```
ClearAll["Global`*"]
```

Timoshenko Beam

Stiffness Matrix For Simple Cross-Section

$$\text{Csimple} = \begin{pmatrix} \text{GAKx} & 0 & 0 & 0 & 0 & 0 \\ 0 & \text{GAKy} & 0 & 0 & 0 & 0 \\ 0 & 0 & \text{EA} & 0 & 0 & 0 \\ 0 & 0 & 0 & \text{EIxx} & 0 & 0 \\ 0 & 0 & 0 & 0 & \text{EIyy} & 0 \\ 0 & 0 & 0 & 0 & 0 & \text{GJ} \end{pmatrix};$$

$$\Phi = \begin{pmatrix} -u' & 0 & 0 & 0 & \gamma & 0 \\ 0 & v' & 0 & \psi & 0 & 0 \\ 0 & 0 & w' & 0 & 0 & 0 \\ 0 & 0 & 0 & \psi' & 0 & 0 \\ 0 & 0 & 0 & 0 & \gamma' & 0 \\ 0 & 0 & 0 & 0 & 0 & \phi' \end{pmatrix};$$

Element Strain Energy Matrix

```
K = MatrixForm[Simplify[Transpose[Φ].Csimple.Φ]]
```

$$\begin{pmatrix} \text{GAKx}\,(u')^2 & 0 & 0 & 0 & -\text{GAKx}\,\gamma\,u' & 0 \\ 0 & \text{GAKy}\,(v')^2 & 0 & \text{GAKy}\,\psi\,v' & 0 & 0 \\ 0 & 0 & \text{EA}\,(w')^2 & 0 & 0 & 0 \\ 0 & \text{GAKy}\,\psi\,v' & 0 & \text{GAKy}\,\psi^2 + \text{EIxx}\,(\psi')^2 & 0 & 0 \\ -\text{GAKx}\,\gamma\,u' & 0 & 0 & 0 & \text{GAKx}\,\gamma^2 + \text{EIyy}\,(\gamma')^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \text{GJ}\,(\phi')^2 \end{pmatrix}$$

Stiffness Matrix For Complex Cross-Sections

$$
\text{Cgen} = \begin{pmatrix} C11 & C12 & C13 & C14 & C15 & C16 \\ C12 & C22 & C23 & C24 & C25 & C26 \\ C13 & C23 & C33 & C34 & C35 & C36 \\ C14 & C24 & C34 & C44 & C45 & C46 \\ C15 & C25 & C35 & C45 & C55 & C56 \\ C16 & C26 & C36 & C46 & C56 & C66 \end{pmatrix};
$$

$$
\Phi = \begin{pmatrix} -u1'[x] & -u2'[x] & 0 & 0 & 0 & 0 & 0 & 0 & \gamma1[x] & \gamma2[x] & 0 \\ 0 & 0 & v1'[x] & v2'[x] & 0 & 0 & \psi1[x] & \psi2[x] & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & w1'[x] & w2'[x] & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \psi1'[x] & \psi2'[x] & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \gamma1'[x] & \gamma2'[x] & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \phi1'[\text{:} \end{pmatrix}
$$

;

Element Strain Energy Matrix

```
K1 = Expand[Simplify[Transpose[Φ].Cgen.Φ]];
```

Initialize Shape Functions

Linear Shape Functions

$$N1[x\_] = \left(1 - \frac{x}{h}\right);$$

$$N2[x\_] = \frac{x}{h};$$

Substitute Shape Functions Into Strain/Displacement Matrix and Force Vector

```
K1int[x_] = K1 /. {u1'[x] → N1'[x], u2'[x] → N2'[x], v1'[x] → N1'[x],
    v2'[x] → N2'[x], w1'[x] → N1'[x], w2'[x] → N2'[x], ψ1'[x] → N1'[x],
    ψ2'[x] → N2'[x], γ1'[x] → N1'[x], γ2'[x] → N2'[x], ϕ1'[x] → N1'[x],
    ϕ2'[x] → N2'[x], ψ1[x] →  N1[x], ψ2[x] → N2[x], γ1[x] → N1[x], γ2[x] → N2[x]};
MatrixForm[K1int[x]]
```

$$f[x\_] = \begin{matrix} vx\,N1[x] \\ vx\,N2[x] \\ vy\,N1[x] \\ vy\,N2[x] \\ pz\,N1[x] \\ pz\,N2[x] \\ 0 \\ 0 \\ 0 \\ 0 \\ tz\,N1[x] \\ tz\,N2[x] \end{matrix};$$

$$Q = \begin{matrix} Qx1 \\ Qx2 \\ Qy1 \\ Qy2 \\ P1 \\ P2 \\ Mx1 \\ Mx2 \\ My1 \\ My2 \\ T1 \\ T2 \end{matrix};$$

