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Graph Theory

Lecture notes integrated with the book “Graph Theory”, R. Diestel

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Information and Contacts

Personal notes and summaries collected as part of the *Graph Theory* course offered by the degree in Computer Science of the University of Rome "La Sapienza".

Further information and notes can be found at the following link:

<https://github.com/Exyss/university-notes>. Anyone can feel free to report inaccuracies, improvements or requests through the Issue system provided by GitHub itself or by contacting the author privately:

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The notes are constantly being updated, so please check if the changes have already been made in the most recent version.

Suggested prerequisites:

Sufficient knowledge of algorithm design, probability and combinatorics

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1

Introduction to Graph Theory

1.1 Graphs, subgraphs and neighbors

The city of Königsberg (in Prussia), situated along the Pregel River, is divided into four regions: two parts of the mainland and two islands, Kneiphof and Lomse. In the 18th century, these areas were connected by *seven bridges* that spanned the river, crossing it in various directions. As time passed, a fascinating question emerged among residents: could one walk through the city, crossing each of the seven bridges exactly once, and return to the starting point? This became known as the *Seven Bridges of Königsberg problem*.

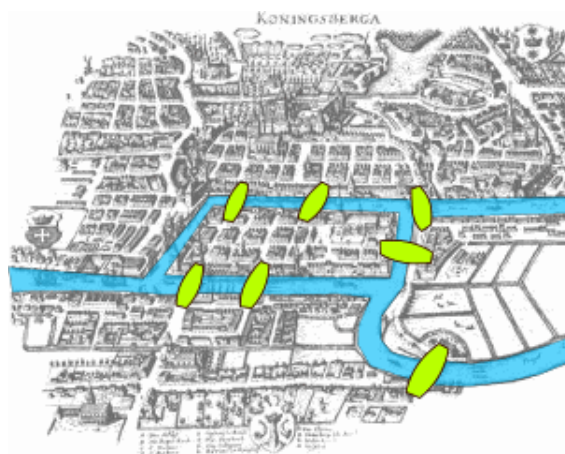


Figure 1.1: The map of Königsberg in the 18th century, showing the layout of the seven bridges.

Though it appeared to be a simple challenge, no one had succeeded in solving it. The riddle became a popular topic of conversation, discussed in markets and taverns alike. Some believed the walk was possible with the right path, while others doubted it could be done. Word of this puzzling problem eventually reached the brilliant Swiss mathematician,

Leonhard Euler. Fascinated by the challenge, Euler sought a solution – not by walking the streets himself, but by abstracting the problem into a more general form. Euler realized that the precise layout of the city itself wasn't essential. What really mattered was how the landmasses were **connected** by the bridges. He began by representing each landmass as a dot and each bridge as a line between them. By doing this, he removed unnecessary details and created a simple yet powerful combinatorial structure, which we now call a **graph**.

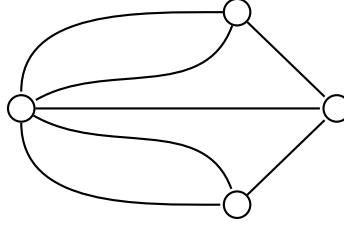


Figure 1.2: The graph drawn by Euler for the Seven Bridges of Königsberg problem.

Through his analysis, Euler discovered a key insight: for a walk to cross each bridge exactly once and return to the starting point, each landmass must be connected to an even number of bridges. In the case of Königsberg, however, every landmass was connected to an odd number of bridges, making such a walk impossible. Euler's solution, published in 1736, was groundbreaking. It not only answered the Königsberg puzzle but also laid the groundwork for an entirely new field of mathematics: *graph theory*. This area of study has since become fundamental to understanding networks, from transportation systems to social media, and even the internet itself. Thus, from a simple riddle about bridges in a small Prussian city, a new mathematical discipline was born—one that continues to influence the world to this day.

Definition 1: Graph

A **graph** is a mathematical structure $G = (V, E)$, where V is the set of vertices (or nodes) of G and E is the set of edges that link the vertices of G .

A graph can be **directed** or **undirected**. In a directed graph the edges are *oriented*, meaning that there is difference between the edge (u, v) – going from u to v – and the edge (v, u) – going from v to u . Formally, we have that:

$$E(G) \subseteq V \times V \setminus \{(u, v) \mid u, v \in V(G)\}$$

In an undirected graph, instead, the edges are *not oriented*, meaning that there not is difference between the edges (u, v) and (v, u) . Formally, we have that:

$$E(G) \subseteq \binom{V(G)}{2} = \{\{u, v\} \mid u, v \in V(G)\}$$

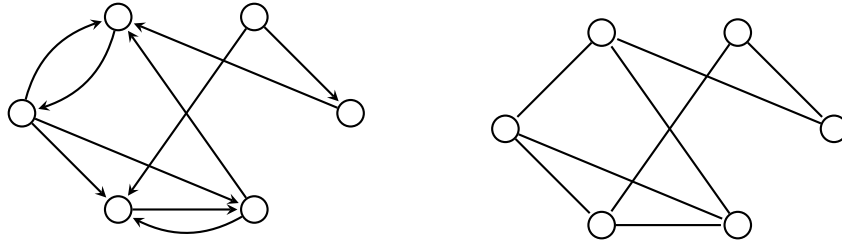


Figure 1.3: A directed graph (left) and an undirected graph (right)

We observe that the definition of graph that we just gave doesn't allow *multiple edges* between two vertices and *loops*, i.e. edges out-going from and in-going to the same vertex. When this is the case, we say that the graph is **simple**. Generally, simple graphs are enough for any model. Sometimes, however, multiple edges and loops are needed – such as in the Seven Bridges of Königsberg problem. Graphs that allow such edges are called **multigraphs**.

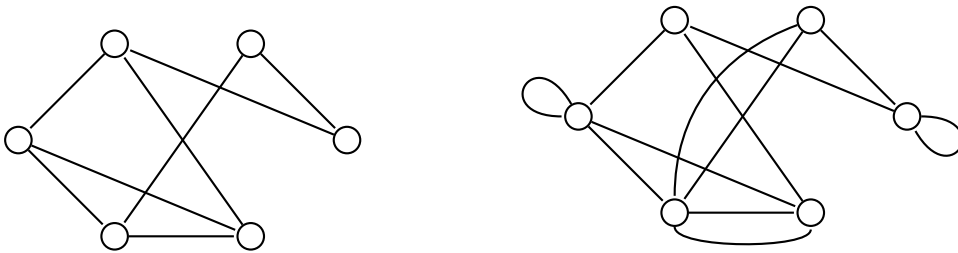


Figure 1.4: A simple graph (left) and a multigraph (right)

From now on, unless stated differently, we'll assume that each graph is simple and undirected. Moreover, to make notation lighter, we will always assume that $|V(G)| = n$, $|E(G)| = m$ and that $xy = \{x, y\}$ (or $xy = (x, y)$ for directed graphs).

