Week 1 – Day 2

Electrical Field & Electric Dipole

Concept 1: Electric Field of Continuous Charge Distribution

Concept 2: Electric Dipole and the Electric Field of a Dipole

Concept 3: Force and Torque on Dipole in an Electric Field

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Hydrogen bond in DNA, Water molecule heated in microwave oven

Reading:

1. University Physics with Modern Physics – Chapter 21
2. Introduction to Electricity and Magnetism – Chapter 1,2

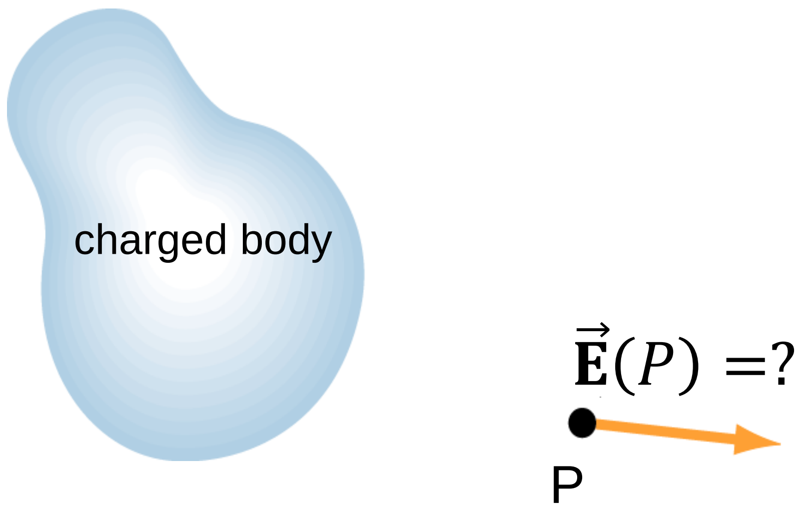
Concept 1: Electric Field of Continuous Charge Distribution

When we have a rigid body with continuous charge distributions, we cannot treat it as a point charge anymore, thus nearby electric field is not a point charge $\vec{E}$ field anymore.

Continuous Charge Distributions

What is the electric field at P due to a charged body with continuous distribution in space?

Note that we cannot treat a sizable body as a point charge anymore, thus the electric field of a point charge, $\vec{\mathrm{E}}_{sp} = k_e \frac{q_s}{r_{sp}^2} \hat{\mathrm{r}}_{sp}$ is no longer valid.



$\vec{\mathbf{E}}(P) = ?$

Continuous Charge Distributions

The strategy is to break out the charged body into many small chunks (that can be treated as a point charge).

We calculate the $\vec{E}_i$ of individual chunks using $\vec{E}$ point charge equation ($\vec{E} = \frac{k_e q}{r^2}\hat{r}$).

We then sum all of the $\vec{E}_i$ (by superposition) to get the total $\vec{E}$.

It sounds like the smaller chunks you break out, the more calculation you need to do.

But don't worry, you can either write code to let computer calculate for you, or for simple geometry of a body, integration is your friend!

Continuous Charge Distributions

- Break distribution into parts:

$$Q = \lim_{N\to\infty} \sum_{i=1}^{N} \Delta q_i \to \int_{body} dq$$

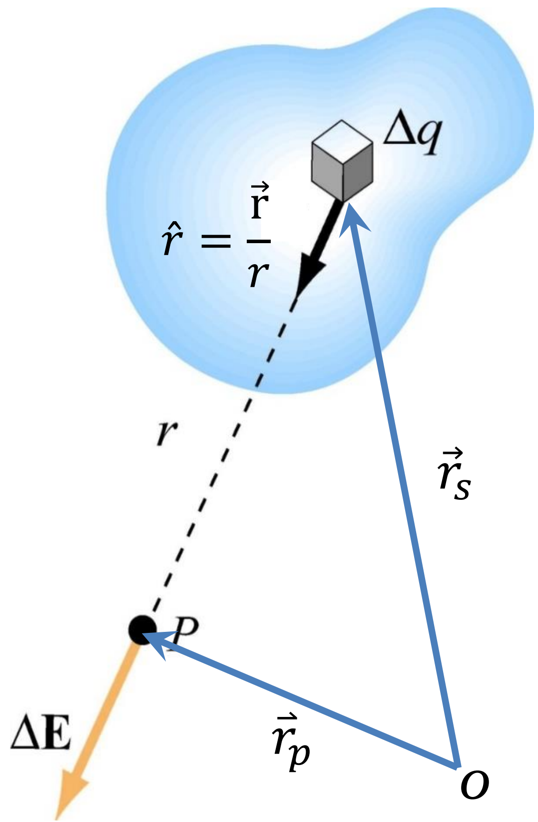
- Electric field at P due to Δq:

$$\Delta\vec{\mathrm{E}} = k_e \frac{\Delta q}{r^3}\vec{\mathrm{r}} \to d\vec{\mathrm{E}} = k_e \frac{dq}{r^3}\vec{\mathrm{r}}$$

- Superposition:

$$\vec{\mathrm{E}} = \sum \Delta\vec{\mathrm{E}} \to \int d\vec{\mathrm{E}} = \int k_e \frac{dq}{r^3}\vec{\mathrm{r}}$$

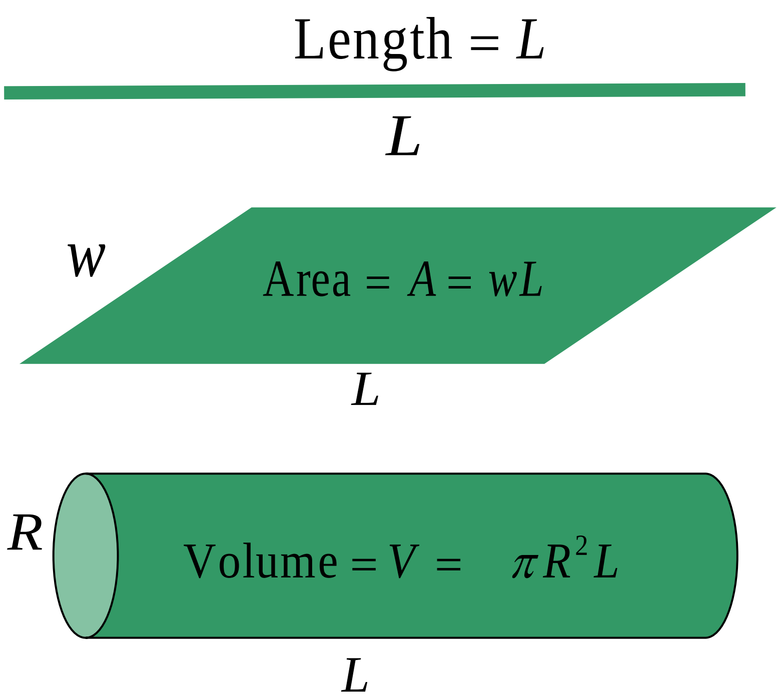
$$\Rightarrow \vec{\mathrm{E}} = \int k_e \frac{dq}{\left|\vec{r}_p - \vec{r}_s\right|^3}(\vec{r}_p - \vec{r}_s)$$



Note:
$\vec{r}_s$: position vector of the source from the origin
$\vec{r}_p$: position vector of the point of interest
$\vec{r} = \vec{r}_p - \vec{r}_s$: vector pointing from the source to the point of interest

Continuous Sources: Charge Density

- We need to convert dq into a geometrical parameter so that we can perform the integration. The conversion factor is charge density.
- Line charge density: λ; surface charge density, σ; volume charge density, ρ



$$dq = \lambda dL$$

e.g. $\lambda = \frac{Q}{L}$ (if uniform)

$$dq = \sigma dA$$

e.g. $\sigma = \frac{Q}{A}$ (if uniform)

$$dq = \rho dV$$

e.g. $\rho = \frac{Q}{V}$ (if uniform)

- Note: λ, σ and ρ are not necessary a constant. If the charge distribution is non-uniform, λ, σ and ρ are not constant, but a function of the geometry.

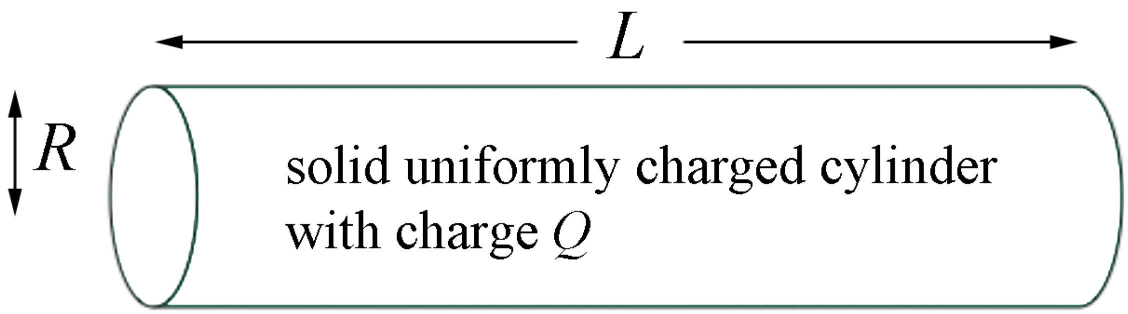
Example: Charge Densities

- A solid cylinder, of length L and radius R, is uniformly charged with total charge Q.

a) What is the volume charge density ρ?

b) What is the linear charge density λ?

c) What is the relationship between these two densities ρ and λ?