$$
\left(\begin{array}{ccccccc}
\frac{C11}{h^2} & -\frac{C11}{h^2} & -\frac{C12}{h^2} & \frac{C12}{h^2} & -\frac{C13}{h^2} & \frac{C13}{h^2} & \\
-\frac{C11}{h^2} & \frac{C11}{h^2} & \frac{C12}{h^2} & -\frac{C12}{h^2} & \frac{C13}{h^2} & -\frac{C13}{h^2} & \\
-\frac{C12}{h^2} & \frac{C12}{h^2} & \frac{C22}{h^2} & -\frac{C22}{h^2} & \frac{C23}{h^2} & -\frac{C23}{h^2} & \\
\frac{C12}{h^2} & -\frac{C12}{h^2} & -\frac{C22}{h^2} & \frac{C22}{h^2} & -\frac{C23}{h^2} & \frac{C23}{h^2} & \\
-\frac{C13}{h^2} & \frac{C13}{h^2} & \frac{C23}{h^2} & -\frac{C23}{h^2} & \frac{C33}{h^2} & -\frac{C33}{h^2} & \\
\frac{C13}{h^2} & -\frac{C13}{h^2} & -\frac{C23}{h^2} & \frac{C23}{h^2} & -\frac{C33}{h^2} & \frac{C33}{h^2} & \\
-\frac{C14}{h^2}+\frac{C12\left(1-\frac{x}{h}\right)}{h} & \frac{C14}{h^2}-\frac{C12\left(1-\frac{x}{h}\right)}{h} & \frac{C24}{h^2}-\frac{C22\left(1-\frac{x}{h}\right)}{h} & -\frac{C24}{h^2}+\frac{C22\left(1-\frac{x}{h}\right)}{h} & \frac{C34}{h^2}-\frac{C23\left(1-\frac{x}{h}\right)}{h} & -\frac{C34}{h^2}+\frac{C23\left(1-\frac{x}{h}\right)}{h} & \\
\frac{C14}{h^2}+\frac{C12\,x}{h^2} & -\frac{C14}{h^2}-\frac{C12\,x}{h^2} & -\frac{C24}{h^2}-\frac{C22\,x}{h^2} & \frac{C24}{h^2}+\frac{C22\,x}{h^2} & -\frac{C34}{h^2}-\frac{C23\,x}{h^2} & \frac{C34}{h^2}+\frac{C23\,x}{h^2} & - \\
-\frac{C15}{h^2}+\frac{C11\left(1-\frac{x}{h}\right)}{h} & \frac{C15}{h^2}-\frac{C11\left(1-\frac{x}{h}\right)}{h} & \frac{C25}{h^2}-\frac{C12\left(1-\frac{x}{h}\right)}{h} & -\frac{C25}{h^2}+\frac{C12\left(1-\frac{x}{h}\right)}{h} & \frac{C35}{h^2}-\frac{C13\left(1-\frac{x}{h}\right)}{h} & -\frac{C35}{h^2}+\frac{C13\left(1-\frac{x}{h}\right)}{h} & \frac{C45}{h^2} \\
\frac{C15}{h^2}+\frac{C11\,x}{h^2} & -\frac{C15}{h^2}-\frac{C11\,x}{h^2} & -\frac{C25}{h^2}-\frac{C12\,x}{h^2} & \frac{C25}{h^2}+\frac{C12\,x}{h^2} & -\frac{C35}{h^2}-\frac{C13\,x}{h^2} & \frac{C35}{h^2}+\frac{C13\,x}{h^2} & - \\
-\frac{C16}{h^2} & \frac{C16}{h^2} & \frac{C26}{h^2} & -\frac{C26}{h^2} & \frac{C36}{h^2} & -\frac{C36}{h^2} & \\
\frac{C16}{h^2} & -\frac{C16}{h^2} & -\frac{C26}{h^2} & \frac{C26}{h^2} & -\frac{C36}{h^2} & \frac{C36}{h^2} &
\end{array}\right.
$$

