EX 5 - Theory + SVM

207380528, 3 02673355

May 28, 2022

Q1 - Kernels and mapping functions

$K(x, y) = (x \cdot y + 1)^3$ function over $\mathbb{R}^2 \times \mathbb{R}^2$. Find Ψ for which K is a kernel

A

We'll expand the right hand side argument to find Ψ.

$$(x \cdot y + 1)^3 = (x \cdot y + 1)^2(x \cdot y + 1) = ((x \cdot y)^2 + 2x \cdot y + 1)(x \cdot y + 1)$$

$$= (x^2 \cdot y^2 + 2x \cdot y + 1)(x \cdot y + 1) = x^3 \cdot y^3 + 2x^2 \cdot y^2 + x \cdot y + x^2 + y^2 + 2x \cdot y + 1$$

$$(x \cdot y)^3 + 3x \cdot y^2 + 3x \cdot y + 1$$

We now will apply $x \in \mathbb{R}^2$ to it and get:

$$(x_1y_1 + x_2y_2)^3 + 3(x_1y_1 + x_2y_2)^2 + 3(x_1y_1 + x_2y_2) + 1$$

And simplify it a bit more:

$$= (x_1y_1+x_2y_2)^2(x_1y_1+x_2y_2)+3((x_1y_1)^2+2x_1y_1x_2y_2+(x_2y_2)^2)+3(x_1y_1+x_2y_2)+1$$

$$= ((x_1y_1)^2+2x_1y_1x_2y_2+(x_2y_2)^2)(x_1y_1+x_2y_2)+3((x_1y_1)^2+2x_1y_1x_2y_2+(x_2y_2)^2))+3(x_1y_1+x_2y_2)+1$$

$$=$$

$$= (x_1y_1)^3 + 2(x_1y_1)^2x_2y_2 + (x_2y_2)^2(x_1y_1) + (x_1y_1)^2(x_2y_2) + 2(x_2y_2)^2x_1y_1 + ...$$

$$...(x_2y_2)^3 + 3((x_1y_1)^2 + 2x_1y_1x_2y_2 + (x_2y_2)^2)) + 3(x_1y_1 + x_2y_2) + 1$$

$$= (x_1y_1)^3 + 3(x_1y_1)^2(x_2y_2) + 3(x_2y_2)^2(x_1y_1) + (x_2y_2)^3 ...$$

$$... + 3(x_1y_1)^2 + 6x_1y_1x_2y_2 + 3(x_2y_2)^2 + 3x_1y_1 + 3x_2y_2 + 1$$

Meaning, for $\Psi(x)$ to afford the kernel$K$ it means:

$$\Psi(x) = x_1^3 + \sqrt{3}x_1^2x_2 + \sqrt{3}x_2^2x_1 + x_2^3 + \sqrt{3}x_1^2 + \sqrt{6}x_1x_2 + \sqrt{3}x_2^2 + \sqrt{3}x_1 + \sqrt{3}x_2 + 1$$

$$\Psi(y) = y_1^3 + \sqrt{3}y_1^2y_2 + \sqrt{3}y_2^2y_1 + y_2^3 + \sqrt{3}y_1^2 + \sqrt{6}y_1y_2 + \sqrt{3}y_2^2 + \sqrt{3}y_1 + \sqrt{3}y_2 + 1$$

B

In class we called it "kernel for the full rational variaties mapping".

C

By using $K(x,y)$ instead of $\Psi(x) \cdot \Psi(y)$, we're moving the calculations to be done in $\mathbb{R}^2$ instead of in $\mathbb{R}^{10}$, so we'll save 8 multiplications per instance.

Q2 - Lagrange multipliers

A

$f(x,y) = 2x - y$. Find the min and max point for f under $g(x,y) = \frac{x^2}{4} + y^2 = 1$.

