(CSE213) Numerical Computing
Sheet 7 − Solutions

Question 1

Consider the function $g(x) = 2x(1 - x)$, which has fixed points at $x = 0$ and at $x = \frac{1}{2}$.

(a) Why should we expect that fixed point iteration, starting even with a value very close to zero, will fail to converge towards $x = 0$?

(b) Why should we expect that fixed point iteration, starting with $p_0 \in [0, 1]$ will converge towards $x = \frac{1}{2}$?

(c) Perform five iterations starting from an arbitrary $p_0 \in [0, 1]$.

(a)

$g'(x) = 2x(-1) + (1 - x)(2) = -2x + 2 - 2x = 2 - 4x$

For the iteration to converge, $|g'(p)| < 1$, where p is the fixed point.
$g'(0) = 2 - 4(0) = 2 \nless 1$, thus, it will fail to converge towards $x = 0$.

(b)

Again, for the iteration to converge, $|g'(p)| < 1$, where p is the fixed point.
$g'(\frac{1}{2}) = 2 - 4(\frac{1}{2}) = 0 < 1$, thus, it will converge towards $x = \frac{1}{2}$.

(c)

Let $p_0 = 0.23782$
$x_{i+1} = 2x_i(1 - x_i)$
$x_1 = 2(0.23782)(1 - 0.23782) = 0.3625232952 \approx 0.36252$
$x_2 = 2(0.36252)(1 - 0.36252) = 0.4621984992 \approx 0.46220$
$x_3 = 2(0.46220)(1 - 0.46220) = 0.49714232 \approx 0.49714$
$x_4 = 2(0.49714)(1 - 0.49714) = 0.4999836408 \approx 0.49998$
$x_5 = 2(0.49998)(1 - 0.49998) = 0.4999999992 \approx 0.5$

Question 2

Verify that $x = \sqrt{a}$ is a fixed point of the function $g(x) = \frac{x^3 + 3xa}{3x^2 + a}$.

In fixed-point iteration method, $g(p) = p$ if p is a fixed point for the function.

$$g(\sqrt{a}) = \frac{(\sqrt{a})^3 + 3(\sqrt{a})a}{3(\sqrt{a})^2 + a} = \frac{a^{3/2} + 3a^{3/2}}{3a + a} = \frac{4a^{3/2}}{4a} = \frac{a^{3/2}}{a} = \sqrt{a}$$

Since $g(\sqrt{a}) = \sqrt{a}$, then $x = \sqrt{a}$ is a fixed point of $g(x)$.

Question 3

Can Newton-Raphson Iteration be used to solve $f(x) = 0$ if $f(x) = \sqrt[3]{x}$?

$$f(x) = \sqrt[3]{x}, \qquad f'(x) = \frac{1}{3}x^{1/3-1} = \frac{1}{3}x^{-2/3} = \frac{1}{3\sqrt[3]{x^2}}$$

Newton-Raphson Iteration Method states that:

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} = x_i - \frac{\sqrt[3]{x_i}}{1/3\sqrt[3]{x_i^2}} = x_i - 3\sqrt[3]{x_i^2} \cdot \sqrt[3]{x_i} = x_i - 3\sqrt[3]{x_i^2 \cdot x_i} = x_i - 3\sqrt[3]{x_i^3} = -2x_i$$

Thus, Newton-Raphson Iteration Method will always diverge, so it cannot be used.

Question 4

Verify that the equation $x^4 - 18x^2 + 45 = 0$ has a root in the interval $[1, 2]$.
Next, perform five iterations of the secant method, using $p_0 = 1$ and $p_1 = 2$.

Given that the exact value of the root is $x = \sqrt{3}$, compute the absolute error in the approximations just obtained.

To verify that there is a root in the interval $[1, 2]$, we look for a sign change between the values of $f(1)$ and $f(2)$.

