10.018 Modelling Space and Systems

Cohort 1.2: Tangent Planes and Directional Derivatives

Term 2, 2021

SINGAPORE UNIVERSITY OF TECHNOLOGY AND DESIGN

Before we start....

To get the most out of this cohort, you should already be familiar with

1. The equation of a tangent line to $f(x)$ (Math I)
2. Meaning of a derivative as *the best linear approximation* (Math I)
3. Vectors (Physics I, Math II lecture)
4. Planes (Physics I, Math II lecture)

as we will be going through

1. Tangent planes in Math II: The 2D analogue of the tangent line
2. Directional derivatives: Differentiating (the restriction of) a multivariate function along some tangent direction

Introduction

The idea of approximating a complicated function by a simpler function is very useful in science and engineering.

Some of the easiest approximations to work with are *linear* approximations.

For example, for a differentiable function of one variable $f(x)$, the tangent *line* at a point $(x_0, f(x_0))$ is given by

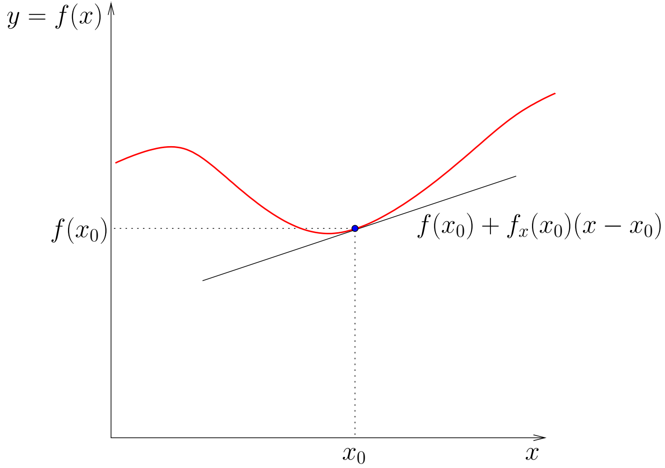
$$y = f(x_0) + f'(x_0)\,(x - x_0).$$

The tangent line approximates the function f near $(x_0, f(x_0))$.

One way to see this is by recalling that $f'(x) = \frac{\mathrm{d}f}{\mathrm{d}x}$ is a limit of $\frac{\Delta f}{\Delta x}$, so if Δx is small, then

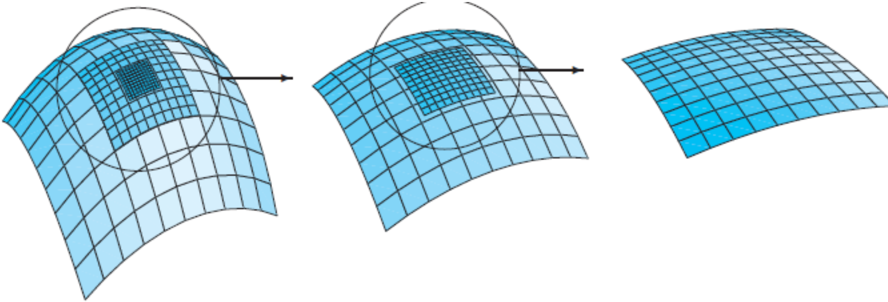
$$\Delta f \approx f'(x_0)\,\Delta x \qquad \text{i.e} \qquad f(x) - f(x_0) \approx f'(x_0)(x - x_0).$$

Tangent line – visualization



Tangent plane

For a function of two variables $f(x, y)$, the corresponding linear approximation is a tangent plane.



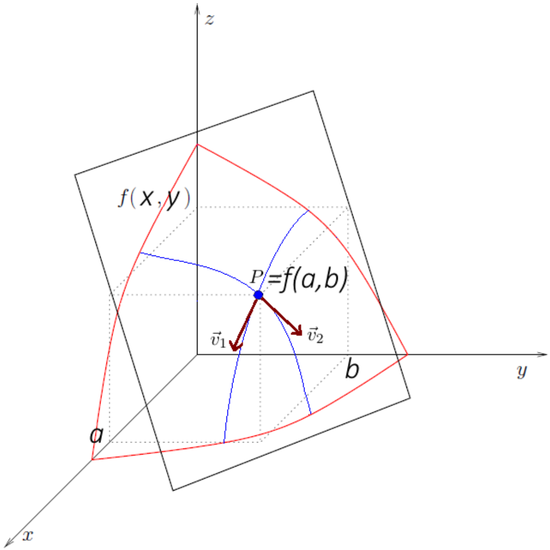
Tangent plane

Geometrically, the tangent plane to a surface at a point $P = (x_0, y_0, f(x_0, y_0))$ is the plane that passes through P, and 'best approximates' the surface near P.

