NP-Completeness of the Steiner Tree Problem

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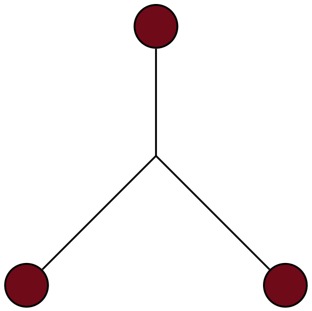
The Steiner Tree Problem

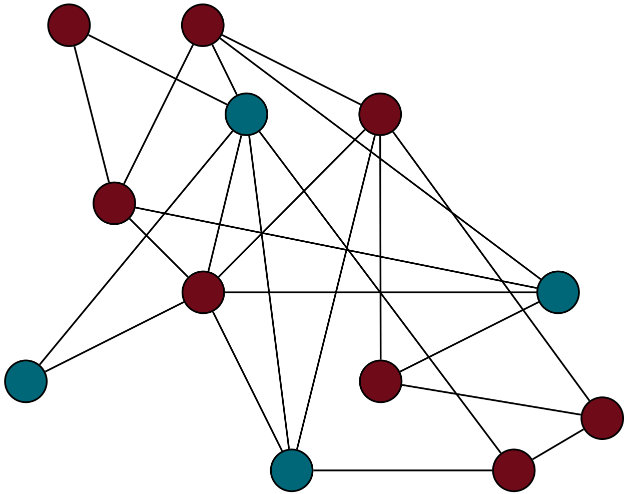
The **Steiner Tree problem** refers to a family of optimization problems, all seeking an optimal interconnection for a specified set of objects under a predefined objective function.

Frequent Objective Function: Minimize the lengths of the interconnections



The Steiner Tree Problem

Given a subset of nodes S ⊆ V(G), a sub-tree T of G is a Steiner Tree for S if S ⊆ V(T)



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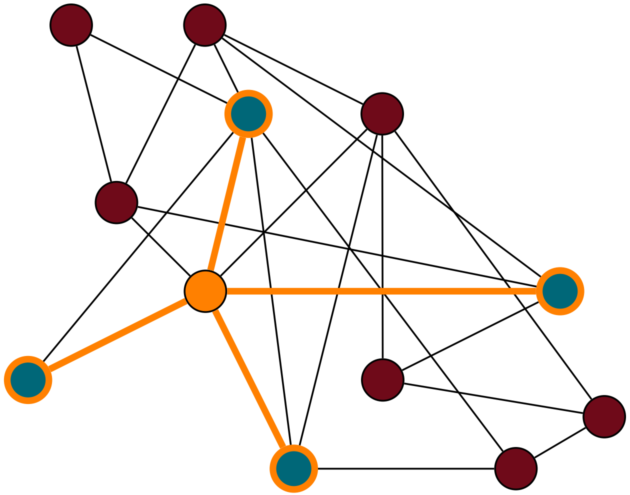


The Steiner Tree Problem

Given a subset of nodes $S \subseteq V(G)$, a sub-tree T of G is a **Steiner Tree** for S if $S \subseteq V(T)$

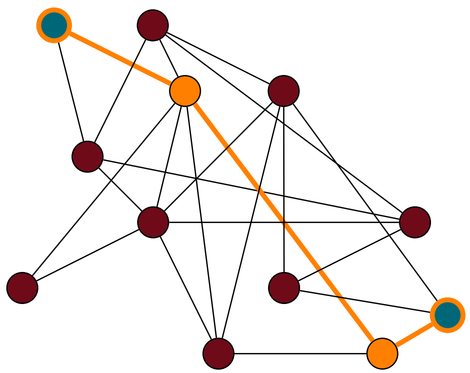
Optimization problem: Given a graph G and subset of nodes $S \subseteq V(G)$, find a Steiner Tree for S (if any) with the minimum number of edges.

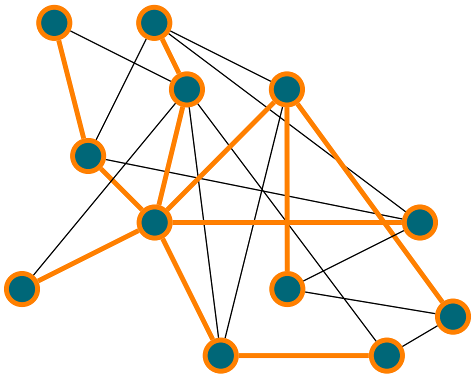
Decision problem: Given a graph G, a subset of nodes $S \subseteq V(G)$ and a value k, is there a Steiner Tree for S with at most k edges?

ST as generalization of other problems

When |S| = 2 the ST problem becomes the Shortest Path Problem

When |S| = n the ST problem becomes the Minimum Spanning Tree problem





ST is in NP

Given $\langle G,S,k \rangle \in ST$, the witness is the Steiner Tree T itself:

1) Check that T is a sub-graph of G
2) Check that T is acyclic
3) Check that $S \subseteq V(T)$
4) Check that $|E(T)| \leq k$

Idea behind the NP-Hardness of ST

Idea: We can use Steiner Trees to “force” some kind of cover

The **Set Cover (SC)** problem is usually a suitable candidate for this idea

$\Longrightarrow$ The decision version is NP-Complete — hardness by reduction from Vertex Cover (VC)

Attempt 1: Reduction from SC to ST

Set Cover

$$U = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$$

$C_1 = \{x_1, x_2, x_4, x_7\}$ $\quad$ $C_2 = \{x_1, x_3, x_5, x_6\}$ $\quad$ $C_3 = \{x_3, x_5, x_7\}$

Conditions:

1) The sub-collection must cover U
2) The sub-collection must have minimal cardinality

Set Cover

Optimization problem: Given a set $U = \{x_1, \ldots, x_{3q}\}$ and a collection of subsets $\mathcal{C} = \{C_1, \ldots, C_m\}$, find the minimal cardinality sub-collection $\mathcal{C}^* \subseteq \mathcal{C}$ that covers U.

