SINGAPORE UNIVERSITY OF TECHNOLOGY AND DESIGN

10.018: Modelling Space and Systems
Term 2, 2021

Homework Week 3

Due Date: 6:30pm, Feb. 16, 2021

Reminder: Part 5 of Math Modeling TO BE SUBMITTED on PIAZZA.

You need to submit **1 page** (pics/screenshots) by uploading on Piazza in your own thread by Mon 6pm, Feb 15th. This will be the last 1% of your MM.

Format of the submission:

- List of team members with their official full names and student IDs.
- Executive Summary of your math modeling problem solution.
  It should restate the problem, state the assumptions made, briefly describe the chosen solution methods, and provide the final results and conclusions, should discuss briefly strengths and weaknesses.

BASIC problems TO BE SUBMITTED.

The BASIC set of problems is designed to be a very easy and straightforward application of the definitions from lectures and cohorts (you might have to do some calculations, but not much). If you have trouble starting any of the questions do consult your cohort instructors (in office hours, via email or via Piazza).

1. *Identifying types of regions I*

   Please identify if the following regions are i) bounded or ii) closed **TOGETHER** with a sketch of the region.

   (a) $\left\{(x, y) \in \mathbb{R}^2 : |x+y| \leq 1\right\}$

   (b) $\left\{(x, y) \in \mathbb{R}^2 : |x|+|y| \leq 1\right\}$

   (c) $\left\{(x, y) \in \mathbb{R}^2 : 1 < x^2+y^2 < 4\right\}$

   Solution: Reminder: The absolute value function $|x|$ can be written as

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{otherwise} \end{cases} \tag{1}$$

   and that for any two numbers

$$a \leq b$$

   we have that

$$-b \leq -a$$

The table below is to give a comprehensive answer as some students gave answer beyond what the question is asking for: closed, not closed, open, not open, bounded, unbounded, neither closed nor open. Note: this is not a english class and we cannot write closed = not open, not closed = open.

Region	Bounded	Unbounded	Closed	Open	Neither closed nor open
a)		$\surd$	$\surd$		
b)	$\surd$		$\surd$		
c)	$\surd$			$\surd$	

(a) We first note that we can write

$$|x+y| \leq 1 \tag{2}$$

as two inequalities

$$x+y \leq 1 \quad \text{when } x+y \geq 0 \tag{3}$$
$$-x-y \leq 1 \quad \text{when } x+y < 0 \tag{4}$$

Now, $-x-y \leq 1$ implies that $x+y \geq -1$. Hence, we have

$$x+y \leq 1 \quad \text{when } x+y \geq 0 \tag{5}$$
$$x+y \geq -1 \quad \text{when } x+y < 0 \tag{6}$$

However, we can write this as the region which is the intersection of these two inequalities without the conditions (check why this is so!)

$$x+y \leq 1 \tag{7}$$
$$x+y \geq -1 \tag{8}$$

Hence we draw the two lines: $x+y=1$ and $x+y=-1$ on our plot. Now, we look at the intersection between the two regions $x+y \leq 1$, and $x+y \geq -1$ to get Figure 1.

This region is **unbounded** and **closed**.

(b) We can proceed similarly as part a) and write

$$|x|+|y| \leq 1 \tag{9}$$

as the four inequalities

$$x+y \leq 1 \tag{10}$$
$$x+y \geq -1 \tag{11}$$
$$x-y \leq 1 \tag{12}$$
$$x-y \geq -1 \tag{13}$$

and look at the intersection between the four regions. We should get something similar to Figure 2.

This region is **bounded** and **closed**.

(c) We know that $x^2+y^2=a^2$ is the equation of a circle, and hence we get Figure 3.

This region is **bounded** and **not closed**.

2. *Identifying types of regions II*

Please identify if the following regions are i) bounded, or ii) closed **TOGETHER** with a sketch of the regions.

