Computer Systems Fundamentals

CSE 232 Computer Systems Fundamentals

Dr. Gamal Fahmy

Logic Gates

Boolean Algebra

Gray Coded Number

0	0000
1	0001
2	0011
3	0010
4	0110
5	0111
6	0101
7	0100
8	1100
9	1101
10	1111
11	1110
12	1010
13	1011
14	1001
15	1000

- Rule of thumb
  - – 1 bit change per increment

0	00
1	01
2	11
3	10

Digital Logic

- **There are three types of logic gates from which we can build all digital computers.**
- **They're all related to semantic logic operators**
- **Consider:**

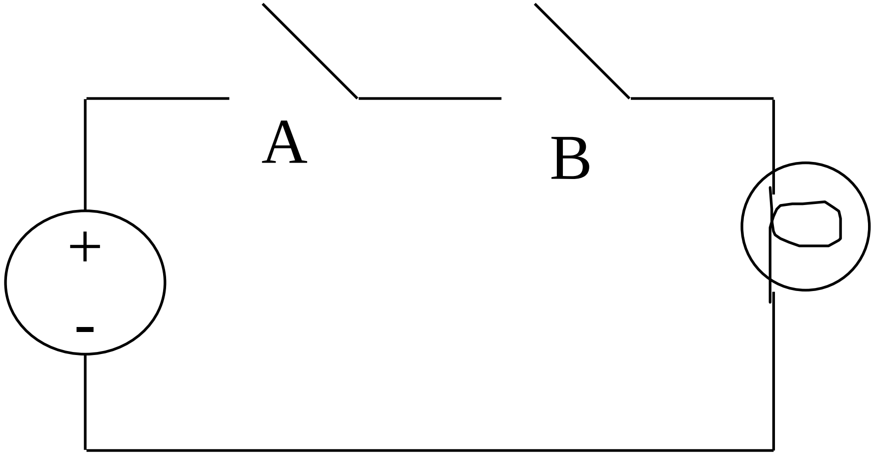
If Ahmed asks Tarek for Tennis AND Tarek accepts, then they'll go for a tennis game.

Digital Logic

- **Let T(*) be the truth value operator:**
  - **g = T(They'll go to a tennis game)**
  - **A = T(Ahmed Asks Tarek)**
  - **B = T(Tarek Accepts)**
- **g=A and B = A * B = A B**
  - **(If, Then provide the semantic equivalent of the equal sign.)**

Digital Logic

- **There is an electrical analog to the AND operator.**
- **True = Switch Closed   True = Light On**



AND Operator
Truth Table

A	B	g=A B
(F) 0	(F) 0	(F) 0
0	1	0
1	0	0
1	1	1

Digital Logic

Consider:
If Ahmed asks Tarek OR Mona Asks Tarek then, he’ll go to a tennis game.

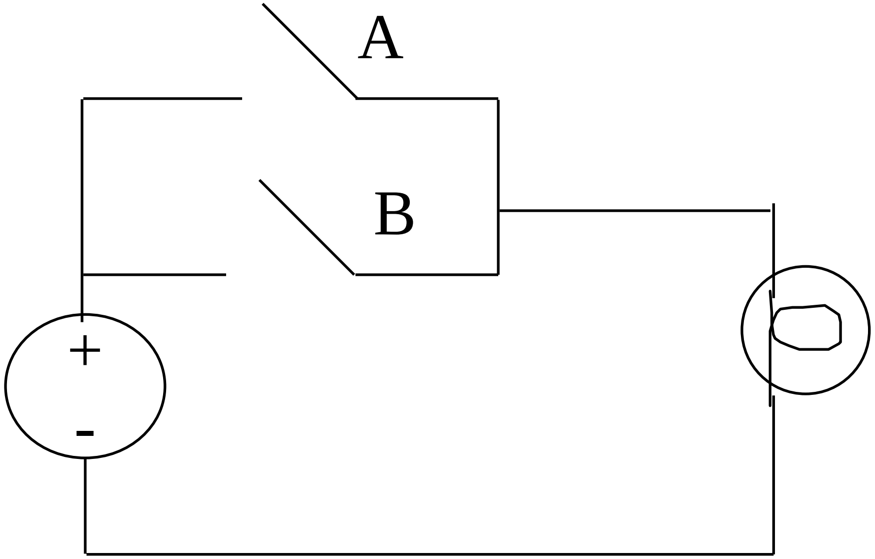
Let T(*) be the truth value operator:

- **g = T (He’ll go to a tennis game)**
- **A = T (Ahmed Asks Tarek)**
- **B = T (Mona Asks Tarek)**

g=A OR B = A + B

Digital Logic

- **There is an electrical analog to the OR operator.**
- **True = Switch Closed True = Light On**

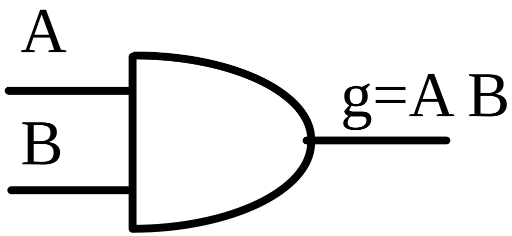


OR Operator
Truth Table

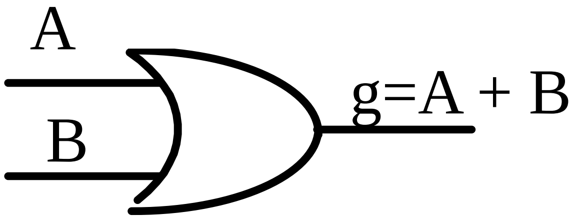
A	B	g=A + B
(F) 0	(F) 0	(F) 0
0	1	1
1	0	1
1	1	1