By plugging in the values, we find that $f(1) = 28$, and $f(2) = -11$, so a root exists.

| i	x_{i-1}	x_i	x_{i+1}	$f(x_{i-1})$	$f(x_i)$	E_a
1	1	2	≈ 1.71795	28	-11	-
2	2	1.71795	≈ 1.73222	-11	≈ 0.58614	≈ 0.01427
3	1.71795	1.73222	≈ 1.73205	≈ 0.58614	≈ -0.00703	≈ 0.00017
4	1.73222	1.73205	≈ 1.73205	≈ -0.00703	≈ 0.00003	0
5	1.73205	1.73205	1.73205	≈ 0.00003	≈ 0.00003	0

Question 5
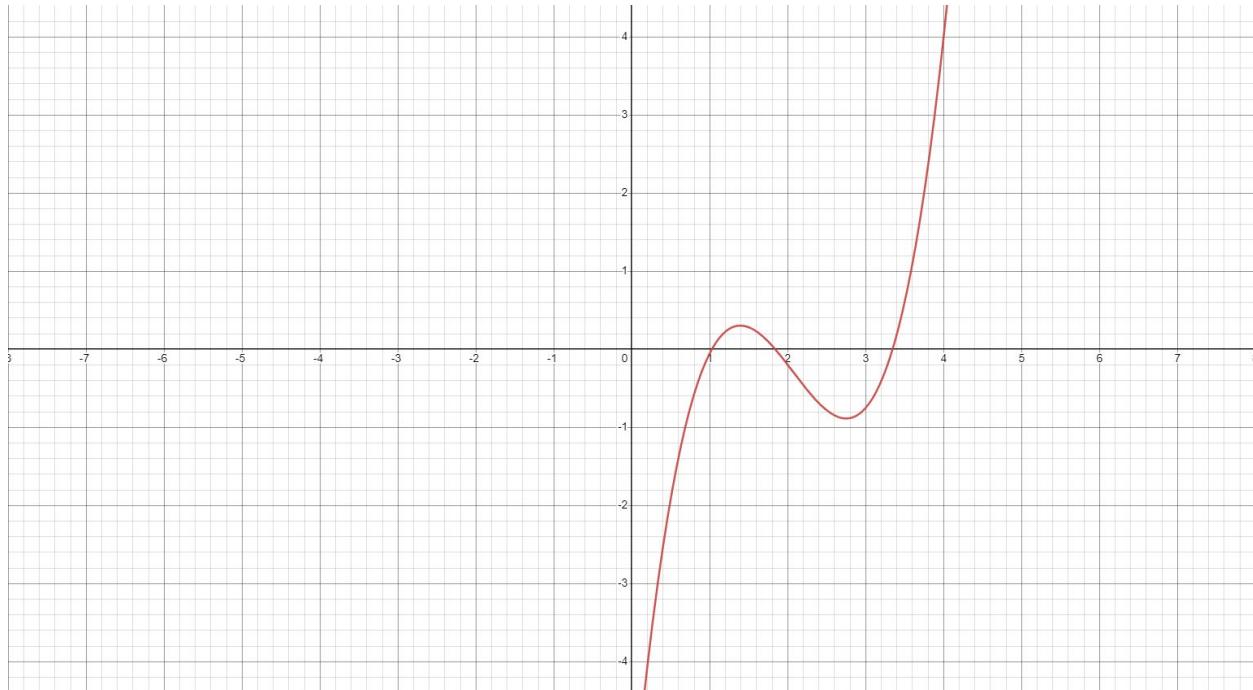
Determine the highest real root of $f(x) = 0.95x^3 - 5.9x^2 + 10.9x - 6 = 0$:

    (a)   Graphically.
    (b)   Using Newton-Raphson Method (3 iterations, $x_0 = 3.55$).

(a)

$f(x) = 0.95x^3 - 5.9x^2 + 10.9x - 6$ is a cubic equation, meaning that it may have three distinct real roots, one distinct real root and two equal real roots, or one real distinct root. We first attempt to try different values for x.

x	$f(x)$
0	-6
0.5	-1.90625
1	-0.05
1.5	0.28125
2	-0.2
2.5	-0.78125
3	-0.75
3.5	0.60625
4	4
4.5	10.14375
5	19.75
5.5	33.53125
6	52.2

According to the graph, the highest real root is around 3.375, since it is almost at the 3.4 mark, but not equal to 3.4.

