10.018: Modelling Space and Systems
Term 2, 2021

Homework Week 5

Due Date: 6:30pm, Tuesday, Mar. 2, 2021

BASIC problems TO BE SUBMITTED.

The BASIC set of problems is designed to be a very easy and straightforward application of the definitions from lectures and cohorts (you might have to do some calculations, but not much). It's a good way to start your homework. If you have trouble starting any of the questions do consult your cohort instructors (in office hours, via email or via Piazza).

I. *(Evaluating double integrals I)*

For these double integrals

- **Draw** the region of integration in the xy-axes (label your axes and equation of lines properly).
- **Evaluate the integrals.**

(a) $\int_0^{\pi/2} \int_0^1 (1 - r^2)\, r^2 \,\mathrm{d}r \,\mathrm{d}\theta$

(b) $\int_0^{2\pi} \int_0^a e^{-r^2} r\, \sin^2(\theta) \,\mathrm{d}r \,\mathrm{d}\theta$

Solution:

(a) We can see from the bounds that this is the quarter circle with radius 1. Recall the equation of a circle with radius r centered at 0, which is $x^2 + y^2 = r^2$. Hence we get $y = \sqrt{1 - x^2}$ to be one of our lines (equivalently, you can write $x = \sqrt{1 - y^2}$). Figure 1 shows us the region of integration.

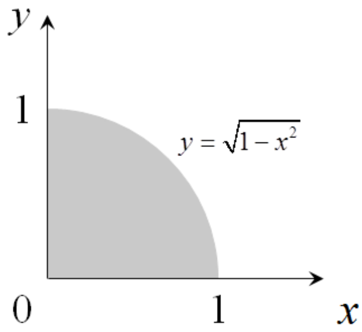

Figure 1: Region of integration for I a)

We can evaluate this integral to get

$$\begin{aligned}\int_0^{\pi/2}\int_0^1 (1-r^2)r^2 \, \mathrm{d}r \, \mathrm{d}\theta &= \int_0^{\pi/2} \left[\frac{1}{3}r^3 - \frac{1}{5}r^5\right]_0^1 \mathrm{d}\theta \\ &= \int_0^{\pi/2} \frac{2}{15} \, \mathrm{d}\theta \\ &= \left[\frac{2}{15}\theta\right]_0^{\pi/2} \\ &= \frac{\pi}{15}\end{aligned}$$

(b) We can see from the bounds that the region of integration is a circle with radius a. Figure 2 shows the region of integration.

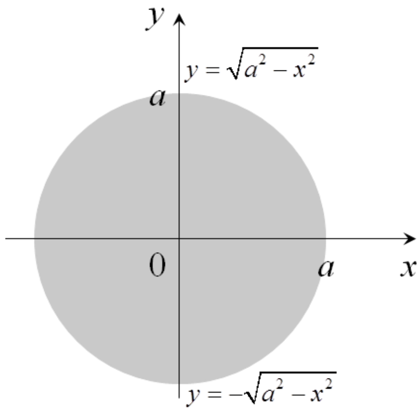

Figure 2: Region of integration for I b)

We can evaluate this integral to get

$$\begin{aligned}\int_0^{2\pi}\int_0^a e^{-r^2} r \sin^2(\theta) \, \mathrm{d}r \, \mathrm{d}\theta &= \int_0^{2\pi} \left[-\frac{1}{2}e^{-r^2}\right]_0^a \sin^2(\theta) \, \mathrm{d}\theta \\ &= \int_0^{2\pi} \left(-\frac{1}{2}e^{-a^2} + \frac{1}{2}\right) \frac{1-\cos(2\theta)}{2} \, \mathrm{d}\theta \\ &= \frac{1}{2}\left(1-e^{-a^2}\right)\left[\frac{1}{2}\left(\theta - \frac{\sin(2\theta)}{2}\right)\right]_0^{2\pi} \\ &= \frac{\pi}{2}\left(1-e^{-a^2}\right)\end{aligned}$$

INTERMEDIATE problems TO BE SUBMITTED.

The INTERMEDIATE set of problems is a *little* harder (but not by much) than the BASIC one. If you have trouble starting any of the questions do consult your cohort instructors (in office hours, via email or via Piazza).

