10.018 Modelling Space and Systems

Lecture 6

Line Integrals and Vector Fields

Term 2, 2021

SINGAPORE UNIVERSITY OF TECHNOLOGY AND DESIGN

Before we start....

To get the most out of this lecture, you should already be familiar with

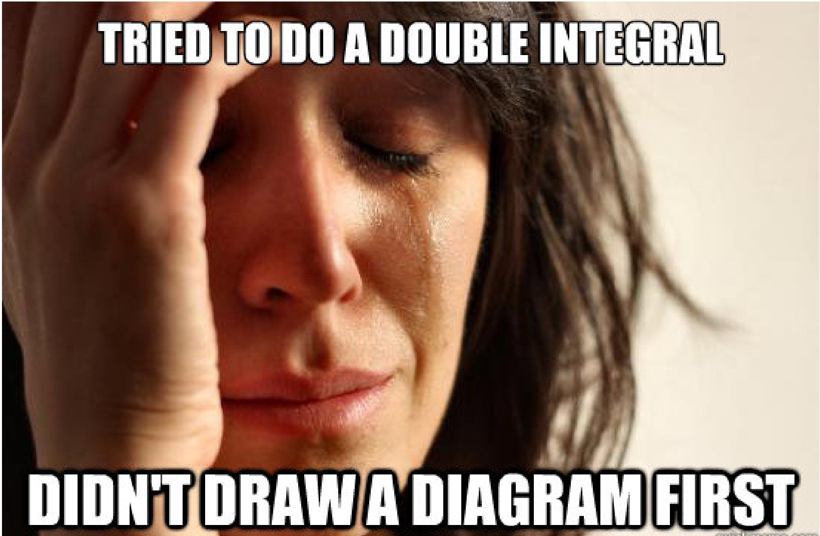
1. Line integrals in Physics, e.g. work done by a force
2. Vector fields in Physics, e.g. gravitational field, electric/magnetic field

as we will be going through

1. Line integrals
2. Vector fields

Hints for the Exam

Double integration could be a confusing topic.



Iterated integrals

Guidelines for evaluating a double integral:

- **Sketch the region** that we are integrating over.
- **Draw arrows** to indicate the direction of integration (horizontal (from left to right) for dx first, vertical (from bottom to top) for dy first).
- Determine whether it is vertically or horizontally simple, then pick the right formula to use.
- Sometimes, doing the iterated integral in a different order can simplify calculations.

Note: the second step could be used when integrating in polar coordinates as well: draw radial arrows for dr, and circular arrows (counter-clockwise) for $d\theta$.

Line integrals – introduction

In addition to double and triple integrals, *line integrals* provide yet another way to extend integration to higher dimensions.

There are two types of line integrals that we will study:

- **line integral of a scalar field**

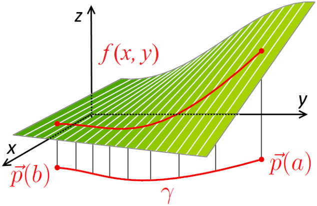
  Notation: $$\int_\gamma f(\vec{x})\,\mathrm{d}s$$

- **line integral of a vector field**

  Notation: $$\int_\gamma \vec{F}(\vec{p})\cdot\mathrm{d}\vec{p}$$

Line integrals – introduction

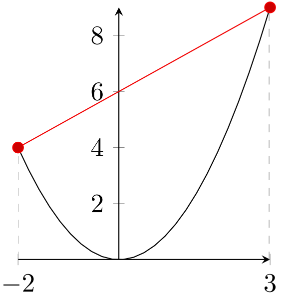
In 3D, a **line integral of a scalar field** gives the signed cross-sectional area bounded by a surface f and a curve γ (i. e. curved **line**) in the xy-plane.



More generally, a line integral can be defined for a function $f : \mathbb{R}^n \to \mathbb{R}$, which we call a *scalar field*, and a curve γ in $\mathbb{R}^n$.

The curve γ is parametrized by a (one-to-one) function $\vec{p}(t)$, and its endpoints are given by $\vec{p}(a)$ and $\vec{p}(b)$.