Integrate the to get 12 x 12 Stiffness Matrix and Force Vector for Straight Beam

```
Jac = 2/h;
xtran[ξ_] = h/2 (ξ + 1);
(* Single point Gauss Integration weighting coefficient *)
w = 2;
(*MatrixForm[Expand[w Kint1[xtran[0]] 1/Jac /.
   {C11→GAKx,C12→0,C13→0,C14→0,C15→0,C16→0,C22→GAKy,C23→0,C24→0,C25→0,C26→0,
    C33→EA,C34→0,C35→0,C36→0,C44→EIxx,C45→0,C46→0,C55→EIyy,C56→0,C66→GJ}]]*)
Ketmp = Expand[w K1int[xtran[0]] 1/Jac];
MatrixForm[Ketmp]
Fetmp = Simplify[Integrate[f[x], {x, 0, h}] + Q];
MatrixForm[Fetmp]
```

$$\begin{pmatrix}
\frac{C11}{h} & -\frac{C11}{h} & -\frac{C12}{h} & \frac{C12}{h} & -\frac{C13}{h} & \frac{C13}{h} & \frac{C12}{2}-\frac{C14}{h} & \frac{C12}{2}\cdots \\
-\frac{C11}{h} & \frac{C11}{h} & \frac{C12}{h} & -\frac{C12}{h} & \frac{C13}{h} & -\frac{C13}{h} & -\frac{C12}{2}+\frac{C14}{h} & -\frac{C12}{2}\cdots \\
-\frac{C12}{h} & \frac{C12}{h} & \frac{C22}{h} & -\frac{C22}{h} & \frac{C23}{h} & -\frac{C23}{h} & -\frac{C22}{2}+\frac{C24}{h} & -\frac{C22}{2}\cdots \\
\frac{C12}{h} & -\frac{C12}{h} & -\frac{C22}{h} & \frac{C22}{h} & -\frac{C23}{h} & \frac{C23}{h} & \frac{C22}{2}-\frac{C24}{h} & \frac{C22}{2}\cdots \\
-\frac{C13}{h} & \frac{C13}{h} & \frac{C23}{h} & -\frac{C23}{h} & \frac{C33}{h} & -\frac{C33}{h} & -\frac{C23}{2}+\frac{C34}{h} & -\frac{C23}{2}\cdots \\
\frac{C13}{h} & -\frac{C13}{h} & -\frac{C23}{h} & \frac{C23}{h} & -\frac{C33}{h} & \frac{C33}{h} & \frac{C23}{2}-\frac{C34}{h} & \frac{C23}{2}\cdots \\
\frac{C12}{2}-\frac{C14}{h} & -\frac{C12}{2}+\frac{C14}{h} & -\frac{C22}{2}+\frac{C24}{h} & \frac{C22}{2}-\frac{C24}{h} & -\frac{C23}{2}+\frac{C34}{h} & \frac{C23}{2}-\frac{C34}{h} & -C24+\frac{C44}{h}+\frac{C22\,h}{4} & -\frac{C44}{h}\cdots \\
\frac{C12}{2}+\frac{C14}{h} & -\frac{C12}{2}-\frac{C14}{h} & -\frac{C22}{2}-\frac{C24}{h} & \frac{C22}{2}+\frac{C24}{h} & -\frac{C23}{2}-\frac{C34}{h} & \frac{C23}{2}+\frac{C34}{h} & -\frac{C44}{h}+\frac{C22\,h}{4} & C24+\cdots \\
\frac{C11}{2}-\frac{C15}{h} & -\frac{C11}{2}+\frac{C15}{h} & -\frac{C12}{2}+\frac{C25}{h} & \frac{C12}{2}-\frac{C25}{h} & -\frac{C13}{2}+\frac{C35}{h} & \frac{C13}{2}-\frac{C35}{h} & -\frac{C14}{2}-\frac{C25}{2}+\frac{C45}{h}+\frac{C12\,h}{4} & \frac{C14}{2}-\frac{C25}{2}-\cdots \\
\frac{C11}{2}+\frac{C15}{h} & -\frac{C11}{2}-\frac{C15}{h} & -\frac{C12}{2}-\frac{C25}{h} & \frac{C12}{2}+\frac{C25}{h} & -\frac{C13}{2}-\frac{C35}{h} & \frac{C13}{2}+\frac{C35}{h} & -\frac{C14}{2}+\frac{C25}{2}-\frac{C45}{h}+\frac{C12\,h}{4} & \frac{C14}{2}+\frac{C25}{2}+\cdots \\
-\frac{C16}{h} & \frac{C16}{h} & \frac{C26}{h} & -\frac{C26}{h} & \frac{C36}{h} & -\frac{C36}{h} & -\frac{C26}{2}+\frac{C46}{h} & -\frac{C26}{2}\cdots \\
\frac{C16}{h} & -\frac{C16}{h} & -\frac{C26}{h} & \frac{C26}{h} & -\frac{C36}{h} & \frac{C36}{h} & \frac{C26}{2}-\frac{C46}{h} & \frac{C26}{2}\cdots
\end{pmatrix}$$

$$\begin{pmatrix} Qx1+\frac{h\,vx}{2} \\ Qx2+\frac{h\,vx}{2} \\ Qy1+\frac{h\,vy}{2} \\ Qy2+\frac{h\,vy}{2} \\ P1+\frac{h\,pz}{2} \\ P2+\frac{h\,pz}{2} \\ Mx1 \\ Mx2 \\ My1 \\ My2 \\ T1+\frac{h\,tz}{2} \\ T2+\frac{h\,tz}{2} \end{pmatrix}$$