What is the equation of the tangent plane? At the point (x_0, y_0), the x-slope of the graph of f is the partial derivative $f_x(x_0, y_0)$ and the y-slope is $f_y(x_0, y_0)$.

Notation: (x_0, y_0) denotes the arbitrary point P, and different textbooks or references may denote this point as (a, b), (α, β), (u, v), etc.

Tangent plane – visualization



Tangent plane – equation

Therefore, from the equation of the plane shown in the lecture

The equation of the *tangent plane* to a surface $z = f(x, y)$, at a point $P = (x_0,\ y_0,\ f(x_0, y_0))$, is given by

$$z = f(x_0, y_0) + f_x(x_0, y_0)\,(x - x_0) + f_y(x_0, y_0)\,(y - y_0).$$

It is common to denote $f(x_0, y_0)$ by z_0, so this equation can also be written as

$$z - z_0 = f_x(x_0, y_0)\,(x - x_0) + f_y(x_0, y_0)\,(y - y_0).$$

Tangent plane – example

Since the tangent plane approximates the surface f, $z - z_0$ approximates Δf, so we have

$$\Delta f \approx f_x(x_0, y_0)\,\Delta x + f_y(x_0, y_0)\,\Delta y.$$

This is analogous to the single variable case: $\Delta f \approx f'(x_0)\,\Delta x$.

Example

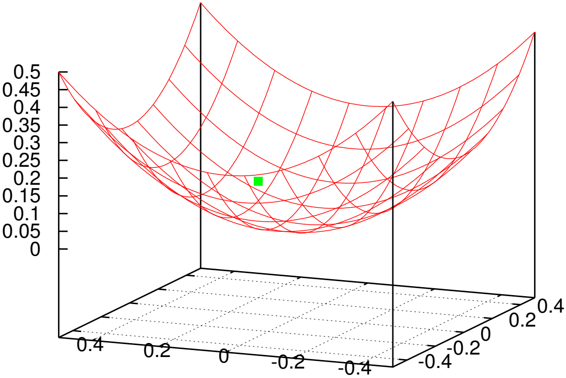
Find the tangent plane to $f(x, y) = x^2 + y^2$ at the point $(0.2, 0.2)$. We first compute $f_x = 2x$, $f_y = 2y$. The tangent plane is given by

$$\begin{aligned} z &= f(a, b) + f_x(a, b)\,(x - a) + f_y(a, b)\,(y - b) \\ &= 0.08 + f_x(0.2, 0.2)\,(x - 0.2) + f_y(0.2, 0.2)\,(y - 0.2) \\ &= 0.08 + 0.4\,(x - 0.2) + 0.4\,(y - 0.2) \\ &= 0.4x + 0.4y - 0.08. \end{aligned}$$

Surface – zooming in

Graph of $x^2 + y^2$, zooming in towards the point $(0.2, 0.2)$:

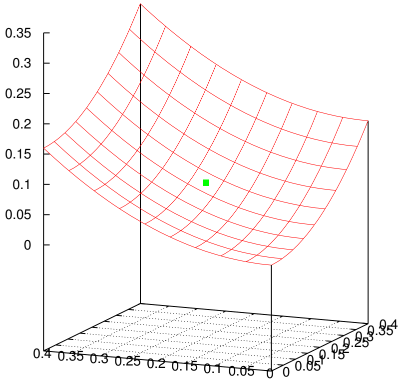
Showing $[-0.5, 0.5] \times [-0.5, 0.5]$:



Surface – zooming in

Graph of $x^2 + y^2$, zooming in towards the point $(0.2, 0.2)$:

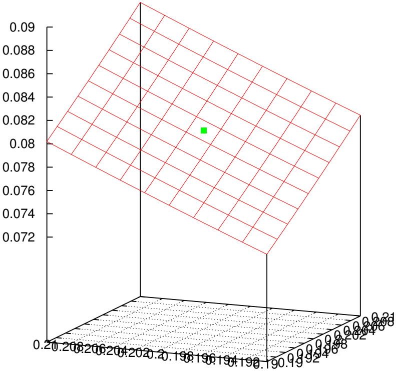
Showing $[0, 0.4] \times [0, 0.4]$:



Surface – zooming in

Graph of $x^2 + y^2$, zooming in towards the point $(0.2, 0.2)$:

Showing $[0.19, 0.21] \times [0.19, 0.21]$:



Activity 1 (10 minutes)

A student is asked to find the tangent plane to $z = x^3 - y^2$ at $(a, b) = (2, 3)$.

The answer he got is $z = 3x^2(x-2) - 2y(y-3) - 1$.

Is his answer wrong? If yes, then point out exactly where it is wrong, and find the correct answer.

Activity 1 (solution)

$z = 3x^2(x-2) - 2y(y-3) - 1$ is not a tangent plane because it is not the equation of a plane.

To find the tangent plane, we can use the formula

$$z = f(a,b) + f_x(a,b)\,(x-a) + f_y(a,b)\,(y-b).$$

For this question, $f(x,y) = x^3 - y^2$, $f_x = 3x^2$ and $f_y = -2y$.
Also, $f(a,b) = f(2,3) = 2^3 - 3^2 = -1$.

The key point to note is that the formula asks for $f_x(a,b)$ and $f_y(a,b)$, not just f_x and f_y.

We have $f_x(a,b) = 3 \times 2^2 = 12$ and $f_y(a,b) = -2 \times 3 = -6$.

So the correct answer is

$$z = -1 + 12(x-2) - 6(y-3) = 12x - 6y - 7.$$

Activity 2 (15 minutes)

A cylinder is measured to have radius 2 cm and height 5 cm, but the measurements are only precise up to 0.1 cm.

Let x denote the radius, y the height, and $f(x, y)$ the volume of the cylinder.

(1) **Using the tangent plane approximation**, estimate the maximum error in computing the volume of the cylinder.

(2) How does this estimate compare with the actual maximum error? It is achieved if the true radius is 2.1 cm and the true height is 5.1 cm.

Activity 2 (solution)

The volume is given by $f(x, y) = \pi x^2 y$.

We wish to see how the volume changes (Δf) if x and y were to change by small amounts.

$$\begin{aligned}\Delta f &\approx f_x(a, b)\,\Delta x + f_y(a, b)\,\Delta y \\ &= 2\pi ab\,\Delta x + \pi a^2\,\Delta y \\ &= 20\pi\,\Delta x + 4\pi\,\Delta y.\end{aligned}$$

As we are given that $|\Delta x| \leq 0.1$ and $|\Delta y| \leq 0.1$, the absolute value of the above expression for Δf is maximized when $\Delta x = \Delta y = \pm 0.1$. So the maximum error is

$$|\Delta f| \approx 2.4\pi \approx 7.54 \text{ cm}^3.$$

This is reasonably close to the actual maximum error, given by

$$|f(2.1, 5.1) - f(2, 5)| \approx 7.83 \text{ cm}^3.$$

Break

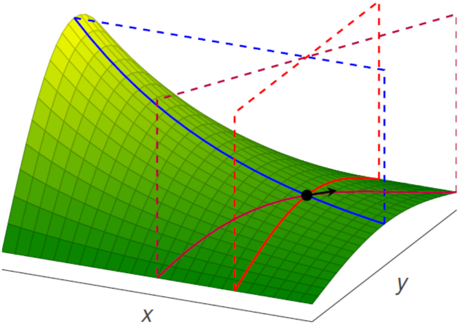
5 min break

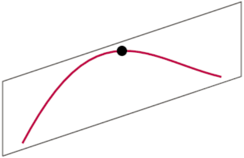
Don't be late.

Directions

The partial derivatives f_x and f_y tell us the rate of change (slope) of the function f along the x and y directions.

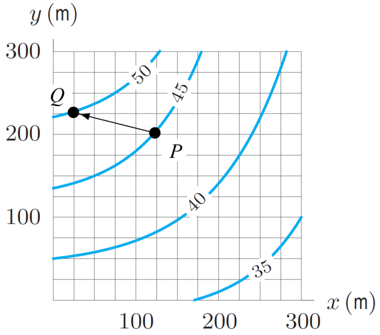
How can we find the rate of change along some other direction?





