



NP-Completeness of the Steiner Tree Problem

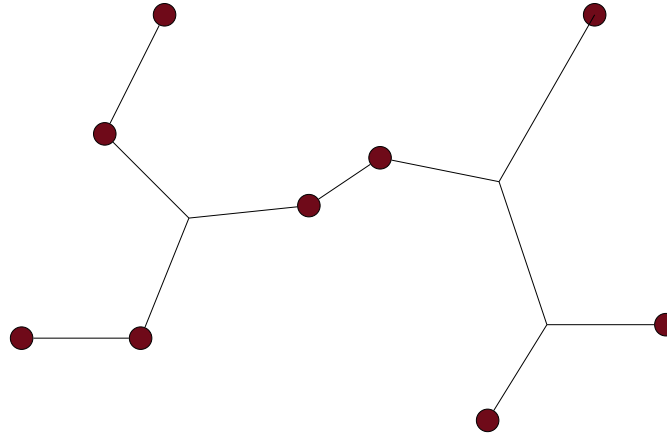
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The Steiner Tree Problem

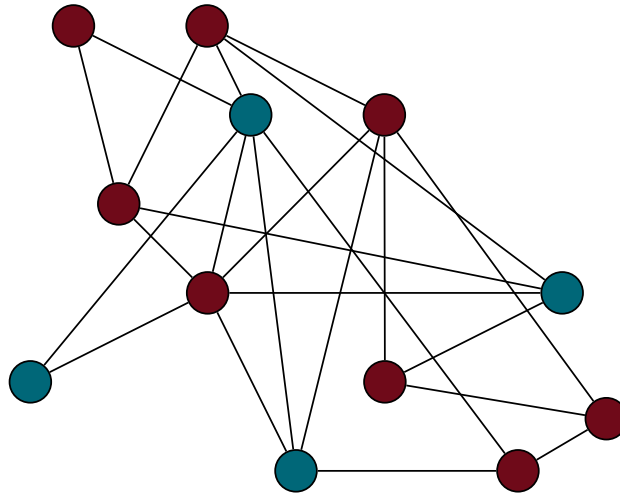
The **Steiner Tree problem** refers to a family of optimization problems, all seeking an optimal interconnection for a specified set of objects under a predefined objective function.

Frequent Objective Function: Minimize the lengths of the interconnections



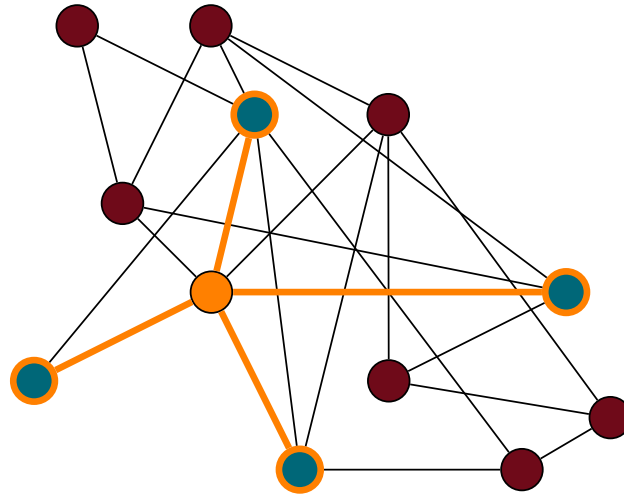
The Steiner Tree Problem

Given a subset of nodes $S \subseteq V(G)$, a sub-tree T of G is a **Steiner Tree** for S if $S \subseteq V(T)$



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Optimization problem: Given a graph G and subset of nodes $S \subseteq V(G)$, find a Steiner Tree for S (if any) with the minimum number of edges.