12 x 12 Element Stiffness Matrix for a beam with a Simple Cross-Section:

```
MatrixForm[Ketmp /. {C11 → GAKx, C12 → 0, C13 → 0, C14 → 0, C15 → 0, C16 → 0,
   C22 → GAKy, C23 → 0, C24 → 0, C25 → 0, C26 → 0, C33 → EA, C34 → 0, C35 → 0,
   C36 → 0, C44 → EIxx, C45 → 0, C46 → 0, C55 → EIyy, C56 → 0, C66 → GJ}]
```

$$\begin{pmatrix}
\frac{GAKx}{h} & -\frac{GAKx}{h} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{GAKx}{2} & \frac{GAKx}{2} \\
-\frac{GAKx}{h} & \frac{GAKx}{h} & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{GAKx}{2} & -\frac{GAKx}{2} \\
0 & 0 & \frac{GAKy}{h} & -\frac{GAKy}{h} & 0 & 0 & -\frac{GAKy}{2} & -\frac{GAKy}{2} & 0 & 0 \\
0 & 0 & -\frac{GAKy}{h} & \frac{GAKy}{h} & 0 & 0 & \frac{GAKy}{2} & \frac{GAKy}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{EA}{h} & -\frac{EA}{h} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{EA}{h} & \frac{EA}{h} & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{GAKy}{2} & \frac{GAKy}{2} & 0 & 0 & \frac{EIxx}{h}+\frac{GAKy\,h}{4} & -\frac{EIxx}{h}+\frac{GAKy\,h}{4} & 0 & 0 \\
0 & 0 & -\frac{GAKy}{2} & \frac{GAKy}{2} & 0 & 0 & -\frac{EIxx}{h}+\frac{GAKy\,h}{4} & \frac{EIxx}{h}+\frac{GAKy\,h}{4} & 0 & 0 \\
\frac{GAKx}{2} & -\frac{GAKx}{2} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{EIyy}{h}+\frac{GAKx\,h}{4} & -\frac{EIyy}{h}+\frac{GAKx\,h}{4} \\
\frac{GAKx}{2} & -\frac{GAKx}{2} & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{EIyy}{h}+\frac{GAKx\,h}{4} & \frac{EIyy}{h}+\frac{GAKx\,h}{4} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

Geometric Stiffness Matrix

Strain Energy for Ritz type formulation:

$$\Phi g = \begin{pmatrix} wx' & 0 & 0 & 0 & 0 & 0 \\ 0 & wy' & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix};$$

$$Pmat = \begin{pmatrix} P & 0 & 0 & 0 & 0 & 0 \\ 0 & P & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix};$$

```
Kg = MatrixForm[P Simplify[Φg. Transpose[Φg]]]
```

$$\begin{pmatrix} P\,(wx')^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & P\,(wy')^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Strain Energy for Finite Element formulation:

$$\Phi g = \begin{pmatrix} N1'[x] & N2'[x] & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & N1'[x] & N2'[x] & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix};$$

```
MatrixForm[Simplify[Transpose[Φg].Pmat.Φg]]
Kgint[x_] = Simplify[Transpose[Φg].Pmat.Φg];
```

$$\begin{pmatrix} \frac{P}{h^2} & -\frac{P}{h^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{P}{h^2} & \frac{P}{h^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{P}{h^2} & -\frac{P}{h^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{P}{h^2} & \frac{P}{h^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Generate Geometrix Stiffness Matrix and Integrate

```
Kgtmp = Integrate[Kgint[x], {x, 0, h}];
MatrixForm[Kgtmp]
```

$$\begin{pmatrix} \frac{P}{h} & -\frac{P}{h} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{P}{h} & \frac{P}{h} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{P}{h} & -\frac{P}{h} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{P}{h} & \frac{P}{h} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Geometric Stiffness Matrix

