Mathematical Logic in Computer Science
Homework 1 2024-25

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April 13, 2025

Question 1 (Compactness). *Let $\mathcal{L} = \{E(x, y)\}$ be the language of graphs.*

1. *For each fixed $n \in \mathbb{N}$ write a sentence C_n such that for any graph $\mathcal{G}$, $\mathcal{G} \models C_n$ if and only if $\mathcal{G}$ contains an n-clique.*
2. *Prove using Compactness that the property of being finitely colorable is not expressible by a theory in $\mathcal{L}$ over the class of graphs.*

Solution to Question 1. For any $n \in \mathbb{N}$, consider the following sentence C_n:

$$C_n \equiv \exists x_1 \dots \exists x_n \bigwedge_{i=1}^{n} \bigwedge_{\substack{j=1 \\ i \neq j}}^{n} (E(x_i, x_j) \land \neg(x_i = x_j))$$

Claim 1.1: $\mathcal{G} \models C_n$ if and only if $\mathcal{G}$ contains an n-clique

Proof of Claim 1.1. Suppose that $\mathcal{G} \models C_n$. Let $v_1, \dots, v_n$ be the variables forming the assignment that satisfies C_n, i.e. $\begin{pmatrix} x_1 & \cdots & x_n \\ v_1 & \cdots & v_n \end{pmatrix}$. Then, $\{v_1, \dots, v_n\}$ form an n-clique since every pair of vertices is adjacent to each other and they are distinct. Vice versa, suppose that $\mathcal{G}$ contains an n-clique $\{u_1, \dots, u_n\}$. Then, the assignment formed by $\begin{pmatrix} x_1 & \cdots & x_n \\ u_1 & \cdots & u_n \end{pmatrix}$ satisfies the non-quantified formula inside C_n, concluding that $\mathcal{G} \models C_n$. $\square$

Consider now the theory $T = \{C_n \mid n \in \mathbb{N}\}$, expressing the property of containing an n-clique for any $n \in \mathbb{N}$. By way of contradiction, suppose that there is a theory T' expressing the property of being finitely colorable.

Claim 1.2: $T^* = T \cup T'$ is finitely satisfiable.

Proof of Claim 1.2. Fix $S \underset{fin}{\subseteq} T^*$ and consider the set $S - T' = \{C_{i_1}, \dots, C_{i_\ell}\}$. Let $M = \max(i_1, \dots, i_\ell)$. Then, the complete graph $\mathcal{K}_M$ on M vertices is M-colorable, hence finitely colorable, and contains an M-clique, which implies that it also contains an h-clique for any $h \leq M$. Thus, $\mathcal{K}_M \models S - T'$ and $\mathcal{K}_M \models T'$. Since it satisfies T', it also satisfies any subset of T' that may be inside S, concluding that $\mathcal{K}_M \models S$. $\square$

It's easy to see that if $\mathcal{G}$ is k-colorable then it doesn't contain a $k + 1$ clique, concluding that T^* is unsatisfiable. However, by Compactness we know that T^* is finitely satisfiable if and only if it is satisfiable, raising a contradiction. Thus, the only possibility is that T' doesn't exist. $\square$

Question 2 (Games and non-expressibility).