Example - Solution

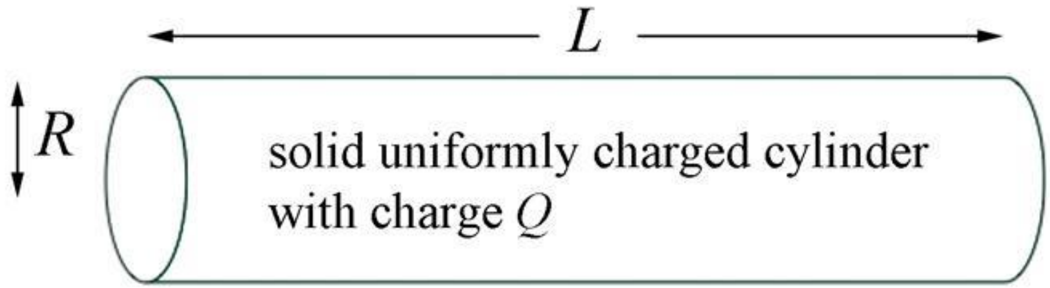
- A solid cylinder, of length *L* and radius *R*, is uniformly charged with total charge *Q*.

a) volume charge density ρ is $\frac{Q}{\pi R^2 L}$.

Note that ρ is a constant if ρ is uniform.

a) linear charge density λ is $\frac{Q}{L}$

b) these two densities ρ and λ are related by $\lambda = \rho \pi R^2$



Strategy of Using the Equation

$$\vec{\mathrm{E}} = \int k_e \frac{dq}{\left|\vec{r}_p - \vec{r}_s\right|^3} (\vec{r}_p - \vec{r}_s)$$

1. Write dq in terms of geometrical parameters ($dl, dA \; or \; dV$) using charge density.
   a) We need to write $dl, dA \; or \; dV$ based on the coordinate system we setup.
   b) It should be a geometrical variable as dq varies in position.
2. Find $\vec{r}_p$ and $\vec{r}_s$ based on the coordinate system that you set.

   Note:
   – We should always setup the coordinate system first to define $\vec{r}_p$ and $\vec{r}_s$.
   – Note that $\vec{r}_s$ is a variable (that you may need to introduce yourself), as it varies when you move the source from a point to another.
3. Calculate $\vec{r}_p - \vec{r}_s$ and $\left|\vec{r}_p - \vec{r}_s\right|$ to substitute into the equation.
4. Perform your integration skill to solve the problem (This step is really just math, but it is crucial).

Concept Question 1.1: Electric Field of a Rod

- A rod of length *L* lies along the *x*-axis with its left end at the origin. The rod has a *uniform* charge density λ. Which of the following expressions best describes the electric field at the point *P*

A. $\vec{\mathrm{E}}(P) = -k_e \int_{x\prime=0}^{x\prime=L} \frac{\lambda dx'}{(x'+d)^3}\,\hat{\imath}$

B. $\vec{\mathrm{E}}(P) = k_e \int_{x\prime=0}^{x\prime=L} \frac{\lambda dx'}{(x'+d)^3}\,\hat{\imath}$

C. $\vec{\mathrm{E}}(P) = -k_e \int_{x\prime=0}^{x\prime=L} \frac{\lambda dx'}{(x'+d)^2}\,\hat{\imath}$

D. $\vec{\mathrm{E}}(P) = k_e \int_{x\prime=0}^{x\prime=L} \frac{\lambda dx'}{(x'+d)^2}\,\hat{\imath}$

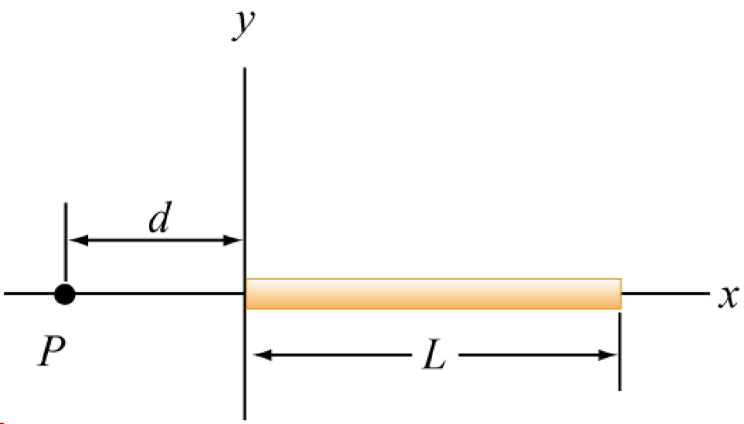
E. $\vec{\mathrm{E}}(P) = -k_e \frac{\lambda L}{(L+d)^2}\hat{\imath}$

F. $\vec{\mathrm{E}}(P) = k_e \frac{\lambda L}{(L+d)^2}\hat{\imath}$

G. $\vec{\mathrm{E}}(P) = -k_e \frac{\lambda L}{d^2}\hat{\imath}$

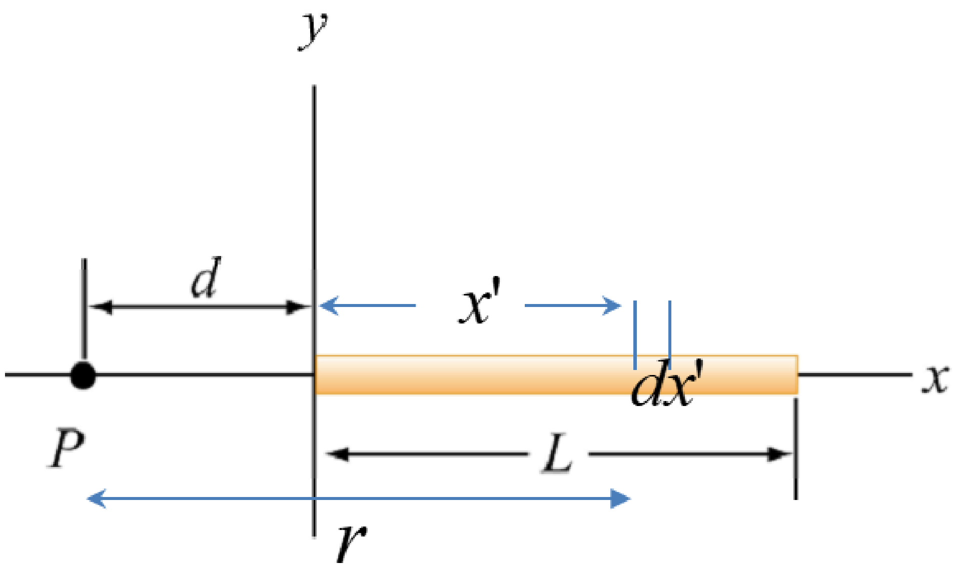
H. $\vec{\mathrm{E}}(P) = k_e \frac{\lambda L}{d^2}\hat{\imath}$

I. None of the above



Concept Question 1.1: Solution

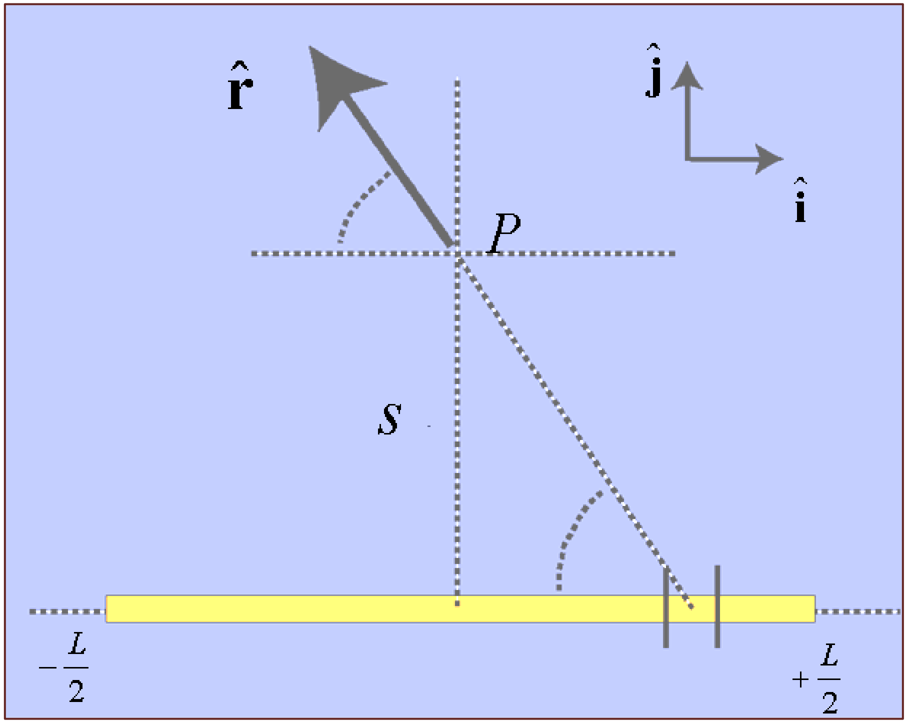
- $dq = \lambda dx'$
- $\vec{r}_{sp} = -(x' + d)\hat{\imath}$
- Plug into the equation:
- $\vec{\mathrm{E}} = \int k_e \frac{dq}{|\vec{r}_p - \vec{r}_s|^3}(\vec{r}_p - \vec{r}_s)$



- Answer:
- C. $\vec{\mathrm{E}}(P) = -k_e \int_{x'=0}^{x'=L} \frac{\lambda dx'}{(x'+d)^2}\,\hat{\imath}$