Mass Matrix For a Ritz Method Formulation:

$$\mathbf{M} = \begin{pmatrix} m & 0 & 0 & 0 & 0 & -m\,ym \\ 0 & m & 0 & 0 & 0 & m\,xm \\ 0 & 0 & m & m\,ym & -m\,xm & 0 \\ 0 & 0 & m\,ym & Ixx & -Ixy & 0 \\ 0 & 0 & -m\,xm & -Ixy & Iyy & 0 \\ -m\,ym & m\,xm & 0 & 0 & 0 & Ixx + Iyy \end{pmatrix};$$

$$\mathbf{M} = \begin{pmatrix} M11 & 0 & 0 & 0 & 0 & M16 \\ 0 & M11 & 0 & 0 & 0 & M26 \\ 0 & 0 & M11 & -M16 & -M26 & 0 \\ 0 & 0 & -M16 & M44 & M45 & 0 \\ 0 & 0 & -M26 & M45 & M55 & 0 \\ M16 & M26 & 0 & 0 & 0 & M66 \end{pmatrix};$$

$$\Phi = \begin{pmatrix} u1[x] & u2[x] & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & v1[x] & v2[x] & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & w1[x] & w2[x] & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \psi1[x] & \psi2[x] & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \gamma1[x] & \gamma2[x] & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \phi1[x] & \phi2[x] \end{pmatrix};$$

```
Mmat = Expand[Simplify[Transpose[Φ].M.Φ]];
MatrixForm[Mmat]
```

$$\begin{pmatrix} M11\,u1[x]^2 & M11\,u1[x]\,u2[x] & 0 & 0 & 0 & \\ M11\,u1[x]\,u2[x] & M11\,u2[x]^2 & 0 & 0 & 0 & \\ 0 & 0 & M11\,v1[x]^2 & M11\,v1[x]\,v2[x] & 0 & \\ 0 & 0 & M11\,v1[x]\,v2[x] & M11\,v2[x]^2 & 0 & \\ 0 & 0 & 0 & 0 & M11\,w1[x]^2 & M11\,w \\ 0 & 0 & 0 & 0 & M11\,w1[x]\,w2[x] & M1 \\ 0 & 0 & 0 & 0 & -M16\,w1[x]\,\psi1[x] & -M16\,w \\ 0 & 0 & 0 & 0 & -M16\,w1[x]\,\psi2[x] & -M16\,w \\ 0 & 0 & 0 & 0 & -M26\,w1[x]\,\gamma1[x] & -M26\,w \\ 0 & 0 & 0 & 0 & -M26\,w1[x]\,\gamma2[x] & -M26\,w \\ M16\,u1[x]\,\phi1[x] & M16\,u2[x]\,\phi1[x] & M26\,v1[x]\,\phi1[x] & M26\,v2[x]\,\phi1[x] & 0 & \\ M16\,u1[x]\,\phi2[x] & M16\,u2[x]\,\phi2[x] & M26\,v1[x]\,\phi2[x] & M26\,v2[x]\,\phi2[x] & 0 & \end{pmatrix}$$

Substitute Shape Functions Into Strain/Displacement Matrix and Force Vector

```
Mint[x_] = Mmat /. {u1[x] → N1[x], u2[x] → N2[x], v1[x] → N1[x],
    v2[x] → N2[x], w1[x] → N1[x], w2[x] → N2[x], ψ1[x] → N1[x],
    ψ2[x] → N2[x], γ1[x] → N1[x], γ2[x] → N2[x], ϕ1[x] → N1[x], ϕ2[x] → N2[x],
    ψ1[x]  →  N1[x], ψ2[x] → N2[x], γ1[x] → N1[x], γ2[x] → N2[x]};
MatrixForm[
 Mint[
  x]]
```

