Mathematical Logic in Computer Science
Homework 2 2024-25

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Question 1 (Basic Concepts)**.** *Let R be a transitive relation on a finite set W. Prove that R is well-founded iff R is irreflexive. (R is called well-founded if there are no infinite paths $\ldots Rs_2Rs_1Rs_0$.)*

Solution. Suppose that R is reflexive. Then, for any element $s \in W$ we can form a trivial infinite path of the form $\ldots RsRsRsRs$ by taking the loop infinitely many times. Thus, R cannot be well-founded.

Vice versa, suppose that R is irreflexive. By way of contradiction, suppose that R is not well-founded. Let $P = \ldots Rs_2Rs_1Rs_0$ be an infinite path on W. Since W is finite, the path P has to eventually loop, meaning that $\exists i, j$ with $i \leq |W| \leq j$ such that $s_iRs_j \ldots s_{i+1}Rs_iR \ldots Rs_2Rs_1Rs_0$. By transitivity, we get that s_iRs_i, contradicting the irreflexivity of R. Thus, R must be well-founded. $\square$

Question 2 (Models and Frames)**.** *Consider the basic temporal language and the frames $(\mathbb{Z}, <), (\mathbb{Q}, <)$, and $(\mathbb{R}, >)$ (the integer, rational, and real numbers, respectively, all ordered by the usual less-than relation $<$). In this exercise we use $E\phi$ to abbreviate $P\phi \vee \phi \vee F\phi$ and $A\phi$ to abbreviate $H\phi \vee \phi \vee G\phi$. Which of the following formulas are valid on these frames?*

1. $GGp \to p$
2. $(p \wedge Hp) \to FHp$
3. $(Ep \wedge E\neg p \wedge A(p \to Hp) \wedge A(\neg p \to G\neg p)) \to E(Hp \wedge G\neg p)$

Solution. Let $\mathcal{Z} = (\mathbb{Z}, <)$, $\mathcal{Q} = (\mathbb{Q}, <)$, and $\mathcal{R} = (\mathbb{R}, >)$. The following table summarizes the validity of the formulas for each model.

	$\mathcal{Z}$	$\mathcal{Q}$	$\mathcal{R}$
(1)	×	×	×
(2)	✓	×	×
(3)	×	×	×

First, we give an informal idea behind each result:

1. $GGp \to p$ doesn't hold in any of the models. Knowing that p will be true for every value following the current one doesn't ensure that it also holds for the current value.

2. $(p \wedge Hp) \rightarrow FHp$ holds only in $\mathcal{Z}$. This is due to the non-density of $\mathbb{Z}$, thus if p holds for the current value n and all previous values $n' < n$, p will hold for all $n'' < n+1$. Without density, there may be values between n and $n+1$ for which p doesn't hold.

3. $(Ep \wedge E\neg p \wedge A(p \rightarrow Hp) \wedge A(\neg p \rightarrow G\neg p)) \rightarrow E(Hp \wedge G\neg p)$ doesn't hold in any of the models. The premise of the implication states that there are two instants x, y for which p holds for any value x' with $x' < x$ and it doesn't hold for any value y' with $y > y'$. The conclusion of the implication states that there is an instant z for which p is true for any value that comes before z and false for any value that comes after z. However, there may be values s, t between x and y for which p holds on s and it doesn't on t, making at least one of the two conclusions false for any value z.

We start by restricting our interest to $\mathcal{Z}$:

- We prove that the formula $GGp \rightarrow p$ is not valid in $\mathcal{Z}$ by giving a model that doesn't satisfy it. Let $\mathfrak{M}_{\mathcal{Z}} = (\mathcal{Z}, V)$ be a model such that $V(p) = \{v \in \mathbb{Z} \mid 0 < v\}$. We observe that:

$$\begin{aligned} \mathfrak{M}_{\mathcal{Z}}, 0 \models GGp &\iff \forall x \in \mathbb{Z} \text{ with } 0 < x,\ \ \mathfrak{M}_{\mathcal{Z}}, x \models Gp \\ &\iff \forall x, y \in \mathbb{Z} \ \text{ with } 0 < x < y,\ \ \mathfrak{M}_{\mathcal{Z}}, y \models p \\ &\iff \forall x, y \in \mathbb{Z} \ \text{ with } 0 < x < y,\ \ y \in V(p) \end{aligned}$$

  which is true by choice of y itself. However, we have that $\mathfrak{M}_{\mathcal{Z}}, 0 \not\models p$ because $0 \notin V(p)$, concluding that $\mathfrak{M}_{\mathcal{Z}}, 0 \not\models GGp \rightarrow p$

