# 10.018 Modelling Space and Systems

## Cohort 1.1: Multivariable Functions

Term 2, 2021

SINGAPORE UNIVERSITY OF TECHNOLOGY AND DESIGN

# Before we start....

To get the most out of this cohort, you should already be familiar with

1. Plotting of 2D graphs using first and second derivatives (Math I)
2. Familiarity with chain rule and product rule in computing derivatives in single variable calculus (Math I)

as we will be learning

1. Level curves in Math II: The 2D analogue of sketching graphs
2. Partial differentiation of a function: The 2D analogue of taking the derivatives of a function

# Introduction

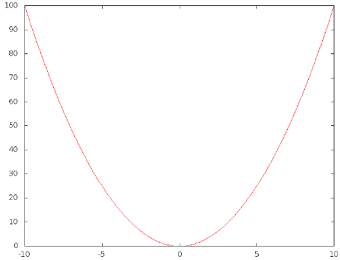
We are already familiar with functions of one variable, denoted by $y = f(x)$.

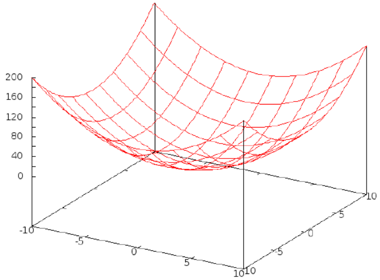
Now we study functions of more than one variable, denoted by $z = f(x_1, x_2, \ldots, x_n)$, or simply $z = f(\vec{x})$.

A function of two variables is often written as $z = f(x, y)$.
Example: let $z$ denote the volume of a cylinder, $x$ denote its radius and $y$ denote its height. Then $z = f(x, y) = \pi x^2 y$.

The graph of a function of two variables is a **surface** in 3-dimensional space.

# Graphs


100
90
80
70
60
50
40
30
20
10
0
-10
-5
0
5
10


200
160
120
80
40
0
-10
-5
0
5
10
10
5
0
-5
-10

# A Comparison Between Functions

| Function | $y = g(x)$ | $z = f(x, y)$ |
|---|---|---|
| Independent variables | $x$ | $x, y$ |
| Dependent variable | $y$ | $z$ |
| Type of graph | **Curve** in 2D space | **Surface** in 3D space |

# Graphs

How would you start to graph the function $z = x^2 + y^2$? If this is tricky, let's consider what happens in the one variable case. How did you draw the function $y = x^2$ back in school? One way to do this is by taking various values of $x$ and finding the corresponding values of $y$:

| $x$ | -5 | -4 | -3 | -2 | -1 | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $y$ | 25 | 16 | 9 | 4 | 1 | 0 | 1 | 4 | 9 | 16 | 25 |

You might want to try this for functions of two variables $z = f(x, y)$ in the next activity. But there is a faster way.

# Activity 1 (10 minutes)

(1) Consider the function $z = f(x, y) = \sqrt{x^2 + y^2}$.

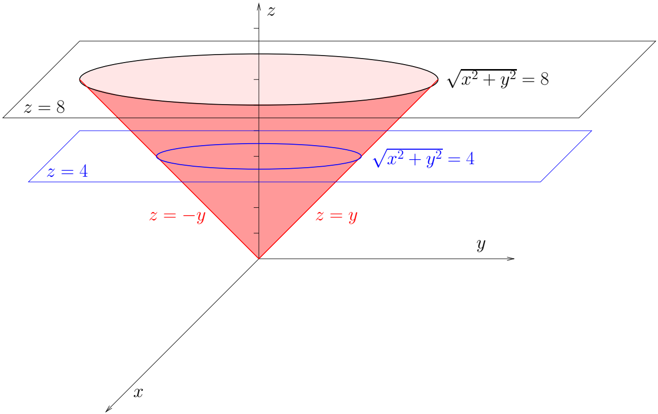
(a) What does the intersection between the graph of $f(x, y)$ and the plane $z = 4$ look like? What about with the plane $z = 8$?

(b) Sketch the graph of $f(x, y)$ in 3-dimensional space.

(2) Sketch the graph of $z = x^2$ in 3-dimensional space.

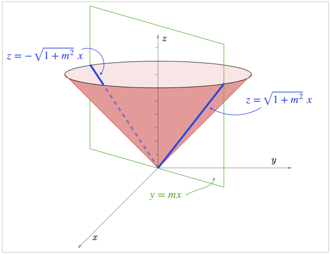
# Activity 1 (solution)

(1a) The intersection between $z = \sqrt{x^2+y^2}$ and $z = c$ (where $c > 0$) can be found by eliminating $z$ from the equations, resulting in $\sqrt{x^2+y^2} = c \iff x^2+y^2 = c^2$, which is the equation of a circle.