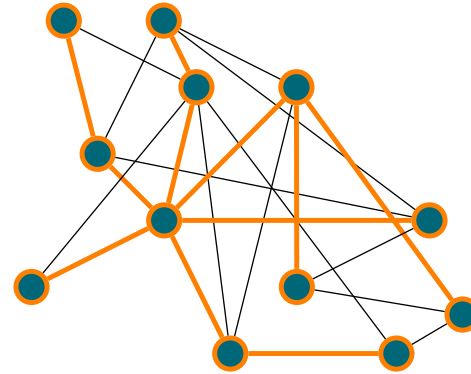
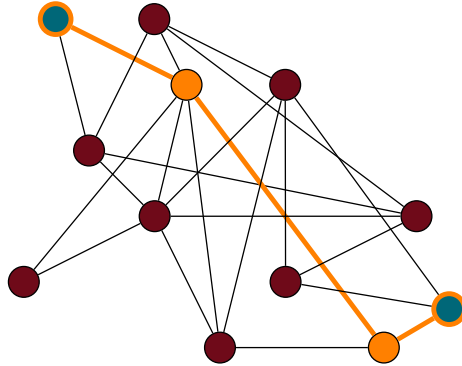
Decision problem: Given a graph G , a subset of nodes $S \subseteq V(G)$ and a value k , is there a Steiner Tree for S with at most k edges?



ST as generalization of other problems

When $|S| = 2$ the ST problem becomes the **Shortest Path Problem**

When $|S| = n$ the ST problem becomes the **Minimum Spanning Tree problem**





ST is in NP

Given $\langle G, S, k \rangle \in ST$, the witness is the Steiner Tree T itself:

- 1) Check that T is a sub-graph of G
- 2) Check that T is acyclic
- 3) Check that $S \subseteq V(T)$
- 4) Check that $|E(T)| \leq k$





Idea behind the NP-Hardness of ST

Idea: We can use Steiner Trees to “force” some kind of cover

The **Set Cover (SC)** problem is usually a suitable candidate for this idea

⇒ The decision version is NP-Complete — hardness by reduction from Vertex Cover (VC)



Set Cover

$$U = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$$

$$C_1 = \{x_1, x_2, x_4, x_7\}$$

$$C_2 = \{x_1, x_3, x_5, x_6\}$$

$$C_3 = \{x_3, x_5, x_7\}$$

Conditions:

- 1) The sub-collection must cover U
- 2) The sub-collection must have minimal cardinality





Set Cover

Optimization problem: Given a set $U = \{x_1, \dots, x_{3q}\}$ and a collection of subsets $\mathcal{C} = \{C_1, \dots, C_m\}$, find the minimal cardinality sub-collection $\mathcal{C}^* \subseteq \mathcal{C}$ that covers U .

Decision problem: Given a set $U = \{x_1, \dots, x_{3q}\}$, a collection of subsets $\mathcal{C} = \{C_1, \dots, C_m\}$ and an integer $k > 0$, is there a sub-collection $\mathcal{C}^* \subseteq \mathcal{C}$ that covers U and such that $|\mathcal{C}^*| \leq k$?

Attempt: Reduction from Set Cover to Steiner Tree



Attempt reduction from SC to ST

Constructing the graph G :

- 1) For every sub-set C_j we add a node u_j
- 2) For every element $x_j \in U$ we add a node v_j — if $x_j \in C_i$ then we add the edge (u_i, v_j)
- 3) We add a special node z and the edge (z, u_i) for all $i \in [m]$

Set of terminal nodes: $S = \{z, x_1, \dots, x_{3q}\}$



Attempt reduction from SC to ST

Example:

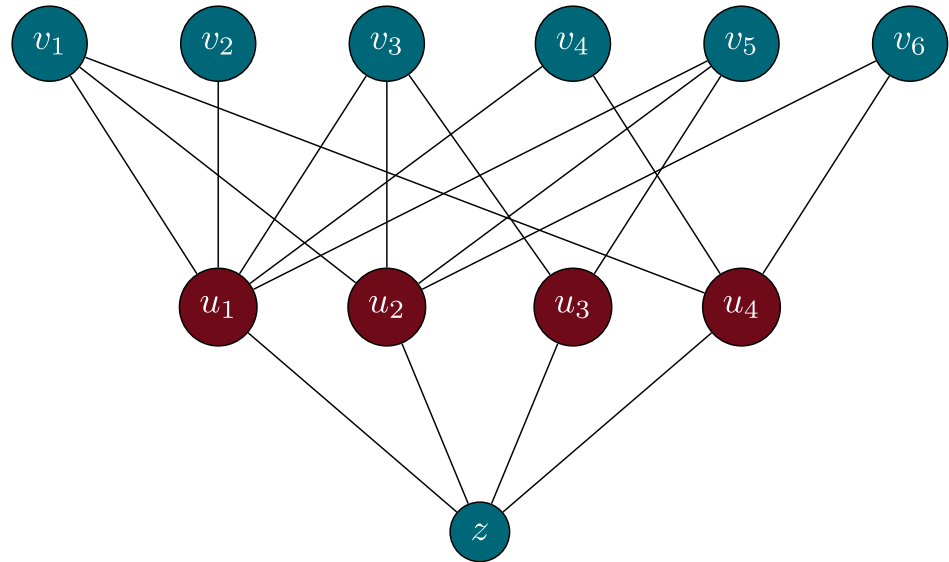
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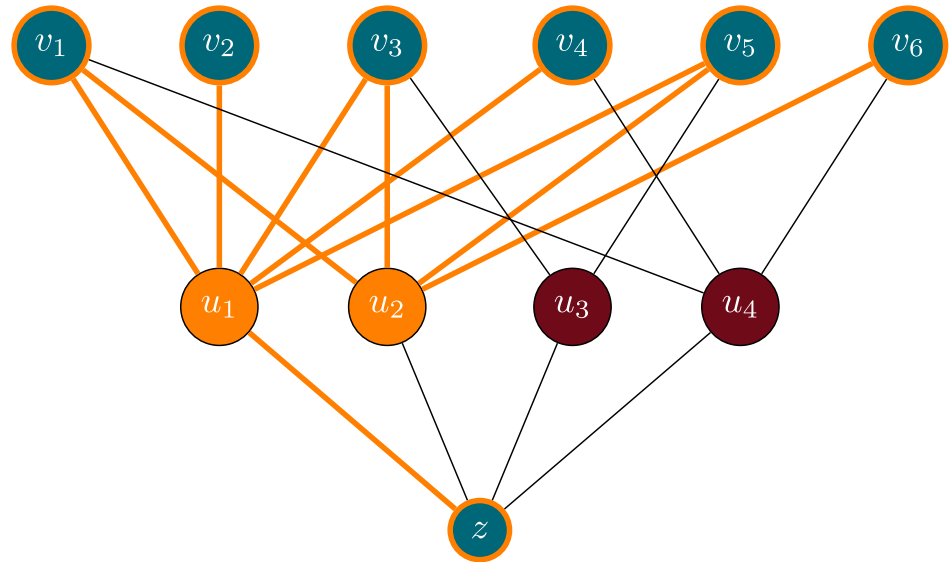
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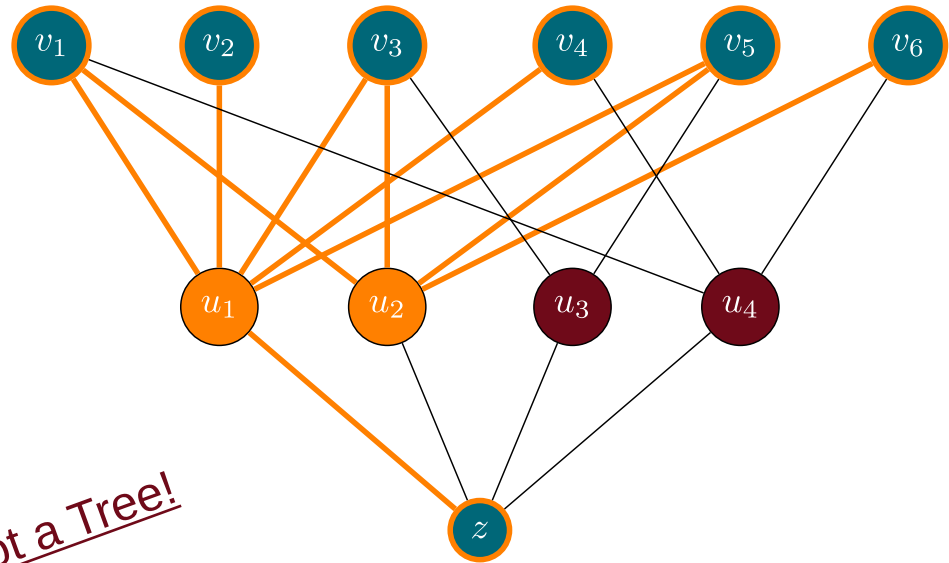
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Exact Cover

