Week 10 - Day 1

Modelling of a Rechargeable Battery

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RC and RL Circuit Analysis

Concept 1: RC Circuit Analysis – Charging and Discharging Characteristic of a Capacitor in DC Circuit

Concept 2: RL Circuits Analysis

Reading:

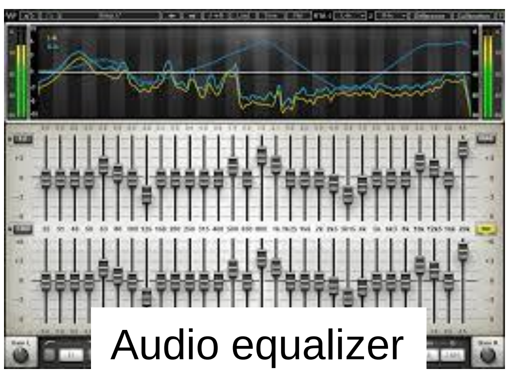
University Physics with Modern Physics – Chapter 30

Introduction to Electricity and Magnetism – Chapter 11

Concept 1: RC Circuit Analysis

Application: RC Circuits

- We can find capacitors (or RC circuits) in almost every modern circuitry.
- Examples: it has been applied in camera flashlight (LC oscillator to boost up voltage), pacemaker (oscillator/pulse generator circuits), traffic lights (timer circuits to control timing) , equalizers in audio equipment (filter circuits), and so on.
- We shall study the charging and discharging behavior of a simple RC circuit.
- A simple RC circuit can be used to model a typical charging and discharging profile of a rechargeable battery at constant voltage mode.

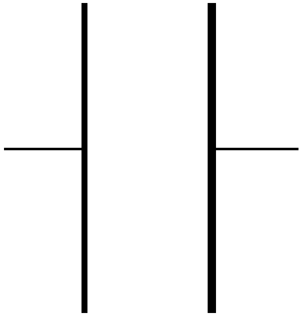

Audio equalizer


Pacemaker

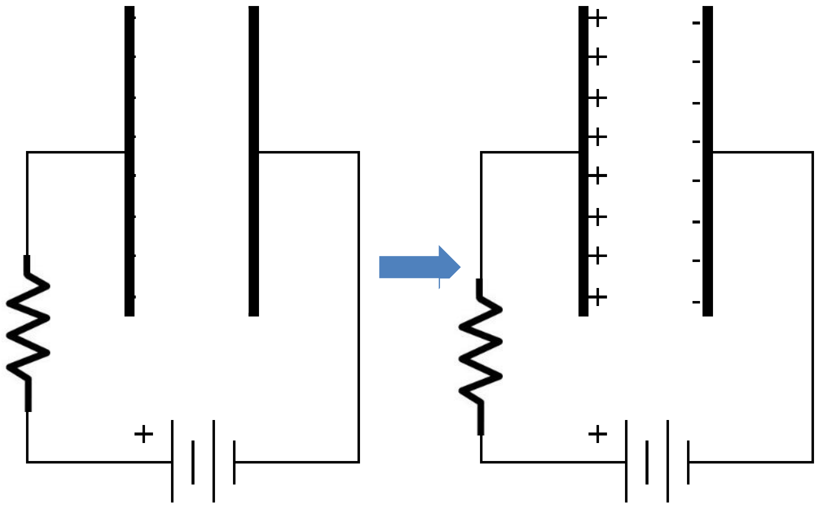

Camera flash

Some intuition on Capacitors in a circuit

Suppose you have an isolated **uncharged** capacitor



Because there are no charges, there is no electric field between the two plates, which implies no potential difference, and also no energy stored in the capacitor. The capacitor can be thought of as a (peculiar) empty battery!



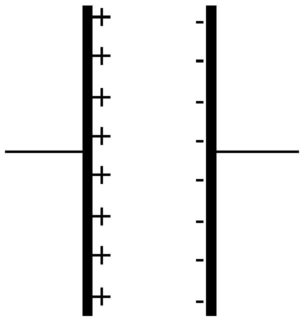
We now connect the two sides of the capacitor through a resistance and a battery. At first the battery will push a current through the wires (electrons from the battery will go to one of the plates, while electrons from the other plate will go to the battery).

This starts to form an electric field in the capacitor and a potential difference that opposes the battery. The more charges are transferred, the stronger is the opposition from the capacitor, and the smaller is the current.

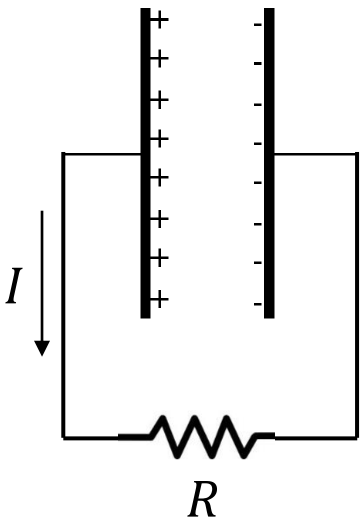
We can expect that at the beginning the current is larger, and then it decreases as the capacitor is charged and stores energy (we can then detach this capacitor from this circuit and use it somewhere else at a later time as we shall do in the next slide).

Some intuition on Capacitors in a circuit

Suppose you have an isolated **charged** capacitor



Because of the positive charges on one side and negative on the other, there is an electric field between the two plates, which implies a potential difference, and also energy stored in the capacitor. The capacitor can be thought of as a (peculiar) charged battery!



We now connect the two sides of the capacitor through a resistance. The electrons now have a way to move from one plate to the other, and they really want to do that because of the difference in potential, but

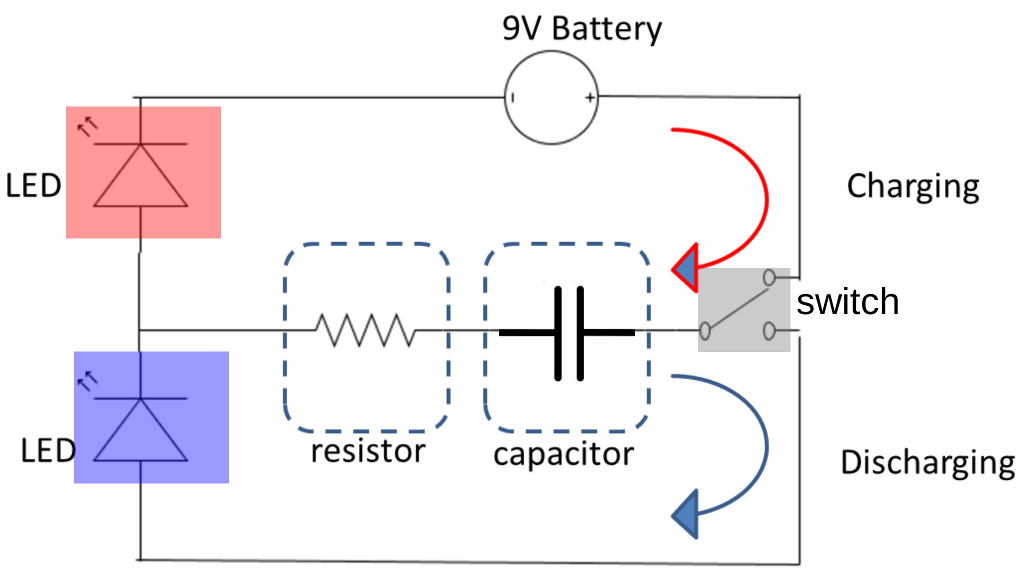
1) they will go through the resistance and energy will be lost there,
2) as the electrons leave the capacitor, the charge stored in the capacitor is lesser, (i) the electric field, (ii) the potential difference and (iii) the energy stored decrease. This means that the more they move towards the other plate, the lesser they "want" to go.

We can thus expect, as we connect a resistance, that there will first be a strong (counter-clockwise) current, and then the current diminishes.