Definition 2: Subgraph

Let G be a graph. If H is a graph such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ then H is a **subgraph** of G , written as $H \subseteq G$. A subgraph $H \subseteq G$ is said to be **induced** when for all edges $xy \in E(G)$ such that $u, v \in V(H)$ it holds that $xy \in E(H)$.

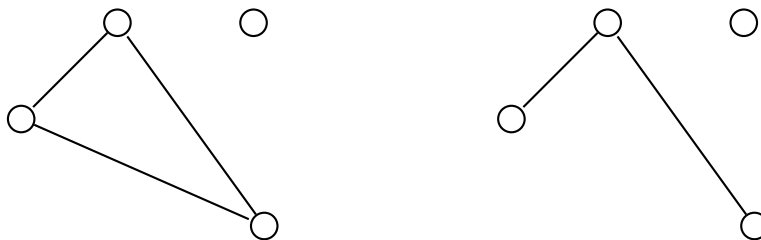


Figure 1.5: Both graphs are a subgraph of the simple graph shown on the left in [Figure 1.4](#). The left subgraph is induced, while the right one is not.

We observe that, by definition, if a subgraph $H \subseteq G$ is induced then H is the unique induced subgraph for the vertices $V(H)$. Hence, when talking about an induced graphs we can consider his set of vertices. Given a subset of vertices $X \subseteq V(G)$, we denote with $G[X]$ the unique induced subgraph of G such that $V(G[X]) = X$, where $E(G[X]) = \{xy \in E(G) \mid x, y \in X\}$.

Definition 3: Adjacency, neighborhood and independence

Given a graph G and two nodes $x, y \in V(G)$, we say that x, y are **adjacent** to each other, written as $x \sim y$, if $xy \in E(G)$. For any edge $xy \in E(G)$, we say that xy is incident to x and y . The set of all vertices adjacent to x in G is called **neighborhood**, written as $N_G(x)$, is defined as:

$$N_G(x) = \{y \in V(G) \mid x \sim y\}$$

A subset of vertices X such that for each $x, y \in X$ it holds that $x \not\sim y$ is called **independent set**.

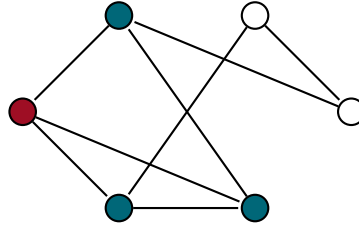


Figure 1.6: The blue nodes form the neighborhood of the red node. The red node and the white nodes form an independent set.

Definition 4: Degree

Let G be a graph. Given a vertex $x \in V(G)$, the **degree** of x over G is defined as $\deg_G(x) = |N_G(x)|$. The minimum and maximum degree of G are respectively denoted as $\delta(G)$ and $\Delta(G)$.

$$\delta(G) = \min_{x \in V(G)} \deg_G(x)$$

$$\Delta(G) = \max_{x \in V(G)} \deg_G(x)$$

We say that a graph is **k-regular** if $\delta = \Delta = k$, i.e. every node has degree k

When the context makes it clear, we'll simply write $\deg(x)$ instead of $\deg_G(x)$. The minimum and maximum degree of a graph are two very powerful theorem-proving tools: large portion of results are proven by reasoning on the degree of each node, often proving that some condition does or does not hold. The most basic result involving the degree of a graph is known as the **Handshaking lemma**, which can be stated in two equivalent forms.

Lemma 1: Handshaking lemma

For every graph G it holds that:

$$\sum_{x \in V(G)} \deg(x) = 2m$$

Equivalently, for every graph G the number of odd-degree vertices is even.

Proof. It's easy to see that every edge $xy \in E(G)$ is counted exactly two times through $\deg(x)$ and $\deg(y)$, implying that:

$$\sum_{x \in V(G)} \deg(x) = 2m$$

Consider now the subset $X \subseteq V(G)$ containing all the vertices of even degree. We observe that:

$$2m = \sum_{x \in V(G)} \deg(x) = \sum_{x \in X} \deg(x) + \sum_{x' \in V(G) - X} \deg(x')$$

Let $a = \sum_{x \in X} \deg(x)$ and $b = \sum_{x' \in X} \deg(x')$. Since the degrees of the vertices in X are even, we know that a is even. Hence, in order for $2m = a + b$ to hold, b must also be even. However, since each degree in $V(G) - X$ is odd, in order for b to be even it must hold that $|V(G) - X|$ is even. \square

1.2 Paths, walks, cycles and trees

After discussing the more general concept of subgraph, we can now focus on particular types of structures that can be usually found in graphs. These sub-structures are the real fundamental tool of reasoning for graph properties.

Definition 5: Path

A **path** is a graph P such that $V(P) = \{x_1, \dots, x_n\}$ and $E(P) = \{x_0x_1, x_1x_2, \dots, x_{n-1}x_n\}$. The length of a path is defined as the number of edges that form it. A path with n vertices is denoted as P_n .

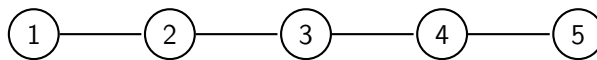


Figure 1.7: The path P_5 .

Definition 6: Walk

Let G be a graph. A **walk** on G is defined as a sequence $x_0e_1x_1e_2\dots e_{k-1}e_kx_k$ where $x_0, \dots, x_k \in V(G)$ and $e_1, \dots, e_k \in E(G)$. The length of a walk is defined as the number of edges that form it.

When the first and last vertices are equal, i.e. $x_0 = x_k$, we say that the walk is **closed**. We observe that, by definition, a walk allows edges and vertices to be repeated in the sequence. When no vertices in a walk are repeated, the walk corresponds to a path.

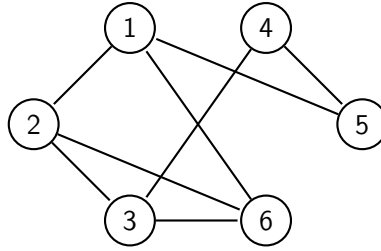


Figure 1.8: The sequence $1\{1,2\}2\{2,6\}6\{6,1\}1\{1,5\}5$ forms a walk on G , but not a path. The sequence $1\{1,2\}2\{2,6\}6\{6,3\}3\{3,4\}4$, instead, forms a path on G .

Proposition 1: Paths and walks

Let G be a graph. Given the vertices $x, y \in V(G)$, there is a path from x to y if and only if there is a walk from x to y .

Proof. Since every path is also a walk, the first direction is trivial. Let $W = x_0e_1x_1e_2\dots e_{k-1}e_kx_k$ be the shortest walk, i.e. the one with minimum length, such that $x = x_0$ and $y = x_k$. By way of contradiction, suppose that W is not a path. Hence, at least one edge in W must repeat at least once. Let $i, j \in [k]$ be two indices such that $e_i = e_j$. Then, the following sequence W' is a walk from x to y with fewer edges than W , raising a contradiction. Hence, W must be a path.

$$W' = x_0e_1x_1e_2\dots e_ix_ie_{j+1}x_{j+1}e_{j+2}\dots e_{k-1}e_kx_k$$

□

Proposition 2

The longest path in any graph has length at least δ .