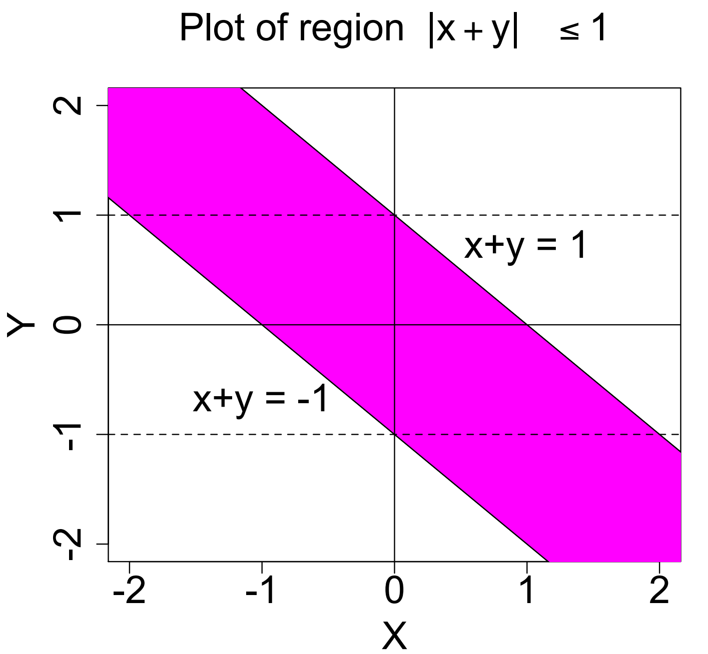

Figure 1: The plot of the region $\{(x, y) \in \mathbb{R}^2 : |x + y| \leq 1\}$

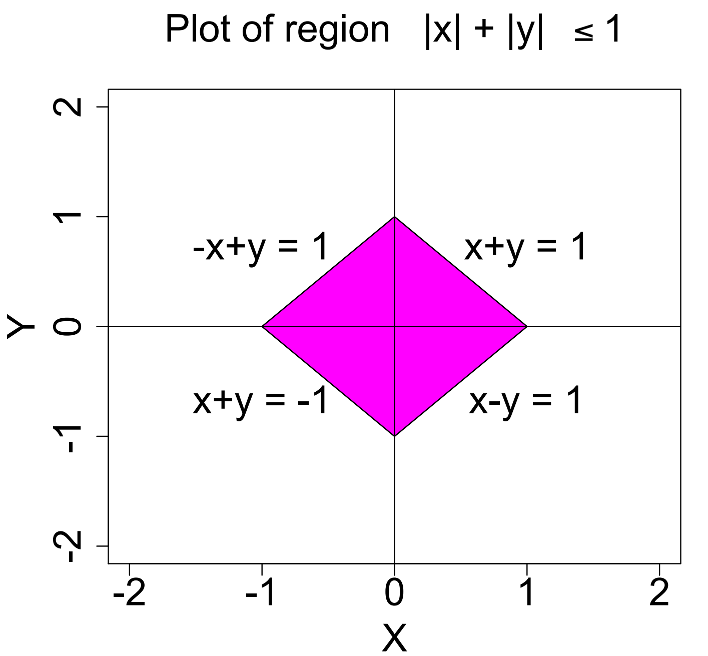

Figure 2: The plot of the region $\{(x, y) \in \mathbb{R}^2 : |x| + |y| \leq 1\}$

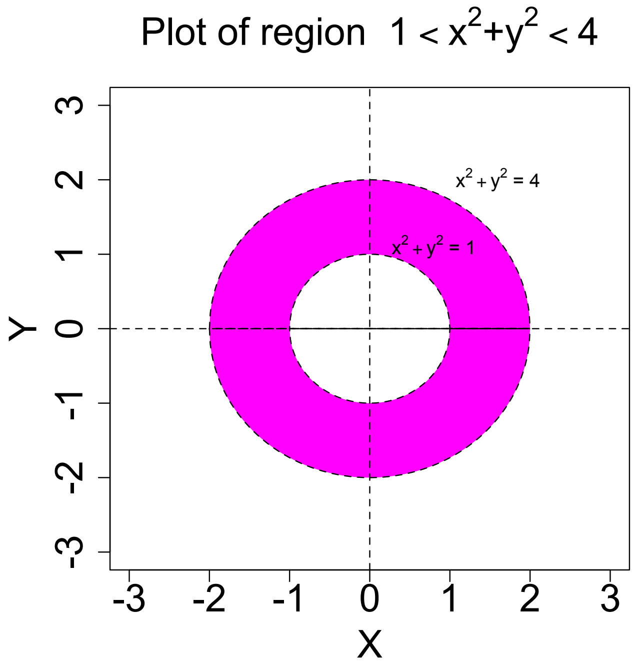

Figure 3: The plot of the region $\{(x, y) \in \mathbb{R}^2 : 1 < x^2 + y^2 < 4\}$