# Activity 1 (solution, continued)

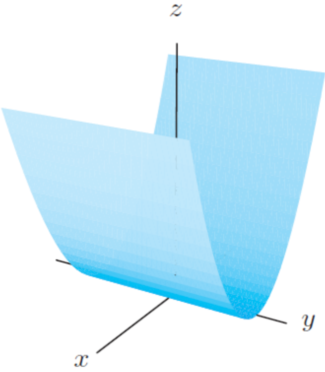
(1b) The graph is an infinite, inverted cone with vertex at the origin:




To see that the boundaries of this surface are really 'straight', we set $y = mx$ (it corresponds to the plane containing the $z$-axis and line $y = mx$ in $xy$-plane), which gives $z = \sqrt{x^2 + (mx)^2}$ $= \sqrt{1+m^2}\,|x|$. This shows that along every direction starting at the origin in the $xy$-plane, $z$ is (piecewise) linear and hence straight.

# Activity 1 (solution, continued)

(2) Cross-section with vertical planes with $y$ fixed produce the same parabola, $z = x^2$.



# Level curves

Many two-variable functions are difficult to sketch.

To more easily visualize the function $z = f(x, y)$, we can sketch the curve

$$f(x, y) = c$$

in the $xy$-plane, for any value of $c$ in the range of $f$. Such a curve is called a level curve / level set (or a level hypersurface for $n$ variables).

Points on a given level curve have the same function value.

A collection of level curves is called a contour map. A contour map can give a good idea of the shape of the graph in 3D-space. Recall Activity 1 Part 2: the level curves are lines parallel to the $y$-axis: $x = \pm\sqrt{c}$.

# Contour maps

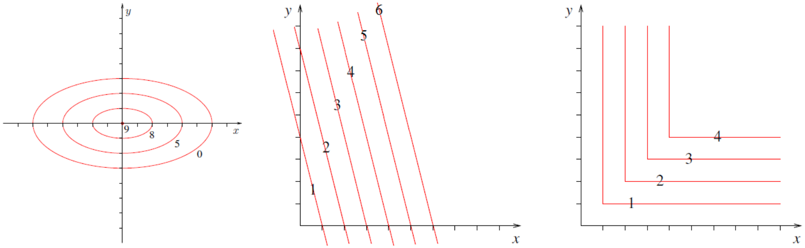


Image credit: (L) maps.google.com.sg (R) www.weather.gov.sg

Left: contour map of the Elevation$(x, y)$, right: contour map of the Rainfall$(x, y)$.

Other examples of level curves and contours: isobar (a line that joins points of equal pressure), isotherm (a line that joins points of equal temperature); equipotential line (joins points of equal electric potential).

# Activity 2 (15 minutes)



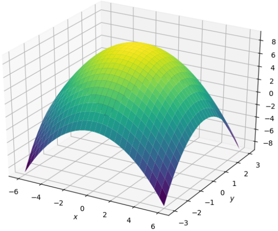

For each plot above, find a function $z = f(x, y)$ that generates the contour map shown.

You can either sketch each function in 3D, or find an exact expression for it.
**Hint:** general equation for an ellipse centered at the origin with x-intercepts $\pm a$ and y-intercepts $\pm b$ is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

# Activity 2 (solution)

The level curve $f(x, y) = c$ for $c = 8$ is an ellipse with semi-major axis 2 and semi-minor axis 1, so its equation is $x^2/4 + y^2 = 1$.
The level curve $f(x, y) = c$ for $c = 0$ is an ellipse with semi-major axis 6 and semi-minor axis 3, so its equation is $x^2/36 + y^2/9 = 1$, or $x^2/4 + y^2 = 9$.
We can perform similar calculations for the other level curves. We observe that the level curve $f(x, y) = c$ satisfies $x^2/4 + y^2 = 9 - c$, therefore

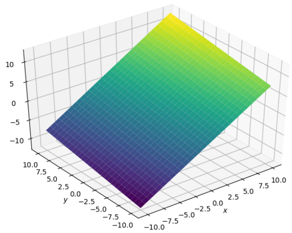
$$z = f(x, y) = 9 - (x^2/4 + y^2).$$



# Activity 2 (solution, continued)

The level curve corresponding to $c = 1$ is a straight line with $x$-intercept 1 and $y$-intercept 4, so its equation is $y = 4 - 4x$. Similar calculations show that the level curve $f(x, y) = c$ satisfies $y = 4c - 4x$, therefore

$$f(x, y) = x + \frac{y}{4}.$$

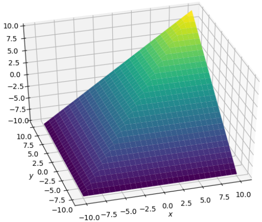


# Activity 2 (solution, continued)

The level curve for $c = 1$ consists of points of the form $(1, y)$ where $y \geq 1$, as well as points of the form $(x, 1)$ where $x \geq 1$. Hence its equation can be written as $\min(x, y) = 1$.