Proof. If $\delta = 1$ then the longest path is trivially made by one single edge. Suppose now that $\delta \geq 2$, implying that there are at least two vertices in G . Let P be the longest path in G and let x_1, \dots, x_k be its vertices.

Claim: $N(x_k) \subseteq \{x_1, \dots, x_{k-1}\}$.

Proof. By way of contradiction, suppose that there is a vertex $x' \in N(x_k)$ such that $x' \notin \{x_1, \dots, x_{k-1}\}$. Then, since $x_k \sim x'$, there must be an edge $x_k x'$, implying that the path $P \cup x_k x'$ is longer than P , raising a contradiction. \square

Through the claim we easily conclude that $\delta \leq |N(x_k)| \leq k$, meaning that P has length at least δ . \square

Definition 7: Cycle

A cycle is a graph C such that $V(C) = \{x_1, \dots, x_n\}$ and $E(G) = \{x_0 x_1, x_1 x_2, \dots, x_{n-1} x_n, x_n x_1\}$. The length of a cycle is defined as the number of edges that form it. A cycle with n vertices is denoted as C_n .

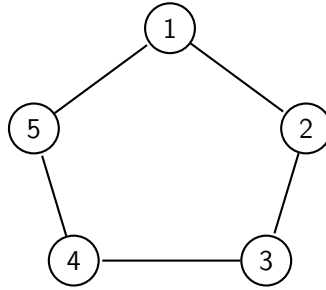


Figure 1.9: The cycle C_5 .

Proposition 3

In any graph such that $\delta \geq 2$ there is a cycle of length at least $\delta + 1$.

Proof. Let P be the longest path in G and let x_1, \dots, x_k be its vertices. Through an argument equal to the claim of [Proposition 2](#), we know that $N(x_k) \subseteq \{x_1, \dots, x_{k-1}\}$. Let $i \in [k-1]$ be the minimal index such that $x_i \in N(x_k)$. Since every neighbor of x_k must be inside P and the graph is simple, it must hold that $i \geq \delta$, meaning that the vertices $x_i, x_{i+1}, \dots, x_{k-1}, x_k, x_i$ form the cycle C_{i+1} . \square

Definition 8: Connectivity, components and distance

Let G be a graph. Two nodes $x, y \in V(G)$ are said to be linked if there is a path from x to y . If every pair of nodes in G is linked, we say that G is **connected**. If a connected subgraph H of G is maximal – meaning that no other edges can be added to it while preserving connectivity – H is called **component** of G . The **distance** $\text{dist}_G(x, y)$ between two nodes x, y is the length of the shortest path connecting them.

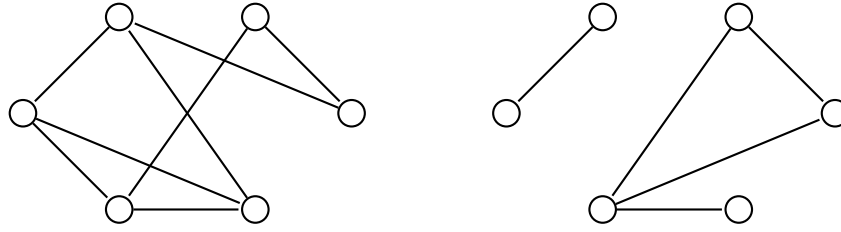


Figure 1.10: A connected graph (left) and a disconnected graph (right). The connected graph has an unique component, while the disconnected graph has two components.

Lemma 2

Let G be a graph. If G is connected and C is a cycle in G then for all $e \in E(C)$ it holds that $G - \{e\}$ is still connected.

Proof. Fix an edge $e \in E(C)$. Let $P = x_0e_1x_1 \dots e_kx_k$ be a path in G from x to y . If $e \notin E(P)$ then P is also a path in $G - \{e\}$, preserving the connectivity of x and y . Suppose now that $e \in E(P)$. Given $C = z_0f_1z_1 \dots f_\ell z_\ell$, without loss of generality assume that $e = e_i = f_1$. Then, the following sequence W is a walk from x to y in $G - \{e\}$ – we cannot be sure that W is a path since P may intersect C on multiple edges.

$$W = x_0e_1x_1 \dots x_i f_\ell z_{\ell-1} \dots f_2 z_2 e_{i+1} x_{i+1} \dots e_k x_k$$

By [Proposition 1](#), we know that since W is a walk from x to y in $G - \{e\}$ there must also be a path from x to y in $G - \{e\}$, preserving connectivity. \square

Definition 9: Tree

A **tree** is an connected acyclic subgraph. Any vertex in a tree with degree 1 is called leaf. A rooted tree is a tree with a chosen node called **root**. If every component of a graph is a tree, the graph is referred to as a **forest**.

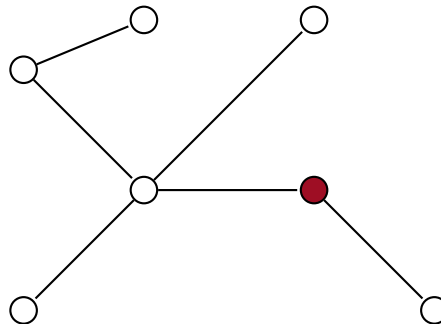


Figure 1.11: A rooted tree. The red node has been chosen as the root.

When the tree is rooted in $r \in V(T)$, we often use the concept of **ancestor** and **parent**. Given two nodes $a, x \in V(T)$, we say that a is an ancestor of x if x lies on a path from r

to x . If p is an ancestor of x and $px \in E(T)$, we say that p is the parent of x . The **least common ancestor (LCA)** between two vertices $x, y \in V(T)$ is the ancestor $z \in V(T)$ shared by x and y that minimizes the value of $\text{dist}(r, z)$. We observe that every pair of vertices of a tree must have a LCA since the root is an ancestor of every node.

Theorem 1: Equivalent definitions of tree

Given a graph T , the following statements are equivalent:

1. T is a tree
2. Every vertex pair of T is connected by an unique path
3. T is minimally connected
4. T is maximally acyclic

Proof. We'll proceed by proving a chain of implications.