$$\begin{pmatrix}
M11\left(1-\frac{x}{h}\right)^2 & \frac{M11\,x\left(1-\frac{x}{h}\right)}{h} & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{M11\,x\left(1-\frac{x}{h}\right)}{h} & \frac{M11\,x^2}{h^2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & M11\left(1-\frac{x}{h}\right)^2 & \frac{M11\,x\left(1-\frac{x}{h}\right)}{h} & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{M11\,x\left(1-\frac{x}{h}\right)}{h} & \frac{M11\,x^2}{h^2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & M11\left(1-\frac{x}{h}\right)^2 & \frac{M11\,x\left(1-\frac{x}{h}\right)}{h} & -M16\left(1-\frac{x}{h}\right)^2 & -\frac{M16\,x}{} \\
0 & 0 & 0 & 0 & \frac{M11\,x\left(1-\frac{x}{h}\right)}{h} & \frac{M11\,x^2}{h^2} & -\frac{M16\,x\left(1-\frac{x}{h}\right)}{h} & -\frac{M1}{h} \\
0 & 0 & 0 & 0 & -M16\left(1-\frac{x}{h}\right)^2 & -\frac{M16\,x\left(1-\frac{x}{h}\right)}{h} & M44\left(1-\frac{x}{h}\right)^2 & \frac{M44\,x\,(}{h} \\
0 & 0 & 0 & 0 & -\frac{M16\,x\left(1-\frac{x}{h}\right)}{h} & -\frac{M16\,x^2}{h^2} & \frac{M44\,x\left(1-\frac{x}{h}\right)}{h} & \frac{M44}{h^2} \\
0 & 0 & 0 & 0 & -M26\left(1-\frac{x}{h}\right)^2 & -\frac{M26\,x\left(1-\frac{x}{h}\right)}{h} & M45\left(1-\frac{x}{h}\right)^2 & \frac{M45\,x\,(}{h} \\
0 & 0 & 0 & 0 & -\frac{M26\,x\left(1-\frac{x}{h}\right)}{h} & -\frac{M26\,x^2}{h^2} & \frac{M45\,x\left(1-\frac{x}{h}\right)}{h} & \frac{M45}{h^2} \\
M16\left(1-\frac{x}{h}\right)^2 & \frac{M16\,x\left(1-\frac{x}{h}\right)}{h} & M26\left(1-\frac{x}{h}\right)^2 & \frac{M26\,x\left(1-\frac{x}{h}\right)}{h} & 0 & 0 & 0 & 0 \\
\frac{M16\,x\left(1-\frac{x}{h}\right)}{h} & \frac{M16\,x^2}{h^2} & \frac{M26\,x\left(1-\frac{x}{h}\right)}{h} & \frac{M26\,x^2}{h^2} & 0 & 0 & 0 & 0
\end{pmatrix}$$

Integrate the to get 12 x 12 Mass Matrix Straight Beam

```
Metmp = ∫_0^h Mint[x] dx;
MatrixForm[Metmp]
```

$$\begin{pmatrix}
\frac{h M11}{3} & \frac{h M11}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{h M16}{3} & \frac{h M16}{6} \\
\frac{h M11}{6} & \frac{h M11}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{h M16}{6} & \frac{h M16}{3} \\
0 & 0 & \frac{h M11}{3} & \frac{h M11}{6} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{h M26}{3} & \frac{h M26}{6} \\
0 & 0 & \frac{h M11}{6} & \frac{h M11}{3} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{h M26}{6} & \frac{h M26}{3} \\
0 & 0 & 0 & 0 & \frac{h M11}{3} & \frac{h M11}{6} & -\frac{h M16}{3} & -\frac{h M16}{6} & -\frac{h M26}{3} & -\frac{h M26}{6} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{h M11}{6} & \frac{h M11}{3} & -\frac{h M16}{6} & -\frac{h M16}{3} & -\frac{h M26}{6} & -\frac{h M26}{3} & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{h M16}{3} & -\frac{h M16}{6} & \frac{h M44}{3} & \frac{h M44}{6} & \frac{h M45}{3} & \frac{h M45}{6} & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{h M16}{6} & -\frac{h M16}{3} & \frac{h M44}{6} & \frac{h M44}{3} & \frac{h M45}{6} & \frac{h M45}{3} & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{h M26}{3} & -\frac{h M26}{6} & \frac{h M45}{3} & \frac{h M45}{6} & \frac{h M55}{3} & \frac{h M55}{6} & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{h M26}{6} & -\frac{h M26}{3} & \frac{h M45}{6} & \frac{h M45}{3} & \frac{h M55}{6} & \frac{h M55}{3} & 0 & 0 \\
\frac{h M16}{3} & \frac{h M16}{6} & \frac{h M26}{3} & \frac{h M26}{6} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{h M66}{3} & \frac{h M66}{6} \\
\frac{h M16}{6} & \frac{h M16}{3} & \frac{h M26}{6} & \frac{h M26}{3} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{h M66}{6} & \frac{h M66}{3}
\end{pmatrix}$$

Reorder All Matricies for Easier Global Assembly

Reorder Vector:

```
reorder = {1, 3, 5, 7, 9, 11, 2, 4, 6, 8, 10, 12};
```

12 x 12 RIE Timoshenko Beam Stiffness Matrix:

$$\text{Ke} = \begin{pmatrix} 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0 \end{pmatrix};$$

$$\text{Fe} = \begin{pmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0 \end{pmatrix};$$

```
Do[Do[Ke[[i, j]] = Expand[Ketmp[[reorder[[i]], reorder[[j]]]]], {i, 12}], {j, 12}]
Do[Fe[[i]] = Fetmp[[reorder[[i]]]], {i, 12}]
MatrixForm[Ke]
MatrixForm[Fe]
```

