Week 6 - Day 2

Electromagnet

Concept 1: Ampere's Law

More Applications: Solenoid/Relay

Reading:

University Physics with Modern Physics – Chapter 28

Introduction to Electricity and Magnetism – Chapter 9

Concept 1: Ampere’s Law

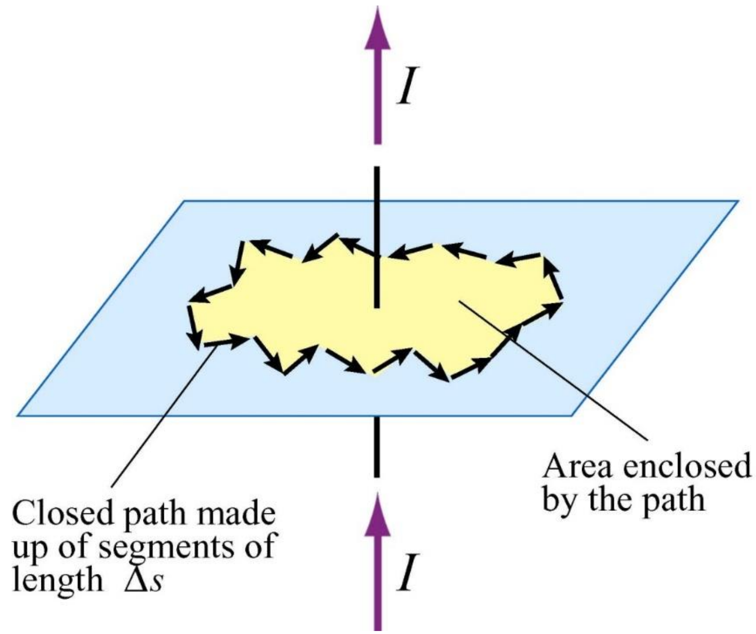
Ampere

- André-Marie Ampère (1775 –1836), a French physicist and mathematician is regarded as one of the main founders of the science of classical electromagnetism, which he referred to as "electrodynamics".
- Ampere continued Ørsted’s work, and showed that pair of parallel current carrying conductors repel or attract depend on direction of current.
- Ampere discovered “Ampere’s Law”.
- The SI unit of electric current, Ampere (A), is named after him.
- The official definition of the ampere: One ampere of current through each of two parallel conductors of infinite length, separated by one meter in empty space free of other magnetic fields, causes a force of exactly $2 \times 10^{-7} N/m$ on each conductor, using the equation $\frac{\mu_o I_1 I_2}{2\pi r}$.
- This also explains why magnetic permeability μ_0 = $4\pi x 10^{-7}$ T.m/A

André-Marie Ampère
(1775 – 1836)

Ampere's Law: The Idea

- In order to have a B field around a loop, there must be current punching through the loop.



Ampere's Law

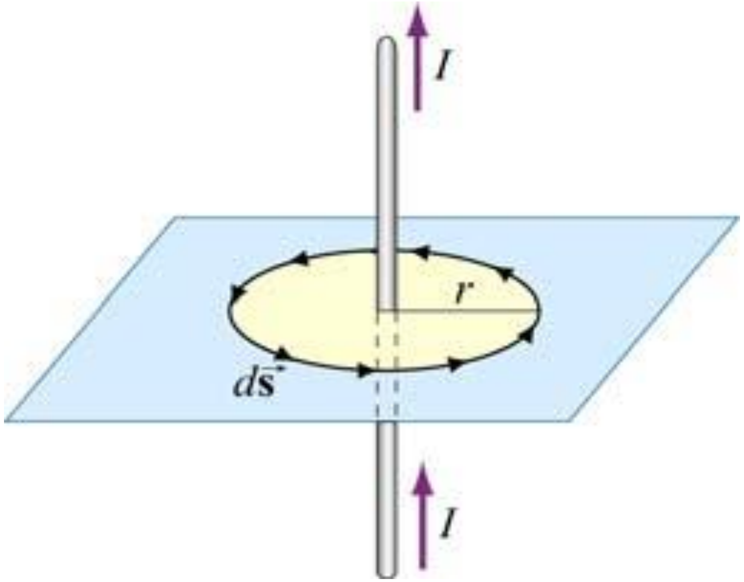
- Consider a closed path of a circle of radius r around a long current carrying wire.

$$\vec{\mathbf{B}} = \frac{\mu_0 I}{2\pi r}\hat{\theta} \qquad \text{(circumferential)}$$

$$\oint \vec{\mathbf{B}} \cdot d\vec{s} = \oint \mathbf{B} ds = \frac{\mu_0 I}{2\pi r}(2\pi r)$$

$$= \mu_0 I$$

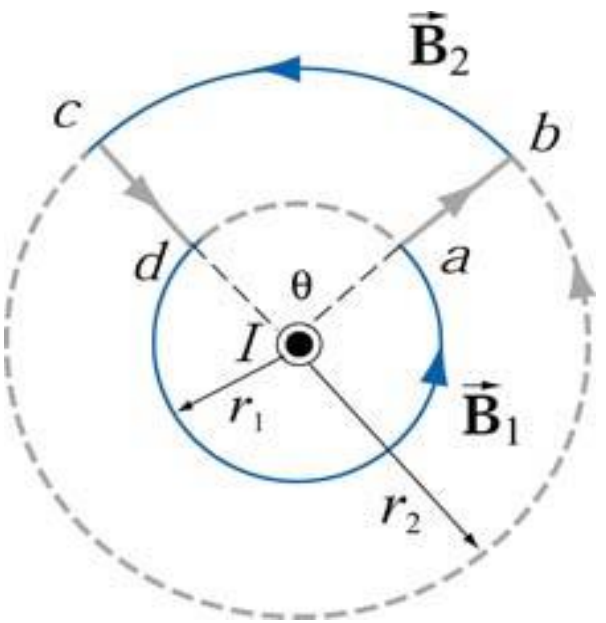
$$\Rightarrow \oint \vec{\mathbf{B}} \cdot d\vec{s} = \mu_0 I$$



- Is this result true for any closed path?