(a) $\left\{(x, y) \in \mathbb{R}^2 : 2x \geq y \geq x\right\}$

(b) $\left\{(x, y) \in \mathbb{R}^2 : 0 < x < y^2 \text{ AND } |y| \geq 1\right\}$

(c) $\left\{(x, y) \in \mathbb{R}^2 : |xy| < 4\right\}$

Solution:

The table below is to give a comprehensive answer as some students gave answer beyond what the question is asking for: closed, not closed, open, not open, bounded, unbounded, neither closed nor open. Note: this is not a english class and we cannot write closed = not open, not closed = open.

Region	Bounded	Unbounded	Closed	Open	Neither closed nor open
a)		$\surd$	$\surd$		
b)		$\surd$			$\surd$
c)		$\surd$		$\surd$	

(a) We get Figure 4.

This region is **unbounded**, and **closed**.

(b) We get Figure 5.

This region is **unbounded**, and **not closed** (in this case, actually it is neither closed nor open).

(c) We get Figure 6.

This region is **unbounded**, and **not closed**.

INTERMEDIATE problems TO BE SUBMITTED.

The INTERMEDIATE set of problems is a *little* harder (but not by much) than the BASIC one. If you have trouble starting any of the questions do consult your cohort instructors (in office hours, via email

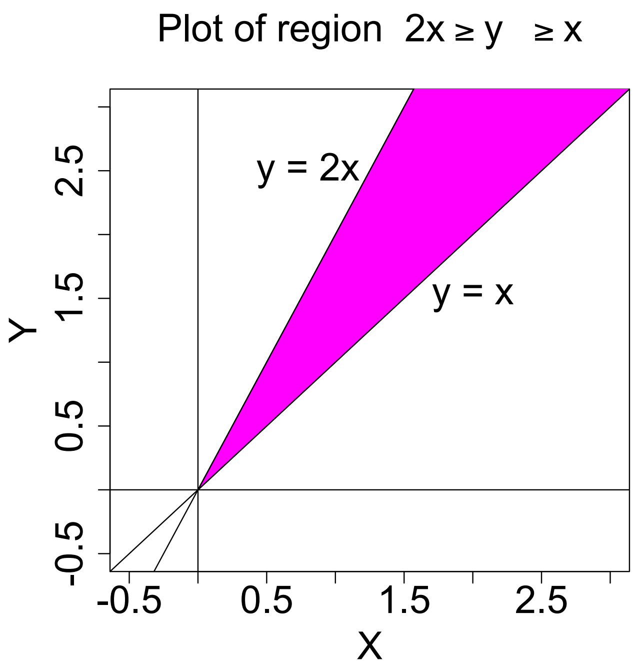

Figure 4: The plot of the region $\{(x, y) \in \mathbb{R}^2 : 2x \geq y \geq x\}$