- 1 \implies 2 Without loss of generality, assume that every pair of nodes has at least one path, since otherwise T is not connected, hence not a tree. Suppose now that there are two vertices $x, y \in V(T)$ that have at least two different paths P, Q from x to y . While traversing P from x to y , let z be the first node such that $z \in V(P) \cap V(Q)$. Similarly, let $w \in V(P) \cap V(Q)$ be the first node encountered while traversing Q from y to x . We observe that since $x, y \in V(P) \cap V(Q)$, the vertices z, w always exist. Given the subpath $P' \subseteq P$ from w to z and the subpath $Q' \subseteq Q$ from z to w , the graph $Q' \cup P'$ is a cycle in T , meaning that T cannot be a tree. By contrapositive, we get that if T is a tree then every pair of vertices is connected by an unique path.
- 2 \implies 3 Suppose that every vertex pair of T is connected by an unique path. Then, T is clearly connected. By way of contradiction, suppose that there is an edge $e \in E(G)$ such that $T - \{e\}$ is still connected. Then, the edge e must be part of at least one of the unique paths connecting two nodes, meaning that such path cannot exist inside $T - \{e\}$, implying that it is not connected. Hence, T must be minimally connected.
- 3 \implies 4 Suppose that T is minimally connected but not maximally acyclic. By way of contradiction, suppose that T has a cycle. Then, by [Lemma 2](#) we know that we can remove an edge of the cycle from T and keep it connected, contradicting the fact that T is minimally connected. Hence, T must be acyclic. Pick two vertices $x, y \in V(T)$. Since T is connected, we know that there is a path P connecting x to y . Then, if we were to add the edge xy , the subgraph $P \cup \{xy\}$ would be a cycle in $T \cup \{xy\}$. Thus, T is maximally acyclic.
- 4 \implies 1 Suppose that T is maximally acyclic. Fix a pair of vertices $x, y \in V(T)$. Since T is maximally acyclic, we know that adding the edge xy makes $T \cup \{xy\}$ cyclic. Let C be the cycle in $T \cup \{xy\}$ containing xy . Then, $C - \{xy\}$ must be a path in T from x to y .

□

Lemma 3

Let T be a tree. Then:

1. If T has at least two nodes then T has at least a leaf.
2. If $x \in V(T)$ is a leaf then $T - \{x\}$ is still a tree

Proof.

1. By way of contradiction, suppose that T is a tree without leaves. Then, we have that $\delta \geq 2$. However, by [Proposition 3](#), in T there must be a cycle with length at least $\delta + 1$, contradicting the very definition of tree. Hence, T must have at least a leaf.
2. Let x be a leaf of T . Since T is acyclic, $T - \{x\}$ is also clearly acyclic. By way of contradiction, suppose that $T - \{x\}$ is not connected. Then, there are at least two vertices $u, v \in V(T - \{x\})$ for which there is a path P between them in T but not in $T - \{x\}$. Since by removing x the vertices u, v became disconnected, the vertex x must lie on P . Moreover, since $u, v \neq x$, the vertex x must be an internal node of the path, meaning that it must have degree 2, contradicting the very definition of leaf.

□

Definition 10: Spanning tree

Given a graph G , we say that $T \subseteq G$ is a **spanning tree** of G if T is a tree and $V(T) = V(G)$.

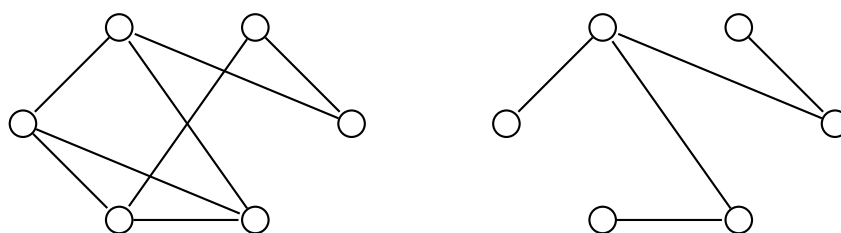


Figure 1.12: A graph (left) and one of its spanning trees (right)

Lemma 4

Every connected graph has a spanning tree.

Proof. If G is a tree then clearly it is its own spanning tree. Suppose now that G is a connected graph that is not a tree, meaning that it is acyclic. Then, through [Proposition 3](#), we can keep removing the edges e_1, \dots, e_k from every cycle of G until we reach a minimally connected subgraph T such that $V(T) = V(G)$. By [Theorem 1](#), we know that T must be a tree. □

Theorem 2

Let T be a connected graph. Then, T is a tree if and only if it has $n - 1$ edges.

Proof. We proceed by induction on n . If $n = 1$ then T is trivially the tree with 0 edges. If $n > 1$, instead, through the previous lemma we know that T must have at least a leaf $x \in V(T)$ for which $T - \{x\}$ is still a tree. By inductive hypothesis, we know that $T - \{x\}$ has $n - 2$ edges. Hence, by adding the unique edge incident to x , we get that T has $n - 1$ edges.

Vice versa, by way of contradiction, suppose that T is connected, has $n - 1$ edges but that it is not acyclic. Then, through the previous lemma we know that T must have a spanning tree T' . Moreover, since T has a cycle and T' does not, we know that T' must have fewer edges than T . However, we have just proven that every tree must have $n - 1$ edges. Hence, we get that $n - 1 = |E(T')| < |E(T)| = n - 1$, which is a contradiction. Hence, T must also be acyclic. \square

1.3 Complete graphs and bipartite graphs

Definition 11: Complete graph

A **complete graph** is a graph where every pair of vertices is adjacent to each other. A complete graph with n vertices is denoted as K_n . An induced subgraph that is complete is called **clique**.

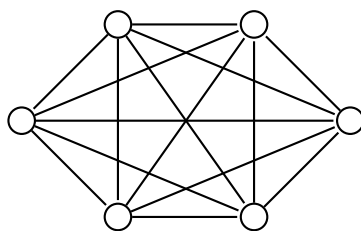


Figure 1.13: The complete graph K_6

Definition 12: Bipartite graph

A **bipartite graph** is a graph with a subset of vertices $X \subseteq V(G)$ such that for every edge $\{u, v\} \in E(G)$ it holds that $u \in X$ and $v \in V(G) - X$. Equivalently, we have that both X and $V(G) - X$ are independent sets. The pair $(X, V(G) - X)$ is called **bipartition** of the graph.

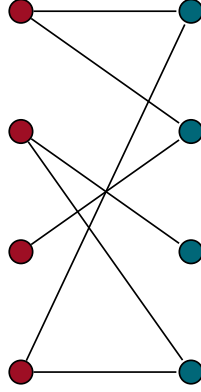


Figure 1.14: The red and blue nodes form a bipartition of the graph.

It's easy to see that no complete graph can be bipartitioned since it's impossible to separate the nodes into two independent sets. In fact, by definition, cliques are the very opposite of the concept of independent set. Through the following lemma we can extend the claim on complete graphs to cliques: if G contains a clique of any cardinality then G cannot be bipartitioned.

Proposition 4

G is bipartite if and only if every subgraph of G is bipartite

Proof. The converse implication trivially holds. Let H be a subgraph of G . Then, any bipartition $(X, V(G) - X)$ of G induces a bipartition $(X \cap V(H), V(H) - X)$ over H . \square

Lemma 5

G is bipartite if and only if every component of G is bipartite

Proof. The direct implication trivially follows from the previous proposition. Given the components H_1, \dots, H_k of G , let $(X_1, V(H_1) - X_1), \dots, (X_k, V(H_k) - X_k)$ be the bipartitions of the components. Then, since each component is by definition disjoint from the others, the pair $(X, V(G) - X)$ where $X = \bigcup_{i \in [k]} X_i$ is a bipartition of G . \square

We observe that the above characterizations of bipartite graphs are useful but still “weak”. A stronger characterization can be achieved through the following theorem. Given a path P and two vertices $x, y \in V(P)$, we denote the subpath of P from x to y with xPy .

