"Sapienza" University of Rome
Faculty of Information Engineering, Informatics and Statistics
Department of Computer Science

Graph Theory

Lecture notes integrated with the book "Graph Theory", R. Diestel

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Contents

Information and Contacts

Personal notes and summaries collected as part of the *Graph Theory* course offered by the degree in Computer Science of the University of Rome "La Sapienza".

Further information and notes can be found at the following link:
https://github.com/Exyss/university-notes. Anyone can feel free to report inaccuracies, improvements or requests through the Issue system provided by GitHub itself or by contacting the author privately:

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The notes are constantly being updated, so please check if the changes have already been made in the most recent version.

Suggested prerequisites:

Sufficient knowledge of algorithm design, probability and combinatorics

1

Introduction to Graph Theory

1.1 Graphs, subgraphs and neighbors

The city of Königsberg (in Prussia), situated along the Pregel River, is divided into four regions: two parts of the mainland and two islands, Kneiphof and Lomse. In the 18th century, these areas were connected by *seven bridges* that spanned the river, crossing it in various directions. As time passed, a fascinating question emerged among residents: could one walk through the city, crossing each of the seven bridges exactly once, and return to the starting point? This became known as the *Seven Bridges of Königsberg problem.*

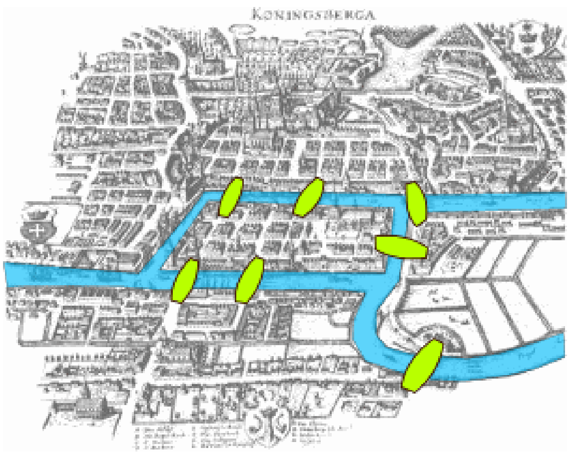


Figure 1.1: The map of Königsberg in the 18th century, showing the layout of the seven bridges.

Though it appeared to be a simple challenge, no one had succeeded in solving it. The riddle became a popular topic of conversation, discussed in markets and taverns alike. Some believed the walk was possible with the right path, while others doubted it could be done. Word of this puzzling problem eventually reached the brilliant Swiss mathematician, Leonhard Euler. Fascinated by the challenge, Euler sought a solution – not by walking the streets himself, but by abstracting the problem into a more general form. Euler realized that the precise layout of the city itself wasn't essential. What really mattered was how the landmasses were **connected** by the bridges. He began by representing each landmass as a dot and each bridge as a line between them. By doing this, he removed unnecessary details and created a simple yet powerful combinatorial structure, which we now call a **graph**.

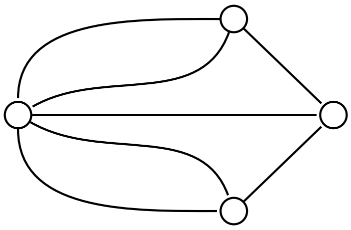


Figure 1.2: The graph drawn by Euler for the Seven Bridges of Königsberg problem.

Through his analysis, Euler discovered a key insight: for a walk to cross each bridge exactly once and return to the starting point, each landmass must be connected to an even number of bridges. In the case of Königsberg, however, every landmass was connected to an odd number of bridges, making such a walk impossible. Euler's solution, published in 1736, was groundbreaking. It not only answered the Königsberg puzzle but also laid the groundwork for an entirely new field of mathematics: *graph theory*. This area of study has since become fundamental to understanding networks, from transportation systems to social media, and even the internet itself. Thus, from a simple riddle about bridges in a small Prussian city, a new mathematical discipline was born—one that continues to influence the world to this day.

Definition 1.1: Graph

A **graph** is a mathematical structure $G = (V, E)$, where V is the set of vertices (or nodes) of G and E is the set of edges that link the vertices of G.

A graph can be **directed** or **undirected**. In a directed graph the edges are *oriented*, meaning that there is difference between the edge (u, v) – going from u to v – and the edge (v, u) – going from v to u. Formally, we have that:

$$E(G) \subseteq V \times V\{(u, v) \mid u, v \in V(G)\}$$

In an undirected graph, instead, the edges are *not oriented*, meaning that there not is difference between the edges (u, v) and (v, u). Formally, we have that:

$$E(G) \subseteq \binom{V(G)}{2} = \{\{u, v\} \mid u, v \in V(G)\}$$

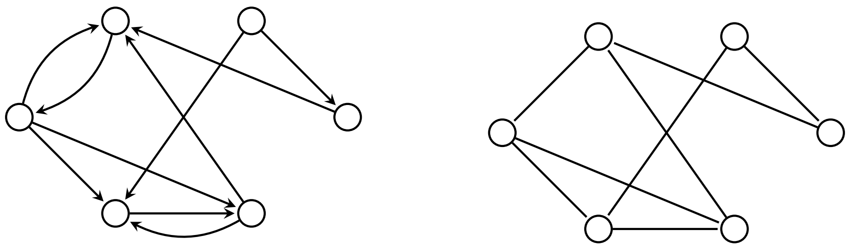


Figure 1.3: A directed graph (left) and an undirected graph (right)

We observe that the definition of graph that we just gave doesn't allow *multiple edges* between two vertices and *loops*, i.e. edges out-going frm and in-going to the same vertex. When this is the case, we say that the graph is **simple**. Generally, simple graphs are enough for any model. Sometimes, however, multiple edges and loops are needed – such as in the Seven Bridges of Königsberg problem. Graphs that allow such edges are called **multigraphs**.

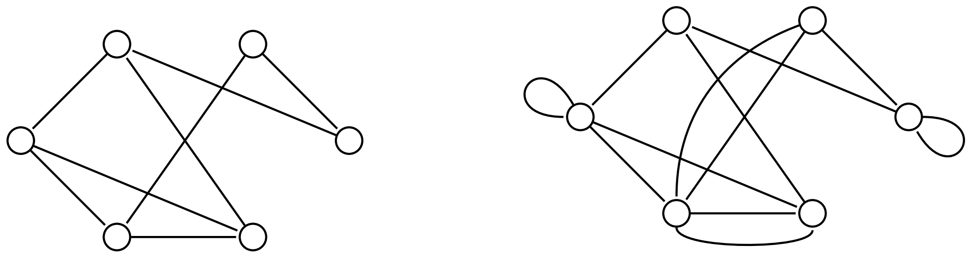


Figure 1.4: A simple graph (left) and a multigraph (right)

From now on, unless stated differently, we'll assume that each graph is simple and undirected. Moreover, to make notation lighter, we will always assume that $|V(G)| = n, |E(G)| = m$ and that $xy = \{x, y\}$ (or $xy = (x, y)$ for directed graphs).

Definition 1.2: Subgraph

Let G be a graph. If H is a graph such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ then H is a **subgraph** of G, written as $H \subseteq G$. A subgraph $H \subseteq G$ is said to be **induced** when for all edges $xy \in E(G)$ such that $u, v \in V(H)$ it holds that $xy \in E(H)$.

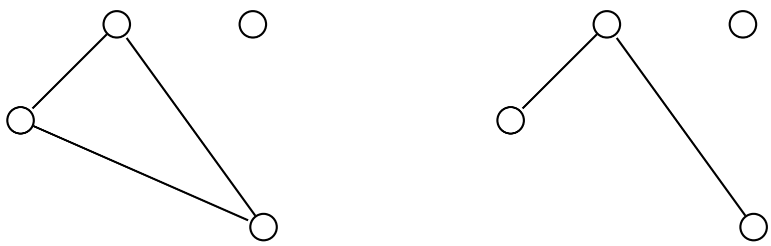


Figure 1.5: Both graphs are a subgraph of the simple graph shown on the left in Figure 1.4. The left subgraph is induced, while the right one is not.

We observe that, by definition, if a subgraph $H \subseteq G$ is induced then H is the unique induced subgraph for the vertices $V(H)$. Hence, when talking about an induced graphs we can consider his set of vertices. Given a subset of vertices $X \subseteq V(G)$, we denote with $G[X]$ the unique induced subgraph of G such that $V(G[X]) = X$, where $E(G[X]) = \{xy \in E(G) \mid x, y \in X\}$.

Definition 1.3: Adjacency, neighborhood and independence

Given a graph G and two nodes $x, y \in V(G)$, we say that x, y are **adjacent** to each other, written as $x \sim y$, if $xy \in E(G)$. For any edge $xy \in E(G)$, we say that xy is incident to x and y. The set of all vertices adjacent to x in G is called **neighborhood**, written as $N_G(x)$, is defined as:

$$N_G(x) = \{y \in V(G) \mid x \sim y\}$$

A subset of vertices X such that for each $x, y \in X$ it holds that $x \not\sim y$ is called **independent set**.

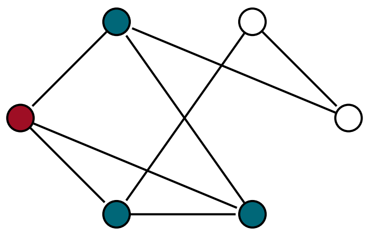


Figure 1.6: The blue nodes form the neighborhood of the red node. The red node and the white nodes form an independent set.

Definition 1.4: Degree

Let G be a graph. Given a vertex $x \in V(G)$, the **degree** of x over G is defined as $\deg_G(x) = |N_G(x)|$. The minimum and maximum degree of G are respectively denoted as $\delta(G)$ and $\Delta(G)$.

$$\delta(G) = \min_{x \in V(G)} \deg_G(x)$$

$$\Delta(G) = \max_{x \in V(G)} \deg_G(x)$$

We say that a graph is **k-regular** if $\delta = \Delta = k$, i.e. every node has degree k

When the context makes it clear, we'll simply write $\deg(x)$ instead of $\deg_G(x)$. The minimum and maximum degree of a graph are two very powerful theorem-proving tools: large portion of results are proven by reasoning on the degree of each node, often proving that some condition does or does not hold. The most basic result involving the degree of a graph is known as the **Handshaking lemma**, which can be stated in two equivalent forms.

Lemma 1.1: Handshaking lemma

For every graph G it holds that:

$$\sum_{x \in V(G)} \deg(x) = 2m$$

Equivalently, for every graph G the number of odd-degree vertices is even.

Proof. It's easy to see that every edge $xy \in E(G)$ is counted exactly two times through $\deg(x)$ and $\deg(y)$, implying that:

$$\sum_{x \in V(G)} \deg(x) = 2m$$

Consider now the subset $X \subseteq V(G)$ containing all the vertices of even degree. We observe that:

$$2m = \sum_{x \in V(G)} \deg(x) = \sum_{x \in X} \deg(x) + \sum_{x' \in V(G)-X} \deg(x')$$

Let $a = \sum_{x \in X} \deg(x)$ and $b = \sum_{x' \in X} \deg(x')$. Since the degrees of the vertices in X are even, we know that a is even. Hence, in order for $2m = a + b$ to hold, b must also be even. However, since each degree in $V(G) - X$ is odd, in order for b to be even it must hold that $|V(G) - X|$ is even. $\square$

1.2 Paths, walks, cycles and trees

After discussing the more general concept of subgraph, we can now focus on particular types of structures that can be usually found in graphs. These sub-structures are the real fundamental tool of reasoning for graph properties.

Definition 1.5: Path

A **path** is a graph P such that $V(P) = \{x_1, \ldots, x_n\}$ and $E(G) = \{x_0x_1, x_1x_2, \ldots, x_{n-1}x_n\}$. The length of a path is defined as the number of edges that form it. A path with n vertices is denoted as P_n.

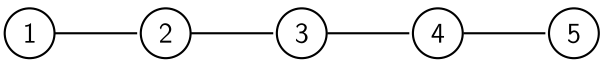


Figure 1.7: The path P_5.

Definition 1.6: Walk

Let G be a graph. A **walk** on G is defined as a sequence $x_0e_1x_1e_2\dots e_{k-1}e_kx_k$ where $x_0,\dots,x_k \in V(G)$ and $e_1,\dots,e_k \in E(G)$. The length of a walk is defined as the number of edges that form it.

When the first and last vertices are equal, i.e. $x_0 = x_k$, we say that the walk is **closed**. We observe that, by definition, a walk allows edges and vertices to be repeated in the sequence. When no vertices in a walk are repeated, the walk corresponds to a path.

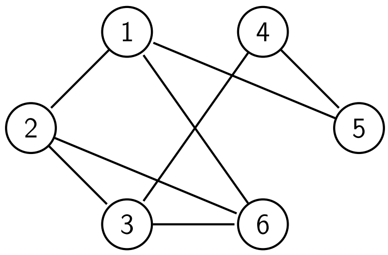


Figure 1.8: The sequence $1\{1,2\}2\{2,6\}6\{6,1\}1\{1,5\}5$ forms a walk on G, but not a path. The sequence $1\{1,2\}2\{2,6\}6\{6,3\}3\{3,4\}4$, instead, forms a path on G.

Proposition 1.1: Paths and walks

Let G be a graph. Given the vertices $x,y \in V(G)$, there is a path from x to y if and only if there is a walk from x to y.

Proof. Since every path is also a walk, the first direction is trivial. Let $W = x_0e_1x_1e_2\dots e_{k-1}e_kx_k$ be the shortest walk. i.e the one with minimum length, such that $x = x_0$ and $y = x_k$. By way of contradiction, suppose that W is not a path. Hence, at least one edge in W must repeat at least once. Let $i,j \in [k]$ be two indices such that $e_i = e_j$. Then, the following sequence W' is a walk from x to y with fewer edges than W, raising a contradiction. Hence, W must be a path.

$$W' = x_0e_1x_1e_2\dots e_ix_ie_{j+1}x_{j+1}e_{j+2}\dots e_{k-1}e_kx_k$$

□

Proposition 1.2

The longest path in any graph has length at least δ.

Proof. If $\delta = 1$ then the longest path is trivially made by one single edge. Suppose now that $\delta \geq 2$, implying that there are at least two vertices in G. Let P be the longest path in G and let $x_1,\dots,x_k$ be its vertices.

Claim: $N(x_k) \subseteq \{x_1,\dots,x_{k-1}\}$.

Proof. By way of contradiction, suppose that there is a vertex $x' \in N(x_k)$ such that $x' \notin \{x_1, \dots, x_{k-1}\}$. Then, since $x_k \sim x'$, there must be an edge $x_k x'$, implying that the path $P \cup x_k x'$ is longer than P, raising a contradiction. □

Through the claim we easily conclude that $\delta \leq |N(x_k)| \leq k$, meaning that P has length at least δ. □

Definition 1.7: Cycle

A cycle is a graph C such that $V(C) = \{x_1, \dots, x_n\}$ and $E(G) = \{x_0x_1, x_1x_2, \dots, x_{n-1}x_n, x_nx_1\}$. The length of a cycle is defined as the number of edges that form it. A cycle with n vertices is denoted as C_n.

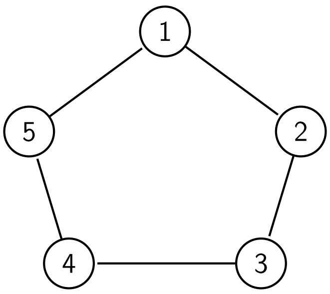


Figure 1.9: The cycle C_5.

Proposition 1.3

In any graph such that $\delta \geq 2$ there is a cycle of length at least $\delta + 1$.

Proof. Let P be the longest path in G and let $x_1, \dots, x_k$ be its vertices. Through an argument equal to the claim of Proposition 1.2, we know that $N(x_k) \subseteq \{x_1, \dots, x_{k-1}\}$. Let $i \in [k-1]$ be the minimal index such that $x_i \in N(x_k)$. Since every neighbor of x_k must be inside P and the graph is simple, it must hold that $i \geq \delta$, meaning that the vertices $x_i, x_{i+1}, \dots, x_{k-1}, x_k, x_i$ form the cycle C_{i+1}. □

Definition 1.8: Connectivity, components and distance

Let G be a graph. Two nodes $x, y \in V(G)$ are said to be linked if there is a path from x to y. If every pair of nodes in G is linked, we say that G is **connected**. If a connected subgraph H of G is maximal – meaning that no other edges can be added to it while preserving connectivity – H is called **component** of G. The **distance** $\mathrm{dist}_G(x, y)$ between two nodes x, y is the length of the shortest path connecting them.

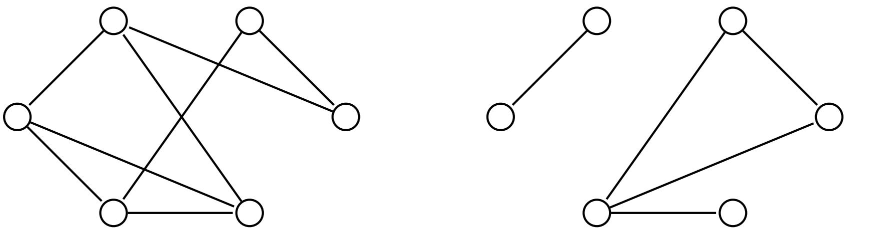


Figure 1.10: A connected graph (left) and a disconnected graph (right). The connected graph has an unique component, while the disconnected graph has two components.

Lemma 1.2

Let G be a graph. If G is connected and C is a cycle in G then for all $e \in E(C)$ it holds that $G - \{e\}$ is still connected.

Proof. Fix an edge $e \in E(C)$. Let $P = x_0 e_1 x_1 \dots e_k x_k$ be a path in G from x to y. If $e \notin E(P)$ then P is also a path in $G - \{e\}$, preserving the connectivity of x and y. Suppose now that $e \in E(P)$. Given $C = z_0 f_1 z_1 \dots f_\ell z_\ell$, without loss of generality assume that $e = e_i = f_1$. Then, the following sequence W is a walk from x to y in $G - \{e\}$ – we cannot be sure that W is a path since P may intersect C on multiple edges.

$$W = x_0 e_1 x_1 \dots x_i f_\ell z_{\ell-1} \dots f_2 z_2 e_{i+1} x_{i+1} \dots e_k x_k$$

By Proposition 1.1, we know that since W is a walk from x to y in $G - \{e\}$ there must also be a path from x to y in $G - \{e\}$, preserving connectivity. □

Definition 1.9: Tree

A **tree** is an connected acyclic subgraph. Any vertex in a tree with degree 1 is called leaf. A rooted tree is a tree with a chosen node called **root**. If every component of a graph is a tree, the graph is referred to as a **forest**.

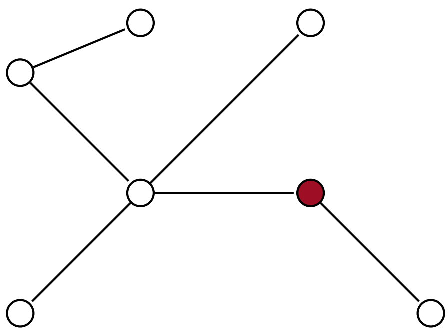


Figure 1.11: A rooted tree. The red node has been chosen as the root.

When the tree is rooted in $r \in V(T)$, we often use the concept of **ancestor** and **parent**. Given two nodes $a, x \in V(T)$, we say that a is an ancestor of x if x lies on a path from r

to x. If p is an ancestor of x and $px \in E(T)$, we say that p is the parent of x. The **least common ancestor (LCA)** between two vertices $x, y \in V(T)$ is the ancestor $z \in V(T)$ shared by x and y that minimizes the value of $\text{dist}(r, z)$. We observe that every pair of vertices of a tree must have a LCA since the root is an ancestor of every node.

Theorem 1.1: Equivalent definitions of tree

Given a graph T, the following statements are equivalent:

1. T is a tree
2. Every vertex pair of T is connected by an unique path
3. T is minimally connected
4. T is maximally acyclic

Proof. We'll proceed by proving a chain of implications.

$1 \implies 2$ Without loss of generality, assume that every pair of nodes has at least one path, since otherwise T is not connected, hence not a tree. Suppose now that there are two vertices $x, y \in V(T)$ that have at least two different paths P, Q from x to y. While traversing P from x to y, let z be the first node such that $z \in V(P) \cap V(Q)$. Similarly, let $w \in V(P) \cap V(Q)$ be the first node encountered while traversing Q from y to x. We observe that since $x, y \in V(P) \cap V(Q)$, the vertices z, w always exist. Given the subpath $P' \subseteq P$ from w to z and the subpath $Q' \subseteq Q$ from z to w, the graph $Q' \cup P'$ is a cycle in T, meaning that T cannot be a tree. By contrapositive, we get that if T is a tree then every pair of vertices is connected by an unique path.

$2 \implies 3$ Suppose that every vertex pair of T is connected by an unique path. Then, T is clearly connected. By way of contradiction, suppose that there is an edge $e \in E(G)$ such that $T - \{e\}$ is still connected. Then, the edge e must be part of at least one of the unique paths connecting two nodes, meaning that such path cannot exist inside $T - \{e\}$, implying that it is not connected. Hence, T must be minimally connected.

$3 \implies 4$ Suppose that T is minimally connected but not maximally acyclic. By way of contradiction, suppose that T has a cycle. Then, by Lemma 1.2 we know that we can remove an edge of the cycle from T and keep it connected, contradicting the fact that T is minimally connected. Hence, T must be acyclic. Pick two vertices $x, y \in V(T)$. Since T is connected, we know that there is a path P connecting x to y. Then, if we were to add the edge xy, the subgraph $P \cup \{xy\}$ would be a cycle in $T \cup \{xy\}$. Thus, T is maximally acyclic.

$4 \implies 1$ Suppose that T is maximally acyclic. Fix a pair of vertices $x, y \in V(T)$. Since T is maximally acyclic, we know that adding the edge xy makes $T \cup \{xy\}$ cyclic. Let C be the cycle in $T \cup \{xy\}$ containing xy. Then, $C - \{xy\}$ must be a path in T from x to y.

□

Lemma 1.3

Let T be a tree. Then:

1. If T has at least two nodes then T has at least a leaf.
2. If $x \in V(T)$ is a leaf then $T - \{x\}$ is still a tree

Proof.

1. By way of contradiction, suppose that T is a tree without leaves. Then, we have that $\delta \geq 2$. However, by Proposition 1.3, in T there must be a cycle with length at least $\delta + 1$, contradicting the very definition of tree. Hence, T must have at least a leaf.
2. Let x be a leaf of T. Since T is acyclic, $T - \{x\}$ is also clearly acyclic. By way of contradiction, suppose that $T - \{x\}$ is not connected. Then, there are at least two vertices $u, v \in V(T - \{x\})$ for which there is a path P between them in T but not in $T - \{x\}$. Since by removing x the vertices u, v became disconnected, the vertex x must lie on P. Moreover, since $u, v \neq x$, the vertex x must be an internal node of the path, meaning that it must have degree 2, contradicting the very definition of leaf.

$\square$

Definition 1.10: Spanning tree

Given a graph G, we say that $T \subseteq G$ is a **spanning tree** of G if T is a tree and $V(T) = V(G)$.

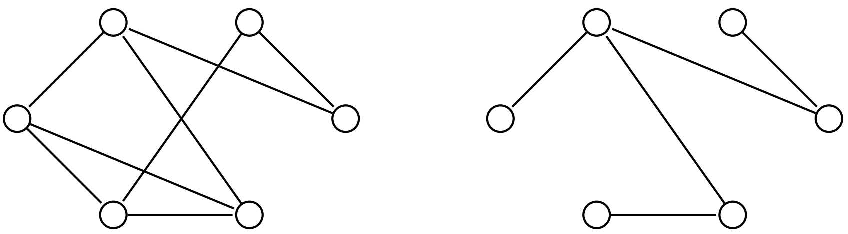


Figure 1.12: A graph (left) and one of its spanning trees (right)

Lemma 1.4

Every connected graph has a spanning tree.

Proof. If G is a tree then clearly it is its own spanning tree. Suppose now that G is a connected graph that is not a tree, meaning that it is acyclic. Then, through Proposition 1.3, we can keep removing the edges $e_1, \dots, e_k$ from every cycle of G until we reach a minimally connected subgraph T such that $V(T) = V(G)$. By Theorem 1.1, we know that T must be a tree. $\square$

Theorem 1.2

Let T be a connected graph. Then, T is a tree if and only if it has $n-1$ edges.

Proof. We proceed by induction on n. If $n = 1$ then T is trivially the tree with 0 edges. If $n > 1$, instead, through the previous lemma we know that T must have at least a leaf $x \in V(T)$ for which $T - \{x\}$ is still a tree. By inductive hypothesis, we know that $T - \{x\}$ has $n-2$ edges. Hence, by adding the unique edge incident to x, we get that T has $n-1$ edges.

Vice versa, by way of contradiction, suppose that T is connected, has $n-1$ edges but that it is not acyclic. Then, through the previous lemma we know that T must have a spanning tree T'. Moreover, since T has a cycle and T' does not, we know that T' must have fewer edges than T. However, we have just proven that every tree must have $n-1$ edges. Hence, we get that $n-1 = |E(T')| < |E(T)| = n-1$, which is a contradiction. Hence, T must also be acyclic. $\square$

1.3 Complete graphs and bipartite graphs

Definition 1.11: Complete graph

A **complete graph** is a graph where every pair of vertices is adjacent to each other. A complete graph with n vertices in denoted as K_n. An induced subgraph that is complete is called **clique**.

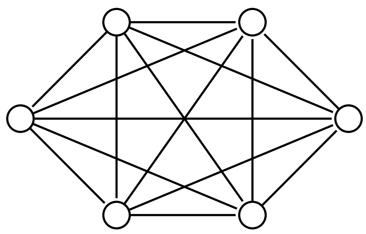


Figure 1.13: The complete graph K_6

Definition 1.12: Bipartite graph

A **bipartite graph** is a graph with a subset of vertices $X \subseteq V(G)$ such that for every edge $\{u, v\} \in E(G)$ it holds that $u \in X$ and $v \in V(G) - X$. Equivalently, we have that both X and $V(G) - X$ are independent sets.

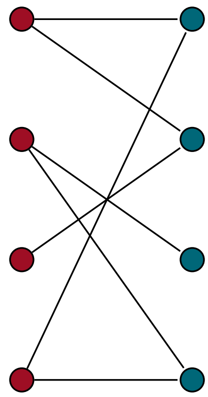


Figure 1.14: The red and blue nodes form a bipartition of the graph.

It's easy to see that no complete graph can be bipartitioned since it's impossible to separates the nodes into two independent sets. In fact, by definition, cliques are the very opposite of the concept of independent set. Through the following lemma we can extend the claim on complete graphs to cliques: if G contains a clique of any cardinality then G cannot be bipartitioned.

Proposition 1.4

G is bipartite if and only if every subgraph of G is bipartite

Proof. The converse implication trivially holds. Let H be a subgraph of G. Then, any bipartition $(X, V(G) - X)$ of G induces a bipartition $(X \cap V(H), V(H) - X)$ over H. □

Lemma 1.5

G is bipartite if and only if every component of G is bipartite

Proof. The direct implication trivially follows from the previous proposition. Given the components $H_1, \ldots, H_k$ of G, let $(X_1, V(H_1) - X_1), \ldots, (X_k, V(H_k) - X_k)$ be the bipartitions of the components. Then, since each component is by definition disjoint from the others, the pair $(X, V(G) - X)$ where $X = \bigcup_{i \in [k]} X_i$ is a bipartition of G. □

We observe that the above characterizations of bipartite graphs are useful but still "weak". A stronger characterization can be achieved through the following theorem. Given a path P and two vertices $x, y \in V(P)$, we denote the subpath of P from x to y with xPy.

Theorem 1.3

G is bipartite if and only if it doesn't contain odd cycles.

Proof. Through the above proposition, it's sufficient to prove that for all $k \in \mathbb{N}$, C_{2k+1} is not bipartite. Fix $k \in \mathbb{N}$. By way of contradiction, suppose that there is bipartition

$(X, V(G) - X)$ of C_{2k+1}. Let $x_1, \dots, x_{2k+1}x_1$ be the edges of C_{2k+1}. Without loss of generality, assume that $x_1 \in X, x_2 \notin X, x_3 \in X, \dots, x_{2k} \notin X$. Then, if $x_{2k+1} \in X$ then $x_{2k+1}x_1 \subseteq X$, while if $x_{2k+1} \notin X$ then $x_{2k}x_{2k+1} \subseteq V(G) - X$. In both cases, we get that the pair cannot be a bipartition.

Vice versa, suppose that G is not bipartite. Then, through the previous lemma at least one component H of G is not bipartite. Let T be a spanning tree of H and fix $r \in V(T)$ as its root. Let $X = \{x \in V(T) \mid \mathrm{dist}_T(r, x) \text{ is even}\}$. By definition, $(X, V(T) - X)$ for a bipartition of T. Hence, since T is bipartite but H isn't, there must be an edge $xy \in E(H)$ such that either $x, y \in X$ or $x, y \notin X$. Let P_x and P_y respectively be the paths in T from x to r and from y to r. Let z be the LCA of x and y in T.

Claim: $C = zP_xx \cup zP_yy \cup xy$ is an odd cycle

Proof of the claim. Since either $x, y \in X$ or $x, y \notin X$ holds, we know that $\mathrm{dist}(r, x)$ and $\mathrm{dist}(r, y)$ must be either both even or both odd. Hence, the lengths of P_x and P_y must share the same parity. Moreover, since z is the LCA of x and y w have that $rP_xz = rP_yz$. Thus, the lengths of zP_xx and zP_yy must also share the same parity. This concludes that $zP_xx \cup zP_yy \cup xy$ must be an odd cycle. □

Since C is an odd cycle and it is a subgraph of T, and thus of G, we conclude that G contains an odd cycle. □

1.4 Eulerian tours and Hamiltonian paths

At the start of this chapter, we introduced the Seven Bridges of Königsberg problem, which led to the emergence of graph theory as a branch of combinatorics. In modern graph theory, the problem is formalized through the concept of *Eulerian tour.*

Definition 1.13: Eulerian tour

An Eulerian tour over a graph G is closed walk that traverses every edge of G exactly once.

To solve the problem, Euler [Eul41] proved the following theorem, which implies that the answer is "no" since the multigraph that models the problem contains some odd-degree vertices.

Theorem 1.4: Euler's theorem

A graph (or multigraph) has an Eulerian tour if and only if G is connected and every vertex has even degree

Proof. By way of contradiction, suppose that G has an Eulerian tour W. By way of contradiction, suppose that G is not connected. Then, we get an easy contradiction: if G is disconnected then W cannot traverse every edge of the graph. Hence, G must be connected. Again, by way of contradiction suppose that G has at least one odd-degree vertex $x \in V(G)$. Let $\deg(x) = 2k+1$. Since W is a closed walk, we can assume without loss of generality that x is the first vertex of the walk. When traversing W starting from x, one of the edges incident to x is crossed, hence we have $2k$ incident edges left. Every time the tour returns to x, two edges are crossed – one in-going and one out-going. Hence, in order to cover all the edges, the tour has to return to x for k times. However, this implies that there are no more edges left to close the tour on x, raising a contradiction. Hence, the vertex x cannot exist.

