10.018 Modelling Space and Systems

Cohort 6.1

Line Integrals, Vector Fields

Term 2, 2021

SINGAPORE UNIVERSITY OF TECHNOLOGY AND DESIGN

Before we start....

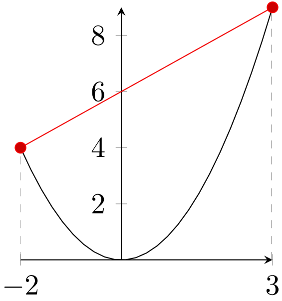
To get the most out of this lecture, you should already be familiar with

1. Line integrals in Physics, and Modelling Space and Systems lecture 6
2. Vector fields in Physics, and Modelling Space and Systems lecture 6

as we will be going through

1. Line integrals
2. Vector fields

From the Lecture: Parametrizing a curve



For example, in $\mathbb{R}^2$, the segment of a **parabola** starting at $(-2,4)$, passing through $(0,0)$ and ending at $(3,9)$ can be parametrized by

$$\vec{p}(t) = \begin{bmatrix} t \\ t^2 \end{bmatrix}, \quad t \in [-2,3].$$

The **straight line** segment from $(-2,4)$ to $(3,9)$ can be parametrized by

$$\vec{p}(t) = \begin{bmatrix} -2 \\ 4 \end{bmatrix} + t \begin{bmatrix} 3-(-2) \\ 9-4 \end{bmatrix} = \begin{bmatrix} -2+5t \\ 4+5t \end{bmatrix}, \; t \in [0,1].$$

Activity 1 (10 minutes)

(1) Parametrize the straight line segment from $(3, 7)$ to $(5, 3)$ as a function of $t \in [0, 1]$, moving at constant speed along the segment.

(2) Find another parametrization of the same line segment, but this time the parametrized function moves slower when t is near $t = 0$ and faster when t is near $t = 1$ sec. There are many possible answers here.

Activity 1 (solution)

(1) We can take

$$\vec{p}_1(t) = [3,\ 7] + t\,[5-3,\ 3-7] = [3+2t,\ 7-4t], \quad t \in [0,1].$$

If we think of t as time (in seconds), then it should be clear that this parametrization moves at constant speed along the segment (and the speed is given by $\|\vec{p}_1{}'(t)\| = \|[2,-4]\| = 2\sqrt{5}$ units/sec).

(2) There are many possible answers; we can replace t by a function of t, whose derivative increases as t goes from 0 to 1 sec. An example of such a function is t^n, where $n > 1$. So we could for instance set

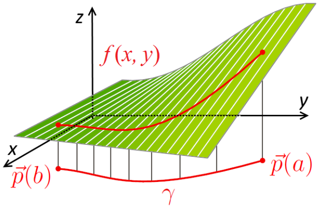
$$\vec{p}_2(t) = [3+2t^2,\ 7-4t^2], \quad t \in [0,1].$$

The speed is given by $\|\vec{p}_2{}'(t)\| = \|[4t, -8t]\| = 4\sqrt{5}t$ units/sec^2.

From the Lecture: Line integrals

In addition to double and triple integrals, *line integrals* provide yet another way to extend integration to higher dimensions.

In 3D, a line integral gives the signed cross-sectional area bounded by a surface f and a curve γ (i. e. curved line) in the xy-plane.



More generally, a line integral can be defined for a function $f : \mathbb{R}^n \to \mathbb{R}$, which we call a **scalar field**,
The curve γ is parametrized by a (one-to-one) function $\vec{p}(t)$, and its endpoints are given by $\vec{p}(a)$ and $\vec{p}(b)$.

