10.018 Modelling Space and Systems

Cohort 4.2

Introduction to Double Integrals

Term 2, 2021

SINGAPORE UNIVERSITY OF TECHNOLOGY AND DESIGN

Before we start....

To get the most out of this cohort, you should already be familiar with

1. Single variable integration

as we will be going through

1. Double integration

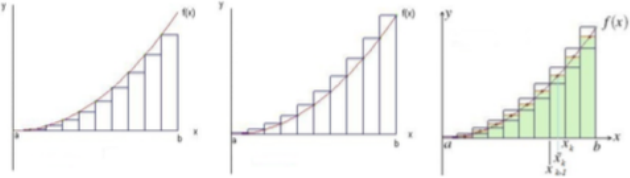
Recall Modelling and Analysis:

Riemann sums

If $f(x)$ is integrable, then the definite integral/signed area is the limit of a Riemann sum:

$$\text{signed area } = \int_a^b f(x)\mathrm{d}x = \lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\Delta x,$$

where n is the number of subintervals, $\Delta x = \frac{b-a}{n}$ is the length of each subinterval and c_k is a point in the k-th interval.



c_k =left end point $\quad$ c_k = right end point $\quad$ c_k = midpoint

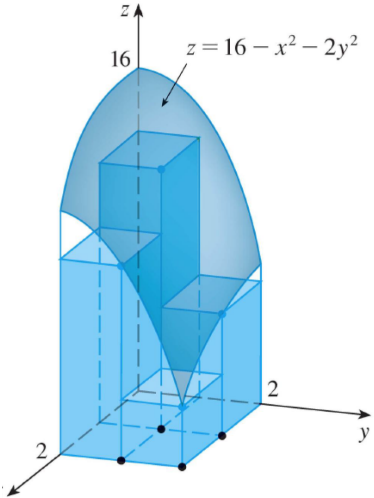
Recall Modelling and Analysis:

Suppose we had the function $y = 10x^2$, and we wanted to find the area from $x = 0$ to $x = 1$. If we had $n = 5$ subintervals with $\Delta x = 0.2$, then we could estimate the area under the curve by summing up the entries in these boxes (area of the rectangles):

Using left endpoint	0	0.08	0.32	0.72	1.28
Using right endpoint	0.08	0.32	0.72	1.28	2
Using midpoint	0.02	0.18	0.50	0.98	1.62

As the number of subintervals goes to infinity, the Riemann sum converges to the integral $\int_0^1 10x^2 \mathrm{d}x$.

Activity 1 / Discussion: The bivariate analogue (10 minutes)



What is the volume under the surface $z = 16 - x^2 - 2y^2$ over the rectangular region where $0 \leq x \leq 2, 0 \leq y \leq 2$.

Estimate this volume by generalizing the approximation of an integral by Riemann sums you learnt in Modelling and Analysis. You can use the picture on the left as an aid.

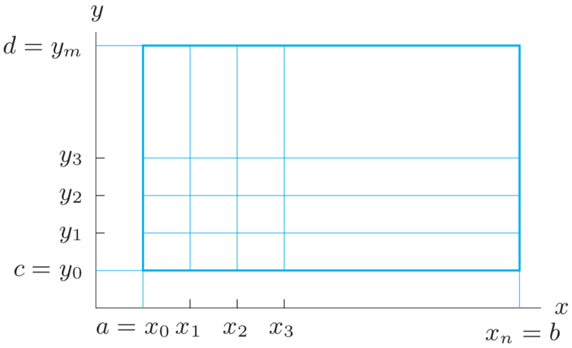
Double Integral on a rectangular region

Given a continuous function $f(x,y)$ defined on a region $a \leq x \leq b$ and $c \leq y \leq d$, we can divide each of the intervals $a \leq x \leq b$ and $c \leq y \leq d$ into n and m subintervals.
In the previous activity, we had $m = n = 2$, and $f(x,y) = 16 - x^2 - 2y^2$, and in the picture, we took the **minimum value** of the function at each corner of the "square".

$$\begin{array}{|c|c|}\hline f(1,2) & f(2,2) \\ \hline f(1,1) & f(2,1) \\ \hline \end{array} \quad = \quad \begin{array}{|c|c|}\hline 7 & 4 \\ \hline 13 & 10 \\ \hline \end{array}$$

Double Integral on a rectangular region

Here is the general picture.