Vice versa, assume that G is connected and every vertex has even degree. Let $W = x_0 e_1 x_1 \dots e_k x_k$ be the longest walk over G with no repeating edges.

Claim 1: $x_k = x_0$, i.e. W is a closed walk

Proof of Claim 1. By way of contradiction, suppose that $x_k \neq x_0$. Let $2\ell + 1$ be the number of edges incident to x_k in W – one edge is given by e_k while 2ℓ edges are given by edges needed to cross the other ℓ the vertices $x_{i_1}, \dots, x_{1_\ell}$ such that $x_i = x_k$. Since x_k has even degree, we know that there must be another edge $\{x_k, y\} \in E(G - W)$, implying that $W \cup \{x_k, y\}$ is walk longer than W, which is absurd. □

Claim 2: W contains every edge of G

Proof of Claim 2. By way of contradiction, suppose that that there is an edge $uv \in E(G - W)$. Fix $i \in [k]$. By connectivity of G, we know that there must be a path P disjoint from x_i to u or from x_i to v. Without loss of generality, assume that P is a path from x_i to u. Since $uv \notin E(W)$, there must be an edge $x_j y \in E(P - W)$ outgoing from W. Since W is a closed walk, we can assume without loss of generality that x_j is the first (and last) vertex of the walk. Then, the walk $W \cup x_j y$ is walk that doesn't repeat any vertices longer than W, raising a contradiction. □

Through the two claims, we conclude that W is an Eulerian tour. □

As a corollary to Euler's theorem, we get that the problem that asks to determine the existence of an Eulerian tour inside the graph lies in the intersection $\mathsf{NP} \cap \mathsf{coNP}$ since the theorem gives us a polynomially verifiable condition both for existence and non-existence of the tour. In fact, as many problems that lie in this intersection, the problem actually lies in the class P.

Another very interesting concept similar to Eulerian tours was formulated by Hamilton: closed walks that touch every node exactly once instead of every edge exactly once. We observe that this condition implies that the closed walk is nothing more than the subgraph P_n. Even though they are similar in concept, finding an **Hamiltonian path** inside a graph is an NP-hard problem, making it intractable. A variant of this concept is the **Hamiltonian cycle**, i.e. a cycle that goes through all the nodes of a graph. The decision of existence of Hamiltonian cycles is also NP-hard.

Definition 1.14: Hamiltonian paths and cycles

An Hamiltonian path over a graph G is a subgraph $P_n \subseteq G$. An Hamiltonian cycle over a graph G is a subgraph $C_n \subseteq G$.

When some conditions are met, an Hamiltonian cycle (hence also an Hamiltonian path) is guaranteed to exist. For instance, Dirac [Dir52] formulated the following theorem.

Theorem 1.5: Dirac's theorem

Let G be a graph. If $\delta \geq \frac{n}{2}$ then there is an Hamiltonian cycle in G.

Proof. First of all, we show that Dirac's condition forces the graph to be connected.

Claim 1: G is connected

Proof of Claim 1. By way of contradiction, suppose that G is not connected. Then, G has at least two components. Hence, the smallest component H of G can have at most $\frac{n}{2}$ edge. However, since $\delta \geq \frac{n}{2}$, each node $x \in H$ must have at least $\frac{n}{2}$ neighbors. Thus, since $\{x\} \cup N(x) \subseteq H$, we get that H must have at least $\frac{n}{2}+1$ nodes, raising a contradiction. □

Let P be the longest path on G an let $x_0, \dots, x_k$ be its vertices.

Claim 2: There is an index ℓ such that $x_0, x_\ell, x_{\ell+1}, \dots, x_{k-1}, x_k, x_{\ell-1}, x_{\ell-2}, \dots, x_1, x_0$ is a cycle

Proof of Claim 2. Using the same argument as Proposition 1.2, we know that $N(x_0), N(x_k) \subseteq \{x_0, \dots, x_k\}$. Let I_0 and I_k be defined as:

$$I_0 = \{i \mid 1 \leq i \leq k, x_i \in N(x_0)\} \qquad I_k = \{i \mid 1 \leq i \leq k, x_{i-1} \in N(k)\}$$

Since $\delta \geq \frac{n}{2}$, we know that $\frac{n}{2} \leq |I_0|, |I_k| \leq n-1$, by the pigeonhole principle there must be at least one index $\ell \in I_0 \cap I_k$, meaning that $x_0 \sim x_\ell$ and $x_k \sim x_{\ell-1}$. Thus, the sequence of nodes $x_0, x_\ell, x_{\ell+1}, \dots, x_{k-1}, x_k, x_{\ell-1}, x_{\ell-2}, \dots, x_1, x_0$ is a cycle. □

Let C be the cycle given by $x_0, x_\ell, x_{\ell+1}, \dots, x_{k-1}, x_k, x_{\ell-1}, x_{\ell-2}, \dots, x_1, x_0$. By way of contradiction, suppose that C has less than n nodes. Given $y \in V(G-C)$, by connectivity of G there must be an index i for which there is a path P connecting x_i and y. However, this implies that the following path

$$P \cup x_i x_{i+1} \cup \ldots \cup x_k x_0 \cup \ldots \cup x_{i-2} x_{i-1}$$

is longer than P, raising a contradiction. Hence, C must be an Hamiltonian cycle. □

We observe that Dirac's condition, i.e. $\delta \geq \frac{n}{2}$ is actually optimal, meaning that we cannot find a better lower bound that guarantees the existence of an Hamiltonian cycle: there are many graphs with $\delta = \frac{n}{2} - 1$ for which there is no Hamiltonian cycle.

1.5 Solved exercises

Problem 1.1

For any graph G, prove that at least two vertices have the same degree.

Solution. By way of contradiction, suppose that no vertices share the same degree. Then, the function $\deg : V \to \{0, \dots, n-1\}$ must be bijective, implying that there are two vertices $u, v \in V$ such that $\deg(u) = 0$ and $\deg(v) = n - 1$, which is absurd since there should both be and not be an edge between u and v. Hence, at least two vertices must share have the same degree. $\square$

Problem 1.2

Show that the components of a graph parition its vertex set.

Solution. Let G be a graph. For each vertex v of G, we denote its compinent with $\text{comp}(v)$. Since each node can reach itself, we get that $v \in \text{comp}(v)$, implying that:

$$V(G) = \bigcup_{v \in V(G)} \text{comp}(v)$$

Consider now any pair of vertices $x, y \in V(G)$. We observe that if $\text{comp}(v) = \text{comp}(y)$ then $\text{comp}(v) \cap \text{comp}(y) \neq \varnothing$ trivially holds since at least x and y lie in the intersection. Vice versa, suppose that $\text{comp}(v) \cap \text{comp}(y) \neq \varnothing$. Then, there is a vertex z that lies in the intersection, implying that there is a path P from x to z and a path Q from z to y. Then, the path $P \cup Q$ goes from x to y, meaning that $\text{comp}(v) = \text{comp}(y)$. This concludes that $\text{comp}(v) \neq \text{comp}(y)$ if and only if $\text{comp}(v) \cap \text{comp}(y) = \varnothing$, proving that the components are pairwise disjoint. $\square$

Problem 1.3

Given a graph G and a cycle C, we say that the edge xy is a chord of C if $x, y \in V(C)$ and $xy \in E(G - C)$. Prove that if $\delta(G) \geq 3$ then there is a cycle in G with a chord.

Proof. Let $P = x_0 x_1 \dots x_k$ be the longest path in G. By choice of P, it must hold that $N(x_k) \subseteq V(P)$, otherwise we could extend P. Since $\delta \geq 3$, we know that x_k has at least three neighbors. Hence, there are at least three vertices $x_i, x_j, x_t \in V(P) \cap N(x_k)$ and we know that one of them must be x_{k-1}. Without loss of generality, let $x_t = x_{k-1}$ and let $i < j$. Then, $C = x_i x_{i+1} \dots x_{j-1} x_j x_{j+1} \dots x_k x_i$ is a cycle with a chord $x_k x_j$. $\square$

Problem 1.4

Let G be a graph with a cycle C and a path P of length at least k between two vertices in C. Prove that G contains a cycle of length at least $\sqrt{k}$.

Solution. We may assume that C has length less than k since otherwise the statement is trivially true. Let $x, y \in V(C)$ be the two endpoints of P. Consider the vertices $V(C) \cap V(P) = \{z_1, \ldots, z_\ell\}$ in which P intersects C while traversing P from x to y, meaning that $z_1 = x$ and $z_\ell = y$. These vertices partition the path P into $\ell - 1$ sub-paths $P_1, \ldots, P_{\ell-1}$, where z_i and z_{i+1} are the endpoints of each P_i. Since $\ell < \sqrt{k}$ due to C having length less than $\sqrt{k}$, we observe that:

$$\max_{i\in[\ell-1]} |E(P_i)| \geq \operatorname*{avg}_{i\in[\ell-1]} |E(P_i)| = \frac{k}{\ell - 1} > \frac{k}{\sqrt{k} - 1} > \sqrt{k}$$

Hence, the longest path P_j between $P_1, \ldots, P_{\ell_1}$ has length greater than $\sqrt{k}$. Moreover, we observe that z_k and z_{j+1} are still connected in $G - C$, implying that there is a path Q connecting them. This concludes that $P_j \cup Q$ is a cycle in G with length at least $\sqrt{k}$. □

Problem 1.5

Let G be a graph. We say that G is k-vertex-connected if $|V(G)| > k$ and for each $X \subseteq V(G)$ with $|X| < k$ the graph $G - X$ is connected. Similarly, we say that G is ℓ-edge-connected if $|V(G)| > 1$ and for each $F \subseteq E(G)$ with $|F| < k$ the graph $G - F$ is connected. We denote with $\kappa(G)$ the maximum value k such that G is k-vertex-connected, while $\lambda(G)$ is the maximum value ℓ such that G is ℓ-edge-connected. Prove that $\kappa(G) \leq \lambda(G)\lambda\delta(G)$ holds for every non-trivial graph.

Solution. Let $X \subseteq V(G)$ be a subset with $|X| = \kappa(G)$ and such that $G - X$ is disconnected. Let $F = \{e \in E(G) \mid |X \cap e| \geq 1\}$. Since removing X disconnects the graph, removing the edges in F also does. Moreover, since $\kappa(G) > 1$ due to G being non-trivial (hence non-disconnected in this case), each vertex in X must have at least one edge, concluding that:

$$\kappa(G) = |X| \leq |F| \leq \lambda(G)$$

By way of contradiction, suppose that $\lambda(G) > \delta(G)$. Then, for any vertex $v \in V(G)$, removing all of its incident edges wouldn't disconnect the graph, which is absurd since v would be an independent node. Hence, it must hold that $\lambda(G) \leq \delta(G)$ □

Problem 1.6

Let G be a connected graph. Prove that there is a path of length at least $\min(2\delta, n - 1)$

Solution. Let $P = x_1 \ldots x_k$ be the longest path in G. We may assume that $k < n$ since otherwise $|E(P)| = n - 1 \geq \min(2\delta, n - 1)$ trivially concludes the proof.

Claim: for all $i \in [k]$ if $x_i \in N(x_1)$ then $x_{i-1} \notin N(x_k)$

Proof of the claim. By way of contradiction, suppose that $x_i \in N(x_1)$ and $x_{i-1} \notin N(x_k)$. Then, the following is a cycle:

$$C = x_1 x_2 \ldots x_{i-2} x_{i-1} x_k x_{k-1} \ldots x_{i+1} x_i x_1$$

Since $k < n$, there is at least one vertex $z \in V(G-P)$. Moreover, by connectivity of G there must be a path Q from z to a vertex x_ℓ of C. Hence, the following path is longer than P, raising a contradiction:

$$Q \cup x_\ell x_{\ell+1} \dots x_{k-1} x_k x_0 x_1 \dots x_{i-1}$$

□

We observe that $N(x_0) \subseteq \{x_2, \dots, x_k\}$ and $N(x_k) \subseteq \{x_1, \dots, x_{k-1}\}$ since otherwise we could get a path longer than P. Therefore, let $N(x_0) = \{x_{i_1}, \dots, x_{i_t}\}$. Through the previous claim, we know that:

$$N(x_k) \subseteq \{x_1, \dots, x_{k-1}\} - \{x_{i_1-1}, \dots, x_{i_t-1}\}$$

implying that:

$$\delta \leq |N(x_k)| \leq k-1-|N(x_0)| \leq k-1-\delta$$

Hence, we have that $k \geq 2\delta + 1$, concluding that $|E(P)| = 2\delta \geq \min(2\delta, n-1)$. □

Problem 1.7

Prove that for every non-trivial tree without nodes of degree 2 the number of leaves in the tree is greater than the number of non-leaves.

Solution. Let T be a tree and let $L = \{\ell \in V(T) \mid \deg(\ell) = 1\}$ be the set of leaves. We observe that since every node with degree 1 is a leaf and there are no nodes of degree 2, every non-leaf node must have degree at least 3. Through the Handshaking lemma we have that:

$$2\,|E| = \sum_{v \in V(G)} = \sum_{\ell \in L} \deg(\ell) + \sum_{v \in V(G)-L} \leq |L| + 3\,|V(G) - L|$$

Moreover, since T is a tree we have that $|E| = n - 1 = |L| + |V(G) - L| - 1$. Thus, we conclude that:

$$2\,|L| + 2\,|V(G) - L| - 2 \leq |L| + 3\,|V(G) - L| \implies |L| \geq |V(G) - L| + 2$$

□

2

Graph matchings

2.1 Maximum matching

In graph theory, a **matching** in a graph is a set of edges that do not have a set of common vertices. In other words, a matching is a graph where each node has either zero or one edge incident to it. Graph matching has applications in flow networks, scheduling and planning, modeling bonds in chemistry, graph coloring, the stable marriage problem, neural networks in artificial intelligence and more.

Definition 2.1: Matching

Given a graph G, a matching over G is a subset $M \subseteq E(G)$ such that $\forall e, e' \in M$ it holds that $e \cap e' = \varnothing$

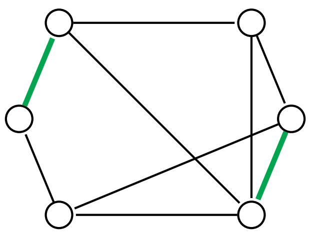


Figure 2.1: The green edges form a matching of the graph.

We're often interested in finding the matching with maximum cardinality. Before proceeding, it's important to distinguish between the concepts of **maximal** and **maximum**. In general, given a property P, a sub-structure X of a structure S is said to be maximal for P over S if $P(X)$ is true and there is no other sub-structure X' of S such that $P(X')$ is true and X is contained inside X'. Instead, X is said to be the maximum for P over S if $P(X)$ is true and there is no other sub-structure X' of S with a higher value for the property $P(X)$. For instance, the matching shown in the above figure is maximal because it cannot be extended with other edges without breaking the matching property, but it's not a maximum matching.

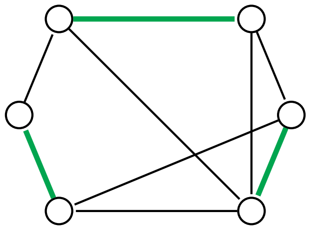


Figure 2.2: Maximum matching for the previous graph.

In particular, we'll focus on matching on **bipartite graphs**. These types of matchings are of particular interest due to how they describe many real-life situations. For instance, the problem of finding the optimal assignment of tasks to a group of employees can be solved by finding a maximum matching: we split the graph in two partitions, one with all the tasks and one with all the employees, connecting each employee to the tasks that he's capable of completing.

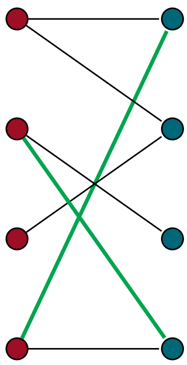


Figure 2.3: A matching on a bipartite graph.

Consider the matching on the bipartite graph shown above. We observe that this matching is neither maximal nor maximum. When the matching is not maximal, we can obtain a matching with greater cardinality simply by extending it.

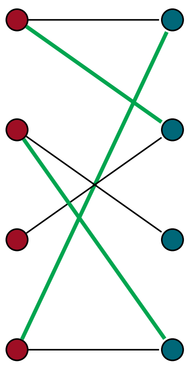


Figure 2.4: A maximal matching on a bipartite graph.

We notice that the above matching is maximal, but not maximum. For instance, we observe that the number of edges outside of the matching that we selected form new matching with more edges than the one that we have considered. Hence, by *swapping* all the edges, we get a matching with higher cardinality. Moreover, this new matching is clearly a maximum one since the number of nodes is equal to twice the number of selected edges.

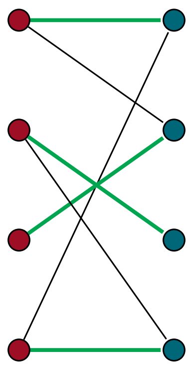


Figure 2.5: The maximum matching obtained by swapping all the edges.

Let's take a closer look to what we just did. We notice that all the edges of the last non-maximum matching actually form a path whose edges alternate between being outside of the matching and inside of the matching. Moreover, both the first and last edge of such path are outside of the matching, hence the number of outside edges is one more than the number of inside edges. We generalize such paths through the concept of *alternating path* and *augmenting path*.

Definition 2.2: Alternating and augmenting path

Let G be a bipartite graph and let $M \subseteq E(G)$ be a matching on G. An M**-alternating path** is a path starting from an unmatched node and whose edges alternate between M and $E(G) - M$. If the path also ends at an unmatched node, the path is said to be M**-augmenting**.

To clarify this definition, let's make things more formal. A path $P = x_0 e_1 x_1 e_2 \dots e_k x_k$ is M-alternating when x_0 is unmatched, meaning that there is no edge $e \in M$ in the matching such that $x_0 \in e$, and whose edges alternate between being outside of the matching and inside of the matching, meaning that $e_1 \notin M, e_2 \in M, e_3 \notin M$ and so on. When x_k is also unmatched, the path is said to be M-augmenting. This name comes from the fact that, in order for x_k to be unmatched, the last edge is not inside the matching, meaning that we have more edges outside of the matching then inside of it. Moreover, since x_0 and x_k are unmatched, by swapping the edges inside of the matching with the edges outside of it we're guaranteed to get a matching (with more edges than the previous one).

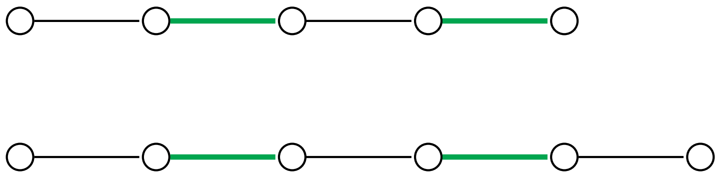


Figure 2.6: An M-alternating path (above) and an M-augmenting path (below).

Lemma 2.1

Let G be a bipartite graph and let $M \subseteq E(G)$ be a matching on G. If P is an M-augmenting path then $M \Delta E(P)$ is matching on G with more edges than M

Proof. First, we recall that the *symmetric difference* $M \Delta E(P)$ is defined as follows:

$$M \Delta E(P) = (M \cup E(P)) - (M \cap E(P))$$

Let $M' = M \Delta E(P)$. We observe that all the edges that aren't shared by M and P are also not in M', hence we can ignore them. Let $P = x_0 e_1 x_1 e_2 \dots e_k x_k$. Since P is augmenting, the number of edges in P that are also in M is less than the number of edges that aren't in M. Hence, we get that $|M'| > |M|$. □

The above lemme gives us an easy way to increase the cardinality of our matching by finding augmenting paths and swapping edges. But how can we be sure that we have reached the maximum matching? Berge [Ber57] proved that the above lemma can indeed be extended: if there are no augmenting paths then the matching is maximum.

Theorem 2.1: Berge's theorem

Let G be a bipartite graph and let $M \subseteq E(G)$ be a matching on G. Then, M is a maximum matching on G if and only if there are no M-augmenting paths.

Proof. One of the two implications directly follows from the previous lemma. For the other implication, we prove the contrapositive. Suppose that M is a non-maximum matching on G. Then, there is at least one matching M' on G such that $|M'| > |M|$. Let $H \subseteq G$ be the graph such that $V(H) = V(G)$ and $E(H) = M \Delta M'$, where Δ denotes the *symmetric difference*. We observe that every edge that is shared among M and M' gets deleted in H, hence we can ignore them.

Claim: for every $x \in V(H)$ it holds that $\deg_H(x) \leq 2$.

Proof of Claim 1. Since M and M' are both matchings on G, each vertex can have at most one edge in M and at most one edge in M'. If they share the same edge then H won't contain such edge. If they don't, x will have degree 2 in H. □

We observe that the above claim has many consequences. In particular, it implies that every component of H must be either a cycle or a path (including trivial paths of one single vertex).

Claim 2: every cycle component of H has even length

Proof of Claim 2. By way of contradiction, suppose that there is a cycle C of odd length. By construction, each component has to alternate between edges of M and M'. Hence, at least one vertex of C must have both edges lying either inside M or M', contradicting the fact that either M or M' is a matching. □

Since $|M'| > |M|$, there must be a component with at least one (and at most one) edge in M'. Since the edges of each component alternate between M and M', by Claim 2 we know that such component cannot be a cycle. Hence, it must be a path component P. In order for P to have more edges in M' than edges in M, the first and last edge must be edges of M', meaning that P is an M-augmenting path. By contrapositive, if there is no M-augmenting path then M is a maximum matching. □

This theorem has been used to construct many algorithms for finding a maximum matching on bipartite graphs. The most famous one is the **Hopcroft-Karp algorithm** [HK73], due to a guaranteed runtime of $O(m\sqrt{n})$. Given a bipartite graph G with bipartition (A, B), the idea behind such algorithms is to run a simultaneous BFS starting from all the unmatched vertices of A, until at least one free node of B is found. Then, we run a DFS over the forest produced by the BFS, starting from the unmatched nodes of B. Each path found through this procedure is an augmenting path, hence their edges can be swapped to increase the cardinality of the matching. The whole process is repeated until no augmenting path is found.

What about non-bipartite graphs? For the general case, problems rise with odd cycles, which never exist in bipartite graphs (Theorem 1.3). Any matching over an odd-length cycle will never cover every vertex of the cycle. This means that in any odd-length cycle there must be a vertex adjacent to two edges which cannot be part of the matching considered. This makes finding the optimal matching hard to find on such graphs. In 1965, Edmonds [Edm65] came up with the **Blossom algorithm**, a real piece of art in the world of algorithm design. The algorithm is based on the repeated contraction and distension of *blossoms*, i.e. a cycles of odd length which contain the maximal number of edges in the current matching. The *blossom lemma* states that any matching found on the graph with every blossom contracted is a maximum matching if and only if it is also a maximum matching for the graph original graph.

The maximum matching problem is highly related to the **Minimum Vertex Cover** problem. This problem involves finding the smallest subset of vertices in a graph such that every edge is incident to at least one vertex in the set.

Definition 2.3: Vertex Cover

Given a graph G, a vertex cover over G is a subset $C \subseteq V(G)$ such that $\forall e \in E(G)$ there is a vertex $v \in C$ such that $v \in e$.

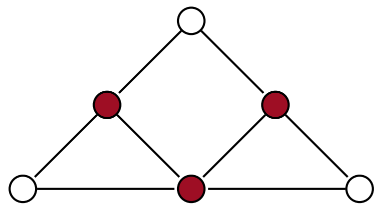


Figure 2.7: The red nodes are the smallest possible vertex cover of the graph.

Proposition 2.1

Given a graph G, for every matching M on G and every vertex cover V of G it holds that $|M| \leq |V|$.

Proof. By definition, we observe that if V is a vertex cover for G then it is also a vertex cover for any subgrpah $G' \subseteq G$. Hence, V is also a vertex cover for any G_M, where $V(G_M) = V(G)$ and $E(G_M) = M$. By definition of matching, in G_M any vertex has either degree 0 or 1. Thus, each vertex of V can cover at most one edge of M, meaning that V has to have at least $|M|$ vertices to cover all the edges of M. □

In bipartite graphs, the above proposition can be strengthened. In 1931, Kőnig [Sza20] proved that the cardinality of the maximum matching and the minimum vertex cover is equal. This implies that the minimum vertex cover can be efficiently found on bipartite graphs by finding the maximum matching.

Theorem 2.2: Kőnig's theorem

Given a bipartite graph G, let M^* and V^* respectively be a maximum matching and a minimum vertex cover on G. Then, it holds that $|M^*| \leq |V^*|$.

Proof. We observe that it suffices to show that there is a (non-minimum) vertex cover V such that $|V| = \mathcal{M}^*$ in order to get $|M^*| \leq |V^*| \leq |V| = |M^*|$. Let (A, B) be the bipartition of G. We define V as the set of vertices such that $\forall ab \in M$, with $a \in A$ and $b \in B$, if there is an alternating path starting from a vertex in A and ending on b then $b \in V$, otherwise $a \in \mathrm{U}$. Fix an edge $xy \in E(G)$ with $x \in A$ and $y \in B$.

Claim: V covers xy

Proof. Suppose that $xy \in M$ then we know that either x or y is in V by definition of U itself, meaning that such edge is covered. Hence, we may assume that $xy \notin M$. We have two cases:

- x is unmatched, meaning that $\nexists uv \in M$ such that $u = x$. Then, y must be matched by M^* to some $u \in A$, since otherwise $M^* \cup \{xy\}$ would be a matching greater than M^*. Hence, the edge uy is an alternating path starting from A and ending on y, meaning that $y \in V$.

- x is matched, meaning that $\exists xv \in M$. Then, by definition of V, either x or v must lie inside V. If $x \in$ U then xy is trivially covered by V. If $v \in V$, instead, by definition of V there must be an alternating path P starting from A and ending at v. This also implies that $P \cup vx \cup xy$ is an alternating path ending on y. However, this implies that y must be matched through some edge $wy \in M$, with $w \neq x$, since otherwise the path $P \cup vx \cup xy$ would be an augmenting path, contradicting the fact that M^* is maximum (Berge's theorem). Thus, we conclude that $y \in V$ since $wy \in M$ and $P \cup vx \cup xy$ is an augmenting path that ends on y.

$\square$

In both cases, we get that either x or y is inside V, concluding that xy is covered by U. Applying the same argument over all edges, we get that V is a vertex cover. $\square$

2.2 Perfect matching

We'll now focus on a particular type of matching on graphs. First, we notice that for any matching M on a graph G (even when G is non-bipartite) it must always hold that:

$$|M| \leq \frac{|V(G)|}{2}$$

When the inequality is satisfied at equality, the matching is said to be **perfect**.We observe that such condition is equivalent to saying that every vertex of the graph is covered by an edge. In fact, the concept of perfect matching can be, in some sense, viewed as the edge-version of a vertex cover, even though there is an additional constraint for the disjointness of the edges.

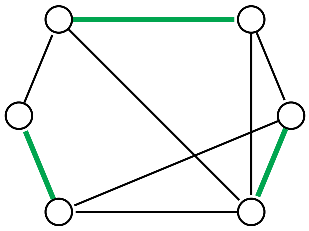


Figure 2.8: A perfect matching.

Definition 2.4: Perfect matching

A matching M on a graph G is said to be perfect if for all vertices of G there is an edge of M covering it.

We observe that every perfect matching is clearly a maximum one, but the contrary doesn't always hold. Moreover, not all graphs can have a perfect matching – trivially, a graph with an isolated vertex cannot have a perfect matching.

For bipartite graphs, the concept of perfect matching has to be slightly tweaked: if (A, B) is the bipartition of the graph and $|A| \neq |B|$, there is no way for a perfect matching to hold. Hence, assuming that $|A| \leq |B|$, we use the concept of A-**perfect matching**, i.e. a matching that covers every vertex of A – notice that if (A, B) is a partition then (B, A) is also a partition, hence we can always assume that $|A| \leq |B|$.

Hall [Hal35] was able to characterize A-perfect matchings on bipartite graphs through the now now called *Hall's Marriage Condition* – the name comes from the original set-theoretic formulation of the problem.

Theorem 2.3: Hall's marriage condition

Let G be a bipartite graph with bipartition (A, B) with $|A| \leq |B|$. Then, G has an A-perfect matching if and only if for all subsets $S \subseteq A$ it holds that $|N(S)| \geq |S|$, where $N(S) = \bigcup_{s \in S} N(s)$.

Proof. Suppose that there is an A-perfect matching M of G. Fix a subset $S \subseteq A$. By definition of M, each node $s \in S$ is matched through M with a different node, therefore $|N(S)| \geq |S|$.

Vice versa, suppose that there is no A-perfect matching M. Let V^* be a minimum vertex cover on G. By Kőnig's theorem, we know that the size of V^* has to be equal to the size of any maximum matching of G. However, since there is no A-perfect matching, any maximum matching has to have size less than $|A|$, meaning that $|V^*| < |A|$. Consider now the set $S = A - V^*$. Since G is bipartite, it must hold that $N(S) = V^* \cap B$. We observe that:

$$|N(S)| \leq |V^* \cap B| = |V^*| - |V^* \cap A|$$

Since $V^* \cap A = A - (A - V^*) = A - S$, we get that:

$$|N(S)| \leq |V^*| - |V^* \cap A| = |V^*| - |A| + |S|$$

Finally, since $|U| < |A|$, we conclude that:

$$|N(S)| \leq |V^*| - |V^* \cap A| = |V^*| - |A| + |S| < |S|$$

□

What if the graph is non-bipartite? Can perfect matchings also be characterized for the general case? The answer is yes! Before proving the result, we'll build an intuition for it. First, we observe that each graph with an odd number of vertices cannot have a perfect matching since there will always be at least one unmatched vertex. This can be extended to components: if a graph contains at least one component with an odd number of vertices then there will always be at least one unmatched vertex inside that component. But what if such components get "connected" through an intermediate vertex x? Then, once such vertex gets matched, the problem still holds: the induced graph $G - x = G[V(G) - \{x\}]$ now contains components with an odd number of vertices.

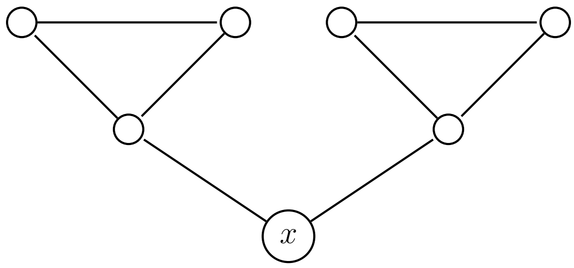


Figure 2.9: Once the vertex x gets matched, the graph $G - \{x\}$ is made of two odd components, making a perfect matching impossible.