1. *(Sketching Region(s) of Integration)*
   In order to evaluate a double integral involving change of variables, we need to look at the current region of integration, before choosing a *good* transformation. In this question, we will use **polar coordinates** as a transformation. For each sub question, please:

   - **Draw** and **shade** the region of integration in the xy-axes (label your axes and equation of lines properly).
   - **Identify** the equation of the (new) lines you need to draw for the region of integration after the change of variables to **polar coordinates**.
   - **Draw** and **shade** the new region of integration (label your axes and equation of lines properly). Note that your new axes will be r and θ.

   Lastly, based on your **drawings**, what do you realize happens to the region of integration after such a transformation?

   (a) Region of integration for $\int_{-2}^{0}\int_{0}^{\sqrt{4-x^2}} f(x,y)\,\mathrm{d}y\,\mathrm{d}x$.

   (b) Region of integration for $\int_{0}^{\frac{\sqrt{2}}{2}}\int_{y}^{\sqrt{1-y^2}} f(x,y)\,\mathrm{d}x\,\mathrm{d}y$.

   Solution:

   (a)

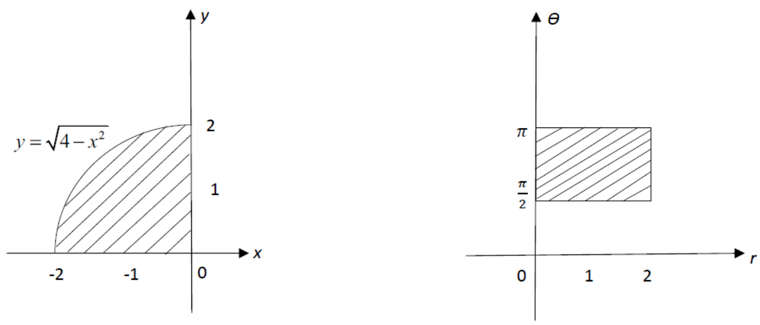

Figure 3: Region of integration (before and after) for part a).

   (b)

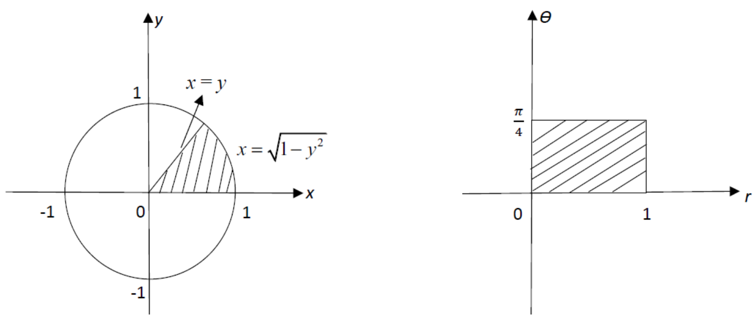

Figure 4: Region of integration (before and after) for part b).

2. *(Reversing order of integration)*
Write the integral $\int_0^4 \int_{\sqrt{y}}^2 \sin(x^3)\, \mathrm{d}x\, \mathrm{d}y$ with the order of integration reversed. Then solve this integral.

Solution: To reverse the order of integration, we need to understand how our initial region of integration looks like. If we sketch it out, we should get a diagram similar to Figure 5.

This figure is both horizontally simple and vertically simple. So let's consider expressing this as a vertically simple region.

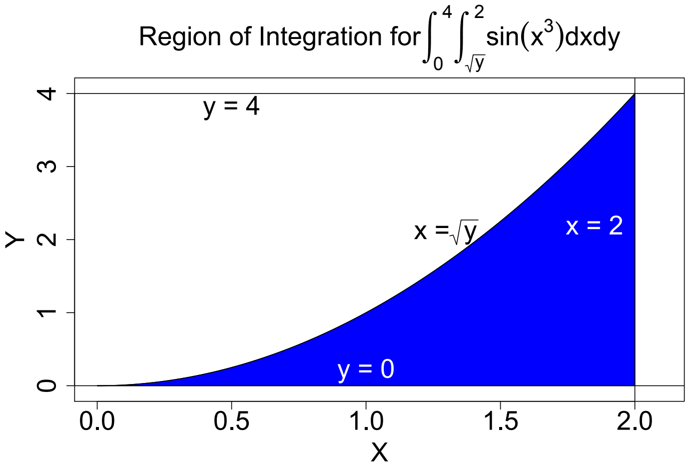

Figure 5: Region of integration for $\int_0^4 \int_{\sqrt{y}}^2 \sin(x^3)\, \mathrm{d}x\, \mathrm{d}y$.