- Lets consider a more complicated closed path :

$$\oint_{abcda} \vec{\mathbf{B}} \cdot d\vec{s} = \int_{ab} \vec{\mathbf{B}} \cdot d\vec{s} + \int_{bc} \vec{\mathbf{B}} \cdot d\vec{s} + \int_{cd} \vec{\mathbf{B}} \cdot d\vec{s} + \int_{da} \vec{\mathbf{B}} \cdot d\vec{s}$$



$$= 0 + B_2(r_2\theta) + 0 + B_1[r_1(2\pi - \theta)]$$

Since $B_1 = \frac{\mu_0 I}{2\pi r_1}$, $B_2 = \frac{\mu_0 I}{2\pi r_2}$

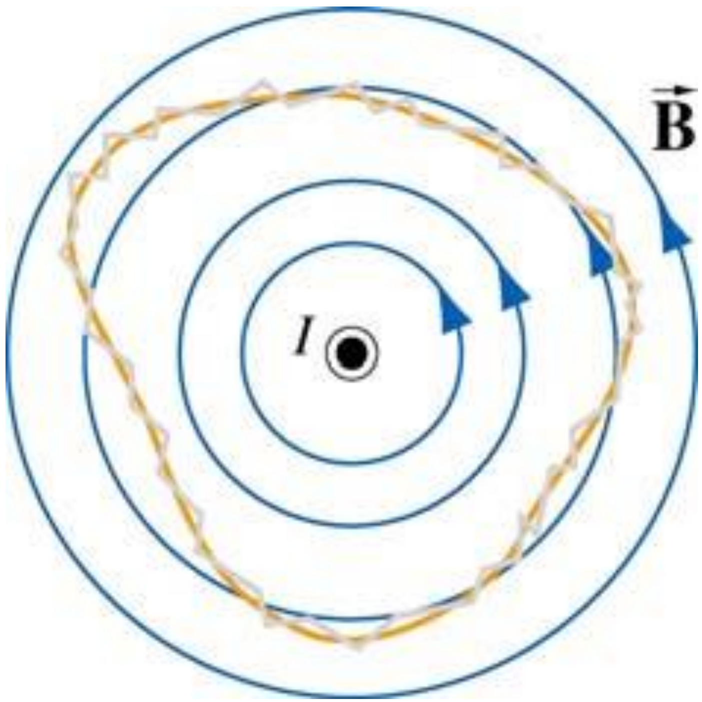
$$\oint_{abcda} \vec{\mathbf{B}} \cdot d\vec{s} = \frac{\mu_0 I}{2\pi r_2}(r_2\theta) + \frac{\mu_0 I}{2\pi r_1}[r_1(2\pi - \theta)]$$

$$= \frac{\mu_0 I}{2\pi}\theta + \frac{\mu_0 I}{2\pi}(2\pi - \theta) = \mu_0 I$$

- Result is the same as the simpler closed path!

Proving Ampere's Law

- By constructing a finer grid of concentric circles and radially outward axis, we can prove the same result is obtained for any closed loop of arbitrary shape.
- This is because any arbitrary path can be constructed by infinite small sections of small paths travel along radial axes and concentric circles.



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Ampere's Law

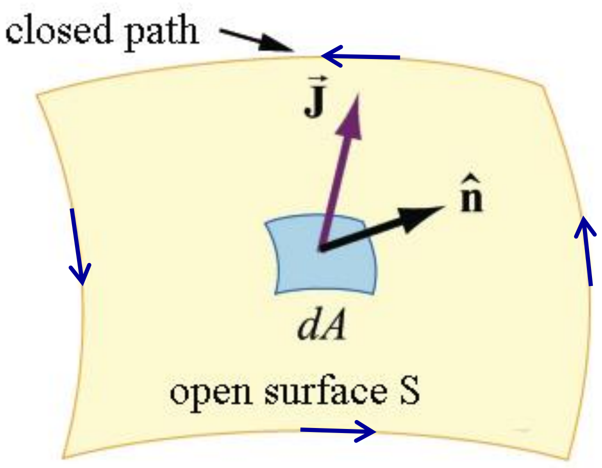
The line integral of about any closed path (an "Amperian loop") is equal to the current enclosed by that path.

$$\oint_{\substack{closed \\ path}} \vec{\mathbf{B}} \cdot d\vec{s} = \mu_0 I_{enc}$$

- I_{enc} is positive by the *Right-Hand Rule*.
- Ampere's Law is analogous to Gauss's Law in electrostatics, it relates field with the sources. (*cf*. "Amperian loop" vs "Gaussian surface")
- Note: Even though Ampere's Law is always true, it is only useful to find a magnetic field when a system possesses certain symmetry:
  - infinitely long straight wire
  - infinitely large sheet
  - infinite solenoid
  - toroid

Current Enclosed : Magnitude

- Current enclosed is the flux of the current density through an open surface S bounded by the closed path.
- $\vec{\mathbf{J}}$ is the current density



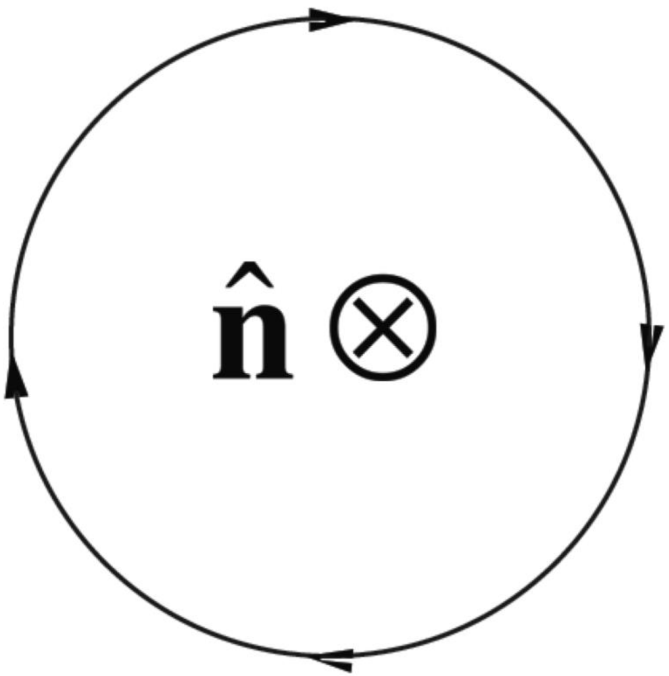
$$I_{enc} = \iint\limits_{\substack{open \\ surface\ S}} \vec{\mathbf{J}} \cdot \hat{n} dA$$

$$\oint\limits_{\substack{closed \\ path}} \vec{\mathbf{B}} \cdot d\vec{s} = \mu_0 I_{enc} = \mu_0 \iint\limits_{\substack{open \\ surface}} \vec{\mathbf{J}} \cdot \hat{n} dA$$

