One Dollar Each Eliminates Envy

One Dollar Each Eliminates Envy

J. Brustle, J. Dippel, V.V. Narayan, M. Suzuki and A. Vetta *

McGill University

December 6, 2019

Abstract

We study the fair division of a collection of m indivisible goods amongst a set of n agents. Whilst envy-free allocations typically do not exist in the indivisible goods setting, envy-freeness can be achieved if some amount of a divisible good (*money*) is introduced. Specifically, Halpern and Shah [11] showed that, given additive valuation functions where the marginal value of each item is at most one dollar for each agent, there always exists an envy-free allocation requiring a subsidy of at most $(n-1)\cdot m$ dollars. The authors also conjectured that a subsidy of $n-1$ dollars is sufficient for additive valuations. We prove this conjecture. In fact, a subsidy of at most one dollar per agent is sufficient to guarantee the existence of an envy-free allocation. Further, we prove that for general monotonic valuation functions an envy-free allocation always exists with a subsidy of at most $2(n-1)$ dollars per agent. In particular, the total subsidy required for monotonic valuations is independent of the number of items.

1 Introduction

We consider the fair division of m indivisible items amongst n agents. Specifically, we desire an allocation where no agent is envious of any other; that is, the value each agent has for its own allocated bundle is at least as great as its value for the bundle of any other agent. This concept, called *envy-freeness*, was introduced by Foley [8]. For divisible goods, Varian [20] explained how to obtain envy-free allocations using the theory of general equilibria: simply share each good equally amongst the agents and then find a competitive equilibrium. Thus envy-free allocations exist for classes of valuation functions where competitive equilibria are guaranteed to exist.

Unfortunately for indivisible goods it is easy to see that envy-free allocations do not exist in general. For example, if the number of agents exceeds the number of items, then in every allocation there is an agent who receives an empty bundle. In classical work, Maskin [15] asked if this impossibility result could be circumvented by the addition of a single divisible good, namely *money*. If so, how much money is needed to eradicate all envy? He considered the case of a market with n agents and $m = n$ goods where each agent can be allocated *at most* one good, and has, without loss of generality, a value of at most one dollar for any specific good. Maskin [15] then showed that an envy-free allocation exists with the addition of $n-1$ dollars into the market. But what happens in the general setting where the number of agents and number of goods may differ and where agents may be allocated more than one good? The purpose of this paper is to understand this case of *multi-unit demand* valuations. In this setting, Halpern and Shah [11] proved that $m \cdot (n-1)$ dollars suffice to support an envy-free allocation when the agents have *additive* valuation functions. Further, they conjectured that, as with the unit-demand setting, there always exists an allocation for which $n-1$ dollars suffice.

The main result in this paper is the verification of this conjecture: for additive valuation functions, precisely $n-1$ dollars is sufficient to guarantee the existence of an envy-free allocation. In fact, our result is stronger in several ways. First, not only is the subsidy at most $n-1$ dollars in total but each agent receives *at most* one dollar in subsidy. Secondly, this allocation is also *envy-free*

*{johannes.brustle,jack.dippel,vishnu.narayan,mashbat.suzuki}@mail.mcgill.ca, adrian.vetta@mcgill.ca

up to one good (EF1) – this settles a second conjecture from [11]. Thirdly, the allocation is *balanced*, that is, the cardinalities of the allocated bundles differ by at most one good. Furthermore, this envy-free allocation can be constructed in polynomial time.

We also study the case of general valuation functions. Requiring only the very mild assumption that the valuation functions are monotone, we prove the perhaps surprising result that envy-free solutions still exist with a subsidy amount that is *independent* of the number of goods m. Specifically, we prove that there is an envy-free allocation where each agent receives a subsidy of at most $2(n-1)$ dollars, which is a total subsidy of $O(n^2)$. Here the envy-free allocation can be constructed in polynomial time given a valuation oracle.

1.1 Related Work

Fair division has been extensively studied over the past six decades. The concept of a fair allocation was formally introduced by Steinhaus [16] via the *cake-cutting problem*: how can a heterogeneous cake be fairly divided among a set of agents? To address this question, it is necessary to first define fairness. The fairness objective of Steinhaus [16] was *proportionality*. An allocation is proportional if every agent is allocated a bundle (or piece of cake) of value at least $\frac{1}{n}$ of its total value for the grand bundle (entire cake). For cake-cutting, and divisible goods in general, when the valuations are additive, envy-freeness implies proportionality. This is because, for any agent and any partition of the cake into n pieces, some piece must be worth at least $\frac{1}{n}$ of the whole cake to that agent. Thus envy-freeness is a stronger fairness guarantee than proportionality. Another classical fairness measure is *equitability*, where all agents should receive bundles of the same value. In the case of divisible goods, Alon [2] showed for additive continuous valuation functions that allocations exist that satisfy proportionality, equitability and envy-freeness simultaneously. Algorithmic methods to obtain envy-free cake divisions for any number of agents are also known; see, for example, Brams and Taylor [5].

More recently, Budish [6] introduced the *maximin share guarantee* inspired by the cut-and-choose protocol. Assume an agent partitions the items into n bundles and then receives the lowest-value bundle. The corresponding value that the agent obtains by selecting its optimal partition is called its *maximin* share. The fairness objective then is to find an allocation where every agent receives a bundle of value at least its maximin share.

Unfortunately, for indivisible goods, there are examples where proportionality, envy-freeness, equitability and the maximin share guarantee are all impossible to achieve. Consequently, there has been much focus on approximate fairness guarantees. One natural approach is the design of approximation algorithms for the maximin share problem. An alternative guarantee is via EF-k allocations [6], where an agent has no envy provided k goods are removed from the bundles of the other agents. Of special interest are the *envy bounded by a single good*, or EF1 allocations. Lipton et al. [14] showed that, when the valuation functions are monotone, an EF1 allocation exists and it can be computed in polynomial time. A large body of recent work on the fair allocation of indivisible goods has focused on achieving these types of approximation guarantee [9, 13, 4, 7].

A parallel line of research considers the use of money in the fair allocation of indivisible goods. This is motivated by the *rent division* problem, where the goal is to allocate n indivisible goods among n agents and divide a fixed total cost, *i.e.* the *rent*, amongst the agents. Su [17] showed that, under mild assumptions, *rental harmony* can be achieved: there is an envy-free division of the goods and the rent. The majority of the literature in this area considers the setting with n unit-demand agents, $m = n$ indivisible items and one divisible good, akin to *money*. Svensson [18] showed that an envy-free and pareto efficient allocation exists under certain conditions. Tadenuma and Thomson [19] study the structure of envy-free allocations of a single indivisible good when monetary compensations are possible. Maskin [15] studies a similar model to [18] under slightly different conditions; he showed that, with sufficient money, an envy-free allocation always exists. Specifically, his results imply that if the agents are unit-demand and their value for each item is at most one dollar, then a total of $n-1$ dollars suffices for envy-freeness. In Aragones [3] and Klijn [12], the authors consider the same model and give polynomial-time algorithms to compute an envy-free allocation with subsidy.

Among the papers that consider a setting with more than n items, most reduce to the above n-item case where at most one good is allocated to each agent. For example, Alkan et al. [1] consider the more general m-item setting and allow the possibility of undesirable objects, but their procedure introduces either "null objects" or "fictitious people" to equalize the number of agents and items before outputting an allocation with single-item bundles. Haake et al. [10] also consider the m-good case and provide a procedure to compute an envy-free allocation with side-payments, but their approach begins by bundling the goods into n sets.

In recent work, Halpern and Shah [11] extend the above models to the multi-demand setting with any number m of indivisible goods. Specifically, they consider the setting in which the n agents have *additive* valuation functions over a set of m items, and, without loss of generality, the value of each item is at most 1. They characterize the *envy-freeable* allocations in terms of the structure of the *envy graph* (see Section 2.1), whose nodes are the agents and whose arc weights represent the envies between pairs of agents. They then study the problem of minimizing the amount of subsidy that is sufficient to guarantee envy-freeness. It is easy to see this *minimum subsidy* can be at least $n-1$ for all envy-freeable allocations. Indeed, consider the case of a single item which each agent values at exactly one dollar; evidently, every agent that does not receive the item must be compensated with a dollar. They present a matching upper bound of $n-1$ dollars for the special cases of *binary* and *identical* additive valuations.

