```
ClearAll["Global`*"]
```

Timoshenko Beam

Stiffness Matrix For Simple Cross-Section

$$\text{Csimple} = \begin{pmatrix} GAKx & 0 & 0 & 0 & 0 & 0 \\ 0 & GAKy & 0 & 0 & 0 & 0 \\ 0 & 0 & EA & 0 & 0 & 0 \\ 0 & 0 & 0 & EIxx & 0 & 0 \\ 0 & 0 & 0 & 0 & EIyy & 0 \\ 0 & 0 & 0 & 0 & 0 & GJ \end{pmatrix};$$

$$\Phi = \begin{pmatrix} u' & 0 & 0 & 0 & -\gamma & 0 \\ 0 & v' & 0 & \psi & 0 & 0 \\ 0 & 0 & w' & 0 & 0 & 0 \\ 0 & 0 & 0 & \psi' & 0 & 0 \\ 0 & 0 & 0 & 0 & \gamma' & 0 \\ 0 & 0 & 0 & 0 & 0 & \phi' \end{pmatrix};$$

Element Strain Energy Matrix

```
K = MatrixForm[Simplify[Transpose[Φ].Csimple.Φ]]
```

$$\begin{pmatrix} GAKx\,(u')^2 & 0 & 0 & 0 & -GAKx\,\gamma\,u' & 0 \\ 0 & GAKy\,(v')^2 & 0 & GAKy\,\psi\,v' & 0 & 0 \\ 0 & 0 & EA\,(w')^2 & 0 & 0 & 0 \\ 0 & GAKy\,\psi\,v' & 0 & GAKy\,\psi^2 + EIxx\,(\psi')^2 & 0 & 0 \\ -GAKx\,\gamma\,u' & 0 & 0 & 0 & GAKx\,\gamma^2 + EIyy\,(\gamma')^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & GJ\,(\phi')^2 \end{pmatrix}$$

Stiffness Matrix For Complex Cross-Sections

```
Cgen = ( C11 C12 C13 C14 C15 C16
         C12 C22 C23 C24 C25 C26
         C13 C23 C33 C34 C35 C36
         C14 C24 C34 C44 C45 C46
         C15 C25 C35 C45 C55 C56
         C16 C26 C36 C46 C56 C66 );

Φ =

  ( u1'[x]  u2'[x]  0       0       0       0       0       0       -γ1[x]  -γ2[x]  0
    0       0       v1'[x]  v2'[x]  0       0       ψ1[x]   ψ2[x]   0       0       0
    0       0       0       0       w1'[x]  w2'[x]  0       0       0       0       0
    0       0       0       0       0       0       ψ1'[x]  ψ2'[x]  0       0       0
    0       0       0       0       0       0       0       0       γ1'[x]  γ2'[x]  0
    0       0       0       0       0       0       0       0       0       0       ϕ1'[x] )
  ;
```

Element Strain Energy Matrix

```
K1 = Expand[Simplify[Transpose[Φ].Cgen.Φ]];
(*MatrixForm[K1]*)
```

Initialize Shape Functions

Linear Shape Functions

```
N1[x_] = (1 - x/h);
N2[x_] = x/h;
```

Substitute Shape Functions Into Strain/Displacement Matrix and Force Vector

```
K1int[x_] =
  K1 /. {u1'[x] → N1'[x], u2'[x] → N2'[x], v1'[x] → N1'[x], v2'[x] → N2'[x],
    w1'[x] → N1'[x], w2'[x] → N2'[x], ψ1'[x] → N1'[x], ψ2'[x] → N2'[x],
    γ1'[x] → N1'[x], γ2'[x] → N2'[x], ϕ1'[x] → N1'[x], ϕ2'[x] → N2'[x],
    ψ1[x] →  N1[x], ψ2[x] → N2[x], γ1[x] → N1[x], γ2[x] → N2[x]};
MatrixForm[K1int[x]]
```

```
         vx N1[x]
         vx N2[x]
         vy N1[x]
         vy N2[x]
         pz N1[x]
         pz N2[x]
f[x_] =  mx N1[x] ;
         mx N2[x]
         my N1[x]
         my N2[x]
         tz N1[x]
         tz N2[x]
```

```
     Qx1
     Qx2
     Qy1
     Qy2
      P1
      P2
Q =  Mx1 ;
     Mx2
     My1
     My2
      T1
      T2
```