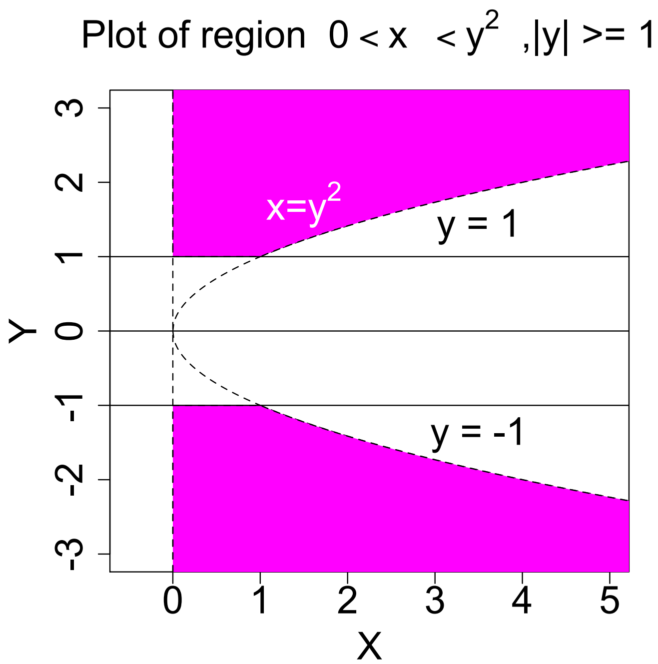

Figure 5: The plot of the region $\{(x, y) \in \mathbb{R}^2 : 0 < x < y^2 \text{ AND } |y| \geq 1\}$

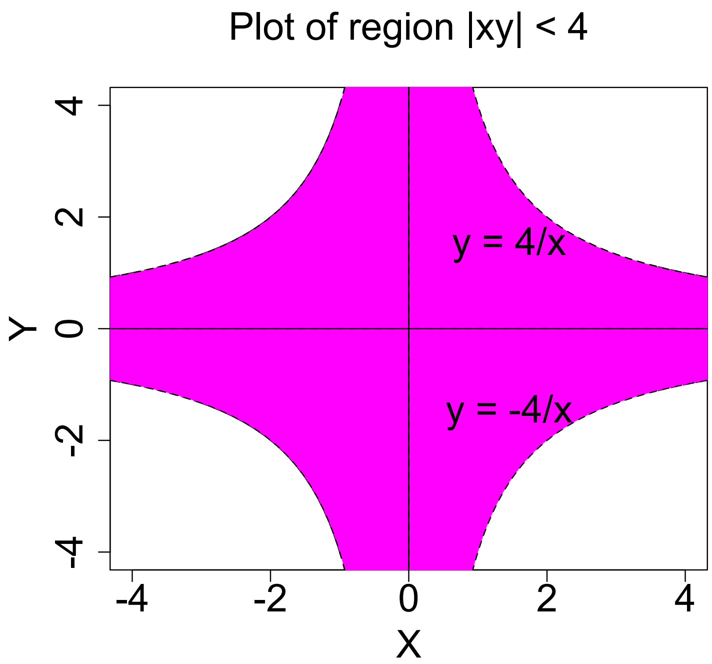

Figure 6: The plot of the region $\{(x, y) \in \mathbb{R}^2 : |xy| < 4\}$

or via Piazza).

3. Let R be the region in $\mathbb{R}^2$ that is bounded by the triangle whose vertices are $A(2, 0)$, $B(0, 2)$, $C(0, -2)$. Find the global maximum and minimum values of

$$f(x, y) = x^2 + y^2 - 2x,$$

with R as the feasible region.

Solution: Since $f(x, y)$ is continuous on the nonempty, closed and bounded region R, by the Extreme Value Theorem, the global maximum is attained at some critical point in R and the global minimum is also attained at some critical point in R. Since $\nabla f = \begin{bmatrix} 2x - 2 & 2y \end{bmatrix}$ is defined everywhere inside the triangle, the critical points of f are its stationary points and the boundary points of R.

Solve $\nabla f = \vec{0}$, the only stationary point is $(1, 0)$, where $f(1, 0) = -1$.

Next we consider the boundary of R, which is the union of the three sides of the triangle. The line segment BC is defined by $x = 0$ and $-2 \leq y \leq 2$. On BC, the function becomes

$$f(0, y) = y^2, \text{ where } -2 \leq y \leq 2.$$

Its minimum is at $y = 0$, where $f(0, 0) = 0$. Its maximum is at $y = -2$ and $y = 2$, where $f(0, -2) = f(0, 2) = 4$.

