NP-Completeness of the Steiner Tree Problem

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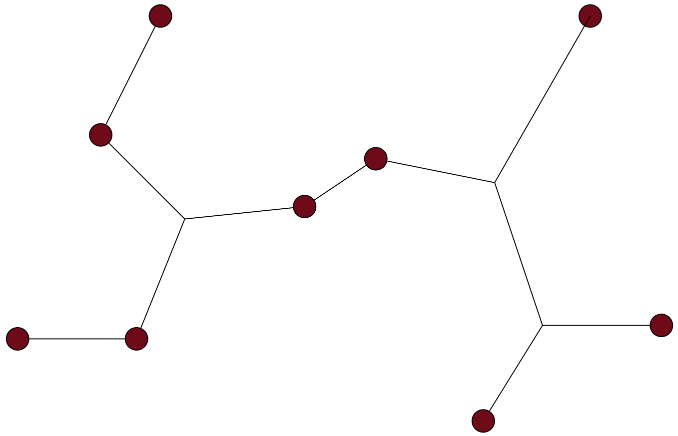
The Steiner Tree Problem

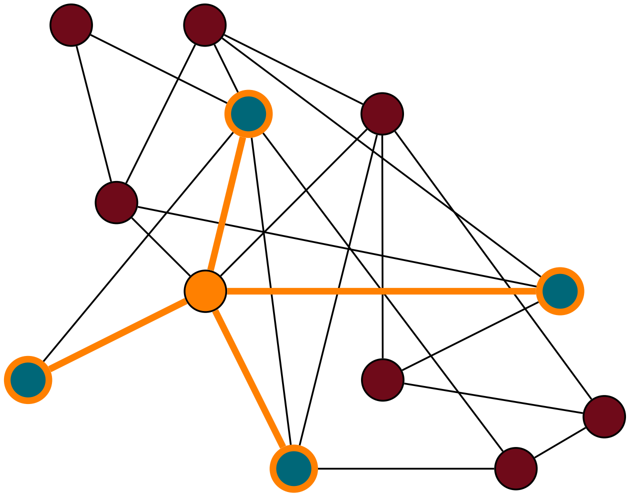
The **Steiner Tree problem** refers to a family of optimization problems, all seeking an optimal interconnection for a specified set of objects under a predefined objective function.

Frequent Objective Function: Minimize the lengths of the interconnections



The Steiner Tree Problem

Given a subset of nodes S ⊆ V(G), a sub-tree T of G is a **Steiner Tree** for S if S ⊆ V(T)



The Steiner Tree Problem

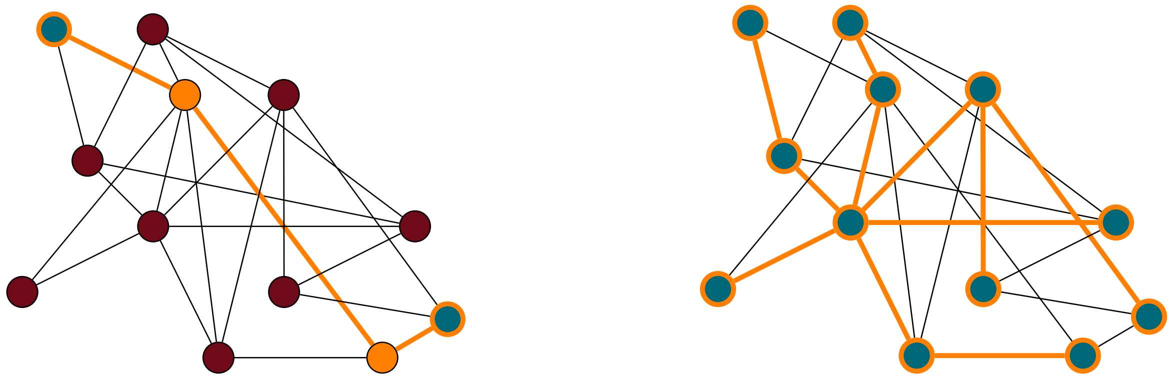
Optimization problem: Given a graph G and subset of nodes $S \subseteq V(G)$, find a Steiner Tree for S (if any) with the minimum number of edges.

Decision problem: Given a graph G, a subset of nodes $S \subseteq V(G)$ and a value k, is there a Steiner Tree for S with at most k edges?