Decision problem: Given a set $U = \{x_1, \ldots, x_{3q}\}$, a collection of subsets $\mathcal{C} = \{C_1, \ldots, C_m\}$ and an integer $k > 0$, is there a sub-collection $\mathcal{C}^* \subseteq \mathcal{C}$ that covers U and such that $|\mathcal{C}^*| \leq k$?

Attempt reduction from SC to ST

Constructing the graph G:

1) For every sub-set C_j we add a node u_j

2) For every element $x_j \in U$ we add a node v_j — if $x_j \in C_i$ then we add the edge (u_i , v_j)

3) We add a special node z and the edge (z, u_i) for all $i \in [m]$

Set of terminal nodes: $S = \{z, x_1, \ldots, x_{3q}\}$

Attempt reduction from SC to ST

Example:

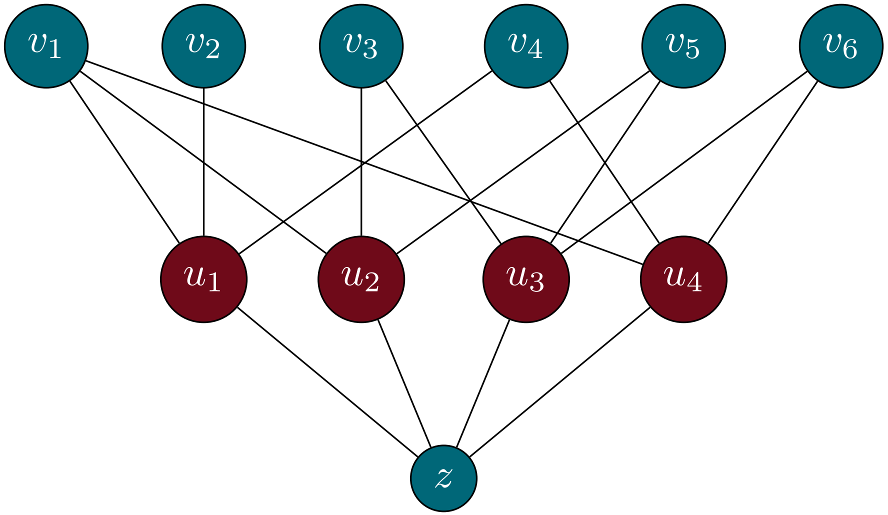
$U = \{x_1, x_2, x_3, x_4, x_5, x_6\}$

$C_1 = \{x_1, x_2, x_4\}$

$C_2 = \{x_1, x_3, x_5\}$

$C_3 = \{x_3, x_5, x_6\}$

$C_4 = \{x_1, x_4, x_6\}$



Attempt reduction from SC to ST

Where does this reduction fail? The number of edges of the output tree is too variable

Examples:

1) If $|U| = n$, $C_i = U - \{x_i\}$ and $k = n$ then T will have n^2 edges since $\mathcal{C}^* = \mathcal{C}$

2) If $|U| = n$, $C_i = \{x_i\}$ and $k = n$ then T will have 2n edges since $\mathcal{C}^* = \mathcal{C}$

$\Longrightarrow$ We cannot find a "good" common value k' for the output triple $\langle G,S,k'\rangle$

We need to force a fixed amount of edges

Exact Cover by 3-Sets

Additional conditions for X3C:

1) $|U| = 3q$ and $|C_i| = 3$ for all $i \in [m]$

2) The sub-collection must be pairwise disjoint

Decision problem: Given a set $U = \{x_1, \ldots, x_{3q}\}$ and a collection of subsets $\mathcal{C} = \{C_1, \ldots, C_m\}$ such that $|C_i| = 3$, is there a sub-collection $\mathcal{C}^* \subseteq \mathcal{C}$ that is a partition of U?

X3C is NP-Complete — hardness by reduction from 3-Dimensional Matching (3DM)

Reduction from X3C to ST

Claim: U has an Exact Cover by 3-Sets over $\mathcal{C}$ if and only if G has a Steiner Tree for S with at most 4q edges

$\Longrightarrow$)

1) If $\mathcal{C}^* \subseteq \mathcal{C}$ is an Exact 3-Cover of U then $|\mathcal{C}^*| = q \longrightarrow$ W.l.o.g. assume $\mathcal{C}^* = \{C_1, \dots, C_q\}$

2) T is made of the edges that have $u_1, \dots, u_q$ as endpoints $\longrightarrow$ T has 4q edges

3) $\mathcal{C}^*$ is an exact cover $\longrightarrow$ T covers S and it is acyclic

Reduction from X3C to ST

Claim: U has an Exact Cover by 3-Sets over $\mathcal{C}$ if and only if G has a Steiner Tree for S with at most 4q edges

$\Longleftarrow$)

1) If T is an ST for S with at most 4q edges $\longrightarrow$ T must contain $u_1, \ldots, u_t$ where $t \leq q$

2) If $t < q$ then S cannot cover all terminals $\longrightarrow$ Contradiction

3) $\mathcal{C}^* = \{C_1, \ldots, C_q\}$ is an Exact 3-Cover of U

References

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- M. W. Bern and R. L. Graham “The Shortest-Network Problem” in Scientific American Magazine Vol. 260 No. 1 (1989). doi:10.1038/scientificamerican0189-84

Thank you for the attention!