Mathematical Logic in Computer Science
Homework 1 2024-25

Simone Bianco, 1986936

Sapienza Università di Roma, Italy

April 18, 2025

Question 1 (Compactness). *Let $\mathcal{L} = \{E(x,y)\}$ be the language of graphs.*

1. *For each fixed $n \in \mathbb{N}$ write a sentence C_n such that for any graph $\mathcal{G}$, $\mathcal{G} \models C_n$ if and only if $\mathcal{G}$ contains an n-clique.*
2. *Prove using Compactness that the property of being finitely colorable is not expressible by a theory in $\mathcal{L}$ over the class of graphs.*

Solution to Question 1. For any $n \in \mathbb{N}$, consider the following sentence C_n:

$$C_n \equiv \exists x_1 \dots \exists x_n \bigwedge_{i=1}^{n} \bigwedge_{\substack{j=1 \\ i \neq j}}^{n} (E(x_i, x_j) \wedge \neg(x_i = x_j))$$

Claim 1.1: $\mathcal{G} \models C_n$ if and only if $\mathcal{G}$ contains an n-clique

Proof of Claim 1.1. Suppose that $\mathcal{G} \models C_n$. Let $v_1, \dots, v_n$ be the variables forming the assignment that satisfies C_n, i.e. $\begin{pmatrix} x_1 & \cdots & x_n \\ v_1 & \cdots & v_n \end{pmatrix}$. Then, $\{v_1, \dots, v_n\}$ form an n-clique since every pair of vertices is adjacent to each other and they are distinct. Vice versa, suppose that $\mathcal{G}$ contains an n-clique $\{u_1, \dots, u_n\}$. Then, the assignment formed by $\begin{pmatrix} x_1 & \cdots & x_n \\ u_1 & \cdots & u_n \end{pmatrix}$ satisfies the non-quantified formula inside C_n, concluding that $\mathcal{G} \models C_n$. □

Consider now the theory $T = \{C_n \mid n \in \mathbb{N}\}$, expressing the property of containing an n-clique for any $n \in \mathbb{N}$. By way of contradiction, suppose that there is a theory T' expressing the property of being finitely colorable.

Claim 1.2: $T^* = T \cup T'$ is finitely satisfiable.

Proof of Claim 1.2. Fix $S \underset{fin}{\subseteq} T^*$ and consider the set $S - T' = \{C_{i_1}, \dots, C_{i_\ell}\}$. Let $M = \max(i_1, \dots, i_\ell)$. Then, the complete graph $\mathcal{K}_M$ on M vertices is M-colorable, hence finitely colorable, and contains an M-clique, which implies that it also contains an h-clique for any $h \leq M$. Thus, $\mathcal{K}_M \models S - T'$ and $\mathcal{K}_M \models T'$. Since it satisfies T', it also satisfies any subset of T' that may be inside S, concluding that $\mathcal{K}_M \models S$. □

It's easy to see that if $\mathcal{G}$ is k-colorable then it doesn't contain a $k+1$ clique, concluding that T^* is unsatisfiable. However, by Compactness we know that T^* is finitely satisfiable if and only if it is satisfiable, raising a contradiction. Thus, the only possibility is that T' doesn't exist. □

Question 2 (Games and non-expressibility). *Consider the structures* $\mathfrak{A} = (\mathbb{N}, \leq)$ *and* $\mathfrak{B} = (\mathbb{Z}, \leq)$*:*

1. *What is the smallest* k *such that* $\mathfrak{A} \equiv_k \mathfrak{B}$ *does not hold?*

2. *Prove that the Duplicator wins every finite moves game if the Spoiler is forced to always play in the same structure (i.e. either always in* $\mathfrak{A}$ *or always in* $\mathfrak{B}$*)*

Solution to Question 2.

1. Through Ehrenfeucht's theorem, we know that $\mathfrak{A} \equiv_k \mathfrak{B}$ if and only if the Duplicator has a strategy to win the k-round game on $\mathfrak{A}, \mathfrak{B}$. It's easy to see that the Spoiler has a trivial strategy to win the 2-round game. On the first round, the Spoiler picks $a_0 = 0_\mathbb{N} \in \mathbb{N}$ and the Duplicator answers with $b_0 \in \mathbb{Z}$. On the second round, the Spoiler can pick any value $b_1 \in \mathbb{Z}$ such that $b_1 < b_0$ in order to win the game since $0_\mathbb{N}$ is the minimal element in $\mathfrak{A}$. In fact, the query $\forall x \exists y\ (y \leq x) \wedge \neg(y = x)$ is enough to distinguish the two structures, hence they cannot be 2-equivalent. Moreover, the Duplicator can always easily win the 1-game by replying with any value to the first choice of the Spoiler. Hence, we conclude that 2 is the smallest value k such that $\mathfrak{A} \equiv_k \mathfrak{B}$ doesn't hold.