$$
\left(\begin{array}{ccccccc}
\frac{C11}{h^2} & -\frac{C11}{h^2} & \frac{C12}{h^2} & -\frac{C12}{h^2} & \frac{C13}{h^2} & -\frac{C13}{h^2} & \\
-\frac{C11}{h^2} & \frac{C11}{h^2} & -\frac{C12}{h^2} & \frac{C12}{h^2} & -\frac{C13}{h^2} & \frac{C13}{h^2} & \\
\frac{C12}{h^2} & -\frac{C12}{h^2} & \frac{C22}{h^2} & -\frac{C22}{h^2} & \frac{C23}{h^2} & -\frac{C23}{h^2} & \\
-\frac{C12}{h^2} & \frac{C12}{h^2} & -\frac{C22}{h^2} & \frac{C22}{h^2} & -\frac{C23}{h^2} & \frac{C23}{h^2} & \\
\frac{C13}{h^2} & -\frac{C13}{h^2} & \frac{C23}{h^2} & -\frac{C23}{h^2} & \frac{C33}{h^2} & -\frac{C33}{h^2} & \\
-\frac{C13}{h^2} & \frac{C13}{h^2} & -\frac{C23}{h^2} & \frac{C23}{h^2} & -\frac{C33}{h^2} & \frac{C33}{h^2} & \\
\frac{C14}{h^2}-\frac{C12\left(1-\frac{x}{h}\right)}{h} & -\frac{C14}{h^2}+\frac{C12\left(1-\frac{x}{h}\right)}{h} & \frac{C24}{h^2}-\frac{C22\left(1-\frac{x}{h}\right)}{h} & -\frac{C24}{h^2}+\frac{C22\left(1-\frac{x}{h}\right)}{h} & \frac{C34}{h^2}-\frac{C23\left(1-\frac{x}{h}\right)}{h} & -\frac{C34}{h^2}+\frac{C23\left(1-\frac{x}{h}\right)}{h} & \\
-\frac{C14}{h^2}-\frac{C12\,x}{h^2} & \frac{C14}{h^2}+\frac{C12\,x}{h^2} & -\frac{C24}{h^2}-\frac{C22\,x}{h^2} & \frac{C24}{h^2}+\frac{C22\,x}{h^2} & -\frac{C34}{h^2}-\frac{C23\,x}{h^2} & \frac{C34}{h^2}+\frac{C23\,x}{h^2} & - \\
\frac{C15}{h^2}+\frac{C11\left(1-\frac{x}{h}\right)}{h} & -\frac{C15}{h^2}-\frac{C11\left(1-\frac{x}{h}\right)}{h} & \frac{C25}{h^2}+\frac{C12\left(1-\frac{x}{h}\right)}{h} & -\frac{C25}{h^2}-\frac{C12\left(1-\frac{x}{h}\right)}{h} & \frac{C35}{h^2}+\frac{C13\left(1-\frac{x}{h}\right)}{h} & -\frac{C35}{h^2}-\frac{C13\left(1-\frac{x}{h}\right)}{h} & \frac{C45}{h^2}+ \\
-\frac{C15}{h^2}+\frac{C11\,x}{h^2} & \frac{C15}{h^2}-\frac{C11\,x}{h^2} & -\frac{C25}{h^2}+\frac{C12\,x}{h^2} & \frac{C25}{h^2}-\frac{C12\,x}{h^2} & -\frac{C35}{h^2}+\frac{C13\,x}{h^2} & \frac{C35}{h^2}-\frac{C13\,x}{h^2} & - \\
\frac{C16}{h^2} & -\frac{C16}{h^2} & \frac{C26}{h^2} & -\frac{C26}{h^2} & \frac{C36}{h^2} & -\frac{C36}{h^2} & \\
-\frac{C16}{h^2} & \frac{C16}{h^2} & -\frac{C26}{h^2} & \frac{C26}{h^2} & -\frac{C36}{h^2} & \frac{C36}{h^2} &
\end{array}\right.
$$

Integrate the to get 12 x 12 Stiffness Matrix and Force Vector for Straight Beam

```
Jac = 2/h;
xtran[ξ_] = h/2 (ξ + 1);
(* Single point Gauss Integration weighting coefficient *)
w = 2;
(*MatrixForm[Expand[w Kint1[xtran[0]] 1/Jac /.
   {C11→GAKx,C12→0,C13→0,C14→0,C15→0,C16→0,C22→GAKy,C23→0,C24→0,C25→0,C26→0,
    C33→EA,C34→0,C35→0,C36→0,C44→EIxx,C45→0,C46→0,C55→EIyy,C56→0,C66→GJ}]]*)
Ketmp = Expand[w K1int[xtran[0]] 1/Jac];
MatrixForm[Ketmp]
Fetmp = Simplify[Integrate[f[x], {x, 0, h}] + Q];
MatrixForm[Fetmp]
```