- We prove that the formula $(p \wedge Hp) \rightarrow FHp$ is valid in $\mathcal{Z}$. Let $\mathfrak{M}_{\mathcal{Z}} = (\mathcal{Z}, V)$ be any model of $\mathcal{Z}$. We observe that:

$$\begin{aligned} \mathfrak{M}_{\mathcal{Z}}, n \models p \wedge Hp &\iff n \in V(p)\ \forall x \in \mathbb{Z} \text{ with } x < n\ \ \mathfrak{M}_{\mathcal{Z}}, x \models p \\ &\iff n \in V(p)\ \forall x \in \mathbb{Z} \text{ with } x < n\ \ x \in V(p) \end{aligned}$$

  Thus, we know that p holds for n and all x such that $x < n$. Moreover, we observe that:

$$\begin{aligned} \mathfrak{M}_{\mathcal{Z}}, n \models FHp &\iff \exists x \in \mathbb{Z} \text{ with } n < x\ \ \mathfrak{M}_{\mathcal{Z}}, x \models Hp \\ &\iff \exists x, y \in \mathbb{Z} \text{ with } n < x\ \text{ and } y < x\ \ \mathfrak{M}_{\mathcal{Z}}, y \models p \\ &\iff \exists x, y \in \mathbb{Z} \text{ with } n < x\ \text{ and } y < x\ \ y \in V(p) \end{aligned}$$

Hence, x must be a successor of n such that p is true for all of x's predecessors. In $\mathbb{Z}$, picking $x = n + 1$ satisfies the formula since we already know that p holds for all y such that $y \leq n$ and there are no elements between n and $n + 1$.

- We prove that the formula $(Ep \wedge E\neg p \wedge A(p \to Hp) \wedge A(\neg p \to G\neg p)) \to E(Hp \wedge G\neg p)$ <u>is not valid</u> in $\mathcal{Z}$. Let $S \equiv Ep \wedge E\neg p \wedge A(p \to Hp) \wedge A(\neg p \to G\neg p)$. Let $\mathfrak{M}_{\mathcal{Z}} = (\mathcal{Z}, V)$ be a model of $\mathcal{Z}$ such that

$$V(p) = \{v \in \mathbb{Z} \mid v < 0\} \cup \{1\}$$

. We observe that $\mathfrak{M}_{\mathcal{Z}}, 0 \models S$ is true when:

  - There are two values $x, y \in \mathbb{Z}$ with $x \in V(p), y \notin V(p)$. This comes from the definition of Ep and $E\neg p$ (e.g. x has to be either less than, equal to or greater than 0)
  - For any value $m \in \mathbb{Z}$, if $m \in V(p)$ then for all $x' \in \mathbb{Z}$ with $x' < m$ then $x' \in V(p)$
  - For any value $m \in \mathbb{Z}$, if $m \notin V(p)$ then for all $y' \in \mathbb{Z}$ with $m < y'$ then $y' \notin V(p)$

  Picking $x = -1$ and $y = 1$ satisfies all the above conditions, thus $\mathfrak{M}_{\mathcal{Z}}, 0 \models S$ is satisfied. However, we have that $\mathfrak{M}_{\mathcal{Z}}, 0 \not\models E(Hp \wedge G\neg p)$ because for all values $z \in \mathbb{Z}$ with $x < z < y$ at least one between Hp and $G\neg p$ is false.

We now consider the model $\mathcal{Q}$:

- We prove that the formula $GGp \to p$ <u>is not valid</u> in $\mathcal{Q}$ by giving a model that doesn't satisfy it. Let $\mathfrak{M}_{\mathcal{Q}} = (\mathcal{Q}, V)$ be a model such that $V(p) = \{v \in \mathbb{Q} \mid 0 < v\}$. We observe that:

$$\begin{aligned}
\mathfrak{M}_{\mathcal{Q}}, 0 \models GGp &\iff \forall x \in \mathbb{Q} \text{ with } 0 < x, \ \ \mathfrak{M}_{\mathcal{Q}}, x \models Gp \\
&\iff \forall x, y \in \mathbb{Q} \ \text{ with } 0 < x < y, \ \ \mathfrak{M}_{\mathcal{Q}}, y \models p \\
&\iff \forall x, y \in \mathbb{Q} \ \text{ with } 0 < x < y, \ \ y \in V(p)
\end{aligned}$$

  which is true by choice of y itself. However, we have that $\mathfrak{M}_{\mathcal{Q}}, 0 \not\models p$ because $0 \notin V(p)$, concluding that $\mathfrak{M}_{\mathcal{Q}}, 0 \not\models GGp \to p$