Gates

- **An AND gate has the electrical schematic:**



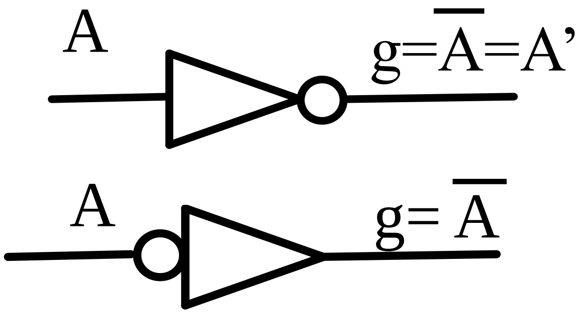
- **An OR gate has the electrical schematic:**



© Gamal Fahmy

Gates

- **One Last Logical operator/gate - NOT**



NOT Operator
Truth Table

A	g = A'
0	1
1	0

- **All digital computers are built using ONLY these three gate types: AND, OR, Inverter**

Digital Logic

- **All of our truth tables had 4 input combinations**
  - **2 Ways to pick the first input**
  - **2 Ways to pick the second input**
  - **2^2 Input Combinations**
  - **How many input combinations for a 3 input gate?**

Digital Logic

- **3 Input AND/OR Truth Tables**

3-input AND
Truth Table

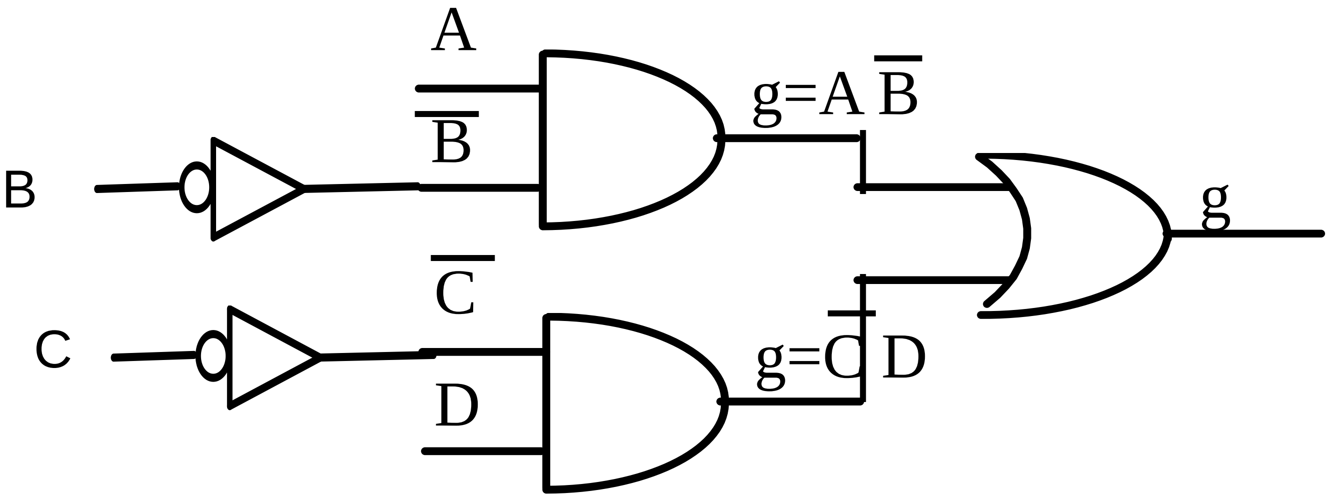
A	B	C	Y
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	0
1	0	0	0
1	0	1	0
1	1	0	0
1	1	1	1

3-Input OR
Truth Table

A	B	C	Y
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	1
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	1

Digital Logic

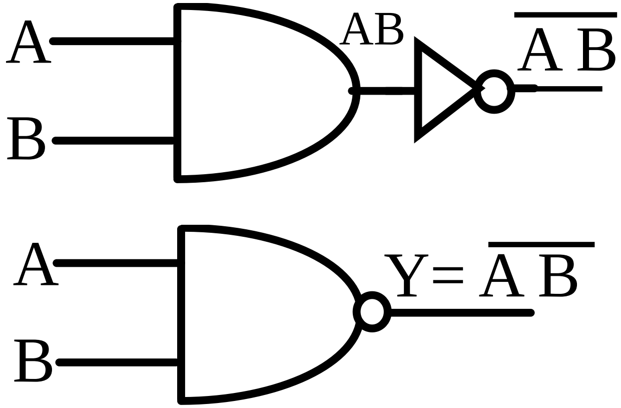
- Draw a Schematic for g=A$\overline{B}$+$\overline{C}$D



© Gamal Fahmy

Digital Logic

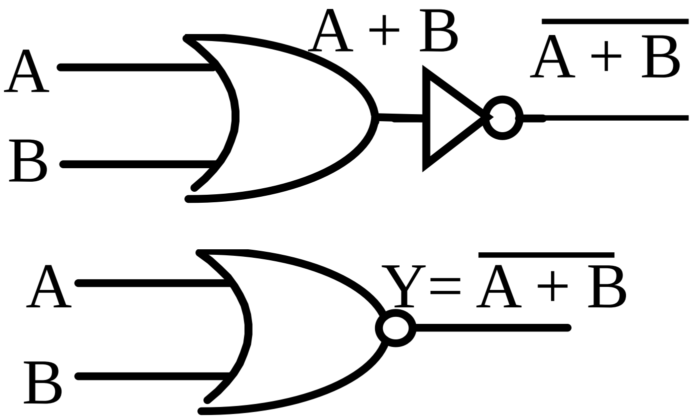
NAND Gate=NOT (AND)



