\nabla R + \lambda \nabla C = \begin{bmatrix} 100 \times \frac{2}{3} \times l^{-\frac{1}{3}}p^{\frac{1}{3}} \\ 100 \times \frac{1}{3} \times l^{\frac{2}{3}}p^{-\frac{2}{3}} \end{bmatrix} + \lambda \begin{bmatrix} 20 \\ 100 \end{bmatrix} = \vec{0}$$

Hence, we have:

$$\begin{cases} \frac{200}{3} \times l^{-\frac{1}{3}}p^{\frac{1}{3}} = -20\lambda & (1) \\ \frac{100}{3} \times l^{\frac{2}{3}}p^{-\frac{2}{3}} = -100\lambda & (2) \\ 20l + 100p - 200,000 = 0 & (3) \end{cases}$$

Dividing (1) by (2) and cancelling λ, we have: $l = 10p$. Substituting $l = 10p$ into (3), we have $p = 666.7$ and $l = 6666.7$; note that these optimization results also satisfy nonnegativity constraints. Substituting these values into (1) or (2), we obtain $\lambda = -1.5472$.

The revenue of this factory is then: $R(l, p) = 100l^{\frac{2}{3}}p^{\frac{1}{3}} = 100 \times 6666.7^{\frac{2}{3}} \times 666.7^{\frac{1}{3}} \approx 309,440(dollars)$

We have provided two justifications below. Either one is sufficient.

Justification 1: We can use the AM-GM inequality to prove that this critical value of \$309, 440 is also the global maximum (when $l = 10p = 6666.7$):

$$\begin{aligned} R = 100\sqrt[3]{l^2 p} &= 100\sqrt[3]{\frac{10l \times 10l \times 100p}{10 \times 10 \times 100}} \\ &= \sqrt[3]{100}\sqrt[3]{10l \times 10l \times 100p} \\ &\leq \sqrt[3]{100} \times \frac{10l + 10l + 100p}{3} = \sqrt[3]{100} \times \frac{20l + 100p}{3} \\ &= \sqrt[3]{100} \times \frac{200,000}{3} \approx 309,440(dollars) \end{aligned}$$

Justification 2: We can use the EVT to to prove that this critical value of \$309, 440 is also the global maximum (when $l = 10p = 6666.7$).

We can see that this must be the maximum using the EVT since the region of the original problem (before dropping the nonnegativity constraints) is closed and bounded and the objective function is continuous. The maximum is either the critical point we have calculated or the two end points of the region. The two end points occur when we spend all investment entirely either on labor or the plastic materials. However, the end points ($\frac{200,000}{20}$, 0) and (0, $\frac{200,000}{100}$) have an objective function value equal to 0. Therefore, they are the global minima of the function over the region. In other words, the critical point we have calculated represents the global maximum.

3. If R is the region $x + y \geq a$, $x^2 + y^2 \leq a^2$, with $a \geq 0$, please draw the region R. Identify if the region is horizontally simple, vertically simple, both, or neither. Write the integral

$$\iint_R x \, \mathrm{d}A$$

as an iterated integral in two different ways. Evaluate one of them.

Solution:

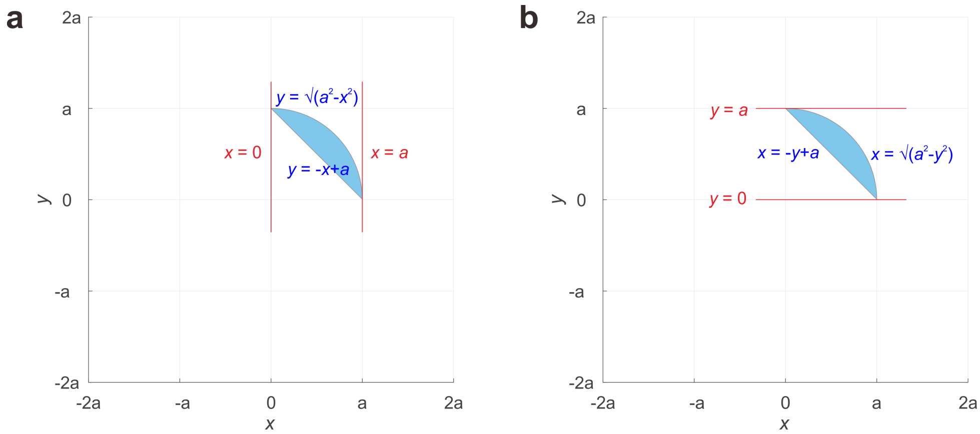

Figure 5: Region of integration and the associated boundaries using two integration methods

It is BOTH horizontally AND vertically simple.