- We prove that the formula $(p \wedge Hp) \to FHp$ <u>is not valid</u> in $\mathcal{Q}$. Let $\mathfrak{M}_{\mathcal{Q}} = (\mathcal{Q}, V)$ be a model of $\mathcal{Q}$ with $V(p) = \{v \in \mathbb{Q} \mid v < 0\} \cup \{0\}$. We observe that:

$$\begin{aligned}
\mathfrak{M}_{\mathcal{Q}}, 0 \models p \wedge Hp &\iff 0 \in V(p) \ \forall x \in \mathbb{Q} \text{ with } x < 0 \ \ \mathfrak{M}_{\mathcal{Q}}, x \models p \\
&\iff n \in V(p) \ \forall x \in \mathbb{Q} \text{ with } x < 0 \ \ x \in V(p)
\end{aligned}$$

which is true by choice of x itself. However, we have that $\mathfrak{M}_{\mathcal{Q}}, 0 \not\models FHp$ because $\forall y \in \mathbb{Q}$ with $0 < y$ there is a value $z \in \mathbb{Q}$ with $0 < z < y$ such that $z \notin V(p)$ due to density of $\mathbb{Q}$.

- We prove that the formula $(Ep \wedge E\neg p \wedge A(p \to Hp) \wedge A(\neg p \to G\neg p)) \to E(Hp \wedge G\neg p)$ <u>is not valid</u> in $\mathcal{Q}$. Let $S \equiv Ep \wedge E\neg p \wedge A(p \to Hp) \wedge A(\neg p \to G\neg p)$. Let $\mathfrak{M}_{\mathcal{Q}} = (\mathcal{Q}, V)$ be a model of $\mathcal{Q}$ such that

  $$V(p) = \{v \in \mathbb{Q} \mid v < 0\} \cup \{1\}$$

  . We observe that $\mathfrak{M}_{\mathcal{Q}}, 0 \models S$ is true when:

  - There are two values $x, y \in \mathbb{Q}$ with $x \in V(p), y \notin V(p)$. This comes from the definition of Ep and $E\neg p$ (e.g. x has to be either less than, equal to or greater than 0)
  - For any value $m \in \mathbb{Q}$, if $m \in V(p)$ then for all $x' \in \mathbb{Q}$ with $x' < m$ then $x' \in V(p)$
  - For any value $m \in \mathbb{Q}$, if $m \notin V(p)$ then for all $y' \in \mathbb{Q}$ with $m < y'$ then $y' \notin V(p)$

  Picking $x = -1$ and $y = 1$ satisfies all the above conditions, thus $\mathfrak{M}_{\mathcal{Q}}, 0 \models S$ is satisfied. However, we have that $\mathfrak{M}_{\mathcal{Q}}, 0 \not\models E(Hp \wedge G\neg p)$ because for all values $z \in \mathbb{Q}$ with $x < z < y$ at least one between Hp and $G\neg p$ is false.

Finally, we consider the model $\mathcal{R}$:

- We prove that the formula $GGp \to p$ <u>is not valid</u> in $\mathcal{R}$ by giving a model that doesn't satisfy it. Let $\mathfrak{M}_{\mathcal{R}} = (\mathcal{R}, V)$ be a model such that $V(p) = \{v \in \mathbb{R} \mid 0 > v\}$. We observe that:

  $$\begin{aligned}
  \mathfrak{M}_{\mathcal{R}}, 0 \models GGp &\iff \forall x \in \mathbb{R} \text{ with } 0 > x,\ \ \mathfrak{M}_{\mathcal{R}}, x \models Gp \\
  &\iff \forall x, y \in \mathbb{R} \ \text{ with } 0 > x > y,\ \ \mathfrak{M}_{\mathcal{R}}, y \models p \\
  &\iff \forall x, y \in \mathbb{R} \ \text{ with } 0 > x > y,\ \ y \in V(p)
  \end{aligned}$$

  which is true by choice of y itself. However, we have that $\mathfrak{M}_{\mathcal{R}}, 0 \not\models p$ because $0 \notin V(p)$, concluding that $\mathfrak{M}_{\mathcal{R}}, 0 \not\models GGp \to p$