$$\left(\begin{array}{lllllll}
\frac{C11}{h} & -\frac{C12}{h} & -\frac{C13}{h} & \frac{C12}{2}-\frac{C14}{h} & \frac{C11}{2}-\frac{C15}{h} & -\frac{C16}{h} & -\frac{C11}{h} \\
-\frac{C12}{h} & \frac{C22}{h} & \frac{C23}{h} & -\frac{C22}{2}+\frac{C24}{h} & -\frac{C12}{2}+\frac{C25}{h} & \frac{C26}{h} & \frac{C12}{h} \\
-\frac{C13}{h} & \frac{C23}{h} & \frac{C33}{h} & -\frac{C23}{2}+\frac{C34}{h} & -\frac{C13}{2}+\frac{C35}{h} & \frac{C36}{h} & \frac{C13}{h} \\
\frac{C12}{2}-\frac{C14}{h} & -\frac{C22}{2}+\frac{C24}{h} & -\frac{C23}{2}+\frac{C34}{h} & -C24+\frac{C44}{h}+\frac{C22\,h}{4} & -\frac{C14}{2}-\frac{C25}{2}+\frac{C45}{h}+\frac{C12\,h}{4} & -\frac{C26}{2}+\frac{C46}{h} & -\frac{C12}{2}+ \\
\frac{C11}{2}-\frac{C15}{h} & -\frac{C12}{2}+\frac{C25}{h} & -\frac{C13}{2}+\frac{C35}{h} & -\frac{C14}{2}-\frac{C25}{2}+\frac{C45}{h}+\frac{C12\,h}{4} & -C15+\frac{C55}{h}+\frac{C11\,h}{4} & -\frac{C16}{2}+\frac{C56}{h} & -\frac{C11}{2}+ \\
-\frac{C16}{h} & \frac{C26}{h} & \frac{C36}{h} & -\frac{C26}{2}+\frac{C46}{h} & -\frac{C16}{2}+\frac{C56}{h} & \frac{C66}{h} & \frac{C16}{h} \\
-\frac{C11}{h} & \frac{C12}{h} & \frac{C13}{h} & -\frac{C12}{2}+\frac{C14}{h} & -\frac{C11}{2}+\frac{C15}{h} & \frac{C16}{h} & \frac{C11}{h} \\
\frac{C12}{h} & -\frac{C22}{h} & -\frac{C23}{h} & \frac{C22}{2}-\frac{C24}{h} & \frac{C12}{2}-\frac{C25}{h} & -\frac{C26}{h} & -\frac{C12}{h} \\
\frac{C13}{h} & -\frac{C23}{h} & -\frac{C33}{h} & \frac{C23}{2}-\frac{C34}{h} & \frac{C13}{2}-\frac{C35}{h} & -\frac{C36}{h} & -\frac{C13}{h} \\
\frac{C12}{2}+\frac{C14}{h} & -\frac{C22}{2}-\frac{C24}{h} & -\frac{C23}{2}-\frac{C34}{h} & -\frac{C44}{h}+\frac{C22\,h}{4} & \frac{C14}{2}-\frac{C25}{2}-\frac{C45}{h}+\frac{C12\,h}{4} & -\frac{C26}{2}-\frac{C46}{h} & -\frac{C12}{2}- \\
\frac{C11}{2}+\frac{C15}{h} & -\frac{C12}{2}-\frac{C25}{h} & -\frac{C13}{2}-\frac{C35}{h} & -\frac{C14}{2}+\frac{C25}{2}-\frac{C45}{h}+\frac{C12\,h}{4} & -\frac{C55}{h}+\frac{C11\,h}{4} & -\frac{C16}{2}-\frac{C56}{h} & -\frac{C11}{2}- \\
\frac{C16}{h} & -\frac{C26}{h} & -\frac{C36}{h} & \frac{C26}{2}-\frac{C46}{h} & \frac{C16}{2}-\frac{C56}{h} & -\frac{C66}{h} & -\frac{C16}{h}
\end{array}\right.$$

$$\begin{pmatrix} Qx1 + \frac{h\,vx}{2} \\ Qy1 + \frac{h\,vy}{2} \\ P1 + \frac{h\,pz}{2} \\ Mx1 \\ My1 \\ T1 + \frac{h\,tz}{2} \\ Qx2 + \frac{h\,vx}{2} \\ Qy2 + \frac{h\,vy}{2} \\ P2 + \frac{h\,pz}{2} \\ Mx2 \\ My2 \\ T2 + \frac{h\,tz}{2} \end{pmatrix}$$