The line segment AC is defined by $y = x - 2$ and $0 \leq x \leq 2$. On AC, the function becomes

$$f(x, x - 2) = x^2 + (x - 2)^2 - 2x = 2x^2 - 6x + 4, \text{ where } 0 \leq x \leq 2.$$

Since $\frac{\mathrm{d}f(x,x-2)}{\mathrm{d}x} = 4x - 6$, its critical points are at $x = 3/2$ (where the gradient is zero), and at $x = 0$ and $x = 2$ (endpoints of $[0, 2]$). Since $f(\frac{3}{2}, \frac{3}{2} - 2) = -\frac{1}{2}$ at $x = 3/2$, and $f(0, 0 - 2) = 4$ at $x = 0$ and $f(2, 2 - 2) = 0$, so its minimum is $f(\frac{3}{2}, \frac{3}{2} - 2) = -\frac{1}{2}$ and its maximum is $f(0, 0 - 2) = 4$.

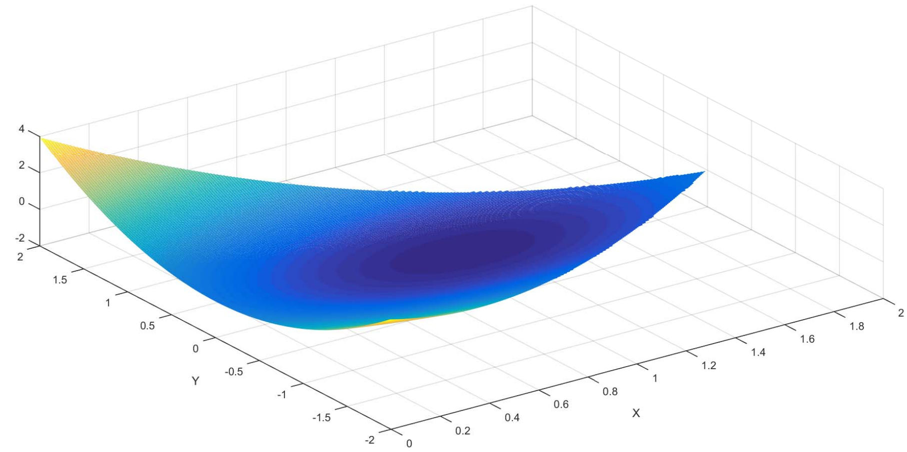

Figure 7: A plot of $f(x, y) = x^2 + y^2 - 2x$

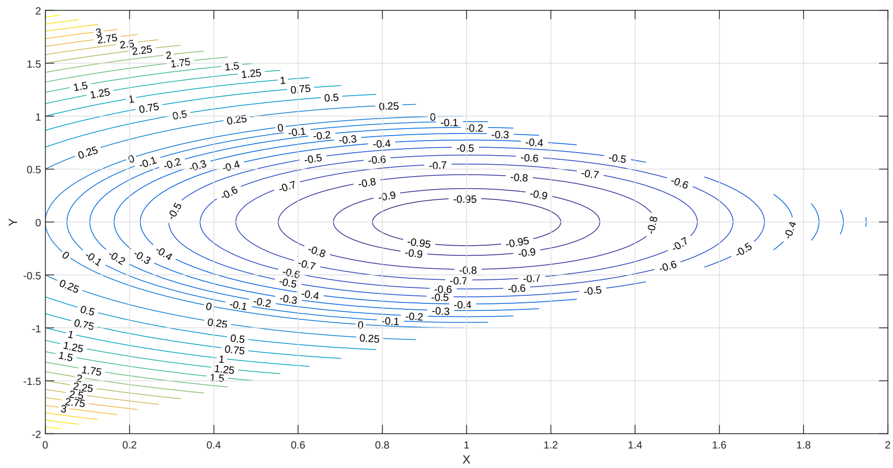

Figure 8: Level curves of $f(x, y) = x^2 + y^2 - 2x$

We can apply the same analysis on the extreme values attained along the line segment AB. Alternatively, we observe that $f(x, y)$ is symmetric in the x axis and AB and AC are symmetric in the x axis, therefore, the extreme values of $f(x, y)$ on AB are also $-\frac{1}{2}$ and 4. Refer to Figures for a function plot and level curves of f.