Current Enclosed : Sign Convention

- Right-Hand Rule determines the direction of path integration and the normal vector of the surface enclosed.
- Integration direction *clockwise* for line integral requires that unit normal points *into page* for open surface integral
- Current positive into page, negative out of page

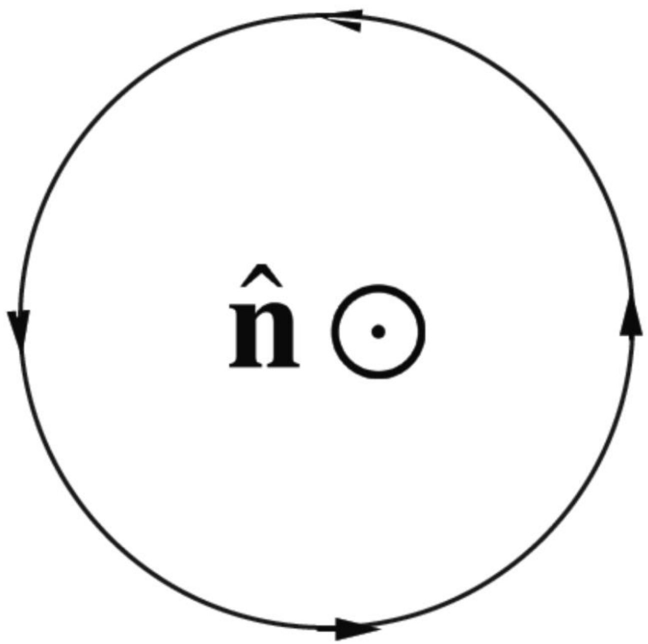
$$\oint_{\substack{closed\\path}} \vec{\mathbf{B}} \cdot d\vec{s} = \mu_0 \iint_{\substack{open\\surface}} \vec{\mathbf{J}} \cdot \hat{n}\, dA$$



Current Enclosed : Sign Convention

- Integration direction *counterclockwise* for line integral requires that unit normal points *out of page* for open surface integral.
- Current positive out of page, negative into page.

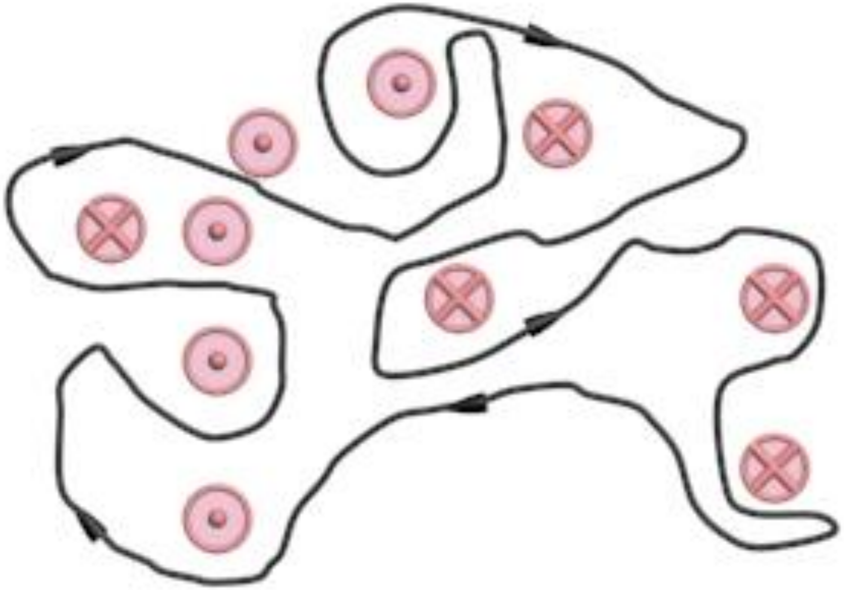
$$\oint_{\substack{\text{closed}\\ \text{path}}} \vec{\mathbf{B}} \cdot d\vec{s} = \mu_0 \iint_{\substack{\text{open}\\ \text{surface}}} \vec{\mathbf{J}} \cdot \hat{n}\, dA$$



Concept Question 1.1

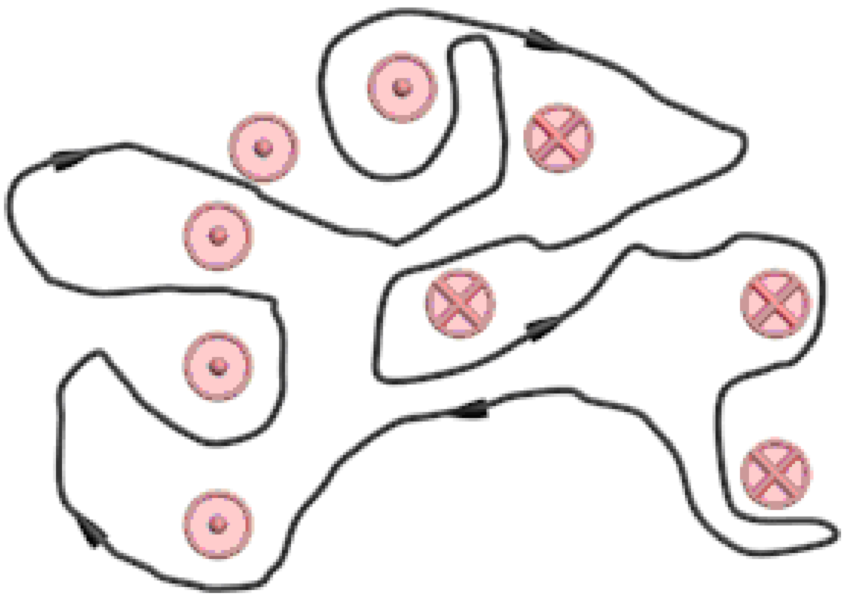
Integrating **B** around the loop shown gives us