Theorem 3: Equivalent definition of bipartite graphs

G is bipartite if and only if it doesn't contain odd cycles.

Proof. Through the above proposition, it's sufficient to prove that for all $k \in \mathbb{N}$, C_{2k+1} is not bipartite. Fix $k \in \mathbb{N}$. By way of contradiction, suppose that there is bipartition

$(X, V(G) - X)$ of C_{2k+1} . Let $x_1, \dots, x_{2k+1}x_1$ be the edges of C_{2k+1} . Without loss of generality, assume that $x_1 \in X, x_2 \notin X, x_3 \in X, \dots, x_{2k} \notin X$. Then, if $x_{2k+1} \in X$ then $x_{2k+1}x_1 \subseteq X$, while if $x_{2k+1} \notin X$ then $x_{2k}x_{2k+1} \subseteq V(G) - X$. In both cases, we get that the pair cannot be a bipartition.

Vice versa, suppose that G is not bipartite. Then, through the previous lemma at least one component H of G is not bipartite. Let T be a spanning tree of H and fix $r \in V(T)$ as its root. Let $X = \{x \in V(T) \mid \text{dist}_T(r, x) \text{ is even}\}$. By definition, $(X, V(T) - X)$ for a bipartition of T . Hence, since T is bipartite but H isn't, there must be an edge $xy \in E(H)$ such that either $x, y \in X$ or $x, y \notin X$. Let P_x and P_y respectively be the paths in T from x to r and from y to r . Let z be the LCA of x and y in T .

Claim: $C = zP_xx \cup zP_yy \cup xy$ is an odd cycle

Proof of the claim. Since either $x, y \in X$ or $x, y \notin X$ holds, we know that $\text{dist}(r, x)$ and $\text{dist}(r, y)$ must be either both even or both odd. Hence, the lengths of P_x and P_y must share the same parity. Moreover, since z is the LCA of x and y we have that $rP_xz = rP_yz$. Thus, the lengths of zP_xx and zP_yy must also share the same parity. This concludes that $zP_xx \cup zP_yy \cup xy$ must be an odd cycle. \square

Since C is an odd cycle and it is a subgraph of T , and thus of G , we conclude that G contains an odd cycle. \square

1.4 Eulerian tours and Hamiltonian paths

At the start of this chapter, we introduced the Seven Bridges of Königsberg problem, which led to the emergence of graph theory as a branch of combinatorics. In modern graph theory, the problem is formalized through the concept of *Eulerian tour*.

Definition 13: Eulerian tour

An Eulerian tour over a graph G is closed walk that traverses every edge of G exactly once.

To solve the problem, Euler [Eul41] proved the following theorem, which implies that the answer is “no” since the multigraph that models the problem contains some odd-degree vertices.

Theorem 4: Euler's theorem

A graph (or multigraph) has an Eulerian tour if and only if G is connected and every vertex has even degree

Proof. By way of contradiction, suppose that G has an Eulerian tour W . By way of contradiction, suppose that G is not connected. Then, we get an easy contradiction: if G is disconnected then W cannot traverse every edge of the graph. Hence, G must be connected. Again, by way of contradiction suppose that G has at least one odd-degree

vertex $x \in V(G)$. Let $\deg(x) = 2k + 1$. Since W is a closed walk, we can assume without loss of generality that x is the first vertex of the walk. When traversing W starting from x , one of the edges incident to x is crossed, hence we have $2k$ incident edges left. Every time the tour returns to x , two edges are crossed – one in-going and one out-going. Hence, in order to cover all the edges, the tour has to return to x for k times. However, this implies that there are no more edges left to close the tour on x , raising a contradiction. Hence, the vertex x cannot exist.

Vice versa, assume that G is connected and every vertex has even degree. Let $W = x_0 e_1 x_1 \dots e_k x_k$ be the longest walk over G with no repeating edges.

Claim 1: $x_k = x_0$, i.e. W is a closed walk

Proof of Claim 1. By way of contradiction, suppose that $x_k \neq x_0$. Let $2\ell + 1$ be the number of edges incident to x_k in W – one edge is given by e_k while 2ℓ edges are given by edges needed to cross the other ℓ the vertices $x_{i_1}, \dots, x_{i_\ell}$ such that $x_i = x_k$. Since x_k has even degree, we know that there must be another edge $\{x_k, y\} \in E(G - W)$, implying that $W \cup \{x_k, y\}$ is walk longer than W , which is absurd. \square

Claim 2: W contains every edge of G

Proof of Claim 2. By way of contradiction, suppose that there is an edge $uv \in E(G - W)$. Fix $i \in [k]$. By connectivity of G , we know that there must be a path P disjoint from x_i to u or from x_i to v . Without loss of generality, assume that P is a path from x_i to u . Since $uv \notin E(W)$, there must be an edge $x_j y \in E(P - W)$ outgoing from W . Since W is a closed walk, we can assume without loss of generality that x_j is the first (and last) vertex of the walk. Then, the walk $W \cup x_j y$ is walk that doesn't repeat any vertices longer than W , raising a contradiction. \square

Through the two claims, we conclude that W is an Eulerian tour. \square

Another very interesting concept similar to Eulerian tours was formulated by Hamilton: closed walks that touch every node exactly once instead of every edge exactly once. We observe that this condition implies that the closed walk is nothing more than the subgraph P_n . Surprisingly, finding Hamiltonian path inside a graph is way harder than finding an Eulerian tour: the former is an NP-hard problem, while the latter can be solved in polynomial time. A variant of this concept is the Hamiltonian cycle, i.e. a cycle that goes through all the nodes of a graph

Definition 14: Hamiltonian paths and cycles

An Hamiltonian path over a graph G is a subgraph $P_n \subseteq G$. An Hamiltonian cycle over a graph G is a subgraph $C_n \subseteq G$.

When some conditions are met, an Hamiltonian cycle (hence also an Hamiltonian path) is guaranteed to exist. For instance, Dirac [Dir52] formulated the following theorem.

Theorem 5: Dirac's theorem

Let G be a graph. If $\delta \geq \frac{n}{2}$ then there is an Hamiltonian cycle in G .

Proof. First of all, we show that Dirac's condition forces the graph to be connected.

Claim 1: G is connected

Proof of Claim 1. By way of contradiction, suppose that G is not connected. Then, G has at least two components. Hence, the smallest component H of G can have at most $\frac{n}{2}$ edge. However, since $\delta \geq \frac{n}{2}$, each node $x \in H$ must have at least $\frac{n}{2}$ neighbors. Thus, since $\{x\} \cup N(x) \subseteq H$, we get that H must have at least $\frac{n}{2} + 1$ nodes, raising a contradiction. \square

Let P be the longest path on G and let x_0, \dots, x_k be its vertices.