2. Consider the case where the Spoiler is forced to always play in $\mathfrak{A}$. Then, the Duplicator has a trivial strategy to win the ∞-game since $\mathbb{Z}$ contains $\mathbb{N}$: every choice $a_i \in \mathbb{N}$ is mapped to $b_i = a_i$ in $\mathbb{Z}$, preserving the order relations. Hence, the Duplicator can also win the k-game for any k.

   In the other case, instead, the Duplicator cannot win the ∞-game since the Spoiler has a strategy to win. On the first round, the Spoiler picks $b_0 = 0_\mathbb{Z}$ and the Duplicator answers with $a_0 \in \mathbb{N}$. Then, for any other i-th round, the Spoiler always picks $b_i = b_{i-1} - 1 \in \mathbb{Z}$. Since there are a finite amount of elements in $\{0, \dots, a_0\} \subset \mathbb{N}$, the Duplicator will eventually run out of options. Nonetheless, the Duplicator still has a strategy to win the k-game for any k. The idea is to map all the elements $z \in \mathbb{Z}$ with $z \geq 0$ to all the $n \in \mathbb{N}$ with $n \geq 2^k$, while reserving all the $n' \in \mathbb{N}$ with $n' < 2^k$ as a way to "simulate density up to k elements" in order to survive at least k queries that ask for a middle element.

   Fix $k \in \mathbb{N}$. Let $a_{-1} = 2^k$ and $b_{-1} = 0$. On each round i, let $a_1, \dots, a_{i-1} \in \mathbb{N}$ and $b_1, \dots, b_{i-1} \in \mathbb{Z}$ be the choices made in the previous rounds. Let $A_i = \{a_j \mid -1 \leq j < i \text{ and } a_j \leq 2^k\}$ and let

$B_i = \{b_j \mid -1 \leq j < i \text{ and } b_j \leq 0\}$. The Duplicator plays the i-th round through the following decision process:

(a) If the Spoiler picks $b_i \in \mathbb{Z}$ such that $b_i \geq 0$ then the Duplicator answers with $a_i \in \mathbb{N}$ such that $a_i = 2^k + b_i$.

(b) If the Spoiler picks $b_i \in \mathbb{Z}$ such that $b_i < 0$ then:

  i. If $b_i < \min(B_{i-1})$ then the Duplicator answers with $a_i = \frac{1}{2}\min(A_{i-1})$

  ii. If $b_i > \min(B_{i-1})$ then the Duplicator answers with $a_i = \frac{1}{2}(a_j + a_t)$, where j, t are two indices such that b_j, b_t are the two values in B_{i-1} that minimize $|b_j - b_t|$ and such that $b_j < b_i < b_t$.

Claim 2.1: For all $i \in \mathbb{N}$, if $i \leq k$ it holds that:

(a) The distance between each pair $x, y \in A_i \cup \{0\}$ is at least 2^{k-i}.

(b) $a_1, \dots, a_i \mapsto b_1, \dots, b_i$ is a partial isomorphism.

Proof of Claim 2.1. Any time the choice made by the Spoiler falls in Case (a) the partial isomorphism is trivially preserved since each positive the integer gets only shifted by 2^k places. This allows us to restrict our focus entirely on choices of Case (b).

We proceed by induction on i. When $i = 1$, we have that $A_0 = \{2^k\}$ and $B_0 = \{0\}$. Hence, since $b_1 < 0 = \min(A_0)$, the Duplicator answers with $a_1 = 2^k - 1$. Since $|a_{-1} - a_1| = |0 - a_1| = 2^{k-1}$ and relations are preserved since no other choices have been made, the claim holds.

Assume now that the claim holds for the i-th round. For the $(i+1)$-th round, we consider the two subcases:

(a) If $b_{i+1} < \min(B_i)$ then by setting $a_{i+1} = \frac{1}{2}\min(A_i)$ we guarantee that b_{i+1}, a_{i+1} become the new minimal elements in both orders, preserving relations. By inductive hypothesis, we know that $|0 - \min(A_i)| \geq 2^{k-i}$, hence $|0 - a_{i+1}| \geq 2^{k-(i+1)}$, while we also get that $|a_j - a_{i+1}| \geq 2^{k-i} \geq 2^{k-(i+1)}$ for all $a_j \in A_i$ since $a_{i+1} < \min(A_i)$.

(b) If $b_{i+1} > \min(B_i)$ then the elements b_j, b_t are guaranteed to exist since $0 > b_{i+1} > \min(B_i)$. The minimality of the distance between a_j and a_t allows us to restrict our interest to proving that the distance claim holds on the latter elements. By inductive hypothesis, we know that $|a_j - a_t| \geq 2^{k-i}$. By assigning a_{i+1} to the element lying exactly in the middle between these two elements, we get that $|a_j - a_{i+1}| \geq 2^{k-(i+1)}$ and $|a_{i+1} - a_t| \geq 2^{k-(i+1)}$. This choice also preserves the order relation.

$\square$

The distance property between each element chosen in $\mathbb{N}$ guarantees that the elements selected by the Duplicator are all distinct for at least k rounds (to be precise,they can survive for $k+1$ rounds), even if the choice of the Spoiler always falls in the second case.

$\square$