Directional derivatives – example

The figure shows level sets of temperature graph at point (x, y).

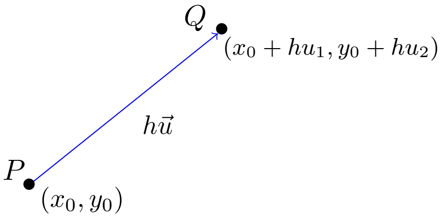


What is the average rate of change of temperature as we walk from P to Q?

At P the temperature is $45°C$, at Q the temperature is $50°C$. From P to Q we walked $\sqrt{100^2 + 25^2} \approx 103$ meters. Since the temperature rises by $5°C$ as we move 103 meters, so the average rate of change is $5°C/103m \approx 0.05°C/m$.

Concept Check: How is this example similar (or different) to finding the average rate of change in Math I?

Directional derivatives



Suppose we want to compute the rate of change of $f(x, y)$ at point $P = (x_0, y_0)$ in the direction of the **unit vector** $\vec{u} = [u_1, u_2]$.

Consider $Q = (x_0 + hu_1, y_0 + hu_2)$, where $h > 0$, whose displacement from P is $h\vec{u}$. Since $\|\vec{u}\| = 1$, the distance from P to Q is h. Thus,

$$\begin{array}{l}\text{Average rate of change} \\ \text{in } f \text{ from } P \text{ to } Q\end{array} = \frac{\text{Change in } f}{\text{Distance from } P \text{ to } Q} = \frac{f(x_0 + h\,u_1, y_0 + h\,u_2) - f(x_0, y_0)}{h}.$$

Directional derivatives

To find *instantaneous* rate of change we let $h \to 0$:

Definition (Directional derivative of f at (x_0, y_0) in the direction of the unit vector $\vec{u}$):

$$D_{\vec{u}} f(x_0, y_0) = \begin{array}{l}\text{Rate of change}\\ \text{of } f \text{ in direction}\\ \text{of } \vec{u} \text{ at } (x_0, y_0)\end{array} = \lim_{h \to 0} \frac{f(x_0 + h\,u_1,\; y_0 + h\,u_2) - f(x_0, y_0)}{h}$$

provided the limit exists.

What if we are given a vector which it <u>not a unit vector</u> $\vec{v}$?
In this case we need to make it unit length by dividing by its length, i.e. we will take the directional derivative with respect to $\vec{u}$

$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|}.$$

Directional derivatives – formula

There is an easier way of finding the directional derivative without taking the limit. Consider

$$D_{\vec{u}} f(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0 + h\,u_1,\, y_0 + h\,u_2) - f(x_0, y_0)}{h} = \lim_{h\to 0} \frac{\Delta f}{h}$$

From the tangent plane approximation

$$\Delta f \approx f_x\,\Delta x + f_y\,\Delta y = f_x\,hu_1 + f_y\,hu_2$$

Plugging this into the definition:

$$\frac{\Delta f}{h} \approx \frac{f_x(x_0, y_0)\,hu_1 + f_y(x_0, y_0)\,hu_2}{h} = f_x(x_0, y_0)\,u_1 + f_y(x_0, y_0)\,u_2$$

This approximation becomes exact as $h \to 0$ and thus we have

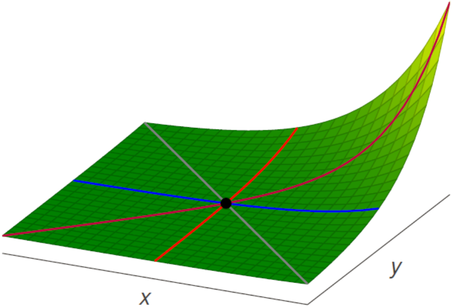
Directional derivative formula

For a *unit vector* $\vec{u} = [u_1, u_2]$, the directional derivative of f in the direction of $\vec{u}$ is given by

$$D_{\vec{u}} f(x_0, y_0) = f_x(x_0, y_0)\,u_1 + f_y(x_0, y_0)\,u_2.$$

Example

Let $f(x, y) = e^{x+y}$. Find the directional derivative of f at point (x, y) in the direction of $\vec{u} = \frac{1}{\sqrt{2}}[1, -1]$.



$$
\begin{aligned}
f_x &= f_y = e^{x+y} \\
D_{\vec{u}} f &= \frac{1}{\sqrt{2}} f_x + (-\frac{1}{\sqrt{2}}) f_y \\
&= \frac{1}{\sqrt{2}} e^{x+y} + (-\frac{1}{\sqrt{2}}) e^{x+y} \\
&= 0
\end{aligned}
$$

Self-check: what do you think are the level sets of e^{x+y}?