Claim 2: There is an index ℓ such that $x_0, \dots, x_k, x_\ell, x_{\ell+1}, x_0$ is a cycle

Proof of Claim 2. Using the same argument as [Proposition 2](#), we know that $N(x_0), N(x_k) \subseteq \{x_0, \dots, x_k\}$. Let I_0 and I_k be defined as:

$$I_0 = \{i \mid 1 \leq i \leq k, x_i \in N(x_0)\} \quad I_k = \{i \mid 1 \leq i \leq k, x_{i-1} \in N(x_k)\}$$

Since $\delta \geq \frac{n}{2}$, we know that $\frac{n}{2} \geq |I_0|, |I_k| \leq n - 1$, by the pigeonhole principle there must be at least one index $\ell \in I_0 \cap I_k$, meaning that $x_0 \sim x_\ell$ and $x_k \sim x_{\ell+1}$. Thus, the sequence of nodes $x_0, \dots, x_k, x_\ell, x_{\ell+1}, x_0$ is a cycle. \square

Let C be the cycle given by $x_0, \dots, x_k, x_\ell, x_{\ell+1}, x_0$. By way of contradiction, suppose that C has less than n nodes. Given $y \in V(G - C)$, by connectivity of G there must be an index i for which there is a path P connecting x_i and y . However, this implies that the following path

$$P \cup x_i x_{i+1} \cup \dots \cup x_k x_0 \cup \dots \cup x_{i-2} x_{i-1}$$

is longer than P , raising a contradiction. Hence, C must be an Hamiltonian path. \square

We observe that Dirac's condition, i.e. $\delta \geq \frac{n}{2}$ is actually optimal, meaning that we cannot find a better lower bound that guarantees the existence of an Hamiltonian cycle. For instance, there are many graphs with $\delta = \frac{n}{2} - 1$ for which there is no Hamiltonian cycle.

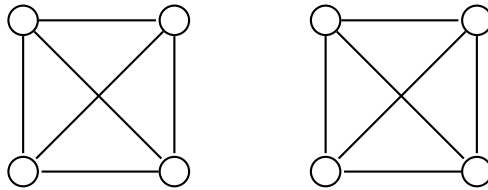


Figure 1.15: Example of a graph with $\delta = \frac{n}{2} - 1$ without Hamiltonian cycles.

1.5 Solved exercises

Problem 1

For any graph G , prove that at least two vertices have the same degree.

Proof. By way of contradiction, suppose that no vertices share the same degree. Then, the function $\deg : V \rightarrow \{0, \dots, n-1\}$ must be bijective, implying that there are two vertices $u, v \in V$ such that $\deg(u) = 0$ and $\deg(v) = n-1$, which is absurd since there should both be and not be an edge between u and v . Hence, at least two vertices must share have the same degree. \square

Graph matchings

2.1 Maximum matching

In graph theory, a **matching** in a graph is a set of edges that do not have a set of common vertices. In other words, a matching is a graph where each node has either zero or one edge incident to it. Graph matching has applications in flow networks, scheduling and planning, modeling bonds in chemistry, graph coloring, the stable marriage problem, neural networks in artificial intelligence and more.

Definition 15: Matching

Given a graph G , a matching over G is a subset $M \subseteq E(G)$ such that $\forall e, e' \in M$ it holds that $e \cap e' = \emptyset$

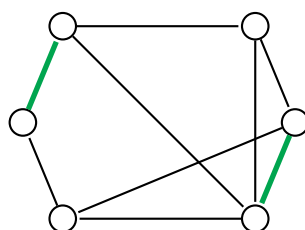


Figure 2.1: The green edges form a matching of the graph.

We're often interested in finding the matching with maximum cardinality. Before proceeding, it's important to distinguish between the concepts of **maximal** and **maximum**. In general, given a property P , a sub-structure X of a structure S is said to be maximal for P over S if $P(X)$ is true and there is no other sub-structure X' of S such that $P(X')$ is true and X is contained inside X' . Instead, X is said to be the maximum for P over S if $P(X)$ is true and there is no other sub-structure X' of S with a higher value for the property $P(X)$. For instance, the matching shown in the above figure is maximal because

it cannot be extended with other edges without breaking the matching property, but it's not a maximum matching.

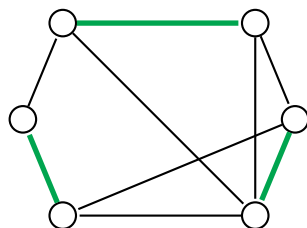


Figure 2.2: Maximum matching for the previous graph.

In particular, we'll focus on matching on **bipartite graphs**. These types of matchings are of particular interest due to how they describe many real-life situations. For instance, the problem of finding the optimal assignment of tasks to a group of employees can be solved by finding a maximum matching: we split the graph in two partitions, one with all the tasks and one with all the employees, connecting each employee to the tasks that he's capable of completing.

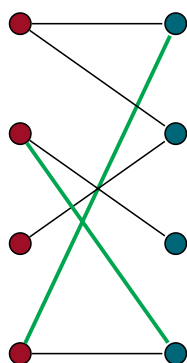


Figure 2.3: A matching on a bipartite graph.

Consider the matching on the bipartite graph shown above. We observe that this matching is neither maximal nor maximum. When the matching is not maximal, we can obtain a matching with greater cardinality simply by extending it.

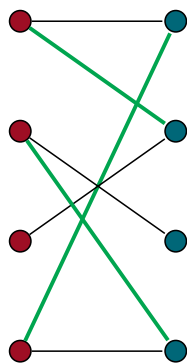


Figure 2.4: A maximal matching on a bipartite graph.

We notice that the above matching is maximal, but not maximum. For instance, we observe that the number of edges outside of the matching that we selected form new matching with more edges than the one that we have considered. Hence, by *swapping* all the edges, we get a matching with higher cardinality. Moreover, this new matching is clearly a maximum one since the number of nodes is equal to twice the number of selected edges.

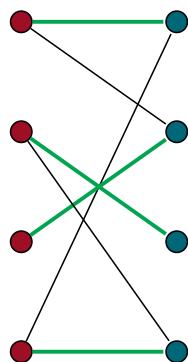


Figure 2.5: The maximum matching obtained by swapping all the edges.

Let's take a closer look to what we just did. We notice that all the edges of the last non-maximum matching actually form a path whose edges alternate between being outside of the matching and inside of the matching. Moreover, both the first and last edge of such path are outside of the matching, hence the number of outside edges is one more than the number of inside edges. We generalize such paths through the concept of *alternating path* and *augmenting path*.

Definition 16: Alternating and augmenting path

Let G be a bipartite graph and let $M \subseteq E(G)$ be a matching on G . An **M -alternating path** is a path starting from an unmatched node and whose edges alternate between M and $E(G) - M$. If the path also ends at an unmatched node, the path is said to be **M -augmenting**.

To clarify this definition, let's make things more formal. A path $P = x_0 e_1 x_1 e_2 \dots e_k x_k$ is M -alternating when x_0 is unmatched, meaning that there is no edge $e \in M$ in the matching such that $x_0 \in e$, and whose edges alternate between being outside of the matching and inside of the matching, meaning that $e_1 \notin M, e_2 \in M, e_3 \notin M$ and so on. When x_k is also unmatched, the path is said to be M -augmenting. This name comes from the fact that, in order for x_k to be unmatched, the last edge is not inside the matching, meaning that we have more edges outside of the matching than inside of it. Moreover, since x_0 and x_k are unmatched, by swapping the edges inside of the matching with the edges outside of it we're guaranteed to get a matching (with more edges than the previous one).