In conclusion, the minimum value of $f(x, y)$ is -1, which is attained at $(1, 0)$. The maximum value of $f(x, y)$ is 4, which is attained at $(0, 2)$ and $(0, -2)$.

Here is an alternative argument to solve for the global minimizers of $f(x, y)$. Complete the square

$$\begin{aligned} f(x, y) &= x^2 + y^2 - 2x \\ &= (x - 1)^2 + y^2 - 1 \\ &\geq -1. \end{aligned}$$

The lower bound -1 is attained where $x - 1 = y = 0$, i.e. at $(x, y) = (1, 0)$, therefore, $(1, 0)$ is the only global minimizer.

4. *A rectangular cardboard box without a lid is to have a volume of* $32,000$ *cm*^{3}*. Find the dimensions that minimize the amount of cardboard used. Is it a global minimum? Justify.*

Solution: Let the length and width of the rectangular base be x dm and y dm. Since the volume of the cardboard box is 32 dm^3, its height must be $\frac{32}{xy}$ dm. The amount of cardboard used is, therefore, $f(x,y) = xy + \frac{64}{y} + \frac{64}{x}$ with $x > 0$ and $y > 0$.

To find the minimum of $f(x,y)$, we compute

$$\nabla f(x,y) = [y - \frac{64}{x^2}, x - \frac{64}{y^2}]^T.$$

The stationary points satisfy $x^2y = 64$ and $xy^2 = 64$. Dividing the two equations, we have

$$\frac{x}{y} = 1 \text{ or } x = y,$$

which gives $x = 4$ dm and $y = 4$ dm and $f(4,4) = 48$ dm^3.

Next, we show that $(x,y) = (4,4)$ is a global minimizer of $f(x,y)$ using two methods.

Method 1: $f(x,y)$ is a sum of three positive terms because $x > 0$ and $y > 0$. By the AM-GM inequality,

$$\begin{aligned} f(x,y) &= xy + \frac{64}{y} + \frac{64}{x} \\ &\geq 3(xy \cdot \frac{64}{y} \cdot \frac{64}{x})^{\frac{1}{3}} \\ &\geq 3(64)^{\frac{2}{3}} \\ &= 48. \end{aligned}$$

Since the lower bound on $f(x,y)$, 48, is attained at $(x,y) = (4,4)$, $(4,4)$ is a global minimizer of $f(x,y)$ and the global minimum of $f(x,y)$ is 48.

Method 2: Consider the region R enclosed by $x = 1$, $y = 1$, and $xy = 48$. It is a closed and bounded region where $f(x,y)$ is continuous. By EVT, there exists a global minimum of $f(x,y)$ restricted to R. The only stationary point $(4,4)$ is in R. Along the boundary $x = 1$, $f(1,y) = y + \frac{64}{y} + 64 > 48$; along the boundary $y = 1$, $f(x,1) = x + \frac{64}{x} + 64 > 48$; along the boundary $xy = 48$, $f(x, \frac{48}{x}) = 48 + \frac{4x}{3} + \frac{64}{x} > 48$. Therefore the minimum value of $f(x,y)$ restricted to R is 48.

Outside region R with $x > 0$ and $y > 0$, at least one of the three conditions is satisfied: (a) $x < 1$, (b) $y < 1$, and (c) $xy > 48$. In all three cases, we can show, using similar arguments applied to the three boundaries $x = 1$, $y = 1$, and $xy = 48$, that $f(x,y) > 48$. Thus, $f(x,y)$ attains global minimum at $(4,4)$.

Alternative solution: Let the length and width of the rectangular base be x and y. Let the height of the box be h. Then $xyh = 32000$ cm^3.