$$\begin{pmatrix}
\frac{C11}{h} & -\frac{C11}{h} & \frac{C12}{h} & -\frac{C12}{h} & \frac{C13}{h} & -\frac{C13}{h} & -\frac{C12}{2}+\frac{C14}{h} & -\ldots \\
-\frac{C11}{h} & \frac{C11}{h} & -\frac{C12}{h} & \frac{C12}{h} & -\frac{C13}{h} & \frac{C13}{h} & \frac{C12}{2}-\frac{C14}{h} & C\ldots \\
\frac{C12}{h} & -\frac{C12}{h} & \frac{C22}{h} & -\frac{C22}{h} & \frac{C23}{h} & -\frac{C23}{h} & -\frac{C22}{2}+\frac{C24}{h} & -\ldots \\
-\frac{C12}{h} & \frac{C12}{h} & -\frac{C22}{h} & \frac{C22}{h} & -\frac{C23}{h} & \frac{C23}{h} & \frac{C22}{2}-\frac{C24}{h} & C\ldots \\
\frac{C13}{h} & -\frac{C13}{h} & \frac{C23}{h} & -\frac{C23}{h} & \frac{C33}{h} & -\frac{C33}{h} & -\frac{C23}{2}+\frac{C34}{h} & -\ldots \\
-\frac{C13}{h} & \frac{C13}{h} & -\frac{C23}{h} & \frac{C23}{h} & -\frac{C33}{h} & \frac{C33}{h} & \frac{C23}{2}-\frac{C34}{h} & C\ldots \\
-\frac{C12}{2}+\frac{C14}{h} & \frac{C12}{2}-\frac{C14}{h} & -\frac{C22}{2}+\frac{C24}{h} & \frac{C22}{2}-\frac{C24}{h} & -\frac{C23}{2}+\frac{C34}{h} & \frac{C23}{2}-\frac{C34}{h} & -C24+\frac{C44}{h}+\frac{C22\,h}{4} & -\ldots \\
-\frac{C12}{2}-\frac{C14}{h} & \frac{C12}{2}+\frac{C14}{h} & -\frac{C22}{2}-\frac{C24}{h} & \frac{C22}{2}+\frac{C24}{h} & -\frac{C23}{2}-\frac{C34}{h} & \frac{C23}{2}+\frac{C34}{h} & -\frac{C44}{h}+\frac{C22\,h}{4} & C24\ldots \\
\frac{C11}{2}+\frac{C15}{h} & -\frac{C11}{2}-\frac{C15}{h} & \frac{C12}{2}+\frac{C25}{h} & -\frac{C12}{2}-\frac{C25}{h} & \frac{C13}{2}+\frac{C35}{h} & -\frac{C13}{2}-\frac{C35}{h} & \frac{C14}{2}-\frac{C25}{2}+\frac{C45}{h}-\frac{C12\,h}{4} & -\frac{C14}{2}-\ldots \\
\frac{C11}{2}-\frac{C15}{h} & -\frac{C11}{2}+\frac{C15}{h} & \frac{C12}{2}-\frac{C25}{h} & -\frac{C12}{2}+\frac{C25}{h} & \frac{C13}{2}-\frac{C35}{h} & -\frac{C13}{2}+\frac{C35}{h} & \frac{C14}{2}+\frac{C25}{2}-\frac{C45}{h}-\frac{C12\,h}{4} & -\frac{C14}{2}+\ldots \\
\frac{C16}{h} & -\frac{C16}{h} & \frac{C26}{h} & -\frac{C26}{h} & \frac{C36}{h} & -\frac{C36}{h} & -\frac{C26}{2}+\frac{C46}{h} & -\ldots \\
-\frac{C16}{h} & \frac{C16}{h} & -\frac{C26}{h} & \frac{C26}{h} & -\frac{C36}{h} & \frac{C36}{h} & \frac{C26}{2}-\frac{C46}{h} & C\ldots
\end{pmatrix}$$

$$\begin{pmatrix}
Qx1+\frac{h\,vx}{2} \\
Qx2+\frac{h\,vx}{2} \\
Qy1+\frac{h\,vy}{2} \\
Qy2+\frac{h\,vy}{2} \\
P1+\frac{h\,pz}{2} \\
P2+\frac{h\,pz}{2} \\
\frac{h\,mx}{2}+Mx1 \\
\frac{h\,mx}{2}+Mx2 \\
\frac{h\,my}{2}+My1 \\
\frac{h\,my}{2}+My2 \\
T1+\frac{h\,tz}{2} \\
T2+\frac{h\,tz}{2}
\end{pmatrix}$$

12 x 12 Element Stiffness Matrix for a beam with a Simple Cross-Section:

```
MatrixForm[Ketmp /. {C11 → GAKx, C12 → 0, C13 → 0, C14 → 0, C15 → 0, C16 → 0,
   C22 → GAKy, C23 → 0, C24 → 0, C25 → 0, C26 → 0, C33 → EA, C34 → 0, C35 → 0,
   C36 → 0, C44 → EIxx, C45 → 0, C46 → 0, C55 → EIyy, C56 → 0, C66 → GJ}]
```

$$\left(\begin{array}{cccccccccc}
\frac{GAKx}{h} & -\frac{GAKx}{h} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{GAKx}{2} & \frac{GAKx}{2} \\
-\frac{GAKx}{h} & \frac{GAKx}{h} & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{GAKx}{2} & -\frac{GAKx}{2} \\
0 & 0 & \frac{GAKy}{h} & -\frac{GAKy}{h} & 0 & 0 & -\frac{GAKy}{2} & -\frac{GAKy}{2} & 0 & 0 \\
0 & 0 & -\frac{GAKy}{h} & \frac{GAKy}{h} & 0 & 0 & \frac{GAKy}{2} & \frac{GAKy}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{EA}{h} & -\frac{EA}{h} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{EA}{h} & \frac{EA}{h} & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{GAKy}{2} & \frac{GAKy}{2} & 0 & 0 & \frac{EIxx}{h}+\frac{GAKy\,h}{4} & -\frac{EIxx}{h}+\frac{GAKy\,h}{4} & 0 & 0 \\
0 & 0 & -\frac{GAKy}{2} & \frac{GAKy}{2} & 0 & 0 & -\frac{EIxx}{h}+\frac{GAKy\,h}{4} & \frac{EIxx}{h}+\frac{GAKy\,h}{4} & 0 & 0 \\
\frac{GAKx}{2} & -\frac{GAKx}{2} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{EIyy}{h}+\frac{GAKx\,h}{4} & -\frac{EIyy}{h}+\frac{GAKx\,h}{4} \\
\frac{GAKx}{2} & -\frac{GAKx}{2} & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{EIyy}{h}+\frac{GAKx\,h}{4} & \frac{EIyy}{h}+\frac{GAKx\,h}{4} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right.$$