But what if we add another edge? In that case, we would have enough vertices to complete the perfect matching.

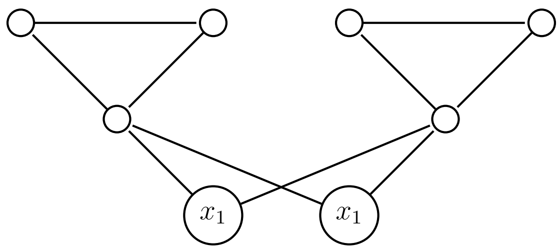


Figure 2.10: Once the vertex x_1 gets matched with an edge touching one of the two odd components, x_2 can be matched through the edges touching the other component.

Thus, it's easy to see that if there is a subset of vertices $S \subseteq V(G)$ such that $\mathcal{O}(G-S) > |S|$ then there cannot be a perfect matching, where $\mathcal{O}(G)$ is the number of odd components of G. To make things easier to read, we say that a graph satisfies the **perfect matching condition** when for all subsets $S \subseteq V(G)$ it holds that $\mathcal{O}(G - S) \leq |S|$. In 1947, Tutte [LP09] proved that such condition perfectly characterizes the existence of a perfect matching. To show the result, we first prove the following lemma.

Lemma 2.2

Let G be a graph and let G' be a super-graph of G, i.e. $G \subseteq G'$, obtained by adding edges to G and keeping the same vertices. Then, if G' doesn't satisfy the perfect matching condition then G also doesn't.

Proof. Suppose that G' doesn't satisfy the condition, meaning that $\exists S \subseteq V(G)$ such that $\mathcal{O}(G' - S) > |S|$. Then, since G' is obtained by adding edges, the number of odd components in G' may remain the same of G or decrease if two previously disconnected components are now connected in G', meaning that:

$$|S| < \mathcal{O}(G' - S) \leq \mathcal{O}(G - S)$$

and thus that G also doesn't satisfy the condition. $\square$

Theorem 2.4: Tutte's theorem

Let G be a graph. Then, G has a perfect matching if and only if for all subsets $S \subseteq V(G)$ it holds that $\mathcal{O}(G - S) \leq |S|$.

Proof. Suppose that there is a perfect matching M on G. Fix a subset $S \subseteq V(G)$. Then, every odd component of $G - S$ sends at least one matching edge to a distinct vertex of S, meaning that $|S| \geq \mathcal{O}(G - S)$.

Vice versa, suppose that there is no perfect matching on G. Let G' be the maximal supergraph of G that doesn't have a perfect matching obtained by adding edges and keeping the same vertices. We say that a set X satisfies the *clique-adjacency condition* if every component of $G'[V(G') - X]$ is a clique and every $u \in X$ is adjacent to every $v \notin X$

Claim 1: if there is a subset $S' \subseteq V(G')$ that satisfies the clique-adjacency condition then G doesn't satisfy the perfect matching condition.

Proof of Claim 1. Assume that S' is a subset satisfying the clique-adjacency condition. We observe that, since very component of $G'[(V(G)) - S']$ is a clique, for each even component we can always find a perfect matching restricted to it. For odd components, instead, we can always find a perfect matching over all its nodes except for one. Let $M_1, \dots, M_k$ be the perfect matching defined on the even components and let $M'_1, \dots, M'_h$ be the p.m. defined on the odd components.

If S' violates the perfect matching condition on G then we're done. Otherwise, if S' doesn't violate it then we know that $|S'| \geq \mathcal{O}(G - S')$. Hence, since every vertex of S' is adjacent to every vertex that isn't in S', we can match the unmatched vertex of each odd component of $G'[V(G) - S']$ with a vertex of S'. We observe that the remaining vertices of S' must form a clique, since otherwise we could add edges to G' and preserve the absence of a perfect matching (the only way to prevent this is if all the edges are already in the graph). Moreover, this clique must have an odd number of nodes, since otherwise we could form a perfect matching on G'. Hence, G' must have an odd number of total vertices (thus also G does), meaning that the empty set $\varnothing$ violates the perfect matching condition on G. Hence, at least one subset of G must violate the perfect matching condition. □

Let $S = \{v \in V(G') \mid \deg_{G'}(v) = n - 1\}$. By way of contradiction, suppose that that S doesn't satisfy the clique-adjacency condition. Then, there is a component of $G'[V(G') - X]$ that isn't a clique or there is a $u \in S$ and a $v \notin S$ such that $u \not\sim v$. However, by definition of S, the second case cannot hold, hence there must be a component that isn't a clique, meaning that there are at least two vertices x, y in that component such that $x \not\sim y$.

Let P be the shortest path from x to y in $G'[V(G') - S]$. Let a, b, c be the first vertices of P. We observe that $a \not\sim c$ since otherwise P wouldn't be the shortest path. Moreover, since P is a path in $G'[V(G') - S]$, we know that $b \notin S$. Hence, there must be a vertex $d \in V(G)$ such that $b \not\sim d$. In other words, we obtain a kite-shaped subgraph (see Figure 2.11).

By maximality of G', we know that $G' \cup \{ac\}$ must have a perfect matching M_1. Likewise, $G' \cup \{bd\}$ must have a perfect matching M_2. Furthermore, it must hold that $ac \in M_1$ and $bd \in M_2$ since otherwise M_1 and M_2 would be perfect matchings on G'.

Claim 2: $(M_1 - \{ac\}) \cup (M_2 - \{bd\})$ contains a perfect matching on G'

Proof of Claim 2. Let $\overline{M} = (M_1 - \{ac\})\Delta(M_2 - \{bd\})$. Since M_1 and M_2 are two matchings, every vertex of $\overline{M}$ has degree at most 2. Hence, every component of $\overline{M}$ must be either an alternating path or an alternating cycle. Moreover, we observe that the only vertices that have degree 1 are a, b, c, d. Thus, these vertices must be the endpoints of two components P_1, P_2 that are alternating paths. We observe that, since $ac \in M_1$ and $bd \in M_2$, one of such paths must be an M_1-alternating path, while the other must be an M_2-alternating path. Without loss of generality, let P_1 be the M_1-alternating path and let P_2 be the M_2-alternating path. We notice that $M' = (M_1 \cup M_2) - (E(P_1) \cup E(P_2))$ is a perfect matching on $G - (V(P_1) \cup V(P_1))$. We have three cases:

1. P_1 is a path from a to c and P_2 is a path from b to d. Then, $(M_1 \cap E(P_2)) \cup (M_2 \cap E(P_1)) \cup M'$ is a perfect matching on G'
2. P_1 is a path from a to b and P_2 is a path from c to d. Then, $(M_1 \cap E(P_2)) \cup (M_2 \cap E(P_1)) \cup M' \cup \{bd\}$ is a perfect matching on G'
3. P_1 is a path from a to d and P_2 is a path from b to d. Then, $(M_1 \cap E(P_2)) \cup (M_2 \cap E(P_1)) \cup M' \cup \{ac\}$ is a perfect matching on G'

□

Since $(M_1 - \{ac\}) \cup (M_2 - \{bd\})$ always contains a perfect matching on G', we conclude that S it is impossible for S to not satisfy the clique-adjacency condition. Thus, through Claim 1 and the previous lemma we know that both G' and G violate the perfect matching condition. □

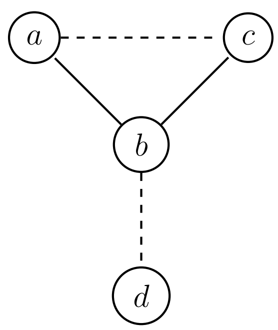


Figure 2.11: The kite-shaped subgraph yield by Tutte's argument. The dashed edges represent non-existing edges.

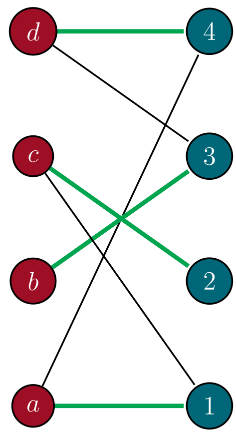
2.3 Stable matching

We'll now focus on a more advanced type of maximum, i.e. the **Stable Marriage** problem. This problem acts as a generalization of the common maximum matching problem, where the entities involved can have some *preferences* for their possible partners. In general, a matching on a bipartite graph is said to be **stable** if no vertex is unmatched and there is no other possible partner outside of the matching that they would prefer to be matched with. The preferences of each vertex $v \in V(G)$ are described through a bijective weight function $w_v : V \to [|X|]$, where $X = B$ if $v \in A$ and $X = A$ otherwise.

Definition 2.5: Stable matching

Let G be a bipartite graph and let $\{w_v\}_{v \in V(G)}$ be a family of preference functions. A matching M on G is said to be **stable** if for all edges $ab \in E(G)$ at least one of the following holds:

- a is matched and $\nexists ab' \in M$ such that $w_a(b) > w_a(b')$
- b is matched and $\nexists a'b \in M$ such that $w_b(a) > w_a(a')$



$w_a : 4231$	$w_1 : acbd$
$w_b : 4123$	$w_2 : abdc$
$w_c : 1243$	$w_3 : bcda$
$w_d : 3241$	$w_4 : dcba$

Figure 2.12: A stable matching. The preferences are represented through ordered lists

We observe that the definition of stable matching implies that the trivial empty matching is <u>not stable</u>, since at least one of the two endpoints of every edge has to be matched. Surprisingly, Gale and Shapley [GS13] proved that in any bipartite graphs it is always possible to find a stable matching independently of all preferences and entities involved. The result was first proved in the context of mathematical economics, solving the problem of optimal matching processes, such as college admission, job recruiting, hospital admitting and many more, awarding the two authors the *2012 Nobel Prize in Economics*. Before proving the result, we give some a “name” to some properties in order to make things more readable:

- Given two matchings M, M' on a graph G, we say that M' is **better** than M if for all $ab \in M'$ there is at least an edge $a'b \in M$ such that $w_b(a') \geq w_b(a)$ and there is at least an edge $a''b \in M'$ such that $w_b(a) > w_b(a'')$. In other words, there is at least one edge that is strictly more preferable, while all the other edges are at least as good as the ones in M.
- We say that a vertex $a \in A$ is **acceptable** for $b \in B$ if either b is unmatched or b is matched to $a' \in A$ but $w_b(a) > w_b(a')$. In other words, b is either unmatched or matched to a partner less preferred than a.
- We say that $a \in A$ is **happy** if either a is unmatched or a is matched to $b \in B$ and for all $b' \in B$ that find a acceptable it holds that $w_a(b) \geq w_a(b')$. In other words, a is either unmatched or has reached its optimal partner.

Theorem 2.5: Gale-Shapley theorem

Every bipartite graph with a family of preference functions $\{w_v\}_{v \in V(G)}$ has a stable matching.

Proof. We give a constructive proof. The idea is to construct a sequence of matchings $M_0, \dots, M_k$ such that for each matching it holds that every vertex of A is happy inside it, but only the last matching is stable. First, we claim the following.

Claim 1: if M is not stable and every vertex of A is happy in M then there is no $a \in A$ such that a is unmatched and acceptable for some $b \in B$

Proof of Claim 1. Suppose that M is not stable and that every vertex of A is happy in M. Then, by definition of unstable matching, there must be at least one edge $ab \in E(G)$ for which one of the following four cases holds:

1. a is unmatched and b is unmatched. Then, a is acceptable for b since the latter is unmatched.
2. a is unmatched and $\exists a'b \in M$ such that $w_b(a) > w_b(a')$. Then, a is acceptable for b since the latter is matched to a' but a is a more preferred partner for b.
3. $\exists ab' \in M$ such that $w_a(b) > w_a(b')$ and b is unmatched. Then, a is acceptable to b since the latter is unmatched, contradicting the fact that a is happy in M since a is matched to b' but prefers b.
4. $\exists ab' \in M$ such that $w_a(b) > w_a(b')$ and $\exists a'b \in M$ such that $w_b(a) > w_b(a')$. Then, a is acceptable to b since a is a more preferred partner for b, contradicting the fact that a is happy in M since a is matched to b' but prefers b.

Hence, the only possibilities are the first and second case, concluding that a is unmatched and acceptable for b. $\square$

We'll now define the extension procedure. Let $M_0 = \varnothing$. Since the matching is empty, each vertex of A is unmatched, hence happy. Moreover, by definition M_0 cannot be stable. This will act as our base case. Fix $i \in [k]$ and suppose that M_i is unstable and every vertex inside it is happy. Then, by Claim 1 there must be a vertex $a_{i+1} \in A$ that is unmatched in M_i and must have at least one $b \in B$ that finds a acceptable. Let b_{i+1} be a vertex of B that finds a acceptable and that maximizes the value of $w_{a+1}(b_{i+1})$. We set M_{i+1} as the matching obtained by removing the potential already existing partner a' of b_{i+1} in M_i and by adding the edge $a_{i+1}b_{i+1}$

$$M_{i+1} = \begin{cases} M_i \cup \{a_{i+1}b_{i+1}\} & \text{if } \nexists a'b_{i+1} \in M_i \\ (M_i \cup \{a_{i+1}b_{i+1}\}) - \{a'b_{i+1}\} & \text{otherwise} \end{cases}$$

We claim that the extended matching is better than the previous and preserves happiness for all vertices.

Claim 2: M_{i+1} is better than M_i and every vertex of A is happy in M_{i+1}.

Proof of Claim 2. By construction, every edge $xy \in M_i$ such that $y \neq b_{i+1}$ is also inside M_{i+1}, hence every such edge preserves happiness and the overall preferences of each vertex. If $\nexists a'b_{i+1} \in M_i$ then M_{i+1} is trivially better than M_i. If $\exists a'b_{i+1} \in M_i$, instead, the new edge $a_{i+1}b_{i+1}$ since a_{i+1} is acceptable for b, meaning that M_{i+1} is better than M_i also in this case. Moreover, by choice of b_{i+1}, the vertex a_{i+1} is now matched with an optimal partner. Hence, the happiness of a_{i+1} is preserved: a_{i+1} was happy due to it being unmatched, but now it's happy due to being matched with the best partner. Moreover, the happiness of the potential vertex a' is also happy since it became unmatched. $\square$

Since every matching in the sequence is better than the previous ones, we know that such sequence cannot cycle. Moreover, since there is a finite number of possible matchings, the sequence will eventually reach a matching where every vertex of A is matched. Let M_k be such matching. Then, by contrapositive of Claim 1, we know that M_k is either stable or has a vertex that isn't happy. However, by construction of the sequence, we know that every vertex in M_k is happy, thus the only possibility is that M_k is stable. $\square$

2.4 Solved exercises

Problem 2.1

Let G be a bipartite k-regular graph with bipartition (A, B). Prove that:

1. $|A| = |B|$
2. G has a perfect matching

Solution. Given any set X, let E_X denote the subset of edges with an endpoint in X. Since the graph is k-regular, we have that $|E_A| = k\,|A|$ and $|E_B| = k\,|B|$. Moreover, since the graph is bipartite, we have that $E_A = E_B$. Hence, we get that $k\,|A| = k\,|B|$ and thus that $|A| = |B|$. Consider now a subset $S \subseteq A$. We observe that $E_S \subseteq E_{N(S)}$ by definition of neighborhood. Hence, we have that $k\,|S| = |E_S| \leq \left|E_{N(S)}\right| = k\,|N(S)|$ and thus that $|S| \leq |N(S)|$. By Hall's marriage condition, we conclude that G has a perfect matching. $\square$

3

Connectivity and Structure

3.1 Disjoint paths, hitting set and separations

Many tools of graph theory enable us to study the structure of the underlying graph when some conditions are met. One such tool is **Menger's theorem**, which can be used to derive many results regarding the structure of graphs. We'll start by defining the tools needed to discuss such theorem.

Definition 3.1: (A, B) path

Let G be a graph. Given two subsets $A, B \subseteq V(G)$, an (A, B) path is a path with one endpoint in A, one endpoint in B and no internal vertices in $A \cup B$

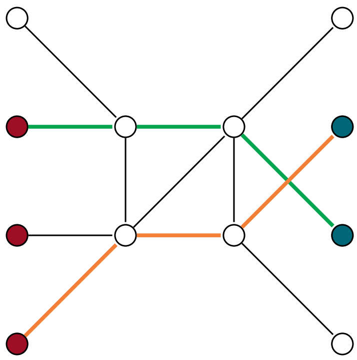


Figure 3.1: The blue nodes form the set A, while the red nodes form the set B. The colored paths form the maximum number of vertex-disjoint (A, B) paths.

We're interested in finding the maximum number of **vertex-disjoint (A, B) paths**, i.e. paths that share no common edges. In many cases, we can identify a "bottleneck" called **hitting set**, i.e. a set of vertices where all disjoint paths are forced to pass, such as the middle nodes of the graph in Figure 3.1.

Definition 3.2: (A, B) **Hitting set**

Let G be a graph. Given two subsets $A, B \subseteq V(G)$, an (A, B) **hitting set** is a subset $X \subseteq V(G)$ such that every (A, B) path intersects X.

Proposition 3.1

Let G be a graph and let $k \in \mathbb{N}$. Given two subsets $A, B \subseteq V(G)$, if there is an (A, B) hitting set of size k then there are at most k vertex-disjoint (A, B) paths.

Proof. Let X be an (A, B) hitting set of size k. By definition, every (A, B) path must have at least one vertex in H. Hence, there can be at most k vertex-disjoint (A, B) paths. □

The above proposition clearly implies that the maximum number of disjoint (A, B) paths is upper bounded by the cardinality of the minimum (A, B) hitting set. However, finding hitting sets is no easy task. A more convenient way to reason about hitting sets are **graph separations**.

Definition 3.3: (A, B) **Separation**

Let G be a graph. Given two subsets $A, B \subseteq V(G)$, an (A, B) **separation** is a pair of subsets $X, Y \subseteq V(G)$ for which all of the following conditions holds:

1. $A \subseteq X$ and $B \subseteq Y$
2. $X \cup Y = V(G)$
3. $\nexists uv \in E(G)$ such that $x \in X - Y$ and $y \in Y - X$

The **order** of an (A, B) separation is defined as $|X \cap Y|$.

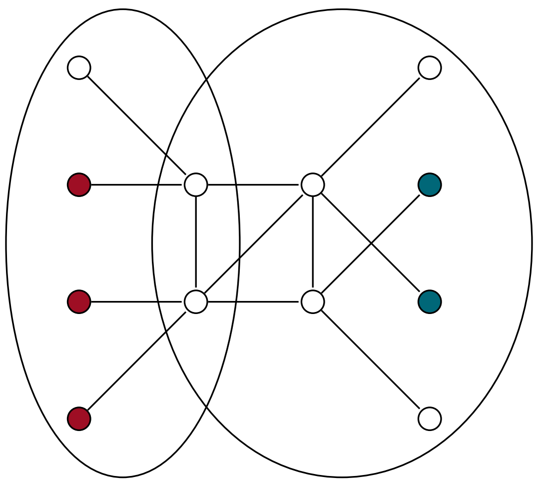


Figure 3.2: An (A, B) separation of order 2.

From the very definition of separation, it's easy to see that every (A, B) must pass through the intersection between X and Y since there are no edges that go from $X - Y$ to $Y - X$. Moreover, for any hitting set there is always a separation whose intersection is the hitting set itself, making the two concepts "equivalent" for our purposes.

Proposition 3.2

Let G be a graph. Given two subsets $A, B \subseteq V(G)$, it holds that:

- If X, Y form an (A, B) separation then $X \cap Y$ is an (A, B) hitting set.
- If Z is an (A, B) hitting set then there is an (A, B) separation P, Q such that $P \cap Q = Z$.

Proof. The first point follows from the definition of (A, B) separation. Suppose now that Z is an hitting set. Let $H_1, \ldots, H_t$ be the components of $G - Z$. We observe that no component has an edge $\{u, v\}$ such that $u \in A - Z$ and $v \in B - Z$, since otherwise there would be an (A, B) path that doesn't intersect Z. Let P, Q be the two subsets defined as:

$$P = Z \cup \bigcup_{\substack{H_i \text{ s.t.} \\ A \cap H_i = \varnothing}} V(H)$$

$$Q = Z \cup \bigcup_{\substack{H_i \text{ s.t.} \\ A \cap H_i \neq \varnothing}} V(H)$$

By the above observation, P contains all the vertices in $B - Z$, while Q contains all the vertices in $V(G) - (B - Z)$, concluding that $P \cup Q = V(G)$ and $P \cap Q = Z$. $\square$

The two previous propositions easily imply that the minimum order of an (A, B) separation is equal to the minimum cardinality of an hitting set, implying that the former also gives an upper bound on the maximum number of vertex-disjoint (A, B) paths. Menger [Men27] proved that the lower bound also holds, meaning that the two optimization problems are equivalent. To prove Menger's result, we'll prove a stronger version of the theorem.

Theorem 3.1: Menger's theorem (stronger version)

Let G be a graph and let $k \in \mathbb{N}$. Given two subsets $A, B \subseteq V(G)$, there are either at least k vertex-disjoint (A, B) paths or there is an (A, B) separation of order less than k.

Proof. We proceed by induction of $|E(G)|$. If $|E(G)| = 0$ then the only possible (A, B) paths are individual nodes in $A \cap B$. If $|A \cap B| \geq k$ then we have at least k vertex-disjoint (A, B) paths. If $|A \cap B| < k$, instead, $A \cap B$ is an hitting set, concluding by the previous proposition that there is an (A, B) separation of order less then k.

Assume that the lemma holds for every graph with at most $m-1$ edges. Suppose that $|E(G)| = m$ and fix an edge $xy \in E(G)$. By inductive hypothesis, we know that $G - \{xy\}$ has either at least k vertex-disjoint (A, B) paths or there is an (A, B) separation of order less than k. If the first case holds then G also has at least k vertex-disjoint (A, B) paths, hence we may focus on the second case. Let X, Y be an (A, B) separation in $G - \{xy\}$ of order less than k. We observe that if $x, y \in X$ or $x, y \in Y$ then X, Y is also a separation for G.

Suppose now that $x \in X - Y$ and $y \in Y - X$. Then, X and $Y \cup \{x\}$ is either still a separation of order less than k or it becomes a separation of order k with $k-1$ (A, B) paths. Since the first case concludes the proof, we assume the second one to hold. Consider the graphs $G[X]$ and $G[Y]$ and apply the inductive hypothesis on them. If both $G[X]$ and $G[y]$ have k disjoint paths then for the former they must be paths from A to $(X \cap Y) \cup \{x\}$, while for the latter they must be paths from $(X \cap Y) \cup \{y\}$ to B. By joining these paths, we get at least k disjoint paths in G. If at least one of them doesn't have these paths, then it must have a separation X', Y' of order less than k. Without loss of generality, assume $G[X]$ to be the one with such property, meaning that $A \subseteq X'$ and $(X \cap Y) \cup \{x\} \subseteq Y'$. Then, X' and $Y \cup Y'$ for a separation of G of order less than k. $\square$

Corollary 3.1: Menger's theorem (original version)

Let G be a graph. Given two subsets $A, B \subseteq V(G)$, the maximum number of (A, B) vertex-disjoint paths is equal to the minimum order of an (A, B) separation.

Proof. Let M be the maximum number of (A, B) vertex-disjoint paths and let m be the minimum order of an (A, B) separation. Since each separation given an upper bound on the number of paths, we know that $M \leq m$. By way of contradiction, suppose that $M < m$. Then, since there are less than m paths, by the strong version of Menger's theorem, there must be a separation of order less than m, which is absurd by choice of m. Hence, we conclude that $M \geq m$. $\square$

It's easy to see that the weak version of Menger's theorem also implies the strong version of the theorem, making them effectively equivalent. For our purposes, we'll be more interested in the stronger formulation of the theorem. In particular, this version is in many cases sufficient to find derive information about the structure of a graph when it is unknown. We'll see many results of this form in the following sections.

3.2 k-connectivity and topological minors

Connectivity clearly plays a fundamental role in graph theory: in almost every case, we're interested in graphs that are connected. However, the concept of connectivity is not enough to inquiry in the structure of a graph since it just tells us that each vertex can be reached by every other node. For these reasons, we give a generalization of the standard concept of connectivity. A graph is said to be **k-connected** if it has at least k vertices and removing less than k vertices keeps it connected.

Definition 3.4: k-connectivity

Given a graph G, we say that G is k-connected if $|V(G)| \geq k$ and for each $X \subseteq V(G)$ with $|X| < k$ it holds that $G - X$ is still connected.

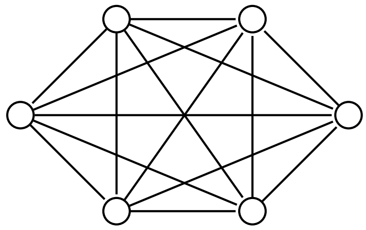


Figure 3.3: The complete graph K_6 is 5-connected. In general, K_{t+1} is t-connected.

We observe that the standard notion of connectivity is trivially equivalent to the notion of 1-connectivity since the only subset of vertices that has less than 1 vertex is the empty set. Not-so-surprisingly, k-connectivity is strictly related to Menger's theorem. This relation is expressed through the following theorem.

Theorem 3.2: Menger's theorem (connectivity version)

A graph G is k-connected if and only if for each distinct $x, y \in V(G)$ there are at least k internally disjoint (x, y) paths.

Note: two paths are internally disjoint if their internal vertices don't intersect.

Proof. Suppose that G is k-connected. For any distinct pair $x, y \in V(G)$ we have two cases:

- If $x \not\sim y$ then, since G is k-connected, it must hold that $\delta \geq k$, thus $N(x)$ and $N(y)$ contain at least k vertices. By Theorem 3.1, there are at least k disjoint $N(x), N(y)$ paths. By adding the edges to x and y on each of these paths, we get at least k internally disjoint (x, y) paths.
- If $x \sim y$ then $G - \{xy\}$ is $k - 1$ connected. Again, by Theorem 3.1, there are at least $k - 1$ disjoint $N(x), N(y)$ paths. By adding the edges to x and y on each of these paths, we get at least k internally disjoint (x, y) paths. The final path is given by the edge xy.

Vice versa, if a graph G contains k internally disjoint paths between any two vertices, then $|G| > k$ and G cannot be separated by fewer than k vertices, thus G is k-connected. □

In general, when a graph is k-connected, lot's of information on the structure of the graph can be inferred, such as the existence of paths, cycles and topological minors. To give an idea behind how these structural information can be derived, we'll list a series of results

Proposition 3.3

Let G be a k-connected graph. Then:

1. $\delta(G) \geq k$
2. Given $A, B \subseteq V(G)$ with $|A|, |B| \geq k$, there are at least k vertex-disjoint (A, B) paths.
3. Given $x \in V(G)$ and $Y \subseteq V(G)$ with $|Y| \geq k$, there are at least k paths $P_1, \ldots, P_k$ from x to Y that intersect only in x.
4. If $k \geq 2$, given $x_1, \ldots, x_k \in V(G)$ there is a cycle C that passes through $x_1, \ldots, x_k$ in some order.
5. If $k = 10t$ for some $t \in \mathbb{N}$, given $x_1, \ldots, x_k \in V(G)$ there is a cycle C that passes through $x_1, \ldots, x_k$ in any order.

Proof.

1. By way of contradiction, suppose that there is a vertex $x \in V(G)$ such that $\deg(x) < k$. Then, the set $N(x)$ is a set with less than k vertices that disconnects G, contradicting the k-connectivity of G
2. By contrapositive, suppose that there are two subsets A, B with $|A|, |B| \geq k$ and less than k disjoint (A, B) paths. By Theorem 3.1, there exists an (A, B) separation X, Y of order less than k. Thus, $X \cap Y$ is an hitting set with less than k vertices. Moreover, we know that $|X| \geq |A| \geq k$ and $|X \cap Y| < k$, thus there is at least one vertex in $X - Y$, concluding that such vertex is disconnected in $G - (X \cap Y)$, hence G is not k-connected.
3. Proof similar to the previous point.
4. Let $X = \{x_1, \ldots, x_k\}$. Let C be a cycle containing as many vertices chosen from X as possible. By way of contradiction, suppose that there is a vertex $x_\ell \in X$ that isn't covered by C. By the previous point, there are at least k paths $P_1, \ldots, P_k$ from x_ℓ to $V(C)$ that intersect only in x_ℓ. Since C contains at most $k-1$ vertices chosen from $x_1, \ldots, x_k$, by the pigeonhole principle there must be a subpath Q in C that doesn't intersect X, meaning that $X \subseteq C - Q$. Moreover, there are at least two paths P_i, P_j with the second endpoint in Q. Hence, $(C - Q) \cup (Q \cap P_1) \cup (Q \cap P_2) \cup P_1 \cup P_2$ form a cycle C' with more vertices taken from X than C, raising a contradiction.
5. Omitted.

□

In some cases, k-connectivity can also be used to establish the existence of more complex structures. This is the case of **topological minors**. Given an edge xy, an **edge subdivision** of xy is obtained by decomposing the edge into two edges xz and xy, where z is a new vertex. A **subdivision** of a graph G is a graph obtained by subdividing at least one edge of G.

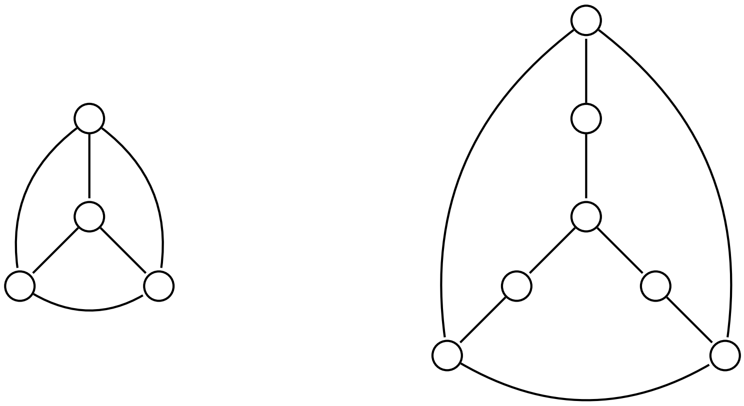


Figure 3.4: The graph K_4 (left) and one of its subdivisions of K_4 (right).

Definition 3.5: Topological minor

Let G be a graph. A graph H is a **topological minor** of G if G has a subgraph that is a subdivision of H.

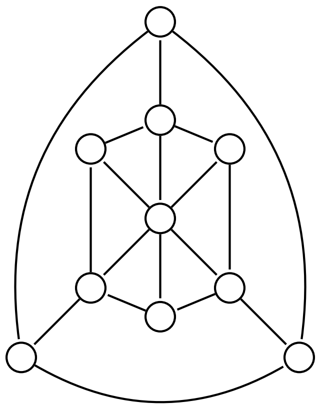


Figure 3.5: A graph containing K_4 as a topological minor.

Proposition 3.4

If G is a 3-connected graph then it contains K_4 as a topological minor.

