10.018: Modelling Space and Systems
Term 2, 2021

Homework Week 6

Due Date: $+\infty$

This is a practice problem set. You DO NOT submit it.

BASIC problems.

The BASIC set of problems is designed to be a very easy and straightforward application of the definitions from lectures and cohorts (you might have to do some calculations, but not much). It's a good way to start your MSS homework. If you have trouble starting any of the questions do consult your cohort instructors (in office hours, via email or via Piazza).

1.

   (a) Parametrize the straight line segment through $P = (1, 0)$ and $Q = (2, 1)$.

   (b) Parametrize the straight line segment through $P = (2, 5)$ and $Q = (12, 9)$.

   Solution:

   (a) The parameterization is in terms of the displacement vector $\overrightarrow{OP} = [1, 0]$ from the origin to the point P and the displacement vector $\overrightarrow{PQ} = [1, 1]$ from P to Q. $\vec{p}(t) = \overrightarrow{OP} + t\overrightarrow{PQ}$, or, expressed in coordinates, $\vec{p}(t) = [1 + 1t, 0 + 1t]$. $t = 0$ corresponds to $\overrightarrow{OP} + 0\overrightarrow{PQ} = \overrightarrow{OP}$, and $t = 1$ corresponds to $\overrightarrow{OP} + 1\overrightarrow{PQ} = \overrightarrow{OQ}$.

   (b) The parameterization is in terms of the displacement vector $\overrightarrow{OP} = [2, 5]$ from the origin to the point P and the displacement vector $\overrightarrow{PQ} = [10, 4]$ from P to Q. $\vec{p}(t) = \overrightarrow{OP} + t\overrightarrow{PQ}$, or, expressed in coordinates, $\vec{p}(t) = [2 + 10t, 5 + 4t]$.

2. *Find the length of the curve $y = \frac{2}{3}x^{3/2}$ from $P = (0, 0)$ to $Q = (a, \frac{2}{3}a^{3/2})$.*

   Solution: The parametrized curve is $\vec{p}(t) = [t, \frac{2}{3}t^{3/2}]$, where $t \in [0, a]$. The arc length is

$$\begin{aligned}
\int_\gamma 1 ds &= \int_\gamma \|\vec{p'}(t)\| dt \\
&= \int_0^a \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \\
&= \int_0^a \sqrt{1^2 + \left(t^{\frac{1}{2}}\right)^2} dt \\
&= \int_0^a \sqrt{1 + t} dt \\
&= \frac{2}{3}\left[(1 + a)^{\frac{3}{2}} - 1\right]
\end{aligned}$$

3.

(a) *Find the divergence of* $\vec{F}(x,y) = [x^2 - y^2, 2xy]$.

(b) *Find the divergence and curl of* $\vec{G}(x,y,z) = [3x^2 - \sin(xz), 0, -\sin(xz)]$.

Solution:

(a) $\operatorname{div}\vec{F} = \frac{\partial}{\partial x}(x^2 - y^2) + \frac{\partial}{\partial y}(2xy) = 2x + 2x = 4x$

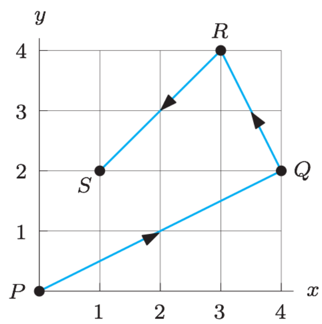
(b) $\operatorname{div}\vec{G} = \frac{\partial}{\partial x}(3x^2 - \sin(xz)) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(-\sin(xz)) = 6x - z\cos(xz) + 0 - x\cos(xz) = 6x - z\cos(xz) - x\cos(xz)$

curl $\vec{G} = [\frac{\partial}{\partial y}(-\sin(xz)) - \frac{\partial}{\partial z}(0), -(\frac{\partial}{\partial x}(-\sin(xz)) - \frac{\partial}{\partial z}(3x^2 - \sin(xz)), \frac{\partial}{\partial x}(0) - \frac{\partial}{\partial y}(3x^2 - \sin(xz))] = [0, (z-x)\cos(xz), 0]$