Geometric Stiffness Matrix

Strain Energy for Ritz type formulation:

$$\Phi g = \begin{pmatrix} wx' & 0 & 0 & 0 & 0 & 0 \\ 0 & wy' & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix};$$

$$Pmat = \begin{pmatrix} P & 0 & 0 & 0 & 0 & 0 \\ 0 & P & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix};$$

```
Kg = MatrixForm[P Simplify[Φg.Transpose[Φg]]]
```

$$\begin{pmatrix} P\,(wx')^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & P\,(wy')^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Strain Energy for Finite Element formulation:

$$\Phi g = \begin{pmatrix} N1'[x] & N2'[x] & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & N1'[x] & N2'[x] & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix};$$

```
MatrixForm[Simplify[Transpose[Φg].Pmat.Φg]]
Kgint[x_] = Simplify[Transpose[Φg].Pmat.Φg];
```

$$\begin{pmatrix} \frac{P}{h^2} & -\frac{P}{h^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{P}{h^2} & \frac{P}{h^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{P}{h^2} & -\frac{P}{h^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{P}{h^2} & \frac{P}{h^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Generate Geometrix Stiffness Matrix and Integrate

```
Kgtmp = Integrate[Kgint[x], {x, 0, h}];
MatrixForm[Kgtmp]
```

$$\begin{pmatrix} \frac{P}{h} & -\frac{P}{h} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{P}{h} & \frac{P}{h} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{P}{h} & -\frac{P}{h} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{P}{h} & \frac{P}{h} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Geometric Stiffness Matrix

Mass Matrix For a Ritz Method Formulation:

$$M = \begin{pmatrix} m & 0 & 0 & 0 & 0 & -m\,ym \\ 0 & m & 0 & 0 & 0 & m\,xm \\ 0 & 0 & m & m\,ym & -m\,xm & 0 \\ 0 & 0 & m\,ym & Ixx & -Ixy & 0 \\ 0 & 0 & -m\,xm & -Ixy & Iyy & 0 \\ -m\,ym & m\,xm & 0 & 0 & 0 & Ixx+Iyy \end{pmatrix};$$

$$M = \begin{pmatrix} M11 & 0 & 0 & 0 & 0 & M16 \\ 0 & M11 & 0 & 0 & 0 & M26 \\ 0 & 0 & M11 & -M16 & -M26 & 0 \\ 0 & 0 & -M16 & M44 & M45 & 0 \\ 0 & 0 & -M26 & M45 & M55 & 0 \\ M16 & M26 & 0 & 0 & 0 & M66 \end{pmatrix};$$

$$\Phi = \begin{pmatrix} u1[x] & u2[x] & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & v1[x] & v2[x] & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & w1[x] & w2[x] & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \psi1[x] & \psi2[x] & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \gamma1[x] & \gamma2[x] & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \phi1[x] & \phi2[x] \end{pmatrix};$$

```
Mmat = Expand[Simplify[Transpose[Φ].M.Φ]];
MatrixForm[Mmat]
```

$$\begin{pmatrix} M11\,u1[x]^2 & M11\,u1[x]\,u2[x] & 0 & 0 & 0 & \text{M11 w} \\ M11\,u1[x]\,u2[x] & M11\,u2[x]^2 & 0 & 0 & 0 & \\ 0 & 0 & M11\,v1[x]^2 & M11\,v1[x]\,v2[x] & 0 & \\ 0 & 0 & M11\,v1[x]\,v2[x] & M11\,v2[x]^2 & 0 & \\ 0 & 0 & 0 & 0 & M11\,w1[x]^2 & \text{M11 w} \\ 0 & 0 & 0 & 0 & M11\,w1[x]\,w2[x] & \text{M1} \\ 0 & 0 & 0 & 0 & -M16\,w1[x]\,\psi1[x] & -M16 \\ 0 & 0 & 0 & 0 & -M16\,w1[x]\,\psi2[x] & -M16 \\ 0 & 0 & 0 & 0 & -M26\,w1[x]\,\gamma1[x] & -M26 \\ 0 & 0 & 0 & 0 & -M26\,w1[x]\,\gamma2[x] & -M26 \\ M16\,u1[x]\,\phi1[x] & M16\,u2[x]\,\phi1[x] & M26\,v1[x]\,\phi1[x] & M26\,v2[x]\,\phi1[x] & 0 & \\ M16\,u1[x]\,\phi2[x] & M16\,u2[x]\,\phi2[x] & M26\,v1[x]\,\phi2[x] & M26\,v2[x]\,\phi2[x] & 0 & \end{pmatrix}$$