Double Integral on a rectangular region

More formally, the area of each subrectangle $\Delta A = \Delta x \Delta y$, where $\Delta x = (b-a)/n$ is the width of each subdivision on the x axis, and $\Delta y = (d-c)/m$ is the width of each subdivision on the y axis.

Suppose we let

- M_{ij} to be the point which gives the maximum value on the function on each rectangle (what are they in Activity 1?)
- L_{ij} to be the point which gives the minimum value on the function on each rectangle (this is what we had in Activity 1)
- (u_{ij}, v_{ij}) be any point in the ij^{th} subrectangle

Double Integral on a rectangular region

We therefore have

$$\sum_{i,j} L_{ij}\Delta x\Delta j \leq \sum_{i,j} f(u_{ij}, v_{ij})\Delta x\Delta y \leq \sum_{i,j} M_{ij}\Delta x\Delta y$$

If we take the limit of the subintervals as n and m go to infinity, we can show that the upper and lower bounds tend to the same limit. By the Squeeze Theorem (Modelling and Analysis), then the middle term tends to the same limit.

Double Integral on a rectangular region

Let $R = [a, b] \times [c, d] = \{(x, y) \in \mathbb{R} : a \leq x \leq b$ and $c \leq y \leq d\}$ be a rectangular region.

Double Integral over a rectangular region R

The definite integral of a continuous function f over the region $R = [a, b] \times [c, d]$ is

$$\iint_R f \; dA = \lim_{\Delta x, \Delta y \to 0} \sum_{i,j} f(u_{ij}, v_{ij}) \Delta x \Delta y$$

where (u_{ij}, v_{ij}) is some point in the $(i, j)^{\text{th}}$ subrectangle.

Computing the value of double integrals

Remember that in Modelling and Analysis, we had

$$\int_a^b f(x)\ \mathrm{d}x = \lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\Delta x$$

We can look at the discrete (summation) case for inspiration.

Activity 2 (5 mins)

Suppose we wanted to approximate $\iint\limits_{[0,1]\times[0,1]} 16 - x^2 - 2y^2 \, \mathrm{d}A$ and we partition $[0, 1]$ into 5 subintervals along each axis for a total of 25 subrectangles. We compute $f(u_{ij}, v_{ij})\Delta x \Delta y$ given in the following table.

	$\mathbf{x = 0.2}$	$\mathbf{x = 0.4}$	$\mathbf{x = 0.6}$	$\mathbf{x = 0.8}$	$\mathbf{x = 1.0}$	Total
$\mathbf{y = 1.0}$	0.5584	0.5536	0.5456	0.5344	0.5200	
$\mathbf{y = 0.8}$	0.5872	0.5824	0.5744	0.5632	0.5488	
$\mathbf{y = 0.6}$	0.6096	0.6048	0.5968	0.5856	0.5712	
$\mathbf{y = 0.4}$	0.6256	0.6208	0.6128	0.6016	0.5872	
$\mathbf{y = 0.2}$	0.6352	0.6304	0.6224	0.6112	0.5968	
Total						

Activity 2 continued (5 mins)

1. Sum each row. What is the interpretation of the sum of each row?
2. Sum each column. What is the interpretation of the sum of each column?
3. Sum up the row sums
4. Sum up the column sums
5. Verify that the sum of the row sums equals to the sum of the column sums.

Activity 2 (solution)

	$\mathbf{x = 0.2}$	$\mathbf{x = 0.4}$	$\mathbf{x = 0.6}$	$\mathbf{x = 0.8}$	$\mathbf{x = 1.0}$	y total
$\mathbf{y = 1.0}$	0.5584	0.5536	0.5456	0.5344	0.5200	2.712
$\mathbf{y = 0.8}$	0.5872	0.5824	0.5744	0.5632	0.5488	2.856
$\mathbf{y = 0.6}$	0.6096	0.6048	0.5968	0.5856	0.5712	2.968
$\mathbf{y = 0.4}$	0.6256	0.6208	0.6128	0.6016	0.5872	3.048
$\mathbf{y = 0.2}$	0.6352	0.6304	0.6224	0.6112	0.5968	3.096
x total	3.016	2.992	2.952	2.896	2.824	14.68