NAND
Truth Table

A	B	Y
0	0	1
0	1	1
1	0	1
1	1	0

NOR Gate = NOT(OR)

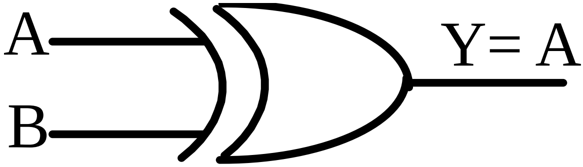


NOR
Truth Table

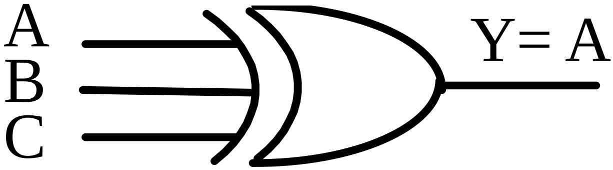
A	B	Y
0	0	1
0	1	0
1	0	0
1	1	0

Digital Logic

- Exclusive OR:Output = 1 if odd number of inputs = 1



$Y = A \oplus B = A\,\overline{B} + \overline{A}\,B$



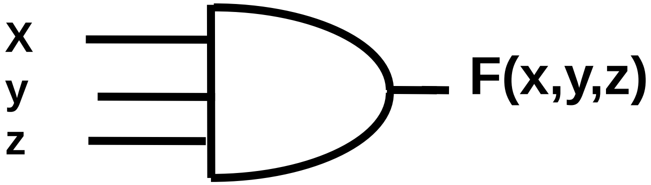
$Y = A \oplus B \oplus C$

3-Input XOR
Truth Table

A	B	C	Y
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	1

Boolean Algebra

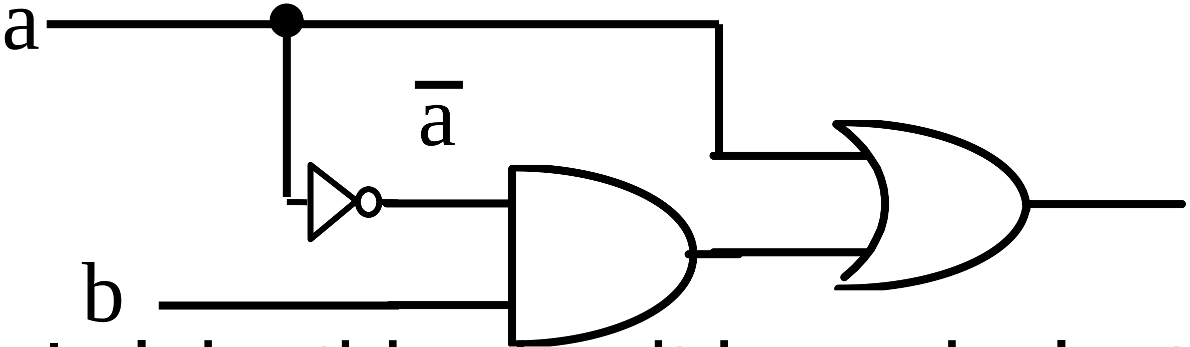
1. It is representing inputs/ variables x,y,z and output as function on the inputs F(x,y,z)
2. + is OR
3. ● is AND
4. ‘ is NOT



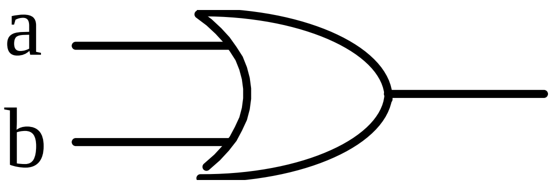
© Gamal Fahmy

Boolean Algebra

- Why study Boolean Algebra? Consider:



- I claim this circuit is equivalent to:



- Could use a truth table.
- For a 10 Variable function need 2^{10} lines.

Boolean Algebra

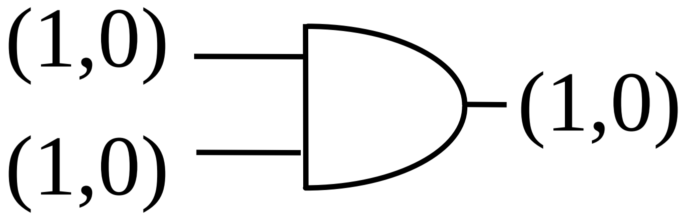
- George Boole (1815-1864), a mathematician sought to formalize logic using mathematical notation.
- He developed a consistent set of laws that were sufficient to define a new type of algebra: Boolean Algebra cf. Linear Algebra
- Many of the rules are the same as Linear Algebra. Some are different.

Boolean Algebra

- There are 7 fundamental laws, that tell us what operations are valid in Boolean algebra.