Demonstration: RC circuit and Capacitor Charging

Observation

- When first charging the capacitor (the switch is in the position illustrated below), the red LED lights up and gradually dims down.
- It shows highest current flows through the circuit at the beginning (when the capacitor has no charge). When the capacitor is fully charged, no more current flows through the circuit.
- After you change the position of the switch, you will be discharging the capacitor, the blue LED lights up and gradually dims down.



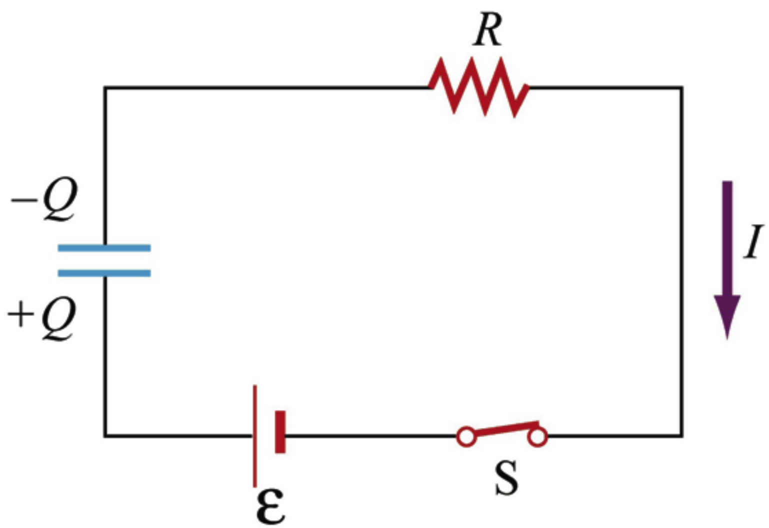
Concept Question 1.1: RC Circuit

An uncharged capacitor is connected to a battery, resistor and switch. At $t = 0$, the switch is closed. The current in the circuit right after the switch is closed $(t = 0^+)$ is

A. Nearly zero

B. At a maximum then decreasing

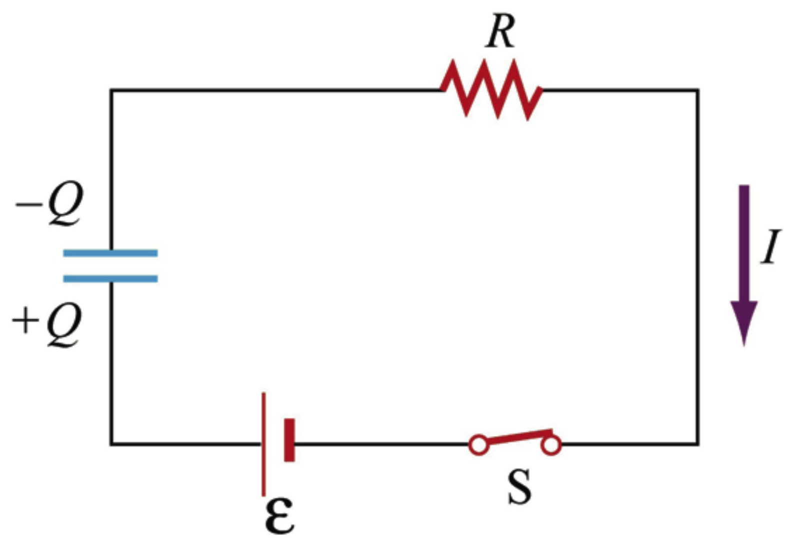
C. Nearly constant but non-zero



Multiple Choice

Concept Question 1.1 (Solution)

- Answer: B. At a maximum then decreasing
- When the switch is closed at $t = 0$, the capacitor looks like a short circuit. At $t = 0^+$, the current $I = \frac{\varepsilon}{R}$ (Ohm's Law).
- But very quickly, the current will be decreasing, while the capacitor is charging.



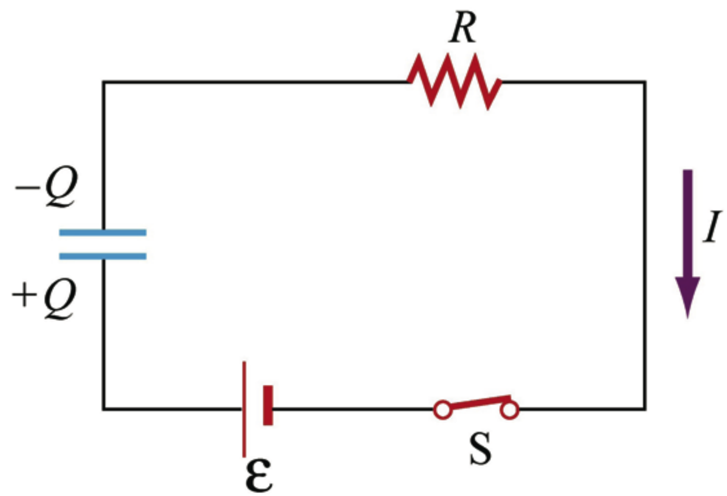
Concept Question 1.2: RC Circuit

An uncharged capacitor is connected to a battery, resistor and switch. The switch is initially open, but it is closed at t = 0. A very long time after the switch is closed, the current in the circuit is

A. Nearly zero

B. At a maximum and decreasing

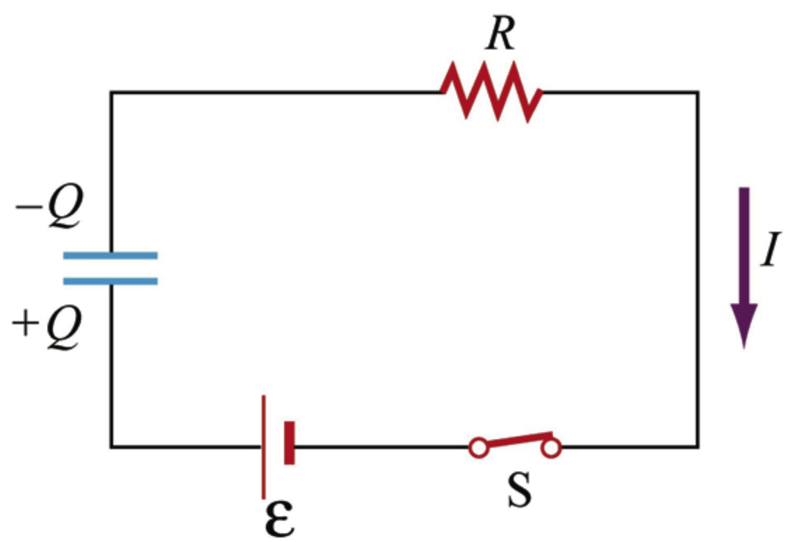
C. Nearly constant but non-zero



Multiple Choice

Concept Question 1.2 (Solution)

- Answer: A. After a long time, the current is 0
- Eventually the capacitor gets "completely charged" – the voltage increase provided by the battery is equal to the voltage drop across the capacitor. The voltage drop across the resistor at this point is 0 – no current is flowing.
- Note: Fully charged capacitor looks like an open circuit.