Proof. Since G is 3-connected, we know that it has at least a vertex and that $\delta(G) \geq 3$. Fix $x \in V(G)$. Since $G - x$ has $\delta(G - x) \geq 2$, by Proposition 1.3 we know that it contains a cycle C of length at least 3. By Proposition 3.3, we know that there are three paths P_1, P_2, P_3 from x to $V(C)$ that intersect only in x. Then, $P_1 \cup P_2 \cup P_3 \cup C$ form a subdivision of K_4 in G. $\square$

3.3 Disjoint cycles and feedback vertex sets

Similarly to Euler's theorem, as a corollary to Menger's theorem we get that the problem that asks to determine the existence of k disjoint paths in a graph lies in the intersection $\mathsf{NP} \cap \mathsf{coNP}$ since the theorem gives us a polynomially verifiable condition both for existence and non-existence of the tour. In fact, the problem actually lies in the class P by reduction to the *max-flow problem*.

A very similar problem asks to deterine the existence of k disjoint cycles in a graph. This problem, however, is NP-complete. Nonetheless, many results are able to assert the existence of a number of disjoint cycles when some conditions are met. These results will be discussed in the following section.

Proposition 3.5

Let G be a multigraph. If G is 3-regular then it contains a cycle of length at most $2\lceil \log n \rceil$

Proof. If G contains a loop or parallel edges (or multi-edges), the proposition trivially holds. Thus, we may assume to be working with simple graphs. Fix a vertex $v \in V(G)$. Consider a BFS on G that starts from v and goes on until no cycles of length at most $2\lceil \log n \rceil$ are encountered. Let T be the tree produced by such BFS. Since G is 3-regular, v will have three children in T and each of those children will be the root of a binary tree inside in T. In other words, all the other nodes in the tree (except the leaves) will have two children.

If the BFS stops before visiting the whole graph, the proposition holds. Hence, we may assume that the BFS doesn't stop, implying that $V(T) = V(G)$. By 3-regularity of G, we know that $\delta = 3$, hence by Proposition 1.3 we get that there must be a cycle in G. By definition of BFS, this cycle must connect two leaves $x, y \in V(T)$ through an edge, otherwise the algorithm wouldn't have taken the shortest path on one of the two branches. Therefore, this cycle must be formed by the edge xy and the paths from x to z and from y to z, where z is the *lowest common ancestor* of x and y.

Since T contains n nodes and by definition of BFS it is a balanced tree, the height of T is at most $\lceil \log n \rceil - 1$. Hence, the two paths can have length at most $\lceil \log n \rceil - 1$, concluding that the cycle has length at most $2\lceil \log n \rceil - 1 \leq 2\lceil \log n \rceil$ □

The above proposition works only for 3-regular graphs, but out final result will involve any type of graph. We give a definition of graphs that are not too far away from being 3-regular.

Definition 3.6: 3-graph with discrepancy d

A 3-graph G of discrepancy d is a multigraph such that $\Delta \leq 3$ and:

$$\sum_{v \in V(G)} 3 - \deg(v) \leq d$$

By definition, a 3-graph of discrepancty 0 is a 3-regular graph.

We define the inverse operation of *edge subdivision* as **vertex suppression**. Given a vertex v of degree 2, suppressing v means to remove v and join its two neighbors with an edge. For verteices of degree 1, instead, suppressing v means removing v.

Proposition 3.6

Let G be a 3-graph with discrepancy d. Then, there is a sequence of at most d operations chosen from:

- suppression of a vertex of degree 2
- suppression of a vertex of degree 1
- deletion of a vertex of degree 0

that results in a 3-regular graph.

Proof. We proceed by induction on d. When $d = 0$, the initial graph already has discrepancy 0 so no operations have to be applied. Assume the inductive hypothesis for $d - 1$ and consider the case d. Since G has discrepancy at least 1, at least one vertex v has degree at most 2. We have three cases:

- If $\deg(v) = 0$ then by removing v we get a graph with discrepancy $d - 3$ since the degree difference decreases by $3 + 0$ due to removal of v:

$$\sum_{v \in V(G-x)} 3 - \deg(v) = -3 + \sum_{v \in V(G)} 3 - \deg(v) \leq d - 3 \leq d - 1$$

- If $\deg(v) = 1$ then by suppressing v we get a graph H with discrepancy $d - 1$ since the degree difference decreases by $3 - 1$ due to the removal of v and the degree of its neighbor gets decreased by 1:

$$\sum_{v \in V(H)} 3 - \deg(v) = -1 + \sum_{v \in V(G)} 3 - \deg(v) \leq d - 1$$

- If $\deg(v) = 2$ then by suppressing v we get a graph H with discrepancy $d - 1$ since the degree difference decreases by $3 - 2$ due to the removal of v and the degree of its neighbors is unchanged:

$$\sum_{v \in V(H)} 3 - \deg(v) = -1 + \sum_{v \in V(G)} 3 - \deg(v) \leq d - 1$$

In all three cases, by applying the inductive hypothesis we conclude the proof. □

Lemma 3.1

There is a constant $c > 0$ such that for any $k \in \mathbb{N}$ for any 3-regular multigraph G with $|V(H)| \geq ck\log(k+1)$ then there are at least k vertex-disjoint cycles in G.

Proof. We proceed by induction on k. When $k = 1$, since G is 3-regular we have that $\delta(G) = 3$. By Proposition 1.3, we conclude that G contains a cycle. Assume the inductive hypothesis for $k-1$ and consider the case k. Again, since G is 3-regular it must contain at least a cycle. Let C be the shortest cycle in G. Since $|V(G)| \geq ck\log(k+1)$, by Proposition 3.5 we have that:

$$|V(C)| \leq 2\lceil \log(ck\log(k+1)) \rceil \leq 2\log(ck\log(k+1)) + 2$$

Consider now the graph $G - C$. Since we removed edges, this graph is a 3-graph with discrepancy d for some constant d. By the previous proposition, we know that $G - C$ can be suppressed into a 3-regular graph G'. Moreover, we have that:

$$|V(G')| \geq |V(G)| - 2|V(C)| \geq ck\log(k+1) - 4\log(ck\log(k+1)) + 4 \geq c(k-1)\log k$$

Thus, by inductive hypothesis we have that G' contains at least $k-1$ disjoint cycles, which implies that also $G - C$ does. Since C is disjoint from all those cycles, we get that G contains at leasts k disjoint cycles. □

Erdös and Pósa [EP65] proved that the number of disjoint cycles in a graph is strictly related the size of **feedback vertex sets**, that being subset of vertices that intersect every cycle of the graph G.

Definition 3.7: Feedback vertex set

Let G be a graph. A subset $X \subseteq V(G)$ is called **feedback vertex set (FVS)** if every cycle of G contains a vertex of X. Equivalently, $G - X$ is acyclic.

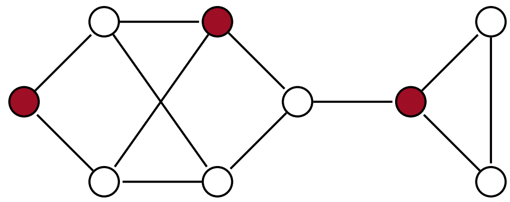


Figure 3.6: The ref vertices form a feedback vertex set.

Theorem 3.3: Erdős-Pósa theorem

There is a constant $c > 0$ such that any $k \in \mathbb{N}$ and any graph G it holds that G either has at least k vertex-disjoint cycles or there is a FVS X such that $|X| \leq ck \log k$.

Proof. If G contains at least k disjoint cycles, the theorem is trivially true. Moreover If G is acyclic, the empty set is an FVS that satisfies the theorem. Thus, we may assume that G contains at least one cycle but less than k disjoint cycles.

Let H be a maximal subgraph of G such that $2 \leq \delta(H) \leq \Delta(H) \leq 3$. We observe that such subgraph H always exists since G contains at least one cycle. Let $C_{i_1}, \ldots, C_{i_t}$ be the cycle components of H, i.e. the components of H that are cycles. For each $j \in [t]$, let x_j be a vertex of C_{i_j}. Let $W = \{x_j \mid j \in t\}$. Since G has less than k disjoint cycles, H contains less than k cycle components, implying that $|W| \leq k + 1$.

Let H' be the subgraph obtained by removing all the cycle components from H. Then, H' is a subdivision of a 3-regular graph. Furthermore, since G contains less than k disjoint cycles, by the previous lemma H must contain less than $c'k \log(k+1)$ vertices, for some constant $c' > 0$. Let $\mathcal{U} = \{v \in V(H') \mid \deg_{H'}(v) = 3\}$. We have that $|\mathcal{U}| < c'k \log(k+1)$.

Claim 1: every cycle of G intersects H.

Proof of Claim 1. By way of contradiction, suppose that there is a cycle C of G that doesn't intersect H. Then, since every vertex of C has degree 2, $H \cup C$ contradicts the maximality of H. □

Claim 2: there are no paths in $G - (\mathcal{U} - W)$ of length at least 1 such that all of the following hold:

- both endpoints are in $H - (\mathcal{U} - W)$
- no internal vertex is in H
- no edge is in H

Proof of Claim 2. By way of contradiction, suppose that there is a path P with all of these properties. Then, since every vertex of P has degree 2, no edge and no internal vetex of P lies in H, the graph $H \cup C$ contradicts the maximality of H. □

Claim 3: every cycle of $G - (\mathcal{U} \cup W)$ intersects H in exactly one vertex.

Proof of Claim 3. Through Claim 1, we know that every cycle of G must intersect H in at least 1 vertex, hence this also holds for cycles of $G - (\mathcal{U} \cup W)$. By way of contradiction, suppose that there is a cycle C of $G - (\mathcal{U} \cup W)$ that intersects H in at least two vertices. The cycle C can be partitioned into edge disjoint paths $P_1, \ldots, P_\ell$ such that the ends of each P_i are in $C \cap H$ and no internal vertex of P_i is in H. Since C has at least two vertices, we know that $\ell \geq 2$.

We observe that it may be possible for some P_i to be a single edge in H, but at least one of $P_1, \dots, P_\ell$ must be. However, since $C \subseteq G - (\mathcal{U} - W)$ and since $H - (\mathcal{U} - W)$ is acyclic, this single edge path violates Claim 2, raising a contradiction. □

Let $Z \subseteq V(H - (\mathcal{U} - W))$ be the set of vertices z for which there is a cycle C of $G - (\mathcal{U} \cup W)$ such that $C \cap H = \{z\}$. We observe that given $z, z' \in Z$ such that $z \neq z'$, the two cycles C, C' in $G - (\mathcal{U} \cup W)$ such that $C \cap H = \{z\}$ and $C' \cap H = \{z'\}$ must be disjoint, otherwise there would be a path from v_1 to v_2 that contradicts Claim 2. Furthermore, since there are at most $k - 1$ cycles, we conclude that $|Z| \leq k - 1$.

By construction of $\mathcal{U}, W, Z$ and by all the claims that we proved, we get that $X = \mathcal{U} \cup W \cup Z$ is a FVS of G. Moreover, we have that:

$$\begin{aligned}
|X| &\leq |\mathcal{U}| + |W| + |Z| \\
&< c' k \log(k + 1) + k - 1 + k - 1 \\
&\leq ck \log k
\end{aligned}$$

for some $c > 0$, concluding the proof. □

3.4 Directed graphs and path covers

Up until now, we have discussed only *undirected graphs*. In this section, we'll shift our focus to *directed graphs* (or *digraphs*), that being graphs whose edges are oriented, implying that the edge (x, y) is not equal to the edge (y, x). We gave a formal definition of directed graphs in Section 1.1. In particular, we'll focus on determining the existence of **path covers**, i.e. sets of disjoint directed paths that cover every vertex of the graph.

Definition 3.8: Path cover

Let G be a digraph. A **path cover** for G is a set $\mathcal{P}$ of disjoint paths such that $V(G) = \cup_{P \in \mathcal{P}} V(P)$.

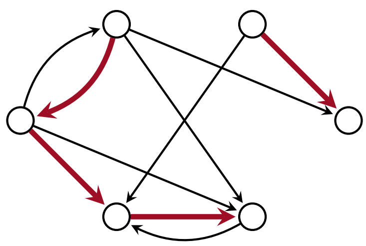


Figure 3.7: The red paths form a path cover.

We observe that any digraph contains a trivial path cover, that being the cover formed by the set of every single-vertex path. We're interested in finding the minimum cardinality path cover. Gallai and Milgram [GM60] proved that path covers are strictly related to *independent sets*.

Theorem 3.4: Gallai-Milgram theorem

For every digraph G, there is a path cover $\mathcal{P}$ of G and an independent set X such that $|X \cap P| = 1$ for all $P \in \mathcal{P}$.

We prove a slightly stronger result that directly implies the above theorem. Given a path cover $\mathcal{P}$, we define $\text{ter}(\mathcal{P})$ as the set of terminal nodes of the paths in $\mathcal{P}$.

$$\text{ter}(\mathcal{P}) = \{v \in V(G) \mid \exists P \in \mathcal{P} \text{ s.t. } v \in V(P), \text{our} - \deg_P(v) = 0\}$$

We say that a path cover $\mathcal{P}$ is *minimal by inclusion* if there is no other path cover $\mathcal{P}'$ whose terminal nodes are a strict subset of the terminal nodes of the former, i.e. $\text{ter}(P') \subset \text{ter}(P)$.

Proposition 3.7

Let G be a digraph and let $\mathcal{P}$ be a path cover of G that is minimal by inclusion. Then, there is an independent set X such that $|X \cap P| = 1$ for all $P \in \mathcal{P}$.

Proof. We proceed by induction on n. If $n = 1$, we trivially have that the single node in G forms both the path cover and the independent set. Assume the inductive hypothesis and consider $n > 1$. We may assume that $\text{ter}(\mathcal{P})$ is not an independent set, since otherwise the proof trivially follows.

Let $\mathcal{P} = \{P_1, P_2, \dots, P_k\}$. For each $i \in [k]$, let x_i be the terminal of P_i. Since $\text{ter}(\mathcal{P})$ is not an independent set, there is at least one edge between the terminals of two paths in the cover. Without loss of generality, assume that such edge is $(x_2, x_1) \in E(G)$. Consider the set $\mathcal{P}' = \{P_1 - x_1, P_2, \dots, P_k\}$. This set is clearly a path cover for $G - x$.

Claim: $\mathcal{P}'$ is minimal by inclusion for $G - x$.

Proof of the claim. By way of contradiction, suppose that $\mathcal{P}'$ is not minimal by inclusion for $G - x$. Then, there must be another path cover $\mathcal{P}^*$ for $G - x$ such that $\text{ter}(\mathcal{P}^*) \subset \text{ter}(\mathcal{P}')$.

We observe that the path P_1 must contain at least 2 nodes, otherwise $\mathcal{P}'' = (\mathcal{P} - \{P_1\}) \cup \{P_2 \cup (x_2, x_1)\}$ is a path cover for G such that $\text{ter}(P'') \subset \text{ter}(\mathcal{P})$, contradicting the minimality by inclusion of $\mathcal{P}$. Thus, the node x_1' that is adjacent to x_1 in P_1 must be the new terminal of $P_1 - x_1$, concluding that $\text{ter}(\mathcal{P}') = \{x_1', x_2, \dots, x_k\}$.

Since $\text{ter}(\mathcal{P}^*) \subset \{x_1', x_2, \dots, x_k\}$, we hvae three cases:

1. $x_1' \in \text{ter}(\mathcal{P}^*)$. Then, there is a $x_j \in \text{ter}(\mathcal{P}') - \text{ter}(\mathcal{P}^*)$ such that $x_j \neq x_1'$. Let Q_1' be the path in $\mathcal{P}^*$ ending in x_1'. Then, the set

   $$\mathcal{P}^{\#} = (\mathcal{P} - Q_1') \cup (Q_1' \cup (x_1', x_1))$$

   is a path cover for G with $\text{ter}(P^{\#}) \subset \text{ter}(\mathcal{P})$, contradicting the minimality by inclusion for G of $\mathcal{P}$.

2. $x'_1 \notin \text{ter}(\mathcal{P}^*)$ and $x_2 \in \text{ter}(\mathcal{P}^*)$. Let Q_2 be the path in $\mathcal{P}^*$ ending in x_2. Then, the set
$$\mathcal{P}^\# = (\mathcal{P} - Q_2) \cup (Q_2 \cup (x_2, x_1))$$
is a path cover for G with $\text{ter}(P^\#) \subset \text{ter}(\mathcal{P})$, contradicting the minimality by inclusion for G of $\mathcal{P}$.

3. $x'_1 \notin \text{ter}(\mathcal{P}^*)$ and $x_2 \notin \text{ter}(\mathcal{P}^*)$. Then, the set
$$\mathcal{P}^\# = \mathcal{P} \cup \{x_1\}$$
is a path cover for G with $\text{ter}(P^\#) \subset \text{ter}(\mathcal{P})$, contradicting the minimality by inclusion for G of $\mathcal{P}$.

□

Since $\text{ter}(\mathcal{P}')$ is minimal by inclusion for $G - x$, by inductive hypothesis we get that there is an independent set X such that $|X \cap P_i| = 1$ for all $i \in [k]$, concluding that X also satisfies the theorem for G. □

Corollary 3.2

Let G be a digraph. Then, the minimum cardinality of a path cover for G is at most the maximum cardinality of an independent set for G.

Proof. By way of contradiction, suppose that the minimum cardinality of a path cover $\mathcal{P}$ for G is greater than the maximum cardinality of an independent set X for G. Then, by the previous theorem, there must be an independent set X' such that $|X' \cap P| = 1$ for all $P \in \mathcal{P}$. Since every path in $\mathcal{P}$ is disjoint, X' must have at least $|\mathcal{P}|$ vertices, implying that $|X| < |\mathcal{P}| \leq |X'|$ and thus contradicting the choice of X. □

The above corollary gives us a lower bound for the maximum independent set problem, which is known to be NP-hard. However, it's easy to see that a graph contains an Hamiltoniam path if and only if there is a path cover made of one single path, concluding that the former problem can be reduced to the latter, making the minimum path cover problem also NP-hard.

Nonetheless, the Gallai-Milgram theorem still has some applications in other branches of mathematics. For instance, it can be used to get a short proof of a result originally proved – using different techniques – by Dilworth [Dil50] involving *partially ordered sets* (or *posets*). Given a poset $(S, \prec)$, we define a **chain** as a sequence of pairwise comparable elements, such as $a_1 \prec a_2 \prec \ldots \prec a_k$. We define an **anti-chair** as a set of incomparable elements, i.e. such that $\forall x, y$ it holds that $x \not\prec y$ and $y \not\prec y$.

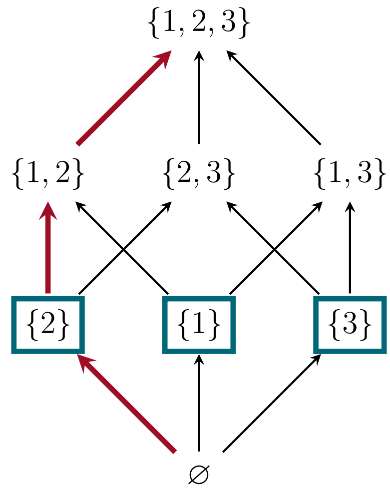


Figure 3.8: The poset $(\{1, 2, 3\}, \subseteq)$ (transitive and reflexive edges are omitted). The red path forms a chain, while the blue rectangles form an anti-chain.

Theorem 3.5: Dilworth's theorem

In every poset $(S, \prec)$, the minimum number of chains covering S is equal to the maximum cardinality of an antichain.

Proof. Let $(S, \prec)$ be a poset. Let $c(S)$ denote the minimum number of chains that cover S and let $a(S)$ be the maximum cardinality of an anti-chain in S. We construct the graph G_S such that $V(G_S) = S$ and $E(G) = \{(x, y) \mid x \prec y\}$. By construction, each chain in S corresponds to a path in G_S. Similarly, each anti-chain in S corresponds to a set of disconnected nodes, i.e. nodes for which there is no path that connects them. By definition, each set of disconnected nodes is also an inpendendent set. Thus, by the Gallai-Milgram theorem we get that:

$$c(S) = |\mathcal{P}^*| \leq |X^*| \leq a(S)$$

where $\mathcal{P}^*$ is a minimum path cover and X^* is a maximum independent set. Moreover, for each path cover there must be at least a path for each node in a set of disconnected nodes, concluding that $c(S) \geq a(S)$ also holds. $\square$

3.5 Solved exercises

Problem 3.1

Prove that any t-connected subgraph G with $n \geq t+2$ contains $K_{2,t}$ as a topological minor.

Solution. We may assume that $G \neq K_n$ since otherwise G contains $K_{2,t}$ as a subgraph, hence as a topological minor. Let $x, y \in V(G)$ be two vertices such that $x \not\sim y$. Since G is t-connected, we know that $\delta \geq t$, hence $|N(x)|, |N(y)| \geq t$. By Menger's theorem, there are at least t disjoint paths from $N(x)$ to $N(y)$ since otherwise we would have a separation that contradicts the t-connectivity of G.

Let $P_1, \dots, P_t$ be t of those paths. For each $i \in [t]$, let x_i be the endpoint of P_i lying in $N(x)$ and let y_i be the one lying in $N(y)$. Then, the graph given by

$$\bigcup_{i=1}^{t} xx_i \cup P_i \cup y_i y$$

is a subdivision of $K_{2,t}$ since each internal vertex of $P_i \cup \{y_i, y\}$ can be suppressed, obtaining an edge from y to x_i. □

Problem 3.2

Let G be a 3-connected subgraph and let xy be an edge of G. Prove that G/xy is 3-connected if and only if $G - \{x, y\}$ is 2-connected.

Solution. We use the characterization of k-connectivity given by Theorem 3.2. Let z be the vertex given by the contraction of xy. We split the proof in two claims.

Claim 1: if G/xy is 3-connected then $G - \{x, y\}$ is 2-connected.

Proof of Claim 1. Suppose that G/xy is 3-connected. Fix two vertices $u, v \in V(G - \{x, y\})$. Since G/xy is 3-connected and $z, w \in V(G - \{x, y\}) \subseteq V(G/xy)$, there are at least 3 internally disjoint paths P_1, P_2, P_3 from u to v. If $z \notin V(P_1 \cup P_2 \cup P_3)$ then the three paths are also paths of $G - \{x, y\}$. Hence, we may assume that $z \in V(P_1 \cup P_2 \cup P_3)$.

Then, exactly one of the paths passes through z since they are internally disjoint. Without loss of generality, let P_1 be such path. Let P_1^{xy} denote the path induced by P_1 in G after de-contracting xy. Since $z \in V(P_1)$, at least one between x and y lies inside P_1^{xy}. Moreover, the second vertex cannot lie inside P_2^{xy}, P_3^{xy}, since otherwise P_1, P_2, P_3 wouldn't be internally disjoint. This concludes that $P_1^{xy}, P_2^{xy}, P_3^{xy}$ are three internally disjoint paths in $G - \{x, y\}$ that connect u and v, concluding that $G - \{x, y\}$ is 2-connected. □

Claim 2: if $G - \{x, y\}$ is 2-connected then G/xy is 3-connected

Proof of Claim 2. Suppose that $G - \{x, y\}$ is 2-connected. Fix two vertices $s, t \in V(G)$. Since G is 3-connected, there are at least 3 internally disjoint paths Q_1, Q_2, Q_3 from s to

t. Since $G - \{x, y\}$ is 2-connected, at least one of such paths must be impossible to take in $G - \{x, y\}$, which can happen only if it passes through x or y. Moreover, there can be at least another such path since otherwise Q_1, Q_2, Q_3 wouldn't be internally disjoint.

Suppose that there is only one path crossing x or y. Without loss of generality, assume that Q_1 is such path, implying that Q_2, Q_3 are paths of G/xy from s to t. If $x, y \in V(Q_1)$, the path Q_1/xy forms the third path in G/xy from s to t. If $x \in V(Q_1)$ and $y \notin V(Q_1)$, the path $(Q_1 - \{x\}) \cup \{z\}$ forms the third path in G/xy from s to t. The case $x \in V(Q_1)$ and $y \notin V(Q_1)$ is similar to the latter one.

Suppose now that there are two paths crossing x or y. Without loss of generality, assume that Q_1, Q_2 are such paths, implying that Q_3 is a path of G/xy from s to t. We also assume that Q_1 is the path crossing x and that Q_2 is the one crossing y. Then, there must be another path Q' in G that goes from s to t, since otherwise $G - \{x, y\}$ wouldn't be 2-connected due to Q_1 and Q_2 being impossible to take. Moreover, by contracting xy the two paths get joined together, but we're still able to traverse them. Thus, $Q'' = (Q_1 - \{x\}) \cup \{z\}$ is another path from s to t, for a total of 3 internally disjoint paths, that being Q_3, Q' and Q''. □

□

Problem 3.3

Prove that every 3-connected graph has a cycle of even lenght. Show that there is a 3-connected graph that has no cycle of odd lenght.

Proof. Let G be a 3-connected graph and consider two nodes $x, y \in V(G)$. By Proposition 3.3, there are at least three internally disjoint paths P_1, P_2, P_3 from x to y. We may assume that $|P_1 \cup P_2| = 2k + 1$ and $|P_2 \cup P_3| = 2h + 1$, i.e. that they are odd cycles, otherwise the result trivially follows. However, this implies that:

$$|P_1 \cup P_3| = |P_1 \cup P_2| + |P_2 \cup P_3| - 2\,|P_2| = 2k + 1 + 2h + 1 - 2\,|P_2| = 2(k + h + 1 + |P_2|)$$

concluding that it is an even cycle. □

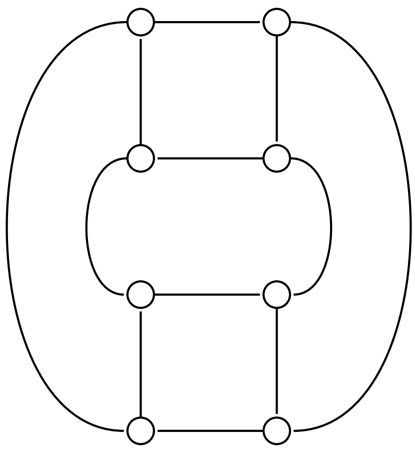


Figure 3.9: A 3-connected graph without odd length cycles.

Problem 3.4

Given a graph G, let $\alpha(G)$ denote the largest size of a set of independent vertices in G. Prove that the vertices of G can be covered by at most $\alpha(G)$ disjoint subgraphs each being either a cycle, an edge or a single vertex.

Proof. We proceed by induction on $\alpha(G)$. When $\alpha(G) = 1$, the graph G must be the complete graph K_n. Then, G can be covered by a single vertex (if $n = 1$), a single edge (if $n = 2$) or a single cycle (if $n \geq 3$). Assume the inductive hypothesis and consider a graph with $\alpha(G) > 1$.

Claim: for all $v \in V(G)$ with $\deg(v) \geq 1$ it holds that $\alpha(G - N(v)) \leq \alpha(G) - 1$

Proof of the claim. Let X be a maximum independent set in $G - N(v)$. Then, by adding v to X we get an independent set $X \cup \{v\}$ in G, implying that:

$$\alpha(G) \geq |X \cup \{v\}| = \alpha(G - N(v)) + 1$$

□

Suppose that $\delta = 0$. Then, there is an isolated vertex $x \in V(G)$. It's easy to see that $\alpha(G - x) \leq \alpha(G) - 1$ since any independent set of $G - x$ can be extended to G by adding x. Hence, by inductive hypothesis we get that $G - x$ has a cover $\mathcal{H} = \{H_1, \dots, H_k\}$ made of disjoint cycles, edges and single vertices with $k \leq \alpha(G) - 1$. Then, $\mathcal{H} \cup \{x\}$ is a cover for G of $k + 1 \leq \alpha(G)$ subgraphs.

Suppose now that $\delta = 1$. Then, there must be an edge $xy \in E(G)$ such that $\deg(y) = 1$. Through the claim, we know that $\alpha(G - \{x, y\}) \leq \alpha(G - N(y)) \leq \alpha(G) - 1$. Hence, by inductive hypothesis we have that $G - \{x, y\}$ has a cover $\mathcal{Q} = \{Q_1, \dots, Q_h\}$ with $h \leq \alpha(G) - 1$ subgraphs. Then, $\mathcal{Q} \cup \{xy\}$ is a cover for G of $h + 1 \leq \alpha(G)$ subgraphs.

Consider now the case where $\delta \geq 2$. Let $P = x_1, \dots, x_k$ be the longest path in G. By choice of P, it must hold that $N(x_k) \subseteq V(P)$, otherwise we could extend P. Let x_i be the vertex in $N(x_k) \cap V(P)$ with smallest index i. We observe that the cycle $C = x_i, x_{i+1}, \dots, x_k, x_i$ contains all the neighbors of x_k by choice of i. Hence, through the claim we have that $\alpha(G - C) \leq \alpha(G - N(x)) \leq \alpha(G) - 1$. Thus, by inductive hypothesis $G - C$ has a cover $\mathcal{R} = \{R_1, \dots, R_t\}$ with $t \leq \alpha(G) - 1$ subgraphs. Then, $\mathcal{R} \cup \{C\}$ is a cover for G of $t + 1 \leq \alpha(G)$ subgraphs.

□

Problem 3.5

Prove that every pair of vertices in a 2-connected graph lie on a shared cycle without using Menger's theorem.

Proof. Fix two vertices $u, v \in V(G)$ and let $P = x_1, \dots, x_k$ be the longest path in G that contains u, v. By choice of P, we have that $N(x_1), N(x_k) \subseteq V(P)$.

Claim: there are two vertices x_i, x_j with $i < j$ such that $x_i \in N(x_k)$ and $x_j \in N(x_1)$.

Proof of the claim. By way of contradiction, suppose that no such vertices exist. Then, there must be a vertex $x_\ell \in V(P)$ such that $\forall x_i \in N(x_k)$ and $\forall x_j \in N(x_1)$ it holds that $i \leq \ell \leq j$. However, removing such vertex x_ℓ would disconnect x_1 from x_k, contradicting the 2-connectivity of G. □

Let x_i, x_j be two vertices chosen according to the claim. Consider the cycles $C_1 = x_1 x_2 \dots x_j x_1$ and $C_2 = x_i x_{i+1} \dots x_k x_i$. We may assume that u and v don't lie both on C_1 or both on C_2, otherwise we're done. Without loss of generality, suppose that $u \in V(C_1)$ and $v \in V(C_2)$. Then, the cycle $C = x_1 x_2 \dots x_j x_k x_{k-1} \dots x_{i+1} x_i x_1$ contains both u and v, concluding the proof. □

Problem 3.6

Show that every k-connected graph ($k \geq 2$) with at least $2k$ vertices contains a cycle of length at least $2k$.

Proof. Since G is k-connected, we know that $\delta \geq k \geq 2$, hence G contains at least a cycle. Let C be longest cycle in G. We may assume that C has less than $2k$ vertices, otherwise we're done. Since G has at least $2k$ vertices, there is at least one vertex $v \in V(G - C)$. By Menger's theorem, there are at least k paths $P_1, \dots, P_k$ from v to C that intersect only on v.

