10.018 Modelling Space and Systems

Lecture 5

Double Integration, Triple Integration, Change of Variables

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SINGAPORE UNIVERSITY OF TECHNOLOGY AND DESIGN

Before we start....

To get the most out of this lecture, you should already be familiar with

1. Basics of double integration and iterated integrals (last week)

as we will be going through

1. Triple integrals (briefly)
2. Applications of integration
3. Change of variables in double integrals

Review of Last Week

We looked at three types of regions where we could compute double integrals over

- Over a rectangular region R
- Over a horizontally simple region R
- Over a vertically simple region R

Review of last week

Using the idea of slices, we have the following results:

If R is vertically simple, then

$$\iint\limits_R f(x,y)\,\mathrm{d}A = \int_a^b \int_{y_1(x)}^{y_2(x)} f(x,y)\,\mathrm{d}y\,\mathrm{d}x.$$

If R is horizontally simple, then

$$\iint\limits_R f(x,y)\,\mathrm{d}A = \int_c^d \int_{x_1(y)}^{x_2(y)} f(x,y)\,\mathrm{d}x\,\mathrm{d}y.$$

Some guidelines for evaluating a double integral:

- **Sketch the region that we are integrating over**.
- Determine whether it is vertically or horizontally simple, then pick the right formula to use.
- Sometimes, doing the iterated integral in a different order can simplify calculations.

Triple Integrals

In Week 1 lecture (slide 12), mentioned that understanding concepts for 1 dimension (Modelling and Analysis) makes the jump to 2 dimensions (Modelling Space and Systems) easier.

Triple integrals then follow “naturally” from double integrals.

The Double and Triple Integral (picture)

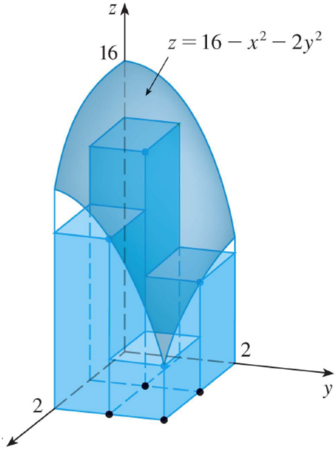


Figure: Double integral

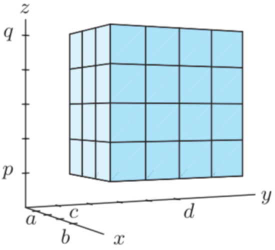


Figure: Triple integral

Triple Integration

Similarly, we can write

$$\iiint\limits_R f \, \mathrm{d}V = \int_p^q \left(\int_c^d \left(\int_a^b f(x,y,z) \, \mathrm{d}x \right) \, \mathrm{d}y \right) \, \mathrm{d}z$$

and interchange the order of integration (for constants a, b, c, d, q, p).

Strongly recommended exercise before Week 5 Cohort 2: "Mimic" or follow what we did in Week 4 Cohort 2 for double integrals and come up with the above equation for triple integrals. Perhaps go further?

Moving On

What if our region R (for double integration) isn't rectangular, or horizontally or vertically simple?

- Change of variables (done in lecture)
- Different coordinate systems (done in cohort)

But let's pause for a while and figure out why we're interested in integration.

An Brief Exposition on Numerical Integration

It is certainly true that numerical integration methods exist which can compute integrals.

- Trapezium rule
- Simpson's rule

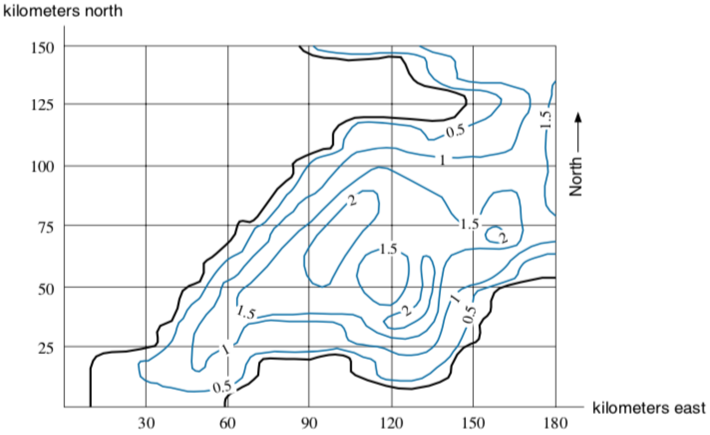
An Brief Exposition on Numerical Integration

It is certainly true that numerical integration methods exist which can compute integrals.