The cardboard used is $A(x,y,h) = xy + 2xh + 2yh$. Hence, by the AM-GM Inequality:

$$\begin{aligned} A(x,y,z) &= xy + 2xh + 2yh \\ &\geq 3\sqrt[3]{(xy)(2xh)(2yh)} = 3\sqrt[3]{4(xyh)^2} = 3\sqrt[3]{4(32000 \text{ cm}^3)^2} = 4800 \text{ cm}^2, \end{aligned}$$

with equality when $xy = 2xh = 2yh$. From $xy = 2xh$, we get $x = 2h$; from $xy = 2yh$, we get $y = 2h$. Then 32000 cm^3 $= xyh = (2h)(2h)(h)$ whose solution is $h = 20$ cm. Thus $(x,y,h) = (40 \text{ cm}, 40 \text{ cm}, 20 \text{ cm})$ is the only possible global minimiser. Since $A(40 \text{ cm}, 40 \text{ cm}, 20 \text{ cm}) = 4800 \text{ cm}^2$, therefore the global minimum area of the cardboard used is 4800 cm^2 when the length and width of the base are both 40 cm and the height is 20 cm.

Challenging problems [OPTIONAL].

5. Consider the problem $\min\{f(x,y) : (x,y) \in \mathbb{R}^2\}$, where

$$f(x,y) = (y - x^2)(y - 2x^2).$$

   (a) Show that $(0,0)$ is a stationary point of f.

   (b) Show that the second derivative test is inconclusive at $(0,0)$.

   (c) Show that the origin is a local minimizer of f along any line that passes through the origin (i.e., a line with equation $y = mx$ for some $m \in \mathbb{R}$).

   (d) Show that the origin is not a local minimizer of f.

   (e) How do you reconcile parts (c) and (d)?

Solution:

(a) Since $f_x = 8x^3 - 6xy$ and $f_y = 2y - 3x^2$, thus $f_x(0,0) = f_y(0,0) = 0$, so $(0,0)$ is a stationary point.

(b) Since $f_{xx} = 24x^2 - 6y$ and $f_{xy} = -6x = f_{yx}$ and $f_{yy} = 2$, hence $D(0,0) = f_{xx}(0,0)f_{yy}(0,0) - f_{xy}(0,0)f_{yx}(0,0) = 0 \cdot 2 - 0 \cdot 0 = 0$. So the second derivative test is inconclusive at $(0,0)$.

(c) Let $g(x) = f(x, mx) = (mx - x^2)(mx - 2x^2) = m^2x^2 - 3mx^3 + 2x^4$. $g'(x) = 2m^2x - 9mx^2 + 8x^3$ and $g'(0) = 0$. $g''(x) = 2m^2 - 18mx + 24x^2$ and $g''(0) = 2m^2$. When $m \neq 0$, $g''(0) > 0$, therefore, $x = 0$ is a local minimizer of g. When $m = 0$, $g'(x) = 8x^3$, which changes its sign from negative to positive when x increases from negative to positive, therefore, $x = 0$ is a minimizer to g.

(d) Let $y = 1.5\epsilon^2$ and $x = \epsilon$ for some $\epsilon > 0$. Note that the point $(\epsilon, 1.5\epsilon^2)$ can be made arbitrarily close to $(0,0)$ if ϵ is sufficiently small. Since $f(\epsilon, 1.5\epsilon^2) = -0.25\epsilon^4 < 0 = f(0,0)$, so $(0,0)$ is not a local minimizer of f.

(e) $(0,0)$ is a local minimizer of f along any straight line that passes through it, but a local maximizer along the curve defined by $y = 1.5x^2$. See Figures 9 and 10 for a function plot and level curves of f. The level curve $f(x,y) = 0$ consists of two parabolas $y = x^2$ and $y = 2x^2$. Above $y = 2x^2$ and below $y = x^2$, $f(x,y) > 0$. Between these two parabolas, $f(x,y) < 0$. Any straight line that passes through the origin, except the line $y = 0$, passes a region above $y = 2x^2$ and a region below $y = x^2$ near the origin. Therefore, $f(x,y)$ has a local minimum along those lines. On the other hand, a parabola $y = kx^2$ for $1 < k < 2$, passes through the regions between $y = x^2$ and $y = 2x^2$ near the origin. Therefore, $f(x,y)$ has a local maximum along the parabola.