Similar equations can be computed for the other level curves, and we conclude that

$$f(x, y) = \min(x, y).$$



# Break

## 5 min break

Don't be late.

# Derivatives

Recall that for a function of a single variable $y = f(x)$, the derivative of $f$ with respect to $x$ is:

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}.$$

For a function $f(x, y)$ of two variables, we can take the derivative with respect to $x$, or with respect to $y$.

**Definitions**: the partial derivative of $f$ with respect to $x$ is:

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x,\, y) - f(x, y)}{\Delta x}$$

and the partial derivative with respect to $y$ is:

$$\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{\Delta f}{\Delta y} = \lim_{\Delta y \to 0} \frac{f(x,\, y+\Delta y) - f(x, y)}{\Delta y}.$$

# Partial derivatives

Another way to denote $\frac{\partial f}{\partial x}$ is $f_x$; similarly, another way to denote $\frac{\partial f}{\partial y}$ is $f_y$.

In practice, partial derivatives are usually easy to compute!

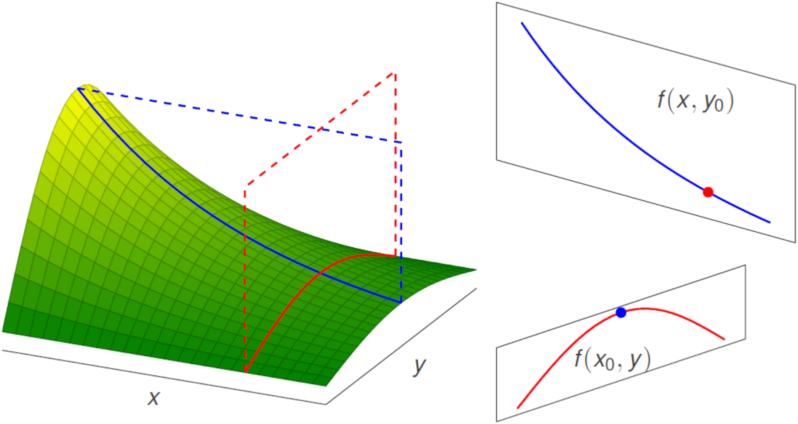
To find $\frac{\partial f}{\partial x}$, simply treat $y$ as a constant, then take the derivative of $f$ with respect to $x$.

To find $\frac{\partial f}{\partial y}$, simply treat $x$ as a constant, then take the derivative of $f$ with respect to $y$.

For a function $f(x_1, x_2, \ldots, x_n)$, the partial derivatives with respect to $x_i$ are similarly defined.

The symbol $\partial$ is pronounced as 'partial' **in this context**, otherwise is pronounced as 'del'.

# Partial derivatives – visualization




**Self-check:** In which figure above the slope of the curve at $(x_0, y_0)$ equals to $f_x(x_0, y_0)$?

# Partial derivatives – example

Let

$$f(x,y) = \frac{1}{x^2+3y^2}.$$

To find $\frac{\partial f}{\partial x}$, treat $y$ as a constant and use the *chain rule*:

$$\frac{\partial f}{\partial x} = f_x = -\frac{1}{(x^2+3y^2)^2} \cdot 2x = \frac{-2x}{(x^2+3y^2)^2}.$$

To find $\frac{\partial f}{\partial y}$, treat $x$ as a constant and use the *chain rule*:

$$\frac{\partial f}{\partial y} = f_y = -\frac{1}{(x^2+3y^2)^2} \cdot 6y = \frac{-6y}{(x^2+3y^2)^2}.$$

# Second order partial derivatives

The *second order partial derivatives* of a function $f(x, y)$ are defined as

$$\frac{\partial}{\partial x}\frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} f_x = f_{xx} \qquad \frac{\partial}{\partial y}\frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y} f_x = f_{xy}$$

$$\frac{\partial}{\partial x}\frac{\partial f}{\partial y} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} f_y = f_{yx} \qquad \frac{\partial}{\partial y}\frac{\partial f}{\partial y} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y} f_y = f_{yy}$$

You can remember this as the partial derivatives are written from the right to the left in the $\partial$ notation, while the subscripts of $f$ are from left to right.