From the Lecture: Line integrals – formula

Line integral of a scalar field

For a function $f : \mathbb{R}^n \to \mathbb{R}$, the line integral along a curve γ parametrized by $\vec{p} : [a, b] \to \mathbb{R}^n$ is given by

$$\int_\gamma f(\vec{x})\,\mathrm{d}s = \int_a^b f\left(\vec{p}(t)\right) \|\vec{p}\,'(t)\|\,\mathrm{d}t.$$

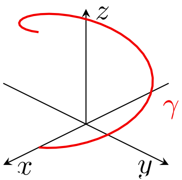
1. Find the parametrization $\vec{p}(t)$ of the curve γ, identify a and b.
2. Plug the parametrized curve into $f(\vec{x})$.
3. Compute $\|\vec{p}\,'(t)\|$.
4. Evaluate the integral as per above.

Activity 2 (10 minutes)

(1) Integrate $f(x, y, z) = y\sin(z)$ along the curve γ, parametrized by

$$\vec{p}(t) = [\cos(t),\, \sin(t),\, t], \quad t \in [0, 2\pi].$$

(2) Find the length of the curve. What function do you need to integrate along the curve to get its length?



Activity 2 (solution)

(1) Here $f(\vec{p}(t)) = \sin(t)\sin(t)$, $\vec{p}\,'(t) = [-\sin(t), \cos(t), 1]$, $a = 0$, $b = 2\pi$.

$$\begin{aligned}
\int_\gamma y\sin(z)\,\mathrm{d}s &= \int_a^b f(\vec{p}(t))\,\|\vec{p}\,'(t)\|\,\mathrm{d}t \\
&= \int_0^{2\pi} (\sin t)^2 \sqrt{(-\sin t)^2 + (\cos t)^2 + 1^2}\,\mathrm{d}t \\
&= \int_0^{2\pi} \frac{1-\cos(2t)}{2}\,\sqrt{2}\,\mathrm{d}t \\
&= \frac{\sqrt{2}}{2}\left[t - \frac{1}{2}\sin(2t)\right]_0^{2\pi} = \sqrt{2}\,\pi.
\end{aligned}$$

(2) The length of the curve can be found by

$$\int_\gamma 1\,\mathrm{d}s = \int_a^b 1\,\|\vec{p}\,'(t)\|\,\mathrm{d}t = \int_0^{2\pi} \sqrt{(-\sin t)^2 + (\cos t)^2 + 1^2}\,\mathrm{d}t = 2\sqrt{2}\,\pi \text{ unit.}$$

Break

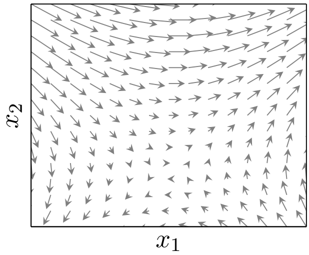
If you haven't done the mid-term survey please do it now ;-)

5 min break

Don't be late

Vector fields – introduction

Definition: a vector field $\vec{F} : \mathbb{R}^n \to \mathbb{R}^n$ is vector-valued function that associates a vector in $\mathbb{R}^n$ to each point of its domain.



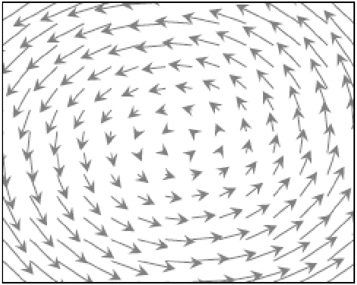
In $\mathbb{R}^2$:

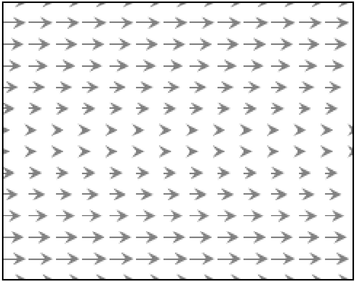
$$\vec{F}(x_1, x_2) = \vec{F}(\vec{x}) = \begin{bmatrix} F_1(\vec{x}) \\ F_2(\vec{x}) \end{bmatrix}.$$

At each point, you can imagine a vector field as describing a flow (of a fluid) with both magnitude and direction.