Claim: there are least two paths P_i, P_j whose endpoints in C are adjacent

Proof. By way of contradiction, suppose that no such paths exist. Then, for each pair of paths their enpoints in C must be at least at distance 2 in C, implying that C must have at least $2k$ vertices, raising a contradiction. □

Let P_i, P_j be the two paths chosen according to the claim and let x_i, x_j be their endpoints in C. Then, $(C - \{x_i x_j\}) \cup P_i \cup P_j$ form a cycle longer than C, raising a contradiction. □

4

Extremal graph theory

4.1 Existence of cliques as subgraphs

An important question in graph theory is determining the existance of cliques, i.e. subgraphs K_n, inside a given graph. In general, finding a maximum size clique is an NP-hard problem. This makes us inquiry on which conditions should be met to guarantee the existance of a K_n subgraph. Turán [Tur41] was able to give an exact upper bound the number of edges that can be included in an undirected graph in order for it to not contain a clique of a given size. It is one of the central results of **extremal graph theory**, an area studying the largest or smallest graphs with given properties.

Definition 4.1: Edge maximum graph with no K_r

Let G be a graph. We say that G has the **edge maximum with no k_r subgraph on n vertices property**, written as $\text{ex}(n, r)$, if for all graphs G' with $|V(G)| = |V(G')|$ and $|E(G)| < |E(G')|$ it holds that G' has K_r as a subgraph.

By definition, if G is a graph on n vertices such that $|E(G)| > \text{ex}(n, r)$ then G contains K_r as a subgraph. To prove Turán's theorem, we start by giving a generalization of the concept of bipartite graph, where, instead of partitioning the graph into two sets without edges across them, the graph gets split into more than two sets.

Definition 4.2: r-partite graph

An **r-partite graph** is a graph whose vertices can be partitioned into r independent sets. A **complete r-partite graph** is an r-partite graph with partition $(X_1, \dots, X_n)$ such that $\forall x \in X_i, x' \in X_j$ it holds that $x \sim x'$. The complete r-partite graph with partitions of $n_1, \dots, n_r$ vertices is denoted with $K_{n_1,\dots,n_r}$.

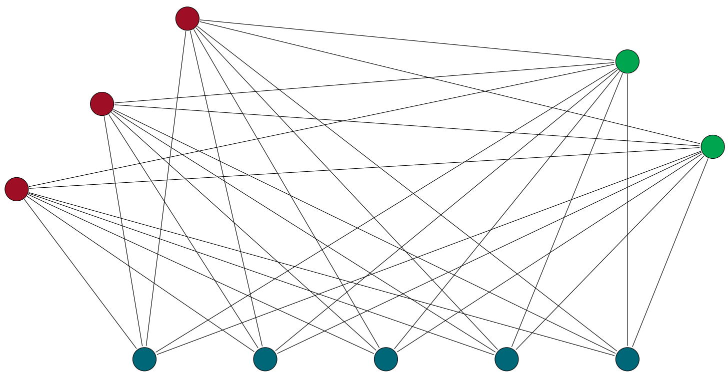


Figure 4.1: The complete 3-partite graph $K_{3,5,2}$

Proposition 4.1

Any $(r-1)$-partite graph doesn't contain K_r as a subgraph.

Proof. Let $X_1, \dots, X_{r-1}$ be the partition on G. By way of contradiction, suppose that G contains K_r as a subgraph. Let $Y = \{y_1, \dots, y_r\}$ be the set of vertices inducing such subgraph, i.e. $G[Y] = K_r$. By the pigeonhole principle, at least two vertices y_i, y_j must fall inside the same independent set, thus no edge between them can exist, raising a contradiction. □

Definition 4.3: Turán graph

The **Turán graph**, written as $T_{n,r}$ is a complete r-partite graph on n vertices with a partition $X_1, \dots X_r$ such that for each $i, j \in [r]$ it holds that $||X_i| - |X_j|| \leq 1$. We denote with $t(n, r)$ the number of edges in $T_{n,r}$.

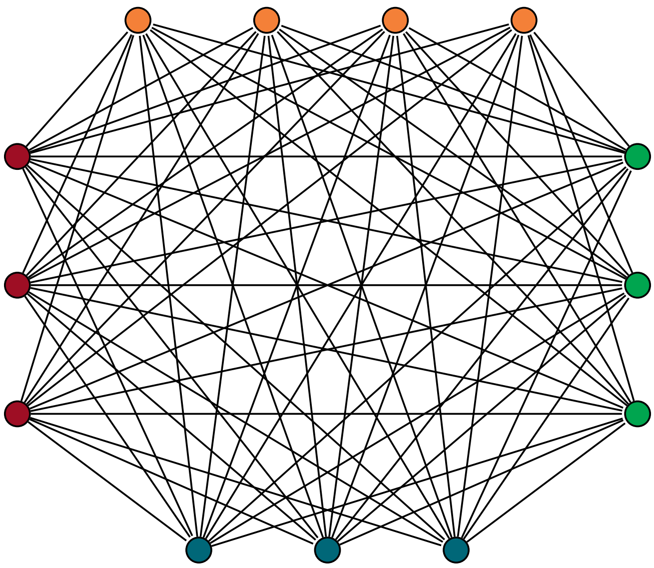


Figure 4.2: The Turán graph $T_{13,4}$.

We observe that $t(n, r) \sim \binom{n}{2}\binom{r-1}{r}$ for $n \to +\infty$. Moreover, we also notice that, by definition, if $n \leq r$ then $T_{n,r}$ can be parititoned into n independent sets that contain 1 vertex and $r - n$ empty sets, implying that $T_{n,r} = K_n$. From the previous proposition, we know that the Turán graph gives a *lower bound* on the maximum number of edges that a graph can have before inducing K_n as a subgraph. Turán's theorem not only proved that his graph also gives an upper bound, but he also proved that any graph having the $\mathrm{ex}(n, r)$ property must be equal to his graph.

Theorem 4.1: Turán's theorem

If G is a graph with the $\mathrm{ex}(n, r)$ property then $G = T_{n,r-1}$.

Proof. The proof is split in two parts: proving that if G is $(r-1)$-partite then it must be equal to Turán's graph and the proving that it is $(r-1)$-partite.

Claim 1: if G is $(r-1)$-partite then $G = T_{n,r-1}$.

Proof of Claim 1. Since G is maximum, we know that G must be a complete $(r-1)$-bipartite graph, otherwise we could add an edge between two of the $r-1$ independent sets and still guarantee the non-existence of an $(r-1)$-clique.

Let $X_1, \dots, X_{r-1}$ be the partition on G and suppose by way of contradiction that $G \neq T_{n,r-1}$. Then, there are two sets X_i, X_j such that $||X_i| - |X_j|| > 1$. Hence, we can assume without loss of generality that $|X_i| < |X_j|$. Fix a vertex $y \in X_j$. Let G' be the graph obtained by removing all the edges from y to X_i, moving y to X_i and then adding an edge from y to each vertex that is left in X_j. Since G doesn't contain K_r, by construction G' also doesn't. However, G' has strictly more edges than G, raising a contradiction.

$$|E(G')| = |E(G)| - |X_i| + |X_j| - 1 > |E(G)|$$

□

Claim 2: For any $n' \geq r'$ then $t(n', r') = t(n', r') + (n' - r')(r' - 1) + \binom{r'}{2}$.

Proof of Claim 2. Let $X_1, \dots, X_{r'}$ be the partition of $T_{n',r'}$. Since $n' \geq r'$, for each $i \in [r']$ we can fix a vertex $z_i \in X_i$. Let $Z = \{z_1, \dots, z_{r'}\}$. Since $T_{n',r'}$ is a complete r-partite graph, we get that $G[Z] = K_{r'}$ and that $G - Z = T_{n'-r',r'}$. Hence, we have that $|G[Z]| = \binom{r'}{2}$ and $|T_{n'-r',r'}| = t(n' - r', r')$.

Additionally, for each $i \in [r']$ each vertex $z \in X_i$ has exactly $r - 1$ neighbors in Z – that being $z_1, \dots, z_{i-1}, z_{i+1}, \dots, z_{r'}$. The union of $G[Z]$, $G - Z$ and these final edges compose the graph $T_{n',r'}$, concluding that:

$$t(n', r') = t(n', r') + (n' - r')(r' - 1) + \binom{r'}{2}$$

□

We'll now prove that G is $(r-1)$-partite. We proceed by induction on n. Our base case is given by $n \leq r-1$: in this case we trivially have that $G = K_n = T_{n,r-1}$, thus $|E(G)| = t(n, r-1)$. Assume the inductive hypothesis and consider the case $n > r-1$. Using the same argument as Claim 1, we know that, we know that G must be a complete $(r-1)$-bipartite graph. Hence, there must be an induced $(r-1)$-clique. Let $Y = \{y_1, \dots, y_{r-1}\}$ be the vertices that induce such clique, i.e. $G[Y] = K_{r-1}$. Since G has no r-clique, any $v \in V(G) - Y$ has at most $r-2$ neighbors in Y.

Since $G - Y$ is a graph on $n-(r-1)$ vertices without K_r, we know that $|E(G-Y)| \leq |E(H)|$ where H is a graph with the $\mathrm{ex}(n-r, r-1)$ property. By inductive hypothesis, we know that $H = T_{n-(r-1),r-1}$, hence $|E(G-Y)| \leq |E(H)| = t(n-(r-1), r-1)$. Putting it all togheter, we get that:

$$|E(G)| \leq \binom{r-1}{2} + (n-(r-1))(r-2) + t(n-(r-1), r-1)$$

By Claim 2, we conclude that $|E(G)| \leq t(n, r-1)$. Moreover, by Proposition 4.1, we know that the Turán graph $T_{n,r-1}$ gives a lower bound on the edges of G since the latter has the $\mathrm{ex}(n, r)$ property, concluding that $|E(G)| = t(n, r-1)$. Hence, we conclude that the previous inequality is actually tight.

$$|E(G)| = \binom{r-1}{2} + (n-(r-1))(r-2) + t(n-(r-1), r-1)$$

Moreover, this also implies that any $v \in V(G) - Y$ has actually exactly $r-2$ neighbors in Y. For each $i \in [r-1]$, let $X_i = \{x \in V(G-Y) \mid x \nsim y_i\} \cup \{y_i\}$.

Claim 3: For all $i \in [r-1]$ the set X_i is an independent set.

Proof of Claim 3. By way of contradiction suppose that there are two vertices $z, z' \in X_i$ such that $z \sim z'$. Then, since z, z' are adjacent to all the vertices in $Y - \{y_i\}$, we get that $G[(Y - \{y_i\}) \cup \{z, z'\}] = K_r$, raising a contradiction. □

Since G it's easy to see that $X_1, \dots, X_{r-1}$ partition G into $(r-1)$-independent sets. By Claim 1, this concludes that $G = T_{n,r-1}$. □

Corollary 4.1: Existence of K_r as a subgraph

Any graph G such that $|E(G)| > t(n, r-1)$ contains K_r as a subgraph.

4.2 Existence of k-connected subgraphs

After discussing extremal edge conditions that guarantee the existence of K_n as a clique, we shift our focus to extremal conditions that guarantee the existence of a k-connected subgraph. Conditions that guarantee the existence of such subgraphs are of particular interest – this shouldn't come as a surprise since we showed how k-connectivity has an important role in inducing structures in graphs. We discuss Mader [Mad72] extremal result on this topic.

Theorem 4.2: Mader's theorem

If G is a graph such that $n \geq 2k$ and $|E(G)| \geq (2k-1)(n-k)$ then G contains a k-connected subgraph.

Proof. We proceed by induction on n. When $n = 2k$, we have that:

$$|E(G)| \geq (2k-1)(n-k) = (2k-1)k = \frac{(2k-1)2k}{2} = \binom{2k}{2}$$

meaning that G has the maximum number of edges possible, concluding that $G = K_{2k}$ and thus that it is $(2k-1)$-connected (hence also k-connected). Assume the inductive hypothesis and consider the case with $n > 2k$ vertices. We may assume that G is k-connected, since otherwise the thesis trivially holds.

Claim 1: if $\delta(G) < 2k$ then G has a k-connected subgraph.

Proof of Claim 1. Suppose that $\delta(G) < 2k$. Then, there is at least a vertex v such that $\deg(v) \leq 2k-1$, implying that $G - v$ has $n - 1 \geq 2k$ vertices and edge count:

$$|E(G-v)| \geq |E(G)| - (2k-1) \geq (2k-1)(n-k) - (2k-1) = (2k-1)((n-1)-k)$$

Hence, we can apply the inductive hypothesis, concluding that $G - v$ has a k-connected subgraph and thus G also does. □

Clam 1 allows us to also assume that $\delta(G) \geq 2k$, since otherwise the thesis holds. Since G is also not k-connected, there is an (A, B)-separation such that $A - B, B - A = \varnothing$ and $|A \cap B| < k$. Since they form a separation, there are no edges that go from $A - B$ to $B - A$. Hence, since $\delta(G) \geq 2k$ and $|A| \neq \varnothing$, there are at least $2k + 1$ vertices in A. The same also holds for B, thus $|A|, |B| \geq 2k + 1$.

Claim 2: it holds that $|E(G[A])| \leq (2k-1)(|A|-k)-1$ or $|E(G[B])| \leq (2k-1)(|B|-k)-1$

Proof of Claim 2. By way of contradiction, suppose that the claim doesn't hold. Then, since there are no edges from $A - B$ to $B - A$, we have that:

$$\begin{aligned}|E(G)| &= |E(G[A])| + |E(G[B])| - |E(G[A \cap B])| \\ &\leq |E(G[A])| + |E(G[B])| \\ &\leq (2k-1)(|A| + |b| - 2k) - 2\end{aligned}$$

By the inclusion-exclusion principle, we know that $|A|+|B| = |A \cup B|+|A \cap B|$, implying that:

$$\begin{aligned}|E(G)| &\leq (2k-1)(|A|+|b|-2k)-2 \\ &= (2k-1)(|A \cup B|+|A \cap B|-2k)-2 \\ &< (2k-1)(n+(k-1)-2k)-2 \\ &< (2k-1)(n-k)\end{aligned}$$

which contradicts the initial assumption the amount of edges in G. □

Since $n > |A|, |B| \geq 2k+1$, Claim 2 implies that the inductive hypothesis can be applied on $G[A]$ or $G[B]$, concluding that G also contains a k-connected subgraph. □

Together with Proposition 3.4, Mader's theorem immediately gives us the following corollary.

Corollary 4.2

Any graph G with $n \geq 6$ and $|E(G)| \geq 5n$ contains K_4 as a topological minor.

Bollobás and Thomason [BT96] proved that the above corollary can be extended to any k-clique, with a small blow-up on the edge bound.

Theorem 4.3: Bollobás-Thomason theorem

There is a contant $c > 0$ such that for all $r \in \mathbb{N}$ any graph G with $|E(G)| \geq c\mathbb{R}^2 n$ contains K_r as a topological minor.

Proof. Omitted. □

We observe that the edge bound condition given by Mader's theorem is not strict, meaning that it is not known if there are graphs with $(2k-1)(n-k)-1$ edges that don't contain a k-connected subgraph.

Open Question 4.1

For any $k \in \mathbb{N}$, what is the smallest value c_k such that all graphs with n vertices and $|E(G)| \geq c_k n + f(k)$ for some function $f(k)$ contain a k-connected subgraph?

Madir's theorem gives the bound $c_k \leq 2k-1$ and there are known graphs that give lower bounds, such as the **split graph** formed of a $(k-1)$-clique and an independent set of $n-(k-1)$ vertices that is fully connected to the clique.

Definition 4.4: Split graph

A **split graph** is a graph whose vertices can be partitioned into a clique and an independent set. A **complete split graph** is a split graph whose independent set is fully connected to the clique.

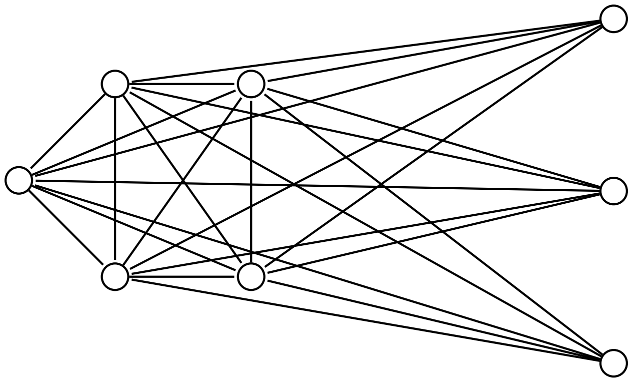


Figure 4.3: The complete split graph with a 5-clique and an independent set of 3 vertices

Any complete split graph with a $(k-1)$-clique and an independent set of $n-(k-1)$ vertices has as edge count:

$$|E(G)| = (k-1)(n-(k-1)) + \binom{k-1}{2}$$

Moreover, such split graphs are clearly not k-connected since removing the $(k-1)$-cliques disconnects it, implying that $c_k \geq k-1$.

4.3 Existence of cliques as minors

In Section 3.2 we introduced the concept of topological minor through subdivisions. In this section, we give a more general definition of the concept of **minors**, a fundamental structure studied in graph theory.

Consider a graph G and an edge $xy \in E(G)$. A **contraction** of xy on G refers to the process of merging the two vertices x and y into a new vertex z, moving all the edges of x and y to z and deleting any **parralel edge** (or *multi-edges*) that formed during the process. The contacted graph is denoted with G/xy.

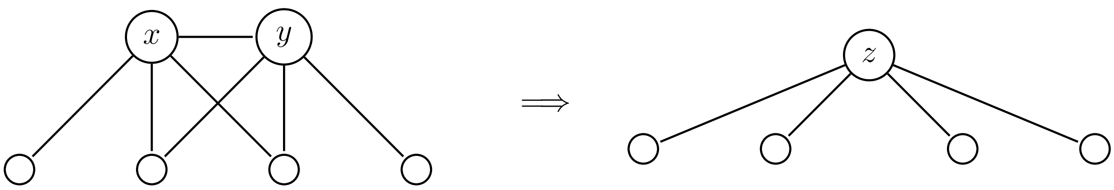


Figure 4.4: Example of edge contraction.

Definition 4.5: Minor

Let G be a graph. A graph H is a **minor** of G if H can be obtained from G by removing vertices, edges and by contracting edges.

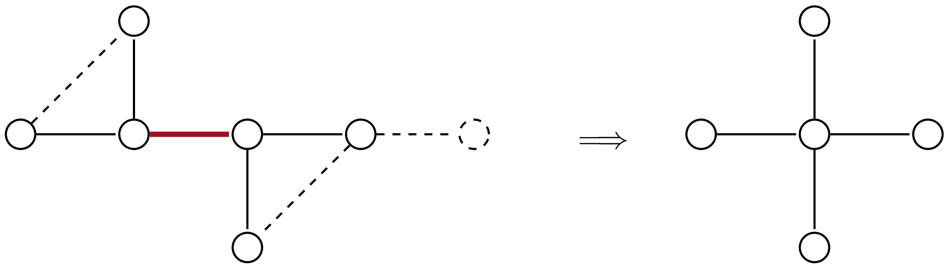
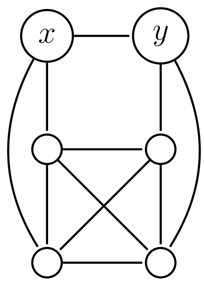


Figure 4.5: By removing the dashed edges and vertices and contracting the red edge, we get the cross graph as a minor

It's easy to see that topological minors are minors. Consider a graph G that has H as topological minor, meaning that it contains a graph H' that is a subdivision of H. If we remove from G all the edges and vertices that are not part of H', we are left with a subdivision of H. Through edge contraction, each edge subdivision can be reversed, obtaining H as a minor. This allows us to form some sort of **structure hierarchy** in graphs.

$$H \text{ ind. subgraph of } G \implies H \text{ subgraph of } G \implies H \text{ top. minor of } ofG \implies H \text{ minor of } G$$

It's easy to see that the first three levels of this hierarchy are strict inclusions, i.e. not all subgraphs are induced and not all topological minors are subgraphs. The same also holds for the last level of the hierarchy. For instance, consider the following graph.



If we contract the edge xy, we obtain K_5 as a minor. However, K_5 is not a topological minor of G: every subdivision of K_5 has at least 5 vertices of degree 4, but G has only 4 vertices of degree 4.

Theorem 4.4: Extremal conditions for k-cliques as minors

Let p be an integer $2 \leq p \leq 7$. If G is a graph such that $|E(G)| \geq (p-2)n - \binom{p-1}{2} + 1$ then G contains K_p as minor.

Proof. The theorem is proved individually for each value of p.

- For $p = 2$, we have that:

$$|E(G)| \geq (2-2)n - \binom{2-1}{2} + 1 = 1$$

Since a single edge corresponds to K_2, we conclude that G contains K_2 as a subgraph and hence as a minor.

- For $p = 3$, we have that:

$$|E(G)| \geq (3-2)n - \binom{3-1}{2} + 1 = n - 1 + 1 = n$$

Since a forest has at most $n-1$ edges, we get that G must contain at least a cycle. This cycle can be contracted into K_3, concluding that G contains K_3 as a minor.

- For $p = 4$, we have that:

$$|E(G)| \geq (4-2)n - \binom{4-1}{2} + 1 = 2n - 3 + 1 = 2n - 2$$

We proceed by induction on $n+m$. We observe that if $n = 3$ then $|E(G)| \geq 2 \cdot 3 - 2 = 4$ which is impossible with only 3 vertices. Hence, to satisfy the edge bound n must be at least 4. Therefore, for the base case assume that $n = 4$, which means that

$$|E(G)| \geq 2 \cdot 4 - 2 = 6 = \binom{4}{2}$$

concluding that $G = K_4$, thus it contains itself as a minor. Assume the inductive hypothesis and consider the case $n + m + 1$. Suppose that $|E(G)| > 2n - 2$. Then, for any edge $e \in E(G)$ it holds that $|E(G-e)| \geq 2n - 2$. By inductive hypothesis, we conclude that $G - e$ contains K_4 as a minor, hence G also does.

Consider now the case where $|E(G)| = 2n-2$. Then, for any edge $e \in E(G)$ we have that G/e has $n-1$ vertices. Again, if $|E(G/e)| \geq 2(n-1) - 2$ then by inductive hypothesis G/e contains K_4 as a minor, hence G also does. Therefore, we may assume that $|E(G)| = 2n - 2$ and that $|E(G/e)| < 2(n-1) - 2$ for all $e \in E(G)$.

Claim 1: every edge $e \in E(G)$ is contained inside two distinct 3-cliques.

Proof of Claim 1. Fix an edge $e \in E(G)$. Since $|E(G)| = 2n - 2$ and $|E(G/e)| < 2(n-1) - 2$, at least 3 edges must have been deleted during the contraction. One of those edges is clearly the edge e, while the other two must be parallel edges that have been removed afterwards. However, this can happen only if the edge e is contained inside two distinct 3-cliques (see Figure 4.6). By applying the same reasoning on all edges, we conclude the claim. □

Since every edge of G is contained inside two 3-cliques, we get that $\delta(G) \geq 3$. By way of contradiction, suppose that $\delta(G) \geq 4$. Then, by the Handshaking lemma we get that:

$$|E(G)| \geq \frac{4}{2}n = 2n$$

contradicting the assumption for which $|E(G)| = 2n-2$. Thus, we get that $\delta(G) = 3$, implying that there is at least one vertex $v \in V(G)$ such that $\deg(v) = 3$.

Claim 2: $N(v) \cup \{v\} = K_4$

Proof of Claim 2. By way of contradiction, suppose that there are two vertices $x, y \in N(v)$ such that $xy \notin E(G)$. By Claim 1, the edge vx must be contained inside two distinct 3-cliques. However, since $xy \notin E(G)$, these two cliques cannot contain both x and y. Without loss of generality, suppose that one of the first cliques is formed by the vertices $\{v, x, z\}$, where z is the third neighbor of v. Then, in order to form the second clique, there must be another vertex w in order to form the clique $\{v, x, w\}$, contradicting the fact that $\deg(v) = 3$. This concludes that for all $x, y \in N(V)$ it holds that $x \sim y$, forming a 3-clique. By adding the three edges connecting v to $N(v)$, we get a 4-clique. □

Since v and its neighborhood form a K_4 clique, we conclude that G contains K_4 as a minor.

- For $p = 5$, we have that:

$$|E(G)| \geq (5-2)n - \binom{5-1}{2} + 1 = 3n - 6 + 1 = 3n - 5$$

We proceed by induction on $n + m$. We observe that if $n = 3$ then $|E(G)| \geq 3 \cdot 3 - 5 = 4$ which is impossible with only 3 vertices. The same goes for the case $n = 4$. Hence, to satisfy the edge bound n must be at least 5. Therefore, for the base case assume that $n = 5$, which means that

$$|E(G)| \geq 3 \cdot 5 - 5 = 10 = \binom{5}{2}$$

concluding that $G = K_5$, thus it contains itself as a minor. Assume the inductive hypothesis and consider the case $n + m + 1$. Suppose that $|E(G)| > 3n - 5$. Then, for any edge $e \in E(G)$ it holds that $|E(G - e)| \geq 3n - 5$. By inductive hypothesis, we conclude that $G - e$ contains K_5 as a minor, hence G also does.

Consider now the case where $|E(G)| = 3n-5$. Then, for any edge $e \in E(G)$ we have that G/e has $n - 1$ vertices. Again, if $|E(G/e)| \geq 3(n-1) - 5$ then by inductive hypothesis G/e contains K_4 as a minor, hence G also does. Therefore, we may assume that $|E(G)| = 3n - 5$ and that $|E(G/e)| < 3(n-1) - 5$ for all $e \in E(G)$.

Through the same reasoning as Claim 1 of the case $p = 4$, we get that every edge $e \in E(G)$ is contained inside three distinct 3-cliques. Since every edge of G is contained inside three 3-cliques, we get that $\delta(G) \geq 4$. By way of contradiction, suppose that $\delta(G) \geq 6$. Then, by the Handshaking lemma we get that:

$$|E(G)| \geq \frac{6}{2}n = 3n$$

contradicting the assumption for which $|E(G)| = 3n-5$. Thus, we get that $\delta(G) \leq 5$. Let v a vertex of minimum degree in G, implying that $\deg(v) \in \{4, 5\}$. We have two sub-cases:

1. $\deg(v) = 4$. Through the same reasoning as Claim 2 of the case $p = 5$ we conclude that $N(v) \cup \{v\} = K_5$.

2. $\deg(v) = 5$. Unlike the above case, we cannot directly conclude that v and its neighborhood form a 5-clique since in this case we have a total of 6 edges. However, the same reasoning still allows us to deduce that $\delta(G[N(v)]) \geq 3$ and that $\Delta(G[N(v)]) \leq 4$.

   If $\delta(G[N(v)]) = 4$ then $G[N(v)] = K_5$, concluding the proof. Hence, we may assume that $\delta(G[N(v)]) = 3$. Since $|N(v)| = 5$, the Handshaking lemma gurantees that at least one neighbor of v has degree 4, since otherwise we would have an odd number of odd-degree vertices, which is impossible. This implies that $N(v)$ is missing either 1 or 2 edges from being a 5-clique.

   If there is only one *anti-edge*, that being a missing edge, contracting any of the edges xy adjacent to it will induce that $G[N(v)]/xy = K_4$, hence $G[N(v) \cup v]/xy = K_5$. Otherwise, if there are two anti-edges, contracting any of the edges zw that are adjacent to both of them will induce that $G[N(v)]/zw = K_4$, hence $G[N(v) \cup v]/zw = K_5$.

- For $p = 6$ and $p = 7$, the proof is similar to $p = 4$ and $p = 5$

□

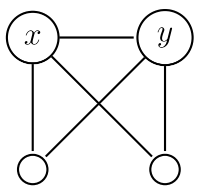


Figure 4.6: The two 3-cliques containing the edge e in the case $p = 4$ of the previous proof.

The cases with $p \in \{2, 3, 4, 5\}$ were first proven by Dirac [Dir52] and later independently by Gyori [Gyo82], while the cases $p \in \{6, 7\}$ were proven by Mader [Mad72]. Jorgensen [Jor94] proved that the theorem also extends to $p = 8$, with the exception of an infinite family of graphs for which it doesn't hold. Similarly, Song and Thomas [ST06] proved that the theorem extends to $p = 9$ with the exception of two infinite families of graphs, while Zhu [Zhu21] proved that for $p = 10$ the theorem holds except for a few infinite families. Both of the latter results were proven with the assistance of a computer.

In a series of papers, Kostochka [Kos82] and Thomason [Tho01] were able to prove the existence of a constant for which the theorem holds in every graph, with a different edge bound. This theorem can be seen as a generalization of theorem Theorem 4.3. Thomason also proved that this bound is optimal since there are some *Erdős-Rényi random graphs* that have $c'p\sqrt{\log p}n$ edges and no K_p minor, where $c' < c$. Interestingly, for these graphs it also holds that n is in function of p.

Theorem 4.5: Kostochaka-Thomason theorem

There is a constant $c > 0$ such that any graph G with $|E(G)| \geq cp\sqrt{\log p}n$ contains K_p as a minor.

Proof. Omitted. □

Seymour and Thomas conjectured that the missing condition for Theorem 4.4 to hold for every $p \geq 2$ is the $(p-2)$-connectivity of the graph.

Conjecture 4.1: Seymour-Thomas conjecture

For all $p \in \mathbb{N}$ there is a constant $N_p > 0$ such that any $(p-2)$-connected graph G with $n \geq N_p$ and $|E(G)| \geq (p-2)n - \binom{p-1}{2} + 1$ contains K_p as a minor.

The closest result for this conjecture was proven by Böhme, Kawarabayashi, Maharry, et al. [BKM+09].

Theorem 4.6: Linear connectivity and clique minors

There is a constant $c > 0$ such that for all $p \in \mathbb{N}$ there is another constant $N_p > 0$ for which any $c(p+1)$-connected graph G with $n \geq N_p$ contains K_p as a minor.

Proof. Omitted. □

4.4 Ramsey numbers

In previous sections we focused on finding extremal number of edges that guarantee the presence of some structure inside a graph. In this section, we'll discuss extremal number of nodes. In particular, we focus on the minimal number of nodes that force a graph to contain either a *t-clique* or a *t-indset*, i.e. an independent set of t nodes. This extremal number of nodes is known in the literature as **Ramsey number**, named after F. P. Ramsey, the pioneer and main researcher of this problem. Ramsey numbers are of fundamental interest in graph theory due to being one of the hardest question in the whole field.

Definition 4.6: Ramsey number

Given $t \in \mathbb{N}$, we define the **Ramsey number** $R(t)$ as the minimum value such that all graphs with at least $R(t)$ nodes contain a t-clique or a t-indset.