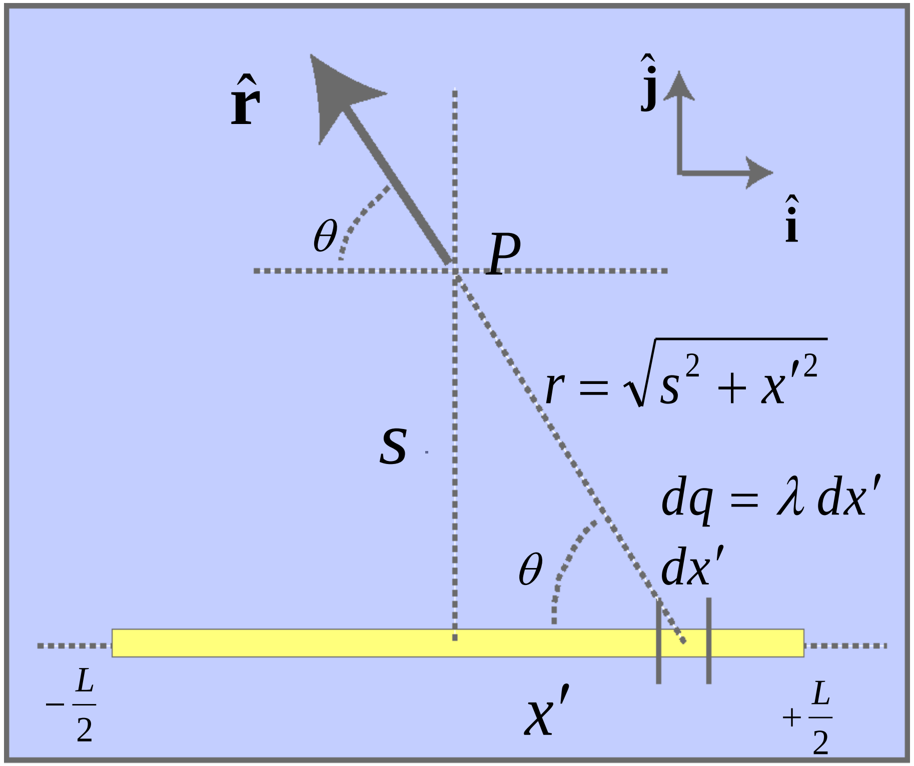
Case Problem 1.1: Line of Charge

- Point *P* lies on perpendicular bisector of uniformly charged line of length *L*, a distance *s* away. The charge on the line is *Q*. Find an **integral expression** for the direction and magnitude of the electric field at *P*.



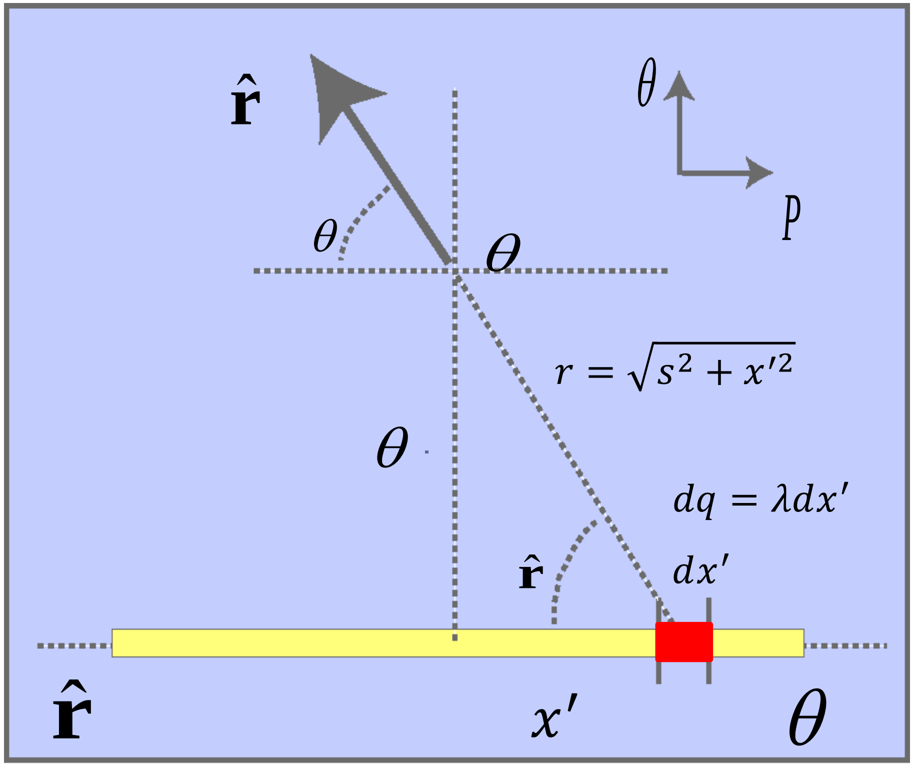
Hint:

- Typically give the integration variable (x') a "primed" variable name.



$$\vec{E} = \int d\vec{E} = \int k_e \frac{dq}{r^2} \hat{r}$$

Case Problem 1.1: Solution



Step 1: Draw diagram.

Step 2: By symmetry, the field at point P is going to be along the y-axis (up if the line is positively charged).

Step 3: Label the picture entirely.

Step 4: Set up the basic equations:

$$dq = \lambda dx';$$

$$\vec{\mathrm{r}} = -x'\hat{\imath} + s\hat{\jmath}; \qquad r = \sqrt{x'^2 + s^2}$$

$$d\vec{\mathrm{E}} = k_e \frac{dq}{r^3}\,\vec{\mathrm{r}} \rightarrow dE_y = k_e \frac{dq}{r^3}\,s$$

- **Step 5**: Substitute and integrate

$$E_y = \int dE_y = k_e \int \frac{dq}{r^3}\, s \; = k_e \int_{x\prime=-L/2}^{x\prime=L/2} \frac{\lambda dx'}{(x'^2 + s^2)^{3/2}}\, s$$

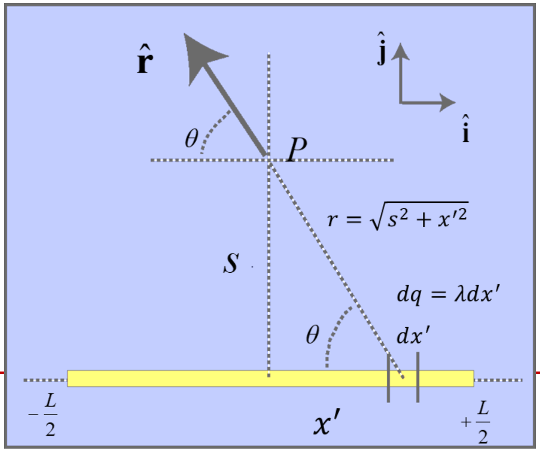
$$= \frac{k_e s Q}{L} \int_{x\prime=-L/2}^{x\prime=L/2} \frac{dx'}{(x'^2 + s^2)^{3/2}}$$

Solving this integration is not straight forwards, unfortunately. But it has been figured it out by mathematicians.

In the exam, you may not need to solve the integration of this kind as the integration table is given. But for now, let's see how we can solve it by hand.

From the geometry,

- $x' = s\cot\theta \quad \Rightarrow dx' = -s d\theta/\sin^2\theta \;\; \& \frac{s}{\sqrt{x\prime^2+s^2}} = \sin\theta$
- $\therefore \quad \frac{1}{(x\prime^2+s^2)^{3/2}} = \frac{\sin^3\theta}{s^3}$



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- The integral becomes

$$E_y = \frac{k_e sQ}{L}\int_{x\prime=-L/_2}^{x\prime=L/_2} \frac{dx'}{(x'^2+s^2)^{3/2}} = \frac{k_e sQ}{L}\int_{\theta_i}^{\theta_f} \frac{-s\sin^3\theta\, d\theta}{\sin^2\theta\, s^3} = -\frac{k_e Q}{sL}\int_{\theta_i}^{\theta_f} \sin\theta\, d\theta$$

$$= \frac{k_e Q}{sL}\left(\cos\theta_f - \cos\theta_i\right)$$

- $\cos\theta_f = \frac{L/2}{\sqrt{(L/2)^2+s^2}}$ & $\cos\theta_i = \frac{-L/2}{\sqrt{(L/2)^2+s^2}}$
- $E_y = \frac{k_e Q}{sL}\left(\frac{L/2}{\sqrt{(L/2)^2+s^2}} - \frac{-L/2}{\sqrt{(L/2)^2+s^2}}\right) = \frac{k_e Q}{s}\frac{1}{\sqrt{(L/2)^2+s^2}}$

- **Step 6**: Clean Up
- $\vec{\mathrm{E}} = \frac{k_e Q}{s}\frac{1}{\sqrt{(L/2)^2+s^2}}\,\hat{\mathrm{j}}$

- **Step 7**: Check Limits (for the sake of verification and deeper insights)
- Limits*: s >> L (far away)* and *s << L (close)*
- $\lim_{s \gg L} \vec{E} \rightarrow \frac{k_e Q}{s^2}\,\hat{\mathrm{j}}$ Looks like the E field of a point charge if we are far away
- $\lim_{s \ll L} \vec{E} \rightarrow 2\frac{k_e Q}{Ls}\,\hat{\mathrm{j}} = 2\frac{k_e \lambda}{s}\,\hat{\mathrm{j}}$ Looks like E field of an infinite charged line if we are close

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Concept Question 1.2: Electric Field of a Ring

- A uniformly charged ring of radius a has total charge Q. Which of the following expressions best describes the electric field at the point *P* located at the center of the ring?