More generally, they prove that, for additive valuations, an envy-freeable allocation always exists if the total subsidy at least is $m \cdot (n-1)$ dollars. But, based on the experimental analysis of over $100,000$ synthetic instances and over $3,000$ real-world instances of fair division, Halpern and Shah [11] conjecture that this upper bound can be improved to $n-1$ dollars. That is, for agents with additive valuations an envy-freeable allocation that requires a subsidy of at most $n-1$ always exists. In addition, they conjecture that an allocation exists that is both envy-freeable (with perhaps a much larger subsidy) *and* EF1 for the fair division problem with additive valuations.

Conjecture 1.1. *[11] For additive valuations, there is an envy-freeable allocation that requires a total subsidy of at most $n-1$ dollars.*

Conjecture 1.2. *[11] For additive valuations, there is an envy-freeable allocation that is EF1.*

1.2 Our Results

In this work we settle both Conjecture 1.1 and Conjecture 1.2. In fact, our main result is even stronger in a several ways. To wit, we show that, for any instance with additive valuations, there is an allocation that is simultaneously envy-freeable, EF1, balanced, *and* requires a total subsidy of at most $n-1$ dollars. Moreover, we present an algorithm that computes such an allocation in polynomial time. Our bound not only applies to the total subsidy, but to each individual payment – the payment made to each agent in this allocation is at most one dollar! Formally, in Sections 3 and 4 we prove the following theorem.

Theorem 1.3. *For additive valuations there is an envy-freeable allocation where the subsidy to each agent is at most one dollar. (This allocation is also EF1, balanced, and can be computed in polynomial time.)*

It is easy to see that, when minimizing the total subsidy, at least one agent will not receive a subsidy. Thus Theorem 1.3 implies that the total subsidy required is indeed at most $n-1$ dollars.

In Section 5 we consider the general setting where the agents have arbitrary monotone valuation functions. Analogously, without loss of generality, we may scale the valuations so that marginal value of each item for any agent never exceeds one dollar. We show that there is an envy-freeable allocation in which the subsidy required is at most $2(n-1)$ dollars per agent. Thus, the total subsidy required to ensure the existence of an envy-free allocation at most $O(n^2)$. Note that the assumption of monotonicity is extremely mild and so the valuations the agents have for bundles of items may range from 0 to $\Omega(m)$ in quite an arbitrary manner. Consequently, it is somewhat remarkable that the total subsidy required to ensure the existence of an envy-free allocation is independent of the number of items m. In particular, when m is large the subsidy required is negligible in terms of m and thus, typically, also negligible in terms of the values of the allocated bundles. In this case, given a valuation oracle for each agents, the corresponding envy-free allocation and subsidies can be computed in polynomial time. Specifically, in Section 5 we prove:

Theorem 1.4. *For monotonic valuations there is an envy-freeable allocation where the subsidy to each agent is at most* $2(n-1)$ *dollars. (Given a valuation oracle, this allocation can be computed in polynomial time.)*

In effect, our work implies that there is, in fact, a much stronger connection between the classical divisible goods (cake-cutting) setting and the indivisible goods setting than was previously known. While the classical guarantees (envy-freeness and proportionality) can be achieved with divisible goods, for the indivisible-goods setting much of the recent literature focuses on achieving weaker fairness properties. We show that by simply introducing a small subsidy that only depends on the number of agents, the much stronger classical guarantees *can* be achieved in the indivisible goods setting. Moreover, allocations that give these classical guarantees with a small bounded subsidy can be efficiently found.

2 The Fair Division with Subsidy Problem

There is a set $I = \{1, 2, \ldots, n\}$ of agents and a set $J = \{1, 2, \ldots, m\}$ of indivisible goods (items). Each agent $i \in I$ has a valuation function v_i over the set of items. That is, for each bundle $S \subseteq J$ of items, agent i has value $v_i(S)$. We make the standard assumptions that the valuation functions are *monotonic*, that is, $v_i(S) \leq v_i(T)$ when $S \subseteq T$, and that $v_i(\emptyset) = 0$. An agent i and valuation function v_i are *additive* if, for each item $j \in J$, agent i has value $v_i(j) = v_i(\{j\})$, and for any collection $S \subseteq J$, agent i has value $v_i(S) = \sum_{j \in S} v_i(j)$. We denote the vector of valuation functions by $\boldsymbol{v} = (v_1, \ldots, v_n)$, and call $\boldsymbol{v}$ a *valuation profile*. Additionally, without loss of generality we scale each agent i's valuation function so that the maximum marginal value of any item j is at most 1. Specifically, for additive valuations, this implies $v_i(j) \leq 1$ for every agent i and item j.

An *allocation* is an ordered partition $\mathcal{A} = \{A_1, \ldots, A_n\}$ of the set of items into n bundles. Agent i receives the (possibly empty) bundle A_i in the allocation $\mathcal{A}$. The allocation $\mathcal{A}$ is *envy-free* if

$$v_i(A_i) \;\geq\; v_i(A_k) \qquad \forall i \in I, \forall k \in I.$$

That is, for any pair of agents i and k, agent i prefers its own bundle A_i over the bundle A_k. In the (envy-free) *fair division problem* the objective is to find an envy-free allocation of the items.

Unfortunately, this objective is generally impossible to satisfy. A natural relaxation of the objective arises by incorporating subsidies. Specifically, let $\boldsymbol{p} = (p_1, \ldots, p_n)$ be a non-negative subsidy vector, where agent i receives a payment $p_i \geq 0$. An *allocation with payments* $(\mathcal{A}, \boldsymbol{p})$ is then envy-free if

$$v_i(A_i) + p_i \geq v_i(A_k) + p_k \qquad \forall i \in I, \forall k \in I.$$

That is, each agent prefers its bundle plus payment over the bundle plus payment of every other agent. In the *fair division with subsidy problem* the objective is to find an envy-free allocation with payments whose total subsidy $\sum_{i\in I} p_i$ is minimized.

2.1 Envy-Freeability and the Envy Graph

For any fixed allocation $\mathcal{A}$, a payment vector $\boldsymbol{p}$ such that $\{\mathcal{A}, \boldsymbol{p}\}$ is envy-free does not always exist. To see this, consider an instance with a single item and agents $I = \{1, 2\}$ with values $v_1 < v_2$ for the item. Now take the fixed allocation where the item is given to agent 1. It follows that agent 2 must receive a payment of at least v_2 to eliminate its envy. But then, because $v_2 > v_1$, agent 1 is envious of the bundle plus payment allocated to agent 2. Thus, no payment vector can eliminate the envy of both agents for this allocation.

We call an allocation $\mathcal{A}$ *envy-freeable* if there exists a payment vector $\boldsymbol{p} = (p_1, \ldots, p_n)$ such that $\{\mathcal{A}, \boldsymbol{p}\}$ is envy-free. There is a nice graphical characterization for the envy-freeability of an allocation $\mathcal{A}$. The *envy graph*, denoted $G_{\mathcal{A}}$, for an allocation $\mathcal{A}$ is a complete directed graph with vertex set I. For any pair of agents $i, k \in I$ the weight of arc (i, k) in $G_{\mathcal{A}}$ is the envy agent i has for agent k under the allocation $\mathcal{A}$, that is, $w_{\mathcal{A}}(i, k) \; = \; v_i(A_k) - v_i(A_i)$.

An allocation is envy-freeable if and only if its envy graph does not contain a positive-weight directed cycle. More generally, Halpern and Shah [11] obtained the following theorem; we include their proof in order to familiarize the reader with the structure of envy-freeable allocations.

Theorem 2.1. *[11] The following statements are equivalent.*

(a) The allocation $\mathcal{A}$ is envy-freeable.