- We prove that the formula $(p \wedge Hp) \to FHp$ <u>is not valid</u> in $\mathcal{R}$. Let $\mathfrak{M}_{\mathcal{R}} = (\mathcal{R}, V)$ be a model of $\mathcal{R}$ with $V(p) = \{v \in \mathbb{R} \mid v > 0\} \cup \{0\}$. We observe that:

  $$\begin{aligned}
  \mathfrak{M}_{\mathcal{R}}, 0 \models p \wedge Hp &\iff 0 \in V(p)\ \forall x \in \mathbb{R} \text{ with } x > 0\ \ \mathfrak{M}_{\mathcal{R}}, x \models p \\
  &\iff n \in V(p)\ \forall x \in \mathbb{R} \text{ with } x > 0\ \ x \in V(p)
  \end{aligned}$$

which is true by choice of x itself. However, we have that $\mathfrak{M}_{\mathcal{R}}, 0 \not\models FHp$ because $\forall y \in \mathbb{R}$ with $0 > y$ there is a value $z \in \mathbb{R}$ with $0 > z > y$ such that $z \notin V(p)$ due to density of $\mathbb{R}$.

- We prove that the formula $(Ep \wedge E\neg p \wedge A(p \to Hp) \wedge A(\neg p \to G\neg p)) \to E(Hp \wedge G\neg p)$ is not valid in $\mathcal{R}$. Let $S \equiv Ep \wedge E\neg p \wedge A(p \to Hp) \wedge A(\neg p \to G\neg p)$. Let $\mathfrak{M}_{\mathcal{R}} = (\mathcal{R}, V)$ be a model of $\mathcal{R}$ such that

$$V(p) = \{v \in \mathbb{R} \mid v > 0\} \cup \{-1\}$$

. We observe that $\mathfrak{M}_{\mathcal{R}}, 0 \models S$ is true when:

  - There are two values $x, y \in \mathbb{R}$ with $x \in V(p), y \notin V(p)$. This comes from the definition of Ep and $E\neg p$ (e.g. x has to be either greater than, equal to or less than 0)
  - For any value $m \in \mathbb{R}$, if $m \in V(p)$ then for all $x' \in \mathbb{R}$ with $x' > m$ then $x' \in V(p)$
  - For any value $m \in \mathbb{R}$, if $m \notin V(p)$ then for all $y' \in \mathbb{R}$ with $m > y'$ then $y' \notin V(p)$

  Picking $x = 1$ and $y = -1$ satisfies all the above conditions, thus $\mathfrak{M}_{\mathcal{R}}, 0 \models S$ is satisfied. However, we have that $\mathfrak{M}_{\mathcal{R}}, 0 \not\models E(Hp \wedge G\neg p)$ because for all values $z \in \mathbb{R}$ with $x > z > y$ at least one between Hp and $G\neg p$ is false.

$\square$

Question 3 (General frames). *Consider the structure $g = (\mathbb{N}, C, A)$ where A is the collection of finite and co-finite subsets of $\mathbb{N}$, and C is defined by:*

$$C(n_1, n_2, n_3) \iff n_1 \le n_2 + n_3 \text{ and } n_2 \le n_1 + n_3 \text{ and } n_3 \le n_1 + n_2$$

If C is the accessibility relation of a dyadic modal operator, show that g is a general frame.

Solution. In order to prove that g is a general frame, we have to prove that A is closed under union, complement and modal operators. Closure under complement is easily implied by the very definition of A: for any $X \in A$, if X is finite then $\overline{X}$ is co-finite, otherwise if X is co-finite then $\overline{X}$ is finite.

We now prove that A is closed under union. Fix two subsets $X, Y \in A$. We may assume that at least one of X and Y is co-finite, otherwise we trivially get that $X \cup Y$ is finite and thus that $X \cup Y \in A$. Without loss of generality, assume that X is co-finite, implying that $\overline{X}$ is finite. By De Morgan's theorem we have that $\overline{X \cup Y} = \overline{X} \cap \overline{Y}$. Since $\overline{X}$ is finite, $\overline{X} \cap \overline{Y}$ must also be finite, concluding that $X \cup Y$ is co-finite and thus that $X \cup Y \in A$.

Lastly, we prove that A is closed under Δ_C, i.e. the dyadic modal operator associated with C. To prove this, we have to show that for each $X_1, X_2 \in A$ it holds that $m_{\Delta_C}(X_1, X_2) \in A$, where:

$$m_{\Delta_C}(X_1, X_2) = \{w \in \mathbb{N} \mid \exists (x_1, x_2) \in X_1 \times X_2 \;\; C(w, x_1, x_2)\}$$

Fix two subsets $X_1, X_2 \in A$. If both X_1 and X_2 are finite, both subsets have a maximum under $\leq$, implying that $m_{\Delta_C}(X_1, X_2)$ is finite since for each pair $(x_1, x_2) \in X_1 \times X_2$ there are only finitely many values $w \in \mathbb{N}$ for which $w \leq x_1 + x_2$ holds.