Substitute Shape Functions Into Strain/Displacement Matrix and Force Vector

```
Mint[x_] = Mmat /. {u1[x] → N1[x], u2[x] → N2[x], v1[x] → N1[x],
    v2[x] → N2[x], w1[x] → N1[x], w2[x] → N2[x], ψ1[x] → N1[x],
    ψ2[x] → N2[x], γ1[x] → N1[x], γ2[x] → N2[x], ϕ1[x] → N1[x], ϕ2[x] → N2[x],
    ψ1[x]  →  N1[x], ψ2[x] → N2[x], γ1[x] → N1[x], γ2[x] → N2[x]};
MatrixForm[
 Mint[
  x]]
```

$$\left(\begin{array}{ccccccccc}
\text{M11}\left(1-\frac{x}{h}\right)^2 & \frac{\text{M11}\,x\left(1-\frac{x}{h}\right)}{h} & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{\text{M11}\,x\left(1-\frac{x}{h}\right)}{h} & \frac{\text{M11}\,x^2}{h^2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \text{M11}\left(1-\frac{x}{h}\right)^2 & \frac{\text{M11}\,x\left(1-\frac{x}{h}\right)}{h} & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{\text{M11}\,x\left(1-\frac{x}{h}\right)}{h} & \frac{\text{M11}\,x^2}{h^2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \text{M11}\left(1-\frac{x}{h}\right)^2 & \frac{\text{M11}\,x\left(1-\frac{x}{h}\right)}{h} & -\text{M16}\left(1-\frac{x}{h}\right)^2 & -\frac{\text{M16}\,x\cdots}{\cdots} \\
0 & 0 & 0 & 0 & \frac{\text{M11}\,x\left(1-\frac{x}{h}\right)}{h} & \frac{\text{M11}\,x^2}{h^2} & -\frac{\text{M16}\,x\left(1-\frac{x}{h}\right)}{h} & -\frac{\text{M1}\cdots}{h} \\
0 & 0 & 0 & 0 & -\text{M16}\left(1-\frac{x}{h}\right)^2 & -\frac{\text{M16}\,x\left(1-\frac{x}{h}\right)}{h} & \text{M44}\left(1-\frac{x}{h}\right)^2 & \frac{\text{M44}\,x\,(\cdots}{h} \\
0 & 0 & 0 & 0 & -\frac{\text{M16}\,x\left(1-\frac{x}{h}\right)}{h} & -\frac{\text{M16}\,x^2}{h^2} & \frac{\text{M44}\,x\left(1-\frac{x}{h}\right)}{h} & \frac{\text{M44}\cdots}{h^2} \\
0 & 0 & 0 & 0 & -\text{M26}\left(1-\frac{x}{h}\right)^2 & -\frac{\text{M26}\,x\left(1-\frac{x}{h}\right)}{h} & \text{M45}\left(1-\frac{x}{h}\right)^2 & \frac{\text{M45}\,x\,(\cdots}{h} \\
0 & 0 & 0 & 0 & -\frac{\text{M26}\,x\left(1-\frac{x}{h}\right)}{h} & -\frac{\text{M26}\,x^2}{h^2} & \frac{\text{M45}\,x\left(1-\frac{x}{h}\right)}{h} & \frac{\text{M45}\cdots}{h^2} \\
\text{M16}\left(1-\frac{x}{h}\right)^2 & \frac{\text{M16}\,x\left(1-\frac{x}{h}\right)}{h} & \text{M26}\left(1-\frac{x}{h}\right)^2 & \frac{\text{M26}\,x\left(1-\frac{x}{h}\right)}{h} & 0 & 0 & 0 & 0 \\
\frac{\text{M16}\,x\left(1-\frac{x}{h}\right)}{h} & \frac{\text{M16}\,x^2}{h^2} & \frac{\text{M26}\,x\left(1-\frac{x}{h}\right)}{h} & \frac{\text{M26}\,x^2}{h^2} & 0 & 0 & 0 & 0
\end{array}\right.$$

Integrate the to get 12 x 12 Mass Matrix Straight Beam

```
Metmp = ∫_0^h Mint[x] dx;
MatrixForm[Metmp]
```