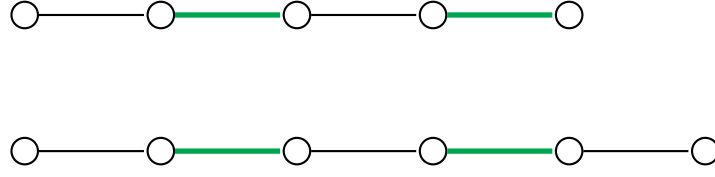


Figure 2.6: An M -alternating path (above) and an M -augmenting path (below).

Lemma 6

Let G be a bipartite graph and let $M \subseteq E(G)$ be a matching on G . If P is an M -augmenting path then $M \Delta E(P)$ is matching on G with more edges than M

Proof. First, we recall that the symmetric difference $M \Delta E(P)$ is defined as follows:

$$M \Delta E(P) = (M \cup E(P)) - (M \cap E(P))$$

Let $M' = M \Delta E(P)$. We observe that all the edges that aren't shared by M and P are also not in M' , hence we can ignore them. Let $P = x_0 e_1 x_1 e_2 \dots e_k x_k$. Since P is augmenting, the number of edges in P that are also in M is less than the number of edges that aren't in M . Hence, we get that $|M'| > |M|$. \square

The above lemma gives us an easy way to increase the cardinality of our matching by finding augmenting paths and swapping edges. But how can we be sure that we have reached the maximum matching? Berge [Ber57] proved that the above lemma can indeed be extended: if there are no augmenting paths then the matching is maximum.

Theorem 6: Berge's theorem

Let G be a bipartite graph and let $M \subseteq E(G)$ be a matching on G . Then, M is a maximum matching on G if and only if there are no M -augmenting paths.

Proof. One of the two implications directly comes from the previous lemma. For the other implication, we prove the contrapositive. Suppose that M is a non-maximum matching on G . Then, there is at least one matching M' on G such that $|M'| > |M|$. Let $H \subseteq G$ be the *multigraph* such that $V(H) = V(G)$ and $E(H) = M \Delta M'$. We observe that every edge that is shared among M and M' gets deleted in H , hence we can ignore them.

Claim: for every $x \in V(H)$ it holds that $\deg_H(x) \leq 2$.

Proof of Claim 1. Since M and M' are both matchings on G , each vertex can have at most one edge in M and at most one edge in M' . Since H is a multigraph, at most both such edges can be in H . \square

We observe that the above claim has many consequences. In particular, it implies that every component of H must be either a cycle or a path (including trivial paths of one single vertex).

Claim 2: every cycle component of H has even length

Proof of Claim 2. By way of contradiction, suppose that there is a cycle C of odd length. By construction, each component has to alternate between edges of M and M' . Hence, at least one vertex of C must have both edges lying either inside M or M' , contradicting the fact that either M or M' is a matching. \square

Since $|M'| > |M|$, there must be a component with at least one (and at most one) edge in M' . Since the edges of each component alternate between M and M' , by Claim 2 we know that such component cannot be a cycle. Hence, it must be a path component P . In order for P to have more edges in M' than edges in M , the first and last edge must be edges of M' , meaning that P is an M -augmenting path. By contrapositive, if there is no M -augmenting path then M is a maximum matching. \square

This theorem has been used to construct many algorithms for finding a maximum matching on bipartite graphs. The most famous one is the **Hopcroft-Karp algorithm** [HK73], due to a guaranteed runtime of $O(m\sqrt{n})$. Given a bipartite graph G with bipartition (A, B) , the idea behind such algorithms is to run a simultaneous BFS starting from all the unmatched vertices of A , until at least one free node of B is found. Then, we run a DFS over the forest produced by the BFS, starting from the unmatched nodes of B . Each path found through this procedure is an augmenting path, hence their edges can be swapped to increase the cardinality of the matching. The whole process is repeated until no augmenting path is found.

What about non-bipartite graphs? For the general case, problems arise with odd cycles, which never exist in bipartite graphs (Theorem 3). Any matching over an odd-length cycle will never cover every vertex of the cycle. This means that in any odd-length cycle there must be a vertex adjacent to two edges which cannot be part of the matching considered. This makes finding the optimal matching hard to find on such graphs. In 1965, Edmonds [Edm65] came up with the **Blossom algorithm**, a real piece of art in the world of algorithm design. The algorithm is based on the repeated contraction and distension of *blossoms*, i.e. a cycles of odd length which contain the maximal number of edges in the current matching. The *blossom lemma* states that any matching found on the graph with every blossom contracted is a maximum matching if and only if it is also a maximum matching for the graph original graph.

2.1.1 König's theorem

The maximum matching problem is highly related to the **Minimum Vertex Cover** problem. This problem involves finding the smallest subset of vertices in a graph such that every edge is incident to at least one vertex in the set.

Definition 17: Vertex Cover

Given a graph G , a vertex cover over G is a subset $C \subseteq V(G)$ such that $\forall e \in E(G)$ there is a vertex $v \in C$ such that $v \in e$.

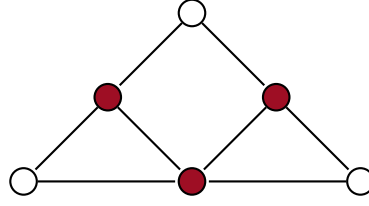


Figure 2.7: The red nodes are the smallest possible vertex cover of the graph.

Proposition 5

Given a graph G , for every matching M on G and every vertex cover V of G it holds that $|M| \leq |V|$.

Proof. By definition, we observe that if V is a vertex cover for G then it is also a vertex cover for any subgraph $G' \subseteq G$. Hence, V is also a vertex cover for any G_M , where $V(G_M) = V(G)$ and $E(G_M) = M$. By definition of matching, in G_M any vertex has either degree 0 or 1. Thus, each vertex of V can cover at most one edge of M , meaning that V has to have at least $|M|$ vertices to cover all the edges of M . \square

In bipartite graphs, the above proposition can be strengthened. In 1931, Kőnig [Sza20] proved that the cardinality of the maximum matching and the minimum vertex cover is equal. This implies that the minimum vertex cover can be efficiently found on bipartite graphs by finding the maximum matching.

Theorem 7: Kőnig's theorem

Given a bipartite graph G , let M^* and V^* respectively be a maximum matching and a minimum vertex cover on G . Then, it holds that $|M^*| \leq |V^*|$.

Proof. We observe that it suffices to show that there is a (non-minimum) vertex cover V such that $|V| = |M^*|$ in order to get $|M^*| \leq |V^*| \leq |V| = |M^*|$. Let (A, B) be the bipartition of G . We define V as the set of vertices such that $\forall ab \in M^*$, with $a \in A$ and $b \in B$, if there is an alternating path starting from a vertex in A and ending on b then $b \in V$, otherwise $a \in U$. Fix an edge $xy \in E(G)$ with $x \in A$ and $y \in B$.