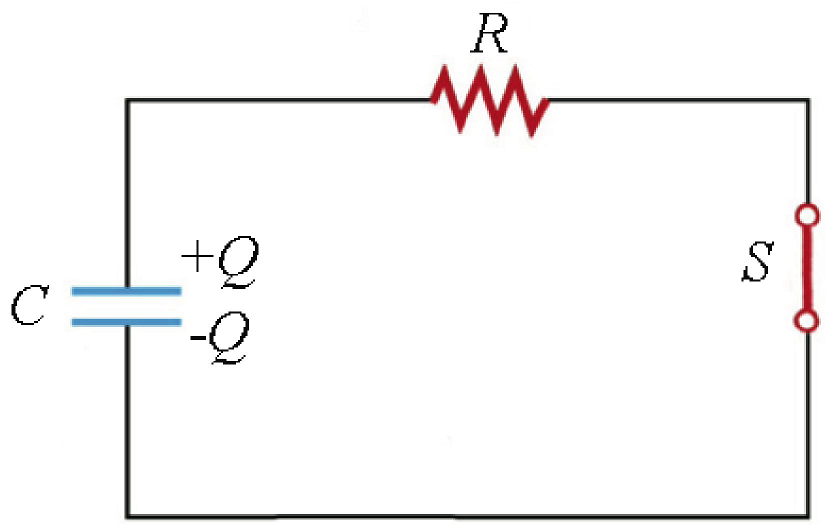
Concept Question 1.3:

The capacitor is initially fully charged to Q, with $\Delta V_c = \varepsilon$. At $t = 0$, the switch is closed. The current in the circuit is

A. Nearly zero

B. At a maximum and decreasing

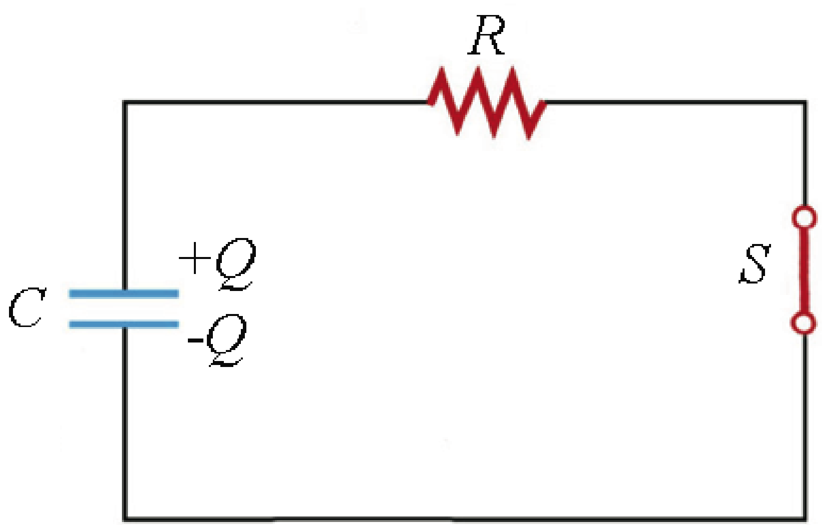
C. Nearly constant but non-zero



Multiple Choice

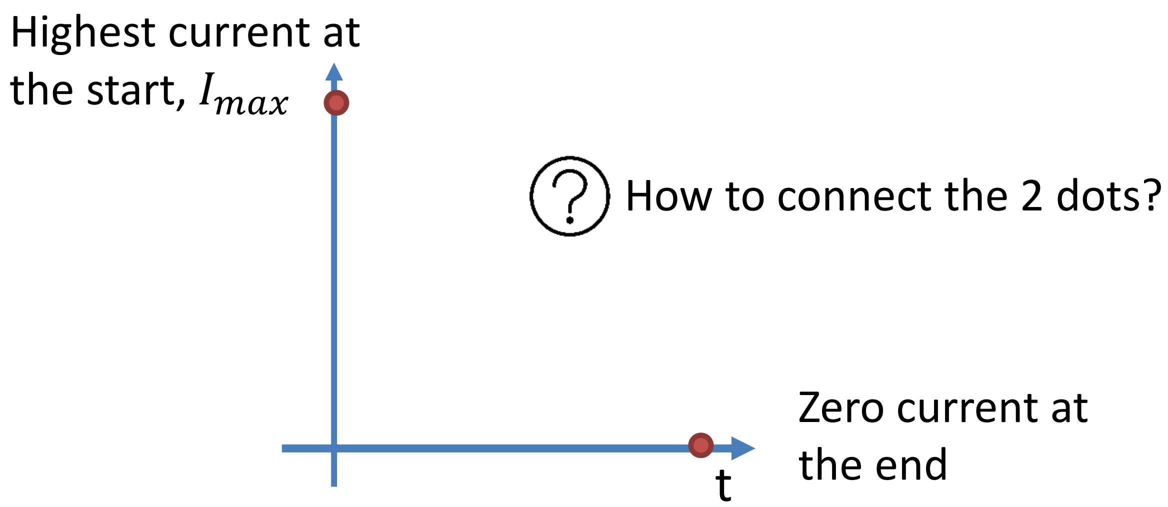
Concept Question 1.3 (Solution)

- Answer: B. At a maximum and decreasing
- At $t = 0^+$, right after the switch is closed, the fully charged capacitor acts like a battery that supplies a voltage ε. The current $I(t = 0^+) = \frac{\varepsilon}{R}$.
- However, while the capacitor is discharging (losing charge through the circuit), the voltage ΔV_c can no longer sustain at ε but decreasing over time. So do the current I.



Now we know:

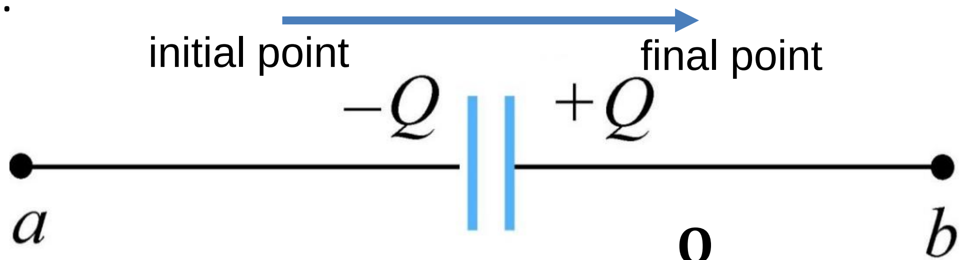
- Regardless charging or discharging condition, it is always the highest current flows at the start and zero current at the end.
- Question: How much the current flows during the process? What is the current flow vs. time?
- Strategy: We apply KVL to get the circuit equation, which we can try to solve how much the charge stored in the capacitor as a function of time. Then by differentiating Q(t), we can get I(t).



Sign Conventions of Capacitor

The positively charged plate has a higher potential and vice versa.

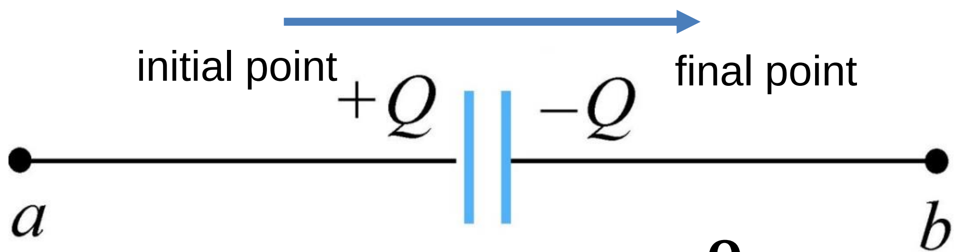
When moving across a capacitor from the negatively to positively charged plate, the change in potential **increases**.



$V_b > V_a$

$$\Delta V = V_b - V_a = +\frac{\mathbf{Q}}{\mathbf{C}}$$

Likewise, when moving across a capacitor from positively to negatively charged plate, the change in potential **decreases**:



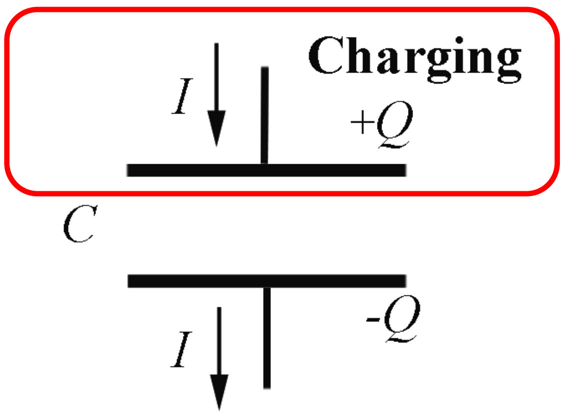
$V_b < V_a$

$$\Delta V = V_b - V_a = -\frac{\mathbf{Q}}{\mathbf{C}}$$

Tip: think of the direction of the electric field (from positive to negative charges). The potential increases in the opposite direction of the electric field.