(b)

$$f(x) = 0.95x^3 - 5.9x^2 + 10.9x - 6$$
$$f'(x) = 2.85x^2 - 11.8x + 10.9$$

i	x_i	x_{i+1}	$f(x_i)$	$f'(x_i)$	ε_a
1	3.55	3.379072909 ≈ 3.37907	0.84218125 ≈ 0.84218	4.927125 ≈ 4.92713	-
2	3.37907	3.345872107 ≈ 3.34587	0.1184663757 ≈ 0.11847	3.568599085 ≈ 3.56860	≈ 0.99227%
3	3.34587	3.34464559 ≈ 3.34465	0.00406586187 ≈ 0.00407	3.324045262 ≈ 3.32405	≈ 0.03648%

After three iterations of the Newton-Raphson Method, the highest root of the function can be said to be approximately equal to 3.34465.

Question 6

For each of the functions given below, use the secant method to approximate all real roots.

Use an absolute tolerance of 10^{-4} as a stopping condition.
Hint: You can plot the functions to get the initial guesses.
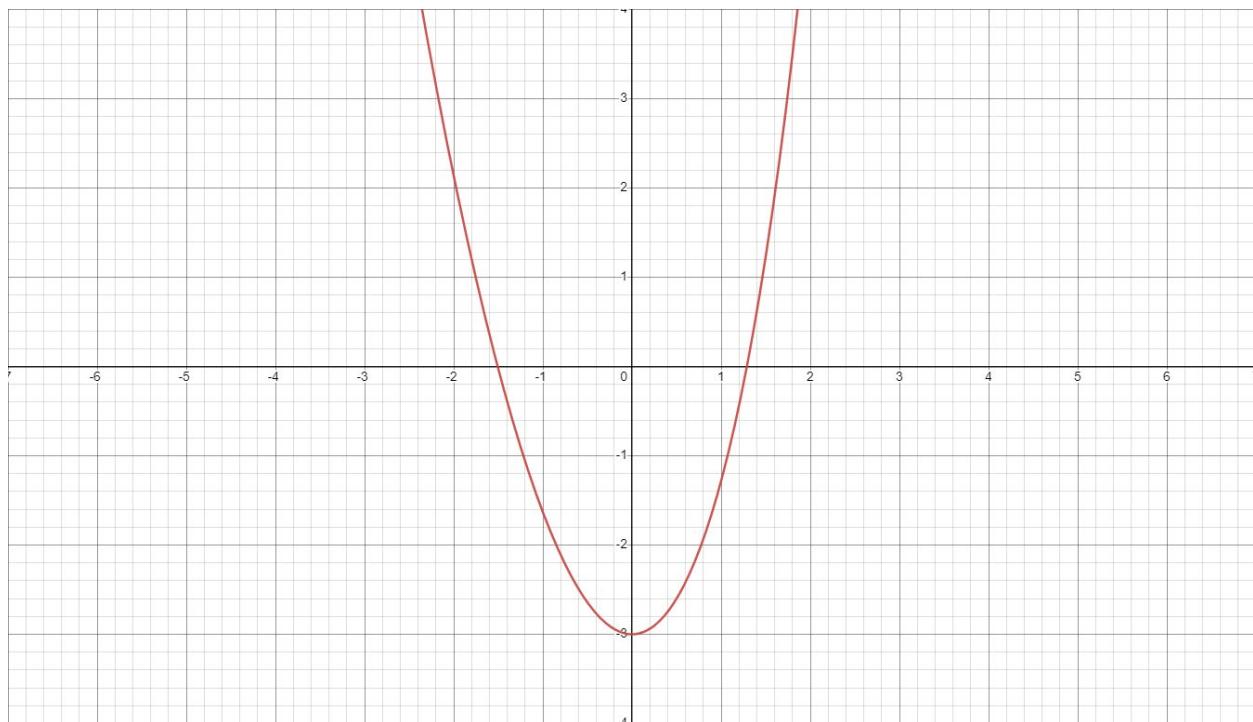
(a) $f(x) = e^x + x^2 - x - 4$
(b) $f(x) = x^3 - x^2 - 10x + 7$