Method 1:

$$\iint_R x \, \mathrm{d}A = \int_0^a \int_{-x+a}^{\sqrt{a^2-x^2}} x \, \mathrm{d}y\mathrm{d}x$$

Method 2:

$$\begin{aligned}\iint_R x\,\mathrm{d}A &= \int_0^a \int_{a-y}^{\sqrt{a^2-y^2}} x\,\mathrm{d}x\mathrm{d}y \\ &= \int_0^a \frac{1}{2}x^2\Bigg|_{a-y}^{\sqrt{a^2-y^2}} \mathrm{d}y \\ &= \frac{1}{2}\int_0^a (2ay-2y^2)\mathrm{d}y \\ &= (\frac{ay^2}{2}-\frac{y^3}{3})\Bigg|_0^a = \frac{a^3}{6}\end{aligned}$$

Challenging problems [OPTIONAL].

CAUTION! This problem will cause the death of some neurons, but they die anyway!!!

1. Find the critical points (i.e., points that satisfy the first-order necessary conditions) of $f(x,y,z) = x^3+y^3+z^3$, subject to the following constraints:
   (a) $g(x,y,z) = x^2+y^2+z^2 = 1$ (you should get **fourteen** critical points);
   (b) $g(x,y,z) = x^2+y^2+z^2 = 1$ and $h(x,y,z) = x+y+z = 0$ (you should get **six** critical points).

   Hint: if we have multiple constrains, we can still use the method of Lagrange Multipliers.

   Solution:

   (a) By the method of Lagrange multipliers, we have: $\nabla f + \lambda \nabla g = \vec{0}$ and $g(x,y,z) = 0$

$$\begin{bmatrix}3x^2\\3y^2\\3z^2\end{bmatrix} + \lambda\begin{bmatrix}2x\\2y\\2z\end{bmatrix} = \vec{0}. \quad \text{and} \quad x^2+y^2+z^2=1$$

$$\begin{cases}3x^2 = -2\lambda x & (1)\\ 3y^2 = -2\lambda y & (2)\\ 3z^2 = -2\lambda z & (3)\\ x^2+y^2+z^2 = 1 & (4)\end{cases}$$

   We notice that $\lambda \neq 0$, otherwise equations (1), (2), (3) imply $(x,y,z) = (0,0,0)$, which is not a point on the constraint. So (1) + (2) + (3) and (4) given:

$$-2\lambda(x+y+z) = 3(x^2+y^2+z^2) = 3\times 1 = 3$$

   Hence, we have:

$$x+y+z = -\tfrac{3}{2\lambda} \qquad (5)$$

   Next, we will solve for x, y, z by cases.

   Case 1: $x,y,z \neq 0$. Equations (1), (2), (3) give $x=y=z=-\frac{2}{3}\lambda$.

   Substituting this into equation (5), we get:

$$-2\lambda = -\frac{3}{2\lambda} \implies \lambda = \pm\frac{\sqrt{3}}{2},\ x=y=z=\mp\frac{1}{\sqrt{3}}$$

So we get $(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$and $(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}})$.

Case 2: Exactly one of x, y, and z is 0. Without loss of generality, suppose $x = 0$. Then equations (2), (3) give $y = z = -\frac{2}{3}\lambda$. Substituting this into equation (5), we get

$$-\frac{4}{3}\lambda = -\frac{3}{2\lambda} \implies \lambda = \pm\frac{\sqrt{3}}{2\sqrt{2}} \implies y = z = \mp\frac{1}{\sqrt{2}}$$

So we get $(0, \pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}})$. Similarly, for $y = 0$, we have $(\pm\frac{1}{\sqrt{2}}, 0, \pm\frac{1}{\sqrt{2}})$; for $z = 0$, we have $(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}, 0)$.

