Stochastic Process week 3 Exercise

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Chap 4: 20, 22, 23, 32, 38, 45, 47

20 A transition probability matrix $\mathbf{P}$ is said to be doubly stochastic is the sum over each column equals one; that is,

$$\sum_i P_{ij} = 1, \quad \text{for all } j$$

If such a chain is irreducible and consists of $M+1$ states $0, 1, \cdots, M$, show that the long-run proportions are given by

$$\pi_j = \frac{1}{M+1}, \quad j = 0, 1, \cdots, M$$

Solution:

Since that $\pi = \pi\mathbf{P}$, for all j we have:

$$\begin{cases} \pi_j = \sum_i \pi_i P_{ij}, & \forall j \\ \sum_i P_{ij} = \sum_j P_{ij} = 1 \\ \sum_i \pi_i = 1 \end{cases}$$

22 Let Y_n be the sum of n independent rolls of a fair die. Find

$$\lim_{n\to\infty} P\left\{Y_n \text{ is a multiple of 13 }\right\}$$

Solution:

Let $\pi_1 = (0, 1/6, 1/6, 1/6, 1/6, 1/6, 1/6, 0, 0, 0, 0, 0, 0)$ and The transition matrix is:

$$\mathbf{P}=\begin{bmatrix}
0 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6\\
1/6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6\\
1/6 & 1/6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/6 & 1/6 & 1/6 & 1/6\\
1/6 & 1/6 & 1/6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/6 & 1/6 & 1/6\\
1/6 & 1/6 & 1/6 & 1/6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/6 & 1/6\\
1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/6\\
1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

Which satisfy exercise 20. Then for a stationary distribution $\pi_i = \frac{1}{13}$

23 In a good weather year the number of storms is Poisson distributed with mean 1; in a bad year it is Poisson distributed with mean 3. Suppose that any year's weather conditions depends on past years only through the previous year's condition. Suppose that a good year is equally likely to be followed by either a good or a bad year, and that a bad year is twice as likely to be followed by a bad year as by a good year. Suppose that last year-call it year 0-was a good year.

(a) Find the expected total number of storms in the next two years (that is, in years 1 and 2).

(b) Find the probability there are no storms in year 3.

(c) Find the long-run average number of storms per year.

(d) Find the proportion of years that have no storms.

Solution:

Let $X_i, i = 0, 1, 2, \cdots$ denote the weather, 0 for good and 1 for bad weather. The transition matrix is

$$\mathbf{P} = \begin{bmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{bmatrix}$$

And $\pi_0 = (1, 0)$. Let Y_i denote the #storms of year i, and

$$Y_i | X_i \sim \begin{cases} \text{Pois}(1), & X_i = 0 \\ \text{Pois}(3), & X_i = 1 \end{cases}$$

(a)

$$
\begin{aligned}
&\mathrm{E}\left(Y_1+Y_2\right)\\
=&\mathrm{E}\left(Y_1\right)+\mathrm{E}\left(Y_2\right)\\
=&p_{00}\times 1+p_{01}\times 3+\left(p_{00}^2+p_{01}p_{10}\right)\times 1+\left(p_{00}p_{01}+p_{01}p_{11}\right)\times 3\\
=&\frac{25}{6}
\end{aligned}
$$

(b)

$$
\begin{aligned}
&P(Y_3=0)\\
=&P(Y_3=0|X_3=0)P(X_3=0)+P(Y_3=0|X_3=1)P(X_3=1)\\
=&e^{-1}(p_{00}^3+2p_{00}p_{01}p_{10}+p_{01}p_{11}p_{10})+e^{-3}(3p_{00}^2p_{01}+p_{01}^2p_{10})\\
=&\frac{29}{72}e^{-1}+\frac{11}{24}e^{-3}
\end{aligned}
$$

(c) Let $\pi=\mathbf{P}\pi$, $\pi=(2/5,3/5)$. So,

$$
\begin{aligned}
\mathrm{E}\left(Y\right)&=\mathrm{E}\left(Y|X=0\right)P(X=0)+\mathrm{E}\left(Y|X=1\right)P(X=1)\\
&=1\times\frac{2}{5}+3\times\frac{3}{5}=\frac{11}{5}
\end{aligned}
$$

(d)

$$
\begin{aligned}
&P(Y=0)\\
=&P(Y=0|X=0)P(X=0)+P(Y=0|X=1)P(X=1)\\
=&\frac{2}{5}e^{-1}+\frac{3}{5}e^{-3}
\end{aligned}
$$

32 Each of two switches is either on or off during a day. On day n, each switch will independently be on with probability

$$[1+\text{ number of on switches during day } n-1]/4$$

For instance, if both switches are on during day $n-1$, then each will independently be on during day n with probability $3/4$. What fraction of days are both switches on? What fraction are both off?

Solution:

There are three state:0,1,2 switch(es) is on. So, the probability vector be π, the transition matrix is

$$\mathbf{P} = \begin{bmatrix} 9/16 & 6/16 & 1/16 \\ 1/4 & 1/2 & 1/4 \\ 1/16 & 6/16 & 9/16 \end{bmatrix}$$

The stationary distribution is $\pi = (3/8, 1/4, 3/8)$.

38 Capa plays either one or two chess games every day, with the number of games that she plays on successive days being a Markov chain with transition probabilities

$$P_{1,1} = .2, \quad P_{1,2} = .8 \quad P_{2,1} = .4, \quad P_{2,2} = .6$$

Capa wins each game with probability p. Suppose she plays two games on Monday.

(a) What is the probability that she wins all the games she plays on Tuesday?

(b) What is the expected number of games that she plays on Wednesday?

(c) In the long run, on what proportion of days does Capa win all her games.

Solution:

(a)

$$P = P_{11}p + P_{12}p^2 = 0.2p + 0.8p^2$$

(b)

$$E(X) = 1 \times \left(P_{11}^2 + P_{12}P_{21}\right) + 2 \times (P_{12}P_{22} + P_{11}P_{12}) = 1.48$$

(c) Let π denote the stationary distribution, $\pi = \pi\mathbf{P}$. Then $\pi = (1/3, 2/3)$

45 Consider an irreducible finite Markov chain with states $0, 1, \cdots N$.

(a) Starting in state i, what is the probability the process will ever visit state j? Explain.

(b) Let $x_i = P\{\text{visit state N before state } 0|\text{start in } i\}$. Compute a set of linear equations that the x_i satisfy $i = 0, 1, \cdots, N$.

(c) If $\sum_j jP_{ij} = i$ for $i = 1, \cdots, N-1$, show that $x_i = i/N$ is a solution to the equations in part (b).

Solution:

(a) Let the first visit probability in time t be f_{ij}^t, the acquiring probability is $\sum_{t=1}^{\infty} f_{ij}^t$.

(b)

$$\begin{cases} x_0 = 0 \\ x_N = 1 \\ x_i = \sum_j jP_{ij}/N \end{cases}$$

47 Let $\{X_n, n \geq 0\}$ denote an ergodic Markov chain with limiting probabilities π_i. Define the process $\{Y_n, n \geq 1\}$ by $Y_n = (X_{n-1}, X_n)$. That is, Y_n keeps track of the last two states of the original chain. Is $\{Y_n, n \geq 1\}$ a Markov chain? If so, determine its transition probabilities and find

$$\lim_{n \to \infty} P\{Y_n = (i, j)\}$$

Solution:

It is a Markov chain. To show that, Y_n only depends on X_n, and can be derived from y_{n-1}.

$$\lim_{n \to \infty} P\{Y_n = (i, j)\} = |\pi_j - \pi_i|$$