(b) The allocation $\mathcal{A}$ maximizes (utilitarian) welfare across all reassignments of its bundles to agents: for every permutation π of $I = [n]$, we have $\sum_{i\in I} v_i(A_i) \geq \sum_{i\in I} v_i(A_{\pi(i)})$.

(c) The envy graph $G_{\mathcal{A}}$ contains no positive-weight directed cycles.

Proof.
(a) $\Rightarrow$ *(b)*: Let $\mathcal{A} = \{A_1, \ldots, A_n\}$ be envy-freeable. Then, by definition, there exists a payment vector $\boldsymbol{p}$ such that $v_i(A_i) + p_i \geq v_i(A_k) + p_k$, for any pair of agents i and k. Rearranging, we have $v_i(A_k) - v_i(A_i) \leq p_i - p_k$. Then, for any permutation π of $I = [n]$

$$\sum_{i\in I} \big(v_i(A_{\pi(i)}) - v_i(A_i)\big) \;\leq\; \sum_{i\in I} \big(p_i - p_{\pi(i)}\big) \;=\; \sum_{i\in I} p_i - \sum_{i\in I} p_{\pi(i)} \;=\; 0.$$

Thus the allocation $\mathcal{A}$ maximizes welfare over all reassignments of its bundles.
(b) $\Rightarrow$ *(c)*: Assume $\mathcal{A}$ maximizes welfare over all reassignments of its bundles and take a directed cycle C in the envy graph $G_{\mathcal{A}}$. Without loss of generality $C = \{1, 2, \ldots, r\}$ for some $r \geq 2$. Now define a permutation π_C of I according to the following rules: (i) $\pi_C(i) = i + 1$ for each $i \leq r - 1$, (ii) $\pi_C(r) = 1$, and (iii) $\pi_C(i) = i$ otherwise. Then the weight of the cycle C in the envy graph

satisfies

$$\begin{aligned}
w_{\mathcal{A}}(C) &= \sum_{(i,k)\in C} w_{\mathcal{A}}(i,k) \\
&= \sum_{i=1}^{r-1} (v_i(A_{i+1}) - v_i(A_i)) + (v_r(A_1) - v_r(A_r)) \\
&= \sum_{i=1}^{r-1} (v_i(A_{i+1}) - v_i(A_i)) + (v_r(A_1) - v_r(A_r)) + \sum_{i=r+1}^{n} (v_i(A_i) - v_i(A_i)) \\
&= \sum_{i\in I} v_i(A_{\pi(i)}) - v_i(A_i) \\
&\le 0.
\end{aligned}$$

The inequality holds as $\mathcal{A}$ maximizes welfare over all bundle reassignments. Thus C has non-positive weight.

$(c) \Rightarrow (a)$: Assume the envy graph $G_{\mathcal{A}}$ contains no positive-weight directed cycles. Let $\ell_{G_{\mathcal{A}}}(i)$ be the maximum weight of any path (including the empty path) that starts at vertex i in $G_{\mathcal{A}}$. For each agent $i \in I$, set its payment $p_i = \ell_{G_{\mathcal{A}}}(i)$. Observe that $p_i \ge 0$ as the empty path has weight zero. The corresponding pair $(\mathcal{A}, \boldsymbol{p})$ is then envy-free. To see this, recall that there are no positive-weight cycles. Therefore, for any pair of agents i and k, we have

$$p_i \;=\; \ell_{G_{\mathcal{A}}}(i) \;\ge\; w_{\mathcal{A}}(i,k) + \ell_{G_{\mathcal{A}}}(k) \;=\; (v_i(A_k) - v_i(A_i)) + p_k.$$

Thus $v_i(A_i) + p_i \ge v_i(A_k) + p_k$ and the allocation $\mathcal{A}$ is envy-freeable. □

Theorem 2.1 is important for two reasons. First, whilst an allocation $\mathcal{A} = \{A_1, A_2, \dots, A_n\}$ need not be envy-freeable, Condition (b) tells us that there is some permutation π of the bundles in $\mathcal{A}$ such that the resultant allocation, $\mathcal{A}^\pi = \{A_{\pi(1)}, A_{\pi(2)}, \dots, A_{\pi(n)}\}$, is envy-freeable! For example, consider again the simple one-item, two-agent instance above. If the item is allocated to agent 1 then the weight on the arc $(1,2)$ is $-v_1$ and the weight on the arc $(2,1)$ is v_2. Because $v_1 < v_2$, the envy graph has a positive-weight directed cycle $\{1,2\}$ and so, by Theorem 2.1, this allocation is not envy-freeable. However, suppose we fix the bundles and find a utility-maximizing reallocation of these fixed bundles. This reallocation assigns the item to agent 2 and now there is no positive-weight directed cycle in the resultant envy-free graph; consequently this allocation is envy-freeable by providing a subsidy in the range $[v_1, v_2]$ to agent 1.

Second, to calculate the subsidy vector $\boldsymbol{p}$ associated with an envy-freeable allocation, such as $\mathcal{A}^\pi$, it suffices to calculate the maximum-weight paths beginning at each vertex in its envy graph. (In fact, it is straightforward to prove that the heaviest-path weights *lower bound* the payment to each agent in any envy-free payment vector of an envy-freeable allocation [11].) Note that given any payment vector that eliminates envy, we may uniformly increase or decrease the payments to all agents while maintaining envy-freeness. As a consequence, in the payment vector that minimizes the total subsidy, there is at least one agent that receives a payment of 0. Together these arguments give the following very useful observation.

Observation 1. *For any envy-freeable allocation $\mathcal{A}$, the minimum total subsidy required is at most $(n-1)\cdot \ell^{\max}_{G_{\mathcal{A}}}$, where $\ell^{\max}_{G_{\mathcal{A}}}$ is the maximum weight of a directed path in the envy graph $G_{\mathcal{A}}$.*

Halpern and Shah [11] then prove:

Theorem 2.2. *[11] For any envy-freeable allocation $\mathcal{A}$, the minimum total subsidy required is at most $(n-1)\cdot m$.*

Proof. In a minimum subsidy vector, at least one agent requires no subsidy. Thus it suffices to show that the subsidy to any agent i is at most m. By Observation 1, it suffices to show that the heaviest path weight starting at any vertex is at most m. Without loss of generality, let the heaviest path be $P = \{1, 2, \ldots, r\}$. The subsidy made to agent 1 can then be upper bounded by

$$\ell_{G_{\mathcal{A}}}(1) \;=\; \sum_{(i,k)\in P} w_{\mathcal{A}}(i,k) \;=\; \sum_{i=1}^{r-1} (v_i(A_{i+1}) - v_i(A_i)) \;\leq\; \sum_{i=1}^{r-1} v_i(A_{i+1}) \;\leq\; \sum_{i=1}^{r-1} |A_{i+1}| \;\leq\; |J| \;=\; m.$$

Here the second inequality holds because each agent has value at most one for any item. The third inequality is due to the fact that for the allocation $\mathcal{A}$ the bundles $\{A_1, A_2, \ldots, A_n\}$ are disjoint. Consequently $p_i \leq m$ for each agent, as required. □

For an arbitrary envy-freeable allocation $\mathcal{A}$ the bound in Theorem 2.2 is tight. To see this, consider the example where every agent has value 1 for each item, and the grand bundle (containing all items) is given to agent 1. This allocation is envy-freeable, and here each of the other $n-1$ agents requires a subsidy of m for envy-freeness. Ergo, to provide an improved bound on the total subsidy, we cannot consider any generic envy-freeable allocation. Instead, our task is find a specific envy-freeable allocation where the heaviest paths in the associated envy graph have much smaller weight. In particular, for the case of additive agents, we want that these path weights are at most 1 rather than at most m. This is our goal in the subsequent sections of the paper.

Before doing this, let us briefly discuss some computational aspects. Theorem 2.1 provides efficient methods to test if a given allocation is envy-freeable. For example, this can be achieved via a maximum-weight bipartite matching algorithm to verify Condition (b). Alternatively, Condition (c) can be tested in polynomial time using the Floyd-Warshall algorithm.[1] Finally, given an arbitrary non-envy-freeable allocation $\mathcal{A}$, one can efficiently find a corresponding envy-freeable allocation $\mathcal{A}^{\pi}$ by fixing the n bundles of the given allocation and computing a maximum-weight bipartite matching between the agents and the bundles.