It's easy to see that $R(1) = 1$ and $R(2) = 2$ since a single vertex is both a 1-clique and a 1-indset and two vertices can be either a 2-clique or a 2-indset. For the value $R(3)$, instead, by definition we have that $R(3) \geq 3$ since otherwise we could never have a 3-clique or a 3-indset. Moreover, we can also prove that $R(3) > 5$ since we can find graphs with 3, 4 and 5 vertices that don't contain neither a 3-clique nor a 3-indset.

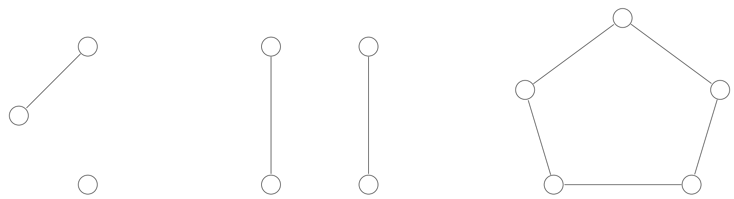


Figure 4.7: The three graphs with 3 (left), 4 (middle) and 5 (right) vertices giving the $R(3) > 5$ lower bound.

We observe that this lower bound is actually tight, meaning that $R(3) = 6$.

Proposition 4.2

$R(3) = 6$

Proof. Through the previous counter examples we know that $R(3) \geq 6$, hence it suffices to prove that $R(3) \leq 6$.

Since every graph with more than 6 vertices contains a subgraph with 6 vertices, we can restrict our focus on graphs with 6 vertices. Let G be one such graph. By way of contradiction, suppose that G contains neither a 3-clique nor a 3-indset.

Claim: $\Delta \leq 2$

Proof of the claim. Fix a vertex $v \in V(G)$. Each pair of vertices $x, y \in N(v)$ must be non-adjacent, since otherwise v, x, y form a 3-clique, raising a contradiction. Hence, $N(x)$ must be an independent set, implying that $|N(x)| < 3$ since otherwise G contains a 3-indset. $\square$

Fix a vertex $u \in V(G)$. By the claim, we know that $|N(u)| < 3$, thus $\left|\overline{N(u) \cup \{u\}}\right| = 6 - |N(u) \cup \{u\}| \geq 3$. We have two cases:

1. $\exists x, y \in \overline{N(u) \cup \{u\}}$ such that $x \nsim y$. Then, u, x, y form a 3-indset.
2. $\forall x, y \in \overline{N(u) \cup \{u\}}$ it holds that $x \sim y$. Then, $\overline{N(u) \cup \{u\}}$ contains a 3-clique.

In both cases, the initial assumption is contradicted. $\square$

For the case $t = 4$, Kalbfleisch [Kal65] proved the lower bound $R(4) > 17$ through a graph with 17 vertices that contains neither a 4-clique nor a 4-indset. This graph uses Z_17 as vertex set, meaning that $V(G) = \{x_0, \ldots, x_{16}\}$, where $x_i x_j \in E(G)$ if and only if $i - j \equiv \pm 2^i \pmod{17}$, with $0 \leq i \leq 3$.

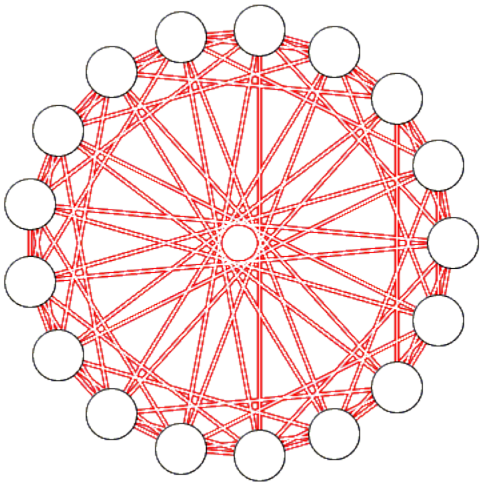


Figure 4.8: Kalbfleisch's graph giving the $R(4) > 17$ lower bound.

Again, this lower bound can be shown to be tight. To achieve this result, we first give a generalization of the Ramsey number.

Definition 4.7: Complementary graph

Given a graph G, we define the **complementary graph** as the graph $\overline{G}$ such that $V(\overline{G}) = V(G)$ and $E(\overline{G}) = \overline{E(G)}$.

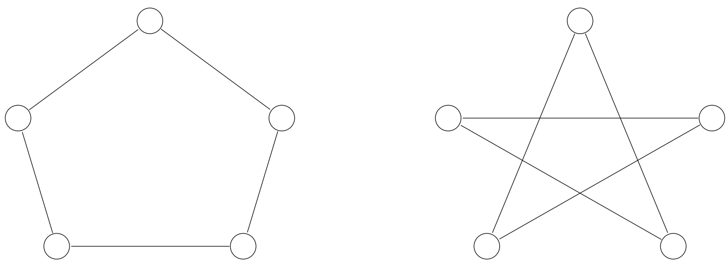


Figure 4.9: The cycle C_5 and the graph $\overline{C_5}$

It's easy to see that, by definition, the complement of a complementary graph is equal to the former graph, i.e. $\overline{\overline{G}} = G$. Moreover, any t-clique in G corresponds to a t-indset in $\overline{G}$. This allows to reformulate the definition of $R(t)$: it is the minimal value such that one between G and $\overline{G}$ contains a t-clique. We now give a more general formulation of the Ramsey number.

Definition 4.8: Generalized Ramsey number

Given $s, t \in \mathbb{N}$, we define the **Generalized Ramsey number** $R(s,t)$ as the minimum value such that for all graphs G with at least $R(s,t)$ vertices it holds that G contains an s-clique or $\overline{G}$ contains a t-clique.

By definition, we have that $R(t) = R(t,t)$. Moreover, through the previous observations, we get that $R(s,t) = R(t,s)$. This symmetry of the general Ramsey number can be used to easily derive upper bounds.

Lemma 4.1

For any $t \in \mathbb{N}$, it holds that $R(t) \leq 2R(t, t-1)$.

Proof. Let $k = R(t, t-1)$ and let G be a graph with $n = 2k$ nodes. Given any vertex $v \in V(G)$, we observe that v has either at least t neighbors or at least t non-neighbors, i.e. either $|N(v)| \geq t$ or $\left|\overline{N(v) \cup \{v\}}\right|$. We recall that $k = R(t, t-1) = R(t-1, t)$. Then, in the first case we get that $N(v)$ contains a $(t-1)$-clique or a t-indset, thus that $\overline{N(v) \cup \{v\}}$ contains a t-clique or a t-indset. In the second case, instead, we get that $\overline{N(v) \cup \{v\}}$ contains a t-clique or a $(t-1)$-indset, thus that $\overline{N(v)}$ contains a t-clique or a t-indset. □

Proposition 4.3

$R(4,3) \leq 9$

Proof. Since every graph with more than 9 vertices contains a subgraph with 9 vertices, we can restrict our focus on graphs with 9 vertices. Let G be one such graph. By way of contradiction, suppose that G contains neither a 4-clique nor a 3-indset.

Claim 1: $\Delta \leq 5$.

Proof of Claim 1. Fix a vertex $v \in V(G)$. By way of contradiction, suppose that $|N(v)| \geq 6$. Then, by Proposition 4.2 $N(v)$ contains a 3-clique or a 3-indset. In the first case, we get that $N(v) \cup v$ contains a 4-clique, thus G also does. In the second case, $N(v)$ contains a 3-indset, thus G also does. Both cases raise a contradiction. □

Claim 2: $\delta \geq 5$

Proof of Claim 2. Fix a vertex $v \in V(G)$. By way of contradiction, suppose that $|N(v)| \leq 4$. Then, we have that $\left|\overline{N(v) \cup \{v\}}\right| = 9 - |N(v) \cup \{v\}| \geq 4$. We have two cases:

1. $\exists x, y \in \overline{N(u) \cup \{u\}}$ such that $x \not\sim y$. Then, u, x, y form a 3-indset.
2. $\forall x, y \in \overline{N(u) \cup \{u\}}$ it holds that $x \sim y$. Then, $\overline{N(u) \cup \{u\}}$ contains a 4-clique.

In both cases, the initial assumption is contradicted. □

Since $\delta = \Delta = 5$, we get that 5 is 5-regular. However, by the Handshaking lemma, there cannot be an odd-regular graph with an odd number of vertices:

$$2\,|E(G)| = \sum_{v \in V(G)} \deg(v) = 5 \cdot 9 = 45$$

This concludes that such graph G cannot exist. □

Corollary 4.3

$R(4) = 18$

Proof. Follows from the above lemma, the above proposition and the lower bound given in Figure 4.8 □

Using ideas similar to the above proposition, one can prove that $R(5, 4) \leq 25$, which immediately gives the upper bound $R(5) \leq 50$. However, this upper bound is not always tight. In fact, very recenlty Angeltveit and McKay [AM24] proved that $R(5) \leq 46$. Moreover, also known that $R(5) \geq 43$ Exoo [Exo89]. Even though $R(5)$ lies between these 4 values, finding the actual value is still a very hard task. In particular, this task is even hard to reach with computer assisted proofs (there are at least $2^{\binom{43}{2}} = 2^{903}$ graphs to analyze).

For $R(6)$, the current known bounds are $102 \leq R(6) \leq 160$. To give an intuition behind the hardness of this problem, Paul Erdős said the following: "imagine an alien force, vastly more powerful than us, landing on Earth and demanding the value of $R(5)$ or they will destroy our planet. In that case, we should marshal all our computers and all our mathematicians and attempt to find the value. If instead they ask for the value of $R(6)$, we should attempt to destroy the aliens."

What about the general case? Can we even be sure that $R(t)$ exists for every value of t? This question was positively answered by Ramsey [Ram30] in the paper that gave birth to this field of graph theory.

Theorem 4.7: Ramsey's theorem

For any $t \in \mathbb{N}$, the value $R(t)$ exists and is bounded by $R(t) \leq 2^{2t-3}$ when $t \geq 2$.

Proof. We already know that $R(1) = 1$ and $R(2) = 2$, hence they exist. Fix $t \in \mathbb{N}$ with $t \geq 2$ and let G be a graph with 2^{2t-3} nodes.

Claim: there is a sequence of sets $X_1, \ldots, X_{2t-3}$ and vertices $x_1, \ldots, x_{2t-3}$ such that for each $i \in [2t-2]$ it holds that:

1. $x_i \in X_i$ and $|X_i| \geq 2^{2t-2-i}$
2. $X_{i+1} \subseteq X_i - \{x_i\}$
3. x_i is either adjacent to every vertex of X_{i+1} or non-adjacent to all vertices of X_{i+1}

Proof. We construct the sequences in an inductive way. For $i = 1$, we set $X_1 = V(G)$ and choose x_1 as any vertex of the graph. The set X_2 is then given by the largest set among $N(x_1)$ and $\overline{N(x_1) \cup \{x_1\}}$. By construction, the two last properties are satisfied. For the first property, we observe that:

$$N(x-1) \cup (\overline{N(x_1) \cup \{x_1\}}) = X_1 - \{x_1\}$$

thus:

$$\max(N(x_1), \overline{N(x_1) \cup \{x_1\}}) \geq \frac{2^{2t-3} - 1}{2} \geq 2^{2t-2-1}$$

Assume now that the sequence is defined up to some $i < 2t - 2$. The set X_{i+1} is then given by the largest set among $N(x_i) \cap X_i$ and $(\overline{N(x_i) \cup \{x_i\}}) \cap X_i$. Again, the two last properties are satisfied by construction of X_{i+1}. For the first property, we observe that:

$$(N(x_i) \cap X_i) \cup ((\overline{N(x_i) \cup \{x_i\}}) \cap X_i) = X_i - \{x_i\}$$

thus:

$$\max(N(x_i) \cap X_i, (\overline{N(x_i) \cup \{x_i\}}) \cap X_i) \geq \frac{2^{2t-2-i}}{2} = 2^{2t-2-(i+1)}$$

□

Claim: for each $i, j \in [2t - 2]$ such that $i < j$ it holds that:

- If $x_i \sim x_j$ then $x_i \sim x_{j'}$ for all $j' \in [i + 1, 2t - 2]$
- If $x_i \not\sim x_j$ then $x_i \not\sim x_{j'}$ for all $j' \in [i + 1, 2t - 2]$

Proof. Given any $j' \in [i + 1, 2t - 2]$, by construction of the sequence we have that $x_{j'} \in X_{j'} \subseteq X_j \subseteq X_{i+1}$. Hence, by construction of X_{i+1}, we know that x_i is either adjacent to all the vertices in X_{i+1} or non-adjacent to all of them. Thus, since $j \in [i + 1, 2t - 2]$ also holds, if $x_i \sim x_j$ then $x_i \sim x_{j'}$, otherwise $x_i \not\sim x_{j'}$. □

Consider now the set $K = N(x_{2t-2}) \cup \{x_1, \dots, x_{2t-2}\}$. If $|K| \geq t - 1$ then the Claim 2 ensures that $K \cup \{x_{2t-2}\}$ contains a clique of size t. $|K| \leq t - 2$, instead, by Claim 2 $\overline{N(x_{2t-2})}$ contains an independent set of size $2t - 2 - |K| \geq 2t - 2 - (t - 2) = t$. □

Ramsey's theorem gives the upper bound $R(t) \leq 2^{2t-3} = \frac{4^t}{8}$. Erdös and Szckeres [ES87] gave an asymptotic lower and upper bound on the problem:

$$\sqrt{2^t} \sim (1 + o(1)) \frac{t}{\sqrt{2}e} 2^{\frac{t}{2}} \leq R(t) \leq (1 + o(1)) \frac{4^{t-1}}{\sqrt{\pi t}} \sim 4^t$$

4.5 Solved exercises

Problem 4.1

Let $\mathrm{ex}(n, H)$ be the minimal value such that for every graph G with at least n vertices and at least $\mathrm{ex}(n, H)$ edges it holds that G contains H as a subgraph.

1. Prove that $\mathrm{ex}(n, K_{1,3}) = n + 1$
2. For all $t \geq 4$, find the value $\mathrm{ex}(n, K_{1,t})$ and prove its correctness

Solution. First, we observe that G contains $K_{1,t}$ as a subgraph if and only if G has a vertex of degree at least t. This also implies that $n \geq t + 1$ implicitely holds. Then, we also observe that, by the Handshaking lemma, if $|E(G)| = n + 1 = \frac{n}{2}(3 - 1) + 1$ then it's guaranteed that at least one vertex will have degree greater than 2. Hence, we get that $\mathrm{ex}(n, K_{1,3}) \leq n + 1$. Then lower bound $\mathrm{ex}(n, K_{1,3}) > n$, instead, is given by the cycle C_n, which has n edges and no vertex of degree at least 3. This concludes that $\mathrm{ex}(n, K_{1,3}) = n + 1$. We now generalize the upper bound to any $t \geq 3$.

Claim 1: for all $t \geq 3$ it holds that $\mathrm{ex}(n, K_{1,t}) \leq \frac{n}{2}(t - 1) + 1$.

Proof of Claim 1. Fix $t \geq 3$. Let G be any graph such that $|E(G)| \geq \frac{n}{2}(t - 1) + 1$. Through some algebraic manipulation we get that:

$$\begin{aligned}
\max_{v \in V(G)} \deg(v) &\geq \underset{v \in V(G)}{\mathrm{avg}} \deg(v) \\
&= \frac{1}{n} \sum_{v \in V(G)} \deg(v) \\
&= \frac{2\,|E(G)|}{n} \\
&= \frac{2}{n}\left(\frac{n}{2}(t-1) + 1\right) \\
&= t - 1 + \frac{2}{n} \\
&> t - 1
\end{aligned}$$

Hence, the vertex with maximum degree has degree at least t, concluding that G contains $K_{1,t}$ as a subgraph. □

For the lower bound, we generalize the result through an almost complete bipartite graph instead of a cycle.

Claim 2: for all $t \geq 3$ it holds that $\mathrm{ex}(n, K_{1,t}) \geq \frac{n}{2}(t - 1) + 1$.

Proof of Claim 2. Fix $t \geq 3$. Let G be the bipartite graph with bipartition (X, Y) such that $X = \{x_1, \ldots, x_{t-1}\}$, $Y = \{y_1, \ldots, y_{t-1}\}$ and for all $i \in [t - 1]$ we have that $N(x_i) = Y - \{y_i\}$ (which also implies that $N(y_i) = X - \{x_i\}$). By construction, every vertex has degree $t-1$, hence $K_{1,t}$ is not a subgraph of G. Moreover, given that $n = 2(t-1)$, we have that:

$$|E(G)| = (t-1)^2 = \frac{2(t-1)}{2}(t-1) = \frac{n}{2}(t-1)$$

concluding that $\text{ex}(n, K_{1,t}) > \frac{n}{2}(t-1)$ □

□

Problem 4.2

Let $\text{ex}(n, H)$ be the minimal value such that for every graph G with at least n vertices and at least $\text{ex}(n, H)$ edges it holds that G contains H as a subgraph. Find the value of $\text{ex}(n, P_4)$ and prove its correctness.

Solution. The lower bound $\text{ex}(n, P_4) > n$ is easily given by a graph made of two K_3 components. We prove that this bound is optimal, i.e. that $\text{ex}(n, P_4) = n + 1$.

Let G be a graph with $|E(G)| \geq n + 1$. By way of contradiction, suppose that G doesn't contain P_4 as a subgraph. Since any forest graph has $n - 1$ edges, we know that G must have at least a component with a cycle. Moreover, we observe that every component containing a cycle must have exactly 3 nodes, otherwise it would contain P_4 as a subgraph. Hence, every component of G must be either a 3-clique or a tree. Let $C_1, \dots, C_k$ be the 3-clique components of G and let $T_1, \dots, T_h$ be the tree components of G. We have that:

$$\begin{aligned}
|E(G)| &= \sum_{i \in [k]} |E(C_i)| + \sum_{j \in [h]} |E(T_j)| \\
&= 3k + \sum_{j \in [h]} |V(T_j)| - 1 \\
&= 3k + \sum_{j \in [h]} |V(T_j)| - \sum_{j \in [h]} 1 \\
&= 3k + (n - 3k) - h \\
&= n - h
\end{aligned}$$

Since $h \geq 0$, we get that $|E(G)| \leq n$, raising a contradiction. This concludes that G must contain P_4 as a subgraph. □

Problem 4.3

Let $R(H)$ be the minimal value such that for every graph G with at least $R(H)$ vertices it holds that G contains H as a subgraph.

1. Find the value of $R(P_4)$ and prove its correctness.
2. Find the value of $R(P_5)$ and prove its correctness.
3. Prove a general lower bound for $R(P_n)$ for all $n \in \mathbb{N}$

Proof. We start by giving a general lower bound for P_n. Let $q = \left\lfloor \frac{n}{2} \right\rfloor - 1$ and consider the graph $G = K_{n-1} \cup K_q$. It's easy to see that $P_n \not\subseteq G$. Moreover, the complement of G corresponds to $\overline{G} = K_{n-1,q}$, which also doesn't contain P_n. Hence, we get that for all $n \in \mathbb{N}$ it holds that $R(P_n) \geq n - 1 + q + 1 = n + \left\lfloor \frac{n}{2} \right\rfloor - 1$.

For the case $n = 4$, we get that $R(P_4) \geq 5$. We now prove that this bound is tight. Let G be a graph with $n = 5$ vertices. Since $G \cup \overline{G} = K_5$ and $|E(K_5)| = \binom{5}{2} = 10$, exactly one between G and $\overline{G}$ has at least 5 edges. Without loss of generality, let G be such graph. Then, G cannot be a forest since $|E(G)| \geq n$, implying that there is a cycle C.

By way of contradiction, suppose that neither G nor $\overline{G}$ contains P_4. Then, we get that $|C| = 3$ since otherwise $P_4 \subseteq C$. Let $C = c_1, c_2, c_3$ and let $V(G - C) = \{x_1, x_2\}$. We observe that $\forall i \in [3]$ it holds that $x_1 \not\sim c_i$ and $x_2 \not\sim c_i$ since otherwise $C \cup x_1$ or $C \cup x_2$ would contain P_4. However, this implies that $\overline{G} = K_{3,2}$, which contains P_4, raising a contradiction. Hence, at least one between G and $\overline{G}$ must contain P_4.

For the case $n = 5$, we get that $R(P_5) \geq 6$. We prove that this bound is also tight. Let G be a graph with $n = 6$ vertices. Let G be a graph with $n = 6$ vertices. Since $G \cup \overline{G} = K_6$ and $|E(K_6)| = \binom{6}{2} = 15$, exactly one between G and $\overline{G}$ has at least 8 edges. Without loss of generality, let G be such graph. Then, G cannot be a forest since $|E(G)| \geq n$, implying that there is a cycle C.

By way of contradiction, suppose that neither G nor $\overline{G}$ contains P_5. Then, we get that $|C| \leq 4$ since otherwise $P_5 \subseteq C$. Let $X = V(G - C)$. We have two cases:

1. $|C| = 4$. Let $C = c_1, c_2, c_3, c_4$ and let $X = \{x_1, x_2\}$. We observe that none of x_1 and x_2 can be adjacent to a vertex in C, since otherwise we get that $P_5 \subseteq G$. However, this forces that $c_1 x_1 c_2 x_2 c_3$ is a P_5 anti-path in $\overline{G}$, raising a contradiction.

2. $|C| = 3$. Let $C = c_1, c_2, c_3$ and let $X = \{x_1, x_2, x_3\}$. We observe that it cannot hold that none of the vertices of X are non-adjacent to at least a vertex of C since otherwise $P_5 \subseteq K_{3,3} \subseteq \overline{G}$. Hence, without loss of generality, assume that $x_1 \sim c_1$. Now, for all $i \in \{2, 3\}$ it must hold that $x_2 \not\sim c_i$ and $x_3 \not\sim c_i$ since otherwise $P_5 \subseteq G$. This also implies that $x_2 \sim c_1$ and $x_3 \sim c_1$, otherwise $c_2 x_2 c_1 x_3 c_3$ is a P_5 anti-path in $\overline{G}$. In turn, we get that for all $j \in \{2, 3\}$ it must hold that $x_j \not\sim x_1$, otherwise $c_2 c_3 c_1 x_i x_1$ is a P_5 path in G. However, this forces that $c_2 x_2 x_1 x_3 c_3$ is a P_5 anti-path in $\overline{G}$, raising a contradiction.

In both cases, we get a contradiction, concluding that P_5 must exist inside G. □

Problem 4.4

Show that for any choice of the function $c : E(K_{17}) \to \{\text{red}, \text{green}, \text{blue}\}$ there are three vertices $x, y, z \in V(K_{17})$ such that $c(xy) = c(yz) = c(xz)$, i.e. there is a K_3 subgraph formed of a single color.

Proof. Fix a vertex $u \in V(K_{17})$. Since $\deg(u) = 16$, by the Pidgeonhole Principle there must be at least $\left\lceil \frac{16}{3} \right\rceil$ edges $uv_1, \ldots, uv_6$ incident to v sharing the same color. Without loss of generality, suppose that all of these edges are colored in red. We may assume that

$\forall i, j \in [6]$ it holds that $c(v_i v_j) \neq$ red, otherwise we get a red triangle. Hence, each edge in the graph $G[X]$ with $X = \{v_1, \ldots, v_6\}$ must be colored either in green or blue. This reduces the problem to finding a K_3 subgraph in $G[X]$ that is colored in green or blue, which is equivalent to the Ramsey problem $R(3, 3)$. In particular, since $|X| = 6$, we get that $G[X]$ must contain a triangle that is green or blue. $\square$

5

Graph decompositions

5.1 Block decomposition

Graph decompositions are a way to break down a graph into simpler or more structured parts to better understand its properties, make algorithms more efficient, or solve complex problems. Decompositions are also used to characterize some graph properties. We start by discussing **block decomposition**, used to find the "weak points" in a graph.

Definition 5.1: Block

Let G be a graph. A **block** in G is a maximal conneced subset $H \subseteq G$ without a **cut vertex**, i.e. a vertex x such that $H - x$ is disconnected.

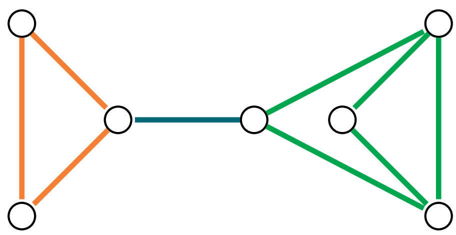


Figure 5.1: A graph and its three blocks.

We observe that any vertex and any edge taken by themselves both form a graph that has no cut vertex, but they may not be the maximal subgraph that contains it, hence we can only conclude that every edge lies inside a block. If a single edge were to be a block, then it must be a **bridge**, that being an edge that is not contained inside a cycle, otherwise the cycle would be a bigger graph without cut vertex that contains the edge.

We also observe that, by definition, any maximal 2-connected subgraph is a block. Moreover, if a block is neither a bridge nor a single vertex, it must be a 2-connected subgraph. This concludes that, in any case, a block is either:

- A single vertex
- A bridge
- A 2-connected subgraph

Proposition 5.1

Let G be a graph. If B_1 and B_2 are blocks of G, then:

- $|V(B_1 \cap B_2)| \leq 1$
- If $\exists v \in V(B_1 \cap B_2)$ then v is a cut vertex of $B_1 \cup B_2$

Proof. By way of contradiction, suppose that $|V(B_1 \cap B_2)| \geq 2$. Then, given $z \in V(B_1 \cup B_2)$, since both B_1 and B_2 have no cut vertices it must hold that $B_1 - z$ and $B_2 - z$ are still connected. Moreover, since $|V((B_1 \cup B_2) - z)| \geq 1$ and both $B_1 - z$ and $B_2 - z$ are connected, we know that $(B_1 - z) \cup (B_2 - z) = (B_1 \cup B_2) - z$ must be connected, concluding that $B_1 \cup B_2$ has no cut vertex. However, the latter is a larger graph without cut vertices that strictly contains B_1 and B_2, contradicting their maximality. This concludes that $|V(B_1 \cap B_2)| \leq 1$.

Suppose now that $|V(B_1 \cap B_2)| = \{v\}$. By way of contradiction, suppose that v is not a cut vertex of $B_1 \cup B_2$. Then, there must be a path P from $B_1 - v$ to $B_2 - v$ connecting the two sugraphs.

Claim: $B_1 \cup B_2 \cup P$ has no cut vertex

Proof of the claim. Since $B_1 - v$ and $B_2 - v$ are connected and they both contain at least a vertex of P, we know that $(B_1 - v) \cup P \cup (B_2 - v) = (B_1 \cup P \cup B_2) - v$ is connected. Consider now any other vertex $x \in V(B_1 \cup B_2 \cup P)$ with $x \neq v$. We have two cases:

1. $x \in V(B_1 \cup B_2)$. Then, it must hold that either $x \in V(B_1 - B_2)$ or $x \in V(B_2 - B_1)$ since $|V(B_1 \cap B_2)| = \{v\}$ and $x \neq v$. Without loss of generality, assume that $x \in V(B_1 - B_2)$. Since B_1 has no cut vertex we know that $B_1 - x$ is still connected. Moreover, we also know that $B_2 - x = B_2$ and that P has at least a vertex in B_2, concluding that $(B_1 - x) \cup P \cup (B_2 - v) = (B_1 \cup P \cup B_2) - x$ is connected.

2. $x \in V(P - (B_1 \cup B_2))$. Then, the graph $P - x$ is made of two path components Q_1, Q_2, one with an enpoint in B_1 and the other with an endpoint in B_2. Since $B_1 - x = B_1$ and $B_2 - x = B_2$, we know that $(B_1 \cup Q_1) - x$ and $(B_2 \cup Q_2) - x$ are both connected graphs, hence $((B_1 \cup Q_1) - x) \cup ((B_2 \cup Q_2) - x) = (B_1 \cup B_2 \cup Q_1 \cup Q_2) - x$ is also connected.

□

The claim implies that $B_1 \cup B_2 \cup P$ is larger graph without cut vertices that strictly contains B_1 and B_2, contradicting their maximality. Thus, v must be a cut vertex of $B_1 \cup B_2$. □

Definition 5.2: Block decomposition

Let G be a graph. The **block decomposition** of G is a graph $\mathcal{B}$ with vertex set $\mathcal{B} \cup \mathcal{Z}$, where:

- $\mathcal{B} = \{B_1, \ldots, B_k\}$ is the set of blocks of G
- $\mathcal{Z}$ is the set of all cut vertices of G
- For all $B_i \in \mathcal{B}$ and $z_j \in \mathcal{Z}$ it holds that $B_i \sim z_j$ if and only if $z_j \in B_i$

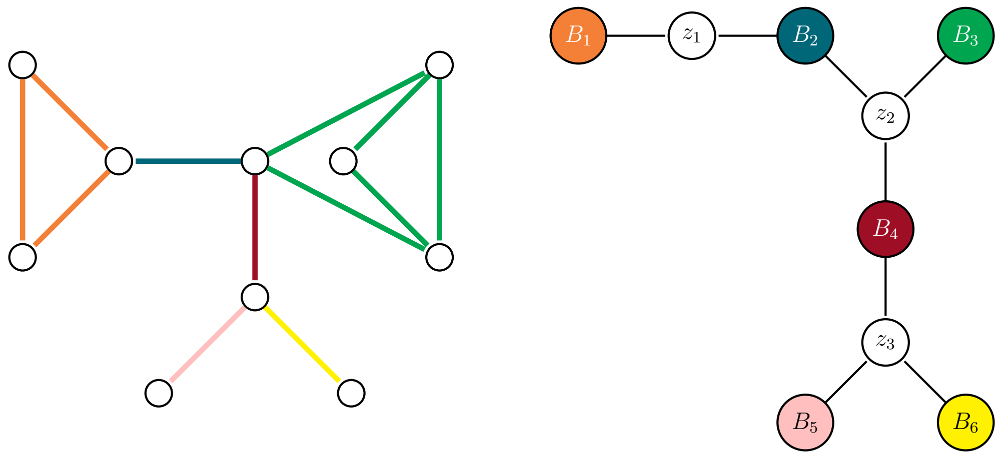


Figure 5.2: A graph and its block decomposition.

Theorem 5.1

The block decomposition of any graph is a forest. Moreover, if the graph is connected, then the block decomposition is a tree.

Proof. The second portion of the theorem follows from the definition of block decomposition and the first portion of the theorem, hence we prove only the latter.

Let G be a graph and let $\mathcal{B}$ be its block decomposition. By way of contradiction, suppose that $\mathcal{B}$ is not a forest, meaning that it contains at least a cycle. Let C be any induced cycle on $\mathcal{B}$, i.e. an induced subgraph that is a cycle without additional edges between its vertices. By definition, $\mathcal{B}$ is bipartite since the block nodes and the cut nodes form two independent sets. Thus, C must also be bipartite, meaning that it must be a cycle of even length by Theorem 1.3. Thus, we get that $C = B_1 z_1 B_2 z_2 \ldots B_{k-1} z_k B_1$. Moreover, since C is induced, we know that $z_i \notin B_j$ for $j \notin \{i, i+1\}$.