(a)

As stated by the given hint, we will attempt to plot the graph by trying values of x.

x	$f(x)$
-3	8.049787068
-2.5	4.832084999
-2	2.135335283
-1.5	-0.02686983985
-1	-1.632120559
-0.5	-2.64346934
0	-3
0.5	-2.601278729
1	-1.281718172
1.5	1.23168907
2	5.389056099
2.5	11.93249396
3	22.08553692

As shown, the roots are in the intervals [-2, -1] and [1, 2].

For the first root, we set $x_0 = -2, x_1 = -1$.
The secant method states that

$$x_{i+1} = x_i - \frac{f(x_i)}{\left[\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}\right]}$$

i	x_{i-1}	x_i	x_{i+1}	$f(x_{i-1})$	$f(x_i)$	E_a
1	-2	-1	≈ -1.43321	≈ 2.13534	≈ -1.63212	-
2	-1	-1.43321	≈ -1.52067	≈ -1.63212	≈ -0.27416	≈ 0.08746
3	-1.43321	-1.52067	≈ -1.50680	≈ -0.27416	≈ 0.05167	≈ 0.01387
4	-1.52067	-1.50680	≈ -1.50710	≈ 0.05167	≈ -0.00114	≈ 0.00030
5	-1.50680	-1.50710	-1.5071	≈ -0.00114	≈ 0.00000	0

After applying the secant method for the first root, we find that it is approximately equal to -1.5071.
For the second root, we set $x_0 = 1, x_1 = 2$.
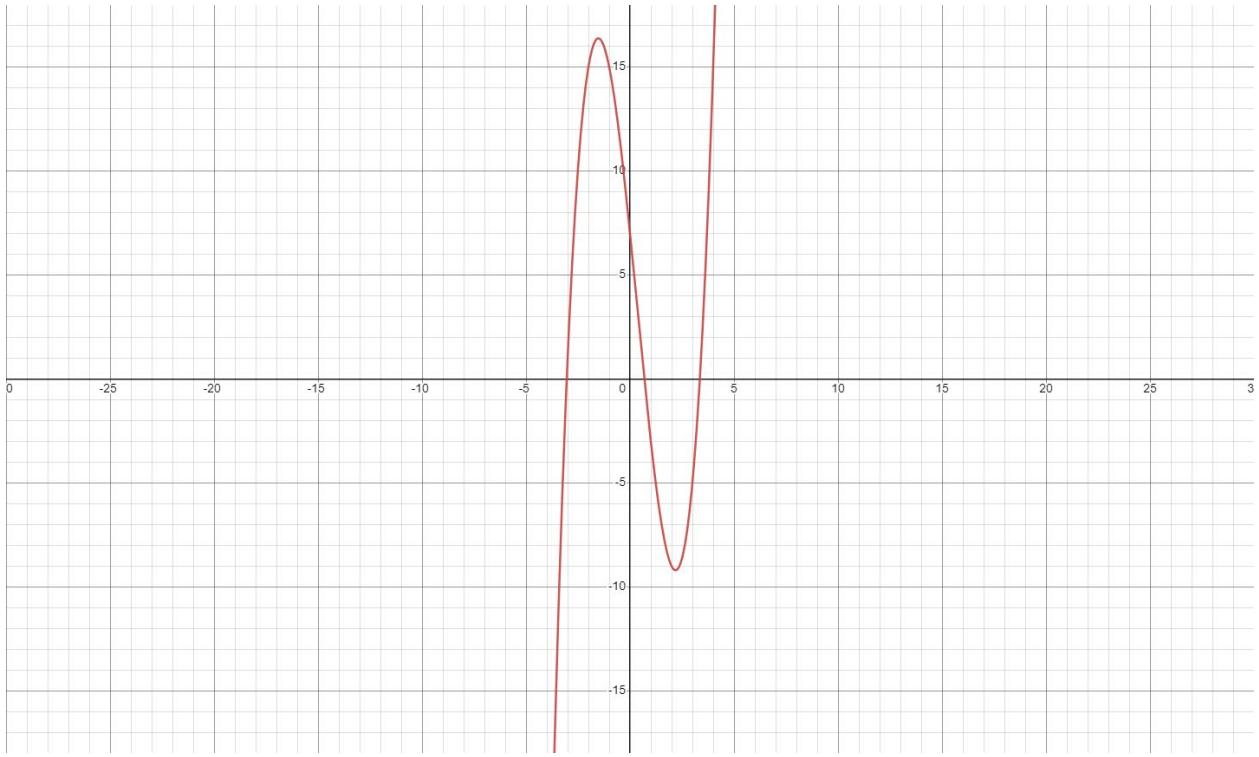
i	x_{i-1}	x_i	x_{i+1}	$f(x_{i-1})$	$f(x_i)$	ε_a
1	1	2	≈ 1.19214	≈ -1.28172	≈ 5.38906	-
2	2	1.19214	≈ 1.25781	≈ 5.38906	≈ -0.47682	≈ 0.06567
3	1.19214	1.25781	≈ 1.29036	≈ -0.47682	≈ -0.15801	≈ 0.03255
4	1.25781	1.29036	≈ 1.28865	≈ -0.15801	≈ 0.00876	≈ 0.00171
5	1.29036	1.28865	≈ 1.28868	≈ 0.00876	≈ -0.00015	≈ 0.00003