3 An Allocation Algorithm for Additive Agents

In this section we present an allocation algorithm for the case of additive agents. Recall our task is to construct an envy-freeable allocation $\mathcal{A}$ with maximum path weight 1 in the envy graph $G_{\mathcal{A}}$. We do this via an allocation algorithm defined on the valuation graph for the instance. The valuation graph H is the complete bipartite graph on vertex sets I and J, where edge (i,j) has weight $v_i(j)$. We denote by $h[\hat{I}, \hat{J}]$ the subgraph of H induced by $\hat{I} \subseteq I$ and $\hat{J} \subseteq J$. The allocation algorithm then proceeds in rounds where each agent is matched to exactly one item in each round. For the first round, we set $J_1 = J$. In round t, we then find a maximum-weight matching M_t in $H[I, J_t]$. If agent i is matched to item $j = \mu_i^t$ then we allocate item μ_i^t to that agent. We then recurse on the remaining items $J_{t+1} = J_t \setminus \cup_{i\in I}\mu_i^t$. The process ends when every item has been allocated. This procedure is formalized via pseudocode in Algorithm 1.

Suppose the algorithm terminates in T rounds. We assume that every agent receives an item in each round. For rounds 1 to $T-1$ this is evident because agent i can be assigned a item for which

[1] In fact, a simple reduction converts the problem of finding minimum payments for a fixed allocation into a shortest-paths problem and any efficient shortest-paths algorithm can be applied.

Algorithm 1: Bounded-Subsidy Algorithm

$A_i \leftarrow \emptyset$ for all $i \in I$;
$t \leftarrow 1$; $J_1 \leftarrow J$;
while $J_t \neq \emptyset$ **do**
    Compute a *maximum-weight matching* $M^t = \{(i, \mu_i^t)\}_{i\in I}$ in $H[I, J_t]$;
    Set $A_i \leftarrow A_i \cup \{\mu_i^t\}$ for all $i \in I$;
    Set $J_{t+1} \leftarrow J_t \setminus \cup_{i\in I}\mu_i^t$;
    $t \leftarrow t + 1$;
end

it has zero value. For round T, we assume there are exactly n items remaining, possibly by adding dummy items of no value to any agent.

This algorithm has many interesting properties. In this section we prove that it outputs an envy-freeable allocation $\mathcal{A}$. Furthermore, the allocation $\mathcal{A}$ is EF1, thus settling Conjecture 1.2. The allocation is also balanced in that (discarding any additional dummy items) the bundles that the agents receive differ in size by at most one item; in particular, each agent receives a bundle of size either $\lfloor \frac{m}{n} \rfloor$ or $\lceil \frac{m}{n} \rceil$. The allocation algorithm also clearly runs in polynomial time.

We also show in this section that any allocation $\mathcal{A}$ that is both envy-freeable and EF1 has a heaviest path weight in the envy graph of weight at most $n-1$. Thus, By Observation 1, the algorithm outputs an allocation that requires a subsidy of at most $(n-1)^2$. As claimed though, the heaviest path weight in $G_{\mathcal{A}}$ is in fact at most one and so the total subsidy needed is at most $n-1$. We defer the proof of this fact, our main result, to Section 4.

3.1 The Allocation Is Envy-freeable

Let's first see that the output allocation $\mathcal{A}$ is envy-freeable.

Lemma 3.1. *The output allocation $\mathcal{A}$ is envy-freeable.*

Proof. Let M^t be the maximum matching found in round t and $\boldsymbol{\mu}^t = \{\mu_1^t, \mu_2^t, \dots, \mu_n^t\}$ the corresponding items allocated in that round. By Theorem 2.1 it suffices to show that no directed cycle in the envy graph corresponding to the final allocation $\mathcal{A}$ has positive weight. Take any directed cycle C in the envy graph $G_{\mathcal{A}}$. Again, we may assume without loss of generality that $C = \{1, 2, \dots, r\}$ for some $r \geq 2$. We have

$$\begin{aligned}
w_{\mathcal{A}}(C) &= \sum_{(i,k)\in C} w_{\mathcal{A}}(i,k) \\
&= \sum_{(i,k)\in C} [v_i(A_k) - v_i(A_i)] \\
&= \sum_{(i,k)\in C} \sum_{t=1}^{T} \left[v_i(\mu_k^t) - v_i(\mu_i^t)\right] \\
&= \sum_{(i,k)\in C} \sum_{t=1}^{T} w_{\boldsymbol{\mu}^t}(i,k) \\
&= \sum_{t=1}^{T} \sum_{(i,k)\in C} w_{\boldsymbol{\mu}^t}(i,k).
\end{aligned}$$

Let π_C be the permutation of I under which $\pi_C(i) = i + 1$ for each $i \leq r - 1$, $\pi_C(r) = 1$, and $\pi_C(i) = i$ otherwise. In each round t, since M_t is a maximum-weight matching, $\sum_{(i,k)\in C} w_{\boldsymbol{\mu}^t}(i,k)$ is non-positive: otherwise, the matching $\hat{M}^t$ obtained by allocating to each agent i the item $\mu^t_{\pi_C(i)}$ has greater weight than M_t, a contradiction. Thus $w_{\mathcal{A}}(C)$ is also non-positive. Consequently, by Theorem 2.1 the allocation produced by the algorithm is envy-freeable. $\square$

3.2 The Allocation Is EF1

We say that an allocation $\mathcal{A}$ satisfies the *envy bounded by a single good* property, and is *EF1*, if for each pair i, k of agents, either $A_k = \emptyset$ or there exists an item $j \in A_k$ such that

$$v_i(A_i) \geq v_i(A_k \setminus \{j\}).$$

Next, let's prove the output allocation $\mathcal{A}$ is EF1.

Lemma 3.2. *The output allocation $\mathcal{A}$ is EF1.*

Proof. Let $\mathcal{A} = \{A_1, \ldots, A_n\}$. Recall, in any round t, the algorithm computes a maximum-weight matching M^t in $H[I, J_t]$ and allocates item μ^t_i to agent i. Thus $A_i = \{\mu^1_i, \ldots, \mu^T_i\}$ is the set of items allocated to agent i. Observe that $v_i(\mu^t_i) \geq v_i(j)$ for any item $j \in J_{t+1}$, the collection of items unallocated at the start of round $t+1$. Otherwise, we can replace the edge (i, μ^t_i) with (i, j) in M_t, to obtain a higher-weight matching in $H[I, J_t]$. Therefore, for any pair of agents i and k, we have

$$\begin{aligned}
v_i(A_i) &= v_i(\{\mu^1_i, \ldots, \mu^T_i\}) \\
&= v_i(\mu^1_i) + \cdots + v_i(\mu^{T-1}_i) + v_i(\mu^T_i) \\
&\geq v_i(\mu^1_i) + \cdots + v_i(\mu^{T-1}_i) \\
&\geq v_i(\mu^2_k) + \cdots + v_i(\mu^T_k) \\
&= v_i(A_k \setminus \{\mu^1_k\}).
\end{aligned}$$

Ergo, the output allocation $\mathcal{A}$ is EF1. $\square$

Claim 3.3. *Let $\mathcal{A}$ be both envy-freeable and EF1. Then the minimum total subsidy required is at most $(n-1)^2$.*

Proof. Since there is an agent that requires no subsidy, it suffices to prove that the maximum path weight in the envy graph $G_{\mathcal{A}}$ is at most $n-1$. But $\mathcal{A}$ is EF1. So agent i envies agent k by at most one, the maximum value of a single item. Thus every arc (i,k) has weight at most one, that is, $w_{\mathcal{A}}(i,k) \leq 1$. The result follows as any path contains at most $n-1$ arcs. $\square$

Since we have shown that the output allocation $\mathcal{A}$ is both envy-freeable and EF1, it immediately follows by Claim 3.3 that it requires a total subsidy of at most $(n-1)^2$.