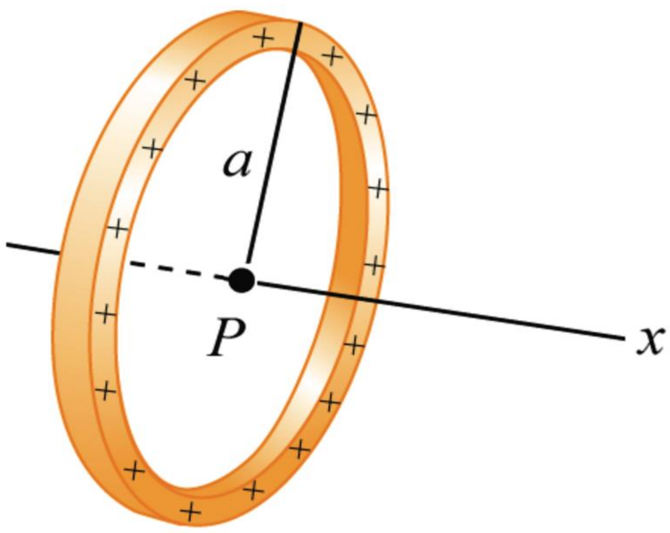
A. $\vec{E}(P) = -k_e \int_{\theta=0}^{\theta=2\pi} \frac{\lambda a d\theta}{a^3}\,\hat{i}$

B. $\vec{E}(P) = k_e \int_{\theta=0}^{\theta=2\pi} \frac{\lambda a d\theta}{a^3}\,\hat{i}$

C. $\vec{E}(P) = k_e \frac{Q}{a^2}\hat{i}$

D. $\vec{E}(P) = -k_e \frac{Q}{a^2}\hat{i}$

E. $\vec{E}(P) = 0$



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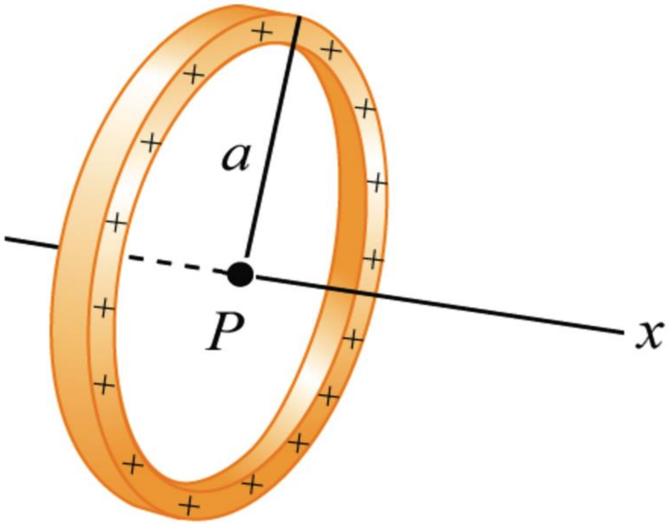
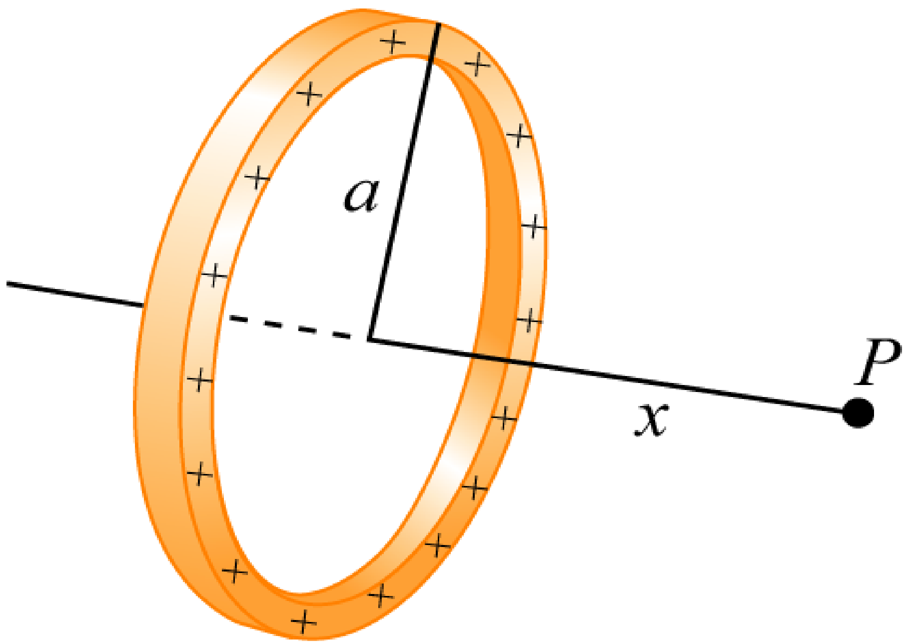
Concept Question 1.2: Solution

- Answer: E. $\vec{\mathrm{E}}(P) = 0$



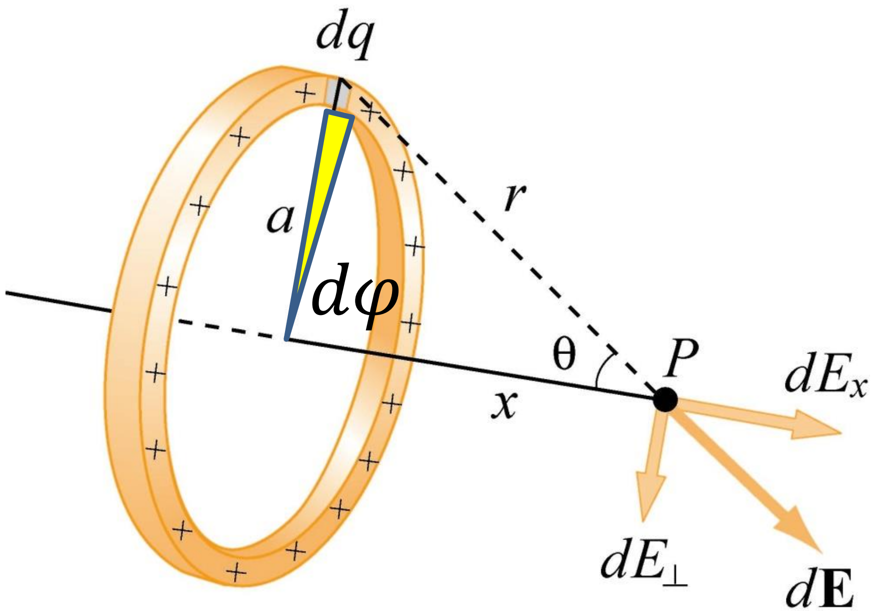
Case Problem 1.2: Ring of Charge

- A ring of radius a is uniformly charged with total charge Q. Find the direction and magnitude of the electric field at the point P lying a distance x from the center of the ring along the axis of symmetry of the ring.

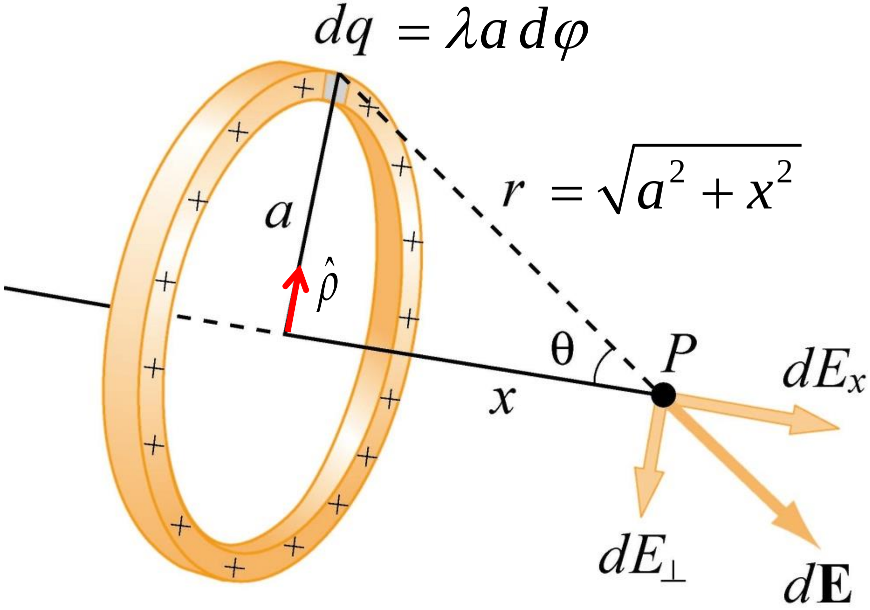


Case Problem 1.2: Solutions

- 1) Draw the diagram according to the problem.
- 2) Think about it, figure out the electric field configuration.
  - $\vec{E}_{\perp} = \oint d\vec{E}_{\perp} = 0$ Symmetry!
- 3) Label the picture entirely and define variables.
- $dq = \lambda dl = \lambda(ad\varphi) \quad r = \sqrt{a^2 + x^2}$