$$
\begin{pmatrix}
\frac{h M11}{3} & \frac{h M11}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{h M16}{3} & \frac{h M16}{6} \\
\frac{h M11}{6} & \frac{h M11}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{h M16}{6} & \frac{h M16}{3} \\
0 & 0 & \frac{h M11}{3} & \frac{h M11}{6} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{h M26}{3} & \frac{h M26}{6} \\
0 & 0 & \frac{h M11}{6} & \frac{h M11}{3} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{h M26}{6} & \frac{h M26}{3} \\
0 & 0 & 0 & 0 & \frac{h M11}{3} & \frac{h M11}{6} & -\frac{h M16}{3} & -\frac{h M16}{6} & -\frac{h M26}{3} & -\frac{h M26}{6} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{h M11}{6} & \frac{h M11}{3} & -\frac{h M16}{6} & -\frac{h M16}{3} & -\frac{h M26}{6} & -\frac{h M26}{3} & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{h M16}{3} & -\frac{h M16}{6} & \frac{h M44}{3} & \frac{h M44}{6} & \frac{h M45}{3} & \frac{h M45}{6} & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{h M16}{6} & -\frac{h M16}{3} & \frac{h M44}{6} & \frac{h M44}{3} & \frac{h M45}{6} & \frac{h M45}{3} & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{h M26}{3} & -\frac{h M26}{6} & \frac{h M45}{3} & \frac{h M45}{6} & \frac{h M55}{3} & \frac{h M55}{6} & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{h M26}{6} & -\frac{h M26}{3} & \frac{h M45}{6} & \frac{h M45}{3} & \frac{h M55}{6} & \frac{h M55}{3} & 0 & 0 \\
\frac{h M16}{3} & \frac{h M16}{6} & \frac{h M26}{3} & \frac{h M26}{6} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{h M66}{3} & \frac{h M66}{6} \\
\frac{h M16}{6} & \frac{h M16}{3} & \frac{h M26}{6} & \frac{h M26}{3} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{h M66}{6} & \frac{h M66}{3}
\end{pmatrix}
$$

Reorder All Matricies for Easier Global Assembly

Reorder Vector:

```
reorder = {1, 3, 5, 7, 9, 11, 2, 4, 6, 8, 10, 12};
```

12 x 12 RIE Timoshenko Beam Stiffness Matrix:

$$\texttt{Ke} = \begin{matrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{matrix};$$

$$\texttt{Fe} = \begin{matrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{matrix};$$

```
Do[Do[Ke[[i, j]] = Expand[Ketmp[[reorder[[i]], reorder[[j]]]]], {i, 12}], {j, 12}]
Do[Fe[[i]] = Fetmp[[reorder[[i]]]], {i, 12}]
MatrixForm[Ke]
MatrixForm[Fe]
```

$$\begin{pmatrix}
\frac{C11}{h} & \frac{C12}{h} & \frac{C13}{h} & -\frac{C12}{2}+\frac{C14}{h} & \frac{C11}{2}+\frac{C15}{h} & \frac{C16}{h} & -\frac{C11}{h} \\
\frac{C12}{h} & \frac{C22}{h} & \frac{C23}{h} & -\frac{C22}{2}+\frac{C24}{h} & \frac{C12}{2}+\frac{C25}{h} & \frac{C26}{h} & -\frac{C12}{h} \\
\frac{C13}{h} & \frac{C23}{h} & \frac{C33}{h} & -\frac{C23}{2}+\frac{C34}{h} & \frac{C13}{2}+\frac{C35}{h} & \frac{C36}{h} & -\frac{C13}{h} \\
-\frac{C12}{2}+\frac{C14}{h} & -\frac{C22}{2}+\frac{C24}{h} & -\frac{C23}{2}+\frac{C34}{h} & -C24+\frac{C44}{h}+\frac{C22\,h}{4} & \frac{C14}{2}-\frac{C25}{2}+\frac{C45}{h}-\frac{C12\,h}{4} & -\frac{C26}{2}+\frac{C46}{h} & \frac{C12}{2}- \\
\frac{C11}{2}+\frac{C15}{h} & \frac{C12}{2}+\frac{C25}{h} & \frac{C13}{2}+\frac{C35}{h} & \frac{C14}{2}-\frac{C25}{2}+\frac{C45}{h}-\frac{C12\,h}{4} & C15+\frac{C55}{h}+\frac{C11\,h}{4} & \frac{C16}{2}+\frac{C56}{h} & -\frac{C11}{2}- \\
\frac{C16}{h} & \frac{C26}{h} & \frac{C36}{h} & -\frac{C26}{2}+\frac{C46}{h} & \frac{C16}{2}+\frac{C56}{h} & \frac{C66}{h} & -\frac{C16}{h} \\
-\frac{C11}{h} & -\frac{C12}{h} & -\frac{C13}{h} & \frac{C12}{2}-\frac{C14}{h} & -\frac{C11}{2}-\frac{C15}{h} & -\frac{C16}{h} & \frac{C11}{h} \\
-\frac{C12}{h} & -\frac{C22}{h} & -\frac{C23}{h} & \frac{C22}{2}-\frac{C24}{h} & -\frac{C12}{2}-\frac{C25}{h} & -\frac{C26}{h} & \frac{C12}{h} \\
-\frac{C13}{h} & -\frac{C23}{h} & -\frac{C33}{h} & \frac{C23}{2}-\frac{C34}{h} & -\frac{C13}{2}-\frac{C35}{h} & -\frac{C36}{h} & \frac{C13}{h} \\
-\frac{C12}{2}-\frac{C14}{h} & -\frac{C22}{2}-\frac{C24}{h} & -\frac{C23}{2}-\frac{C34}{h} & -\frac{C44}{h}+\frac{C22\,h}{4} & -\frac{C14}{2}-\frac{C25}{2}-\frac{C45}{h}-\frac{C12\,h}{4} & -\frac{C26}{2}-\frac{C46}{h} & \frac{C12}{2}+ \\
\frac{C11}{2}-\frac{C15}{h} & \frac{C12}{2}-\frac{C25}{h} & \frac{C13}{2}-\frac{C35}{h} & \frac{C14}{2}+\frac{C25}{2}-\frac{C45}{h}-\frac{C12\,h}{4} & -\frac{C55}{h}+\frac{C11\,h}{4} & \frac{C16}{2}-\frac{C56}{h} & -\frac{C11}{2}+ \\
-\frac{C16}{h} & -\frac{C26}{h} & -\frac{C36}{h} & \frac{C26}{2}-\frac{C46}{h} & -\frac{C16}{2}-\frac{C56}{h} & -\frac{C66}{h} & \frac{C16}{h}
\end{pmatrix}$$