**Theorem (symmetry of second derivatives)**

If $f_{xy}$ and $f_{yx}$ are continuous at a point, then they are *equal* at that point.

# Activity 3 (10 minutes)

For the functions $f(u,v) = u^2 v$ and $g(x,y) = e^{x^2-y}$, compute all the first and second order partial derivatives.

# Activity 3 (solution)

$$f_u = 2uv, \quad f_v = u^2, \quad f_{uv} = 2u = f_{vu}, \quad f_{uu} = 2v, \quad f_{vv} = 0.$$

$$g_x = 2xe^{x^2-y}, \quad g_y = -e^{x^2-y}, \quad g_{xy} = -2xe^{x^2-y} = g_{yx}, \quad g_{yy} = e^{x^2-y}.$$

Using the *product rule*,

$$g_{xx} = (2)(e^{x^2-y}) + (2x)(2xe^{x^2-y}) = 2(1+2x^2)e^{x^2-y}.$$

# Mathematical Modeling

What is a mathematical model?

A mathematical model is a representation of a system or scenario that is used to gain qualitative and/or quantitative understanding of some **real-world problems** and to predict future behavior.

# Math Problem vs Math Modeling Problem

1. (**Math Problem**) The population of Singapore is 5.5 million, and 35% of its citizens recycle their plastic water bottles. If each person uses 9 water bottles per week, how many bottles are recycled each week in Singapore?

2. (**Math Modeling Problem**) How much plastic is recycled in Singapore?

*We usually do not have complete information when trying to solve real-world problems!*

## Mathematical Modeling

Math modeling questions allow you to research real-world problems, using your discoveries to create new knowledge.

Your creativity and how you think about this problem are both valuable in finding a solution to a modeling question. This is part of what makes modeling so interesting and fun!

At this stage, it *doesn't matter what particular math you use in math modeling*. What matters is whether you are able to reason in the math modeling paradigm and whether your model can reflect the reality.

# Math Modeling

Math modeling could be partitioned into the following steps:

- **Defining the Problem Statement.**
- Making Assumptions. Defining Variables.
- Getting a Solution.
- Analysis and Model Assessment.
- Reporting the Results.

**We will be covering each step in cohorts and each step is worth 1% of your final grade.**

We are grading **the process not the product**! I.e. no matter what you submit it'll be given full marks.

BUT, I encourage you to take this seriously and do your best.

This is a practice for Math modeling that you will need to do in 1D project and 2D project as well.

## How would YOU answer this question?

What is the best recycling method for Singapore?

Have a class discussion with your instructor and come up with several more focused, equally-viable questions.

This is the first part of Math Modeling, **Defining the Problem**.

# Guidelines

- Often math modeling questions are vague. Goal: **develop a concise restatement of the question**.
- Focus on subjective words that can be interpreted in different ways or are not easily quantified: *best, efficient, robust, optimal, convenient etc.*
- Explore the problem by doing a combination of research, your own experience and brainstorming, keeping in mind your time constraints.
- Keep an open mind and a positive attitude; you can prune out ideas later that are not realistic.
- Brainstorm as if you had access to any data you need.
- Visual diagrams, such as mind maps, can be a powerful tool leading to the structure of the model.
- In the end, you should have a concise statement that explains **what exactly** the model will measure or predict.

# Math Modeling Activity 4 (1% of the grade)

Stay within your current team of **5 people**.

Designate one student to **submit work on Piazza by midnight today**. This is to give your cohort instructors time to give feedback before the next class.

Indicate in the corner of the whiteboard/paper all the members of your team (Names and Student IDs). And sketch out the mind map/discussion of the problem.
**In the end you should have a precise mathematical reformulation of the problem written.**

The Math Modeling problem will be revealed in the class.

# Math Modeling Activity (1% of the grade)

*See background information (in separate pdf).* The Ministry of Information Protection in a fictitious democracy has hired your team to place a cost on privacy. You and your team are tasked to: create a mathematical model that would describe the "cost" of privacy of electronic communications to the different strata of society.

**Take the picture of your work (make sure names and student IDs of students are visible). And upload as a post within your cohort group on Piazza (read the Piazza instructions).**

Upload **two photos**: a team selfie (wefie), and a picture of your thought process containing the concise restatement of the question.

You won't be graded on the correctness, so don't worry!

## Summary

We have covered:

- Level curves
- Partial derivatives
- Math Modeling: Defining the Problem

Textbook: read Section 19.1 and Section 19.2, then try some of Exercises 19.1.1–19.1.27 and 19.2.1–19.2.32. You may discuss them on Piazza.