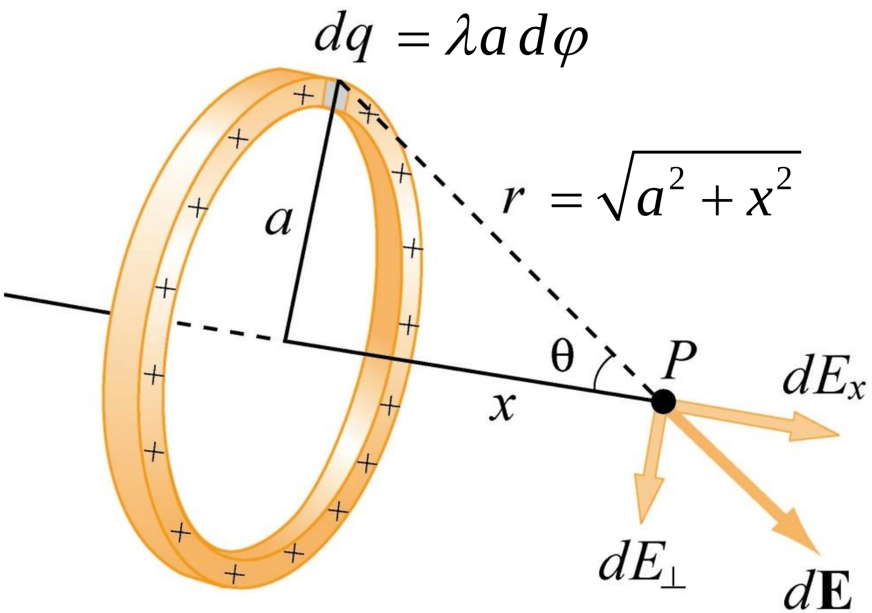


- 4) Write Equation
- $d\vec{\mathrm{E}} = k_e dq \frac{\hat{\mathrm{r}}}{r^2}$
- $\Rightarrow d\vec{\mathrm{E}} = k_e dq \frac{\vec{\mathrm{r}}}{r^3} = \frac{k_e dq}{r^3}(x\hat{\imath} - a\hat{\rho})$
- $\Rightarrow dE_x = k_e dq \frac{x}{r^3}$

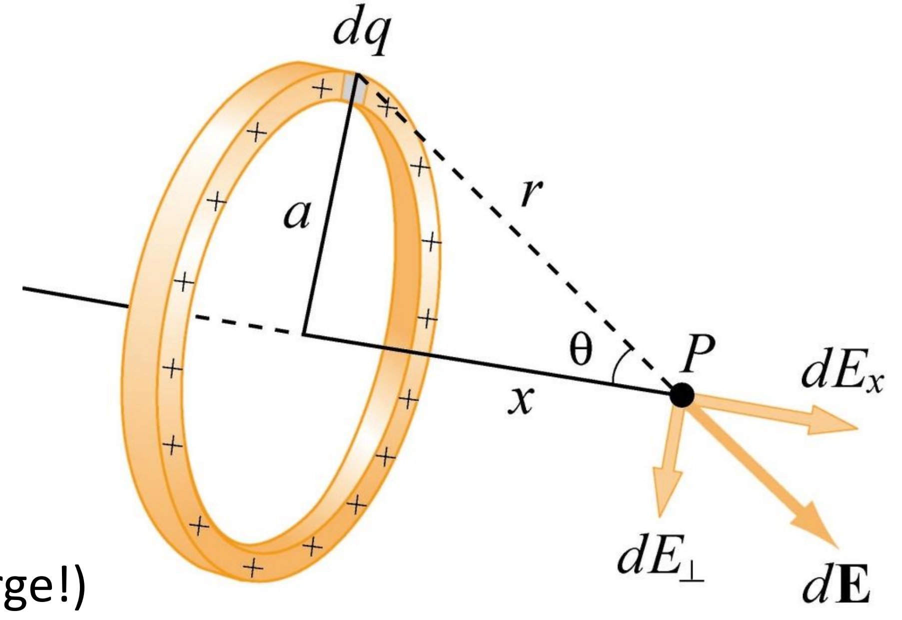


- 5) Substitute and integrate
- $E_x = \int dE_x = \int k_e dq \frac{x}{r^3} = k_e \frac{x}{r^3} \int dq$
- This particular problem is a very special case because everything except dq is constant, and

$$\int dq = \int_0^{2\pi} \lambda a d\varphi = \lambda a \int_0^{2\pi} d\varphi = \lambda a 2\pi = Q$$



- 6) Clean Up
- $E_x = k_e Q \frac{x}{r^3}$
- $\Rightarrow E_x = k_e Q \frac{x}{(a^2+x^2)^{3/2}}$
- $\vec{\mathrm{E}} = k_e Q \frac{x}{(a^2+x^2)^{3/2}} \hat{\mathrm{i}}$

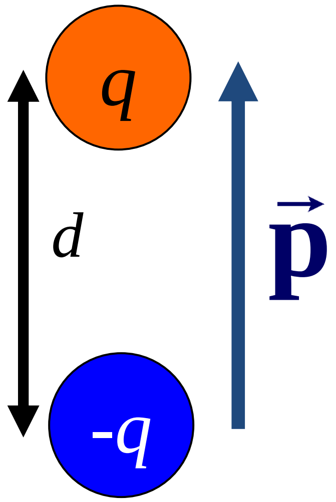


- 7) Check Limit $a \to 0$
- $E_x \to k_e Q \frac{x}{(x^2)^{3/2}} = \frac{k_e Q}{x^2}$ (point charge!)

Concept 2: Electric Dipole and the Electric Field of a Dipole

Electric Dipole

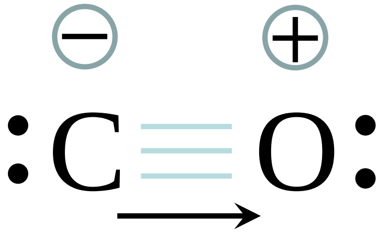
- Two equal but opposite charges $+q$ and $-q$, separated by a distance d
- Dipole moment two charges
- $\vec{P} \equiv qd\,\hat{\mathrm{j}}$
- $\vec{P}$ points from negative to positive charge
- Units: mC
- Dipole moment for neutral charge distribution of N point charges, i.e.



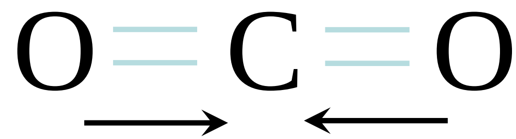
$$\sum_{i=1}^{N} q_i = 0 \Rightarrow \vec{\mathrm{p}} \equiv \sum_{i=1}^{N} q_i \vec{\mathrm{r}}_i$$

Electric Dipole Examples in Nature

- Electric Dipole moment of a CO molecule

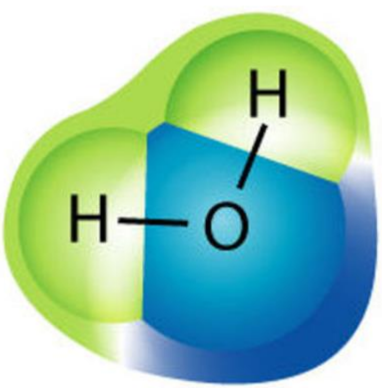


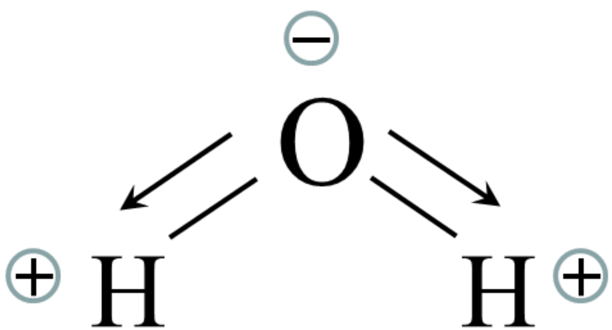
- Electric Dipole moment of a carbon dioxide molecule



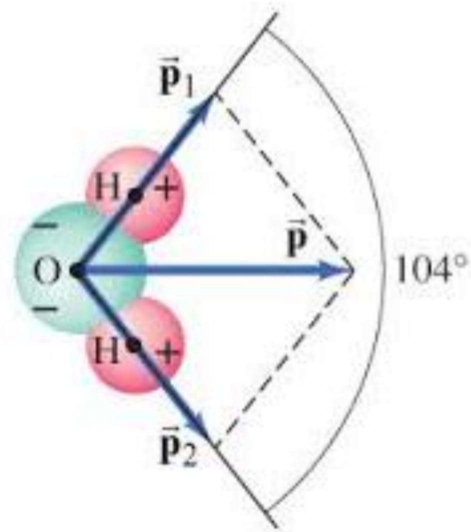
$$\vec{\mathbf{p}} = \vec{\mathbf{p}}_1 + \vec{\mathbf{p}}_2 = 0$$