4 The Subsidy Required Is at Most One per Agent

In this section we complete our analysis of the additive setting. By the EF1 property of the output allocation $G_{\mathcal{A}}$ we have an upper bound of 1 on the weight of any arc in the envy graph $G_{\mathcal{A}}$. But this is insufficient to accomplish our goal of proving that the envy graph has maximum path weight 1. How can we do this? As a thought experiment, imagine that, rather than an upper bound of 1 on each arc weight, we have a lower bound of -1 on each arc weight. The subsequent lemma proves this would be a sufficient condition!

Lemma 4.1. *Let $\mathcal{A}$ be an envy-freeable allocation. If $w_{\mathcal{A}}(i,k) \geq -1$ for every arc (i,k) in the envy graph then the maximum subsidy required is at most one per agent.*

Proof. By Theorem 2.1, as $\mathcal{A}$ is an envy-freeable the envy graph $G_{\mathcal{A}}$ contains no positive-weight cycles. Let P be the maximum-weight path in $G_{\mathcal{A}}$. Without loss of generality, $P = \{1, 2, \ldots, i\}$ with weight $p_1 = \ell_{G_{\mathcal{A}}}(1)$. Now take the directed cycle $C = P \cup (i, 1)$. Because C has non-positive weight and every arc weight is at least -1, we obtain

$$0 \;\geq\; w_{\mathcal{A}}(C) \;=\; \ell_{G_{\mathcal{A}}}(1) + w_{\mathcal{A}}(i,1) \;\geq\; \ell_{G_{\mathcal{A}}}(1) - 1.$$

Therefore $\ell_{G_{\mathcal{A}}}(1) \leq 1$ and the maximum subsidy is at most one. □

At first glance, Lemma 4.1 seems of little use. We already know every arc in the envy graph has weight at most 1. Suppose in addition that every arc weight was at least -1. That is, $1 \geq w_{\mathcal{A}}(i,k) \geq -1$ for each arc (i,k). Consequently, $v_i(A_i) \leq v_i(A_k)+1$ and $v_i(A_k) \leq v_i(A_i)+1$. In instances with a large number of valuable items this means that every agent is essentially indifferent over which bundle in $\mathcal{A}$ they receive. It is unlikely that an allocation with this property even exists for every instance, and certainly not the case that our algorithm outputs such an allocation.

The trick is to apply Lemma 4.1 to a modified fair division instance. In particular we construct, for each agent i, a modified valuation function $\bar{v}_i$ from v_i. We then prove that the allocation $\mathcal{A}^{\boldsymbol{v}}$ output for the original valuation profile $\boldsymbol{v}$ is envy-freeable even for the modified valuation profile $\bar{\boldsymbol{v}}$. Next we show that with this same allocation, every arc weight is at least -1 in the envy graph under the modified valuation profile $\bar{\boldsymbol{v}}$. By Lemma 4.1, this implies that the maximum subsidy required is at most one for the valuation profile $\bar{\boldsymbol{v}}$. To complete the proof we show that the maximum subsidy required by each agent for the original valuation profile $\boldsymbol{v}$ is at most the subsidy required for $\bar{\boldsymbol{v}}$.

4.1 A Modified Valuation Function

Let $\mathcal{A}^{\boldsymbol{v}} = \{A_1^{\boldsymbol{v}}, \ldots, A_n^{\boldsymbol{v}}\}$ be the allocation output by our algorithm under the original valuation profile $\boldsymbol{v}$. We now create the modified valuation profile $\bar{\boldsymbol{v}}$. For each agent i, define $\bar{v}_i$ according to the rule:

$$\begin{aligned}
&\bar{v}_i(\mu_i^t) = v_i(\mu_i^t) && \forall t \leq T\\
&\bar{v}_i(\mu_k^t) = \max\left(v_i(\mu_k^t), v_i(\mu_i^{t+1})\right) && \forall k \in I \setminus \{i\},\ \forall t \leq T-1\\
&\bar{v}_i(\mu_k^T) = v_i(\mu_k^T) && \forall k \in I \setminus \{i\}.
\end{aligned}$$

That is, the value $\bar{v}_i(j)$ remains the same for any item $j \in A_i^{\boldsymbol{v}}$ that was allocated to agent i by the algorithm. For any other item j, the value $\bar{v}_i(j)$ is the maximum of the original value $v_i(j)$ and the value of the item allocated to i by the algorithm in the round that immediately follows the round where j was allocated to some agent.

The following two observations are trivial but will be useful.

Observation 2. *For any agent i and item $j \in A_i^{\boldsymbol{v}}$, we have $v_i(j) = \bar{v}_i(j)$.* □

Observation 3. *For any agent i and item $j \notin A_i^{\boldsymbol{v}}$, we have $v_i(j) \leq \bar{v}_i(j)$.* □

We will show the bound on the subsidy by a sequence of claims based on the proof plan outlined above. First we show that $\mathcal{A}^{\boldsymbol{v}}$ envy-freeable even under the modified valuation profile.

Claim 4.2. *The allocation $\mathcal{A}^{\boldsymbol{v}}$ output under the original valuation profile $\boldsymbol{v}$ is an envy-freeable allocation under the modified valuation profile $\bar{\boldsymbol{v}}$.*

Proof. By Theorem 2.1, to show that the allocation $\mathcal{A}^{\boldsymbol{v}}$ is envy-freeable under the modified valuation profile $\bar{\boldsymbol{v}}$ we must show that there is no positive-weight cycle in the envy graph using the modified values. So suppose cycle C has positive modified weight. To obtain a contradiction, first observe that, in the allocation $\mathcal{A}^{\boldsymbol{v}}$, agent i receives the bundle $\mathcal{A}^{\boldsymbol{v}}_i = \{\mu_i^1, \mu_i^2, \dots, \mu_i^T\}$. Thus with respect to $\bar{\boldsymbol{v}}$ the envy agent i has for agent k is

$$\bar{v}_i(\mathcal{A}^{\boldsymbol{v}}_k) - \bar{v}_i(\mathcal{A}^{\boldsymbol{v}}_i) \;=\; \sum_{t=1}^{T} \bar{v}_i(\mu_k^t) - \sum_{t=1}^{T} \bar{v}_i(\mu_i^t) \;=\; \sum_{t=1}^{T} \left(\bar{v}_i(\mu_k^t) - \bar{v}_i(\mu_i^t)\right). \tag{1}$$

As the envy graph contains a positive-weight cycle C we have, by (1), that

$$0 < \sum_{(i,k)\in C} \bar{v}_i(\mathcal{A}^{\boldsymbol{v}}_k) - \bar{v}_i(\mathcal{A}^{\boldsymbol{v}}_i) \;=\; \sum_{(i,k)\in C} \sum_{t=1}^{T} \left(\bar{v}_i(\mu_k^t) - \bar{v}_i(\mu_i^t)\right) \;=\; \sum_{t=1}^{T} \sum_{(i,k)\in C} \left(\bar{v}_i(\mu_k^t) - \bar{v}_i(\mu_i^t)\right).$$

This implies there exists a round t such that

$$\sum_{(i,k)\in C} \bar{v}_i(\mu_k^t) > \sum_{(i,k)\in C} \bar{v}_i(\mu_i^t). \tag{2}$$

Now M^t is a maximum-weight matching in $H[I, J_t]$ for the original valuation profile $\boldsymbol{v}$. Let $\hat{M}^t$ be the matching formed from M^t by permuting around the cycle C the bundles of the agents in C. But then, by (2), the matching $\hat{M}^t$ has greater weight in $H[I, J_t]$ than the matching M^t for the modified profile $\bar{\boldsymbol{v}}$. Consequently, we will obtain our contradiction if we can prove that M^t is a maximum-weight matching in $H[I, J_t]$ even with respect to $\bar{\boldsymbol{v}}$.