Parametrizing a curve



For example, in $\mathbb{R}^2$, the segment of a **parabola** starting at $(-2,4)$, passing through $(0,0)$ and ending at $(3,9)$ can be parametrized by

$$\vec{p}(t) = \begin{bmatrix} t \\ t^2 \end{bmatrix}, \quad t \in [-2,3].$$

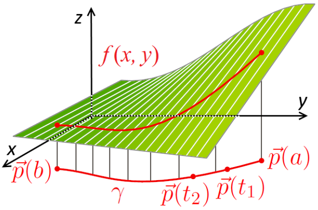
The **straight line** segment from $(-2,4)$ to $(3,9)$ can be parametrized by

$$\vec{p}(t) = \begin{bmatrix} -2 \\ 4 \end{bmatrix} + t \begin{bmatrix} 3-(-2) \\ 9-4 \end{bmatrix} = \begin{bmatrix} -2+5t \\ 4+5t \end{bmatrix}, \ t \in [0,1].$$

As another example, a semicircle with radius 1, centred at $(0,0)$ can be parametrized by

$$\vec{p}(t) = \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}, \quad t \in [0,\pi].$$

Line integrals – definition



In 3D, we can divide the interval $[a, b]$ into n subintervals, and pick a t_i in each subinterval. Let $\vec{p}(t_i)$ be a point on the curve γ, and let the distance between successive points be Δs_i.

Then the line integral is defined as

$$\int_\gamma f(x, y)\,\mathrm{d}s = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i, y_i)\Delta s_i.$$

We note that $\Delta s_i = \|\vec{p}(t_i + \Delta t) - \vec{p}(t_i)\| \approx \|\vec{p}\,'(t_i)\|\Delta t$, which in the limit $(n \to \infty)$ becomes

$$\mathrm{d}s = \|\vec{p}\,'(t)\|\,\mathrm{d}t = \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t,$$

where $\vec{p}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$.

Line integrals – formula

Generalizing this to higher dimensions, we have:

Line integral of a scalar field

For a function $f : \mathbb{R}^n \to \mathbb{R}$, the line integral along a curve γ parametrized by $\vec{p} : [a, b] \to \mathbb{R}^n$ is given by

$$\int_\gamma f(\vec{x})\,\mathrm{d}s = \int_a^b f\left(\vec{p}(t)\right) \|\vec{p}\,'(t)\|\,\mathrm{d}t.$$

1. Find the parametrization $\vec{p}(t)$ of curve γ, identify a and b.
2. Plug the parametrized curve into $f(\vec{x})$.
3. Compute $\|\vec{p}\,'(t)\|$.
4. Evaluate the integral according to the formula above.

Fun fact: the value of the integral **does not depend** on which parametrization of γ we use (this can be shown using integration by substitution).

Line integrals – formula

For instance, for two variables, if we write $\vec{p}(t) = [x(t),\, y(t)]$, then

$$\int_\gamma f(x,y)\,\mathrm{d}s = \int_a^b f\big(x(t),\, y(t)\big)\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t.$$

For three variables, if we write $\vec{p}(t) = [x(t),\, y(t),\, z(t)]$, then

$$\int_\gamma f(x,y,z)\,\mathrm{d}s = \int_a^b f\big(x(t),\, y(t),\, z(t)\big)\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2}\,\mathrm{d}t.$$

Line integrals – example

Evaluate
$$\int_\gamma (xy)^{1/3}\,\mathrm{d}s$$
where γ is the curve $y = x^2$ for $0 \le x \le 1$.

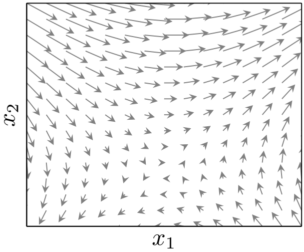
Solution: we can parametrize γ as $\vec{p}(t) = [t,\ t^2],\ \ t \in [0, 1]$. Then $\vec{p}\,'(t) = [1,\ 2t]$ and

$$\begin{aligned}
\int_\gamma (xy)^{1/3}\,\mathrm{d}s &= \int_0^1 (t\,t^2)^{1/3}\sqrt{(1)^2 + (2t)^2}\,\mathrm{d}t \\
&= \int_0^1 t\sqrt{1+4t^2}\,\mathrm{d}t \\
&= \int_0^1 \sqrt{u}\,\frac{1}{8}\frac{\mathrm{d}u}{\mathrm{d}t}\,\mathrm{d}t, \qquad \text{with } u = 1+4t^2 \\
&= \int_1^5 \frac{1}{8}\sqrt{u}\,\mathrm{d}u = \frac{1}{12}\Big[u^{3/2}\Big]_1^5 = \frac{1}{12}\big(5\sqrt{5}-1\big).
\end{aligned}$$

Vector fields – introduction

Definition: a **vector field** $\vec{F} : \mathbb{R}^n \to \mathbb{R}^n$ is vector-valued function that associates a **vector** in $\mathbb{R}^n$ to each point of its domain.

Compare with a **scalar field** (aka a function), $F : \mathbb{R}^n \to \mathbb{R}$ that associates a **scalar** to each point of its domain.