- Electric Dipole moment of a water molecule





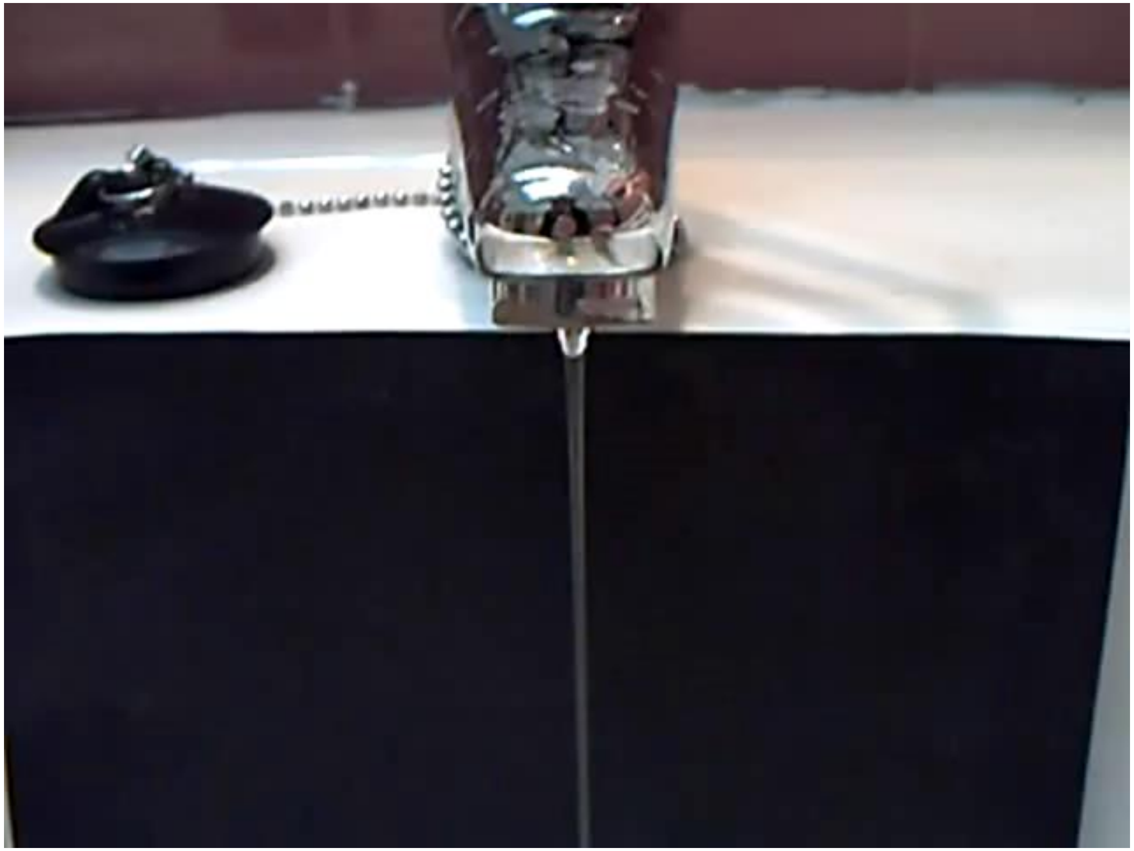
$$\vec{\mathbf{p}} = \vec{\mathbf{p}}_1 + \vec{\mathbf{p}}_2 \neq 0$$



A water molecule can be represented as a vector $\vec{p}$ (dipole moment)

Video Demo:

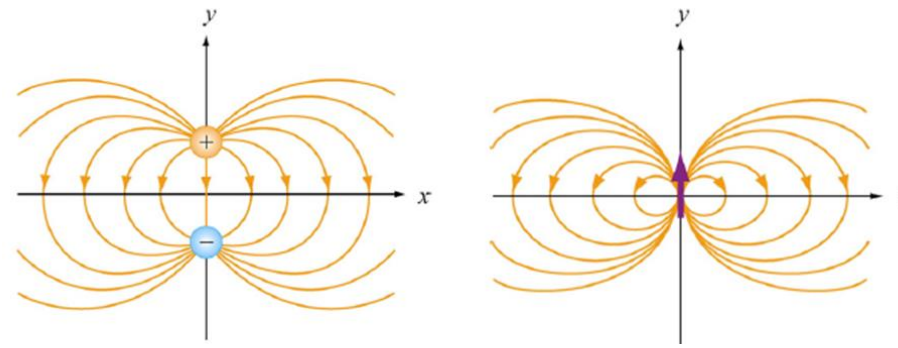
- Water are drawn by an external **(non-uniform) electric field** due to 2 factors:
  - Water molecules are electric dipoles.
  - Water usually contents charged ions that can be deflected by external $\vec{E}$.



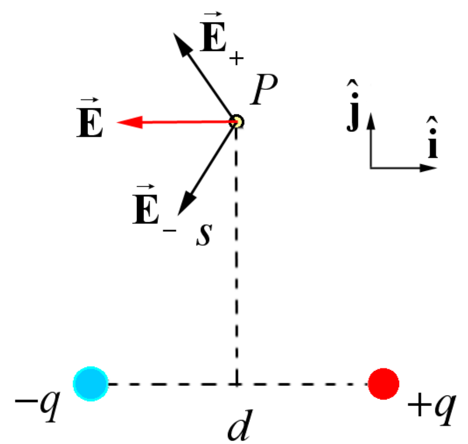
https://youtu.be/7b-w0oWttN0

Electric Field of an Electric Dipole

- Pictorially, a dipole field is illustrated as below.
- Note: both charges can be represented by a dipole vector, $\vec{P} \equiv qd\,\hat{\jmath}$



Recall Case Week 1 Day 1 Problem 3.1



- Electric field at point P, $\vec{\mathrm{E}}_p = k_e \dfrac{qd}{\left[\left(\frac{d}{2}\right)^2 + z^2\right]^{3/2}}(-\hat{\imath})$
- For $z \gg d$,

$$\vec{\mathrm{E}}_p = \frac{k_e qd}{z^3}(-\hat{\imath}) = \frac{k_e P}{z^3}(-\hat{\imath})$$

- This is the dipole electric field at the midpoint of both charges.
- Note: Dipole electric field drops by $1/r^3$, faster than $1/r^2$
- This is quite remarkable. The fact that the charges are slightly separated one from the other, and that their sum is 0, results in a very different dependence of the electric field, $1/r^3$ instead of $1/r^2$.

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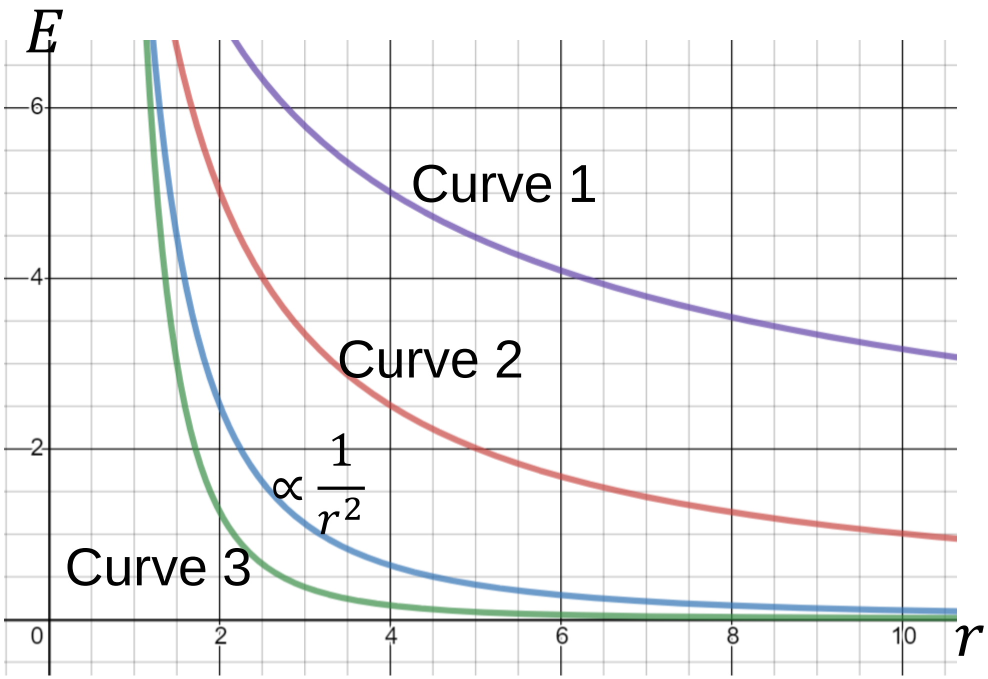
Concept Question 2.1: Dipole Field

- The blue curve represents the magnitude of the electric field of a point charge at a distance r, as $E \propto \frac{1}{r^2}$
- Which curve could possibly be the electric field of a dipole?

A. Curve 1

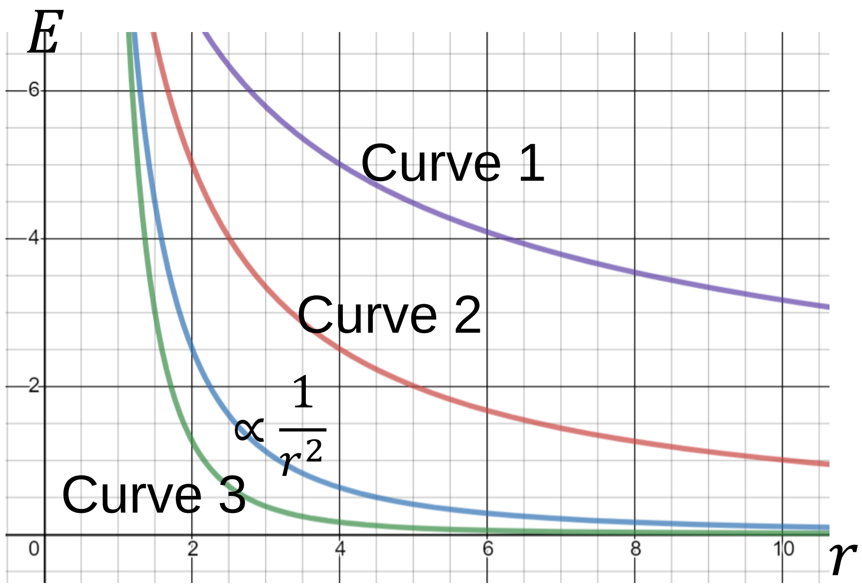
B. Curve 2

C. Curve 3



Concept Question 2.1: Solution

- Answer: C – Curve 3
- The electric field of a dipole falls off by $\frac{1}{r^3}$, more rapidly than $1/r^2$.
- We know this must be a case by thinking about what a dipole looks like from a large distance. To first order, it isn’t there (net charge is 0), so the E-Field must decrease faster than if there were a point charge there.