Hence, we may assume that at least one between X_1, X_2 is co-finite. Without loss of generality assume that X_1 is co-finite. Then, X_1 must be infinite since $X_1 = \mathbb{N} - \overline{X_1}$ and $\overline{X_1}$ is finite, concluding that X_1 has no maximum value. Consider the set:

$$\overline{m_{\Delta_C}(X_1, X_2)} = \{w \in \mathbb{N} \mid \forall (x_1, x_2) \in X_1 \times X_2 \;\; \neg C(w, x_1, x_2)\}$$

Claim 3.1: for any $n_1, n_2, n_3 \in \mathbb{N}$ it holds that $\neg C(n_1, n_2, n_3)$ if and only if exactly one of the following conditions holds:

1. $n_1 > n_2 + n_3$
2. $n_2 > n_1 + n_3$
3. $n_3 > n_1 + n_2$

Proof of Claim 3.1. By definition, we have that

$$\neg C(n_1, n_2, n_3) \iff n_1 > n_2 + n_3 \text{ or } n_2 > n_1 + n_3 \text{ or } n_3 > n_1 + n_2$$

We now observe that:

- It cannot hold that both $n_1 > n_2 + n_3$ and $n_2 > n_1 + n_3$ are true, otherwise:
$$n_1 > n_2 + n_3 > n_1 + 2n_3 \implies 0 > n_3$$
which is impossible since $n_3 \in \mathbb{N}$

- It cannot hold that both $n_1 > n_2 + n_3$ and $n_3 > n_1 + n_2$ are true, otherwise:
$$n_1 > n_2 + n_3 > 2n_2 + n_1 \implies 0 > n_2$$
which is impossible since $n_2 \in \mathbb{N}$

- It cannot hold that both $n_2 > n_1 + n_3$ and $n_3 > n_1 + n_2$ are true, otherwise:
$$n_2 > n_1 + n_3 > 2n_1 + n_2 \implies 0 > n_1$$
which is impossible since $n_1 \in \mathbb{N}$

□

Claim 3.1 allows us to partition $\overline{m_{\Delta_C}(X_1, X_2)}$ into three disjoint subsets Z_1, Z_2, Z_3, where:

- $Z_1 = \{w \in \mathbb{N} \mid \forall(x_1, x_2) \in X_1 \times X_2 \;\; w > x_1 + x_2\}$
- $Z_2 = \{w \in \mathbb{N} \mid \forall(x_1, x_2) \in X_1 \times X_2 \;\; x_1 > w + x_2\}$
- $Z_3 = \{w \in \mathbb{N} \mid \forall(x_1, x_2) \in X_1 \times X_2 \;\; x_2 > w + x_2\}$

Claim 3.2: Z_1, Z_2, Z_3 are finite.

Proof of Claim 3.2. Since X_1 has a minimum, there are only finitely many values $w \in \mathbb{N}$ for which $\min(X_1) > w + x_2$ holds for every $x_2 \in X_2$. Similarly, since X_2 has a minimum there are only finitely many values $w \in \mathbb{N}$ for which $\min(X_2) > w + x_1$ holds for every $x_1 \in X_1$. This concludes that Z_2 and Z_3 are finite.

Consider now the set Z_1. Since X_1 has no maximum, for each $w \in \mathbb{N}$ there is always a pair $(x_1, x_2) \in \mathbb{N}$ with $w < x_1$ such that $w \not> x_1 + x_2$. This concludes that $Z_1 = \varnothing$. □

Since $\overline{m_{\Delta_C}(X_1, X_2)} = Z_1 \cup Z_2 \cup Z_3$, by Claim 3.2 we conclude that $\overline{m_{\Delta_C}(X_1, X_2)}$ is finite and thus that $m_{\Delta_C}(X_1, X_2)$ is co-finite. □

Question 4 (Modal Consequence Relations)**.** *Let Σ be a set of formulas in the basic modal language and let F denote the class of all frames. Show that $\Sigma \models^g_F \phi$ iff $\{\Box^n \sigma \mid \sigma \in \Sigma, n \in \mathbb{N}\} \models_F \phi$.*

Solution. Let $\Pi = \{\Box^n \sigma \mid \sigma \in \Sigma, n \in \mathbb{N}\}$. We recall that $\Sigma \models^g_F \phi$ holds iff:

$$\forall \mathfrak{F} \in F\ \forall \mathfrak{M} \in \text{Model}(\mathfrak{F})\ ((\forall w \in W\ \mathfrak{M}, w \models \Sigma) \to (\forall w \in W\ \mathfrak{M}, w \models \phi))$$

while $\Pi \models_F \phi$ holds iff:

$$\forall \mathfrak{F} \in F\ \forall \mathfrak{M} \in \text{Model}(\mathfrak{F})\ \forall w \in W\ ((\mathfrak{M}, w \models \Sigma) \to (\mathfrak{M}, w \models \phi))$$

where mod($\mathfrak{F}$) is the set of all models of the frame $\mathfrak{F}$.