Claim: V covers xy

Proof. Suppose that $xy \in M^*$ then we know that either x or y is in V by definition of V itself, meaning that such edge is covered. Hence, we may assume that $xy \notin M^*$. We have two cases:

- x is unmatched, meaning that $\nexists uv \in M^*$ such that $u = x$. Then, y must be matched by M^* to some $u \in A$, since otherwise $M^* \cup \{xy\}$ would be a matching greater than M^* . Hence, the edge uy is an alternating path starting from A and ending on y , meaning that $y \in V$.

- x is matched, meaning that $\exists xv \in M$. Then, by definition of V , either x or v must lie inside V . If $x \in U$ then xy is trivially covered by V . If $v \in V$, instead, by definition of V there must be an alternating path P starting from A and ending at v . This also implies that $P \cup vx \cup xy$ is an alternating path ending on y . However, this implies that y must be matched through some edge $wy \in M$, with $w \neq x$, since otherwise the path $P \cup vx \cup xy$ would be an augmenting path, contradicting the fact that M^* is maximum ([Berge's theorem](#)). Thus, we conclude that $y \in V$ since $wy \in M$ and $P \cup vx \cup xy$ is an augmenting path that ends on y .

□

In both cases, we get that either x or y is inside V , concluding that xy is covered by U . Applying the same argument over all edges, we get that V is a vertex cover. □

2.2 Perfect matching

We'll now focus on a particular type of matching on graphs. First, we notice that for any matching M on a graph G (even when G is non-bipartite) it must always hold that:

$$|M| \leq \frac{|V(G)|}{2}$$

When the inequality is satisfied at equality, the matching is said to be **perfect**. We observe that such condition is equivalent to saying that every vertex of the graph is covered by an edge. In fact, the concept of perfect matching can be, in some sense, viewed as the edge-version of a vertex cover, even though there is an additional constraint for the disjointness of the edges.

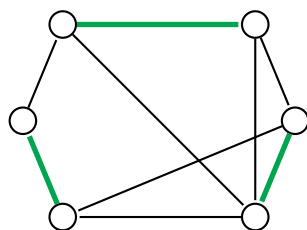


Figure 2.8: A perfect matching.

Definition 18: Perfect matching

A matching M on a graph G is said to be perfect if for all vertices of G there is an edge of M covering it.

We observe that every perfect matching is clearly a maximum one, but the contrary doesn't always hold. Moreover, not all graphs can have a perfect matching – trivially, a graph with an isolated vertex cannot have a perfect matching.

For bipartite graphs, the concept of perfect matching has to be slightly tweaked: if (A, B) is the bipartition of the graph and $|A| \neq |B|$, there is no way for a perfect matching to hold. Hence, assuming that $|A| \leq |B|$, we use the concept of **A -perfect matching**, i.e. a matching that covers every vertex of A – notice that if (A, B) is a partition then (B, A) is also a partition, hence we can always assume that $|A| \leq |B|$.

Hall [Hal35] was able to characterize A -perfect matchings on bipartite graphs through the now now called *Hall's Marriage Condition* – the name comes from the original set-theoretic formulation of the problem.

Theorem 8: Hall's marriage condition

Let G be a bipartite graph with bipartition (A, B) . Then, G has an A -perfect matching if and only if for all subsets $S \subseteq A$ it holds that $|N(S)| \geq |S|$, where $N(S) = \bigcup_{s \in S} N(s)$.

Proof. Suppose that there is an A -perfect matching M of G . Fix a subset $S \subseteq A$. By definition of M , each node $s \in S$ is matched through M with a different node, therefore $|N(S)| \geq |S|$.

Vice versa, suppose that there is no A -perfect matching M . Let V^* be a minimum vertex cover on G . By [Kőnig's theorem](#), we know that the size of V^* has to be equal to the size of any maximum matching of G . However, since there is no A -perfect matching, any maximum matching has to have size less than $|A|$, meaning that $|V^*| < |A|$. Consider now the set $S = A - V^*$. Since G is bipartite, it must hold that $N(S) = V^* \cap B$. We observe that:

$$|N(S)| \leq |V^* \cap B| = |V^*| - |V^* \cap A|$$

Since $V^* \cap A = A - (A - V^*) = A - S$, we get that:

$$|N(S)| \leq |V^*| - |V^* \cap A| = |V^*| - |A| + |S|$$

Finally, since $|U| < |A|$, we conclude that:

$$|N(S)| \leq |V^*| - |V^* \cap A| = |V^*| - |A| + |S| < |S|$$

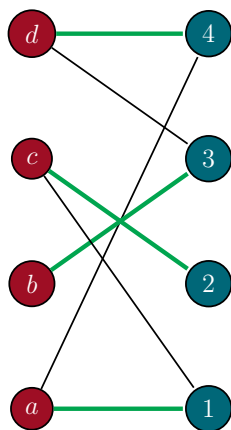
□

2.3 Stable matching

We'll now focus on a more advanced type of maximum, i.e. the **Stable Marriage** problem. This problem acts as a generalization of the common maximum matching problem, where the entities involved can have some *preferences* for their possible partners. In general, a matching on a bipartite graph is said to be **stable** if no vertex is unmatched and there is no other possible partner that they would prefer to be matched with. The preferences of each vertex $v \in V(G)$ are described through a bijective weight function $w_v : V \rightarrow [|X|]$, where $X = B$ if $v \in A$ and $X = A$ otherwise.

Definition 19: Stable matching

Let G be a bipartite graph and let $\{w_v\}_{v \in V(G)}$ be a family of preference functions. A matching M on G is said to be **stable** if there is no edge $ab \notin M$ for which $\exists ay, xb \in M$ such that $w_a(b) > w_a(y)$ and $w_b(a) > w_b(x)$



$w_a : 4231$	$w_1 : acbd$
$w_b : 4123$	$w_2 : abdc$
$w_c : 1243$	$w_3 : bcda$
$w_d : 3241$	$w_4 : dcba$

Figure 2.9: A stable matching. The preferences are represented through ordered lists

2.4 Solved exercises

Problem 2

Let G be a bipartite k -regular graph with bipartition (A, B) . Prove that:

1. $|A| = |B|$
2. G has a perfect matching

Proof. Given any set X , let E_X denote the subset of edges with an endpoint in X . Since the graph is k -regular, we have that $|E_A| = k|A|$ and $|E_B| = k|B|$. Moreover, since the graph is bipartite, we have that $E_A = E_B$. Hence, we get that $k|A| = k|B|$ and thus that $|A| = |B|$. Consider now a subset $S \subseteq A$. We observe that $E_S \subseteq E_{N(S)}$ by definition of neighborhood. Hence, we have that $k|S| = |E_S| \leq |E_{N(S)}| = k|N(S)|$ and thus that $|S| \leq |N(S)|$. By [Hall's marriage condition](#), we conclude that G has a perfect matching. \square

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