“Sapienza” University of Rome
Faculty of Information Engineering,
Informatics and Statistics
Department of Computer Science

Discrete Mathematics

Lecture notes integrated with the book "Discrete Mathematics",
Norman L. Biggs

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Information and Contacts

Personal notes and summaries collected as part of the *Discrete Mathematics* course offered by the degree in Computer Science of the University of Rome "La Sapienza".

Further information and notes can be found at the following link:
https://github.com/Exyss/university-notes. Anyone can feel free to report inaccuracies, improvements or requests through the Issue system provided by GitHub itself or by contacting the author privately:

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The notes are constantly being updated, so please check if the changes have already been made in the most recent version.

Suggested prerequisites:

Preventive learning of material related to the *Algebra* course is recommended

Licence:

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Introduction to Discrete Maths

1.1 Solved exercises

Problem 1

Let $S = \{4n - 3 \mid n \in \mathbb{N}\}$ and let $S_{\mathbb{P}} \subseteq S$ be the set of S-prime numbers, that being the numbers in S that have exactly two factors (1 and itself) in S.

1. Prove that S is closed under multiplication.
2. Are there infinitely many S-prime numbers?
3. Prove that $1617 \in S$ and find two different factorizations of 1617 into S-primes.
4. Find a few more examples of S-integers with more than one factorization.

Solution:

First, we formally define $S_{\mathbb{P}}$ as $S_{\mathbb{P}} = \{x \in S \mid \nexists a, b \in S - \{1, x\} : x = ab\}$. It's easy to notice that $\mathbb{P} \cap S \subseteq S_{\mathbb{P}}$, meaning that if a prime number is also in S then it's an S-prime number.

1. Given $(4a-3), (4b-3) \in S$, we show that:

$$(4a-3)(4b-3) = 16ab - 12a - 12b + 9 =$$

$$16ab - 12a - 12b + 12 - 3 = 4(4ab - 3a - 3b + 4) - 3$$

   Since $ab - 3a - 3b + 4 \in \mathbb{N}$, we conclude that $(4a-3)(4b-3) \in S$.

2. By way of contradiction, we suppose that $S_{\mathbb{P}}$ is finite, meaning that $S_{\mathbb{P}} = \{p_1, \dots, p_n\}$.

   Consider the number $q := 4p_1 \dots p_n - 3$. It's easy to see that $q \in S - S_{\mathbb{P}}$, meaning that q is S-composite and thus that $\exists p_i, p_j \in S_{\mathbb{P}}$ such that $p_i \mid q$ and $p_j \mid q$.

   Without loss of generality, we procede with p_i. By reflection, we have that $p_i \mid p_i$, which implies that $p_i \mid 4p_1 \dots p_n$. Then, since $p_i \mid 4p_1 \dots p_n$ and $p_i \mid q$, it must also divide their difference, which equals 3, implying that $p_i \mid 3$.

   Finally, since $p_i \mid 3$, it must hold that $p_i \leq 3$, implying that $p_i \in \{1, 2, 3\}$. However, if $p_i = 2$ or $p_i = 3$, that would imply that $2 \in S$ or $3 \in S$, which is a contradiction. By the same reasoning, p_i can't be equal to 1 since that would imply that $p_j = 3$ and that $3 \in S$, which is a contradiction. Thus, the set $S_{\mathbb{P}}$ must be infinite.

   Another way to prove this result is by showing that $\forall k \in \mathbb{N}$ it holds that $4 \cdot 2^k - 3 \in S_{\mathbb{P}}$. This can be easily done by way of contradiction. Moreover, this generator of infinite S-prime numbers can be extended to all primes, meaning that $\forall p \in \mathbb{P}$ and $\forall k \in \mathbb{N}$ it holds that $4p^k - 3 \in S_{\mathbb{P}}$.

3. It's easy to see that $1617 = 4 \cdot 405 - 3$, thus $1617 \in S$. We now consider the prime factorization $1617 = 3 \cdot 11 \cdot 7^2$, we notice that $1617 = 33 \cdot 49$ and $1617 = 21 \cdot 77$.

   Since $33 = 4 \cdot 9 - 3$, we get that $33 \in S$. However, since $33 = 3 \cdot 11$ and $3, 7 \notin S$, the number 33 must be S-prime. By the same reasoning, we can show that $49, 21, 77 \in S_{\mathbb{P}}$, giving us two different S-prime factorizations of 1617.

4. Following the structure of the previous example, we can simply replace one of the numbers that form the prime factorization of 1617 with another prime number that isn't in S:

   - The number $441 = 3 \cdot 3 \cdot 7^2 \in S$ can be rewritten as $441 = 9 \cdot 49 = 21 \cdot 21$, where $9, 21, 49 \in S_{\mathbb{P}}$.
   - The number $2789 = 3 \cdot 19 \cdot 7^2 \in S$ can be rewritten as $1029 = 57 \cdot 49 = 21 \cdot 133$, where $21, 57, 133 \in S_{\mathbb{P}}$.