ST as generalization of other problems

When |S| = 2 the ST problem becomes the **Shortest Path Problem**

When |S| = n the ST problem becomes the **Minimum Spanning Tree problem**



ST is in NP

Given $\langle G,S,k\rangle \in ST$, the witness is the Steiner Tree T itself:

1) Check that T is a sub-graph of G
2) Check that T is acyclic
3) Check that $S \subseteq V(T)$
4) Check that $|E(T)| \leq k$

Idea behind the NP-Hardness of ST

Idea: We can use Steiner Trees to “force” some kind of cover

The **Set Cover (SC)** problem is usually a suitable candidate for this idea

$\Longrightarrow$ The decision version is NP-Complete — hardness by reduction from Vertex Cover (VC)

Set Cover

$$U = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$$

$C_1 = \{x_1, x_2, x_4, x_7\}$ $C_2 = \{x_1, x_3, x_5, x_6\}$ $C_3 = \{x_3, x_5, x_7\}$

Conditions:

1) The sub-collection must cover U
2) The sub-collection must have minimal cardinality

Set Cover

Optimization problem: Given a set $U = \{x_1, \dots, x_{3q}\}$ and a collection of subsets $\mathcal{C} = \{C_1, \dots, C_m\}$, find the minimal cardinality sub-collection $\mathcal{C}^* \subseteq \mathcal{C}$ that covers U.

Decision problem: Given a set $U = \{x_1, \dots, x_{3q}\}$, a collection of subsets $\mathcal{C} = \{C_1, \dots, C_m\}$ and an integer $k > 0$, is there a sub-collection $\mathcal{C}^* \subseteq \mathcal{C}$ that covers U and such that $|\mathcal{C}^*| \leq k$?

Attempt reduction from SC to ST

Constructing the graph G:

1) For every sub-set C_j we add a node u_j

2) For every element $x_j \in U$ we add a node v_j — if $x_j \in C_i$ then we add the edge (u_i , v_j)

3) We add a special node z and the edge (z, u_i) for all $i \in [m]$

Set of terminal nodes: $S = \{z, x_1, \ldots, x_{3q}\}$

Attempt reduction from SC to ST

Example:

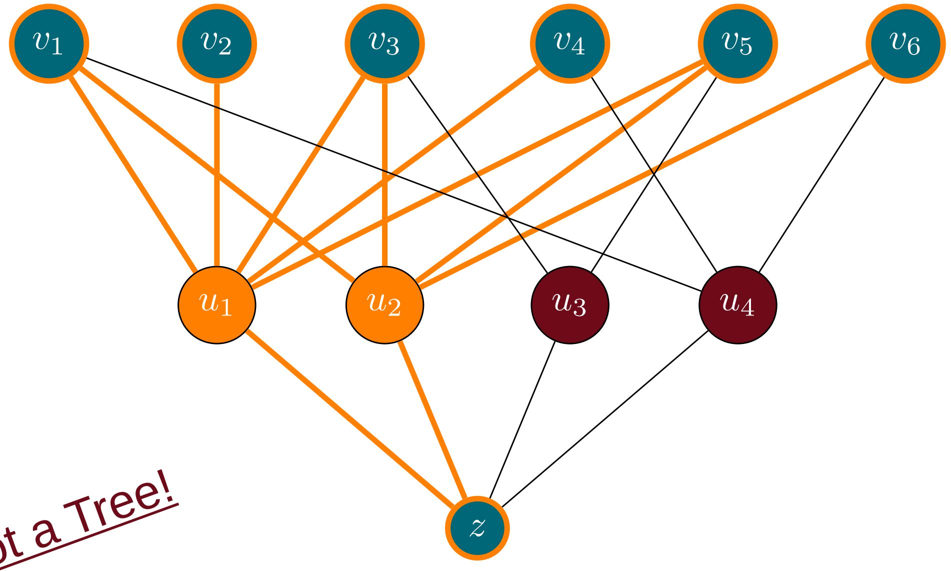
$U = \{x_1, x_2, x_3, x_4, x_5, x_6\}$

$C_1 = \{x_1, x_2, x_3, x_4, x_5\}$

$C_2 = \{x_1, x_3, x_5, x_6\}$

$C_3 = \{x_3, x_5\}$

$C_4 = \{x_1, x_4, x_6\}$



Not a Tree!

Exact Cover

Idea: Use a variant of the Set Cover problem ⟹ **Exact Cover (XC)**

XC is NP-Complete — hardness by reduction from 3-Coloring (3COL)

Conditions of XC:

1) The sub-collection must ~~cover U~~ be a **partition** of U ⟹ No more cycles!

2) The sub-collection must have minimal cardinality

Attempt reduction from XC to ST

Does the reduction still fail? Yes: the number of edges of the output tree is too variable

Examples:

1) If $|U| = n$, $\mathcal{C} = \{\{x_1, \ldots, x_{n-1}\}, \{x_n\}\}$ and $k = n$ then T will have $n+2$ edges
2) If $|U| = n$, $\mathcal{C} = \{\{x_1\}, \ldots, \{x_n\}\}$ and $k = n$ then T will have $2n$ edges

$\Longrightarrow$ We cannot find a "good" common value k' for the output triple $\langle G,S,k'\rangle$

Exact Cover by 3-Sets

Idea: Use a variant of the Exact Cover problem $\Longrightarrow$ **Exact Cover by 3-Sets (X3C)**