- ~~Trapezium rule~~
- ~~Simpson's rule~~

But we simply use Monte Carlo integration for high dimensional integrals (definitely more than two dimensions).

Practical applications of $\iint_R f(x, y)\ \mathrm{d}x\ \mathrm{d}y$



$f(x, y)$: $=$ Population density
R: Region of map we are interested in

Practical applications of $\iint_R f(x,y)\ \mathrm{d}x\ \mathrm{d}y$

Finding the volume of an object.

Example: Sphere of radius a

$f(x,y) := 2\sqrt{a^2 - x^2 - y^2}$
$R :=$ "a circle with radius a"

This was a past year exam question. So no detailed solutions will be provided.

Practical applications of $\iiint_R f(x,y,z)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z$

$f(x,y,z)$: $=$ Density at a point
R: DS (Deformed Sphere)

Practical applications of $\iint_R f(x,y)\,\mathrm{d}x\,\mathrm{d}y$

Interview questions for jobs that require multivariable calculus.

Alice and Bob agree to meet at Crooked Cooks after their cryptography class. Unfortunately, none of them have phones as their friend Eve has a tendency to hack into them. They can arrive uniformly between 20:00 and 21:00. Each person is prepared to wait t minutes before leaving. Find the minimum t (integer) such that the chance they meet is at least $\frac{1}{2}$.

Alice and Bob

Denote Alice's arrival time to be x, and Bob's arrival time to be y.

We can consider the unit square with coordinates $0 \leq x, y \leq 1$, which measures the time in fractions of an hour.

What is the region R of interest in this square?

$R = \left\{0 \leq x, y \leq 1 \mid |x - y| \leq \frac{t}{60}\right\}$ which corresponds to a strip around the diagonal $x = y$.

Hence we have $\iint_R 1 \,\mathrm{d}x \,\mathrm{d}y = 1 - \left(1 - \frac{t}{60}\right)^2$, and we want this to be greater than $\frac{1}{2}$.

This gives $t = 18$ minutes.

Some Comments

The last few slides on applications are "usually" **non-examinable**, because coming up with a proper region R and $f(x, y)$ is hard.

Change of variables (substitution)

Recall *integration by substitution* from single variable calculus: if $x = g(t)$, then

$$\int_{g(a)}^{g(b)} f(x)\,\mathrm{d}x = \int_a^b f(g(t))\,g'(t)\,\mathrm{d}t.$$

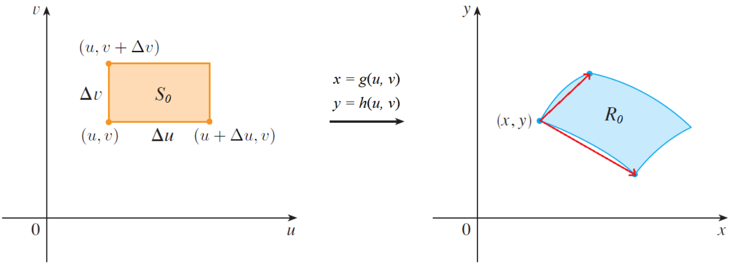
For double integrals, changing to polar coordinates is an example of substitution, also known as *change of variables*. We now consider the general case, where

$$x = g(u,v), \quad y = h(u,v).$$

Consider a small Δu-by-Δv rectangle S_0 in the uv-plane.

Under g and h, S_0 is transformed into a region R_0 in the xy-plane.

Change of variables



When Δu, Δv are small, R_0 is approximately a parallelogram whose edge vectors are shown. These vectors are given by

$$\begin{bmatrix} g(u+\Delta u, v) \\ h(u+\Delta u, v) \end{bmatrix} - \begin{bmatrix} g(u, v) \\ h(u, v) \end{bmatrix} = \begin{bmatrix} \frac{\Delta g}{\Delta u} \\ \frac{\Delta h}{\Delta u} \end{bmatrix} \Delta u,$$

$$\begin{bmatrix} g(u, v+\Delta v) \\ h(u, v+\Delta v) \end{bmatrix} - \begin{bmatrix} g(u, v) \\ h(u, v) \end{bmatrix} = \begin{bmatrix} \frac{\Delta g}{\Delta v} \\ \frac{\Delta h}{\Delta v} \end{bmatrix} \Delta v.$$