Activity 3 (5 minutes)

Find the directional derivative of $f(x,y) = x^3 e^y$ at $(1,0)$, in the direction

(1) $\vec{v} = \left[\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right]$.

(2) $\vec{v} = [3,4]$.

Hint: if in part (2) you got 13, think again!

Activity 3 (solution)

Computing partial derivatives $f_x = 3x^2e^y$, $f_y = x^3e^y$, and $f_x(1,0) = 3$, $f_y(1,0) = 1$. Therefore,

$$
\begin{aligned}
(1)\ D_{\vec{v}}f(1,0) &= f_x(1,0)u_1 + f_y(1,0)u_2 \\
&= 3 \cdot \frac{\sqrt{2}}{2} + 1 \cdot \frac{\sqrt{2}}{2} \\
&= 2\sqrt{2}.
\end{aligned}
$$

(2) Note, $\vec{v}$ is not a unit vector, so we first normalize it:

$$
\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \left[\frac{3}{5}, \frac{4}{5}\right].
$$

$$
\begin{aligned}
D_{\vec{u}}f(1,0) &= f_x(1,0)u_1 + f_y(1,0)u_2 \\
&= 3 \cdot \frac{3}{5} + 1 \cdot \frac{4}{5} \\
&= \frac{13}{5}.
\end{aligned}
$$

Math Modeling

Math modeling could be partitioned into the following steps:

- Defining the Problem Statement.
- **Making Assumptions. Defining Variables.** (1% of your final grade)
- Getting a Solution.
- Analysis and Model Assessment.
- Reporting the Results.

We are grading the process not the product!

Math modeling: Assumptions and Variables

The assumptions tell the reader under what conditions the model is valid.

- Assumptions are necessary! They help simplify the problem and sharpen the focus.
- Assumptions often come naturally from the process of brainstorming and defining the problem statement.
- You should do some preliminary research and may find data to help you make assumptions. In the absence of data, make a *reasonable* assumption and justify the assumption in your write-up.
- Different assumptions can lead to **different, equally valid models**.

Variables

The purpose of a model: predict or quantify something of interest. We refer to these predictions as the **outputs** (or **dependent variables**).

We will also have **independent variables**, or **inputs** to the model.

Some quantities in a model might be held constant, in which case they are referred to as **model parameters**.

Variables

- The problem statement should determine the output of the model. The output variables themselves will be dependent variables.
- The results of the initial brainstorming can provide insight into which variables should be independent variables and which should be fixed model parameters.
- You need to specify units for each variable, because they can reveal relationships between them.
- You will likely need to do some research and make additional assumptions to obtain values for necessary model parameters.

With your instructor: go back to the recycling example and think what the assumptions you have used and what the input/output variables are.

Activity 4. Assumptions and Variables (20 mins)

Based on your Problem Statement (**see your submission and comments on piazza**), list at least **3 main assumptions** that you are using, and **define variables**. State clearly which are the independent variables and which are the dependent variables.

At the end of the class, take **two pictures**:
- a **fresh** team selfie,
- a pic of the whiteboard with your **assumptions** and **variables**,
Update your own Piazza thread created in the previous cohort through the 'edit' function by the day itself. Do not delete your previous work.

Remember, we **are grading the process, not the product!** Put a lot of thought into it, discuss with your groupmates. You will receive feedback on Piazza. If you have trouble coming up with 3 assumptions, 1 or 2 will do. Try your best to be comprehensive and list more.

Summary

We have covered:

- Tangent planes, and how they can be used to approximate a surface.
- Directional derivatives.
- Math Modeling: Making Assumptions. Defining Variables.

Textbook: read Section 19.3 and Section 19.5, then try some of Exercises 19.3.1–19.3.20. You may discuss them on Piazza.