This is true in the final round matching; clearly M^T is a maximum-weight matching in $H[I, J_T]$ because, by definition, $\bar{\boldsymbol{v}}$ and $\boldsymbol{v}$ have the same value for items in J_T. Thus, it remains to prove the statement for each round $t \le T-1$. Now $\boldsymbol{\mu}^t = \{\mu_1^t, \mu_2^t, \dots, \mu_n^t\}$ is the allocation of the items round t. Again, for a contradiction, assume that matching M^t is not maximum in $H[I, J_t]$ for the valuation profile $\bar{\boldsymbol{v}}$. Then, by Theorem 2.1, the envy graph $G_{\boldsymbol{\mu}^t}$ contains a positive-weight directed cycle C. Without loss of generality, let $C = \{1, \dots, r\}$.

We divide our analysis into two cases, depending on whether the weights on the arcs of C change when the valuation profile is modified from $\boldsymbol{v}$ to $\bar{\boldsymbol{v}}$. Specifically, we call an arc $(i, i+1)$ of C *blue* if $\bar{v}_i(\mu_{i+1}^t) = v_i(\mu_{i+1}^t)$, that is, agent i's value for the item allocated to agent $i+1$ does not change when the valuation profile is modified. We call an arc *red* otherwise. Observe that if the arc $(i, i+1)$ of C is red, then $\bar{v}_i(\mu_{i+1}^t) = v_i(\mu_i^{t+1}) > v_i(\mu_{i+1}^t)$, so in the original valuation function agent i strictly prefers the item that it is allocated in round $t+1$ to the item that agent $i+1$ is allocated in round t. In turn, this implies that the weight on any red arc is necessarily negative. We have the following two cases to consider.

(i) *Every arc of C is blue.* Let π_C be the permutation of I under which $\pi_C(i) = i+1$ for each $i \le r-1$, $\pi_C(r) = 1$, and $\pi_C(i) = i$ otherwise. The matching $\mathcal{M}^t$ obtained by allocating to each agent i the item $\mu_{\pi_C(i)}^t$ has greater weight than M^t with respect to the original valuation profile $\boldsymbol{v}$, contradicting the assumption that the algorithm selected a matching of maximum weight.

(ii) *C contains a red arc.* In this case, C can be decomposed into a sequence of d directed paths $P_1, \dots, P_d$ such that each directed path consists of a (possibly empty) sequence of blue arcs followed by exactly one red arc. Figure 1 shows an example of such a decomposition. In the figure, blue arcs are represented by solid lines and red arcs by dashed lines.

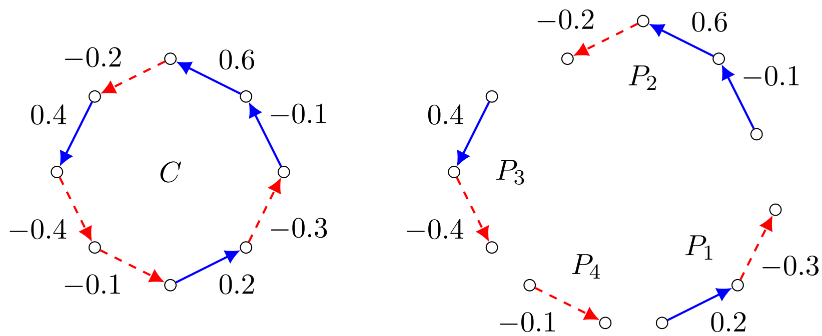

Figure 1: An example showing the decomposition of C into directed paths $P_1, \ldots, P_4$. In this example, P_2 has positive weight.

Now, since C has positive total weight, there is a directed path $P \in \{P_1, \ldots, P_d\}$ of positive total weight. Without loss of generality, let $P = \{1, 2, \ldots, k+1\}$. Thus in the envy graph $G_{\boldsymbol{\mu}^t}$ we have

$$w_{\boldsymbol{\mu}^t}(P) \;=\; \sum_{i=1}^{k} w_{\boldsymbol{\mu}^t}(i, i+1) \;>\; 0. \tag{3}$$

Construct a matching $\mathcal{M}^t = \{(i, \omega_i^t)\}_{i \in I}$ in the following manner. For each agent $i \geq k+1$, set $\omega_i^t = \mu_i^t$; that is, the end-vertex of the path P and all agents not on P are matched to the same item in $\mathcal{M}^t$ as in M^t. For each agent $i \leq k-1$, let $\omega_i^t = \mu_{i+1}^t$, that is in the allocation $\mathcal{M}^t$ agent i receives the item that agent $i+1$ receives in M^t. Finally, for agent k let $\omega_k^t = M_k^{t+1}$; that is, in $\mathcal{M}^t$ agent k receives the item it would have received in the next round in M^{t+1}.

Observe that every item allocated by $\mathcal{M}^t$ was available for allocation in round t and, thus, it was a feasible allocation to select in round t. Next let's compare the relative values of $\mathcal{M}^t$ and M^t under the original valuations $\boldsymbol{v}$. To do this, observe that by definition of $\mathcal{M}^t$ we have

$$\begin{aligned}
v(\mathcal{M}^t) - v(M^t) &= \sum_{i=1}^{k} \big(v_i(\omega_i^t) - v_i(\mu_i^t)\big) \\
&= \sum_{i=1}^{k-1} \big(v_i(\omega_i^t) - v_i(\mu_i^t)\big) + \big(v_k(\omega_k^t) - v_k(\mu_k^t)\big) \\
&= \sum_{i=1}^{k-1} \big(v_i(\mu_{i+1}^t) - v_i(\mu_i^t)\big) + \big(v_k(\mu_k^{t+1}) - v_k(\mu_k^t)\big)\,.
\end{aligned} \tag{4}$$

But $(k, k+1)$ is a red arc in $G_{\boldsymbol{\mu}^t}$. Therefore, it must be the case that $v_k(\mu_k^{t+1}) > v_k(\mu_{k+1}^t)$. Plugging this into (4) gives

$$\begin{aligned}
v(\mathcal{M}^t) - v(M^t) \;&>\; \sum_{i=1}^{k-1} \big(v_i(\mu_{i+1}^t) - v_i(\mu_i^t)\big) + \big(v_k(\mu_{k+1}^t) - v_k(\mu_k^t)\big) \\
&=\; \sum_{i=1}^{k} \big(v_i(\mu_{i+1}^t) - v_i(\mu_i^t)\big)\,.
\end{aligned} \tag{5}$$

But, by definition, $w_{\boldsymbol{\mu}^t}(i, i+1) = v_i(\mu_{i+1}^t) - v_i(\mu_i^t)$. So, together (3) and (5) imply

$$v(\mathcal{M}^t) - v(M^t) \ > \ \sum_{i=1}^{k} w_{\boldsymbol{\mu}^t}(i, i+1) \ > \ 0. \tag{6}$$

Thus $\mathcal{M}^t$ has greater weight than M^t under the original valuations $\boldsymbol{v}$. This contradicts the optimality of M^t. $\square$

Claim 4.2 shows that the allocation $\mathcal{A}^{\boldsymbol{v}}$ produced by the algorithm on the original instance is an envy-freeable allocation in the modified instance. We next show that for this modified valuation profile the subsidy required is at most 1 for each agent. In particular the total subsidy is at most $n-1$.