1. ***Field***: Boolean algebra operates over a field of numbers B with only two elements {0,1}.
2. ***Closure***: For every *a*, *b* in *B*,

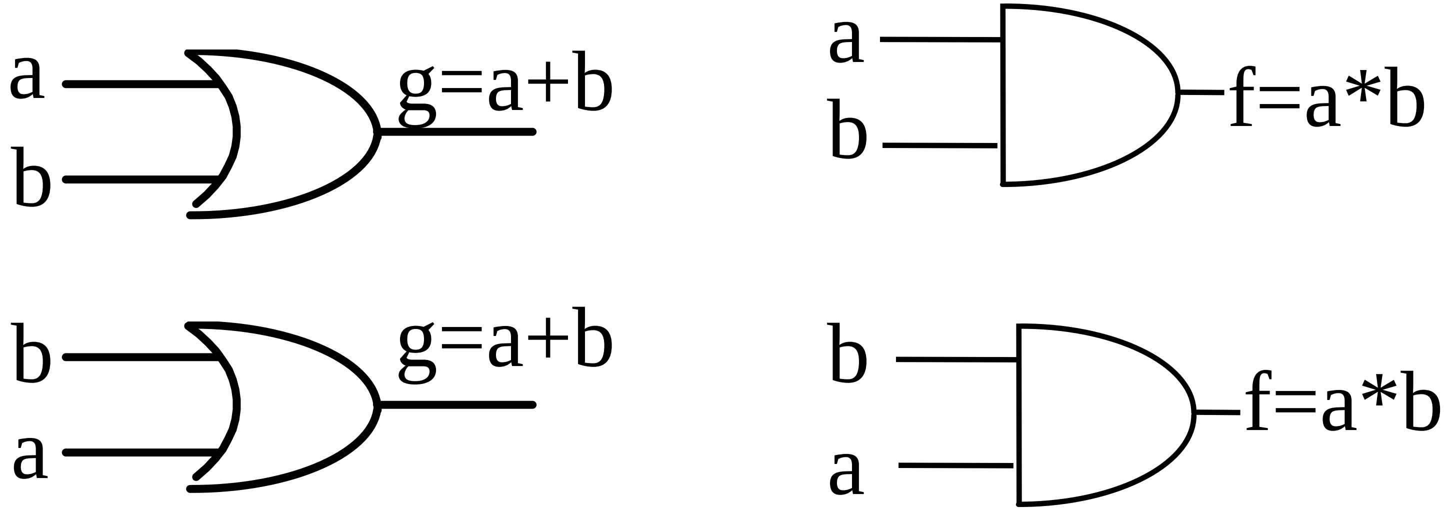
- *a* + *b* is in *B*
- *a***b* is in *B*



Boolean Algebra

3.**Commutative laws**: For every *a*, *b* in *B*,

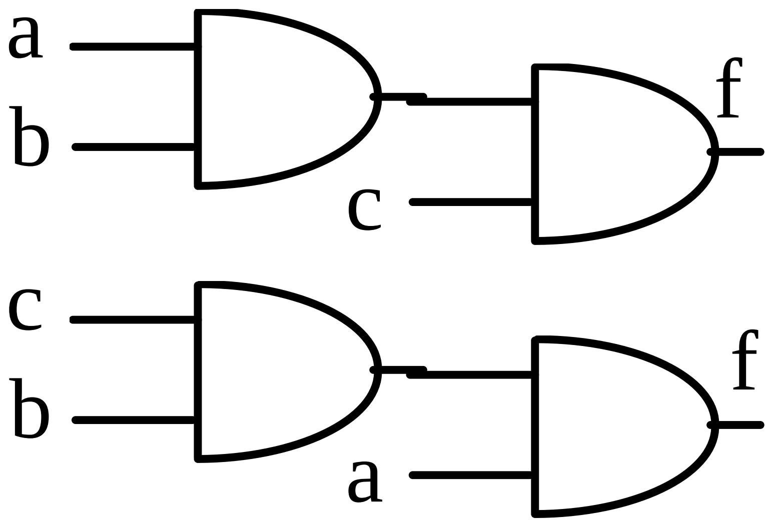
- $a + b = b + a$
- $ab = ba$
  - » Similar to Linear Algebra



Boolean Algebra

4.***Associative laws***: For every *a*, *b*, *c* in *B*,

- $(a + b) + c = a + (b + c) = a + b + c$
- $(ab)\ c = a\ (bc) = abc$
  - » Similar to Linear Algebra



Boolean Algebra

5. ***Distributive laws***: **For every *a*, *b*, *c* in *B*,**

- **$a + (bc) = (a + b)(a + c)$ ('+' Distributes Over '*')**

  (NOT Similar to Linear Algebra.)

- **$a(b + c) = (ab) + (ac)$ ('*' Distributes Over '+')**

  (Similar to Linear Algebra.)

6. ***Complement***: **For each *a* in *B*, there exists an element *a* in *B* (the complement of *a*) such that:**

- ***a.*** **$a + \overline{a} = 1$ (Similar to Mult. Inverse of Linear Algebra)**
- **b.** **$a * \overline{a} = 0$ (Similar to Additive. Inverse of Linear Algebra)**

Boolean Algebra

- *7.* ***Identity***
- a. There exists an identity element with respect to {+}, designated by 0, such that *a* + 0 = *a* for every *a* in *B*.
- (Similar to Additive Ident. of Linear Algebra)

OR Operator Truth Table

A	B	g=A + B
0	**0**	**0**
0	**1**	**1**
1	**0**	**1**
1	1	1

AND Operator Truth Table

A	B	g=A B
0	0	0
0	**1**	**0**
1	**0**	**0**
1	**1**	**1**

b. There exists an identity element with respect to {*}, designated by 1, such that *a**1 = *a* for every a in *B*.