Claim 1: if $\Pi \models_F \phi$ then $\Sigma \models^g_F \phi$

Proof of Claim 1. Assume that $\Pi \models_F \phi$ holds. Fix $\mathfrak{F} = (W, R) \in F$ and a model $\mathfrak{M}$ of $\mathfrak{F}$. Suppose that $\forall w \in W$ it holds that $\mathfrak{M}, w \models \Sigma$. Then, $\forall w \in W, \forall \sigma \in \Sigma$ we have that $\mathfrak{M}, w \models \sigma$. However, this implies that for each $\forall w \in W, \forall \sigma \in \Sigma, n \in \mathbb{N}$ it holds that:

$$\forall w, \forall x_1, \ldots, x_n \text{ with } R(w, x_1), R(x_1, x_2), \ldots, R(x_{n-1}, x_n)\ \ \mathfrak{M}, x_n \models \sigma$$

which is equivalent to saying that $\mathfrak{M}, w \models \Box^n \sigma$, and thus that $\mathfrak{M}, w \models \Pi$ holds for all $w \in W$. By applying the assumption $\Pi \models_F \phi$, we conclude that $\mathfrak{M}, w \models \phi$ for all $w \in W$. Since the argument holds for an arbitrary frame and an arbitrary model, we conclude that $\Sigma \models^g_F \phi$. $\Box$

Claim 2: if $\Sigma \models^g_F \phi$ then $\Pi \models_F \phi$

Proof of Claim 2. Assume $\Sigma \models^g_F \phi$. Fix $\mathfrak{F} = (W, R) \in F$, a model $\mathfrak{M} = (\mathfrak{F}, V)$ and a world $w \in W$.

Let $C \subseteq W \times W$ be the relation of reachability from w in W, meaning that:

$$C(w, w') \iff \exists m \in \mathbb{N}, \exists z_1, \ldots, z_m \in W \quad \text{with } R(w, z_1), \ldots, R(z_m, w')$$

Consider the frame $\mathfrak{F}' = (W', R')$ such that:

$$W' = \{w' \in W \mid C(w, w')\} \qquad R' = R \cap (W' \times W')$$

and consider the model $\mathfrak{M}' = (\mathfrak{F}', V')$, where for all propositional letters p it holds that:

$$V'(p) = V(p) \cap W'$$

Suppose that $\mathfrak{M}, w \models \Pi$. Then, we have that:

$$\begin{aligned}
&\mathfrak{M}, w \models \Pi \\
\iff &\forall \sigma \in \Sigma, \forall n \in \mathbb{N}\ \ \mathfrak{M}, w \models \Box^n \sigma \\
\iff &\forall \sigma \in \Sigma, \forall n \in \mathbb{N}, \forall x_1, \ldots, x_n \in W \text{ with } R(w, x_1), \ldots, R(x_{n-1}, x_n)\ \ \mathfrak{M}, x_n \models \sigma
\end{aligned}$$

Since $W' \subseteq W$, we have that each $x_1, \ldots, x_n$ in the above statement can also be chosen from W'. When taken from W', we get that $R'(w, x_1), \ldots, R'(x_{n-1}, x_n)$ holds by definition of R' and thus that $\mathfrak{M}', x_n \models \sigma$ by definition of $\mathfrak{M}'$.

$$\begin{aligned}
&\mathfrak{M}, w \models \Pi \\
\iff &\forall \sigma \in \Sigma, \forall n \in \mathbb{N}, \forall x_1, \ldots, x_n \in W \text{ with } R(w, x_1), \ldots, R(x_{n-1}, x_n)\ \ \mathfrak{M}, x_n \models \sigma \\
\implies &\forall \sigma \in \Sigma, \forall n \in \mathbb{N}, \forall x_1, \ldots, x_n \in W' \text{ with } R(w, x_1), \ldots, R(x_{n-1}, x_n)\ \ \mathfrak{M}, x_n \models \sigma \\
\implies &\forall \sigma \in \Sigma, \forall n \in \mathbb{N}, \forall x_1, \ldots, x_n \in W' \text{ with } R'(w, x_1), \ldots, R'(x_{n-1}, x_n)\ \ \mathfrak{M}, x_n \models \sigma \\
\implies &\forall \sigma \in \Sigma, \forall n \in \mathbb{N}, \forall x_1, \ldots, x_n \in W' \text{ with } R'(w, x_1), \ldots, R'(x_{n-1}, x_n)\ \ \mathfrak{M}', x_n \models \sigma \\
\implies &\forall \sigma \in \Sigma, \forall w' \in W'\ \ \mathfrak{M}', w \models \sigma \\
\iff &\forall w' \in W'\ \ \mathfrak{M}', w \models \Sigma
\end{aligned}$$