X3C is NP-Complete — hardness by reduction from 3-Dimensional Matching (3DM)

Conditions of X3C:

1) The sub-collection must be a partition of U $\Longrightarrow$ No more cycles!
2) The sub-collection must have minimal cardinality
3) U has **3q elements** and each C_i has **3 elements**

Exact Cover by 3-Sets

Obs. If $|U| = 3q$ and $\mathcal{C}^*$ is an X3C then $|\mathcal{C}^*| = q$

$\Longrightarrow$ All X3Cs have the same cardinality, hence we can drop the minimality constraint

Decision problem: Given a set $U = \{x_1, \dots, x_{3q}\}$ and a collection of subsets $\mathcal{C} = \{C_1, \dots, C_m\}$ such that $|C_i| = 3$, is there a sub-collection $\mathcal{C}^* \subseteq \mathcal{C}$ that is a partition of U?

Reduction from X3C to ST

Example:

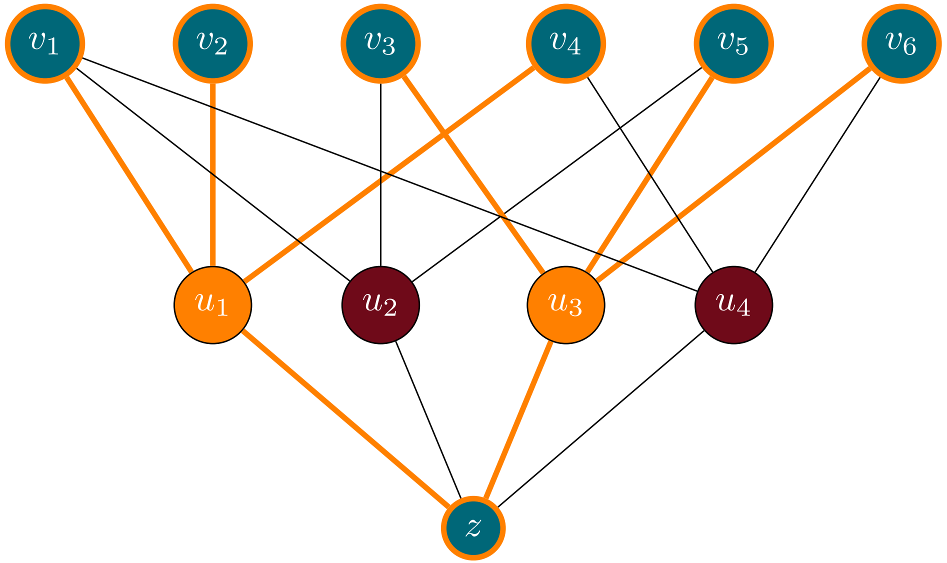
$U = \{x_1, x_2, x_3, x_4, x_5, x_6\}$

$\boxed{C_1 = \{x_1, x_2, x_4\}}$

$C_2 = \{x_1, x_3, x_5\}$

$\boxed{C_3 = \{x_3, x_5, x_6\}}$

$C_4 = \{x_1, x_4, x_6\}$



Reduction from X3C to ST

Claim: U has an Exact Cover by 3-Sets over $\mathcal{C}$ iff G has a Steiner Tree for S with 4q edges

$\Longrightarrow$)

1) If $\mathcal{C}^* \subseteq \mathcal{C}$ is an X3C of U then $|\mathcal{C}^*| = q \longrightarrow$ W.l.o.g. assume $\mathcal{C}^* = \{C_1, \dots, C_q\}$
2) T is made of the edges that have $u_1, \dots, u_q$ as endpoints $\longrightarrow$ T has 4q edges
3) $\mathcal{C}^*$ is an exact cover $\longrightarrow$ T covers S and it is acyclic

Reduction from X3C to ST

Claim: U has an Exact Cover by 3-Sets over $\mathcal{C}$ iff G has a Steiner Tree for S with 4q edges

$\Longleftarrow$)

1) If T is an ST for S with at 4q edges $\longrightarrow$ T must contain $u_1, \ldots, u_t$
2) If $t > q$ then S has more than 4q edges $\longrightarrow$ Contradiction
3) If $t < q$ then S cannot cover all terminals $\longrightarrow$ Contradiction
4) $\mathcal{C}^* = \{C_1, \ldots, C_q\}$ is an Exact 3-Cover of U

References

- M. R. Garey and D. S. Johnson. “The Rectilinear Steiner Tree Problem Is NP-Complete.” SIAM Journal on Applied Mathematics 32, no. 4 (1977). http://www.jstor.org/stable/2100192

- M. W. Bern and R. L. Graham “The Shortest-Network Problem” in Scientific American Magazine Vol. 260 No. 1 (1989). doi:10.1038/scientificamerican0189-84

Thank you for the attention!