(Similar to Multiplicative. Ident. of Linear Algebra)

Summary

Commutative

$a + b = b + a \qquad ab = ba$

Associative

$(a + b) + c = a + (b + c)$

$(ab)\ c = a\ (bc)$

Distributive

$a + (bc) = (a + b)(a + c)$

$a(b + c) = (ab) + (ac)$

Identity

$a + 0 = a \qquad a*1 = a$

Complement

$a + \overline{a} = 1 \qquad a * \overline{a} = 0$

Or with 1 ***And with 0***

$a + 1 = 1 \qquad a *0= 0$

Idempotent

$a + a = a \qquad a * a = a$

Boolean Algebra

Duality Theorem: The Dual of every theorem is a valid theorem. The dual is constructed by interchanging:

1 ⟷ 0

+ ⟷ *

Or With 1: $X+1=1$

Dual of Or With 1: $X*0=0$ (And With 0)

Idempotent: $X+X=X$

Dual of Idempotent: $X*X=X$

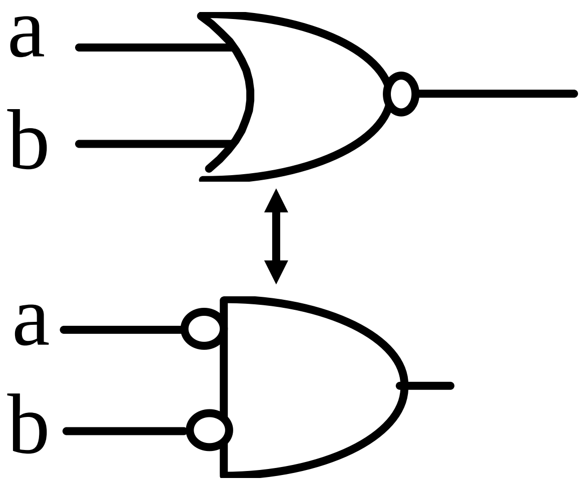
One last useful Theorem: DeMorgan’s Law

Boolean Algebra

Gate
Equivalency

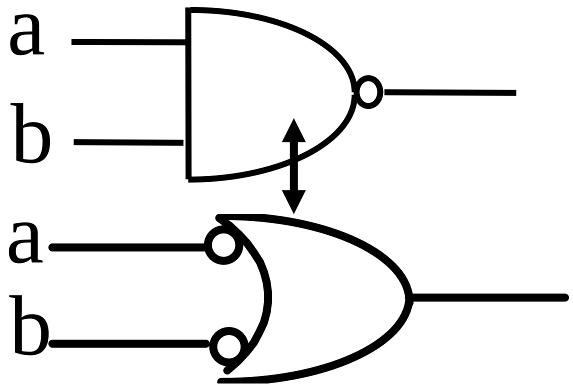
- **DeMorgan's Law**

$$\overline{a}\bullet\overline{b}=\overline{a+b}$$



- **Dual of DeMorgan's Law**

$$\overline{a}+\overline{b}=\overline{a\bullet b}$$



Boolean Algebra

Q:Why is this useful?

A:It allows us to build functions using only one gate type.

Q:Why do we use NAND/NOR gate to build functions rather than AND/OR logic?

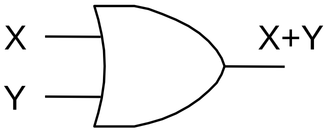
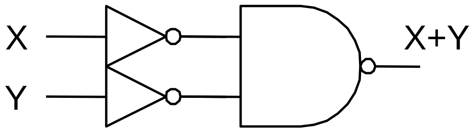
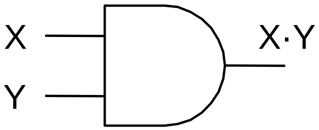
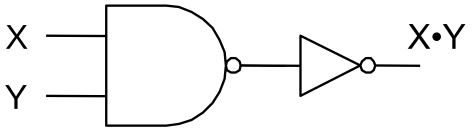
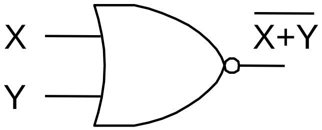
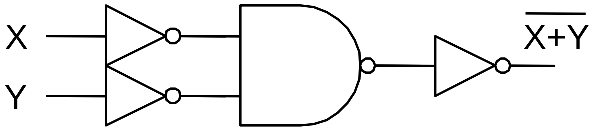
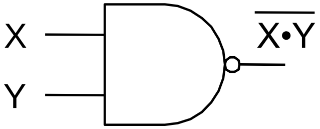
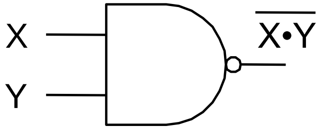
A:NAND/NOR gates are physically smaller and faster.

Q:How does Gate Equivalency help when drawing schematics with one gate type?

A:Let’s look at three approaches.

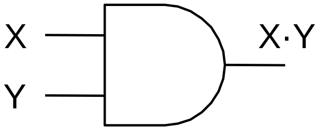
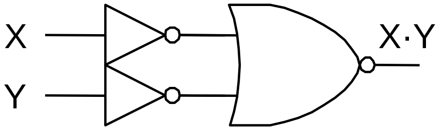
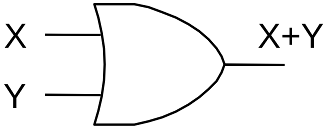
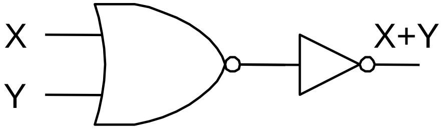
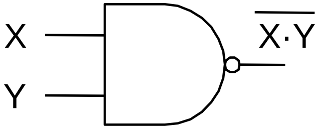
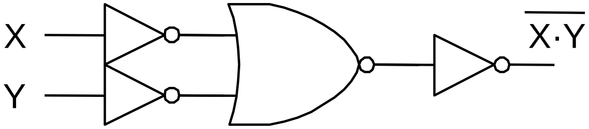
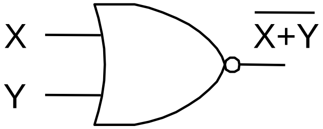
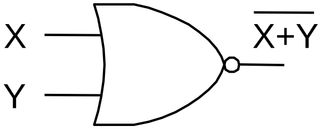
Boolean Algebra