Using the assumption and the fact that $w \in W' \subseteq V'(\phi) \subseteq V(\phi)$, we conclude that:

$$\begin{aligned}
\mathfrak{M}, w \models \Pi &\implies \forall w' \in W'\ \ \mathfrak{M}', w \models \Sigma \\
&\implies \mathfrak{M}', w \models \phi \\
&\implies w \in V'(\phi) \subseteq V(\phi) \\
&\implies \mathfrak{M}, w \models \phi
\end{aligned}$$

$\square$

The two claims conclude the proof. $\square$

Question 5 (Normal Modal Logic)**.** *Let F be a class of frames. Prove that Λ_F is a normal modal logic.*

Solution. We recall that Λ_F is the set of all formulas that are valid for every frame of F. To show that Λ_F is a normal modal logic, we have to prove that:

(1) it contains all tautologies, the formula $\Box(p \to q) \to (\Box p \to \Box q)$ and the formula $\Diamond p \leftrightarrow \neg\Box\neg p$

(2) it is closed under modus ponens, uniform substitution and generalization

Claim 5.1: (1) holds on Λ_F

Proof of Claim 5.1. Fix $\mathfrak{F} = (W, R) \in F$. Since all tautologies are valid on every frame by definition, they are also valid in $\mathfrak{F}$. Likewise, the formula $\Diamond p \leftrightarrow \neg\Box\neg p$ is valid in every frame by definition of $\Diamond$ and $\Box$, thus also in $\mathfrak{F}$. Consider now a model $\mathfrak{M}$ of $\mathfrak{F}$. Given a world $w \in W$, suppose that $\mathfrak{M}, w \models \Box(p \to q)$ and $\mathfrak{M}, w \models \Box p$ then, we have that:

$$\mathfrak{M}, w \models \Box(p \to q) \iff \forall x \in W \text{ with } R(w, x) \;\; \mathfrak{M}, x \models p \to q$$

and that:

$$\mathfrak{M}, w \models \Box p \iff \forall x \in W \text{ with } R(w, x) \;\; \mathfrak{M}, x \models p$$

which implies that $\forall x \in W$ with $R(w, x) \quad \mathfrak{M}, x \models p \wedge (p \to q)$. Since $p \wedge (p \to q) \equiv p \wedge q$, we get that:

$$\iff \forall x \in W \text{ with } R(w, x) \;\; \mathfrak{M}, x \models q \iff \mathfrak{M}, w \models \Box q$$

which concludes that for each $w \in W$ it holds that:

$$\mathfrak{M}, w \models \Box(p \to q) \to (\Box p \to \Box q)$$

Since M was a arbitrary model of $\mathfrak{F}$ and $\mathfrak{F}$ was a arbitrary frame of F, we conclude that the formula is valid on every frame of F. $\Box$

Claim 5.1: (2) holds on Λ_F

Proof of Claim 5.2. We prove that Λ_F is closed under each of the three operations:

- *Modus Ponens*: given two formulas ϕ, ψ, suppose that $\phi \in \Lambda_F$ and $\phi \to \psi \in \Lambda_F$. Then, for every frame $\mathfrak{F} = (W, R) \in F$ it holds that:

$$(\mathfrak{F} \models \phi) \text{ and } \mathfrak{F} \models \phi \to \psi \implies \mathfrak{F} \models \phi \land (\phi \to \psi)$$

  We now observe that:

$$\phi \land (\phi \to \psi) \equiv \phi \land (\neg\phi \lor \psi) \equiv (\phi \land \neg\phi) \lor (\phi \land \psi) \equiv \phi \land \psi$$

  Hence, we that:

$$\mathfrak{F} \models \phi \land (\phi \to \psi) \implies \mathfrak{F} \models \phi \land \psi \implies \mathfrak{F} \models \psi$$

  concluding that $\psi \in \Lambda_F$

- *Generalization*: given a formula ϕ, suppose that $\phi \in \Lambda_F$. Then, for every frame $\mathfrak{F} = (W, R) \in F$ it holds that $\mathfrak{M}, w \models \phi$ for each model $\mathfrak{M}$ of $\mathfrak{F}$ and each $w \in W$. Since every world $w \in W$ satisfies ϕ, we trivially get that $\mathfrak{M}, x \models \phi$ for all $x \in W$ with $R(w, x)$, concluding that $\mathfrak{F} \models \Box\phi$ and thus that $\Box\phi \in \Lambda_F$.