1. a positive number
2. a negative number
3. zero



Concept Question 1.2

Integrating **B** around the loop in the clockwise direction shown gives us



1. a positive number
2. a negative number
3. zero

Biot-Savart Law vs. Ampere's Law

Law	Mathematical Expression	Remarks
Biot-Savart Law	$\vec{\mathbf{B}} = \frac{\mu_0 I}{4\pi} \int \frac{d\vec{s} \times \vec{r}_{sp}}{r_{sp}^3}$	general current source ex: finite wire wire loop
Ampere's law	$\oint_{closed\ path} \vec{\mathbf{B}} \cdot d\vec{s} = \mu_0 I_{enc}$	symmetric current source ex: infinite wire infinite current sheet

How to apply Ampere's Law

1. Identify regions in which to calculate B field.
2. Choose Amperian closed path such that by symmetry B is 0 *or* constant magnitude on the closed path!
3. Calculate

$$\oint_{\substack{closed\\path}} \vec{\mathbf{B}} \cdot d\vec{s} = \begin{cases} \text{B times length} \\ 0 \end{cases}$$

4. Calculate current enclosed:

$$I_{enc} = \iint_{\substack{open\\surface}} \vec{\mathbf{J}} \cdot \hat{n} dA$$

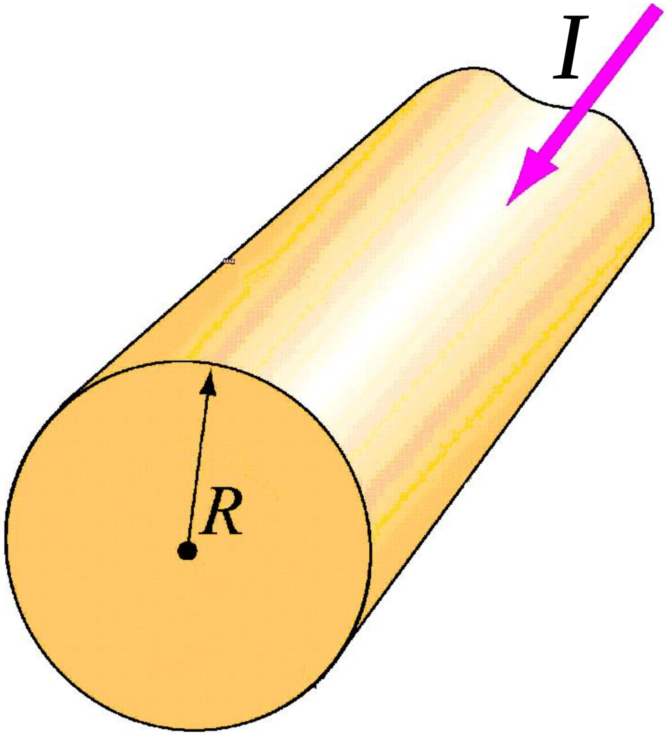
5. Apply Ampere's Law to solve for B: check signs

$$\oint_{\substack{closed\\path}} \vec{\mathbf{B}} \cdot d\vec{s} = \mu_0 I_{enc}$$

Example: Infinite Wire

- A cylindrical conductor has radius R and a uniform current density with total current I. we shall find the direction and magnitude of the magnetic field for the two regions:

  1. outside wire ($r \geq R$)
  2. inside wire ($r < R$)



1. outside wire (r ≥ R)

- Recognize that the infinite long wire is cylindrical symmetric.
- Cylindrical symmetry →
  - ✓ Amperian Circle
  - ✓ B-field counterclockwise

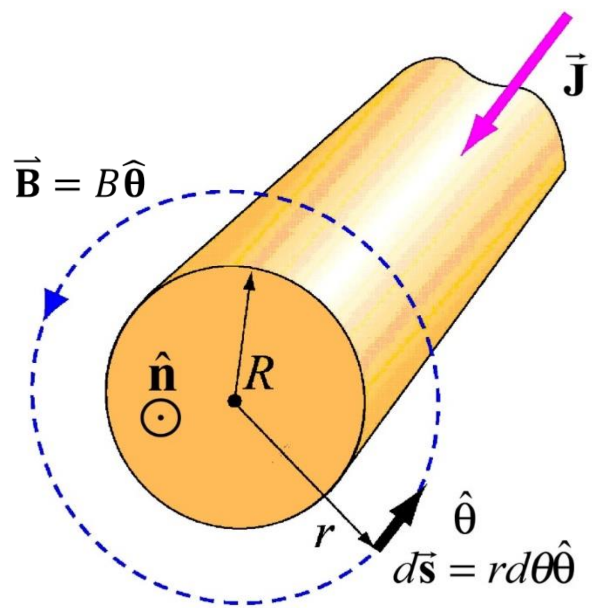
$$\oint \vec{\mathbf{B}} \cdot d\vec{s} = \mu_0 I_{enc}$$

$$\Rightarrow B \oint ds = \mu_0 I$$

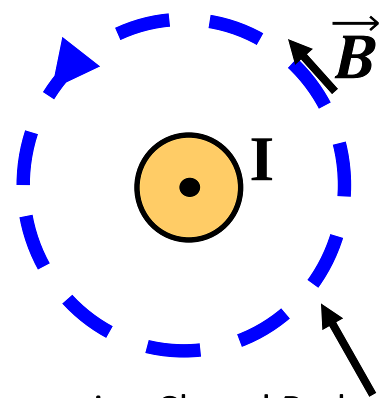
$$\Rightarrow B(2\pi r) = \mu_0 I$$

$$\Rightarrow B = \frac{\mu_0 I}{2\pi r}$$

$$\vec{\mathbf{B}} = \frac{\mu_0 I}{2\pi r} \widehat{\boldsymbol{\theta}}$$



Top view



Amperian Closed Path:

- B is Constant & Parallel
- Current penetrates surface

2. inside wire (r < R)

- For r < R, the amount of current enclosed by the closed loop is proportional to the area enclosed, i.e.