It is straightforward to see that if $x = \sqrt{y}$, then $y = x^2$, so we can reverse the order to get

$$\int_0^4 \int_{\sqrt{y}}^2 \sin(x^3)\, \mathrm{d}x\, \mathrm{d}y = \int_0^2 \int_0^{x^2} \sin(x^3)\, \mathrm{d}y\, \mathrm{d}x$$

Now, we can solve this integral

$$\begin{aligned}\int_0^2 \int_0^{x^2} \sin(x^3)\, \mathrm{d}y\, \mathrm{d}x &= \int_0^2 \left[y \sin(x^3)\right]_0^{x^2} \mathrm{d}x \\ &= \int_0^2 x^2 \sin(x^3)\, \mathrm{d}x \\ &= \left[-\frac{\cos(x^3)}{3}\right]_0^2 \\ &= \frac{1}{3}\left(1 - \cos(8)\right)\end{aligned}$$

3. *(Evaluating double integrals II)*
Evaluate the following integrals in the following sub-questions. You may consider applying change of variables.

   (a) $\int_{-2}^0 \int_0^{\sqrt{4-x^2}} y\, \mathrm{d}y\, \mathrm{d}x$

   (b) $\int_0^{\frac{\sqrt{2}}{2}} \int_y^{\sqrt{1-y^2}} 2x^2\, \mathrm{d}x\, \mathrm{d}y$

Hint: You have done some of this work above.

Solution: First, change the region of integration from the Cartesian coordinates to Polar coordinates as shown in Q1.

(a) Using our new region of integration, we have

$$\begin{aligned}\int_{\pi/2}^{\pi}\int_0^2 r\sin\theta\ r\ \mathrm{d}r\ \mathrm{d}\theta &= \int_{\pi/2}^{\pi}\left[\sin\theta\left(\frac{1}{3}r^3\right)\right]_0^2\ \mathrm{d}\theta\\ &= \frac{8}{3}\int_{\pi/2}^{\pi}\sin\theta\ \mathrm{d}\theta\\ &= \frac{8}{3}\left[-\cos\theta\right]_{\pi/2}^{\pi}\\ &= \frac{8}{3}\end{aligned}$$

(b) Using our new region of integration, we have

$$\begin{aligned}\int_0^{\pi/4}\int_0^1 2r^2(\cos\theta)^2\ r\ \mathrm{d}r\mathrm{d}\theta &= \int_0^{\pi/4}\left[2(\cos\theta)^2\frac{1}{4}r^4\right]_0^1\ \mathrm{d}\theta\\ &= \frac{1}{4}\int_0^{\pi/4}\left(\cos(2\theta)+1\right)\ \mathrm{d}\theta\\ &= \frac{1}{4}\left[\frac{\sin(2\theta)}{2}+\theta\right]_0^{\frac{\pi}{4}}\\ &= \frac{1}{4}\left(\frac{1}{2}+\frac{\pi}{4}\right)\\ &= \frac{1}{8}+\frac{\pi}{16}\end{aligned}$$

4. *(Change of Variables)*
Evaluate $\iint_R\ ye^{(\frac{y}{x})^2}\ \mathrm{d}x\ \mathrm{d}y$, where R is a trapezium in the xy-plane with corners at (0.5, 0), (1, 0), (1, 1), (0.5, 0.5), using the change of variables given by $x = v$ and $y = uv$.
Please follow the following steps.

(a) Step 1: Draw the initial region of integration

(b) Step 2: Draw the new region of integration

(c) Step 3: Solve the integral

(You can check your answer by computing the integral in the xy-plane).

Solution: Figure 6 shows the initial region of integration.