$$\begin{pmatrix} Qx1 + \frac{h\,vx}{2} \\ Qy1 + \frac{h\,vy}{2} \\ P1 + \frac{h\,pz}{2} \\ \frac{h\,mx}{2} + Mx1 \\ \frac{h\,my}{2} + My1 \\ T1 + \frac{h\,tz}{2} \\ Qx2 + \frac{h\,vx}{2} \\ Qy2 + \frac{h\,vy}{2} \\ P2 + \frac{h\,pz}{2} \\ \frac{h\,mx}{2} + Mx2 \\ \frac{h\,my}{2} + My2 \\ T2 + \frac{h\,tz}{2} \end{pmatrix}$$

12 x 12 RIE Timoshenko Beam Geometric Stiffness Matrix:

$$\mathbf{Keg} = \begin{matrix} 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \end{matrix};$$

```
Do[Do[Keg[[i, j]] = Expand[Kgtmp[[reorder[[i]], reorder[[j]]]]], {i, 12}], {j, 12}]
MatrixForm[Keg]
```

$$\begin{pmatrix} \frac{P}{h} & 0 & 0 & 0 & 0 & 0 & -\frac{P}{h} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{P}{h} & 0 & 0 & 0 & 0 & 0 & -\frac{P}{h} & 0 & 0 & 0 & 0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ -\frac{P}{h} & 0 & 0 & 0 & 0 & 0 & \frac{P}{h} & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{P}{h} & 0 & 0 & 0 & 0 & 0 & \frac{P}{h} & 0 & 0 & 0 & 0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \end{pmatrix}$$

12 x 12 Timoshenko Beam Mass Matrix:

$$\mathbf{Me} = \begin{matrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{matrix};$$

```
Do[Do[Me[[i, j]] = Expand[Metmp[[reorder[[i]], reorder[[j]]]]], {i, 12}], {j, 12}]
MatrixForm[Me]
```

$$\begin{pmatrix}
\frac{h\,M11}{3} & 0 & 0 & 0 & 0 & \frac{h\,M16}{3} & \frac{h\,M11}{6} & 0 & 0 & 0 & 0 & \frac{h\,M16}{6} \\
0 & \frac{h\,M11}{3} & 0 & 0 & 0 & \frac{h\,M26}{3} & 0 & \frac{h\,M11}{6} & 0 & 0 & 0 & \frac{h\,M26}{6} \\
0 & 0 & \frac{h\,M11}{3} & -\frac{h\,M16}{3} & -\frac{h\,M26}{3} & 0 & 0 & 0 & \frac{h\,M11}{6} & -\frac{h\,M16}{6} & -\frac{h\,M26}{6} & 0 \\
0 & 0 & -\frac{h\,M16}{3} & \frac{h\,M44}{3} & \frac{h\,M45}{3} & 0 & 0 & 0 & -\frac{h\,M16}{6} & \frac{h\,M44}{6} & \frac{h\,M45}{6} & 0 \\
0 & 0 & -\frac{h\,M26}{3} & \frac{h\,M45}{3} & \frac{h\,M55}{3} & 0 & 0 & 0 & -\frac{h\,M26}{6} & \frac{h\,M45}{6} & \frac{h\,M55}{6} & 0 \\
\frac{h\,M16}{3} & \frac{h\,M26}{3} & 0 & 0 & 0 & \frac{h\,M66}{3} & \frac{h\,M16}{6} & \frac{h\,M26}{6} & 0 & 0 & 0 & \frac{h\,M66}{6} \\
\frac{h\,M11}{6} & 0 & 0 & 0 & 0 & \frac{h\,M16}{6} & \frac{h\,M11}{3} & 0 & 0 & 0 & 0 & \frac{h\,M16}{3} \\
0 & \frac{h\,M11}{6} & 0 & 0 & 0 & \frac{h\,M26}{6} & 0 & \frac{h\,M11}{3} & 0 & 0 & 0 & \frac{h\,M26}{3} \\
0 & 0 & \frac{h\,M11}{6} & -\frac{h\,M16}{6} & -\frac{h\,M26}{6} & 0 & 0 & 0 & \frac{h\,M11}{3} & -\frac{h\,M16}{3} & -\frac{h\,M26}{3} & 0 \\
0 & 0 & -\frac{h\,M16}{6} & \frac{h\,M44}{6} & \frac{h\,M45}{6} & 0 & 0 & 0 & -\frac{h\,M16}{3} & \frac{h\,M44}{3} & \frac{h\,M45}{3} & 0 \\
0 & 0 & -\frac{h\,M26}{6} & \frac{h\,M45}{6} & \frac{h\,M55}{6} & 0 & 0 & 0 & -\frac{h\,M26}{3} & \frac{h\,M45}{3} & \frac{h\,M55}{3} & 0 \\
\frac{h\,M16}{6} & \frac{h\,M26}{6} & 0 & 0 & 0 & \frac{h\,M66}{6} & \frac{h\,M16}{3} & \frac{h\,M26}{3} & 0 & 0 & 0 & \frac{h\,M66}{3}
\end{pmatrix}$$

