Convex Optimization assignment 2

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Week 3

Problem 1 (2.21)

The set of separating hyperplanes. Suppose that C and D are disjoint subsets of $\mathbf{R}^n$. Consider the set of $(a, b) \in \mathbf{R}^{n+1}$ for which $a^T x \leq b$ for all $x \in C$, and $a^T x \geq b$ for all $x \in D$. Show that this set is a convex cone (which is the singleton $\{0\}$ if there is no hyperplane that separates C and D).

Solution:
Let the set contains all the (a, b) be S. Since that for (a, b) be the parameter of two set, $(ka)^T x \leq kb$ for all $x \in C$ and $(ka)^T x \geq kb$ for all $x \in D$, so $(ka, kb) \in S$, S is a cone.
For all $(a, b), (a', b') \in S$, and $\alpha, \beta \in \mathbb{R}^+$, we have

$$(\alpha a + \beta a')^T x \leq (\alpha b + \beta b') \Leftarrow \begin{cases} a^T x \leq b \\ a'^T x \leq b' \end{cases}$$

for all $x \in C$, and

$$(\alpha a + \beta a')^T x \geq (\alpha b + \beta b') \Leftarrow \begin{cases} a^T x \geq b \\ a'^T x \geq b' \end{cases}$$

for all $x \in D$, so S is a convex cone.

Problem 2 (2.24)

(a) Express the closed convex set $\{x \in \mathbb{R}_+^2 \, | x_1 x_2 \geq 1\}$ as an intersection of halfspace.

Solution:
Let the closed convex set be C. Clearly, **bd** $C = \left\{x \in \mathbb{R}_+^2 \, | x_1 x_2 = 1\right\}$. Choosing all the point in **bd** C and apply the supporting hyperplane theorem. That is,

$$C = \bigcup_{p \in \mathbb{R}_+} \left\{x \in \mathbb{R}_+^2 \, | x_2 - \frac{1}{p} \geq -\frac{1}{x_1^2}(x_1 - p)\right\}$$

(b) Let $C = \{x \in \mathbb{R}^n \, | \|x\|_\infty \leq 1\}$, the l_∞ norm unit ball in $\mathbb{R}^n$, and let $\hat{x}$ be the point in the boundary of C. Identify the supporting hyperplane theorem of C at $\hat{x}$ explicitly.

Solution:

Suppose we have $s^T x \geq s^T \hat{x}$ for all $x \in C$, for which

$$\begin{cases} s < 0, \hat{x} = 1 \\ s > 0, \hat{x} = -1 \\ s = 0, \text{otherwise} \end{cases}$$

Problem 3 (2.30,2.31)

$$x \not\prec_K x$$

Proof:

If $x \prec_K x$, we have $x - x = 0 \in \textbf{int}\, K$, which means there exist a tiny ball $B(0, \epsilon) \in K$, and then $\epsilon, -\epsilon \in K$, which is contradicted to the principle "pointed".

K^* is a convex cone although K is not convex.

Proof:

Suppose $x \in K, y, z \in K^*$, so $x^T y \geq 0, x^T z \geq 0$, also $x^T(\alpha y + \beta z) \geq 0$ for which $\alpha, \beta \geq 0, \alpha + \beta = 1$. Then $\alpha y + \beta z \in K^*$, and K^* is convex.

Problem 4 (2.33)

The monotone nonnegative cone. We define the monotone nonnegative cone as

$$K_{\mathrm{m}+} = \{x \in \mathbf{R}^n \mid x_1 \geq x_2 \geq \cdots \geq x_n \geq 0\}.$$

i.e., all nonnegative vectors with components sorted in nonincreasing order.

(a) Show that $K_{\mathrm{m}+}$ is a proper cone.

Solution:

Let $x, y \in K_{m+}$, $tx + (1-t)y$ satisfying

$$tx_1 + (1-t)y_1 \geq tx_1 + (1-t)y_2 \geq tx_2 + (1-t)y_2 \geq tx_3 + (1-t)y_3 \geq \cdots \geq 0$$

So $tx + (1-t)y \in K_{m+}$, and the cone is convex.

The boundary of K_{m+} is $\bigcap_{i,j=1}^{n} \{x \in \mathbb{R}^n\, | x_i = x_j\} \in K_{m+}$, so the cone is closed.

As above, $\textbf{bd}\, K \neq K$, so $\textbf{int}\, K \neq \varnothing$, and the cone is solid.

For all $0 \leq i \leq n$, $x_i \geq 0$, so the cone is pointed.

(b) Find the dual cone $K_{\mathrm{m}+}^*$. Hint. Use the identity

$$\sum_{i=1}^n x_i y_i = (x_1 - x_2)\, y_1 + (x_2 - x_3)\,(y_1 + y_2) + (x_3 - x_4)\,(y_1 + y_2 + y_3) + \cdots$$
$$+ (x_{n-1} - x_n)\,(y_1 + \cdots + y_{n-1}) + x_n\,(y_1 + \cdots + y_n)$$

Solution:
Using the hint, we see that $y^T x \geq 0$ for all $x \in K_{\mathrm{m}+}$ if and only if

$$y_1 \geq 0, \quad y_1 + y_2 \geq 0, \quad \ldots, y_1 + y_2 + \cdots + y_n \geq 0$$

Therefore

$$K_{\mathrm{m}+}^* = \left\{ y \,\middle|\, \sum_{i=1}^k y_i \geq 0, k = 1, \ldots, n \right\}$$

Week 4

Problem 1 (3.6)

Functions and epigraphs. When is the epigraph of a function a halfspace? When is the epigraph of a function a convex cone? When is the epigraph of a function a polyhedron?