12 x 12 RIE Timoshenko Beam Geometric Stiffness Matrix:

```
Keg = 0 0 0 0 0 0 0 0 0 0 0 0
      0 0 0 0 0 0 0 0 0 0 0 0
      0 0 0 0 0 0 0 0 0 0 0 0
      0 0 0 0 0 0 0 0 0 0 0 0
      0 0 0 0 0 0 0 0 0 0 0 0
      0 0 0 0 0 0 0 0 0 0 0 0
      0 0 0 0 0 0 0 0 0 0 0 0
      0 0 0 0 0 0 0 0 0 0 0 0
      0 0 0 0 0 0 0 0 0 0 0 0
      0 0 0 0 0 0 0 0 0 0 0 0
      0 0 0 0 0 0 0 0 0 0 0 0
      0 0 0 0 0 0 0 0 0 0 0 0;
```

```
Do[Do[Keg[[i, j]] = Expand[Kgtmp[[reorder[[i]], reorder[[j]]]]], {i, 12}], {j, 12}]
MatrixForm[Keg]
```

$$\begin{pmatrix} \frac{P}{h} & 0 & 0 & 0 & 0 & 0 & -\frac{P}{h} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{P}{h} & 0 & 0 & 0 & 0 & 0 & -\frac{P}{h} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{P}{h} & 0 & 0 & 0 & 0 & 0 & \frac{P}{h} & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{P}{h} & 0 & 0 & 0 & 0 & 0 & \frac{P}{h} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

12 x 12 Timoshenko Beam Mass Matrix:

$$\mathbf{Me} = \begin{matrix} 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0 \end{matrix};$$

```
Do[Do[Me[[i, j]] = Expand[Metmp[[reorder[[i]], reorder[[j]]]]], {i, 12}], {j, 12}]
MatrixForm[Me]
```

$$\begin{pmatrix}
\frac{h\,M11}{3} & 0 & 0 & 0 & 0 & \frac{h\,M16}{3} & \frac{h\,M11}{6} & 0 & 0 & 0 & 0 & \frac{h\,M16}{6} \\
0 & \frac{h\,M11}{3} & 0 & 0 & 0 & \frac{h\,M26}{3} & 0 & \frac{h\,M11}{6} & 0 & 0 & 0 & \frac{h\,M26}{6} \\
0 & 0 & \frac{h\,M11}{3} & -\frac{h\,M16}{3} & -\frac{h\,M26}{3} & 0 & 0 & 0 & \frac{h\,M11}{6} & -\frac{h\,M16}{6} & -\frac{h\,M26}{6} & 0 \\
0 & 0 & -\frac{h\,M16}{3} & \frac{h\,M44}{3} & \frac{h\,M45}{3} & 0 & 0 & 0 & -\frac{h\,M16}{6} & \frac{h\,M44}{6} & \frac{h\,M45}{6} & 0 \\
0 & 0 & -\frac{h\,M26}{3} & \frac{h\,M45}{3} & \frac{h\,M55}{3} & 0 & 0 & 0 & -\frac{h\,M26}{6} & \frac{h\,M45}{6} & \frac{h\,M55}{6} & 0 \\
\frac{h\,M16}{3} & \frac{h\,M26}{3} & 0 & 0 & 0 & \frac{h\,M66}{3} & \frac{h\,M16}{6} & \frac{h\,M26}{6} & 0 & 0 & 0 & \frac{h\,M66}{6} \\
\frac{h\,M11}{6} & 0 & 0 & 0 & 0 & \frac{h\,M16}{6} & \frac{h\,M11}{3} & 0 & 0 & 0 & 0 & \frac{h\,M16}{3} \\
0 & \frac{h\,M11}{6} & 0 & 0 & 0 & \frac{h\,M26}{6} & 0 & \frac{h\,M11}{3} & 0 & 0 & 0 & \frac{h\,M26}{3} \\
0 & 0 & \frac{h\,M11}{6} & -\frac{h\,M16}{6} & -\frac{h\,M26}{6} & 0 & 0 & 0 & \frac{h\,M11}{3} & -\frac{h\,M16}{3} & -\frac{h\,M26}{3} & 0 \\
0 & 0 & -\frac{h\,M16}{6} & \frac{h\,M44}{6} & \frac{h\,M45}{6} & 0 & 0 & 0 & -\frac{h\,M16}{3} & \frac{h\,M44}{3} & \frac{h\,M45}{3} & 0 \\
0 & 0 & -\frac{h\,M26}{6} & \frac{h\,M45}{6} & \frac{h\,M55}{6} & 0 & 0 & 0 & -\frac{h\,M26}{3} & \frac{h\,M45}{3} & \frac{h\,M55}{3} & 0 \\
\frac{h\,M16}{6} & \frac{h\,M26}{6} & 0 & 0 & 0 & \frac{h\,M66}{6} & \frac{h\,M16}{3} & \frac{h\,M26}{3} & 0 & 0 & 0 & \frac{h\,M66}{3}
\end{pmatrix}$$