Case 3: Exactly two of x, y, and z are 0. Without loss of generality, suppose $y = z = 0$. Then equation (1) gives $x = -\frac{2}{3}\lambda$. Substituting this into equation (5), we get

$$-\frac{2}{3}\lambda = -\frac{3}{2\lambda} \implies \lambda = \pm\frac{3}{2} \implies x = \mp 1$$

So we get $(\pm 1, 0, 0)$. Similarly, we get $(0, \pm 1, 0)$ and $(0, 0, \pm 1)$.

Since the feasible region is a sphere, which is closed and bounded, by the Extreme Value Theorem the global extrema exist and occur at the critical points or at the boundaries (which are empty). By comparing function values at all the critical points, it is found that the global maximum ($f_{max} = 1$) occurs at $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$; the global minimum ($f_{min} = -1$) occurs at $(-1, 0, 0)$, $(0, -1, 0)$ and $(0, 0, -1)$.

(2) By Lagrange multipliers, we have

$$\nabla f + \lambda\nabla g + \mu h = \vec{0} \text{ and } g(x,y) = 0, h(x,y,z) = 0,$$

$$\implies \begin{bmatrix} 3x^2 \\ 3y^2 \\ 3y^2 \end{bmatrix} + \lambda \begin{bmatrix} 2x \\ 2y \\ 2z \end{bmatrix} + \mu \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ and } x^2 + y^2 + z^2 = 1, x + y + z = 0,$$

$$\implies \begin{cases} 3x^2 + 2\lambda x = -\mu & (1) \\ 3y^2 + 2\lambda y = -\mu & (2) \\ 3z^2 + 2\lambda z = -\mu & (3) \\ x^2 + y^2 + z^2 = 1 & (4) \\ x + y + z = 0 & (5) \end{cases}$$

(1)-(2) gives

$$3(x^2 - y^2) + 2\lambda(x - y) = 0$$
$$\implies (x - y)(3x + 3y + 2\lambda) = 0,$$
$$\implies x = y \text{ or } 3x + 3y + 2\lambda = 0.$$

If $x = y$, substituting this into equation (4) and (5), we get

$$2y^2 + z^2 = 1, 2y + z = 0.$$

Solving the above simultaneous equations gives $y = \frac{1}{\sqrt{6}}, z = -\frac{2}{\sqrt{6}}$ and $y = -\frac{1}{\sqrt{6}}, z = \frac{2}{\sqrt{6}}$.

So we have the critical points $\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}\right)$ and $\left(-\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right)$.

If $3x + 3y + 2\lambda = 0$, we look at (2)-(3), which will give us $y = z$ or $3y + 3z + 2\lambda = 0$.

By the symmetry of x, y, z, the case $y = z$ will give us $\left(-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$ and $\left(\frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right)$. So we have to solve the case $3x + 3y + 2\lambda = 0$ and $3y + 3z + 2\lambda = 0$.

Again, we look at (1)-(3), which will give us $x = z$ or $3x + 3z + 2\lambda = 0$. The case $x = z$ will give us $\left(\frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$ and $\left(-\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right)$. So we are left with the final case of $3x + 3y + 2\lambda = 0$, $3y + 3z + 2\lambda = 0$ and $3x + 3z + 2\lambda = 0$. Together with equation (5), this is a linear system of 4 equations with 4 variables. We check that this system is inconsistent with (4) and hence there's no solution in this final case.

In sum, there are 6 critical points. The feasible region is a circle, which is closed and bounded. By the extreme value theorem, the global extrema exist and occur at the critical points or at the boundaries (which are empty). By comparing the values at the critical points, we conclude that the global maximum value ($f_{\max} = \frac{1}{\sqrt{6}}$) occurs at $\left(-\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right)$, $\left(\frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right)$, and $\left(-\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right)$; the global minimum value ($f_{\min} = -\frac{1}{\sqrt{6}}$) occurs at $\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}\right)$, $\left(-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$, and $\left(\frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$.