Remark on Activity 2

Instead of writing $\sum_{i,j} f(u_{ij}, v_{ij})\Delta x\Delta y$, we could instead write this as

$$\sum_{i=1}^{5}\sum_{j=1}^{5} f(u_{ij}, v_{ij})\Delta x\Delta y$$

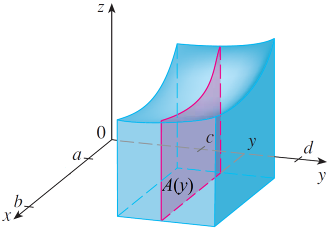
In fact, the order of the summation does not matter, i.e. we could have written this as

$$\sum_{i=1}^{5}\left(\sum_{j=1}^{5} f(u_{ij}, v_{ij})\Delta y\right)\Delta x \quad \text{or} \quad \sum_{j=1}^{5}\left(\sum_{i=1}^{5} f(u_{ij}, v_{ij})\Delta x\right)\Delta y$$

This symmetry of double summation carries over to double integration.

Computing double integrals

To compute a double integral over the rectangle $[a, b] \times [c, d]$, we can first *fix* a value of y, and cut a 'slice' from the solid as shown:



The area of the slice is given by

$$A(y) = \int_a^b f(x, y)\, \mathrm{d}x.$$

And then we integrate with respect to y. We can change the order of integration as well.

Iterated integrals

From the previous slide, we have reduced a double integral into an *iterated integral*, that is, a sequence of two single integrals.

Alternatively, we can first fix a value of x and then cut a slice in the y-direction. As we would expect, computing the iterated integral in this order gives the same answer:

Fubini's theorem

Let R be the rectangular region $[a, b] \times [c, d]$. If $f(x, y)$ is continuous on R, then

$$\iint\limits_R f(x,y)\,\mathrm{d}A = \int_a^b \left(\int_c^d f(x,y)\,\mathrm{d}y \right) \mathrm{d}x = \int_c^d \left(\int_a^b f(x,y)\,\mathrm{d}x \right) \mathrm{d}y.$$

(Iterated integrals are usually written without brackets.)

Properties of double integrals

Just like single integration, double integration is *linear*, where we have

$$\iint\limits_R [f(x,y)+g(x,y)]\,\mathrm{d}A = \iint\limits_R f(x,y)\,\mathrm{d}A + \iint\limits_R g(x,y)\,\mathrm{d}A,$$

$$\iint\limits_R c\,f(x,y)\,\mathrm{d}A = c\iint\limits_R f(x,y)\,\mathrm{d}A,$$

for any real number c.

These properties follow since double summation is linear.

Activity 3 (5 minutes)

Compute the exact volume under the surface

$$f(x, y) = 16 - x^2 - 2y^2,$$

over the region $[0, 2] \times [0, 2]$.

Do this in two ways (dx then dy; dy then dx), and check that your answers agree.

Activity 3 (solution)

$$\int_0^2 \int_0^2 (16 - x^2 - 2y^2)\, \mathrm{d}x\, \mathrm{d}y = \int_0^2 \left[16x - \frac{1}{3}x^3 - 2y^2 x\right]_{x=0}^{x=2} \mathrm{d}y$$
$$= \int_0^2 \left[\frac{88}{3} - 4y^2\right] \mathrm{d}y$$
$$= \left[\frac{88}{3}y - \frac{4}{3}y^3\right]_0^2 = 48.$$

Check by doing the iterated integral in the other order:

$$\int_0^2 \int_0^2 (16 - x^2 - 2y^2)\, \mathrm{d}y\, \mathrm{d}x = \int_0^2 \left[16y - x^2 y - \frac{2}{3}y^3\right]_{y=0}^{y=2} \mathrm{d}x$$
$$= \int_0^2 \left[\frac{80}{3} - 2x^2\right] \mathrm{d}x$$
$$= \left[\frac{80}{3}x - \frac{2}{3}x^3\right]_0^2 = 48.$$

Activity 3.5 (if time permits)

Do you think Fubini's Theorem works if any of the a, b, c, d in the rectangular region $[a, b] \times [c, d]$ goes to ∞ (or $-\infty$)? Why or why not?