After applying the secant method for the second root, we find that it is approximately equal to 1.28868.

(b)

x	$f(x)$
-4	-33
-3.5	-13.125
-3	1
-2.5	10.125
-2	15
-1.5	16.375
-1	15
-0.5	11.625
0	7
0.5	1.875
1	-3
1.5	-6.875
2	-9
2.5	-8.625
3	-5
3.5	2.625
4	15

According to the values found, roots exist in the intervals [-3.5, -3], [0.5, 1] and [3, 3.5].

For the first root, we set $x_0 = -3.5, x_1 = -3$.

i	x_{i-1}	x_i	x_{i+1}	$f(x_{i-1})$	$f(x_i)$	E_a
1	-3.5	-3	≈ -3.03540	-13.125	1	-
2	-3	-3.03540	≈ -3.04282	1	≈ 0.17322	≈ 0.00742
3	-3.03540	-3.04282	≈ -3.04268	≈ 0.17322	≈ -0.00327	≈ 0.00014
4	-3.04282	-3.04268	≈ -3.04268	≈ -0.00327	≈ 0.00007	0

For the second root, we set $x_0 = 0.5, x_1 = 1$.

i	x_{i-1}	x_i	x_{i+1}	$f(x_{i-1})$	$f(x_i)$	E_a
1	0.5	1	≈ 0.69231	1.875	-3	-
2	1	0.69231	≈ 0.68490	-3	≈ -0.07057	≈ 0.00741
3	0.69231	0.68490	≈ 0.68522	≈ -0.07057	≈ 0.00319	≈ 0.00032
4	0.68490	0.68522	0.68522	≈ 0.00319	≈ 0.00000	0

For the third root, we set $x_0 = 3, x_1 = 3.5$.

i	x_{i-1}	x_i	x_{i+1}	$f(x_{i-1})$	$f(x_i)$	E_a
1	3	3.5	≈ 3.32787	-5	2.625	-
2	3.5	3.32787	≈ 3.35533	2.625	≈ -0.49819	≈ 0.02746
3	3.32787	3.35533	≈ 3.35750	≈ -0.49819	≈ -0.03643	≈ 0.00217
4	3.35533	3.35750	≈ 3.35746	≈ -0.03643	≈ 0.00064	≈ 0.00004

As shown above, the roots of the equation are approximately -3.04268, 0.68522 and 3.35746.