In $\mathbb{R}^2$:

$$\vec{F}(x_1, x_2) = \vec{F}(\vec{x}) = \begin{bmatrix} F_1(\vec{x}) \\ F_2(\vec{x}) \end{bmatrix}.$$

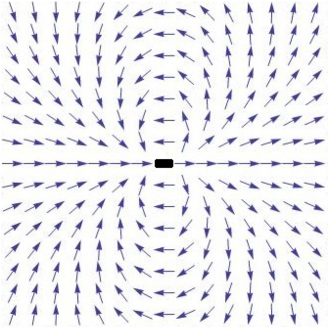
In $\mathbb{R}^n$:

$$\vec{F}(x_1, \ldots, x_n) = \vec{F}(\vec{x}) = \begin{bmatrix} F_1(\vec{x}) \\ \vdots \\ F_n(\vec{x}) \end{bmatrix}.$$

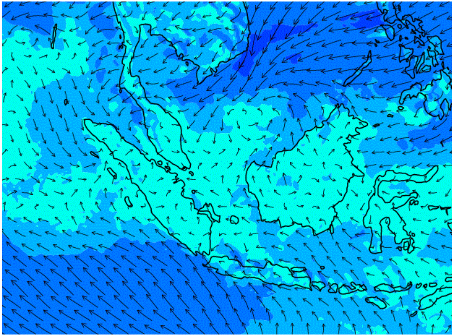
At each point, you can imagine a vector field as describing a flow (of a fluid) with both magnitude and direction.

Vector fields – examples

You are already familiar with vector fields:



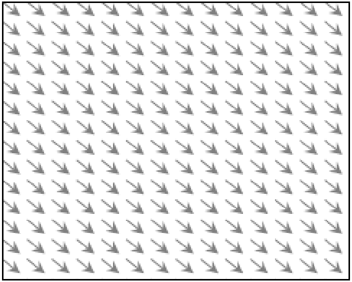
A magnetic field.

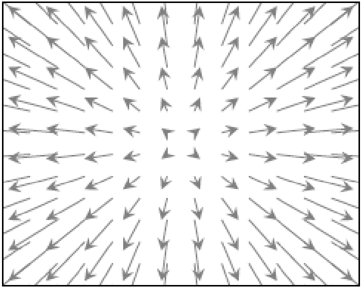


A satellite map of wind velocities.
(Source: www.nea.gov.sg)

Vector fields – more examples

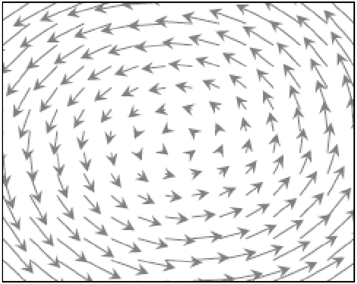
$$\vec{F}(x,y) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad \vec{F}(x,y) = \begin{bmatrix} x \\ y \end{bmatrix}$$

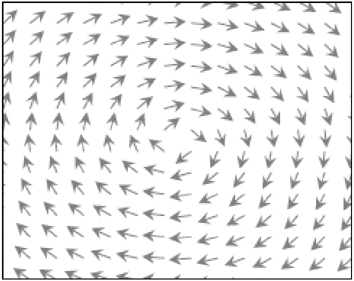




Vector fields – more examples

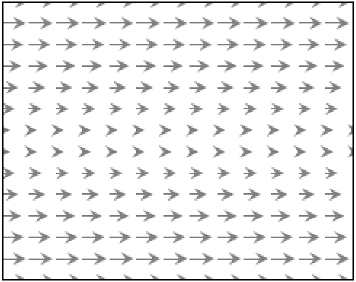
$$\vec{F}(x,y) = \begin{bmatrix} -y \\ x \end{bmatrix} \quad \vec{F}(x,y) = \begin{bmatrix} y/r \\ -x/r \end{bmatrix}$$



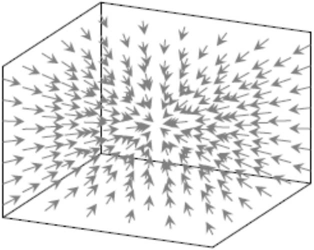


Vector fields – more examples

$$\vec{F}(x,y) = \begin{bmatrix} \sqrt{|y|} \\ 0 \end{bmatrix}$$



$$\vec{F}(x,y,z) = \begin{bmatrix} -x/\rho \\ -y/\rho \\ -z/\rho \end{bmatrix}$$



Divergence and Curl

Important quantities that can be computed from the vector fields are **divergence** and **curl**. They appear in numerous applications in engineering and physics.

They describe properties of the vector fields $\vec{F}$. The definition and physical meaning will be covered in the cohorts.

Introduction

Other than line integral of a scalar field, there is also line integral along a **vector field**.