In this case, the current enclosed by the shared area,

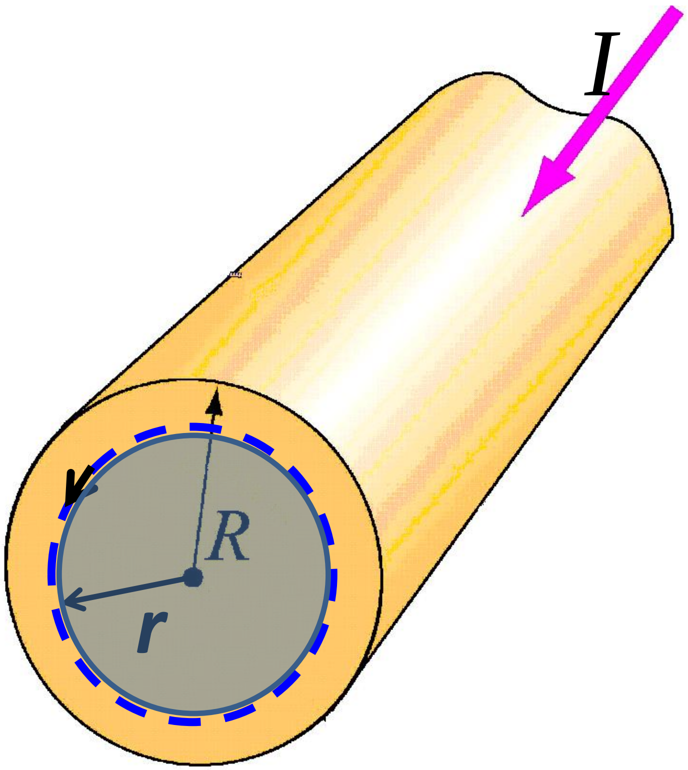
$$I_{enc} = \left[\frac{\pi r^2}{\pi R^2}\right] I$$

$$\oint \vec{\mathbf{B}} \cdot d\vec{s} = \mu_0 I_{enc}$$

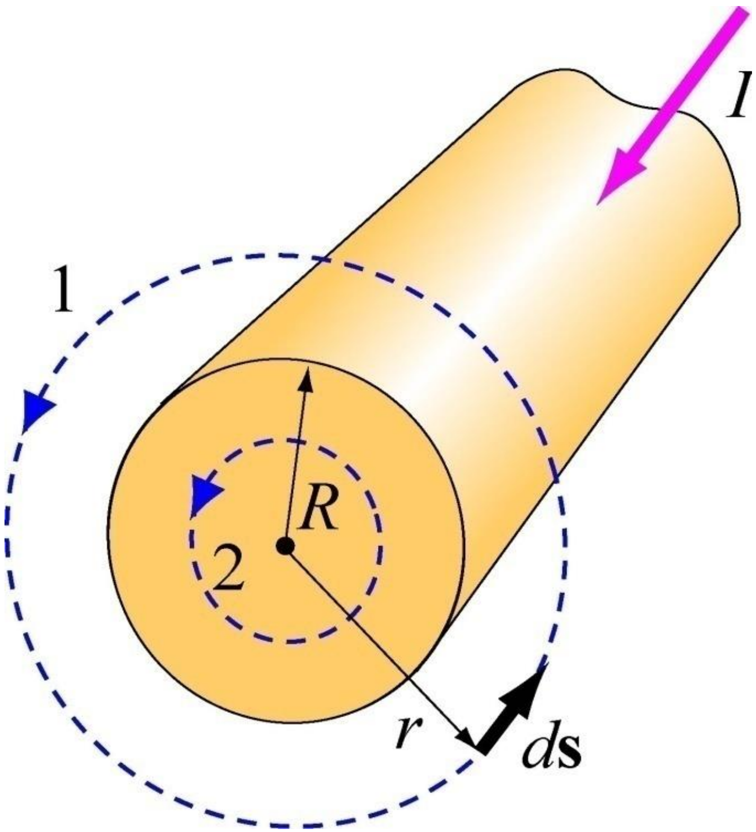
$$\Rightarrow B(2\pi r) = \mu_0 I \left[\frac{\pi r^2}{\pi R^2}\right]$$

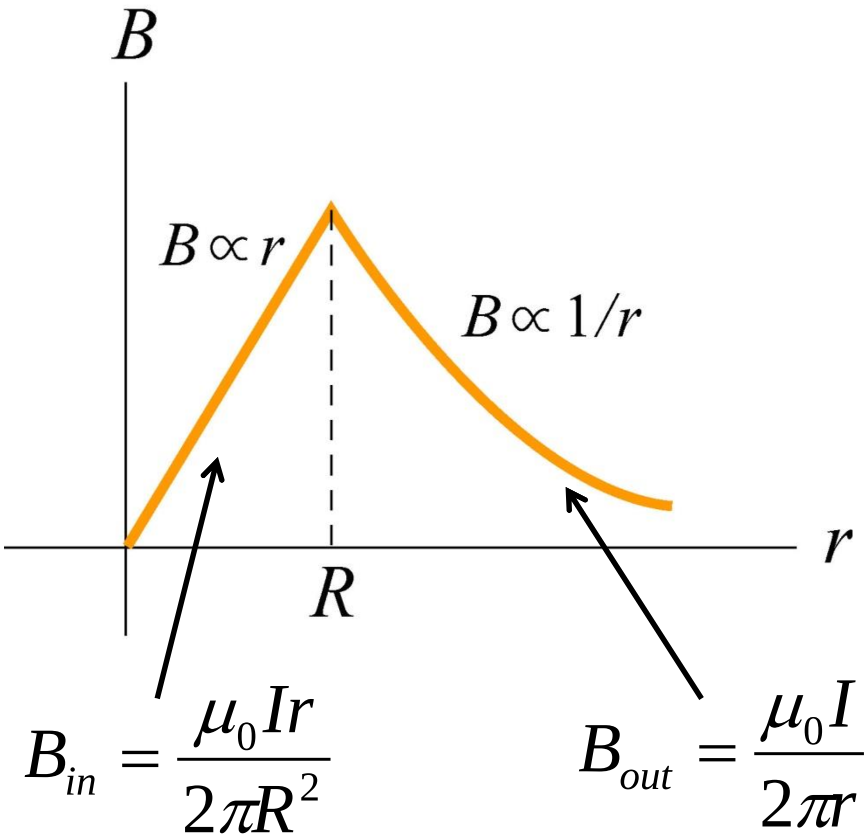
$$\Rightarrow B = \frac{\mu_0 I r}{2\pi R^2}$$

$$\vec{\mathbf{B}} = \frac{\mu_0 I r}{2\pi R^2}\hat{\boldsymbol{\theta}}$$



- Thus, the B field of an infinite wire varies with distance from axis as plotted below,