From $x = v$ and $y = uv$, we get $u = \frac{y}{x}$ and $v = x$. So the four corners are transformed respectively into (0, 0.5), (0, 1), (1, 1), and (1, 0.5), which form a rectangular region as illustrated by Figure 7. One should also check that this change of variables really maps the original trapezium region into the new rectangular region. For instance, the boundary S_2 of the trapezium region can be described by $x = 1$ and $0 \le y \le 1$. Since $u = \frac{y}{x}$ and $v = x$, we obtain $0 \le u \le 1$ and $v = 1$, which corresponds to S_2' of the rectangular region. The other boundaries can be similarly checked. Hence Figure 7 indeed shows the new region of integration.

We can easily get $g_u = 0$, $g_v = 1$, $h_u = v$, and $h_v = u$. Hence we obtain $|g_u h_v - g_v h_u| = |-v| = v$.

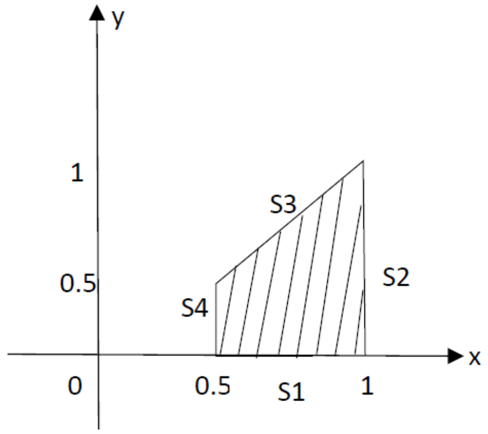

Figure 6: Initial region of integration for $\iint_R ye^{(\frac{y}{x})^2}\,\mathrm{d}x\,\mathrm{d}y$.

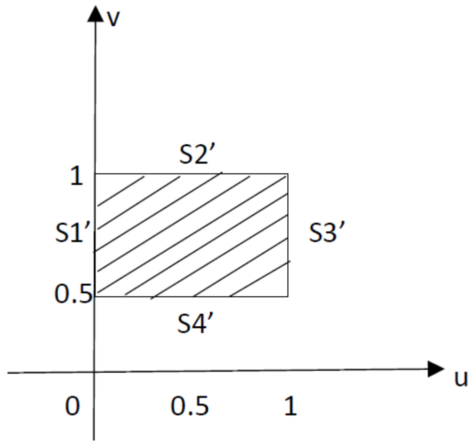

Figure 7: New region of integration $\int_{0.5}^{1}\int_{0}^{1} f(u,v)\,\mathrm{d}u\,\mathrm{d}v$.

Substituting this in, we get

$$\begin{aligned}
\int_{0.5}^{1}\int_{0}^{1} uve^{u^2}\cdot v\,\mathrm{d}u\,\mathrm{d}v &= \int_{0.5}^{1} v^2\left[\frac{1}{2}e^{u^2}\right]_{u=0}^{u=1}\mathrm{d}v \\
&= \frac{1}{2}(e-1)\int_{0.5}^{1} v^2\,\mathrm{d}v \\
&= \frac{1}{2}(e-1)\left[\frac{1}{3}v^3\right]_{v=0.5}^{v=1} \\
&= \frac{7}{48}(e-1) \;=\; \frac{7}{48}e-\frac{7}{48}
\end{aligned}$$

5. *(Triple Integration)*
A trophy is being designed for the SUTD winning team in mathematical modelling. It takes the form of a solid *half ball* H of radius a, with the density of glass depending on the distance r from the centre of the base disc. The density is given by $D = K(2a-r)$ where K is a constant. Find the mass of the half ball, where the mass m is given by $m = \iiint_H D\,\mathrm{d}V$. Hint: which coordinate system do you think will be useful here?