4. *Compute* $\int_\gamma \vec{F} \cdot d\vec{p}$*, where* γ *is the oriented curve in the figure below and* $\vec{F}$ *is a vector field constant on each of the three straight segments of* γ*:*

$$\vec{F} = \begin{cases} [1,0] & \textit{on PQ} \\ [2,-1] & \textit{on QR} \\ [3,1] & \textit{on RS} \end{cases}$$

Hint: You do not need to parametrize the curve γ *to find the answer. Recall the physical meaning of the line integral of a vector field.*



Solution:

The integral $\int_\gamma \vec{F} \cdot d\vec{p}$ is a sum of the line integrals of $\vec{F}$ over each of its three straight segments, which we can compute separately:

$\int_{PQ} \vec{F} \cdot d\vec{p} = \overrightarrow{PQ} \cdot \vec{F} = [4,2] \cdot [1,0] = 4$

$\int_{QR} \vec{F} \cdot d\vec{p} = \overrightarrow{QR} \cdot \vec{F} = [-1,2] \cdot [2,-1] = -4$

$\int_{RS} \vec{F} \cdot d\vec{p} = \overrightarrow{RS} \cdot \vec{F} = [-2,-2] \cdot [3,1] = -8$

$\int_\gamma \vec{F} \cdot d\vec{p} = 4 - 4 - 8 = -8.$

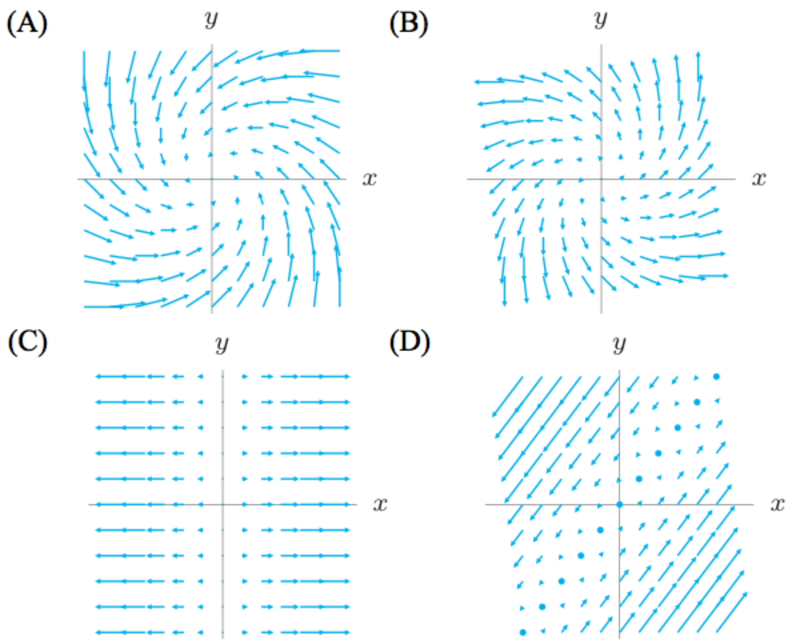
INTERMEDIATE problems.

The INTERMEDIATE set of problems is a *little* harder (but not by much) than the BASIC one. If you have trouble starting any of the questions do consult your cohort instructors (in office hours, via email or via Piazza).

1. *If $\vec{F}$ is a path-independent vector field, with $\int_{(0,0)}^{(1,0)} \vec{F}\cdot d\vec{p} = 5.1$, $\int_{(1,0)}^{(1,1)} \vec{F}\cdot d\vec{p} = 3.2$, and $\int_{(0,1)}^{(1,1)} \vec{F}\cdot d\vec{p} = 4.7$, find*
$$\int_{(0,1)}^{(0,0)} \vec{F}\cdot d\vec{p}.$$
***Notation**: $\int_{(a,b)}^{(c,d)} \vec{F}\cdot d\vec{p}$ indicates a line integral along some path between points (a,b) and (c,d).*

Solution: Since the vector field is path independent, the line integral around the closed curve $(0,0)$ to $(1,0)$ to $(1,1)$ to $(0,1)$ to $(0,0)$ is 0. Thus
$\int_{(0,1)}^{(0,0)} \vec{F}\cdot d\vec{p} = -(\int_{(0,0)}^{(1,0)} \vec{F}\cdot d\vec{p} + \int_{(1,0)}^{(1,1)} \vec{F}\cdot d\vec{p} + \int_{(1,1)}^{(0,1)} \vec{F}\cdot d\vec{p}) = -(5.1+3.2-4.7) = -3.6.$

2.