Charging and Discharging Conditions

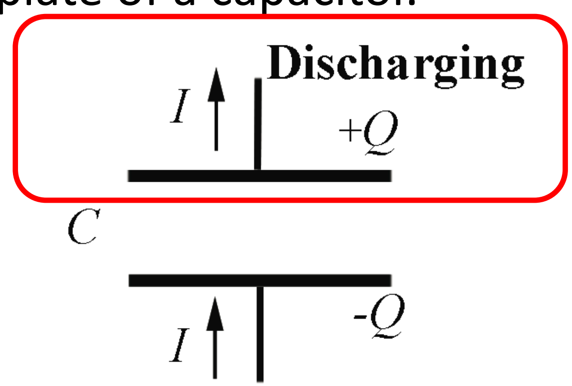
- For charging condition, the direction of current flow is toward the positive plate of a capacitor.



$$I = +\frac{dQ}{dt}$$

Positive sign means the charge in the capacitor is increasing

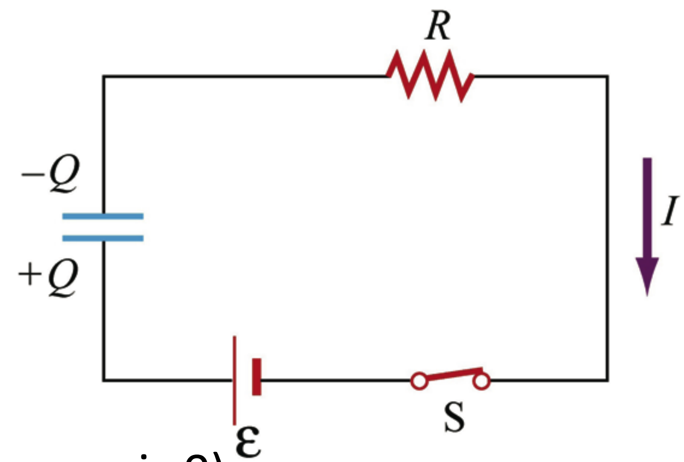
- For discharging condition, the direction of current flow is away from the positive plate of a capacitor.



$$I = -\frac{dQ}{dt}$$

Negative sign means the charge in the capacitor is decreasing

Example: Charging a Capacitor



Using the sign convention and KVL under a charging condition, setup the circuit equation in terms of Q.

Hint: For charging condition, $I = +\frac{dQ}{dt}$

Circulate clockwise or ccw (sum of the potential differences is 0)

$$\sum_i \Delta V_i = \varepsilon - \frac{Q}{C} - IR = 0$$

Substitute $I = +\frac{dQ}{dt}$, then $\varepsilon - \frac{Q}{C} - \frac{dQ}{dt}R = 0$

Hence we get a first order linear inhomogeneous differential equation in Q

$$\frac{dQ}{dt} = -\frac{1}{RC}(Q - C\varepsilon)$$

(you do not need to remember this equation … just think about the two, fairly natural, steps of KVL to derive it)

Solving Q(t), as a function of time

- Using separation of variable, bring variable Q and dQ to 1 side and dt to the other side.

$$\frac{dQ}{dt} = -\frac{1}{RC}(Q - C\varepsilon) \rightarrow \frac{1}{Q - C\varepsilon}dQ = -\frac{1}{RC}\,dt$$

- At t=0, Q in the capacitor is 0. At t, charge is Q(t). Apply integration and the limits for both sides.

$$\int_0^{Q(t)} \frac{1}{Q - C\varepsilon}dQ = \int_0^t -\frac{1}{RC}\,dt$$

$$\ln\left(\frac{Q(t) - C\varepsilon}{-C\varepsilon}\right) = -\frac{t}{RC}$$

$$\frac{Q(t) - C\varepsilon}{-C\varepsilon} = \exp\left(-\frac{t}{RC}\right)$$

$$Q(t) = C\varepsilon\left(1 - \exp\left(-\frac{t}{RC}\right)\right)$$

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Understanding time constant, τ

For the differential equation: $\frac{dQ}{dt} = -\frac{1}{RC}(Q - C\varepsilon)$, the solution when switch is closed at t = 0 (and it is initially uncharged) is

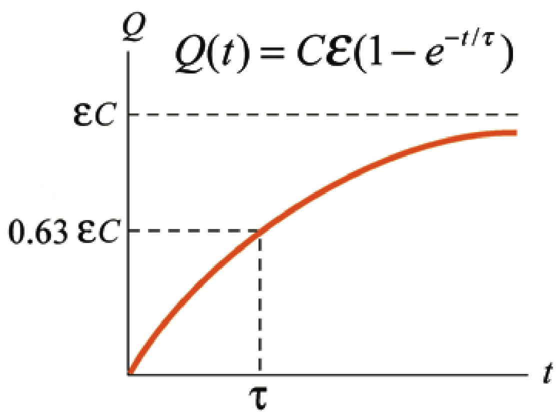
$$Q(t) = C\varepsilon\left(1 - \exp\left(-\frac{t}{RC}\right)\right)$$

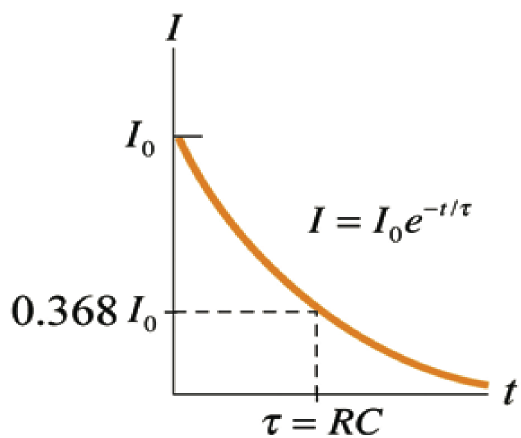
We set RC as τ (check that RC has indeed units of time). Thus, $Q(t) = C\varepsilon\left(1 - e^{-t/\tau}\right)$

The current is the given by the derivative of the charge:

$$I(t) = +\frac{dQ(\mathrm{t})}{dt} \Rightarrow I(t) = \frac{C\varepsilon}{\tau}e^{-t/\tau} = \frac{\varepsilon}{R}e^{-t/\tau} = I_o e^{-t/\tau}$$

- We call $\tau = RC$ as time constant of the RC (resistor-capacitor) circuit (units: seconds).
- It represents time required to reach $(1 - e^{-1}) \approx 63.2\%$ of its final (asymptotic) value in an increasing system OR time required to decrease to $e^{-1} \approx 36.8\%$ of its initial value in a decaying system. The value gives us an intuition how fast the circuit will charge or discharge.

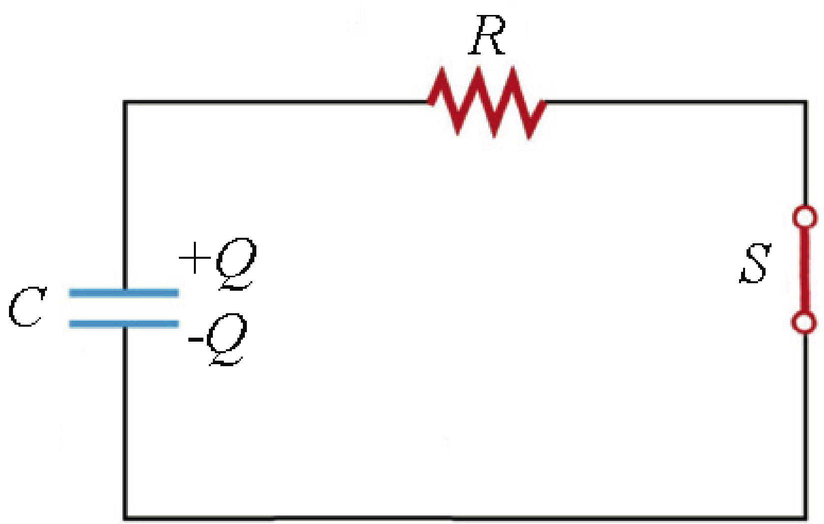




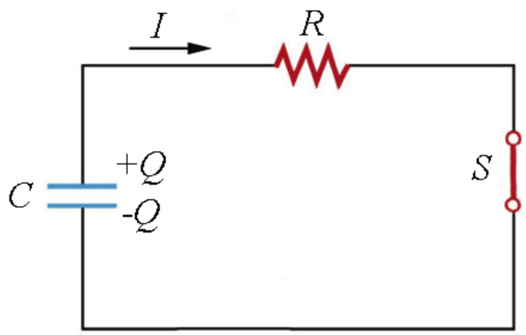
Case Problem 1.1: Discharging A Capacitor

- At t = 0 charge on capacitor is Q_0. What happens when we close switch S at t = 0?