Solution:

Note that for $\iint_D f(x,y)\,\mathrm{d}y\,\mathrm{d}x$, our region of integration is an area, and we multiply by “height"

(z) to get volume. Since the region of integration is a volume, so we multiply by density to get mass. Hence we want to integrate

$$\begin{aligned}
\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^a K(2a-\rho)\rho^2\sin(\phi)\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta &= K\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\left[\frac{2a\rho^3}{3}-\frac{\rho^4}{4}\right]_{\rho=0}^{\rho=a}\sin(\phi)\,\mathrm{d}\phi\,\mathrm{d}\theta \\
&= \frac{5a^4K}{12}\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\sin(\phi)\,\mathrm{d}\phi\,\mathrm{d}\theta \\
&= \frac{5a^4K}{12}\int_0^{2\pi}\left[-\cos(\phi)\right]_0^{\frac{\pi}{2}}\,\mathrm{d}\theta \\
&= \frac{5a^4K}{12}\int_0^{2\pi}1\,\mathrm{d}\theta \\
&= \frac{5}{6}\pi a^4K
\end{aligned}$$

Challenging problems [OPTIONAL].

1. (*Triple Integration*)
Two spheres, one of radius 1, one of radius $\sqrt{2}$, have centers that are 1 unit apart. Write a triple integral, including limits of integration, giving the volume of the region of the smaller sphere that is inside the larger sphere. Evaluate the integral.

Solution: We pick Cartesian coordinates with the smaller sphere centered at the origin, and the larger one centered at $(0, 0, -1)$. A vertical cross-section of the region in the xz-plan is shown in Figure 8. The smaller sphere has equation $x^2 + y^2 + z^2 = 1$. The larger sphere has equation $x^2 + y^2 + (z + 1)^2 = 2$.

Let R denote the region in the xy-plan where the two spheres intersect. The equation of the curve bounding this region can be obtained by solving the system:

$$\begin{aligned} x^2 + y^2 + z^2 &= 1, \\ x^2 + y^2 + (z+1)^2 &= 2. \end{aligned}$$

Subtracting the equations gives

$$\begin{aligned} (z+1)^2 - z^2 &= 1, \\ 2z + 1 &= 1, \\ z &= 0. \end{aligned}$$

Since $z = 0$, the two surfaces intersect in the xy-plane in the circle $x^2 + y^2 = 1$. Thus R is $x^2 + y^2 \leq 1$.

The top part of the larger sphere (above $z = 0$) is given by $z = -1 + \sqrt{2 - x^2 - y^2}$; the bottom half of the smaller sphere is given by $z = -\sqrt{1 - x^2 - y^2}$. Thus the volume is given by

$$\begin{aligned} \text{Volume} &= \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2-y^2}}^{-1+\sqrt{2-x^2-y^2}} \mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x \\ &= \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \left(-1 + \sqrt{2 - x^2 - y^2} + \sqrt{1 - x^2 - y^2}\right) \mathrm{d}y\,\mathrm{d}x. \end{aligned}$$

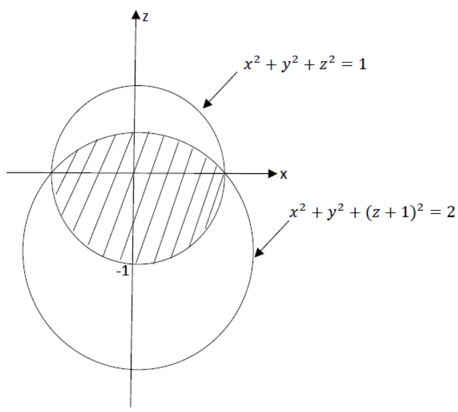

Figure 8: The region D.

We simplify the integral by converting to polar coordinates, given by

$$\begin{aligned}
\text{Volume} &= \int_0^{2\pi} \int_0^1 \left(\sqrt{2-r^2} + \sqrt{1-r^2} - 1\right) r \,\mathrm{d}r \,\mathrm{d}\theta \\
&= \int_0^{2\pi} \left[-\frac{1}{3}\left(2-r^2\right)^{\frac{3}{2}} - \frac{1}{3}\left(1-r^2\right)^{\frac{3}{2}} - \frac{r^2}{2}\right]_0^1 \mathrm{d}\theta \\
&= \int_0^{2\pi} \left(\left(-\frac{1}{3} - \frac{1}{2}\right) - \left(\frac{1}{3}\cdot 2^{\frac{3}{2}} - \frac{1}{3}\right)\right) \mathrm{d}\theta \\
&= 2\pi \frac{4\sqrt{2}-3}{6} \\
&= \pi \frac{4\sqrt{2}-3}{3} = 2.78.
\end{aligned}$$