This question is testing your mathematical intuition. Even if you can't prove that Fubini's Theorem works (or come up with a counter example to why it doesn't work) - you must at least have the intuition to decide whether Fubini's Theorem would work or not.

Activity 3.5 (if time permits)

Consider the following (infinite) table.

$$
\begin{array}{cccccc}
-1 & 0 & 0 & 0 & 0 & \dots \\
1 & -1 & 0 & 0 & 0 & \dots \\
0 & 1 & -1 & 0 & 0 & \dots \\
0 & 0 & 1 & -1 & 0 & \dots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}
$$

1. What is the sum of each row?
2. What is the sum of each column?
3. Is the sum of the row sums equal to the sum of the column sums?
4. What does the result from 1-3 tell you about the applicability of Fubini's Theorem?

Activity 3.5 (solution)

We looked at the discrete case in Activity 2 to build up the intuition for the continuous case. It might help to think of whether a discrete analogue would work or not.

You can see that the total sum of all the **rows** are -1. The first row sums up to -1, but the subsequent rows sum to zero.

But the sum of all the **columns** are zero.

Interchanging the two sums do not give the same result!

Activity 3.5 (solution)

Based on the discrete case, you can in fact deduce that there may be something similar in the continuous case.

In fact, here is a counter example

$$\int_0^\infty \int_0^1 \exp\{-xy\} - xy \exp\{-xy\} \, \mathrm{d}y \, \mathrm{d}x$$

You are not expected to come up with the above counter-example! But you must at least know that "Fubini's Theorem might not work when bounds are infinite".

Activity 3.5 (solution)

Why might this be important?

As **engineering leaders**, you might not be expected to know the fine details of math.

But that does not mean "applying theorems blindly" (garbage in, garbage out).
E.g., if an engineering graduate from SUTD designed and created a new airplane from the math he/she knew, would you be confident to be on its first test flight?

Instead, must have some "intuition" on which **direction** to find a solution (educated guess versus random guess, e.g. "Based on a small case, this is probably the right direction to take", versus "I have no idea how this works, so let's pick any direction").

Break

5 min break

Don't be late.

Double integrals – other regions

On Slide 10, we had the notation

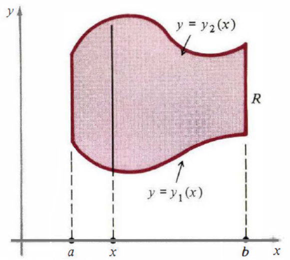
$$\iint\limits_R f \, \mathrm{d}A$$

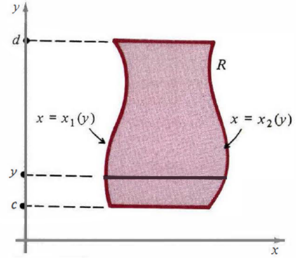
where we saw that $\iint\limits_R$ defined the rectangular region (rectangular grid), and f defined the value of the function in that grid.

In practice, we are not so lucky to have a rectangular region R to integrate over. There will be several examples in the lecture next week to highlight practical problems involving different regions of integration.

But today, we will look at computing the integral where the region R is **vertically simple**, or **horizontally simple**.

Iterated integrals – other regions





A region R is called vertically simple if it can be described by $a \leq x \leq b$, $\quad y_1(x) \leq y \leq y_2(x)$, where $y_1(x)$ and $y_2(x)$ are continuous functions of x on $[a, b]$. (Left figure.)

A region R is called horizontally simple if it can be described by $c \leq y \leq d$, $\quad x_1(y) \leq x \leq x_2(y)$, where $x_1(y)$ and $x_2(y)$ are continuous functions of y on $[c, d]$. (Right figure.)