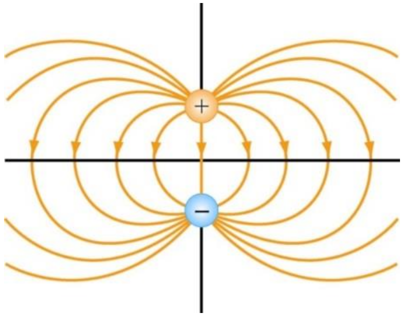
FYI: Point Dipole Approximation (in terms of spherical coordinate)

- For your extra knowledge, we can also rewrite the dipole field in spherical coordinate.
- For distances r>>a, you can show that

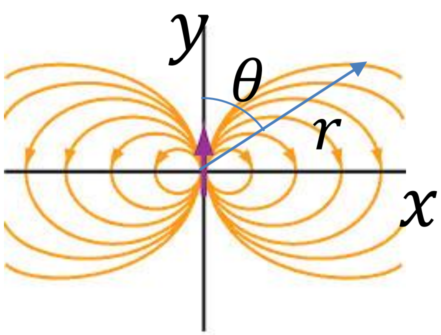
$$E_x(r,\theta) \to \frac{3p}{4\pi\varepsilon_o r^3}\sin(\theta)\cos(\theta)$$

$$E_y(r,\theta) \to \frac{p}{4\pi\varepsilon_o r^3}(3\cos^2\theta - 1)$$

- Where $p = 2qa$ (the definition of dipole, p=qd)
- In the above equations, we notice that $\vec{E}_{dipole} \propto \frac{1}{r^3}$ (again)!



Finite Dipole

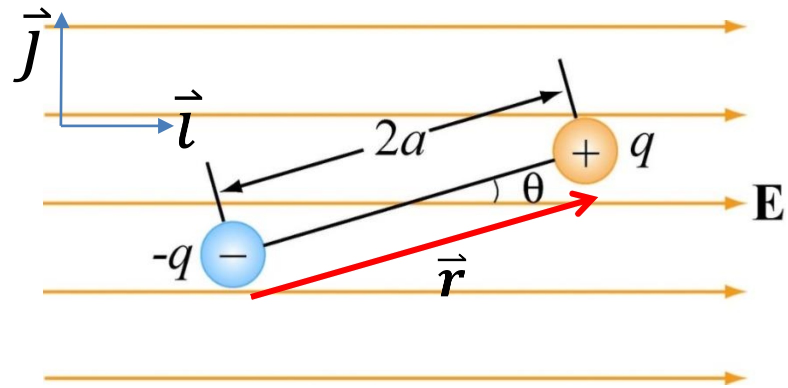


Point Dipole

Concept 3: Force and Torque on Electric Dipole in an Electric Field

Torque on Dipole: $\tau = \vec{p} \times \vec{E}$

Dipole in Uniform Field



- $\vec{\mathrm{E}} = E\,\hat{\mathrm{i}}$
- $\vec{r} \equiv 2a(\cos\theta\ \hat{\mathrm{i}} + \sin\theta\ \hat{\mathrm{j}})$
- $\vec{\mathrm{p}} \equiv \vec{r}q = 2qa(\cos\theta\ \hat{\mathrm{i}} + \sin\theta\ \hat{\mathrm{j}})$

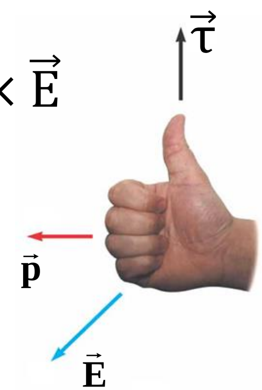
- Total Net Force: $\vec{F}_{net} = \vec{\mathrm{F}}_{+} + \vec{\mathrm{F}}_{-} = q\vec{\mathrm{E}} + (-q)\vec{\mathrm{E}} = 0$
- Total Torque on Dipole:

$$\vec{\tau} = \frac{\vec{r}}{2} \times \vec{F}_{+} + \left(-\frac{\vec{r}}{2}\right) \times \vec{F}_{-}$$
$$= \frac{\vec{r}}{2} \times \vec{F}_{+} + \left(-\frac{\vec{r}}{2}\right) \times -\vec{F}_{+}$$
$$= 2\left(\frac{\vec{r}}{2} \times \vec{F}_{+}\right) = \vec{r} \times \vec{F}_{+}$$

$$\vec{\tau} = \vec{\mathrm{r}} \times \vec{\mathrm{F}} = \vec{\mathrm{r}} \times q\vec{\mathrm{E}}$$
$$= q\vec{\mathrm{r}} \times \vec{\mathrm{E}} = \vec{\mathrm{p}} \times \vec{\mathrm{E}}$$

Recall: Right Hand Rule

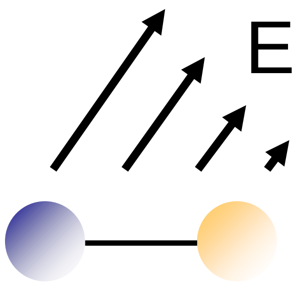
$$\vec{\tau} = \vec{\mathrm{p}} \times \vec{\mathrm{E}}$$



- $\vec{\tau}$ rotates the dipole and tends to align the dipole with the electric field. Once the dipole is aligned in the direction of field, torque vanishes ($\vec{\mathrm{p}} \times \vec{\mathrm{E}}$ will be zero, if p and E are in the same direction).

Concept Question 3.1: Dipole in Non-Uniform Field

A dipole sits in a non-uniform electric field E. Due to the electric field this dipole will feel:



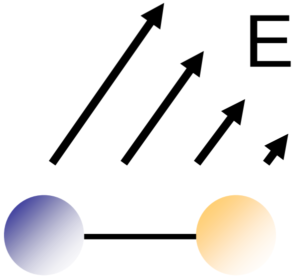
A. force but no torque

B. no force but a torque

C. both a force and a torque

D. neither a force nor a torque

Concept Question 3.1: Solution



Answer: C. both force and torque

Because the field is non-uniform, the forces on the two equal but opposite point charges do not cancel.

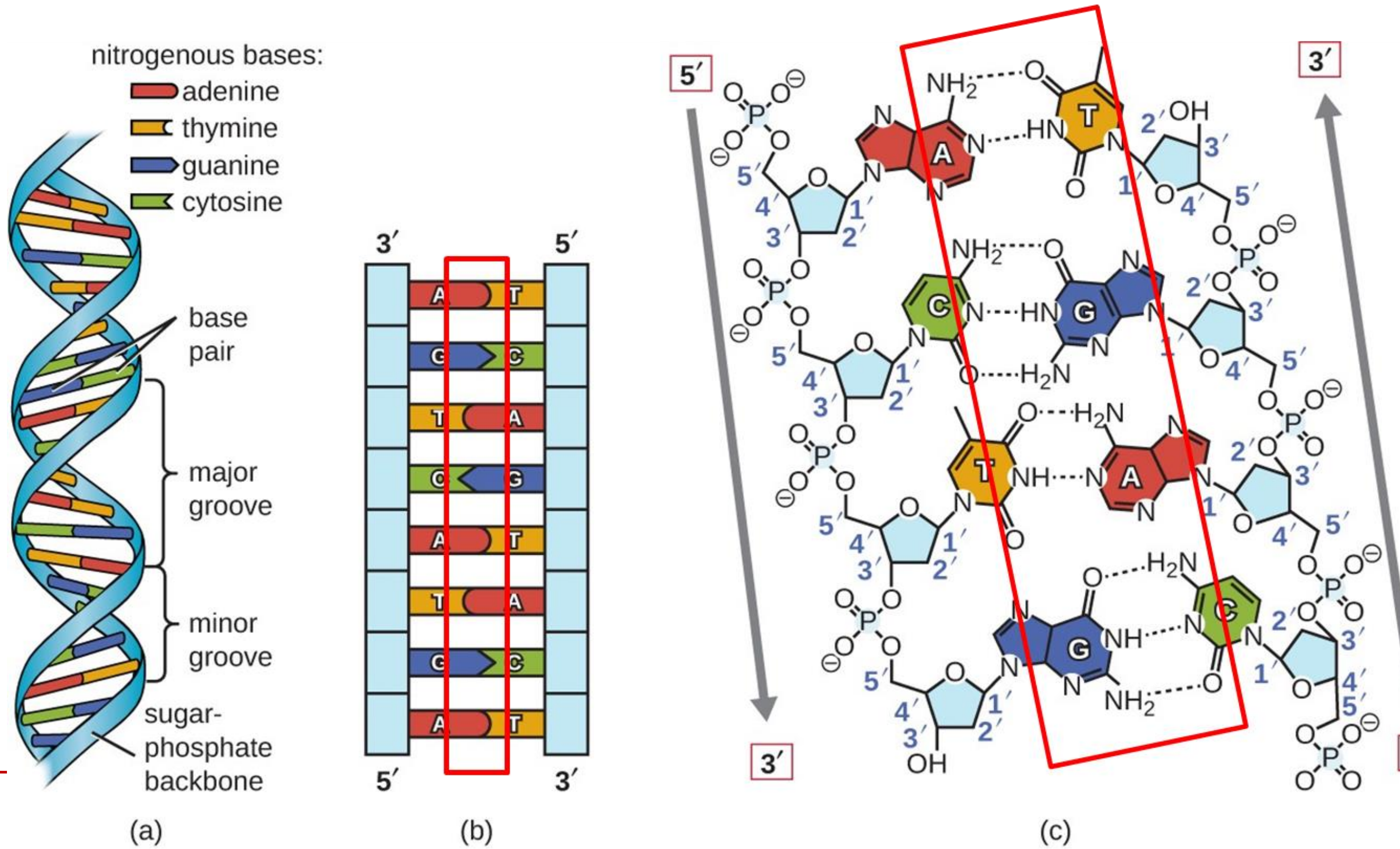
As always, the dipole wants to rotate to align with the field – there is a torque on the dipole as well.

Application: Why is dipole important?