Claim: $\bigcup_{j \in [k]} B_j$ has no cut vertex

Proof. Fix $x \in \bigcup_{j \in [k]} B_j$. Without loss of generality, assume that $x \in B_1$. We have two cases:

- $x \neq z_1$. We proceed by induction on j. Since B_1 has no cut vertex, we know that $B_1 - x$ is connected. For $i > 1$, assume that $\left(\bigcup_{j \in [i-1]} B_j\right) - x$ is connected.

Since $x \notin B_{i+1}$, we know that $B_{i+1} - x$ is connected. Moreover, we know that $\bigcup_{j \in [i-1]}(B_j - x)$ and $B_{i+1} - x$ share a common vertex, that being z_i, we get that $(B_{i+1} - x) \cup \bigcup_{j \in [i-1]}(B_j - x)$ is connected.

- $x = z_1$. Argument similar to the previous point, proceeding on the other side of the cycle.

□

Since C is a cycle on $B_1, \ldots, B_k$, we know that $\bigcup_{j \in [k]} B_j$ is connected. Hence, the latter is a larger connected graph withouth cut vertices that strictly contains B_1, raising a contradiction. □

5.2 Connectivity and decompositions

We'll now focus on a type of decomposition that can be used to easily assert if a graph is 2-connected or not, i.e. the **ear decomposition**. First, we give the defintion of a concept that we've already encoutered in the proof of Menger's theorem. Let G be a graph. Given a subgraph $H \subseteq G$, an H**-path** is a path with:

- Both endpoints in H
- No internal vertex in H
- No edges in H

Definition 5.3: Ear decomposition

An **ear decomposition** of a graph G is a sequence $C, P_1, \ldots, P_k \subseteq G$ such that:

- C is a cycle
- P_i is a $(C \cup P_1 \cup \ldots \cup P_{i-1})$-path
- $G = C \cup P_1 \cup \ldots \cup P_k$

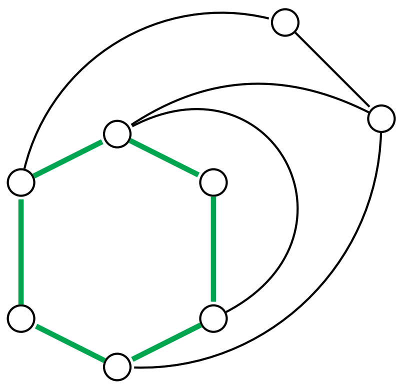


Figure 5.3: A graph represented through an ear decomposition

Ear decompositions are a way to algorithmically characterize the concept of 2-connectivity. In fact, such decomposition can exist if and only if the graph is 2-connected.

Theorem 5.2

A graph has an ear decomposition if and only if it is 2-connected.

Proof. We prove the first direction by induction, proceeding on the *length* of an ear decomposition, that being the number of ears that compose it. If G is a graph with an ear decomposition of length 0, the decomposition is composed of a single cycle, which is 2-connected. Assume the inductive hypothesis holds and consider a graph with an ear decomposition of length $i+1$. Let $C, P_1, \dots, P_{i+1}$ be such decomposition. Given the subgraph $H = C \cup P_1 \cup \dots \cup P_i$, the sequence $C, P_1, \dots, P_i$ is clearly an ear decomposition of H, thus H is 2-connected by inductive hypothesis. Then, for any vertex $x \in V(G)$ we know that $H - x$ is connected and that the two components of $P_{i+1} - x$ are connected to H, thus $G - x = (H \cup P_{i+1}) - x$ is connected, concluding that G is two connected.

Vice versa, suppose that G is 2-connected. Since $\delta \geq 2$, we know that G has a cycle. Thus, at least one subgraph of G has an ear decomposition. Let $H \subseteq G$ be the largest subgraph of G with an ear decomposition and let $C, P_1, \dots, P_k$ be such decomposition. By way of contradiction, suppose that $H \neq G$. We have two cases:

1. H differs from G due to the absence of an edge $e \in E(G)$. Then, the graph $H \cup e$ is bigger than H and has an decomposition $C, P_1, \dots, P_k, e$, contradicting the maximality of H.
2. H differs from G due to the lack of at least one vertex $v \in V(G)$. Then, since G is 2-connected, by Proposition 3.3 we know that there are at least two paths Q_1, Q_2 from v to H. However, this implies that $H \cup e$ is bigger than H and has an decomposition $C, P_1, \dots, P_k, (Q_1 \cup Q_2)$, contradicting the maximality of H.

□

Lemma 5.1

Let G be a 3-connected graph with $n \geq 5$. Then, there is an edge $e \in E(G)$ such that G/e is still 3-connected.

Proof. By way of contradiction, suppose that $\forall xy \in E(G)$ it holds that G/xy is not 3-connected. Then, by Menger's theorem, there is an (A, B) separation of order at most 2 in G/xy.

Claim 1: For any edge $e \in E(G)$ there is a non-trivial separation (A_e, B_e) of order 3 in G such that $e \subseteq A_e \cap B_e$.

Note: non-trivial here means that $A_e - B_e \neq \varnothing$ and $B_e - A_e \neq \varnothing$

Proof of Claim 1. Fix an edge $xy \in E(G)$. By the previous observation, we know that there is an (A, B) separation in G/xy such that $|A \cap B| \leq 2$. Let v_{xy} be the vertex in G/xy obtained by contracting xy. We observe that it must hold that $v_{xy} \in A \cap B$, otherwise $((A - v_{xy}) \cup \{x, y\}, (B - v_{xy}) \cup \{x, y\})$ is a separation of order 2 in G that contradicts its 3-connectivity. Moreover, it also cannot hold that $|A \cap B| < 2$, otherwise by expanding v_{xy} we get that $((A - v_{xy}) \cup \{x, y\}, (B - v_{xy}) \cup \{x, y\})$ is a separation of order 2 in G. Thus, it must hold that $A \cap B = \{v_{xy}, z\}$ for some other vertex z, concluding that $((A - v_{xy}) \cup \{x, y\}, (B - v_{xy}) \cup \{x, y\})$ gives a separation of order 3 in G with $A \cap B = \{x, y, z\}$. □

Of all the separations given by Claim 1, let (A_{xy}, B_{xy}) be the one with minimum value of $|B_{xy}|$. Let $A_{xy} \cap B_{xy} = \{x, y, z\}$. We observe that z must have at least a neighbor $u \in (B_{xy} - A_{xy}) \cup (A_{xy} - B_{xy})$, otherwise $(A_{xy} - z, B_{xy} - z)$ is a separation of order 2 in G, contradicting the 3-connectivity of G. Without loss of generality, assume that $w \in B_{xy} - A_{xy}$. By Claim 1, we now that there is also a separation (A_{zw}, B_{zw}) in G of order 3 such that $z, w \in A_{zw} \cap B_{zw}$. The two separations (A_{xy}, B_{xy}) and (A_{zw}, B_{zw}) induce a grid on the graph that partitions it into 9 areas (see Figure 5.4, we highly suggest to use the figure as a point of reference for the rest of the proof).

We observe that at least one of x and y must lie outside of $A_{zw} \cap B_{zw}$, otherwise we get that $|A_{zw} \cap B_{zw}| = 4$. Without loss of generality, assume that $x \in A_{zw} - B_{zw}$.

Claim 2: $(A_{xy} \cap B_{xy}) - A_{zw} = \varnothing$

Proof of Claim 2. We know that $A_{xy} \cap B_{xy} = \{x, y, z\}$. Moreover, we also know that $x \in A_{zw} - B_{zw}$ and $z \in A_{xy} \cap B_{xy} \cap A_{zw}$, thus $x, y \notin (A_{xy} \cap B_{xy}) - A_{zw}$. This leaves y as the only vertex necessary to be checked.

By way of contradiction, suppose that $y \in B_{zw} - A_{zw}$. Then, since $x \in A_{zw} - B_{zw}$, the edge xy would jump from $A_{zw} - B_{zw}$ to $B_{zw} - A_{zw}$, contradicting the fact that (A_{zw}, B_{zw}) is a separation. Hence, it must hold that $y \notin B_{zw} - A_{zw}$, concluding that $(A_{xy} \cap B_{xy}) - A_{zw} = \varnothing$. □

Claim 3: $(A_{xy} - B_{xy}) - A_{zw} = \varnothing$

Proof of Claim 3. By way of contradiction, suppose that $\exists u \in (A_{xy} - B_{xy}) - A_{zw}$. Since G is connected, u must be connected to x by at least one path. However, since (A_{xy}, B_{xy}) and (A_{zw}, B_{zw}) are separations, the paths cannot jump from $A_{xy} - B_{xy}$ to $B_{xy} - A_{xy}$ or from $A_{zw} - B_{zw}$ to $B_{zw} - A_{zw}$. Hence, all paths are forced to pass through $(A_{xy} \cap B_{xy}) - A_{zw}$ or $(A_{zw} \cap B_{zw}) - B_{xy}$. In particular, by Claim 1 we know that the latter is empty, thus the paths must pass through $(A_{xy} \cap B_{xy}) - A_{zw}$. However, since $A_{zw} \cap B_{zw} = \{z, w, s\}$ for some other vertex s and the paths cannot pass throguh w since $w \in B_{xy} - A_{xy}$, all paths must pass through s or z, concluding that $\{s, z\}$ is a cutset of size 2, contradicting the 3-connectivity of G. □

Claim 4: $(B_{xy} - A_{xy}) - A_{zw} \neq \varnothing$

Proof of Claim 4. By the two previous claims, we know that $(A_{xy} \cap B_{xy}) - A_{zw} = \varnothing$ and $(A_{xy} - B_{xy}) - A_{zw} = \varnothing$. Thus, if it also holds that $(B_{xy} - A_{xy}) - A_{zw} = \varnothing\varnothing$, we get that $B_{xy} - A_{xy} = \varnothing$ contradicting the fact that (A_{xy}, B_{xy}) is a non-trivial separation. □

Claim 5: $(A_{zw} - B_{zw}) - B_{zw} = \varnothing$

Proof of Claim 5. By way of contradiction, suppose that $\exists u \in (A_{zw} - B_{zw}) - B_{zw} = \varnothing$. By Claim 4, we know that $\exists v \in (B_{xy} - A_{xy}) - A_{zw} \neq \varnothing$. Since G is connected, u must be connected to v by at least one path. Through an argument similar to the one of Claim 3, we get that $\{z, w\}$ becomes a cutset of size 2, contradicting the 3-connectivity of G. □

Finally, Claims 3 and 5 conclude that B_{zw} is a strict subset of B_{xy} since $x \notin B_{zw}$, contradicting the minimality of $|B_{xy}|$. □

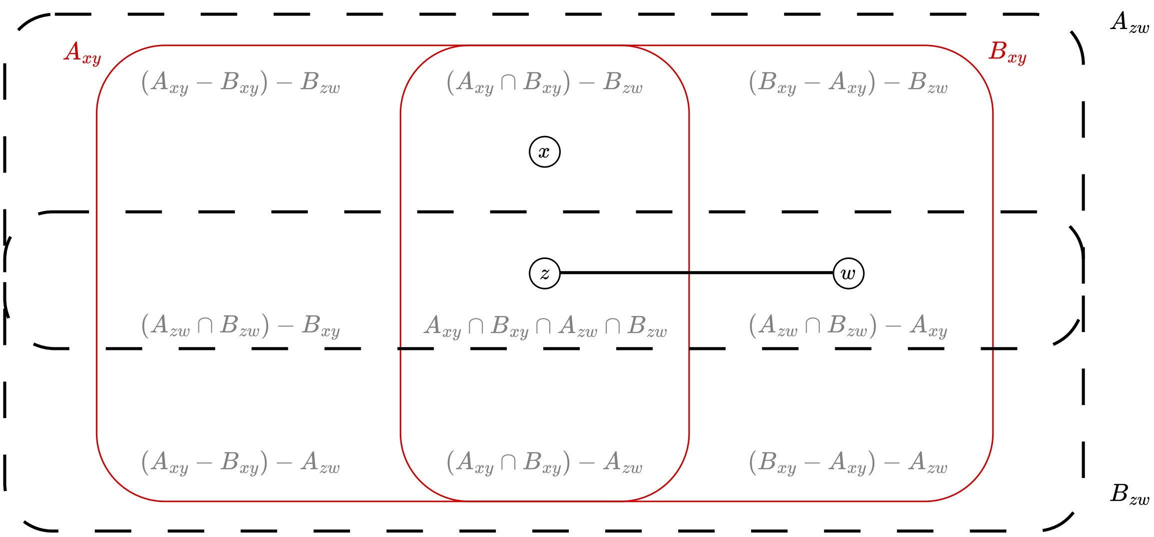


Figure 5.4: The grid induced by the proof of the lemma.

Through this lemma, we can provide an algorithmic construction of 3-connected graphs. The original formulation of such characterization was given by Tutte [Tut61]. In his original idea, it was proven that any 3-connected graph can be constructed starting from K_4, which is known to be 3-connected, and carefully "un-contract" edges that preserve 3-connectivity.

Theorem 5.3: Tutte's construction

A graph G is 3-connected if and only if there is a sequence $G_0, \dots, G_k$ such that:

1. $G_0 = K_4$ and $G_k = G$
2. For all $i \in [k-1]$ there is an edge $xy_{i+1} \in E(G_{i+1})$ such that $\deg(x), \deg(y) \geq 3$ and $G_i = G_{i+1}/xy_{i+1}$

Proof. The first direction follows from the previous lemma. The second direction can be proved by showing that all graphs in the sequence must be 3-connected. □

6 Planar graphs

6.1 Graph drawings

Suppose that there are three houses and three wells. We want to connect each house to each well with a pipe system without making pipes cross each other. It's easy to see that this question is equivalent to trying to draw the graph $K_{3,3}$ in the $\mathbb{R}^2$ plane without making two edges cross each other. After some trial and error, we quickly realize that this task is impossible to solve. When this happens, we say that the graph has not **planar drawing**. Planarity is useful in both theoretical and practical applications because it often simplifies problems. For instance, it is a crucial concept in VLSI circuit design an in computational complexity, where many NP-hard problems become solvable in polynomial time if restricted to planar graphs (e.g. the Hamiltonian path problem).

But how can we prove that something is impossible to draw? This is no easy task. In this section, we'll develop the boilerplate definitions and properties that are necessary to discuss about graph drawings. We start by listing some geometrical definitions of segments and polygons.

Definition 6.1: Segment, polygon, arc

In the space $\mathbb{R}^2$, we define:

- The **straightline segment** between two points $p, q \in \mathbb{R}^2$ as the set $\{\alpha p + (1-\alpha)q \in \mathbb{R}^2 \mid \alpha \in [0,1] \subset \mathbb{R}\}$
- A **polygon** is a subset of $\mathbb{R}^2$ that is the union of finitely many straightline segments and homeomorphic to the unit circle $S^1 = \{x \in \mathbb{R}^2 \mid ||x|| = 1\}$
- An **arc** is a subset of $\mathbb{R}^2$ that is the union of finitely many straightline segments and homeomorphic to the unit interval $[0,1] \subset \mathbb{R}$

Here, the term **homeomorphic** refers to the existance of an *homeomorphism* between the two sets, that being a continuous functions between them whose inverse is also continuous. In particular case of an arc, the images of 0 and 1 under such a homeomorphism are the **endpoints** of the arc. If P is an arc with endpoints p and q, we say that P **links** them and runs between them. The **interior** of P, written as $\mathring{P}$, is the set $\mathring{P} = P - \{p, q\}$.

Definition 6.2: Open and closed set

A set $X \subseteq \mathbb{R}^2$ is said to be:

- **Open** if for all $x \in X$ there is an $\varepsilon \in \mathbb{R}$ such that $B_\varepsilon(x) \subseteq X$.
- **Closed** if it is the complement of an open set in $\mathbb{R}^2$.

The notation $B_\varepsilon(x)$ refers to the **open ball** of radius ε centered at x, i.e. the set $B_\varepsilon(x) = \{y \in \mathbb{R}^2 \mid ||x - y|| < \varepsilon\}$. It's easy to see that by definition the union of two open sets is also open. For closed sets, instead, only the unions of *finitely many* closed sets of them is guaranteed to be closed.

Given an open set $O \subseteq \mathbb{R}^2$, we observe that concept of being linked by an arc that lies inside O induces an equivalence relation on O. The corresponding equivalence classes are called **regions** of O.

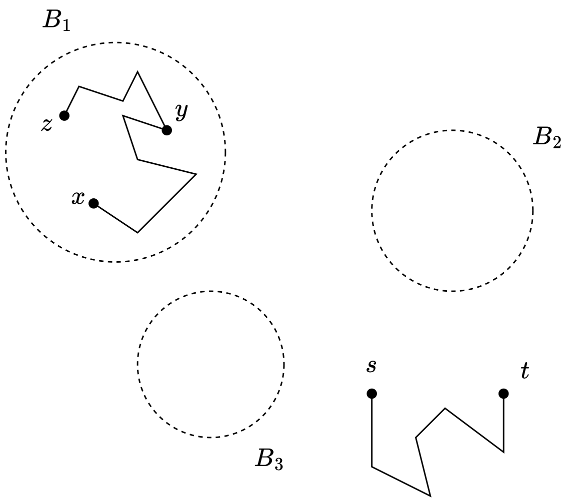


Figure 6.1: The open set $O = B_1 \cup B_2 \cup B_3$ induces 4 regions: B_1, B_2, B_3 and $\overline{O}$

We say that a closed set $X \subseteq \mathbb{R}^2$ is said to **separate** O if $O - X$ has more regions than O. The **frontier** of a set $Y \subseteq \mathbb{R}^2$, written as $\mathrm{front}(Y)$, is the set of points $y \in \mathbb{R}^2$ such that every ball centered on y intersects Y and $\mathbb{R}^2 - Y$.

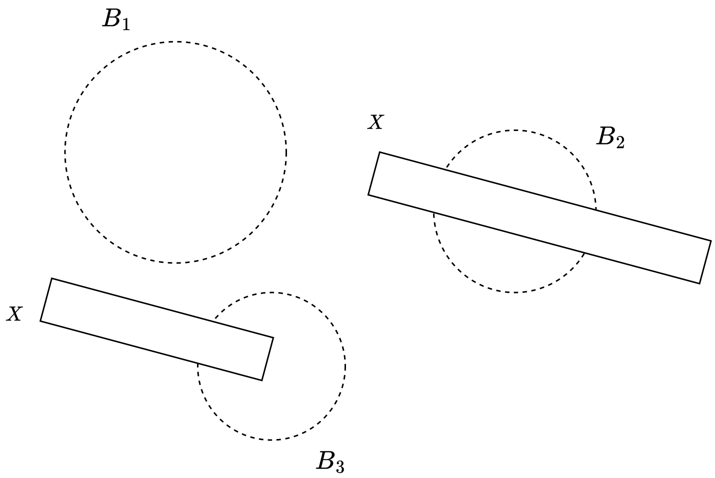


Figure 6.2: The set X separates O since the region B_2 gets split into two regions. The dotted circle is the frontier of B_1.

We're now ready to finally give a definition of planar drawing of a graph. First, we give a very intuitive definition of **plane graph**.

Definition 6.3: Plane graph

A **plane graph** is a pair $G = (V, E)$ such that:

- $V \subseteq \mathbb{R}^2$ is a set of points, called *vertices*
- E is a set of arcs between points of V, called *edges*
- There is no pair of edges in E that share both endpoints.
- The interior of each edge e is disjoint from $V \cup (E - e)$

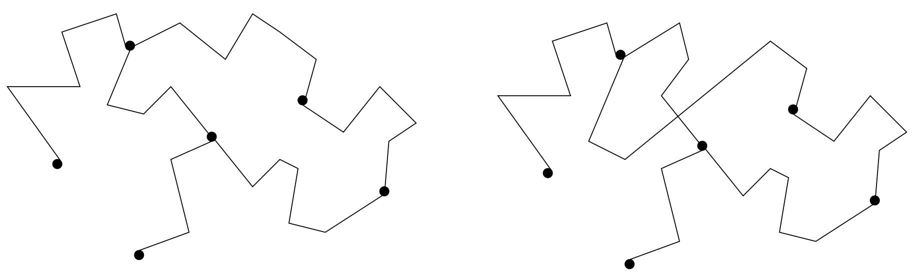


Figure 6.3: Only the left drawing respects the definition of plane graph since in the right one the interiors of two edges are not disjoint.

By definition, every plane graph G corresponds to a closed set of points in $\mathbb{R}^2$ formed by the union of its vertices and its edges. The regions of the open set $\mathbb{R}^2 - G$ are called **faces**. For every plane graph, there is exactly one unbound face, called **outer face**. The other faces are called **inner faces**.

Definition 6.4: Planar graph

A graph G is said to be **planar** if there is an isomorphism between G and a plane graph H, called *drawing of* G. In other words, a graph is said to be planar if it can be represented as a plane graph.

It's easy to see that every plane graph is isomorphic to some planar graph: we can just ignore how the edges are drawn and consider their endpoints. By definition of isomorphism, proving properties on the drawing of a planar graph also proves the property for the graph itself. This allows us to use the various topological results to derive properties of planar graphs.

6.2 Topological properties

In this section we list some topological properties of plane graphs. We start by stating a classical result in topology. Intuitively, this result affirms that any polygon drawn in $\mathbb{R}^2$ has exactly two faces: the inner region of the polygon and the outer region.

Theorem 6.1: Jordan's Curve Theorem

Let $P \subseteq \mathbb{R}^2$ be a polygon. The set $\mathbb{R}^2 - P$ has exactly two regions, both having P as frontier.

Proof. Omitted. $\square$

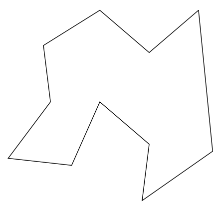


Figure 6.4: The two regions induced by Jordan's Curve Theorem.

Lemma 6.1

Let $P_1, P_2, P_3 \subseteq \mathbb{R}^2$ be pairwise disjoint polygonal arcs linking two points $x, y \in \mathbb{R}^2$. Then, it holds that:

1. $\mathbb{R}^2 - (P_1 \cup P_2 \cup P_3)$ has exactly 3 regions respectively with frontiers $P_1 \cup P_2, P_2 \cup P_3, P_3 \cup P_1$
2. $\mathring{P} \cap \mathring{P}_2 \neq \varnothing$ if P is an arc between $\mathring{P}_1, \mathring{P}_3$ and the interior $\mathring{P}$ lies on the region of $\mathbb{R}^2 - (P_1 \cup P_3)$ that contains $\mathring{P}_2$

Proof. Omitted. $\square$

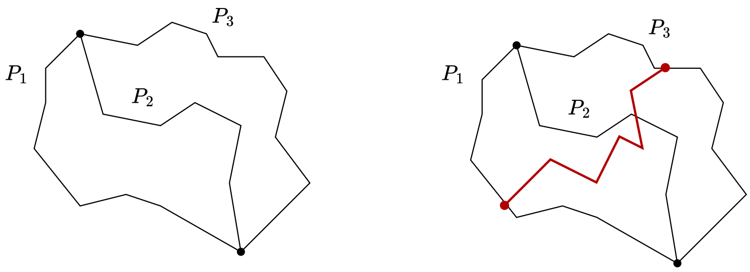


Figure 6.5: The three regions induced by the three paths of the above lemma (left) and the crossing path among them (right).

Lemma 6.2

Let $X_1, X_2 \subseteq \mathbb{R}^2$ be disjoint sets of points made upf of the union of finitely many polygonal arcs and isolated points. Let P be a polygonal arc between X_1, X_2 such that $\mathring{P}$ lies inside a region O of $\mathbb{R}^2 - (X_1 \cup X_2)$. Then, $O - P$ is a region of $\mathbb{R}^2 - (X_1 \cup X_2 \cup P)$. Moreover, the number of total regions remains the same.

Proof. Omitted. $\square$

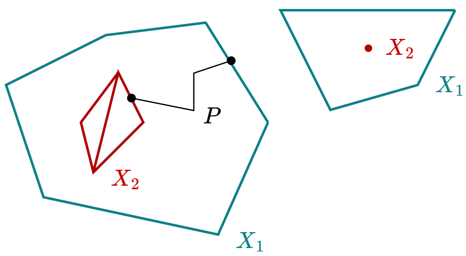


Figure 6.6: The path P connecting the two sets without creating a new region.

Proposition 6.1

Let G be a plane graph. Let f and H respectively be a face and a subplane graph of G. Then, H has a face f' containing f. Moreover, if $\text{front}(f) \subseteq H$ then $f = f'$

Proof. For the first statement, it suffices to show that if x, y are in the same face of G then by definition they're also in the same face of H: since there is a polygonal arc that connects them but avoids G, it also avoids H since $H \subseteq G$.

For the second statement, we prove the contrapositive. Consider any face f' containing f and suppose that $f \neq f'$. Then, since $f \subseteq f'$, there are at least two points $x \in f \cap f'$ and $y \in f' - f$. Since both x and y lie on f', there is a polygonal arc P from x to y must avoid $\text{front}(f')$. Moreover, since $x \in f$ but $y \notin f$ the arc P must not avoid $\text{front}(f)$, otherwise both of x and y lie on f. Hence, $\exists z \in \mathring{P} \cap \text{front}(f)$. However, we also have that $z \notin H$, concluding that $\text{front}(f) \not\subseteq H$. $\square$

Theorem 6.2

A plane forest has exactly one face.

Proof. Let G be a plane forest. We proceed by induction on $|E(G)|$. When $|E(G)| = 0$ the graph is made only of vertices, hence it has only the outer face. Assume tha property holds for all graphs with k edges and suppose that $|E(G)| = k+1$. Fix an edge $xy \in E(G)$. Then, the graph $G-xy$ is a plane forest with k edges, thus by induction has only one face. Since G is a forest, removing the arc xy splits the graph $G-e$ in two disjoint components, otherwise there would be a cycle in G. Then by Lemma 6.2, G has exactly one face, that being $O - \mathring{xy}$. □

Definition 6.5: Compactness

A subset $X \subseteq \mathbb{R}^2$ is said to be **compact** if any covering set (even infinite) of X made of open balls has a finite subcover of X

Proof. Omitted. □

Proposition 6.2

Every straightline segment in $\mathbb{R}^2$ is compact.

Theorem 6.3

Let G be a plane graph. For every edge $e \in E(G)$ it holds that:

1. For every face f of G either $e \subseteq \text{fron}(f)$ or $\mathring{e} \cap \text{fron}(f) = \varnothing$
2. If e lies on a cycle then it is the frontier of exactly two faces
3. If e doesn't lie on a cycle then it is the frontier of exactly one face

Proof. Fix an edge $e \in E(G)$ and fix $x_0 \in \mathring{e}$.

Claim 1: x_0 lies exactly on two faces if e is in a cyele, otherwise on exactly one face

Proof of Claim 1. By definition G is the union of finitely many straightline segments and points. This, there is a small disc D_0 around x_0 such that D_0 intersects at most 2 straightline segments of G. Then, in the disc D_0, the point x is either on the conjunction point of two straightline segments or on a single straightline segment (see Figure 6.7). In both cases, $D_0 - G$ has exactly two regions f_1, f_2 with x_0 on their frontier, implying that there are at most two faces of G with x_0 on their frontier.

If e is in a cycle C, by Theorem 6.1 C induces at least two faces (thus exactly two faces) f_1', f_2' with x_0 on their frontier. Otherwise, by Lemma 6.2 there is exactly one face of G that contains x_0 in the frontier, i.e. f_1', f_2' lie in the same face of G. □

Claim 2: $\forall x, y \in \mathring{e}$ and for all faces f, if $x \in \text{front}(f)$ then $y \in \text{front}(f)$

Proof. Let $f,' f$ be the faces for which $x \in \text{front}(f)$ and $y \in \text{front}(f')$.For each $z \in e$, consider the disc D_z defined as in the previous claim. Through all the infinite discs, we can cover the whole edge e. By compactness, there is a finite subset of discs given by the vertices $z_1, \dots, z_\ell$ that covers e. Following the discs through their intersections, we can define a polygonal arc from f to f', which concludes that $y \in \text{front}(f)$. □

□

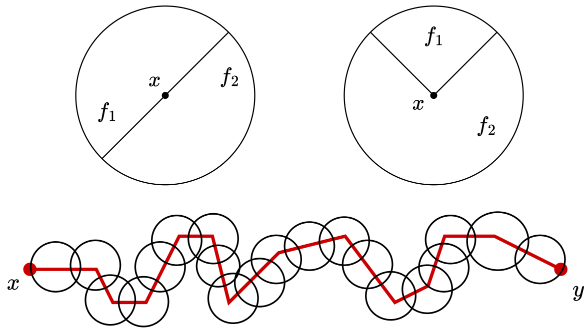


Figure 6.7: The two disc types and the cover used by the above theorem.

Consider a face f on a plane graph G and consider the points on $\text{front}(f) \cap V(G)$. There points correspond to the vertices of G that lie on the frontier of f, thus. All the other points $y \in G - \text{front}(f)$, instead, must lie on some edge $e \in E(G)$ by the previous lemma. Hence, the frontier of any face is actually nothing more than a subset of $\mathbb{R}^2$ that corresponds to a plane subgraph of G, called **boundary**.

Definition 6.6: Boundary

Let G be a plane graph and let f be a face of G. We define the **boundary** of f, written as $\text{bound}(f)$ as the plane subgraph $H \subseteq G$ inducing the face f on G.

Lemma 6.3

If G is a plane graph and two faces distinct f_1, f_2 have the same boundary then G is a cycle and $G = \text{bound}(f_1) = \text{bound}(f_2)$.

Proof. Let $H = \text{bound}(f_1) = \text{bound}(f_2)$. Since $H \subseteq G$, by both statements of Proposition 6.1 it holds that f_1, f_2 are also faces of H. Since H has more than a face, by Theorem 6.2 it contains a cycle C, where f_1, f_2 lie inside distinct faces F_1, F_2 of the cycle C. Moreover, since the two faces f_1, f_2 have all of H as their boundary, no point of H can be inside of F_1 or F_2. However, this can happen only if $H = C$, $f_1 = F_1$ and $f_2 = F_2$. Furthermore, by definition we have that $C \cup F_1 \cup F_2 = \mathbb{R}^2$, hence all points of G must lie on C, concluding that $G = C$ □

Proposition 6.3

Let G be a 2-connected plane graph. Then, every face boundary is a cycle.