Claim 4.3. *For the envy-freeable allocation $\mathcal{A}^{\boldsymbol{v}}$ the subsidy to each agent is at most* 1 *for the modified valuation profile $\bar{\boldsymbol{v}}$.*

Proof. Take the valuation profile $\bar{\boldsymbol{v}}$ and the allocation $\mathcal{A}^{\boldsymbol{v}} = \{A_1^{\boldsymbol{v}}, \dots, A_n^{\boldsymbol{v}}\}$. We claim that for any arc (i,k) its modified weight $\bar{w}_{\boldsymbol{A}^{\boldsymbol{v}}}(i,k)$ in the the envy graph is *at least* -1. To prove this take any pair of agents i and k. Then

$$\begin{aligned}
\bar{w}_{\boldsymbol{A}^{\boldsymbol{v}}}(i,k) &= \bar{v}_i(A_k^{\boldsymbol{v}}) - \bar{v}_i(A_i^{\boldsymbol{v}}) \\
&= \sum_{t=1}^{T} \bar{v}_i(\mu_k^t) - \sum_{t=1}^{T} \bar{v}_i(\mu_i^t) \\
&= \sum_{t=1}^{T} \bar{v}_i(\mu_k^t) - \sum_{t=1}^{T} v_i(\mu_i^t) \\
&= \sum_{t=1}^{T-1} \max(v_i(\mu_k^t), v_i(\mu_i^{t+1})) + v_i(\mu_i^T) - \sum_{t=1}^{T} v_i(\mu_i^t) \\
&\geq \sum_{t=1}^{T-1} v_i(\mu_i^{t+1}) - \sum_{t=1}^{T-1} v_i(\mu_i^t). \qquad (7)
\end{aligned}$$

We can simplify (7) and lower bound it via a telescoping sum:

$$\begin{aligned}
\bar{w}_{\boldsymbol{A}^{\boldsymbol{v}}}(i,k) &\geq \sum_{t=1}^{T-1} \left(v_i(\mu_i^{t+1}) - v_i(\mu_i^t)\right) \\
&= v_i(\mu_i^T) - v_i(\mu_i^1) \\
&\geq -v_i(\mu_i^1) \\
&\geq -1. \qquad (8)
\end{aligned}$$

Now by Claim 4.2, the allocation $\mathcal{A}^{\boldsymbol{v}}$ is envy-freeable with respect to the valuations $\bar{\boldsymbol{v}}$. Applying Lemma 4.1, because the arc weights are lower bounded by -1 the subsidy required per agent is then at most one for the modified valuation profile $\bar{\boldsymbol{v}}$.

Finally, since there is an agent whose payment is 0, the total subsidy required is upper bounded by $n-1$. $\square$

The following claim shows that, for any agent, the subsidy for the original valuation profile is at most the subsidy required for the modified valuation function.

Claim 4.4. *For the allocation $\mathcal{A}^{\boldsymbol{v}}$ the subsidy required by an agent given valuation profile $\boldsymbol{v}$ is at most the subsidy required given valuation profile $\bar{\boldsymbol{v}}$.*

Proof. By Observation 2, $v_i(j) = \bar{v}_i(j)$ for any $j \in A_i^{\boldsymbol{v}}$. Therefore, by additivity,

$$\bar{v}_i(A_i^{\boldsymbol{v}}) \;=\; \sum_{j \in A_i^{\boldsymbol{v}}} \bar{v}_i(j) \;=\; \sum_{j \in A_i^{\boldsymbol{v}}} v_i(j) \;=\; v_i(A_i^{\boldsymbol{v}}). \tag{9}$$

On the other hand, Observation 3 states that $v_i(j) \le \bar{v}_i(j)$ for any $j \notin A_i^{\boldsymbol{v}}$. Thus, for any pair i and k of agents, we have

$$\bar{v}_i(A_k^{\boldsymbol{v}}) \;=\; \sum_{j \in A_k^{\boldsymbol{v}}} \bar{v}_i(j) \;\ge\; \sum_{j \in A_k^{\boldsymbol{v}}} v_i(j) \;=\; v_i(A_k^{\boldsymbol{v}}). \tag{10}$$

Combining (9) and (10) gives

$$\bar{w}_{\boldsymbol{A}^{\boldsymbol{v}}}(i,k) \;=\; \bar{v}_i(A_k^{\boldsymbol{v}}) - \bar{v}_i(A_i^{\boldsymbol{v}}) \;\ge\; v_i(A_k^{\boldsymbol{v}}) - v_i(A_i^{\boldsymbol{v}}) \;=\; w_{\mathcal{A}^{\boldsymbol{v}}}(i,k).$$

Consequently, the weight of any arc (i,k) in the envy graph with the modified valuation profile is at least its weight with the original valuation profile. Therefore the weight of any path in the envy graph is higher with the modified valuation profile than with the original valuation profile. The claim follows. ◻

Together Claims 4.3 and 4.4 give our main result.

Theorem 1.3. *For additive valuations there is an envy-freeable allocation where the subsidy to each agent is at most one dollar. (This allocation is also EF1, balanced, and can be computed in polynomial time.)*

5 Bounding the Subsidy for Monotone Valuations

We now consider the much more general setting where the valuations of the agents are arbitrary monotone functions. That is, the only assumptions we impose are that $v_i(S) \le v_i(T)$ when $S \subseteq T$ and the basic assumption that $v_i(\emptyset) = 0$. Without loss of generality, we may scale the valuations so that the marginal value of each item for any agent never exceeds one dollar. Our goal in this section is to show that there is an envy-freeable allocation in which the total subsidy required for envy-freeness is at most $2(n-1)^2$. In particular, the total subsidy required is independent of of the number of items m. When $m > 2(n-1)$ this bound beats the bound $(n-1) \cdot m$ of [11] for additive valuations described in Theorem 2.2 and, more importantly, it applies to the far more general class of arbitrary monotone valuations.

Our method to compute the desired envy-freeable allocation begins with finding an EF1 allocation. The well-known *envy-cycles* algorithm of Lipton et al. [14] finds such an allocation in polynomial time given oracle access to the valuations, under the same mild conditions on the valuations. For completeness, we briefly describe the envy-cycles algorithm. The algorithm proceeds in a sequence of m rounds, allocating one item in each round. At any point during the algorithm, we denote by G the envy graph corresponding to the current allocation, and by H the subgraph of G that consists of all the agents and only the arcs that have positive weight, that is, positive envy. We call H the *auxiliary graph* of G. The algorithm relies on the following lemma.

Lemma 5.1. *[14] For any partial allocation $\mathcal{A}$ with auxiliary graph H, there is another partial allocation $\mathcal{A}'$ with auxiliary graph H' such that*

- *H' is acyclic.*
- *For each agent i, the maximum weight of an outgoing arc from i is less in $\mathcal{A}'$ than in $\mathcal{A}$.*

The basic idea of the algorithm then is to maintain the following two invariants: (i) at each step, the partial allocation is EF1, and (ii) at the start and end of each round, the auxiliary graph H is acyclic. Since the auxiliary graph is a directed acyclic graph at the start of each round, it has a source vertex. The algorithm simply chooses this vertex and allocates the next item to the corresponding agent. Because no other agent envies this agent before this item is allocated, the envies of the other agents are bounded by the value of this item (so the allocation of this item maintains the EF1 invariant). Next, the algorithm identifies a directed cycle (if one exists) in the auxiliary graph H and redistributes bundles by rotating them around this cycle. It is easy to see that the EF1 guarantee is maintained after this redistribution of the bundles, and that the number of arcs in H strictly decreases. All cycles in H are then eliminated in sequence until H is acyclic and the round ends. When all items have been allocated, the final allocation is EF1.

This immediately raises the question of whether the resulting allocation is envy-freeable. By Claim 3.3, we know that if an allocation is both envy-freeable and EF1, then the total subsidy required for envy-freeness is $(n-1)^2$, since the weight of any path is at most $n-1$. Unfortunately, it is possible that the allocation output by the envy-cycles algorithm is not envy-freeable. However, we show that an EF1 allocation can still be used to produce an envy-freeable allocation that requires only a small increase in the subsidy! Specifically, the following key lemma shows that if we begin by fixing the bundles of an EF1 allocation and then redistribute these bundles to produce an envy-freeable allocation, the weight of any path increases to at most $2(n-1)$. By Theorem 2.1, an envy-freeable allocation can be found by computing a maximum-weight matching.