- In chemistry, intermolecular forces are the forces of attraction or repulsion which act between neighboring particles (atoms, molecules, or ions).
- These forces are weak compared to the intramolecular forces, such as the covalent or ionic bonds between atoms in a molecule.
- **Types of Attractive Intermolecular Forces**
  - Dipole-dipole forces: electrostatic interactions of permanent dipoles in molecules; includes hydrogen bonding.
  - Ion-dipole forces: electrostatic interaction involving a partially charged dipole of one molecule and a fully charged ion.
  - Instantaneous dipole-induced dipole forces or London dispersion forces: forces caused by correlated movements of the electrons in interacting molecules, which are the weakest of intermolecular forces.
- Water molecules attract each other through the special type of dipole-dipole interaction known as hydrogen bonding.
- Hydrogen bonds are incredibly important in biology, because hydrogen bonds keep the DNA bases paired together, helping DNA maintain its unique structure.

Application: Dipole Interaction - Hydrogen bonds in DNA structures

- The bases hold the two strands of DNA together by strong intermolecular forces called hydrogen bonds. Because of the structures of the different bases, adenine (A) always forms hydrogen bonds with thymine (T), whilst guanine (G) always forms hydrogen bonds with cytosine (C). In human DNA, on average there are 150 million base pairs in a single molecule!
- It is the dipole-dipole interaction that forms life!



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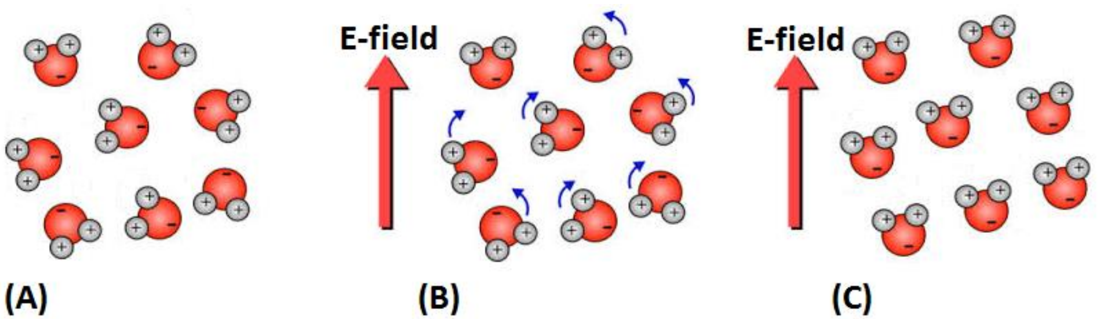
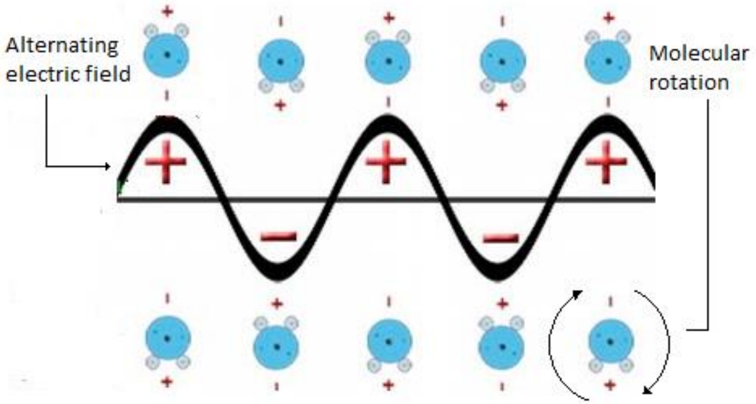
Application: Microwave Oven

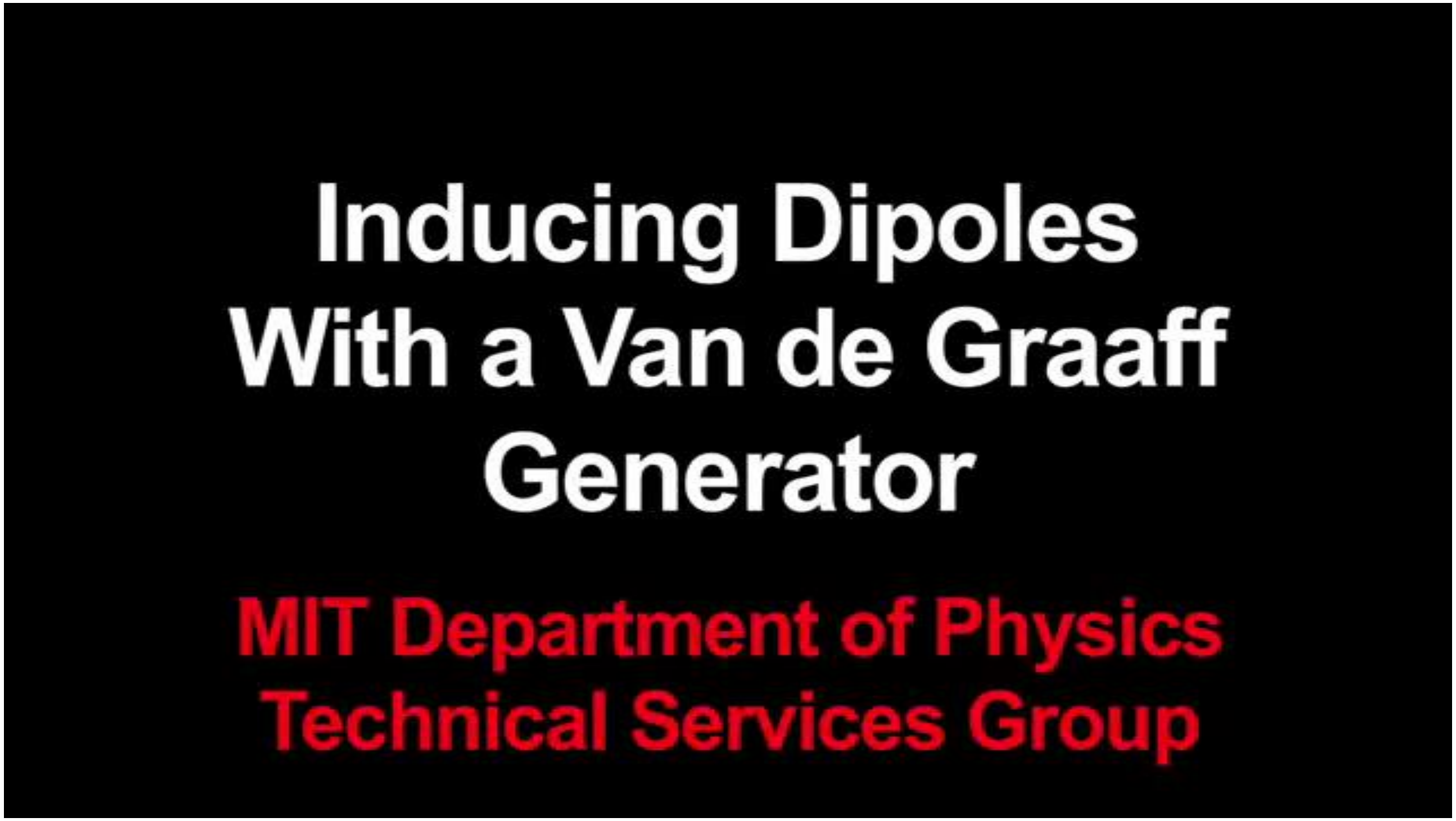
- The fact that water is a polar molecule (carries an electric dipole moment) makes microwave oven works to cook foods that content water.
- A microwave oven, generates 2.45G Hz electromagnetic waves (the wavelength of about 12.2 cm).
- When electromagnetic waves interact with the water molecules in the food, the water molecules physically re-orient (due to the torque!) with the oscillating electric field carried by the wave. For waves at 2.45 GHz, this is occurring over 2 billion times per second!
- The heat energy generated by the physical movement of the water molecules is then transferred throughout the food.



Polarization of water molecules. (A) Disordered water molecules. B) Water molecules rotate to align with E-field. C) Water molecules polarized under E-field.



FYI: Demonstration: Dipole in a Van de Graaff Generator

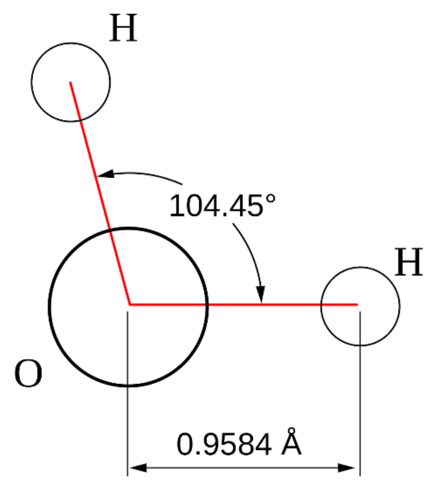


http://tsgphysics.mit.edu/front/?page=demo.php&letnum=D%2022&show=0

In Class Worksheet

Extra Case Problem 2.1:

- Given the dipole moment, $\vec{p}$ of a water molecule is $6.18 \times 10^{-30} Cm$. Estimate the charge separation of O-H bond in a water molecule.



Extra Case Problem 2.1 Solution

- $\vec{p} = 2p_{OH}\cos\left(\frac{104.45^o}{2}\right) = 6.18 \times 10^{-30}Cm$
- $\therefore p_{OH} = 5.04 \times 10^{-30}Cm.$
- By the definition: $p = Qd$
- $5.04 \times 10^{-30}Cm = Q(0.959 \times 10^{-10}m)$
- $Q = 5.26 \times 10^{-20}C = 0.328e$