Mathematical Logic in Computer Science
Homework 2 2024-25

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Question 1 (Basic Concepts)**.** *Let R be a transitive relation on a finite set W. Prove that R is well-founded iff R is irreflexive. (R is called well-founded if there are no infinite paths $\ldots Rs_2Rs_1Rs_0$.)*

Solution. Suppose that R is reflexive. Then, for any element $s \in W$ we can form a trivial infinite path of the form $\ldots RsRsRsRs$ by taking the loop infinitely many times. Thus, R cannot be well-founded.

Vice versa, suppose that R is irreflexive. By way of contradiction, suppose that R is not well-founded. Let $P = \ldots Rs_2Rs_1Rs_0$ be an infinite path on W. Since W is finite, the path P has to eventually loop, meaning that $\exists i, j$ with $i \leq |W| \leq j$ such that $s_iRs_j \ldots s_{i+1}Rs_iR \ldots Rs_2Rs_1Rs_0$. By transitivity, we get that s_iRs_i, contradicting the irreflexivity of R. Thus, R must be well-founded. □

Question 2 (Models and Frames)**.** *Consider the basic temporal language and the frames $(\mathbb{Z}, <)$, $(\mathbb{Q}, <)$, and $(\mathbb{R}, >)$ (the integer, rational, and real numbers, respectively, all ordered by the usual less-than relation $<$). In this exercise we use $E\phi$ to abbreviate $P\phi \vee \phi \vee F\phi$ and $A\phi$ to abbreviate $H\phi \vee \phi \vee G\phi$. Which of the following formulas are valid on these frames?*

1. $GGp \to p$
2. $(p \wedge Hp) \to FHp$
3. $(Ep \wedge E\neg p \wedge A(p \to Hp) \wedge A(\neg p \to G\neg p)) \to E(Hp \wedge G\neg p)$

Solution. Let $\mathcal{Z} = (\mathbb{Z}, <)$, $\mathcal{Q} = (\mathbb{Q}, <)$, and $\mathcal{R} = (\mathbb{R}, >)$. The following table summarizes the validity of the formulas for each model.

	$\mathcal{Z}$	$\mathcal{Q}$	$\mathcal{R}$
(1)	×	×	×
(2)	✓	×	×
(3)	✓	✓	✓

We start by restricting our interest to $\mathcal{Z}$:

- We prove that the formula $GGp \to p$ is not valid in $\mathcal{Z}$ by giving a model that doesn't satisfy it. Let $\mathfrak{M}_{\mathcal{Z}} = (\mathcal{Z}, V)$ be a model such that $V(p) = \{v \in \mathbb{Z} \mid 0 < v\}$. We observe that:

$$\begin{aligned}\mathfrak{M}_{\mathcal{Z}}, 0 \models GGp &\iff \forall x \in \mathbb{Z} \text{ with } 0 < x,\ \ \mathfrak{M}_{\mathcal{Z}}, x \models Gp \\ &\iff \forall x, y \in \mathbb{Z}\ \text{ with } 0 < x < y,\ \ \mathfrak{M}_{\mathcal{Z}}, y \models p \\ &\iff \forall x, y \in \mathbb{Z}\ \text{ with } 0 < x < m,\ \ y \in V(p)\end{aligned}$$

which is true by choice of m itself. However, we have that $\mathfrak{M}_{\mathcal{Z}}, 0 \not\models p$ because $0 \notin V(p)$, concluding that $\mathfrak{M}_{\mathcal{Z}}, 0 \not\models GGp \to p$

- We prove that the formula $(p \land Hp) \to FHp$ is valid in $\mathcal{Z}$. Let $\mathfrak{M}_{\mathcal{Z}} = (\mathcal{Z}, V)$ be any model of $\mathcal{Z}$. We observe that:

$$\begin{aligned}\mathfrak{M}_{\mathcal{Z}}, n \models p \land Hp &\iff n \in V(p)\ \forall x \in \mathbb{Z} \text{ with } x < n \ \ \mathfrak{M}_{\mathcal{Z}}, x \models p \\ &\iff n \in V(p)\ \forall x \in \mathbb{Z} \text{ with } x < n \ \ x \in V(p)\end{aligned}$$

  Thus, we know that p holds for n and all x such that $x < n$. Moreover, we observe that:

$$\begin{aligned}\mathfrak{M}_{\mathcal{Z}}, n \models FHp &\iff \exists x \in \mathbb{Z} \text{ with } n < x \ \ \mathfrak{M}_{\mathcal{Z}}, x \models Hp \\ &\iff \exists x, y \in \mathbb{Z} \text{ with } n < x \ \text{ and } y < x \ \ \mathfrak{M}_{\mathcal{Z}}, y \models p \\ &\iff \exists x, y \in \mathbb{Z} \text{ with } n < x \ \text{ and } y < x \ \ y \in V(p)\end{aligned}$$

  Hence, x must be a successor of n such that p is true for all of x's predecessors. In $\mathbb{Z}$, picking $x = n + 1$ satisfies the formula since we already know that p holds for n and all n such that $y < n$.

- We prove that the formula $(Ep \land E\neg p \land A(p \to Hp) \land A(\neg p \to G\neg p)) \to E(Hp \land G\neg p)$ is valid in $\mathcal{Z}$. Let $\mathfrak{M}_{\mathcal{Z}} = (\mathcal{Z}, V)$ be any model of $\mathcal{Z}$. Let $Q \equiv Ep \land E\neg p \land A(p \to Hp) \land A(\neg p \to G\neg p)$. We observe that $\mathfrak{M}_{\mathcal{Z}}, n \models Q$ holds if all of the following hold for n:

  - There are two values $x, y \in \mathbb{Z}$ with $x \in V(p), y \notin V(p)$ and if $n \in V(p)$
  - For any value $m \in \mathbb{Z}$, if $m \in V(p)$ then for all $x' \in \mathbb{Z}$ with $x' < m$ then $x' \in V(p)$
  - For any value $m \in \mathbb{Z}$, if $m \notin V(p)$ then for all $y' \in \mathbb{Z}$ with $m < y'$ then $y' \notin V(p)$

  In order for the three conditions to not clash, it must hold that:

  - $x < y$
  - For all $x' \in \mathbb{Z}$ with $x' < x$ it holds that $x' \in V(p)$
  - For all $y' \in \mathbb{Z}$ with $y < y'$ it holds that $y'V(p)$

  But this is exactly the condition required by $E(Hp \land G\neg p)$ to be satisfied: x is a value such that p holds for all of its predecessors and

$\square$