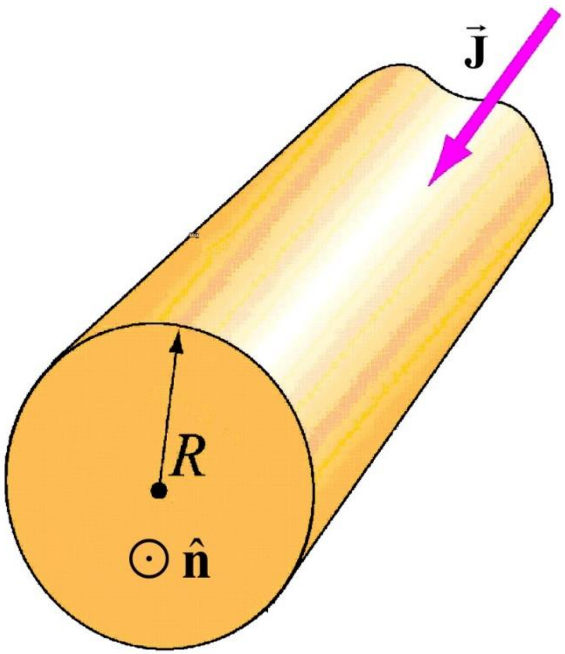
Case Problem 1.1: Non-Uniform Cylindrical Wire

A cylindrical conductor has radius R and a non-uniform current density with total current:

$$\vec{\mathbf{J}} = J_0 \frac{R}{r} \hat{\boldsymbol{n}}$$

Find B everywhere:

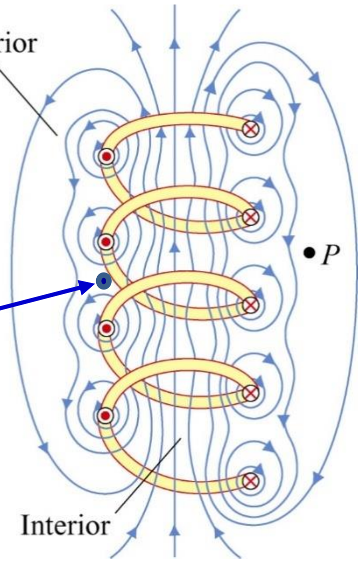
1. outside wire ($r \geq R$)
2. inside wire ($r < R$)



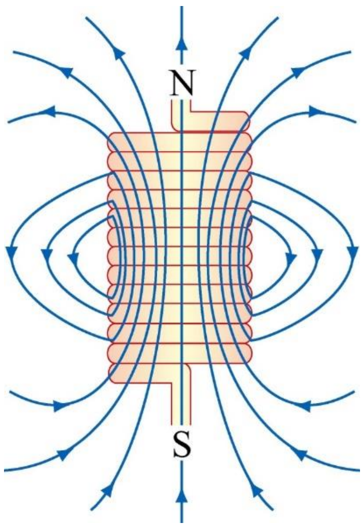
Multiple Wire Loops – Solenoid

- A solenoid is equivalent to multiple current loops stacked on top of each other along the axis. Each loop carries same current.
- Magnetic Field, B of an ideal Solenoid (tightly wound and infinitely long) is uniform inside & zero outside.
- Ampere's Law can help to calculate the B!

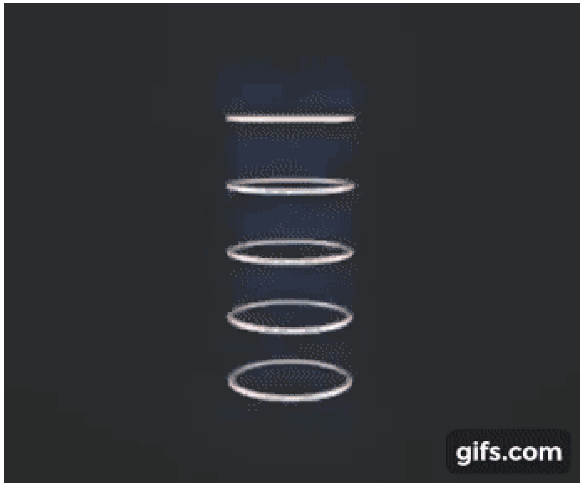
Horizontal components cancel each other right at mid point between 2 adjacent loops.



loosely wound



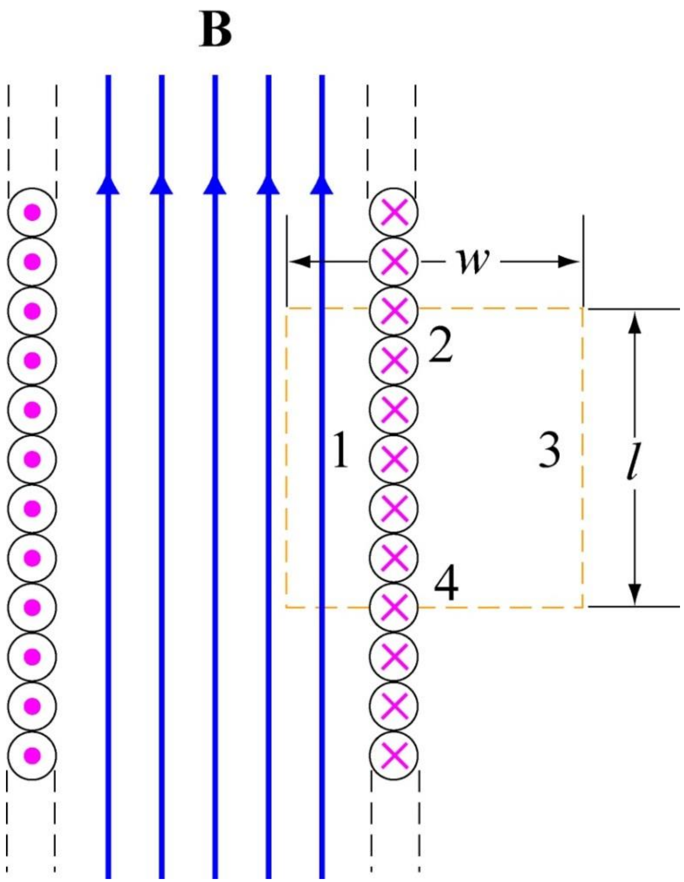
tightly wound



http://youtu.be/GI2Prj4CGZI

Magnetic Field of Ideal Solenoid

- Use Ampere’s law. (Why? What kind of symmetry is exhibited?)