Vector fields – more examples

$$\vec{F}(x,y) = \begin{bmatrix} -y \\ x \end{bmatrix} \qquad \vec{F}(x,y) = \begin{bmatrix} \sqrt{|y|} \\ 0 \end{bmatrix}$$





See Lecture 6 for more examples.

Activity 3 (10 minutes)

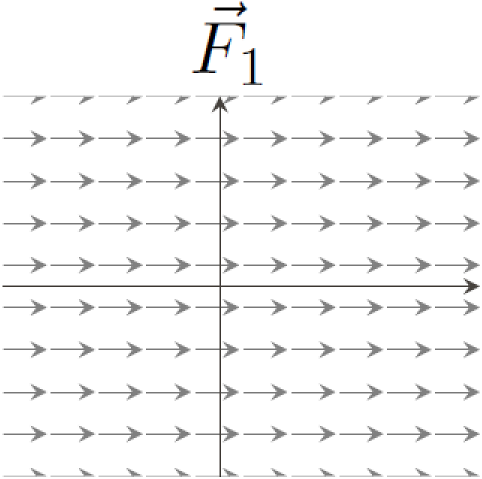
Sketch the following vector fields for $x \in [-1, 1]$ and $y \in [-1, 1]$:

$$\vec{F}_1(x, y) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \qquad \vec{F}_2(x, y) = \begin{bmatrix} x/2 \\ 0 \end{bmatrix}, \qquad \vec{F}_3(x, y) = \begin{bmatrix} 1 \\ x^2/4 \end{bmatrix}.$$

Hints for sketching:

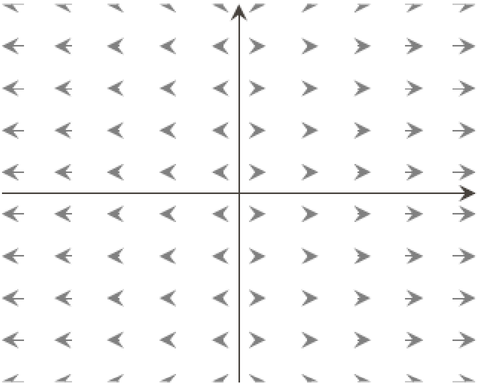
- for each point (x, y) draw a vector $\vec{F}(x, y)$ **attached** to that point.
- Fix one of the coordinates (say, $x = 1$), and draw vector fields for different y's. Then fix another x, and draw more vectors for different y. Notice a pattern and generalize it. (or you can fix y, and draw vectors for various x).
- You don't have to draw to scale, but the general pattern should be obvious (e.g. all of the vectors are of the same length, or length increases as x increases etc)

Activity 3 (solution)

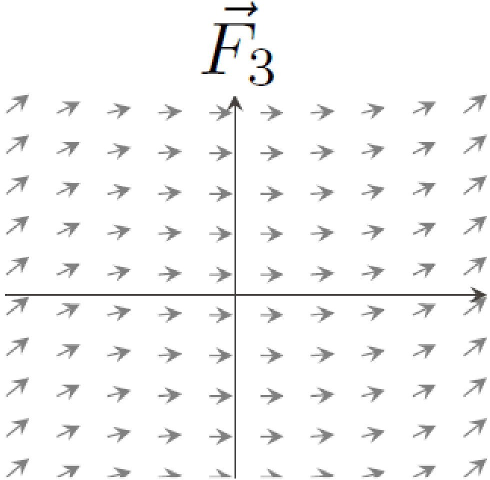


Activity 3 (solution)

$\vec{F}_2$



Activity 3 (solution)



Divergence

Definition: for a vector field $\vec{F} : \mathbb{R}^n \to \mathbb{R}^n$, the divergence of $\vec{F}$ is denoted by div $\vec{F}$ or $\nabla \cdot \vec{F}$, and is defined by (in Cartesian coordinate)

$$\nabla \cdot \vec{F} = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \cdots + \frac{\partial F_n}{\partial x_n}.$$

For instance, if $n = 3$, then

$$\nabla \cdot \vec{F} = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \cdot \begin{bmatrix} F_1(x,y,z) \\ F_2(x,y,z) \\ F_3(x,y,z) \end{bmatrix} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z},$$

which is a *scalar* field (i.e. a function).