Change of Variables

Each parallelogram has an area ΔA, approximately equal to the magnitude of the *cross product* of these vectors*:

$$\Delta A \approx \left\| \begin{bmatrix} \frac{\Delta g}{\Delta u} \\ \frac{\Delta h}{\Delta u} \\ 0 \end{bmatrix} \times \begin{bmatrix} \frac{\Delta g}{\Delta v} \\ \frac{\Delta h}{\Delta v} \\ 0 \end{bmatrix} \right\| \Delta u \Delta v = \left\| \begin{bmatrix} 0 \\ 0 \\ \frac{\Delta g}{\Delta u}\frac{\Delta h}{\Delta v} - \frac{\Delta g}{\Delta v}\frac{\Delta h}{\Delta u} \end{bmatrix} \right\| \Delta u \Delta v.$$

In the limit $\Delta u, \Delta v \to 0$, we obtain

$$\mathrm{d}x\ \mathrm{d}y = \mathrm{d}A = |g_u h_v - g_v h_u|\, \mathrm{d}u\, \mathrm{d}v,$$

* *We have to add zero as the third coordinate because cross-product is defined only for vectors in* $\mathbb{R}^3$

Change of variables formula

Change of variables formula

If $x = g(u, v),\ y = h(u, v)$ are *one-to-one* transformations of a region S in the uv-plane onto a region R in the xy-plane, and g, h have continuous partial derivatives, then

$$\iint\limits_R f(x, y)\,\mathrm{d}x\,\mathrm{d}y = \iint\limits_S f(g(u, v), h(u, v))\,|g_u h_v - g_v h_u|\,\mathrm{d}u\,\mathrm{d}v,$$

The change of variables for transformations involving three or more variables is similarly defined (look at volume of parallelepiped).

Strongly recommended exercise: Look up on how to find the volume of a parallelepiped, and derive the change of variables formula for triple integration.

Change of Variables – generic approach

Suppose we have $\iint_R f(x, y)\ \mathrm{d}x\ \mathrm{d}y$. Then for change of variables:

1. Propose a one-to-one transformation that changes our region of integration R to a region of integration S.
2. Find bounds on region of integration S.
3. Find $|g_u h_v - g_v h_u|$
4. Check to see if we can express our answer in terms of elementary functions. If yes, declare success. Otherwise, repeat Step 1.

Change of Variables – example

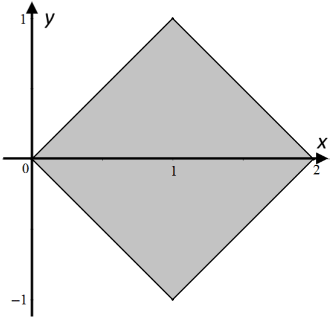
Let R be the square shown, with corners at $(x, y) = (0, 0)$, $(1, 1)$, $(2, 0)$ and $(1, -1)$.

Consider the integral

$$\iint\limits_R f(x, y)\,\mathrm{d}x\,\mathrm{d}y,$$

and the proposed change of variables

$$x = u + v, \quad y = u - v.$$



What region is R transformed into in the uv-plane?

What does the integral look like after the change of variables?

Change of Variables – example, continued

R can be described by: $0 \le x + y \le 2, \quad 0 \le x - y \le 2$.
Since $x + y = 2u$, $x - y = 2v$, these inequalities can be written as

$$0 \le u \le 1, \; 0 \le v \le 1.$$

Therefore R is transformed into a square S with corners at $(0, 0)$, $(1, 0)$, $(1, 1)$ and $(0, 1)$ in the uv-plane.

Partial derivatives are $g_u = g_v = 1, h_u = 1, h_v = -1$, hence
$|g_u h_v - g_v h_u| = 2$
The integral becomes

$$\iint_S f(g(u, v), h(u, v)) \, |g_u h_v - g_v h_u| \, \mathrm{d}u \, \mathrm{d}v = \int_0^1 \int_0^1 2 f(u+v, u-v) \, \mathrm{d}u \, \mathrm{d}v.$$

Note: in any integration problem, computing the quantity $|g_u h_v - g_v h_u|$ is usually easy; the hard part is finding the correct transformation g and h.

Summary

We have covered:

- Triple integrals.
- Applications of double and triple integrals.
- The general change of variables formula.

Textbook: read Section 20.3 to 20.5 (for this entire week at least), then try some of Exercises 20.3.1–20.3.14, Exercises 20.4.1–20.4.41, and Exercises 20.5.1–20.5.30. You may discuss them on Piazza.