- **One approach is to construct AND/OR logic and replace each gate with it's equivalent using the figures shown below.**

Function	Gate	NAND/Inverter Equivalent	
$X+Y$			$X+Y = \overline{\overline{X}\bullet\overline{Y}}$
$X\cdot Y$			$X\bullet Y = \overline{\overline{X\bullet Y}}$
$\overline{X+Y}$			$\overline{X+Y} = \overline{\overline{\overline{X}\bullet\overline{Y}}}$
$\overline{X\bullet Y}$			$\overline{X\bullet Y} = \overline{X\bullet Y}$

Boolean Algebra

- **The drawback to such an approach is that you may use more than the minimal number of gates.**
- **The advantage is that the approach is simple.**

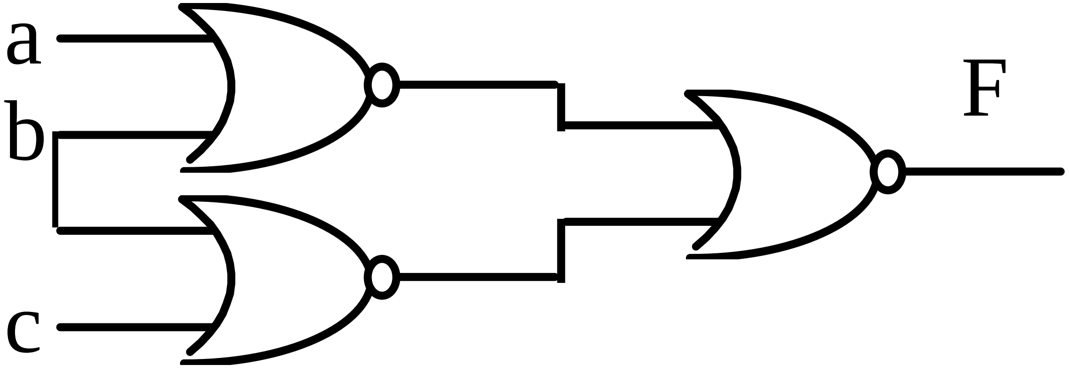
Function	Gate	NOR/Inverter Equivalent	
$X \cdot Y$			$X \cdot Y = \overline{\overline{X}+\overline{Y}}$
$X+Y$			$X+Y = \overline{\overline{X+Y}}$
$\overline{X \cdot Y}$			$\overline{X \cdot Y} = \overline{\overline{\overline{\overline{X}+\overline{Y}}}}$
$\overline{X+Y}$			$\overline{X+Y} = \overline{X+Y}$

Boolean Algebra

- **There are two traditional approaches that yield less costly realizations:**
  - **EX: Using ONLY NOR gates, draw a schematic for the following function: F=(a+b)(b+c)**

Approach #1: Use DeMorgan’s Law:

$$F=\overline{\overline{(a+b)(b+c)}}$$

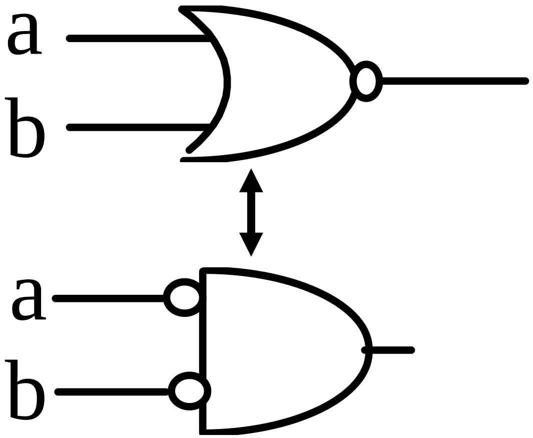


Boolean Algebra

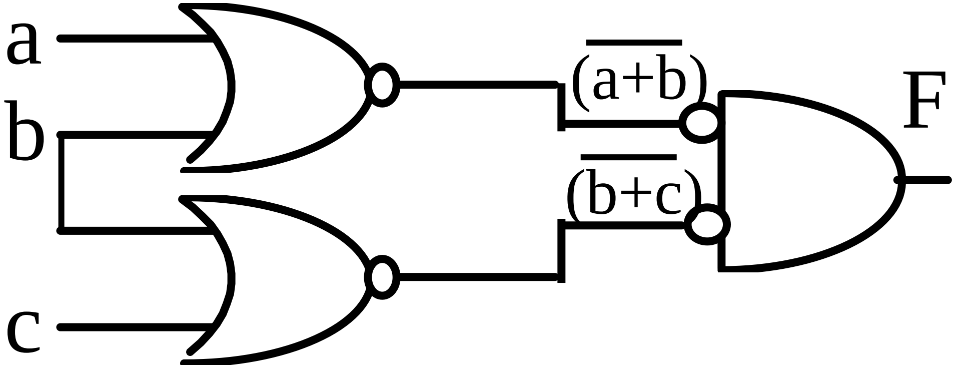
- Drawbacks to using DeMorgan’s Laws:
  - Some functions require many manipulations to get into proper form.
  - During these many manipulations it is easy to make mistakes.

Boolean Algebra

EX: Using ONLY NOR gates, draw a schematic for the following function: F=(a+b)(b+c)



Approach #2:Use Gate Equivalency (Draw gate form that you need. Account for inversions.)