$$\begin{cases} \vec{\mathbf{B}} \perp d\vec{\mathbf{s}} \text{ along sides 2 and 4} \\ \vec{\mathbf{B}} = 0 \text{ along side 3} \end{cases}$$

$$\oint \vec{\mathbf{B}} \cdot d\vec{s} = \int_1 \vec{\mathbf{B}} \cdot d\vec{s} + \int_2 \vec{\mathbf{B}} \cdot d\vec{s} + \int_3 \vec{\mathbf{B}} \cdot d\vec{s} + \int_4 \vec{\mathbf{B}} \cdot d\vec{s}$$

$$I_{enc} = nlI$$

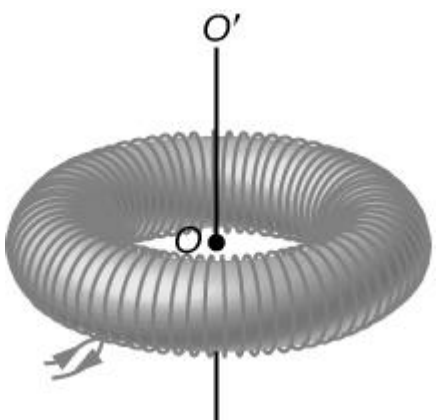
$$n = \frac{N}{L}: \text{ \# of turns per unit length}$$

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = Bl = \mu_0 nlI$$

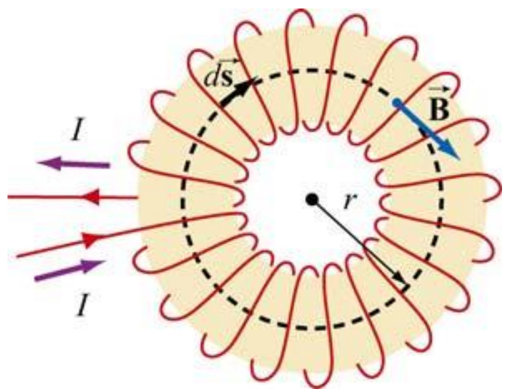
$$\boxed{B = \frac{\mu_0 nlI}{l} = \mu_0 nI}$$

Magnetic Field of a Toroid

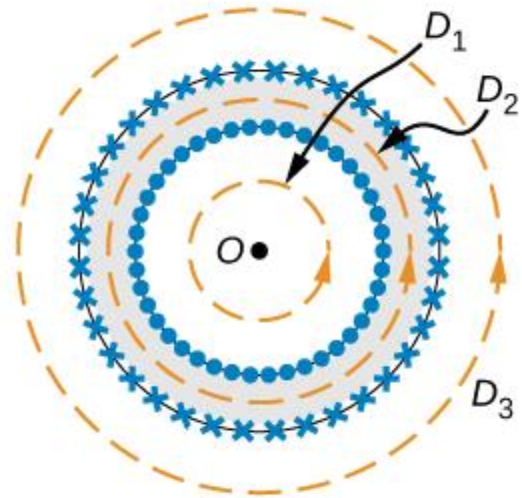
- Consider a toroid which consists of *N* turns.



A toroid is a coil wound into a donut-shaped object.



Top view



In a tightly wound toroid, cylindrical symmetry is a very good approximation. Several paths of integration for Ampère’s law.

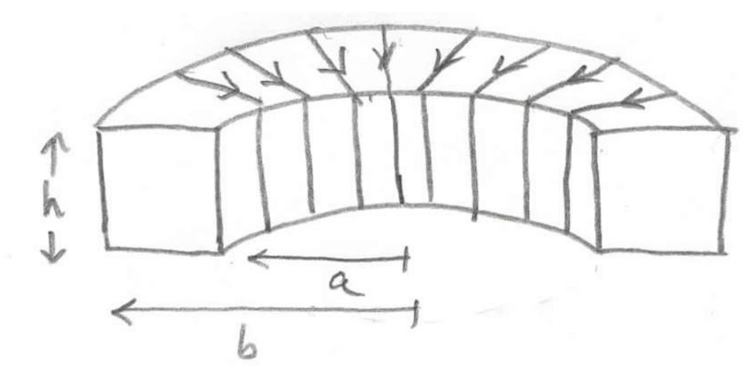
$$\oint \vec{\mathbf{B}} \cdot d\vec{s} = \oint B\, ds = B \oint ds = B(2\pi r) = \mu_0 I_{enc} = \mu_0 N I$$

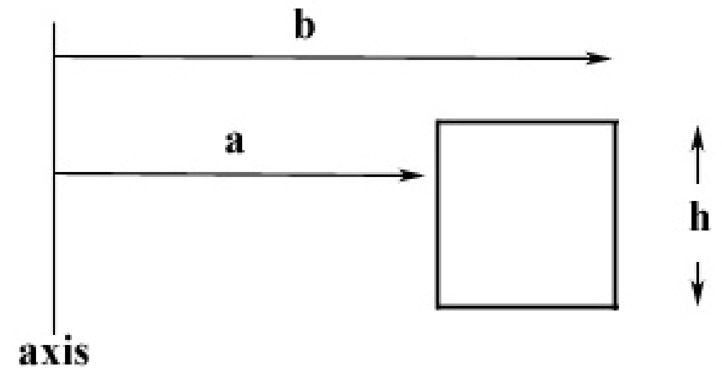
$$\boxed{B = \frac{\mu_0 N I}{2\pi r}}$$

- Unlike solenoid, the magnetic field inside the toroid is non-uniform and decreases as $1/r$.

Case Problem 1.2: Toroid

A toroid has N turns, and an inner radius a, outer radius b, and height h. The toroid has a rectangular cross section shown in the figures below.





When a current I is flowing through the toroid, what are the magnitude and direction of the magnetic field inside the toroid as a function of distance r from the axis of the toroid?