Iterated integrals – other regions

Using the idea of slices, we have the following results:

If R is vertically simple, then

$$\iint\limits_R f(x,y)\,\mathrm{d}A = \int_a^b \int_{y_1(x)}^{y_2(x)} f(x,y)\,\mathrm{d}y\,\mathrm{d}x.$$

If R is horizontally simple, then

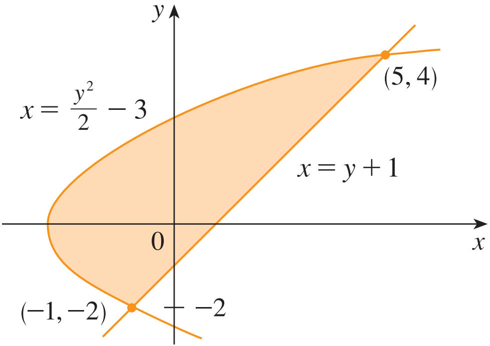
$$\iint\limits_R f(x,y)\,\mathrm{d}A = \int_c^d \int_{x_1(y)}^{x_2(y)} f(x,y)\,\mathrm{d}x\,\mathrm{d}y.$$

Guidelines for evaluating a double integral:

- **Sketch the region that we are integrating over**.
- Determine whether it is vertically or horizontally simple, then pick the correct formula to use.
- Sometimes, doing the iterated integral in a different order can simplify calculations.

Iterated integrals – example

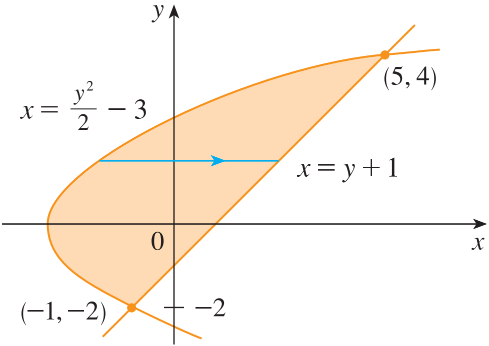
Evaluate $\iint_R xy \, \mathrm{d}A$, where R is the region bounded by $x = y + 1$ and $x = \dfrac{y^2}{2} - 3$.



Question: Is this region horizontally simple or vertically simple?

Iterated integrals – example

Evaluate $\displaystyle\iint_R xy\,\mathrm{d}A$, where R is the region bounded by $x = y + 1$ and $x = \dfrac{y^2}{2} - 3$.



The region is horizontally simple.

Iterated integrals – example, continued

$$\begin{aligned}\iint_R xy\,\mathrm{d}A &= \int_{-2}^{4}\int_{\frac{y^2}{2}-3}^{y+1} xy\,\mathrm{d}x\,\mathrm{d}y\\ &= \int_{-2}^{4}\left[\frac{x^2}{2}y\right]_{\frac{y^2}{2}-3}^{y+1}\mathrm{d}y\\ &= \frac{1}{2}\int_{-2}^{4}\left(-\frac{y^5}{4}+4y^3+2y^2-8y\right)\mathrm{d}y\\ &= 36.\end{aligned}$$

Performing the integration in the other order is more difficult, but will still give the same answer:

$$\iint\limits_R xy\,\mathrm{d}A = \int_{-3}^{-1}\int_{-\sqrt{2x+6}}^{\sqrt{2x+6}} xy\,\mathrm{d}y\,\mathrm{d}x + \int_{-1}^{5}\int_{x-1}^{\sqrt{2x+6}} xy\,\mathrm{d}y\,\mathrm{d}x.$$

Here the limits $\pm\sqrt{2x+6}$ come from solving $x = \dfrac{y^2}{2} - 3$ for y.

Important to check:

Limits on Iterated Integrals

- The limits on the **outer** integral must be **constants**.
- The limits on the **inner** integral can involve only the variable in the outer integral. For example, if the inner integral is with respect to x, its limits can be functions of y.

Compare

$$\int_{-2}^{4} \int_{\frac{y^2}{2}-3}^{y+1} xy \, \mathrm{d}x \, \mathrm{d}y$$

and

$$\int_{-1}^{5} \int_{x-1}^{\sqrt{2x+6}} xy \, \mathrm{d}y \, \mathrm{d}x$$

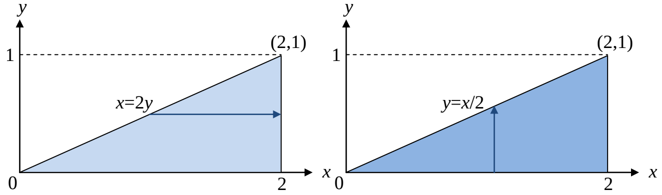
Activity 4 (15 minutes)

Given the integral

$$\iint_R 4\exp\{x^2\}\ \mathrm{d}A = \int_0^1 \int_{2y}^2 4\exp\{x^2\}\,\mathrm{d}x\,\mathrm{d}y.$$

1. Sketch the region of integration R, and identify whether it is horizontally simple or vertically simple.
2. Solve this integral.
   Hint: You may need to do this integral in a different order (use your part 1 to help you)

Activity 4 (solution)



The region of integration is the triangle shown; it is both vertically and horizontally simple.