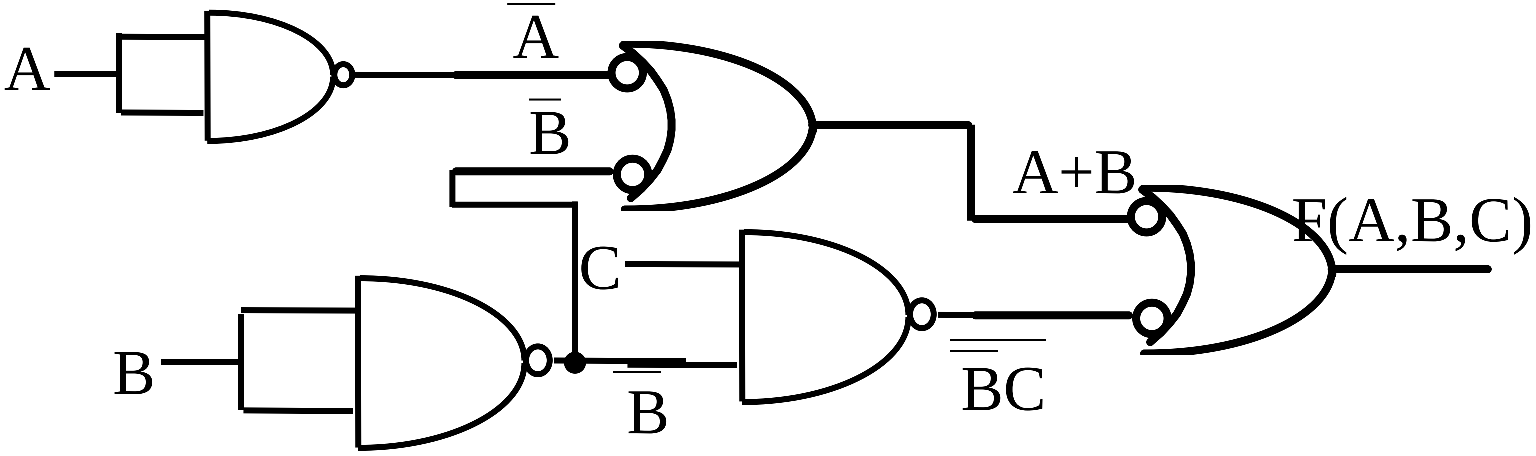
Note that the inversions cancel and we’re left with AND/OR LOGIC.

Boolean Algebra

- General Approach for building functions with 2-input NAND gates:
  - 1. Break function into two pieces.
  - 2. If the pieces form a:
    - AND/NAND function ===> build function with NAND+Inverters/NAND.
    - OR function ===>build with NAND (OR with inverted inputs)
    - NOR (F=$\overline{X+Y}$) function ===>Work with complement of the function (F =$\overline{X+Y}$), which is an OR, then complement F to get $\overline{F}$.
  - 3. Recursively apply these rules

Boolean Algebra

- Ex: Assume you have the forms:
  - $F=(\overline{A+B}) + \overline{B}C$
- & want to find a NAND gate implementation.
- Break the function in the following way.
  - $F(A\ .\ .\ .\ Z)=\overline{g(A\ .\ .\ .\ Z) +}\ h(A\ .\ .\ .\ Z).$



© Gamal Fahmy

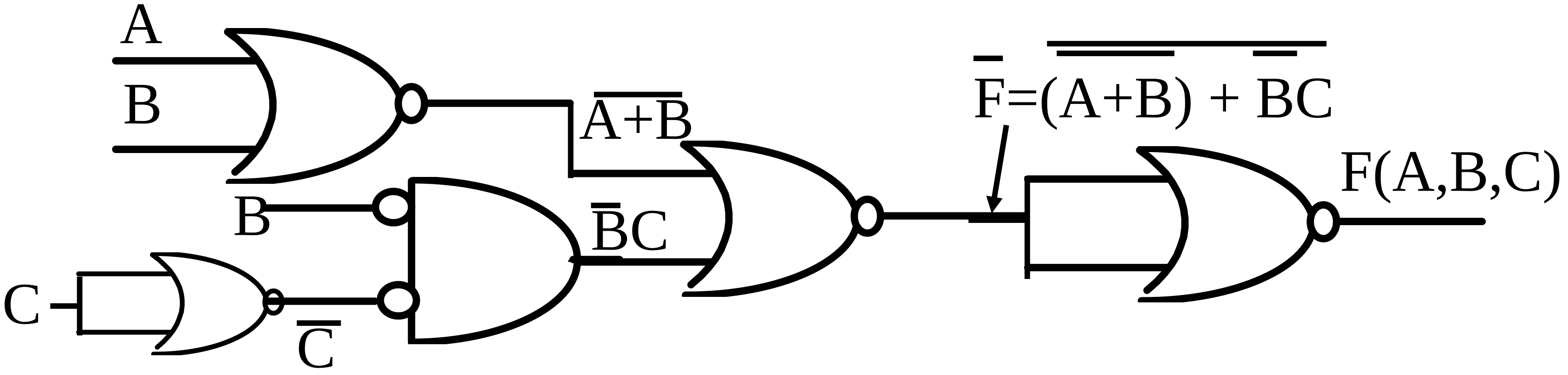
Boolean Algebra

- General Approach for building function with 2-input NOR gates:
  - 1. Break function into two pieces.
  - 2. If the pieces form an:
    - OR/NOR function ===> build function with NOR+Inverters/NOR
    - AND function ===> build function with NOR (AND with inverted inputs)
    - NAND ($F=\overline{X*Y}$) function ===>Work with complement of the function ($\overline{F}$), which is an AND, then complement $\overline{F}$ to get F.
  - 3. Recursively apply these rules.