In $\mathbb{R}^2$, there are 3 vector fields $\vec{F} = [x,y]$, $\vec{G} = [-y,x]$, and $\vec{H} = [x,-y]$. Match the vector fields with their sketches.

(i) $\vec{F} + \vec{G}$

(ii) $\vec{F} + \vec{H}$

(iii) $\vec{G} + \vec{H}$

(iv) $-\vec{F} + \vec{G}$

Solution: i: (B). $\vec{F}(0,y) + \vec{G}(0,y) = [-y,y]$, a vector pointing up to the left if $y > 0$ and down to the right if $y < 0$, as in (B).
ii: (C). $\vec{F}(0,y) + \vec{H}(0,y) = \vec{0}$, the zero vector as in (C).
iii: (D). $\vec{G}(0,y) + \vec{H}(0,y) = [-y,-y]$, a vector pointing down to the left if $y > 0$ and up to the right if $y < 0$, as in (A) and (D). $\vec{G}(x,0) + \vec{H}(x,0) = [x,x]$, a vector pointing up to the right if $x > 0$ and down to the left if $x < 0$, as in (B) and (D). Therefore, it is (D).

iv: (A). $-\vec{F}(x,0)+\vec{G}(x,0)=[-x,x]$, a vector pointing up to the left if $x>0$ and down to the right if $x<0$, as in (A).

3. *If f is a potential function for the two-dimensional vector field $\vec{F}$ then the Fundamental Theorem of Calculus for Line Integrals says that $\int_\gamma \vec{F}\cdot d\vec{p} = f(x,y)-f(0,0)$, where γ is any path from $(0,0)$ to (x,y). The vector field $\vec{F}$ is a conservative field. Prove that the following vector fields are conservative fields by definition and find the potential functions. a, b, and c are constants.*

   (a) $\vec{F}=[ay,ax]$

   (b) $\vec{G}=[abye^{bxy}, c+abxe^{bxy}]$

   Solution:

   (a) $\vec{F}=\nabla f(x,y)=\begin{bmatrix} f_x \\ f_y\end{bmatrix}=\begin{bmatrix} ay \\ ax\end{bmatrix}$
   $f(x,y)=\int f_x dx = axy+p(y)$, where $p(y)$ is a function of y. The derivative of $p(y)$ with respect to x is 0. If we differentiate $axy+p(y)$ with respect to x, we will get back to f_x.
   $f(x,y)=\int f_y dy = axy+q(x)$, where $q(x)$ is a function of x.
   We compare both antiderivatives obtained above and conclude that $p(y)$ and $q(x)$ should be constants. Therefore,
   $$f(x,y)=axy+C.$$

   (b) $\vec{G}=\nabla g(x,y)=\begin{bmatrix} g_x \\ g_y\end{bmatrix}=\begin{bmatrix} abye^{bxy} \\ c+abxe^{bxy}\end{bmatrix}$
   $g(x,y)=\int g_x dx = ae^{bxy}+p(y)$, where $p(y)$ is a function of y.
   $g(x,y)=\int g_y dy = cy+ae^{bxy}+q(x)$, where $q(x)$ is a function of x.
   We compare both antiderivatives obtained above and conclude that $p(y)=cy$ and $q(x)$ should be a constant. Therefore,
   $$f(x,y)=cy+ae^{bxy}+C.$$

4.

   (a) Let $f(x,y)=axy+ax^2y+y^3$. Find div grad f.

   (b) If possible, choose a so that div grad $f=0$ for all x and y.

   Solution:

   (a) Since a is a constant,
   $$\text{div grad } f = \frac{\partial}{\partial x}(\frac{\partial f}{\partial x})+\frac{\partial}{\partial y}(\frac{\partial f}{\partial y}) = \frac{\partial}{\partial x}(ay+2axy)+\frac{\partial}{\partial y}(ax+ax^2+3y^2)=2ay+6y.$$

   (b) Since div grad $f=(2a+6)y$, we have div grad $f=0$ for all x, y if $a=-3$.

5. *Calculate the line integral of $\vec{F}=[-y,x]$ along the following paths in the xy-plane.*

   (a) Straight line segment from the origin to the point $(2,3)$.