1. Setup the circuit equation.
2. Solve for charge in the capacitor, $Q(t)$ and the current flow in the circuit, $I(t)$, as a function of time?

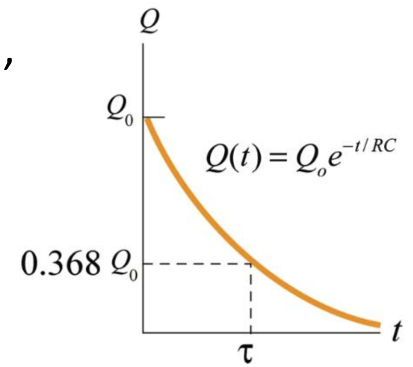


Case Problem 1.1 (Solution)



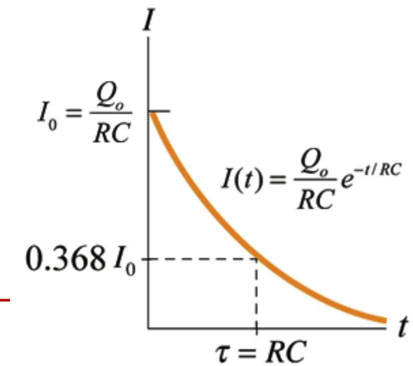
- Circulate clockwise:
- $\sum_i \Delta V_i = \frac{Q}{C} - IR = 0$
- $I = -\frac{dQ}{dt} \Rightarrow \frac{dQ}{dt} = -\frac{Q}{RC}$
- In general, $\frac{dA}{dt} = -\frac{1}{\tau}A$ is called the first order linear differential equation.
- Solving $\frac{dQ}{dt} = -\frac{1}{RC}Q$ using separation of variable (again) $\Rightarrow$ $\frac{1}{Q}dQ = -\frac{1}{RC}dt$.
- Applying the limits, when switch is closed at t = 0, charge in C is Q_o, at time t, charge in C is $Q(t)$.

$$\int_{Q_o}^{Q(t)} \frac{1}{Q}dQ = -\frac{1}{RC}\int_0^t dt \rightarrow \mathrm{Q(t)} = \mathrm{Q_o e}^{-\frac{t}{RC}}$$



- Again, we see RC, thus define $\tau = RC$ as time constant.

$$I = -\frac{dQ}{dt} \Rightarrow I(t) = \frac{Q_o}{\tau}e^{-\frac{t}{\tau}} = \frac{Q_o}{RC}e^{-\frac{t}{\tau}} = \frac{\Delta V_o}{R}e^{-\frac{t}{\tau}} = I_o e^{-\frac{t}{\tau}}$$



- Recall the meaning of τ in a decaying system.

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Important Concepts

For charging:

- At $t = 0$, an empty capacitor looks like a short circuit.
- After long period of time ($t \to \infty$), a fully charged capacitor looks like an open circuit.

For discharging:

- At $t = 0$, the capacitor maintains the same voltage right before the discharge starts.
- After long period of time, the voltage across the capacitor is 0, Q=0, I=0.

Concept Question 1.4: Current Through Capacitor

In the circuit below the switch is closed at $t = 0$. At $t = 0^+$, the current through the (uncharged) capacitor, I_c and the current through the lower resistor, I_R will be:

A. $I_c = 0\ ;\ I_R = 0$

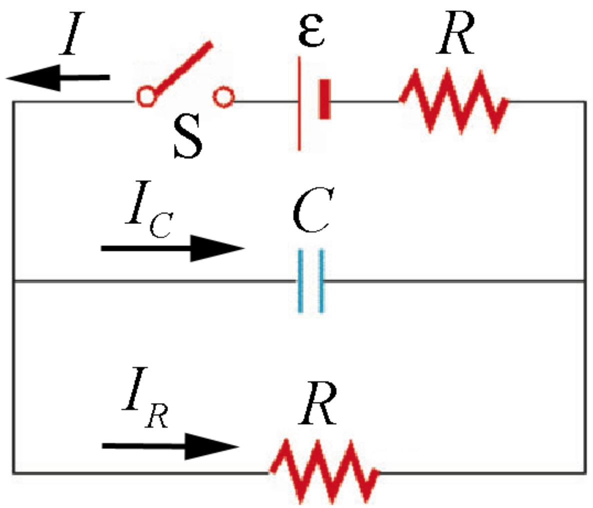
B. $I_c = 0\ ;\ I_R = \frac{\varepsilon}{2R}$

C. $I_c = \frac{\varepsilon}{R}\ ;\ I_R = 0$

D. $I_c = \frac{\varepsilon}{R}\ ;\ I_R = \frac{\varepsilon}{2R}$

E. $I_c = \frac{\varepsilon}{2R}\ ;\ I_R = 0$

F. $I_c = \frac{\varepsilon}{2R}\ ;\ I_R = \frac{\varepsilon}{2R}$

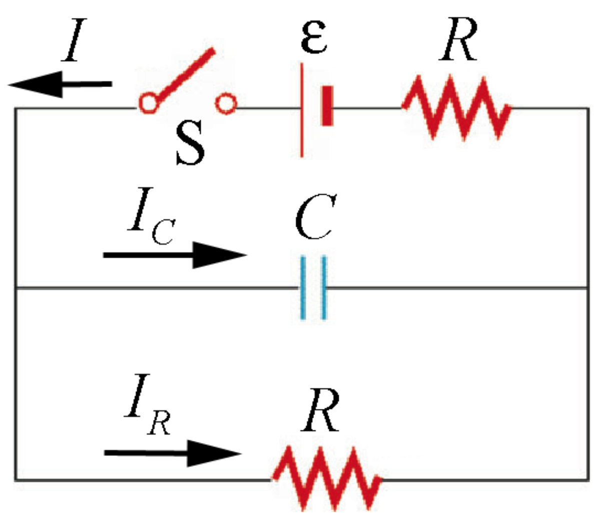


Multiple Choice

Concept Question 1.4 (Solution)

- Answer: C. $I_c = \frac{\varepsilon}{R}$; $I_R = 0$

- The capacitor will act as a short circuit initially at $t = 0^+$. So, all the current will flow through it (and none through the lower resistor).
- The battery only sees the resistor in the upper branch.
- Thus, at $t = 0^+$, $I_C = \frac{\varepsilon}{R}$, $I_R = 0$.