Given a vector field $\vec{F} : \mathbb{R}^n \to \mathbb{R}^n$, a curve γ in $\mathbb{R}^n$, and a parametrization of the curve $\vec{p} : [a, b] \to \mathbb{R}^n$, the *scalar* field

$$\vec{F}(\vec{p}(t)) \cdot \frac{\vec{p}\,'(t)}{\|\vec{p}\,'(t)\|}$$

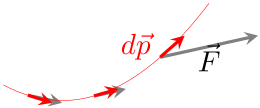
is the **tangential component** of $\vec{F}$ in the direction of, or **along**, γ.

The line integral, in the *scalar* field sense, of this tangential component is

$$\int_a^b \vec{F}(\vec{p}(t)) \cdot \frac{\vec{p}\,'(t)}{\|\vec{p}\,'(t)\|} \|\vec{p}\,'(t)\| \,\mathrm{d}t = \int_a^b \vec{F}(\vec{p}(t)) \cdot \vec{p}\,'(t) \,\mathrm{d}t.$$

This is known as a *line integral along a vector field*; it takes into account how much the curve follows along the field.

Line integral along a vector field



With the notation $\mathrm{d}\vec{p} = [\mathrm{d}x_1, \mathrm{d}x_2, \ldots, \mathrm{d}x_n]$, the **line integral along a vector field** $\vec{F}$ is denoted by $\int_\gamma \vec{F}(\vec{p}) \cdot \mathrm{d}\vec{p}$, and is computed using

$$\int_\gamma \vec{F}(\vec{p}) \cdot \mathrm{d}\vec{p} = \int_a^b \vec{F}\big(\vec{p}(t)\big) \cdot \vec{p}\,'(t)\,\mathrm{d}t.$$

Example from physics: if $\vec{F}$ denotes an electric or gravitational force field, then **the work done on a particle, traveling along a curve** γ (parametrized by $\vec{p}$) is given by

$$W = \int_\gamma \vec{F}(\vec{p}) \cdot \mathrm{d}\vec{p}.$$

Line integral along a vector field – example 1

In $\mathbb{R}^2$, if we write $\vec{F}(x,y) = \begin{bmatrix} F_1(x,y) \\ F_2(x,y) \end{bmatrix}$ and $\mathrm{d}\vec{p} = [\mathrm{d}x,\ \mathrm{d}y]$, then it is customary to write

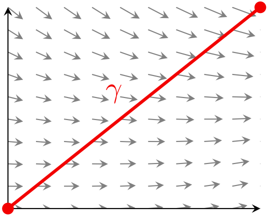
$$\int_\gamma \vec{F}(\vec{p}) \cdot \mathrm{d}\vec{p} = \int_\gamma F_1\,\mathrm{d}x + F_2\,\mathrm{d}y.$$

Example: evaluate $\int_\gamma xy\,\mathrm{d}x + (x-y)\,\mathrm{d}y$, where γ is a segment of $y = x^2$ from $(0,0)$ to $(2,4)$.

Solution: let $\vec{p}(t) = [x(t),\ y(t)] = [t,\ t^2]$, $t \in [0,2]$, then

$$\begin{aligned}\int_\gamma xy\,\mathrm{d}x + (x-y)\,\mathrm{d}y &= \int_0^2 xy\,\frac{\mathrm{d}x}{\mathrm{d}t}\mathrm{d}t + (x-y)\,\frac{\mathrm{d}y}{\mathrm{d}t}\mathrm{d}t \\ &= \int_0^2 t\,t^2\,1\,\mathrm{d}t + (t-t^2)\,2t\,\mathrm{d}t \\ &= \int_0^2 (2t^2 - t^3)\,\mathrm{d}t = \frac{4}{3}.\end{aligned}$$

Line integral along a vector field – example 2



Let $\vec{F}(x,y) = \begin{bmatrix} 3+2xy \\ x^2-3y^2 \end{bmatrix}$, and γ be the curve parametrized by

$$\vec{p}(t) = [t,\, t], \quad t \in [0,1].$$

$$\begin{aligned}
\int_\gamma \vec{F}(\vec{p}) \cdot \mathrm{d}\vec{p} &= \int_a^b \vec{F}(\vec{p}(t)) \cdot \vec{p}\,'(t)\,\mathrm{d}t \\
&= \int_0^1 \begin{bmatrix} 3+2t\,t \\ t^2-3t^2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} \mathrm{d}t \\
&= \int_0^1 \left(3+2t^2+t^2-3t^2\right) \mathrm{d}t = 3.
\end{aligned}$$

As these two examples demonstrate, a line integral along a vector field is no more difficult to evaluate than a single integral.

Summary

We have covered:

- Line integral of scalar fields.
- Parametrization of the curve.
- Line integrals of vector fields.

Textbook: read Sections 21.1 and 21.2, then try some of Exercises 21.1.2–21.1.8 and Exercises 21.2.1–21.2.10.