   (b) Straight line segment from $(2,3)$ to $(0,3)$.

   (c) Counterclockwise around a circle of radius 5 centered at the origin, starting from $(5,0)$ to $(0,-5)$.

   Solution: You can compute $\int_\gamma \vec{F}\cdot d\vec{p}$ by the standard way, but in this case, we can get the values without doing much computation.

(a) Intuitively, the vector field is everywhere perpendicular to the radial line from the origin to $(2, 3)$, so the line integral is 0.
Calculation: The parametrized curve is $\vec{p} = [2t, 3t]$, where $t \in [0, 1]$.

$$\begin{aligned}
&\int_\gamma \vec{F} \cdot \mathrm{d}\vec{p} \\
=&\int_\gamma F_1 \mathrm{d}x + F_2 \mathrm{d}y \\
=&\int_\gamma -y\frac{\mathrm{d}x}{\mathrm{d}t}\mathrm{d}t + x\frac{\mathrm{d}y}{\mathrm{d}t}\mathrm{d}t \\
=&\int_0^1 -3t \times 2\mathrm{d}t + 2t \times 3\mathrm{d}t \\
=&\int_0^1 0\mathrm{d}t \\
=&0.
\end{aligned}$$

(b) Intuitively, since the path is parallel to the x-axis, only the x-component of the vector field contributes to the line integral. The x-component is -3 on this line, and the displacement along this line is -2, so line integral $= [-3, 0] \cdot [-2, 0] = 6$.
Calculation: The parametrized curve is $\vec{p} = [2 - 2t, 3]$, where $t \in [0, 1]$.

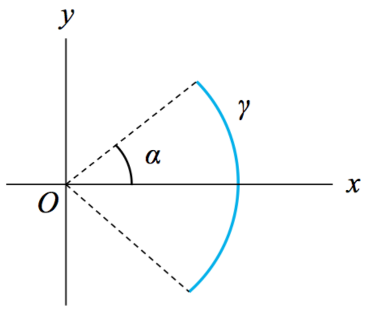
$$\begin{aligned}
&\int_\gamma \vec{F} \cdot \mathrm{d}\vec{p} \\
=&\int_\gamma F_1 \mathrm{d}x + F_2 \mathrm{d}y \\
=&\int_\gamma -y\frac{\mathrm{d}x}{\mathrm{d}t}\mathrm{d}t + x\frac{\mathrm{d}y}{\mathrm{d}t}\mathrm{d}t \\
=&\int_0^1 (-3) \times (-2)\mathrm{d}t + (2 - 2t)0\mathrm{d}t \\
=&\int_0^1 6\mathrm{d}t \\
=&6.
\end{aligned}$$

(c) Intuitively, the circle of radius of 5 has equation $x^2 + y^2 = 25$. On this curve, $||\vec{F}|| = \sqrt{(-y)^2 + x^2} = \sqrt{25} = 5$. In addition, $\vec{F}$ is everywhere tangent to the circle and points in the same direction at $d\vec{p}$, and the path is $3/4$ of the circle. Thus,
line integral $= ||\vec{F}|| \times$ Length of curve $= 5 \times \frac{3}{4} \times 2\pi \times 5 = \frac{75}{2}\pi$.
Calculation: The parametrized curve is $\vec{p} = [5\cos t, 5\sin t]$, where $t \in \left[0, \frac{3}{2}\pi\right]$.

$$
\begin{aligned}
&\int_\gamma \vec{F}\cdot \mathrm{d}\vec{p}\\
=&\int_\gamma F_1\mathrm{d}x + F_2\mathrm{d}y\\
=&\int_\gamma -y\frac{\mathrm{d}x}{\mathrm{d}t}\mathrm{d}t + x\frac{\mathrm{d}y}{\mathrm{d}t}\mathrm{d}t\\
=&\int_0^{\frac{3}{2}\pi} -5\sin t(-5\sin t)\mathrm{d}t + 5\cos t 5\cos t\mathrm{d}t\\
=&\int_0^{\frac{3}{2}\pi} 25\mathrm{d}t\\
=&\frac{75}{2}\pi.
\end{aligned}
$$

6. (a) *Calculate the length of the arc γ (radius R, angle 2α) given below using the line integral.*

   (b) *Calculate the moment of inertia of the arc γ (radius R, angle 2α) with respect to y-axis, assuming its density is uniform everywhere.*



Hint: The moment of inertia is a quantity expressing a body's tendency to resist angular acceleration, which is the sum of the products of the mass of each particle in the body with the square of its distance from the axis of rotation. In this problem, moment of inertia can be computed by $I = \int_\gamma x^2 ds$.