Solution:
For halfspace, the function is affine, so $\left\{x|a^T x + b \geq 0\right\}$ is a halfspace.
For convex cone, let the function satisfy $tf(y) \leq f(ty), t \in \mathbb{R}_+$. Then $tx + (1-t)x' \geq f(ty + (1-t)y')$ and the epigraph is convex cone.
For polyhedron, let the function be piecewisely affine. Then the epigraph is the intersection of many halfspace, which is exactly polyhedron.

Problem 2 (3.17,3.18)

Suppose $p < 1, p \neq 0$. Show that the function

$$f(x) = \left(\sum_{i=1}^n x_i^p\right)^{1/p}$$

with **dom** $f = \mathbb{R}_{++}^n$ is concave.

Solution:
The first order derivative is

$$\nabla f(x) = \left(\frac{f(x)}{x_i}\right)^{1-p}$$

And for $i \neq j$,

$$\nabla^2 f(x) = \frac{1-p}{f(x)} \left(\frac{f(x)^2}{x_i x_j} \right)$$

For $i = j$

$$\nabla^2 f(x) = \frac{1-p}{f(x)} \left(\frac{f(x)^2}{x_i^2} \right)^{1-p} - \frac{1-p}{x_i} \left(\frac{f(x)}{x_i} \right)^{1-p}$$

By Cauchy-Schwarz inequality:

$$y^T \nabla^2 f(x) y = \frac{1-p}{f(x)} \left(\left(\sum_{i=1}^{n} \frac{y_i f(x)^{1-p}}{x_i^{1-p}} \right)^2 - \sum_{i=1}^{n} \frac{y_i^2 f(x)^{2-p}}{x_i^{2-p}} \right) \leq 0$$

Show that $f(x) = \mathbf{tr}\,(X^{-1})$ is convex on $\mathbf{dom}\ f(x) = S_{++}^n$

Solution:

We notate $g(t) = f(Z + tV)$ where $Z + tV \in S_{++}^n$. Suppose that we have diaginal $Z^{-1/2} V Z^{-1/2} = Q \Lambda Q^T$

$$\begin{aligned} g(t) &= \mathbf{tr}\,(Z + tV) \\ &= \mathbf{tr}\left(Z^{-1/2} \left(I + t Z^{-1/2} V Z^{-1/2} \right)^{-1} Z^{-1/2} \right) \\ &= \mathbf{tr}\left(Z^{-1} Q \left(I + t\Lambda \right)^{-1} Q^T \right) \\ &= \mathbf{tr} \sum_{i=1}^{n} \left(Z^{-1} Q Q^T \right)_{ii} (1 + t\lambda) \end{aligned}$$

Since that $1 + t\lambda$ is convex, the weighted sum of convex function preserves convex.

Problem 3 (3.29)

Representation of piecewise-linear convex functions. A convex function $f : \mathbf{R}^n \to \mathbf{R}$, with $\mathrm{dom}\, f = \mathbf{R}^n$, is called piecewise-linear if there exists a partition of $\mathbf{R}^n$ as

$$\mathbf{R}^n = X_1 \cup X_2 \cup \cdots \cup X_L$$

where int $X_i \neq \emptyset$ and int $X_i \cap$ int $X_j = \emptyset$ for $i \neq j$, and a family of affine functions $a_1^T x + b_1, \ldots, a_L^T x + b_L$ such that $f(x) = a_i^T x + b_i$ for $x \in X_i$. Show that this means that $f(x) = \max\left\{ a_1^T x + b_1, \ldots, a_L^T x + b_L \right\}$

Solution:

First we suppose $\overline{X_i} \cap \overline{X_j} = t$. For any $x \in X_i, y \in X_j$, we have:

$$\frac{(y-t)f(y) - (t-x)f(x)}{y-x} \geq f(t)$$

That is:

$$\frac{(y-t)(a_j^T y + b_j) - (t-x)(a_i^T x + b_i)}{y-x} \geq a_i^T t + b_i$$

So

$$a_i^T x + b_i \geq a_j^T x + b_j$$

Consider $\overline{X_i} \cap \overline{X_j} = \varnothing$, we can find the neighborhood and transit it to.

Problem 4 (3.36.1)

Derive the conjugates function of $f(x) = \max_{i=1,2,\cdots,n}\{x_i\}$ on $\mathbb{R}^n$

Solution:

$$f^*(y) = \begin{cases} 0 & \text{if } y \succeq 0, \quad \mathbf{1}^T y = 1 \\ \infty & \text{otherwise} \end{cases}$$