Therefore we have

$$\int_0^1 \int_{2y}^2 4\exp\{x^2\}\,\mathrm{d}x\,\mathrm{d}y = \iint_R 4\exp\{x^2\}\,\mathrm{d}A = \int_0^2 \int_0^{x/2} 4\exp\{x^2\}\,\mathrm{d}y\,\mathrm{d}x.$$

The iterated integral on the left hand side is difficult to evaluate (in fact, $\int e^{x^2}\,\mathrm{d}x$ cannot be written as a finite combination of elementary functions).

Activity 4 (solution, continued)

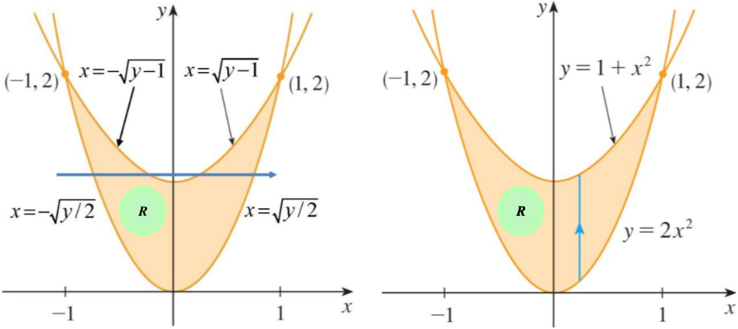
However, after changing the order of integration, the iterated integral on the right hand side is easy to compute:

$$\begin{aligned}
&\int_0^2 \int_0^{\frac{x}{2}} 4\exp\{x^2\}\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_0^2 \left[4y\exp\{x^2\}\right]_0^{\frac{x}{2}} \mathrm{d}x \\
&= \int_0^2 2x\exp\{x^2\}\,\mathrm{d}x \\
&= \left[\exp\{x^2\}\right]_0^2 \\
&= \exp\{4\} - 1.
\end{aligned}$$

Note: the integral in the second last line can be evaluated by substitution, using $t = x^2$.

Activity 5 (15 minutes)

Choose an order of integration to evaluate $\iint_R f(x,y)\,\mathrm{d}A$, where $f(x,y) = x + 2y$ and the region R is shown below.



Activity 5 (solution)

The region is vertically simple.

$$\begin{aligned}
\iint\limits_R f(x,y)\,\mathrm{d}A &= \int_{-1}^{1}\int_{2x^2}^{1+x^2}(x+2y)\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_{-1}^{1}\Big[xy+y^2\Big]_{2x^2}^{1+x^2}\,\mathrm{d}x \\
&= \int_{-1}^{1}(-3x^4-x^3+2x^2+x+1)\,\mathrm{d}x \\
&= \frac{32}{15}.
\end{aligned}$$

The order above is much easier than

$$\int_0^1\int_{-\sqrt{y/2}}^{\sqrt{y/2}} f(x,y)\mathrm{d}x\mathrm{d}y+\int_1^2\int_{-\sqrt{y/2}}^{-\sqrt{y-1}} f(x,y)\mathrm{d}x\mathrm{d}y+\int_1^2\int_{\sqrt{y-1}}^{\sqrt{y/2}} f(x,y)\mathrm{d}x\mathrm{d}y,$$

which still gives the same answer (try it).

Summary

We have covered:

- The limit definition of a double integral.
- Computing double integrals using iterated integrals.
- Vertically and horizontally simple regions.
- Changing the order of integration.

Textbook: read Sections 20.1 and 20.2, then try some of Exercises 20.1.1–20.1.29 and Exercises 20.2.1–20.2.18. You may discuss them on Piazza.