Divergence

Thinking of a vector field as a flow, roughly speaking the **div measures how much the volume of the flow expands/compresses at each point**. See end of the cohort for the derivation.

Above, we denoted by ∇ a vector $\begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix}$. This is done purely for mnemonic purposes. This vector has no meaning and cannot be drawn separately. Indeed, ∇ is a differential operator and has to act on a function.

Divergence – intuition

So for example,

- The rotational vector field $\vec{F}(x,y) = \begin{bmatrix} -y \\ x \end{bmatrix}$ has 0 div.
- A vector field describing the flow of an incompressible fluid has 0 div.
- A vector field describing an explosive blast has positive divergence.
- In physics, divergence of a vector field is related to the source/sink of the vector field, e.g. positive/negative electric point charge is the source/sink of the electric field. See Technological World lecture.

Watch https://youtu.be/C7G4DKTLbaw?t=6m40s from 6:40 – 12:10 for some visualizations.

Curl

Definition: for a vector field $\vec{F} : \mathbb{R}^3 \to \mathbb{R}^3$, the curl of $\vec{F}$ is denoted by curl $\vec{F}$ or $\nabla \times \vec{F}$, and is defined by

$$\nabla \times \vec{F} = \begin{bmatrix} \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \\ \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \\ \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \end{bmatrix}.$$

To remember this formula, think of it as a cross product:

$$\nabla \times \vec{F} = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \times \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix}$$

The curl roughly describes how much an object would spin when placed inside a vector field.

One can think of a 2D vector field as a 3D field with $F_3 = 0$, so its curl is defined by $\nabla \times \vec{F}(x, y) = \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\vec{e}_3$.

Curl – intuition

Thinking of a vector field as the flow velocity of a fluid, the curl gives the *vorticity* of the fluid.

The magnitude of the curl is related to the the angular speed of rotation, and the direction is perpendicular to the plane of maximum rotation.

Watch `https://youtu.be/TVtvlh6KoLo?t=5m40s` from 5:40 – 9:00 for some visualizations.

Div and curl – applications

Gradient, div and curl are the most essential operations in vector analysis. For instance, Maxwell's equations can be written in terms of div and curl:

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \qquad \text{Gauss Law}$$

$$\nabla \cdot \vec{B} = 0 \qquad \text{Gauss Law, No Magnetic Monopole}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \text{Faraday Law}$$

$$\nabla \times \vec{B} = \mu_0 \left(\vec{J} + \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right) \qquad \text{Ampère-Maxwell Law}$$

Speed of light is given by $c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$.

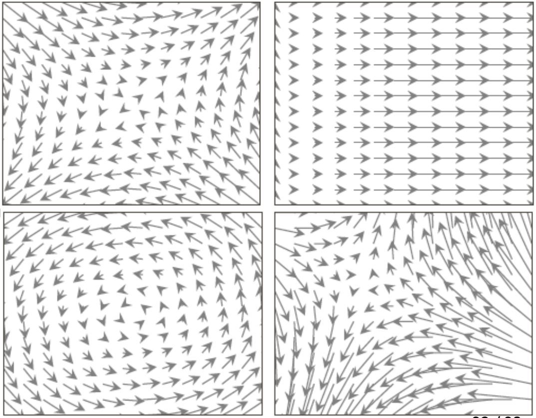
Activity 4 (15 minutes)

(1) For the following vector fields, try to predict (**without computing**) whether the div and curl are positive/zero/negative.