Idea: Use a variant of the Set Cover problem \Rightarrow **Exact Cover (XC)**

XC is NP-Complete — hardness by reduction from 3-Coloring (3COL)

Conditions of XC:

- 1) The sub-collection must ~~cover~~ U be a **partition** of $U \Rightarrow$ No more cycles!
- 2) The sub-collection must have minimal cardinality



Attempt reduction from XC to ST

Does the reduction still fail? Yes: the number of edges of the output tree is too variable

Examples:

- 1) If $|U| = n$, $\mathcal{C} = \{\{x_1, \dots, x_{n-1}\}, \{x_n\}\}$ and $k = n$ then T will have $n+2$ edges
 - 2) If $|U| = n$, $\mathcal{C} = \{\{x_1\}, \dots, \{x_n\}\}$ and $k = n$ then T will have $2n$ edges
- \Rightarrow We cannot find a “good” common value k' for the output triple $\langle G, S, k' \rangle$





Exact Cover by 3-Sets

Idea: Use a variant of the Exact Cover problem \Rightarrow **Exact Cover by 3-Sets (X3C)**

X3C is NP-Complete — hardness by reduction from 3-Dimensional Matching (3DM)

Conditions of X3C:

- 1) The sub-collection must be a partition of $U \Rightarrow$ No more cycles!
- 2) The sub-collection must have minimal cardinality
- 3) U has **3q elements** and each C_i has **3 elements**



Exact Cover by 3-Sets

Obs. If $|U| = 3q$ and \mathcal{C}^* is an X3C then $|\mathcal{C}^*| = q$

\Rightarrow All X3Cs have the same cardinality, hence we can drop the minimality constraint

Decision problem: Given a set $U = \{x_1, \dots, x_{3q}\}$ and a collection of subsets $\mathcal{C} = \{C_1, \dots, C_m\}$ such that $|C_i| = 3$, is there a sub-collection $\mathcal{C}^* \subseteq \mathcal{C}$ that is a partition of U ?



Reduction from X3C to ST

Example:

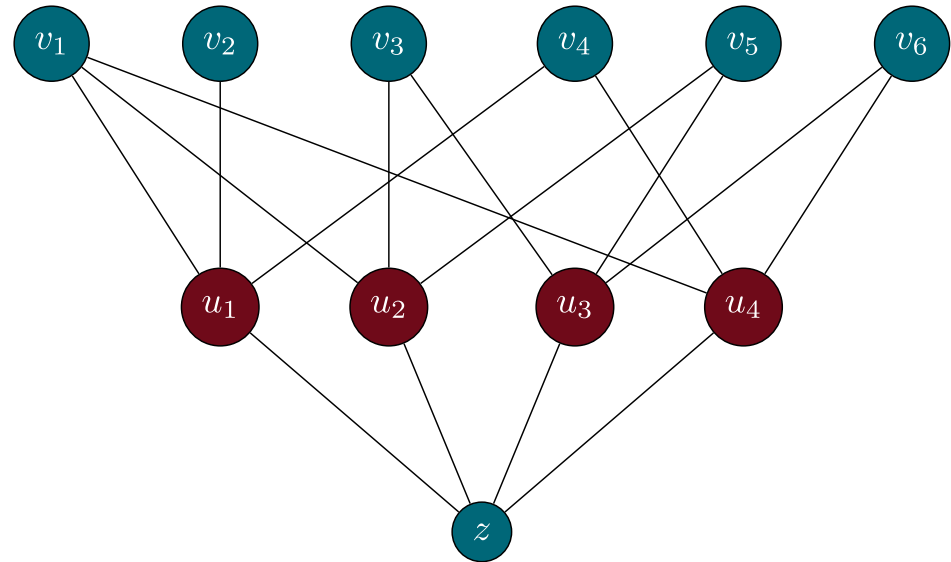
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Reduction from X3C to ST