Solution: Parametrized the arc γ: $x = R\cos t$, $y = R\sin t$, where $-\alpha \le t \le \alpha$.

(a) arc length=$\int_\gamma 1\mathrm{d}s = \int_\gamma 1\|\vec{p}'(t)\|\mathrm{d}t = \int_{-\alpha}^{\alpha} 1\sqrt{(-R\sin t)^2 + (R\cos t)^2}\mathrm{d}t = 2\alpha R.$

(b) $I = \int_\gamma x^2\mathrm{d}s = \int_{-\alpha}^{\alpha} R^2\cos^2 t\sqrt{(-R\sin t)^2 + (R\cos t)^2}\mathrm{d}t = R^3(\alpha + \frac{1}{2}\sin 2\alpha).$

Challenging problems [OPTIONAL].

1. *Two parametrized lines are given.*

   (a) Are $\vec{p_1}$ and $\vec{p_2}$ the same line?

$$\vec{p_1}(t) = [5-3t, 2t, 7+t] \qquad \vec{p_2}(t) = [5-6t, 4t, 7+3t]$$

   (b) Are $\vec{p_3}$ and $\vec{p_4}$ the same line?

$$\vec{p_3}(t) = [5-3t, 1+t, 2t] \qquad \vec{p_4}(t) = [2+6t, 2-2t, 2-4t]$$

Solution:

(a) The coefficients of t in the parameterization show that line $\vec{p_1}$ is parallel to the vector $[-3, 2, 1]$ and line $\vec{p_2}$ is parallel to $[-6, 4, 3]$. Since these vectors are not parallel, so the lines are different.

(b) The coefficients of t in the parameterization show that line $\vec{p_3}$ is parallel to the vector $[-3, 1, 2]$ and line $\vec{p_4}$ is parallel to $[6, -2, -4]$. Since these vectors are parallel, the lines are parallel. To see if they are the same line, check whether they have a common point. A point $(5, 1, 0)$ is on both lines corresponding to $t = 0$ on line $\vec{p_3}$ and $t = \frac{1}{2}$ on line $\vec{p_4}$. Since two lines are parallel and go through a common point, they are the same line.

2. *Let $\vec{F}$ be a path-independent vector field, in physics, the potential function ϕ is usually required to satisfy the equation $\vec{F} = -\nabla\phi$. This problem illustrates the significance of the negative sign.*

(a) *Let the xy-plane represent part of the earth's surface with the z-axis pointing upward. (This scale is small enough that a flat plane is a good approximation to the earth's surface.) Let $\vec{r} = [x, y, z]$, with $z \geq 0$ and x,y,z in meters, be the position vector of a rock of unit mass. The gravitational potential energy function for the rock is $\phi(x, y, z) = gz$, where $g \approx 9.8m/sec^2$. Describe in words the level surfaces of ϕ. Does the potential energy increase or decrease with height above the earth?*
Hint: A ***level surface*** *of a function of three variables, $f(x, y, z)$, is a surface of the form $f(x, y, z) = c$, where c is a constant. The function f can be represented by the family of level surfaces obtained by allowing c to vary.*

(b) *What is the relation between the gravitational vector, $\vec{F}$, and the vector $\nabla\phi$? Explain the significance of the negative sign in the equation $\vec{F} = -\nabla\phi$.*

Solution:

(a) The level surfaces are horizontal planes given by $gz = c$, so $z = c/g$. The potential energy increases with the height above the earth. This means that more energy is stored as 'potential to fall' as height increases.

(b) The gradient of ϕ points upward (in the direction of increasing potential energy), so $\nabla\phi = g\vec{k}$. The gravitational force acts toward the earth in the direction of $-\vec{k}$. So, $\vec{F} = -g\vec{k}$. The negative sign represents the fact that the gravitational force acts in the direction of the decreasing potential energy.