- *Uniform substitution*: given a formula ϕ, suppose that $\phi \in \Lambda_F$ and let $\sigma(\phi)$ a formula obtained by replacing every propositional letter in ϕ with an arbitrary formula. Fix a frame $\mathfrak{F} = (W, R) \in F$ and let $\mathfrak{M}$ be a frame of $\mathfrak{F}$. We construct a new model $\mathfrak{M}' = (\mathfrak{F}, V')$ such that:

$$V'(p) = \{w \in W \mid \mathfrak{M}, w \models \sigma(p)\}$$

  for each propositional letter p.

  We prove by structural induction that for each $w \in W$ it holds that $\mathfrak{M}, w \models \sigma(\phi)$ if and only if $\mathfrak{M}', w \models \phi$. The base case is given by propositional letters, for which the statement holds by definition of $\mathfrak{M}'$. We inductively assume that the statement holds of every subformula of ϕ. Then, on the inductive step, we have three cases:

  1. If $\phi = \neg\psi$ then $\sigma(\phi) = \neg\sigma(\psi)$, thus:

$$\begin{aligned}\mathfrak{M}', w \models \phi &\iff \mathfrak{M}, w \models \neg\psi \\ &\iff \mathfrak{M}', w \not\models \psi \\ &\iff \mathfrak{M}, w \not\models \sigma(\psi) \\ &\iff \mathfrak{M}, w \models \neg\sigma(\psi) \\ &\iff \mathfrak{M}, w \models \sigma(\phi)\end{aligned}$$

2. If $\phi = \psi \to \varphi$ then $\sigma(\phi) = \sigma(\psi) \to \sigma(\varphi)$, thus:

$$\begin{aligned}
\mathfrak{M}', w \models \phi &\iff \mathfrak{M}', w \models \psi \to \varphi \\
&\iff (\mathfrak{M}', w \models \neg\psi) \vee (\mathfrak{M}', w \models \varphi) \\
&\iff (\mathfrak{M}', w \not\models \psi) \vee (\mathfrak{M}', w \models \varphi) \\
&\iff (\mathfrak{M}, w \not\models \sigma(\psi)) \vee (\mathfrak{M}, w \models \sigma(\varphi)) \\
&\iff (\mathfrak{M}, w \models \neg\sigma(\psi)) \vee (\mathfrak{M}, w \models \sigma(\varphi)) \\
&\iff \mathfrak{M}, w \not\models \sigma(\psi) \to \sigma(\varphi) \\
&\iff \mathfrak{M}, w \not\models \sigma(\phi)
\end{aligned}$$

3. If $\phi = \Box\psi$ then $\sigma(\phi) = \Box\sigma(\psi)$, thus:

$$\begin{aligned}
\mathfrak{M}', w \models \phi &\iff \mathfrak{M}', w \models \Box\psi \\
&\iff \forall x \in W \text{ with } R(w, x) \;\; \mathfrak{M}', x \models \psi \\
&\iff \forall x \in W \text{ with } R(w, x) \;\; \mathfrak{M}, x \models \sigma(\psi) \\
&\iff \mathfrak{M}, w \models \Box\sigma(\psi) \\
&\iff \mathfrak{M}, w \models \sigma(\phi)
\end{aligned}$$

Therefore, since for each $w \in W$ it holds that $\mathfrak{M}, w \models \sigma(\phi)$ if and only if $\mathfrak{M}', w \models \phi$ and $\mathfrak{M}'$ is a model of $\mathfrak{F}$, we get that $\mathfrak{M}, w \models \sigma(\phi)$ holds for each $w \in W$. Moreover, since M was a arbitrary model of $\mathfrak{F}$, the argument holds for every model of $\mathfrak{F}$. Likewise, since $\mathfrak{F}$ was a arbitrary frame of F, this holds for every frame of $\mathfrak{F}$, concluding that $\sigma(\phi) \in \Lambda_F$.

$\Box$

The two claims conclude that Λ_F is a normal modal logic. $\Box$