Also $D_{\mathbf{u}}D_{\mathbf{u}}f(x_0,y_0) > 0\, \forall \mathbf{u}$ implies local min at (x_0,y_0) along all directions. But local min at (x_0,y_0) along all directions does not imply $D_{\mathbf{u}}D_{\mathbf{u}}f(x_0,y_0) > 0\, \forall \mathbf{u}$ (that's the example here, $D_{\mathbf{u}}D_{\mathbf{u}}f(0,0) = 0$ along the x axis). And the latter fact, $D_{\mathbf{u}}D_{\mathbf{u}}f(x_0,y_0) > 0\, \forall \mathbf{u}$, is essential in proving that we have a local min of the function.

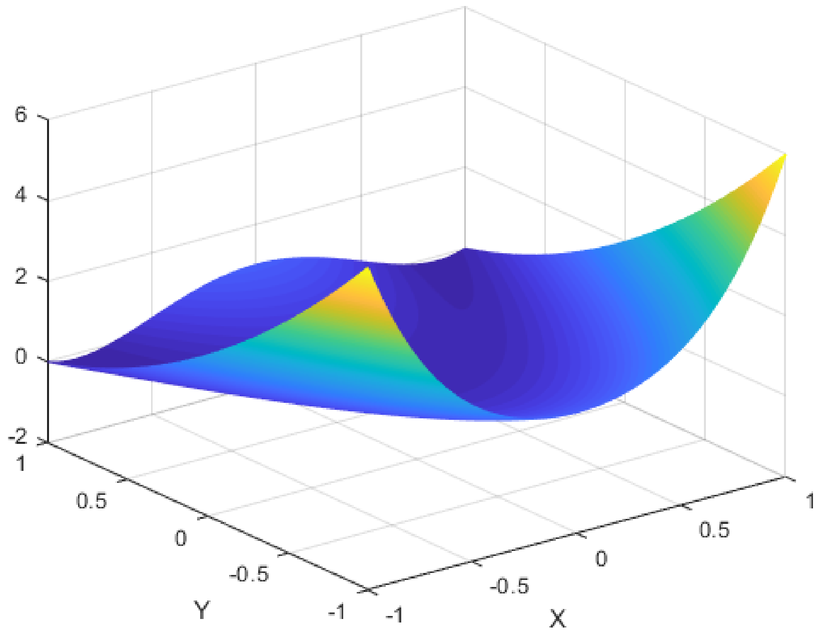

Figure 9: A plot of $f(x, y) = (y - x^2)(y - 2x^2)$

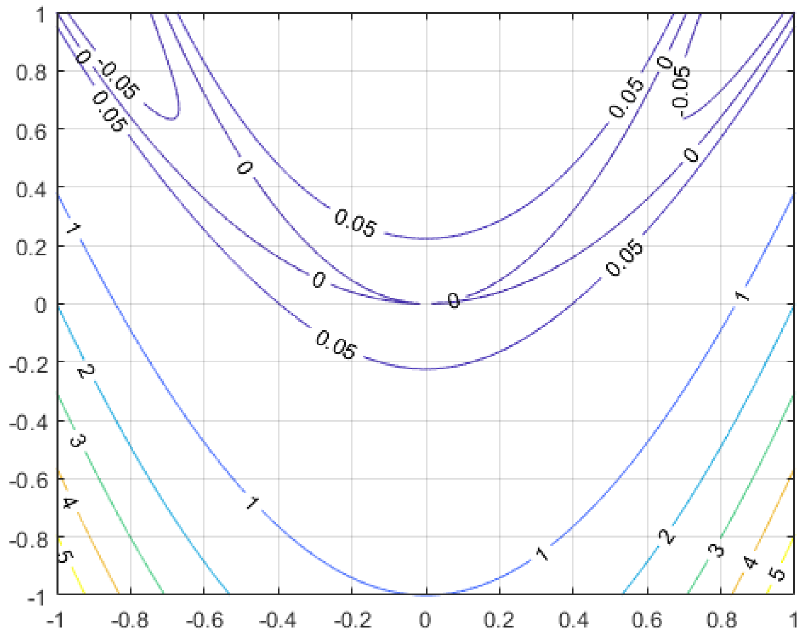

Figure 10: Level curves of $f(x, y) = (y - x^2)(y - 2x^2)$