(2) Compute the exact div and curl of:

$$\vec{F}_1 = \begin{bmatrix} y \\ x \end{bmatrix}, \quad \vec{F}_2 = \begin{bmatrix} x \\ 0 \end{bmatrix},$$

$$\vec{F}_3 = \begin{bmatrix} -y \\ x \end{bmatrix}, \quad \vec{F}_4 = \begin{bmatrix} y - x \\ 2x + 2y \end{bmatrix}.$$



Activity 4 (solution)

$$\nabla \cdot \vec{F}_1 = 0, \quad \nabla \times \vec{F}_1 = \vec{0},$$

which makes sense, as the plot suggests that the flow does not change the area of a small square centred at each point, and also does not cause an object placed in it to spin (about its centre).

$$\nabla \cdot \vec{F}_2 = 1, \quad \nabla \times \vec{F}_2 = \vec{0},$$

since the plot suggests that the flow expands a small square centred at each point, but does not cause an object placed in it to spin.

$$\nabla \cdot \vec{F}_3 = 0, \quad \nabla \times \vec{F}_3 = 2\vec{e}_3,$$

since the plot suggests that the flow does not change area, but does cause an object placed in it to spin.

$$\nabla \cdot \vec{F}_4 = 1, \quad \nabla \times \vec{F}_4 = \vec{e}_3.$$

Activity 5 (PRACTICE)

Try this activity by yourself at home.

Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a scalar field and $F : \mathbb{R}^3 \to \mathbb{R}^3$ be a vector field.

(1) Simplify $\nabla \times (\nabla f)$.

(2) Simplify $\nabla \cdot (\nabla \times F)$.

What assumption(s) do you need to make?

Activity 5 (solution)

(1) Writing ∇f as a column vector:

$$\begin{aligned}\nabla \times (\nabla f) &= \nabla \times \left[f_x,\, f_y,\, f_z\right]^T \\ &= \left[f_{zy} - f_{yz},\, f_{xz} - f_{zx},\, f_{yx} - f_{xy}\right]^T = \vec{0},\end{aligned}$$

assuming that the second order partial derivatives are continuous, and hence equal.

(2)

$$\begin{aligned}\nabla \cdot (\nabla \times F) &= \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right]^T \cdot \left[\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right]^T \\ &= \frac{\partial^2 F_3}{\partial x \partial y} - \frac{\partial^2 F_2}{\partial x \partial z} + \frac{\partial^2 F_1}{\partial y \partial z} - \frac{\partial^2 F_3}{\partial y \partial x} + \frac{\partial^2 F_2}{\partial z \partial x} - \frac{\partial^2 F_1}{\partial z \partial y} \\ &= 0,\end{aligned}$$

again assuming that the second order partial derivatives are continuous.

Summary

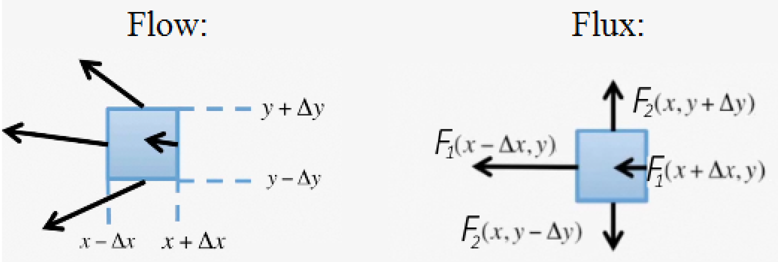
We have covered:

- Line integrals of scalar fields.
- Vector fields.
- Div and curl, and their physical interpretations.

Watch the Concept Vignette videos.

[OPTIONAL] Divergence – intuition

To be more precise, the div measures the net *flux* per unit area or volume. In 2D, consider a vector field $\vec{F}(x,y) = \begin{bmatrix} F_1(x,y) \\ F_2(x,y) \end{bmatrix}$.



Flux per unit area =

$$\frac{\left[F_1(x+\Delta x,y) - F_1(x-\Delta x,y)\right] 2\Delta y}{2\Delta x\, 2\Delta y} + \frac{\left[F_2(x,y+\Delta y) - F_2(x,y-\Delta y)\right] 2\Delta x}{2\Delta x\, 2\Delta y}$$

$$\to \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} = \nabla \cdot \vec{F}.$$