We can write the constraint as

$$\frac{x^2}{4} + y^2 - 1 = 0$$

And write the Lagrangian:

$$L(x,y) = 2x - y + \lambda(\frac{x^2}{4} + y^2 - 1)$$

Next, we'll derive by x, y and λ:

$$(1)\frac{\partial L}{\partial x} = 2 + \frac{2\lambda x}{4} = 0$$
$$\lambda = -4x$$

$$(2)\frac{\partial L}{\partial y} = -1 + 2\lambda y = 0$$

$$\lambda = \frac{1}{2y}$$

We'll put both (1) and (2) togehter and:

$$(3) - 4x = \frac{1}{2y}$$

$$x = -\frac{1}{8y}$$

$$y = -8x$$

$$\frac{\partial L}{\partial \lambda} = \frac{x^2}{4} + y^2 - 1 = 0$$

Plug in (3)

$$\frac{64y^2}{4} + y^2 - 1 = 0$$

$$17y^2 = 1$$

$$y = \pm\frac{1}{\sqrt{17}}$$

$$x = \mp\frac{8}{\sqrt{17}}$$

Plug it back in f:

$$f(-\frac{8}{\sqrt{17}}, \frac{1}{\sqrt{17}}) = -\sqrt{17}$$

$$f(\frac{8}{\sqrt{17}}, -\frac{1}{\sqrt{17}}) = \sqrt{17}$$

It seems that $\sqrt{17}$ is the max point and that $-\sqrt{17}$is the min point

Q3- PAC Learning

The three vectors $u = (\frac{\sqrt{3}}{2}, \frac{1}{2})$, $w = (\frac{\sqrt{3}}{2}, -\frac{1}{2})$, $v = (0, -1)$ define a origin-centered upright around the origin, and which serves as the *true* H. We'll define a cricle around it that bound the triangle. We'll call the the radius of that triangle r^*.

We'll define r, a second cricle that bounds another origin-centered upright triangle with, that bounds another triangle, and the the probability of new data landing in the area between r and r^* is $\leq \epsilon$. We'll note that the new triangle that is bounded by the new circle that is neccasarily inside the origin triangle.

Furthermore, we'll define a third circle, r'^{ϵ}, which is defined by $argminf_r\pi[x_1, x_1 \in A_r] \leq \varepsilon$, which bounds a third origin-centered upright around the origin and represents our hypothesis $L(D)$.

We want to find

$$P(\{D \in X^m : Err(L(D), c)\} > \epsilon) < \delta$$

<u>In the good case</u>, a new instance of the X^m lands in between the the triangle that is bounded by r^{ϵ} and the traingle that is bounded by r. The probability of error is $\leq \epsilon$, we don't have any issue.

<u>In the bad case</u>, the new instances will miss both the traingles that are bounded by r^* and r^{ϵ}.

The probability of that happening is $(1-\epsilon)^m \leq e^{-\epsilon m}$. We can define $m \geq \frac{ln(\frac{1}{\delta})}{\epsilon}$ and get:

$$e^{m\epsilon} \leq e^{-ln(\frac{1}{\delta})} = \delta$$

Which is small enough.

Meaning, the sample complexity here is

$$m(\epsilon, \delta) = \frac{1}{\epsilon} ln \frac{1}{\delta}$$

For example, if $\epsilon = \delta = 0.05$, then $m = \frac{1}{0.05} ln \frac{1}{0.05} \approx 60$.

The time complexity of finding the bounded origin-centered upright around the origin by each of the circles is $O(1)$ as it is not dependent on the sample size. The time complexity of the generating the circle itself is the $O(m)$. Meaning, the time complexity of the solution is $O(m) + O(1) = O(m)$.

Q4 - Confidence Intervals

We'll compute the confidence interval, according to the data provided. We'll take the upper bound of the interval, to find the maximum percentage that we can commit to in 95% confidence.

$$p \in [\hat{p} + 2se, \hat{p} - 2se]$$

$$se = \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

We know that $\hat{p} = 0.2$, $n = 1000$. We'll plug it in and get:

$$se = \sqrt{\frac{0.2(1-0.2)}{1000}} = 0.0126$$

$$p \in [0.175, 0.225]$$

The maximum error, with 95% confidnce, is 22.5%.

Q5 - SVM