Concept Question 1.5: Current Through Capacitor

In the circuit below the switch is closed at $t = 0$. At $t = \infty$ (long after), the current through the capacitor and the voltage across the capacitor will be:

A. $I_c = 0\,;\ \Delta V_c = \varepsilon$

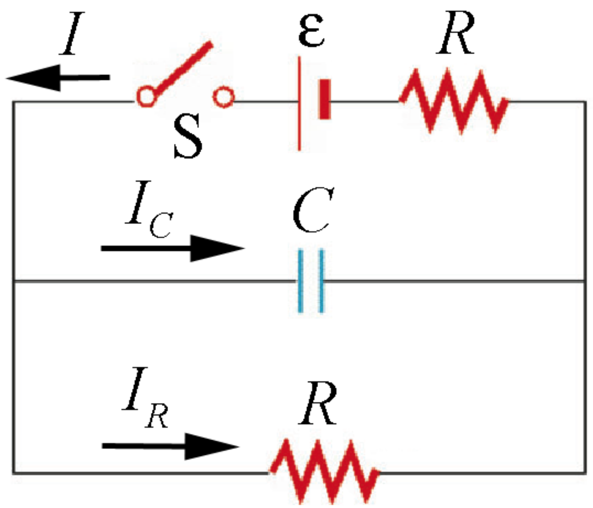
B. $I_c = 0\,;\ \Delta V_c = \frac{\varepsilon}{2}$

C. $I_c = \frac{\varepsilon}{R}\,;\ \Delta V_c = \varepsilon$

D. $I_c = \frac{\varepsilon}{R}\,;\ \Delta V_c = \frac{\varepsilon}{2}$

E. $I_c = \frac{\varepsilon}{2R}\,;\ \Delta V_c = \varepsilon$

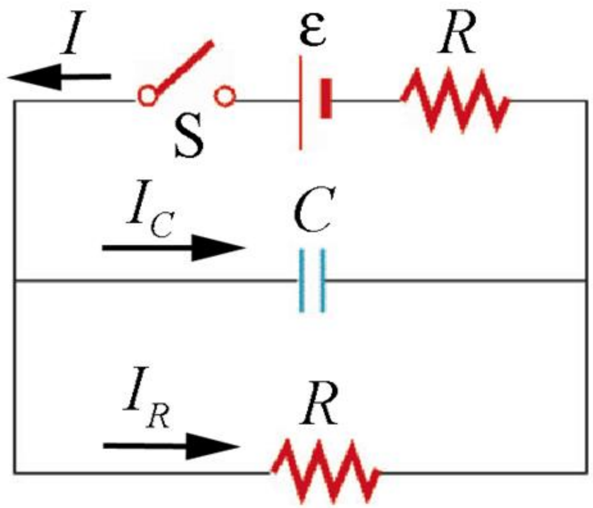
F. $I_c = \frac{\varepsilon}{2R}\,;\ \Delta V_c = \frac{\varepsilon}{2}$



Multiple Choice

Concept Question 1.5 (Solution)

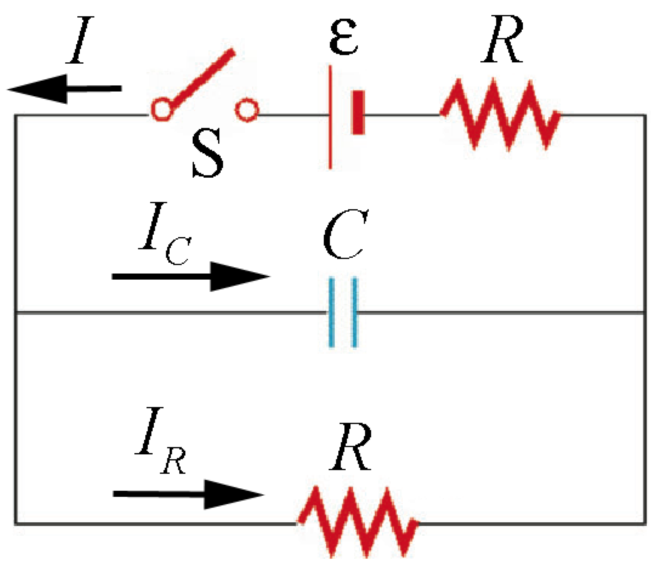
- Answer: B. $I_c = 0$; $\Delta V_c = \frac{\varepsilon}{2}$
- After a long time, the capacitor becomes "fully charged." It will look like a break in the circuit. No more current flows through it.
- This means that all the current flows through the bottom resistor and the battery is sending current through two resistors in series for an effective resistance of 2R: at $t = \infty$, $I_C = 0$, $I_R = \frac{\varepsilon}{2R}$.
- The C is in parallel with a lower R. Thus, ΔV_c is equal to the voltage across R, $\Delta V_R = I_R R$, which is $\frac{\varepsilon}{2}$ (It is literally a potential divider circuit).



Concept Question 1.6: Open Switch in RC Circuit

After the switch has been closed for a very long time, now it is opened. What happens to the current through the lower resistor?

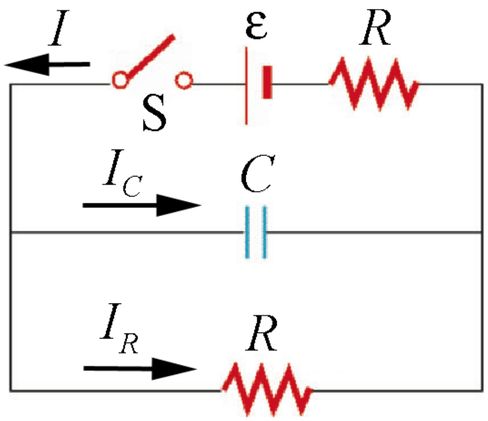
A. It stays the same then decreasing.
B. Same magnitude, flips direction, then decreasing.
C. It is cut in half, same direction, then decreasing.
D. It is cut in half, flips direction, then decreasing.
E. It stays the same for long period of time.
F. Same magnitude, flips direction for long period of time.



Multiple Choice

Concept Question 1.6 (Solution)

- Answer: A. It stays the same then decreasing.
- Immediately the switch is open, the upper branch of the circuit is disconnected. At this instant, the capacitor starts to discharge through the bottom resistor.
- The capacitor has been charged to a potential of $\Delta V_c = \frac{\varepsilon}{2}$, so $I_R = \frac{\Delta V_C}{R} = \frac{\varepsilon}{2R}$ in the same direction as before (its positive terminal is also on the left).
- Note: The current through the capacitor during discharging must flow in the opposite direction of the charging current. Thus, $I_c = -\frac{\varepsilon}{2R}$, right after the switch is open for the discharging.
- The currents decrease while the capacitor is discharging.



Case Problem 1.2: RC Circuit

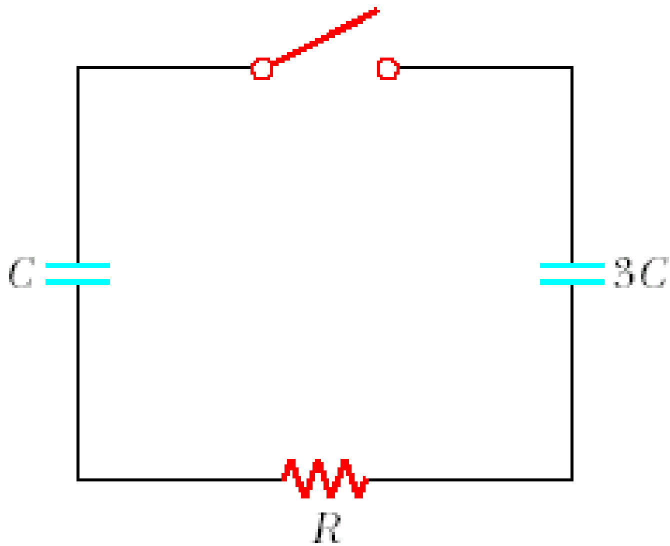
- A charge Q is placed on a capacitor of capacitance C. The capacitor is connected into the circuit shown in the figure below, with an open switch, a resistor, and an initially uncharged capacitor of capacitance $3C$. The switch is then closed, and the circuit comes to equilibrium. In terms of Q and C, find the following:

(a) the charge on each capacitor;

(b) the final potential difference between the plates of each capacitor;

(c) the final energy stored in each capacitor;

(d) the energy dissipated in the resistor.



Case Problem 1.2 (Solution)

(a) The charge on each capacitor;
When the switch is closed, let the charges distributed over $C_1 = C$ and $C_2 = 3C$ be Q_1 and Q_2, respectively, with $Q_1 + Q_2 = Q$. Equilibrium condition implies

$$\Delta V_1 = \Delta V_2 \quad \Rightarrow \quad \frac{Q_1}{C_1} = \frac{Q_2}{C_2}$$

From the two equations, the charges on the plates are found to be $Q_1 = Q/4$ and $Q_2 = 3Q/4$.