12 x 12 Timoshenko Beam Rotation Matrix Testing

```
KetmpRot = Ke /. {C11 → GAKx, C12 → 0, C13 → 0, C14 → 0, C15 → 0, C16 → 0,
    C22 → GAKy, C23 → 0, C24 → 0, C25 → 0, C26 → 0, C33 → EA, C34 → 0, C35 → 0,
    C36 → 0, C44 → EIxx, C45 → 0, C46 → 0, C55 → EIyy, C56 → 0, C66 → GJ};
P1 = {0, 0, 0};
P2 = {0, 0, 1};
nuVec = {0, 1, 0};
zVec = (P2 - P1) / Norm[P2 - P1]
yVec = Cross[zVec, nuVec] / Norm[Cross[zVec, nuVec]]
xVec = Cross[yVec, zVec] / Norm[Cross[yVec, zVec]]
Lambda = Transpose[{xVec, yVec, zVec}]
MatrixForm[Lambda]
```

$$\text{rot} = \begin{matrix} 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0 \end{matrix};$$

```
rot[[1 ;; 3, 1 ;; 3]] = Lambda;
rot[[4 ;; 6, 4 ;; 6]] = Lambda;
rot[[7 ;; 9, 7 ;; 9]] = Lambda;
rot[[10 ;; 12, 10 ;; 12]] = Lambda;
MatrixForm[rot]
MatrixForm[Transpose[rot].KetmpRot.rot]
```

{0, 0, 1}

{-1, 0, 0}

{0, 1, 0}

{{0, -1, 0}, {1, 0, 0}, {0, 0, 1}}

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \frac{GAKy}{h} & 0 & 0 & 0 & \frac{GAKy}{2} & 0 & -\frac{GAKy}{h} & 0 & 0 & 0 & \frac{GAKy}{2} \\ 0 & \frac{GAKx}{h} & 0 & -\frac{GAKx}{2} & 0 & 0 & 0 & -\frac{GAKx}{h} & 0 & -\frac{GAKx}{2} & 0 \\ 0 & 0 & \frac{EA}{h} & 0 & 0 & 0 & 0 & 0 & -\frac{EA}{h} & 0 & 0 \\ 0 & -\frac{GAKx}{2} & 0 & \frac{EIyy}{h}+\frac{GAKx\,h}{4} & 0 & 0 & 0 & \frac{GAKx}{2} & 0 & -\frac{EIyy}{h}+\frac{GAKx\,h}{4} & 0 \\ \frac{GAKy}{2} & 0 & 0 & 0 & \frac{EIxx}{h}+\frac{GAKy\,h}{4} & 0 & -\frac{GAKy}{2} & 0 & 0 & 0 & -\frac{EIxx}{h}+\frac{GAKy\,h}{4} \\ 0 & 0 & 0 & 0 & 0 & \frac{GJ}{h} & 0 & 0 & 0 & 0 & 0 \\ -\frac{GAKy}{h} & 0 & 0 & 0 & -\frac{GAKy}{2} & 0 & \frac{GAKy}{h} & 0 & 0 & 0 & -\frac{GAKy}{2} \\ 0 & -\frac{GAKx}{h} & 0 & \frac{GAKx}{2} & 0 & 0 & 0 & \frac{GAKx}{h} & 0 & \frac{GAKx}{2} & 0 \\ 0 & 0 & -\frac{EA}{h} & 0 & 0 & 0 & 0 & 0 & \frac{EA}{h} & 0 & 0 \\ 0 & -\frac{GAKx}{2} & 0 & -\frac{EIyy}{h}+\frac{GAKx\,h}{4} & 0 & 0 & 0 & \frac{GAKx}{2} & 0 & \frac{EIyy}{h}+\frac{GAKx\,h}{4} & 0 \\ \frac{GAKy}{2} & 0 & 0 & 0 & -\frac{EIxx}{h}+\frac{GAKy\,h}{4} & 0 & -\frac{GAKy}{2} & 0 & 0 & 0 & \frac{EIxx}{h}+\frac{GAKy\,h}{4} \\ 0 & 0 & 0 & 0 & 0 & -\frac{GJ}{h} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$