Boolean Algebra

- EX: Assume you have the forms:
  - $F=\overline{(A+B)} + \overline{B}C$
- **Want to find a NOR gate implementation.**

- Break the function in the following way.
  - $F(A\ .\ .\ .\ Z)=\overline{g(A\ .\ .\ .\ Z)} + h(A\ .\ .\ .\ Z).$



© Gamal Fahmy

Boolean Algebra

The Uniting Theorem:

Minimize the following function:

$F = A\overline{B} + AB$

$F = A(\overline{B} + B)$ Distr.

$F = A*1$ Compl..

$F = A$ Ident..

A	B	F
0	0	0
0	1	0
1	0	1
1	1	1

These two 1's are physically and address adjacent.

Defn: Two 1's are address adjacent if their 'addresses' differ in only one bit position.

Boolean Algebra

We can use this connection between adjacency and the Uniting Thm.to visually implement the uniting Theorem:

Rule of Thumb(Min SOP): Eliminate the literal with CHANGING Address bit and write Canonical SOP for group of 1’s using remaining literals.

A	B	F
0	0	0
0	1	0
1	0	1
1	1	1

$\}$ F=A

Boolean Algebra

TEAMS: Find Min. SOP form for the following Truth Table:

A	B	F
0	0	1
0	1	0
1	0	1
1	1	0

$$F = \overline{A}\,\overline{B} + A\overline{B}$$

Q? Are the 1’s address adjacent?

Boolean Algebra

By analogy we can generate a rule of thumb for generating a Minimum POS form.

Rule of Thumb(Min POS): Eliminate the literal with CHANGING Address bit and write Canonical POS for group of 0's using remaining literals.

Boolean Algebra

Ex: Find Min. POS form for the following Truth Table:

A	B	C	F
0	0	0	0
0	0	1	0
0	1	0	1
0	1	1	0
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

$F=(A+\overline{C})$

Boolean Algebra

Ex: Find Min. POS form for the following Truth Table:

A	B	C	F
0	0	0	0
0	0	1	0
0	1	0	1
0	1	1	0
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

$$F=(A+\overline{C})(B+C)$$

SOP and POS Forms

- We need to be able to translate a truth table description into a logical expression. Two forms are used:
- Sum-of-Products and Product-of-Sums Canonical Form
- Consider first Sum-of-Products Canonical Form Using an Example:

SOP Canonical Form

Term #	A	B	C	F	F1	F2	F3	F4	F1+F2+F3+F4
0	0	0	0	1	1	0	0	0	1
1	0	0	1	1	0	1	0	0	1
2	0	1	0	0	0	0	0	0	0
3	0	1	1	0	0	0	0	0	0
4	1	0	0	0	0	0	0	0	0
5	1	0	1	1	0	0	1	0	1
6	1	1	0	1	0	0	0	1	1
7	1	1	1	0	0	0	0	0	0

- Construct Each Function F1, F2, etc.

$F1 = \overline{A}\,\overline{B}\,\overline{C}$ □ $F2=\overline{A}\,\overline{B}\,C$ □ $F3=A\,\overline{B}\,C$

□ $F4=A\,B\,\overline{C}$

$$F = \overline{A}\,\overline{B}\,\overline{C} + \overline{A}\,\overline{B}\,C + A\overline{B}\,C + AB\overline{C}$$

© Gamal Fahmy

SOP Canonical Form

Term #	A	B	C	F
0	0	0	0	1
1	0	0	1	1
2	0	1	0	0
3	0	1	1	0
4	1	0	0	0
5	1	0	1	1
6	1	1	0	1
7	1	1	1	0

- Minterm: “Product” term containing all literals used by the function.
- Minterm Short-hand Notation:

$$F = \sum m(0,1,5,6)$$

$$F = \overline{A}\,\overline{B}\,\overline{C} + \overline{A}\,\overline{B}C + A\overline{B}C + AB\overline{C}$$

POS Canonical Form

Term #	A	B	C	F	FA	FB	FC	FD	FA*FB*FC*FD
0	0	0	0	1	1	1	1	1	1
1	0	0	1	1	1	1	1	1	1
2	0	1	0	0	0	1	1	1	0
3	0	1	1	0	1	0	1	1	0
4	1	0	0	0	1	1	0	1	0
5	1	0	1	1	1	1	1	1	1
6	1	1	0	1	1	1	1	1	1
7	1	1	1	0	1	1	1	0	0

- Construct Each Function FA, FB, etc.
- $FA=A+\overline{B}+C$
- $FB=A+\overline{B}+\overline{C}$
- $FC=\overline{A}+B+C$
- $FD=\overline{A}+\overline{B}+\overline{C}$
- $F=(A+\overline{B}+C)\,(A+\overline{B}+\overline{C})\,(\overline{A}+B+C)\,(\overline{A}+\overline{B}+\overline{C})$

POS Canonical Form

Term #	A	B	C	F
0	0	0	0	1
1	0	0	1	1
2	0	1	0	0
3	0	1	1	0
4	1	0	0	0
5	1	0	1	1
6	1	1	0	1
7	1	1	1	0

- Maxterm: “Sum” term containing all literals used by the function.
- Maxterm Short-hand Notation:

$$F = \sum m(0,1,5,6) \qquad F = \Pi M(2,3,4,7)$$

- $F=(A+\overline{B}+C)\ (A+\overline{B}+\overline{C})\ (\overline{A}+B+C)\ (\overline{A}+\overline{B}+\overline{C})$
- Compare MinTerm and MaxTerm notation. What do you observe about the union of the integer arguments?

Home work

Problems:

Homework 3,4,5