Question 7

The function $x^3 - 2x^2 - 4x + 8$ has a double root at $x = 2$. Use:

- (a)   the standard Newton-Raphson Method
- (b)   the modified 1 Newton-Raphson Method
- (c)   the modified 2 Newton-Raphson Method

to solve for the root at $x = 2$.

Compare and discuss the rate of convergence using an initial guess of $x_0 = 1.2$.

(a)

$$f(x) = x^3 - 2x^2 - 4x + 8, \qquad f'(x) = 3x^2 - 4x - 4$$

i	x_i	x_{i+1}	$f(x_i)$	$f'(x_i)$
1	1.2	≈ 1.65714	2.048	-4.48
2	1.65714	≈ 1.83700	≈ 0.42991	≈ -2.39022
3	1.83700	≈ 1.92027	≈ 0.10195	≈ -1.22429
4	1.92027	≈ 1.96054	≈ 0.02492	≈ -0.61877
5	1.96054	≈ 1.98038	≈ 0.00617	≈ -0.31101
6	1.98038	≈ 1.99020	≈ 0.00153	≈ -0.15581
7	1.99020	≈ 1.99506	≈ 0.00038	≈ -0.07811
8	1.99506	≈ 1.99759	≈ 0.00010	≈ -0.03945
9	1.99759	≈ 1.99863	≈ 0.00002	≈ -0.01926
10	1.99863	≈ 1.99954	≈ 0.00001	≈ -0.01095
11	1.99954	1.99954	≈ 0.00000	≈ -0.00368

(b)

The modified 1 Newton-Raphson Method states that $x_{i+1} = x_i - m\frac{f(x_i)}{f'(x_i)}$, where m is the multiplicity of the root. In our case, $m = 2$.

i	x_i	x_{i+1}	$f(x_i)$	$f'(x_i)$
1	1.2	≈ 2.11429	2.048	-4.48
2	2.11429	≈ 2.00157	≈ 0.05374	≈ 0.95351
3	2.00157	≈ 1.99998	≈ 0.00001	≈ 0.01257
4	1.99998	1.99998	≈ 0.00000	≈ 0.00016

(c)

The modified 2 Newton-Raphson Method states that

$$x_{i+1} = x_i - \frac{f(x_i)f'(x_i)}{[f'(x_i)]^2 - f(x_i)f''(x_i)}$$

$$f(x) = x^3 - 2x^2 - 4x + 8, \qquad f'(x) = 3x^2 - 4x - 4, \qquad f''(x) = 6x - 4$$

i	x_i	x_{i+1}	$f(x_i)$	$f'(x_i)$	$[f'(x_i)]^2$	$f''(x_i)$
1	1.2	≈ 1.87879	2.048	-4.48	20.0704	3.2
2	1.87879	≈ 1.99806	≈ 0.05699	≈ -0.92560	≈ 0.85674	7.27274
3	1.99806	≈ 2.00193	≈ 0.00002	≈ -0.01551	≈ 0.00024	7.98836
4	2.00193	≈ 2.00096	≈ 0.00001	≈ 0.01545	≈ 0.00024	8.01158
5	2.00096	2.00096	≈ 0.00000	≈ 0.00768	≈ 0.00006	8.00576

After studying all three methods, stopping when it is no longer possible to find a more accurate result, that is when $f(x_i) = 0$, as we are limited by a 5 decimal digit rounding arithmetic, we find that the modified 1 Newton-Raphson Method was the first to converge to the correct answer, converging in 4 iterations, while the slowest was the standard Newton-Raphson Method, which converged in 11 iterations.