Proof. We proceed by induction on n. When $n = 3$ we have that $G = K_3$, thus the statement trivially holds. Assume the inductive hypothesis holds and consider a graph with $n > 3$ nodes. By Theorem 5.2, we know that G has an ear decomposition $C, P_1, \dots, P_k$. We may assume that $k > 0$, otherwise $G = C$ and thus the statement trivially holds. Let $H = C \cup P_1 \cup \ldots \cup P_{k-1}$. Since $C, P_1, \dots, P_{k-1}$ is an ear decomposition of H, we know that the latter is 2-connected. By inductive hypothesis, we get that every face boundary of H is a cycle. Since $H \subseteq G$, we know that P_k lies of a face f of H. Thus, since $\mathrm{bound}(f)$ is a cycle and P_k lies inside it, the latter must split f in two faces f', f'' of G, each bounded by a cycle formed by P_k and a subpath of $\mathrm{bound}(f)$. □

Proposition 6.4

Let G be a 3-connected plane graph and let C be a cycle subgraph of G. Then, C is the boundary of a face of G if and only if C is an induced cycle and $G - C$ is connected.

Proof. Suppose that C is an induced cycle and that $G - C$ is connected. Let f_1 and f_2 be the two faces while considering C as a subgraph. Since $G - C$ is connected, one of the two faces cannot contain vertices of it, otherwise they wouldn't be connected. Without loss of generality, suppose that f_2 doesn't contain vertices of $G - C$. Since f_2 doesn't contain vertices of $G - C$, the only way for a portion of the latter to lie in f_2 is through the existence of an edge of $G - E(C)$, but this cannot happen since C is induced. Hence, Since no portion of $G - C$ lies in f_2, C must be the boundary of f_2.

Vice versa, suppose that C is the boundary of a face f. By way of contradiction, suppose that C is not an induced cycle, i.e. there are two vertices $x, y \in V(C)$ such that $xy \in E(G - C)$. We observe that the edge xy must pass outside of f, i.e. on another face of $\mathbb{R}^2 - C$, otherwise C wouldn't be the boundary of f. Given any pair of vertices $z, w \in V(C)$, they must still be connected in $C - \{x, y\}$ through a path P since G is 3-connected. Again, this path must pass outside of f and it cannot cross xy because P is a path of $C - \{x, y\}$ (hence it doesn't contain xy) and it cannot cross the same vertices of xy by definition of plane graph. However, this contradicts Lemma 6.1 because C, xy and P give 4 faces on G. Hence, we conclude that C must be an induced cycle.

To prove that $G - C$ is connected, fix two vertices $x, y \in V(G - C)$. By Proposition 3.3, there are 3 internally disjoint paths P_1, P_2, P_3 from x to y. Then, by Lemma 6.1, $P_1 \cup P_2 \cup P_3$ has 3 regions. Without loss of generality, assume that f lies in the region bounded by $P_1 \cup P_2$, implying that $C \cap P_3 = \varnothing$ and thus concluding that P_3 still exists in $G - C$. □

We conclude this section on general topological facts and properties with Euler's [Eul58] famous formula that relates the number of vertices, edges and faces on polygonal shapes in $\mathbb{R}^2$, which can be reformulated in terms of plane graphs.

Theorem 6.4: Euler's formula

Let G be a connected plane graph. Then, it holds that $|V(G)| - |E(G)| + |F(G)| = 2$, where $F(G)$ is the set of faces of G.

Proof. Let $|V(G)| = n, |E(G)| = m$ and $|F(G)| = \ell$. We proceed by induction on m. The base case is given when $m = n - 1$: since G is connected, it must be a plane tree, implying that it has only one face. Assume the inductive hypothesis and consider the case $m > n$. Since $m > n$, G cannot be a plane forest, thus it contains a cycle C. Given any edge $e \in E(C)$, by Lemma 1.2 we know that $G - e$ is still connected, hence by inductive hypothesis we have that $|V(G-e)| + |E(G-e)| + |F(G-e)| = 2$, thus $n - (m-1) + |F(G-e)| = 2$. By Theorem 6.3, e must in the frontier of exactly to faces f_1, f_2 of G. However, this implies that $f_1 \cup f_2 \cup \mathring{e}$ is a face of $G - e$, while all the other faces of G are unchanged in $G - e$. Hence, we get that if g is a face of G such that $g \neq f_1, f_2$ then g is also a face of $G - e$, concluding that $|F(G')| = |F(G)| - 1 = \ell - 1$ and thus that $n - (m-1) + \ell = 2$. □

6.3 Planarity conditions

After discussing all the topological properties that we need, we're ready to use them in order to show tools for determining the planarity or non-planarity of a graph, i.e. its ability to be drawn without crossing edges.

Definition 6.7: Triangulation

Let G be a planar graph. We say that G is a triangulation if it has a drawing H such that every face of H is a triangle. Equivalently, G is a triangulation if every induced cycle of G has length 3.

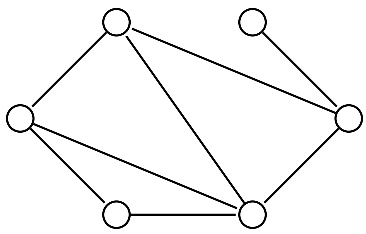


Figure 6.8: A triangulation.

For any plane graph G, we say that it is **maximally plane** if no edges cannot be added to it without breaking the no-crossing-edges condition (thus making it non-plane). From basic geometric facts, it's easy to see that every maximally plane graph must be a triangulation: every polygon can be partitioned into a finite number of triangles.

Theorem 6.5: Planar edge bound

Let G be a plane graph. Then, G has at most $3n-6$ edges. Moreover, G has exactly $3n-6$ edges when G is a triangulation.

Proof. Since every plane graph is a subgraph of a maximally plane graph and every maximally plane graph is a triangulation, we only need to show that every triangulation has $3n-6$ edges. Let G be a triangulation. Let ℓ and m respectively be the number of faces and edges of G. Since every face of G sees 3 edge and every edge of G lies in the frontier of 2 faces by Theorem 6.3, we have that $3\ell = 2m$, hence $\ell = \frac{2}{3}m$. By Theorem 6.4, we conclude that:

$$n - m + \ell = 2 \implies n - m + \frac{2}{3}m = 2 \implies m = 3n - 6$$

□

The above theorem is enough to get our first non-planar graph, that being the graph K_5. In fact, we have that:

$$|E(K_5)| = \binom{5}{2} = \frac{5 \cdot 4}{2} = 10 > 9 = 3 \cdot 5 - 6 = 3 \cdot |V(K_5)| - 6$$

What about our initial graph $K_{3,3}$? Is this edge bound enough to prove that it isn't planar? Sadly, the answer is no. In fact, we have that:

$$|E|\,(K_{3,3}) = 3 \cdot 3 = 9 \leq 12 = 3 \cdot 6 - 6 = 3 \cdot |V|\,(G) - 6$$

To prove that $K_{3,3}$ is not planar, we need a stricter edge bound that holds for planar graphs with the same properties of $K_{3,3}$, i.e. being bipartite and 2-connected.

Proposition 6.5

Let G be a bipartite 2-connected plane graph. Then, G has at most $2n-4$ edges.

Proof. Let ℓ and m respectively be the number of faces and edges of G. Since G is 2-connected, by Theorem 5.2 we know that G has an ear decomposition, hence every edge must lie inside a cycle. By Theorem 6.3, every edge of G lies in the frontier of 2 faces. Moreover, since G is bipartite, there are no cycles of lenght 3 in G. Thus, we get that every edge must lie inside a cycle with at least 4 edges and inside the frontier of 2 faces, implying that $2m \geq 4\ell$, hence $\ell \leq \frac{1}{2}m$. By Theorem 6.4, we conclude that:

$$2 = n - m + \ell = 2 \leq n - m + \frac{1}{2}m \implies 2 \leq n - \frac{m}{2} \implies m \leq 2n - 4$$

□

Corollary 6.1

The graphs K_5 and $K_{3,3}$ are not planar.

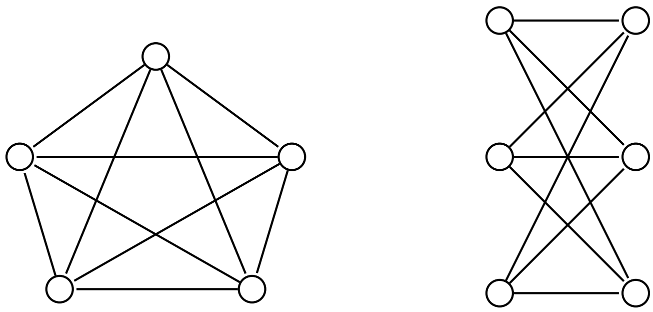


Figure 6.9: The two non-planar graphs K_5 and $K_{3,3}$.

Proposition 6.6

If G is a graph containing K_5 or $K_{3,3}$ as a topological minor then G is non-planar.

Proof. By way of contradiction, suppose that G is planar and let H be a subgraph that of G that is a subdivision of K_5 or $K_{3,3}$. Then, H must also be planar. However, this implies that we could carefully reverse the subdivisions to get a planar representation of K_5 or $K_{3,3}$, raising a contradiction. □

The above result implies that no planar graph contains K_5 or $K_{3,3}$ as a topological minor. Surprisingly, the reverse also holds and it also does for generic minors, not only topological ones. This allows us to reduce the concept of planarity to the non-existence of exactly two graphs minors, which are referred to as **forbidden minors** of the class of planar graphs.

Definition 6.8: Forbidden minor

Let $\mathcal{G}$ be a class of graphs. We say that the graphs $H_1, \ldots, H_k$ are the forbidden minors of $\mathcal{G}$ when for each graph G it holds that $G \in \mathcal{G}$ if and only if G doesn't contain none of $H_1, \ldots, H_k$ as a minor.

To prove that K_5 and $K_{3,3}$ are the forbidden minors of planar graphs, we have to use a different definition of graph minor.

Theorem 6.6: Minor (2nd definition)

Let G and H be two graphs. Then, G contains H as a minor if and only if there is a set of subsets $\mathcal{X} = \{X_v \mid X_v \subseteq V(G), v \in V(H)\}$, called **bags**, such that:

1. For each $u, v \in V(H)$ it holds that $X_u \cap X_v = \varnothing$
2. For each $v \in V(H)$ the graph $G[X_v]$ is connected
3. For each $uv \in E(H)$ there is at least one edge $e_{uv} \in E(G)$ from X_u to X_v

Proof. Suppose that we have a set of bags $\mathcal{X}$ that respect the three conditions. For each vertex $v \in V(H)$, let T_v be a spanning tree on $G[X_v]$. For each $uv \in E(H)$, fix an edge $e_{uv} \in E(G)$ that goes from X_u to X_v. To obtain H from G, we can remove the edge set F and then contract each spanning tree, where:

$$F = E(G) - \left(\bigcup_{v \in V(H)} E(T_v) \cup \bigcup_{uv \in E(H)} e_{uv} \right)$$

Vice versa, suppose that G contains H as minor. Then, there is a sequence of contractions and vertex or edge deletions that yields H starting from G. Let H' be the subgraph of G constructed as follows:

- $V(H')$ is the set of vertices of G that were not removed to obtain H from G, together with the endpoints of the edges that were contracted to obtain H
- $E(H')$ contains the edges of G that were contracted to obtain H

We observe that, by construction, the number of components in H' is exactly $|V(H)|$. For each $v \in V(H)$, we set the bag X_v as the connected component of H' containing v. By definition, all components of H' are pairwise disjoint and internally connected. Then, the edges of H obtained from G are the edges e_{uv} connecting each component of H', i.e. X_u and X_v. $\square$

Proposition 6.7

Let G and H be two graphs. If $\Delta(H) \leq 3$ then G contains H as a topological minor if and only if H as a minor.

Proof. We already know that each topological minor is also a minor, thus the first implication immediately follows. Suppose now that G contains H as a minor. By the previous theorem, there is a set of bags $\mathcal{X} = \{X_v \mid X_v \subseteq V(G), v \in V(H)\}$ given by H. For each edge $uv \in E(H)$, let e_{uv} be the edge of G going from X_u to X_v and let T_u be a spanning tree of $G[X_u]$.

Since $\Delta(H) \leq 3$, there can be at most 3 edges $e_{uv_1}, e_{uv_2}, e_{v_3}$ leaving X_u. Hence, T_u must have at most one vertex of degree at most 3 with three branches going to these edges.

Moreover, we have that $T_u \cup e_{e_{uv_1}} \cup e_{e_{uv_2}} \cup e_{e_{uv_3}}$ is a tree with at most one vertex of degree three. However, this implies that H' is a subdivision of H inside G, where:

$$H' = \bigcup_{v \in V(H)} T_v \cup \bigcup_{uv \in E(H)} e_{uv}$$

□

We observe that the proof of the above lemma fails when $\Delta \geq 4$ because we cannot guarantee that there is a vertex with degree 4 inside of X_u (there are at least four outgoing edges but the spanning tree may have two vertices of degree 3 splitting into 4 paths).

Lemma 6.4

Let G be a graph. Then, G contains K_5 or $K_{3,3}$ as a topological minor if and only if it contains K_5 or $K_{3,3}$ as a minor.

Proof. Again, the first implication immediately follows. Vice versa, we may assume that G doesn't contain $K_{3,3}$ as a minor, otherwise by the previous proposition G contains $K_{3,3}$ also as a topolgical minor. Hence, by way of contradiction we suppose that G contains K_5 as minor but that it doesn't contain neither K_5 nor $K_{3,3}$ as a topological minor.

By the previous theorem, let $\mathcal{X} = \{X_i \mid X_v \subseteq V(G), i \in [5]\}$ be the bags given by the K_5 minor and let $e_{ij} \in G$ be the edges going from X_i to X_j. Then, for each $i \in [5]$ there is a spanning tree T_i of $G[X_i]$ such that $T_i^* = T_i \cup \bigcup_{k \in [5]-\{i\}} e_{ik}$ is either a subdivided $K_{1,4}$ or a subdivision of the graph H containing exactly two vertices of degree 3 and all other vertices of degree 1 (see Figure 6.10).

We observe that it cannot hold that each T_i^* is a subdivided star, otherwise $\bigcup_{i \in [5]} T_i^*$ is a subdivided K_5, which is absurd. Hence, there must be at least one $j \in [5]$ such that T_j^* is a subdivision of H. Then, G contains a subdivision of the graph H' (see Figure 6.10). However, the latter graph contains $K_{3,3}$ as a topological minor, raising another contradiction. □

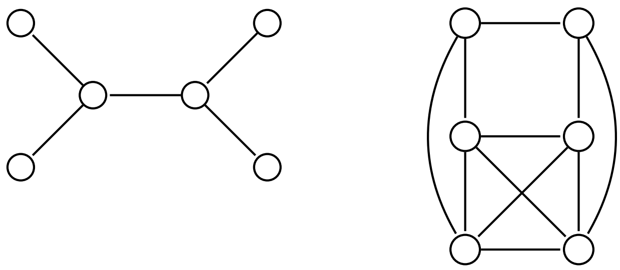


Figure 6.10: The graphs H (left) and H' of the above proof.

The above lemma implies that proving that K_5 and $K_{3,3}$ are the forbidden minors of the class of planar graphs is equivalent to proving that they are the forbidden *topological* minors of said class. The latter result was first proven by Kuratowski [Kur30], while the former was later proven by Wagner [Wag37].

Theorem 6.7: Kuratowski-Wagner theorem

K_5 and $K_{3,3}$ are the forbidden minors of the class of planar graphs.

Proof (for 3-connected graphs). By the previous lemma, we have that G doesn't contain neither K_5 nor $K_{3,3}$ as minors if and only if it doesn't contain them as topological minors. Moreover, we already proved that if G is planar then it doesn't contain neither K_5 nor $K_{3,3}$ as topological minors. By the previous lemma, we also get that it also doesn't contain them as minors.

This leaves us with proving that if no such minors are in the graph then G has a planar drawing. For practical reasons, we restrict this side of the proof to 3-connected graphs. We proceed by induction on n. When $n = 4$, G is always planar since K_4 is planar. Assume the inductive hypothesis and consider a graph G with $n > 4$. By **??**, there is an edge xy such that G/xy is still 3-connected. Since G/xy doesn't contain neither K_5 nor $K_{3,3}$, by inductive hypothesis we get that G/xy is planar.

Let v_{xy} be the new vertex obtained by containg xy in G. Since G/xy is 3-connected, $G/xy - v_{xy}$ is at least 2-connected. Then, by Proposition 6.3 there is a plane graph drawing H of $G/xy - v_{xy}$ with a face that is bounded by a cycle C such that $N(v_{xy}) \subseteq V(C)$. Let $N(v_{xy}) = \{v_1, \dots, v_k\}$. Since v_{xy} is the vertex obtained by contracting xy, each $v_i \in N(v_{xy})$ is adjacent to at least one of x and y in the un-contracted graph G. From now on, we're going to consider the structure of G and the positions of the neighbors of x and y.

Claim 1: if there are two internally disjoint paths P_1, P_2 in C over G such that $N(x) \subseteq V(P_1)$ and $N(y) \subseteq V(P_2)$ then G is planar.

Proof of Claim 1. Since the two paths are internally disjoint, there is a way the draw the edge xy in such a way that the edges going from x to $N(x)$ and from y to $N(y)$ do not intersect. Moreover, since G/xy is also planar, we conclude that G is planar. □

Claim 2: x and y have at most 2 common neighbors in C over G

Proof of Claim 2. By way of contradiction, suppose that $\exists v_1, v_2, v_3 \in V(C) \cap N(x) \cap N(y)$. Then, by contracting C up to forming a K_3 minor having vertices v_1, v_2, v_3, we get K_5 as a minor, raising a contradiction. □

Claim 3: there is no 4-uple of vertices $s, s', t, t' \in V(C)$ over G such that:

- $s \sim x \sim s'$
- $t \sim y \sim t'$

- They occur in C with cyclic order $s \to s' \to t \to t'$

Proof of Claim 3. By way of contradiction, suppose that such 4-uple exists. Then, the two sets $A = \{x, t, t'\}$ and $B = \{y, s, s'\}$ give $K_{3,3}$ as a minor, raising a contradiction. $\square$

Claim 4: both x and y have at least two neighbors in C over G.

Proof of Claim 4. Since G is 3-connected, we know that $\deg_G(x), \deg_G(y) \geq 3$. However, since $x \sim y$, this leaves us with at least two other neighbors in G for both of them. All of these neighbors must also be neighbors of v_{xy}. Hence, considering the fact that G/xy is planar, these neighbors must lie on C. $\square$

We'll now use the four claims to give complete the proof. By Claim 4, we know that both x and y have at least two neighbors in C over G.

Suppose that one of x, y has exactly two neighbors in C over G. Without loss of generality, assume that x is the one that has exactly two neighbors v_1, v_2 on C over G. Then, the vertices v_1, v_2 split the cycle C into two internally disjoint paths P_1, P_2, both having v_1, v_2 as endpoints. By Claim 3, the neighbors of y must either both lie on P_1 or both lie on P_2. In the first case, we get that $N(y) \subseteq V(P_1)$ and $N(x) \subseteq V(P_2)$. In the second case, we get that $N(x) \subseteq V(P_1)$ and $N(y) \subseteq V(P_2)$. For both cases, Claim 1 concludes that G is planar.

Suppose now that both of them have at least three neighbors in C over G. By Claim 2, there must be a vertex $v_1 \in V(C)$ such that $y \sim v_1$ but $x \not\sim v_1$. Let P be the shortest path in $C - v_1$ that contains all the neighbors of x. We observe that, by definition, x must be adjacent to both endpoints of P, otherwise it wouldn't be the shortest path with such property. Again, by Claim 3 y cannot have a neighbor that is internal to P, hence they must all lie on $C - P$. Thus, since $N(x) \subseteq V(P)$ and $N(y) \subseteq V(C - P)$, by Claim 1 we conclude that G is planar. $\square$

6.4 Solved exercises

Problem 6.1

For each $k \in \mathbb{N}$, let G_k be the graph obtained from the graph C_k by adding edges to every pair of vertices of C_k at distance two from each other. Prove that G_k is planar if and only if k is even.

Proof. If k is even, the additional edges form two disjoint cycles D_1, D_2 of length $\frac{k}{2}$. By drawing D_1 on the outer face of C_k and D_2 on its inner face, we get a planar drawing of G_k. Vice versa, suppose that k is odd. Let $V(C) = \{x_1, \dots, x_k\}$ We define the following five bags of vertices:

$$X_1 = \{x_1\} \qquad X_2 = \{x_2\} \qquad X_3 = \{x_3\}$$

$$X_4 = \{4, 6, \dots, k-3, k-1\} \qquad X_5 = \{5, 7, \dots, k-2, k\}$$

It's easy to see that these bags are disjoint from each other and that $G[X_i]$ is connected for every $i \in [5]$. Moreover, for each edge $e_{ij} \in K_5$, there is at least one edge from X_i to X_j. This concludes that K_5 is a minor of G_k. Hence, by Kuratowski's theorem we get that G is not planar. $\square$

Problem 6.2

A graph is called outerplanar if it has a plane drawing in which every vertex lies on the boundary of the outer face. Show that K_4 and $K_{2,3}$ are the forbidden minors of the class of outerplanar graphs.

Proof. First, we observe that, by definition, any outerplanar graph is also planar since they have a plane drawing.

Claim 1: K_4 and $K_{2,3}$ are not outerplanar.

Proof of Claim 1. By way of contradiction, suppose that there is an outerplane drawing H of K_4. Without loss of generality, let $C = x_1, x_2, x_3, x_4$ be the cycle of H with $x_1 \sim x_3$ and $x_2 \sim x_4$. We observe that it cannot hold that both the edges x_1x_3 and x_2x_4 are drawn in the inner face of H, otherwise they would intersect, meaning H wouldn't be a plane drawing. Without loss of generality, assume that x_1x_3 is drawn in the inner face of C and that x_2x_4 is drawn on the outerface. Then, for any possible drawing of x_2x_4, we get that either x_1 or x_3 doesn't lie on the outerface, contradicting the fact that H is an outerplane drawing. This concludes that K_4 has no outerplane drawing.

Similarly, we suppose by way of contradiction that there is an outerplane drawing H' of $K_{2,3}$. Let $X = \{x_1, x_2\}$ and $Y = \{y_1, y_2, y_3\}$ be the bipartitions of $K_{2,3}$. Consider the subplane cycle $C' = x_1, y_1, x_2, y_2, x_1$ of H'. Since H' is an outerplanar drawing, the vertex y_3 must lie on the outer face of C'. Then, for any possible drawing of y_3x_1 and y_3x_2, we get that either y_1 or y_2 doesn't lie on the outerface, contradicting the fact that H' is an outerplane drawing. This concludes that $K_{2,3}$ has no outerplane drawing. $\square$

Claim 1 directly concludes that if G is outerplanar then it doesn't contain neither K_4 nor $K_{2,3}$ as topolgical minors. Moreover, since $\Delta(K_3), \Delta(K_{2,3}) \leq 3$, by Proposition 6.7 we know that a graph doesn't contain neither K_4 nor $K_{2,3}$ as topolgical minors if and only if it doesn't contain them also as minors. This concludes that if G is outerplanar then it doesn't contain neither K_4 nor $K_{2,3}$ as minors.

Vice versa, let G be a graph that doesn't contain neither K_4 nor $K_{2,3}$ as a minor. By way of contradiction, suppose that G is not outerplanar. We may assume that G is at least planar, otherwise by Kuratowski's theorem we get that G contains K_5 or $K_{3,3}$ as minors, contradicting the fact that it doesn't contain neither K_4 nor $K_{2,3}$ as minors.

Let H be any planar drawing of G. Since G is not outerplanar, G must have at least two faces, otherwise every vertex of G would lie on the outerface since the latter always exists. Therefore, since G has at least two faces it cannot be a plane forest, meaning that it contains at least a cycle. Moreover, since H is not an outerplanar drawing, there must be at least a cycle $C = v_1 \dots v_k$ with another vertex x of G that lies inside its inner face. We observe that x must be adjacent to at least one vertex $y \in V(C)$, otherwise x could be drawn on the outerface of C.

Claim 2: there is at least one path P drawn on the outerface of C with endpoints $v_i, v_j \in V(C)$ such that $v_i \not\sim v_j$ and such that $|V(P)| \geq 3$.

Proof of Claim 2. By way of contradiction, suppose that there is no path on the outerface of C with both endpoints in C. Then, we could draw x and the edge xy on the outer face of C and get an outerplanar drawing of G, raising a contradiction. Hence, there is at least one such path P. Again, suppose that the two endpoints of P are adjacent to each other. Then, we could still could draw x and the edge xy on the outer face of C and get an outerplanar drawing of G. Finally, suppose that P is a single edge. Then, we could draw P on the inner face of C and x on the outerface of G, again obtaining an outerplanar drawing of G. $\square$

We observe that Claim 2 also implies that $|C| \geq 4$, since otherwise the two endpoints v_i, v_j of P would be forced to be adjacent. Consider the two subpaths $Q_1 = x_i x_{i+1} \dots x_{j-1}$ and $Q_2 = x_j x_{j+1} \dots x_{i+1}$. Without loss of generality, let Q_1 be the path containing y.

We observe that there cannot be a path from from $P - \{u, v\}$ to $Q_1 - x_i$ or $Q_2 - x_j$, otherwise we would get K_4 as a minor in G, raising a contradiction. However, this implies that we get $K_{2,3}$ as a minor, raising another contradiction. This concludes that if G contains neither K_4 nor $K_{2,3}$ as a minor then G must be outerplanar. $\square$

7

Graph colorings

7.1 Colorings of graphs and maps

An interesting topic in graph theory is **graph colorings**, i.e. the property of a graph of having its vertices colored in such a way that no edge has both endpoints of the same colors. Colorings have many applications, mostly in Computer Science (radio frequency assignment, scheduling, allocation, ...).

Definition 7.1: Graph coloring

Let G be a graph. A **coloring** of G is a function $c : V(G) \to \mathbb{N}$ such that $\forall xy \in E(G)$ it holds that $c(x) \neq c(y)$.

We denote with $\chi(G)$ the minimum number of different colors needed to colors G, i.e. the minimum cardinality of the images of all colorings of G. It's easy to see that for each graph it holds that $\chi \leq n$ since we can trivially assign a different color to each node. A slightly better bound can be obtained considering the maximum degree of the single graph: assigning a different color to a vertex and all of its neighbors is enough.

Proposition 7.1

For all graph G it holds that $\chi(G) \leq \Delta(G) + 1$

Proof. We proceed by induction on n. When $n = 1$, one color suffices. Assume the inductive hypothesis and consider a graph G with $n > 1$ nodes. For any node $v \in V(G)$, by inductive hypothesis we have that:

$$\chi(G - v) \leq \Delta(G - v) + 1 \leq \Delta(G) + 1$$

□

For some special classes of graphs, a constant value upper bound can be obtained. In particular, this is the case of bipartite and planar graphs.

Proposition 7.2

Let G be a graph. Then, G is bipartite if and only if G is 2-colorable.

Proof. Suppose that G has a bipartition (A, B). Then, we simply define the coloring c such that $c(v) = 0$ if $v \in A$ and $c(v) = 1$ in order to obtain a 2-coloring of G.

Vice versa, suppose that G has a 2-coloring c. Without loss of generality, assume that c uses the colors 0 and 1. Then, we can define the bipartition (A, B) such that $A = \{v \in V(G) \mid c(v) = 0\}$ and $B = \{v \in V(G) \mid c(v) = 1\}$. □

For planar graphs, a simple upper bound can be proven through induction and Euler's formula.

Proposition 7.3

If G is planar then $\chi(G) \leq 6$

Proof. We start by claiming a simple property of planar graphs.

Claim: every planar graph has a vertex of degree at most 5

Proof of the claim. By way of contradiction, suppose that $\delta \geq 6$. Then, through the Handshaking lemma we have that:

$$|E| = \frac{1}{2} \sum_{v \in V(G)} \deg(v) \geq \frac{6n}{2} = 3n$$

However, this contradicts Theorem 6.5. □

Now, we proceed by induction on n. When $n = 1$, one color suffices. Assume the inductive hypothesis and consider a graph G with $n > 1$ nodes. Let $x \in V(G)$ be a node with $\deg(x) \leq 5$. By inductive hypothesis we have that $\chi(G - v) \leq 6$. Then, since 6 colors suffice to color $\{x\} \cup N(x)$, we conclude that $\chi(G) \leq 6$. □

Through a more detailed argument, this bound can be reduced to $\chi(G) \leq 5$.

Proposition 7.4

If G is planar then $\chi(G) \leq 5$

Proof. TODO. □

Over the years, the above bound for planar graphs has been improved to $\chi(G) \leq 4$, a result known as the **Four Colors Theorem**. The original proof of this result is the first ever *computer assisted proof* of mathematics, achieved by indentifying a finite subset of special cases and proving through a bruteforce algorithm that each of them has a 4-coloring. This proof was originally considered *flawled* because its correctness cannot be mathematically checked due to too many possibilities of error (a bug in the code, a missing special case, ...). The result was later proven again without the use of a computer after reducing the number of special cases to a very small subset.

Theorem 7.1: Four Colors Theorem

If G is planar then $\chi(G) \leq 4$

Proof. Omitted. $\square$

7.2 Solved exercises

Problem 7.1

We say that a graph G is k-critical if $\chi(G) = k$ and $\chi(G - v) < k$ for all $v \in V(G)$. Find all 3-critical graphs.

Proof. We'll give a characterization of all 3-critical graphs through the following claims.

Claim 1: for all $h \in \mathbb{N}$ the graph C_{2h+1} is 3-critical.

Proof of Claim 1. Fix $h \in \mathbb{N}$. We know that C_{2h+1} is not 2-colorable since it isn't a bipartite graph, hence $\chi(C_{2h+1}) \geq 3$. Moreover, it's easy to see that three colors suffice: starting from a random node, we can color the cycle clockwise alternating tow colors, assigning a third color to the last vertex. This concludes that $\chi(C_{2h+1}) = 3$. By removing a single vertex from the graph we obtain a path, which is always 2-colorable. □

Claim 2: if G is 3-critical then $G = C_{2h+1}$ for some $h \in \mathbb{N}$

Proof of Claim 2. Let G be a 3-critical graph. Since $\chi(G) = 3$, we know that G cannot be a bipartite graph, thus it must contain an odd cycle C. Suppose now that $V(C) \neq V(G)$. Then, there is at least a vertex $v \in V(G - C)$. However, by removing x from G the odd cycle C is still preserved in $G - x$, meaning that $\chi(G - x) \geq 3$, raising a contradiction. Hence, it must hold that $V(C) = V(G)$. Similarly, suppose that $E(C) \neq E(G)$. Then there is at least one edge $uv \in E(G - C)$. Since $V(C) = V(G)$, this edge must split C into two subcycles C' and C'', one having even lenght and the other having odd length. Without loss of generality, let C' be the odd length subcycle. Then, there is a vertex $z \in V(C - C')$ such that removing z in G still preserves the subcycle C' in $G - z$, again meaning that $\chi(G - z) \geq 3$, raising another contradiction. □

The two claims conclude that a graph is 3-critical if and only if it is an odd cycle. □

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