(b) The final potential difference between the plates of each capacitor;
Thus, the final potential difference between the plates of each capacitor is

$$\Delta V_1 = \Delta V_2 = \frac{Q}{4C}$$

(c) The final energy stored in each capacitor;
The energy stored in each capacitor is

$$U_1 = \frac{1}{2}C_1(\Delta V_1)^2 = \frac{1}{2}C\left(\frac{Q}{4C}\right)^2 = \frac{Q^2}{32C}$$

$$U_2 = \frac{1}{2}C_2(\Delta V_2)^2 = \frac{1}{2}(3C)\left(\frac{Q}{4C}\right)^2 = \frac{3Q^2}{32C}$$

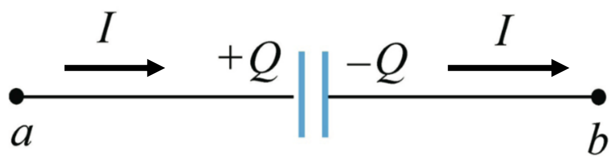
(d) The energy dissipated in the resistor.

The total final energy is $U = U_1 + U_2 = \dfrac{Q^2}{32C} + \dfrac{3Q^2}{32C} = \dfrac{Q^2}{8C}$. The change in potential energy is equal to the energy dissipated in the resistor:

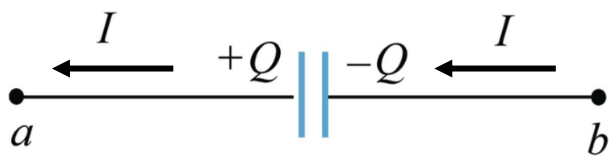
$$U_R = \frac{Q^2}{2C} - \frac{Q^2}{8C} = \frac{3Q^2}{8C}$$

Power of Capacitor

- In the charging condition (current flows to the positive plate), the capacitor **absorbs** power (stores charge).
- In the discharging condition (current flows away from the positive plate), the capacitor **releases** power (loses charge).



charging



discharging

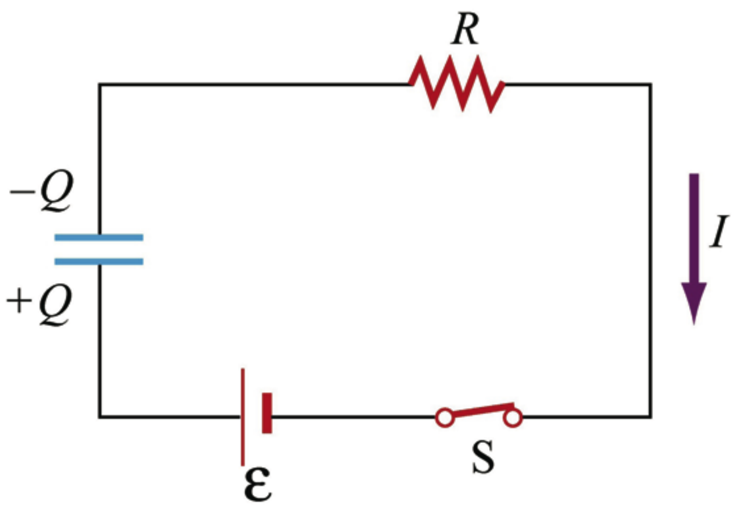
- The magnitude of the power (either absorbing or releasing) can be calculated:

$$|P| = |I(\Delta V)| = \left(\frac{dQ}{dt}\right)\left(\frac{Q}{C}\right) = \frac{d}{dt}\left(\frac{Q^2}{2C}\right) = \frac{dU}{dt}$$

- Note: $dQ^2 = Q\,dQ$

Energy Balance: Circuit Equation

- From KVL: $\varepsilon - \frac{Q}{C} - IR = 0$
- For charging condition , $I = +\frac{dQ}{dt}$
- Multiplying by $I = +\frac{dQ}{dt}$
- $\varepsilon I = I^2 R + \frac{Q}{C}\left(\frac{dQ}{dt}\right) = I^2 R + \frac{d}{dt}\left(\frac{1}{2}\frac{Q^2}{C}\right)$



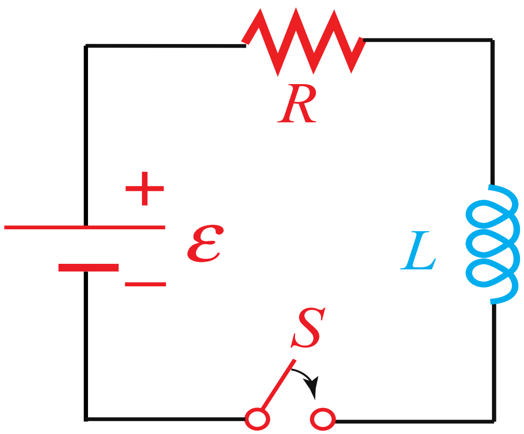
(power delivered by battery) = (power dissipated through resistor)

+ (power absorbed by the capacitor)

Concept 2: RL Circuits

Inductors in Circuits (LR Circuits)

- Inductor: a passive electronic component which is capable of storing energy in a magnetic field when electric current flows through it.
- We can always find inductors in a modern electrical circuit.
- Symbol: 
- Example: Transformer, electric motor, etc. can be modelled as inductors. It is particularly important in AC circuits (not cover in our syllabus).



Faraday's Law: Non-Static Fields

$$\oint \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{s}} = -\frac{d\Phi_B}{dt}$$

- $\vec{\boldsymbol{E}}$ is no longer a static field.
- $\vec{\boldsymbol{E}}$ is time-varying, i.e. $\vec{E}$.
- Note: For static E-field, $\oint \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{s}} = 0$

Recall: Kirchhoff's Voltage Law (2nd Rule)

Recall:

$$\Delta V = -\int \vec{E} \cdot d\vec{s}$$

$$\sum_i \Delta V_i = -\oint \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{s}}$$

Due to static field: $\oint \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{s}} = 0, \sum_i \Delta V_i = 0$. We apply this principle in circuit analysis (KVL)

- But for time-varying electric field $E(t)$,

$$\oint \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{s}} = -\frac{d\Phi_B}{dt}$$

which is definitely not zero when you have a changing magnetic field.

How should we modify Kirchhoff's Voltage law (2nd rule) to accommodate time-varying electric field?

Kirchhoff's Modified Voltage Law

$$\Delta V = -\int \vec{E} \cdot d\vec{s}$$

$$\sum_i \Delta V_i = -\oint \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{s}} = +\frac{d\Phi_B}{dt}$$

$$\sum_i \Delta V_i - \frac{d\Phi_B}{dt} = 0$$

- Consider all the magnetic flux changes are '**localized**' in inductors:

$$\boxed{\sum_i \Delta V_i - L\frac{dI}{dt} = 0}$$

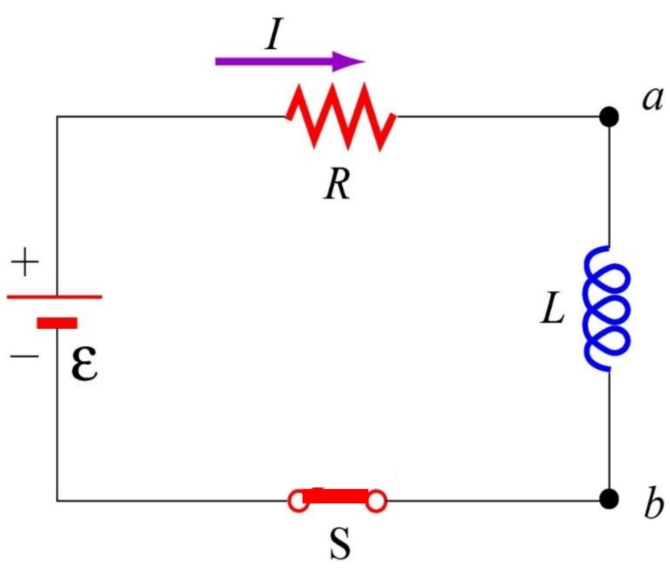
Ideal Inductor

Concept: **emf** generated by an inductor is **NOT** a voltage drop across the inductor!