Lemma 5.2. *Let $\mathcal{A}$ be an EF1 allocation, and $\mathcal{B}$ be the envy-freeable allocation corresponding to a maximum-weight matching between the agents and the bundles of $\mathcal{A}$. Then $\mathcal{B}$ can be made envy-free with a subsidy of at most $2(n-1)$ to each agent.*

Proof. Let $\mathcal{A} = \{A_1, \ldots, A_n\}$ be an EF1 allocation. So, for any pair i and k of agents, $v_i(A_k) - v_i(A_i) \leq 1$. Let π be a permutation of the bundles that maximizes $\sum_i v_i(A_{\pi(i)})$. Then, by Theorem 2.1, the allocation $\mathcal{B} = \{B_1, \ldots, B_n\} = \{A_{\pi(1)}, \ldots, A_{\pi(n)}\}$ is envy-freeable. Next, let P be a directed path in the envy graph $G_{\mathcal{B}}$. Without loss of generality, $P = \{1, 2, \ldots, r\}$ for some $r \geq 2$. Our goal is to show that the weight of P in $G_{\mathcal{B}}$ is at most $2(n-1)$. Clearly, the weight of P in $G_{\mathcal{A}}$ is at most $n-1$. Consider an arc $(i, i+1)$ of P. Since $\mathcal{A}$ is EF1, for any agent k, we have $v_i(A_k) - v_i(A_i) \leq 1$. Now, agent $i+1$ receives the bundle of agent $\pi(i+1)$ in the redistributed allocation $\mathcal{B}$. We have $v_i(A_{\pi(i+1)}) - v_i(A_i) \leq 1$ and, thus, $v_i(B_{i+1}) - v_i(A_i) \leq 1$. It follows that:

$$\begin{aligned}
w_{\mathcal{B}}(P) &= \sum_{(i,k)\in P} w_{\mathcal{B}}(i,k) \\
&= \sum_{i=1}^{r-1} \left(v_i(B_{i+1}) - v_i(B_i)\right) \\
&= \sum_{i=1}^{r-1} \left(v_i(B_{i+1}) - v_i(A_i) + v_i(A_i) - v_i(B_i)\right) \\
&\leq \sum_{i=1}^{r-1} \left(1 + v_i(A_i) - v_i(B_i)\right) \\
&\leq (n-1) + \sum_{i=1}^{r-1} \left(v_i(A_i) - v_i(B_i)\right) \qquad (11)
\end{aligned}$$

To complete the proof, it remains to show that $\sum_{i=1}^{r-1}(v_i(A_i)-v_i(B_i))$ is at most $n-1$. Together with (11), this implies that $w_{\mathcal{B}}(P) \le 2(n-1)$.

Since π maximizes $\sum_i v_i(A_{\pi(i)})$, we have $\sum_i v_i(B_i) \ge \sum_i v_i(A_i)$. The key observation is that, while the sum of values of the bundles received by all agents increases when we redistribute the bundles from $\mathcal{A}$ to $\mathcal{B}$, the value of the bundle received by any single agent increases by *at most* one because $\mathcal{A}$ is EF1. This then constrains the amount by which the total value for any subset of agents can *decrease*. Specifically, let $R \subseteq I$ be the set of agents i that receive a bundle B_i of smaller value than A_i, that is, $R = \{i \in I : v_i(B_i) < v_i(A_i)\}$. Let $S = I \setminus R$, so $S = \{i \in I : v_i(B_i) \ge v_i(A_i)\}$.

Now, we have two cases to consider.

(i) $|R| = 0$.
Then $\sum_{i=1}^{r-1}(v_i(A_i)-v_i(B_i)) \le 0$ and the result follows.

(ii) $|R| \ge 1$.
Then $|S| \le n-1$, and we have

$$\begin{aligned}
\sum_{i\in[r-1]}(v_i(A_i)-v_i(B_i)) &= \sum_{i\in[r-1]\cap R}(v_i(A_i)-v_i(B_i)) + \sum_{i\in[r-1]\cap S}(v_i(A_i)-v_i(B_i)) \\
&\le \sum_{i\in[r-1]\cap R}(v_i(A_i)-v_i(B_i)) \\
&\le \sum_{i\in R}(v_i(A_i)-v_i(B_i)) \\
&\le \sum_{i\in S}(v_i(B_i)-v_i(A_i)) \\
&\le n-1.
\end{aligned}$$

The second to last inequality says that the total decrease in value for agents in R is at most the total increase in value for agents in S (since B is an optimal redistribution of the bundles). The final inequality follows from the fact that $|S| \le n-1$ and for each $i \in S$, $v_i(B_i) - v_i(A_i) \le 1$ since $\mathcal{A}$ is EF1. □

Together, Lemmas 5.1 and 5.2 bound the total subsidy sufficient for envy-freeness when the valuation functions are monotone.

Theorem 1.4. *For monotonic valuations there is an envy-freeable allocation where the subsidy to each agent is at most $2(n-1)$ dollars. (Given a valuation oracle, this allocation can be computed in polynomial time.)* □

References

[1] A. Alkan, G. Demange, and D. Gale. Fair allocation of indivisible goods and criteria of justice. *Econometrica*, 59(4):1023–1039, 1991.

[2] N. Alon. Splitting necklaces. *Advances in Mathematics*, 63:241–253, 1987.

[3] E. Aragones. A derivation of the money Rawlsian solution. *Social Choice and Welfare*, 12(3): 267–276, 1995.

[4] V. Bilò, I. Caragiannis, M. Flammini, A. Igarashi, G. Monaco, D. Peters, C. Vinci, and W. Zwicker. Almost envy-free allocations with connected bundles. In *Proceedings of 10th Innovations in Theoretical Computer Science Conference (ITCS)*, pages 14:1–14:21, 2019.

[5] S. Brams and A. Taylor. An envy-free cake division protocol. *The American Mathematical Monthly*, 102(1):9–18, 1995.

[6] E. Budish. The combinatorial assignment problem: Approximate competitive equilibrium from equal incomes. *Journal of Political Economy*, 119(6):1061–1103, 2011.

[7] I. Caragiannis, D. Kurokawa, H. Moulin, A. Procaccia, N. Shah, and J. Wang. The unreasonable fairness of maximum Nash welfare. *ACM Trans. Econ. Comput.*, 7(3):12:1–32, 2019.

[8] D. Foley. Resource allocation and the public sector. *Yale Econ Essays*, 7(1):45–98, 1967.

[9] M. Ghodsi, M. Hajiaghayi, M. Seddighin, S. Seddighin, and H. Yami. Fair allocation of indivisible goods: Improvements and generalizations. In *Proceedings of the 19th ACM Conference on Economics and Computation (EC)*, pages 539–556, 2018.

[10] C-J. Haake, M. Raith, and F. Su. Bidding for envy-freeness: A procedural approach to n-player fair-division problems. *Social Choice and Welfare*, 19(4):723–749, 2002.

[11] D. Halpern and N. Shah. Fair division with subsidy. In *Proceedings of the 12th International Symposium on Algorithmic Game Theory (SAGT)*, pages 374–389, 2019.

[12] F. Klijn. An algorithm for envy-free allocations in an economy with indivisible objects and money. *Social Choice and Welfare*, 17:201–215, 2000.

[13] D. Kurokawa, A. Procaccia, and J. Wang. Fair enough: Guaranteeing approximate maximin shares. *J. ACM*, 65(2):8:1–27, 2018.

[14] R. Lipton, E. Markakis, E. Mossel, and A. Saberi. On approximately fair allocations of indivisible goods. In *Proceedings of the 5th ACM Conference on Electronic Commerce (EC)*, pages 125–131, 2004.

[15] E. Maskin. On the fair allocation of indivisible goods. In G. Feiwel, editor, *Arrow and the Foundations of the Theory of Economic Policy*, pages 341–349. MacMillan, 1987.

[16] H. Steinhaus. The problem of fair division. *Econometrica*, 16(1):101–104, 1948.

[17] F. Su. Rental harmony: Sperner's lemma in fair division. *The American Mathematical Monthly*, 106(10):930–942, 1999.

[18] L-G. Svensson. Large indivisibles: An analysis with respect to price equilibrium and fairness. *Econometrica*, 51(4):939–954, 1983.

[19] K. Tadenuma and W. Thomson. The fair allocation of an indivisible good when monetary compensations are possible. *Mathematical Social Sciences*, 25(2):117–132, 1993.

[20] H. Varian. Equity, envy and efficiency. *Journal of Economic Theory*, 9:63–91, 1974.