Question 8

Determine the solution of simultaneous nonlinear equations:

$$y = -x^2 + x + 0.5$$
$$y + 5xy = x^2$$

Use Newton-Raphson Method and employ initial guesses of $x = y = 1.25$.

$$y = -x^2 + x + 0.5 \rightarrow y + x^2 - x - 0.5 = 0$$
$$y + 5xy = x^2 \rightarrow x^2 - y - 5xy = 0$$

Newton-Raphson Method for solving systems of non-linear equations states that

i	Steps
	$X_0 = \begin{bmatrix} 1.25 \\ 1.25 \end{bmatrix}, \quad F(X_i) = \begin{bmatrix} y + x^2 - x - 0.5 \\ x^2 - y - 5xy \end{bmatrix} \rightarrow F(X_0) = \begin{bmatrix} 1.0625 \\ -7.5 \end{bmatrix}$ $F'(X_i) = \begin{bmatrix} 2x - 1 & 1 \\ 2x - 5y & -1 - 5x \end{bmatrix} \rightarrow F'(X_0) = \begin{bmatrix} 1.5 & 1 \\ -3.75 & -7.25 \end{bmatrix}$
1	$[F'(X_0)]^{-1} = \dfrac{1}{(1.5)(-7.25) - (1)(-3.75)} \begin{bmatrix} -7.25 & -1 \\ 3.75 & 1.5 \end{bmatrix}$ $= -\dfrac{8}{57} \begin{bmatrix} -7.25 & -1 \\ 3.75 & 1.5 \end{bmatrix} = \begin{bmatrix} {}^{58}\!/\!_{57} & {}^{8}\!/\!_{57} \\ -{}^{10}\!/\!_{19} & -{}^{4}\!/\!_{19} \end{bmatrix}$ $X_1 = \begin{bmatrix} 1.25 \\ 1.25 \end{bmatrix} - \begin{bmatrix} {}^{58}\!/\!_{57} & {}^{8}\!/\!_{57} \\ -{}^{10}\!/\!_{19} & -{}^{4}\!/\!_{19} \end{bmatrix} \begin{bmatrix} 1.0625 \\ -7.5 \end{bmatrix} = \begin{bmatrix} 1.25 \\ 1.25 \end{bmatrix} - \begin{bmatrix} 0.02851 \\ 1.01974 \end{bmatrix} = \begin{bmatrix} 1.22149 \\ 0.23026 \end{bmatrix}$
2	$X_1 = \begin{bmatrix} 1.22149 \\ 0.23026 \end{bmatrix}, \quad F(X_i) = \begin{bmatrix} y + x^2 - x - 0.5 \\ x^2 - y - 5xy \end{bmatrix} \rightarrow F(X_1) = \begin{bmatrix} 0.00081 \\ -0.14452 \end{bmatrix}$ $F'(X_i) = \begin{bmatrix} 2x - 1 & 1 \\ 2x - 5y & -1 - 5x \end{bmatrix} \rightarrow F'(X_1) = \begin{bmatrix} 1.44298 & 1 \\ 1.29168 & -7.10745 \end{bmatrix}$ $[F'(X_1)]^{-1} = \dfrac{1}{(1.44298)(-7.10745) - (1)(1.29168)} \begin{bmatrix} -7.10745 & -1 \\ -1.29168 & 1.44298 \end{bmatrix}$ $= (-0.08660) \begin{bmatrix} -7.10745 & -1 \\ -1.29168 & 1.44298 \end{bmatrix} = \begin{bmatrix} 0.61551 & 0.08660 \\ 0.11186 & -0.12496 \end{bmatrix}$ $X_2 = \begin{bmatrix} 1.22149 \\ 0.23026 \end{bmatrix} - \begin{bmatrix} 0.61551 & 0.08660 \\ 0.11186 & -0.12496 \end{bmatrix} \begin{bmatrix} 0.00081 \\ -0.14452 \end{bmatrix}$ $= \begin{bmatrix} 1.22149 \\ 0.23026 \end{bmatrix} - \begin{bmatrix} -0.01202 \\ 0.01815 \end{bmatrix} = \begin{bmatrix} 1.23351 \\ 0.21211 \end{bmatrix}$