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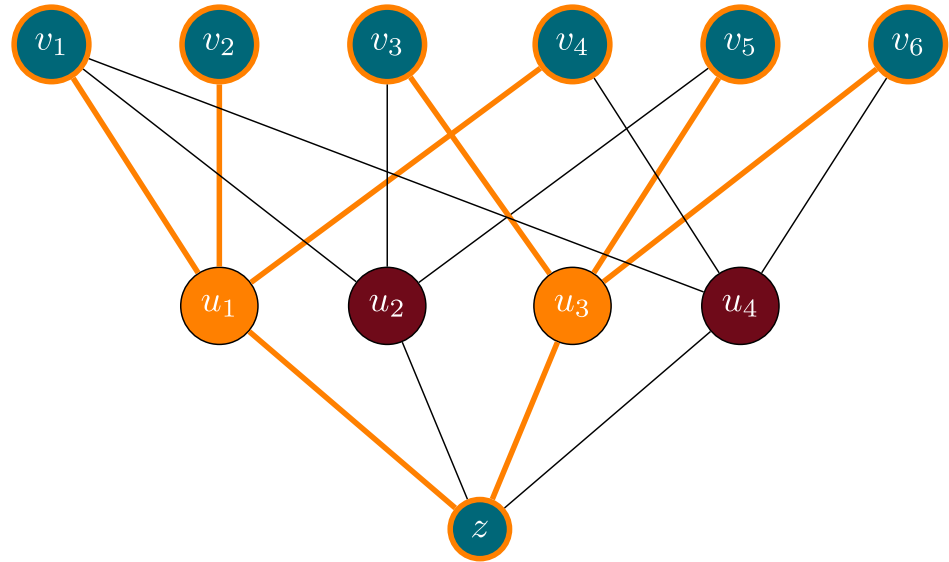
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Reduction from X3C to ST

Claim: U has an Exact Cover by 3-Sets over \mathcal{C} iff G has a Steiner Tree for S with $4q$ edges

\implies)

- 1) If $\mathcal{C}^* \subseteq \mathcal{C}$ is an X3C of U then $|\mathcal{C}^*| = q \rightarrow$ W.l.o.g. assume $\mathcal{C}^* = \{C_1, \dots, C_q\}$
- 2) T is made of the edges that have u_1, \dots, u_q as endpoints $\rightarrow T$ has $4q$ edges
- 3) \mathcal{C}^* is an exact cover $\rightarrow T$ covers S and it is acyclic



Reduction from X3C to ST

Claim: U has an Exact Cover by 3-Sets over \mathcal{C} iff G has a Steiner Tree for S with $4q$ edges

\iff)

- 1) If T is an ST for S with at $4q$ edges $\rightarrow T$ must contain u_1, \dots, u_t
- 2) If $t > q$ then S has more than $4q$ edges \rightarrow Contradiction
- 3) If $t < q$ then S cannot cover all terminals \rightarrow Contradiction
- 4) $\mathcal{C}^* = \{C_1, \dots, C_q\}$ is an Exact 3-Cover of U





References

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- M. W. Bern and R. L. Graham “The Shortest-Network Problem” in Scientific American Magazine Vol. 260 No. 1 (1989). [doi:10.1038/scientificamerican0189-84](https://doi.org/10.1038/scientificamerican0189-84)





**Thank you for the
attention!**