Inductor acts like a battery: $\varepsilon = -L\dfrac{dI}{dt}$

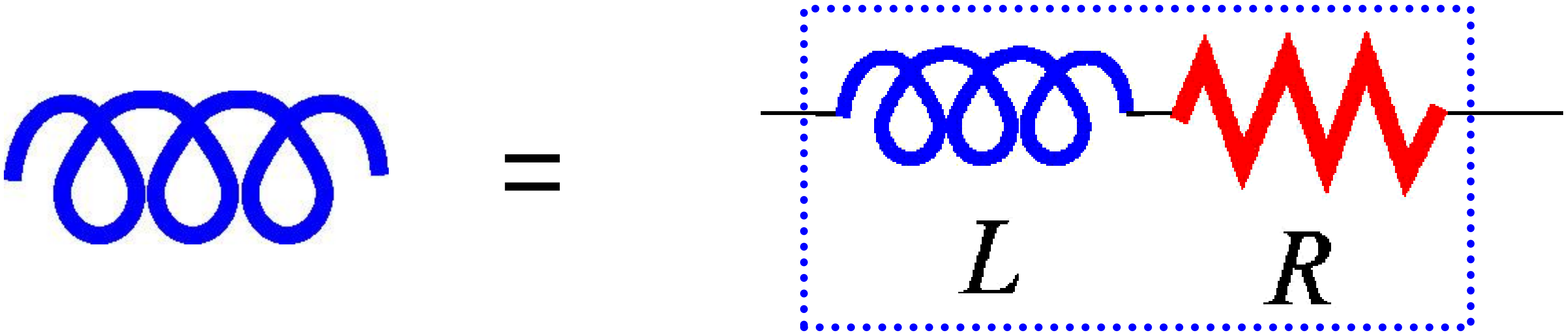
$$\Delta V_L = \int_{\substack{ideal \\ inductor}} \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{s}} = 0$$

- Because resistance is 0, E must be 0!



FYI: Non-Ideal Inductors

- Non-Ideal (Real) Inductor: Not only L but also some R.



- In the direction of current:

$$e = -L\frac{dI}{dt} - IR$$

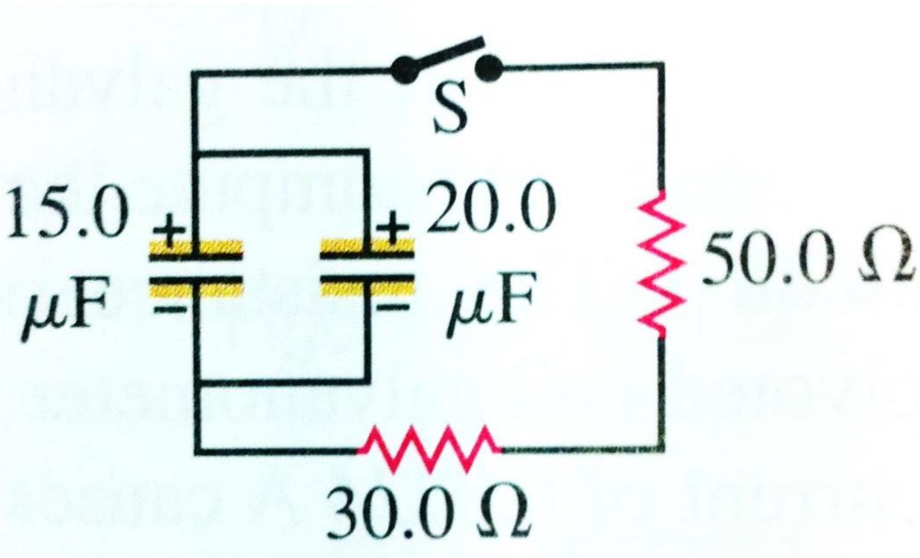
Appendix

Extra Case Problem: RC Circuit

In the circuit below, both capacitors are initially charged to 45 V.

i. What will be the current at the time when $\Delta V_C = 45V$?

ii. How long after closing the switch will the potential across each capacitor be reduced to 10.0 V?



Extra Case Problem (Solution)

Note: The circuit can be simplified into simple RC circuit.

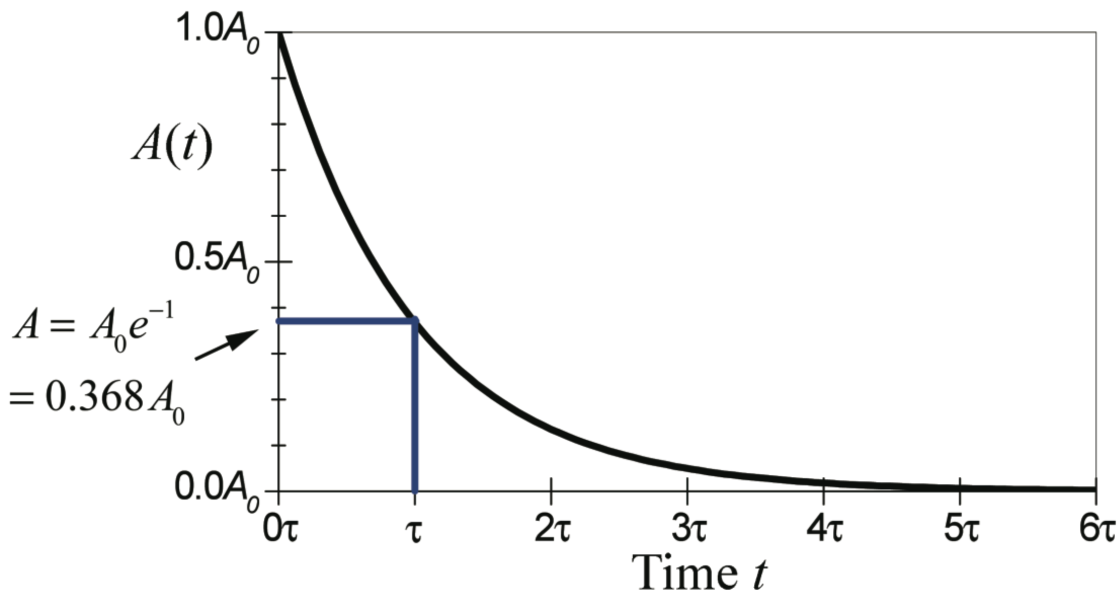
$$C_{total} = C_1 + C_2 = 35\mu\text{F}; R_{eq} = 50\Omega + 30\Omega;\ \ \tau = RC = 2.8\text{ms}$$

- $I = \frac{V}{R} \Rightarrow I_0 = \frac{45\text{V}}{80\Omega} = 0.5625\text{A}$
- $I = \frac{V}{R} \Rightarrow I(t) = \frac{10\text{V}}{80\Omega} = 0.125\text{A}$
- $I(t) = I_0 e^{-t/\tau} = 0.5625\text{A}e^{-t/\tau} = 0.5625e^{-t/2.8ms} = 0.125\text{A}$
- t=4.21 ms

- Alternatively, from $\Delta V_C = \frac{Q}{C}$
- $Q_O = 45V \times 35\mu F = 1.575 \times 10^{-3} C$
- $Q(t) = 10V \times 35\mu F = 3.5 \times 10^{-4} C$
- $Q(t) = Q_o e^{-t/\tau}$
- $3.5 \times 10^{-4} = 1.575 \times 10^{-3} e^{-t/2.8m} \Rightarrow t = 4.21ms$

FYI: Math Review - Exponential Decay

- Consider function *A* where: $\frac{dA}{dt} = -\frac{1}{\tau}A$
- The result is an exponential decay function: $A(t) = A_o e^{-\frac{t}{\tau}}$
- Proof it yourself.



FYI: Math Review - Exponential Behavior

- Slightly modify diff. eq.: $\frac{dA}{dt} = -\frac{1}{\tau}(A - A_f)$
- The result is a "growing" function to A_f:
- $A(t) = A_f\left(1 - e^{-\frac{t}{\tau}}\right)$
- Proof it yourself.