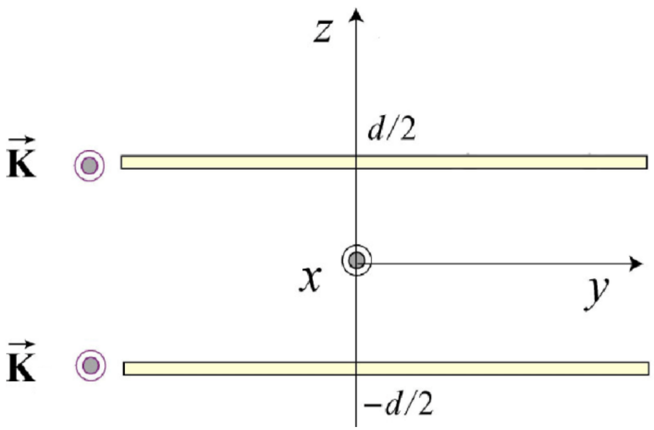
Extra Case Problem 1: Sheets of Current

- Consider two large sheets lying in the *xy*-plane separated by a distance *d* carrying identical surface current densities $\vec{K} = K\hat{\imath}$, as shown in the figure below (the sheets extend infinitely in the *y* direction). You may neglect all edge effects.
- Find the magnetic field everywhere, *i.e.*:

i. above the plates (z > d/2)

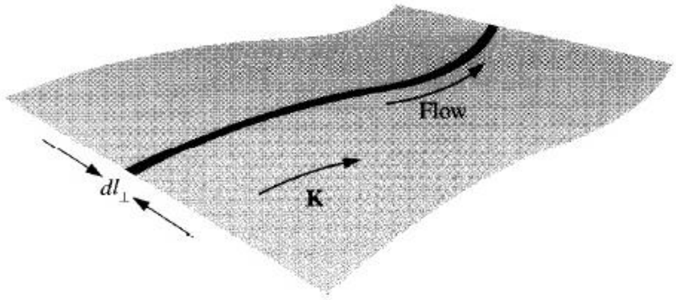
ii. between the plates (−d/2 < z < d/2),

iii. below the plates (z < d/2)



- Note: Surface current density $\vec{K}$ is defined as the current per unit width that is perpendicular to the current flow. $\vec{K} = \frac{dI}{dl_\perp}\hat{n}$, where $\hat{n}$ is the direction of the current flow. Thus, the total sheet current flow,

$$I = \int \vec{\boldsymbol{K}} \cdot \hat{n}\, dl_\perp$$



Summary

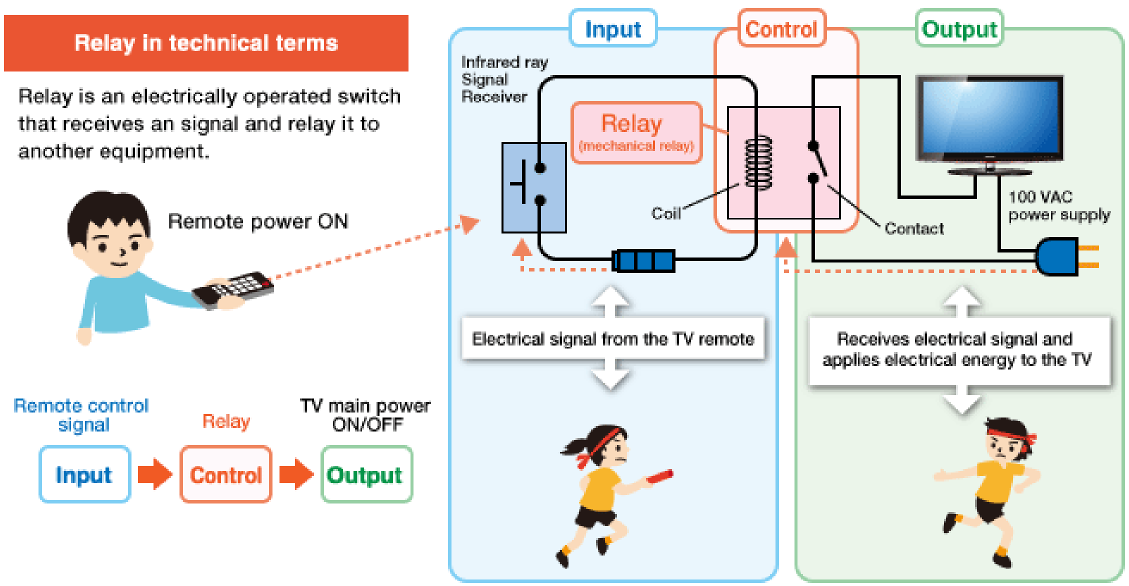
- Ampere's law to magnetic field is as Gauss's law to electric field.
- Ampere's law is true for arbitrary closed paths and current enclosed.

$$\oint_{\substack{closed \\ path}} \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \mu_0 \iint_{\substack{open \\ surface}} \vec{\mathbf{J}} \cdot \hat{\mathbf{n}} dA$$

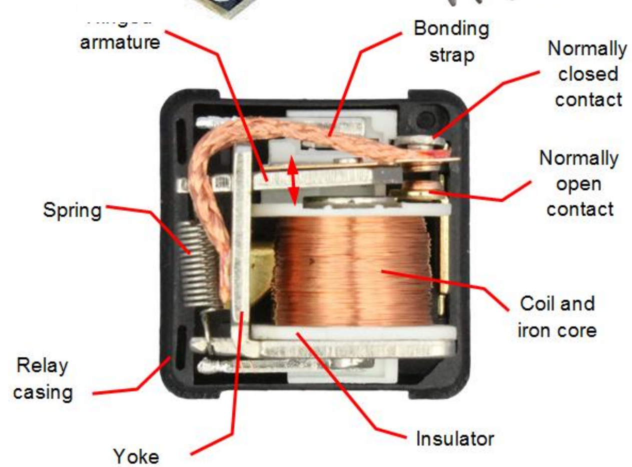
- Direction of loop and normal follows Right Hand Rule.
- Ampere's law is most practical for calculations where magnetic field has high symmetry. Choose a closed path along which the magnetic field is zero or parallel and constant. Magnitude of the magnetic field is given by the total current enclosed divided by the total Amperian path length (where magnetic field is non-zero).

Applications: Electrical Relay

- Directly control a high-power circuit is not always possible. Example: Using Arduino (microcontroller – low power device for signal) to control a standard electric fan or oven (high power devices).
- Relays are electrically operated switches that open and close the circuits by receiving electrical signals from outside sources.
- Relay consists of a coil (solenoid), which receives an electric signal and converts it to a mechanical action and contacts that open and close the electric circuit.



https://www.components.omron.com/relay-basics/basic



In-Class Worksheet