Discrete Mathematics

Chapter 5

Set Theory

SECTION 5.1

Basic Definitions of Set Theory

Basic Definitions

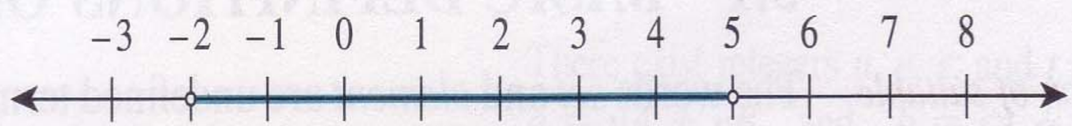
1. Suppose that Ann, Bob, and Cal are three students in a discrete mathematics class. The following sets represent the same set:
   {Ann, Bob, Cal}, {Bob, Cal, Ann}, {Bob, Bob, Ann, Cal, Ann}
2. {Ann} denotes the set whose only element is Ann, whereas the word *Ann* denotes Ann herself. Since these are different
   $$\{ \text{Ann} \} \neq \text{Ann}$$
3. Sets can themselves be elements of other sets. For example {1, {1}} has two elements: the number 1 and the set {1}.
4. Sometimes a set may appear to have more elements than it really has. For every nonnegative integer n, let $U_n = \{-n, n\}$. Then $U_2 = \{-2, 2\}$ and $U_1 = \{-1, 1\}$ each have two elements, but $U_0 = \{-0, 0\} = U_0 = \{0\}$ has only one element since $-0 = 0$

Example

Recall that $\mathbf{R}$ denotes the set of all real numbers, $\mathbf{Z}$ the set of all integers, and $\mathbf{Z}^+$ the set of all positive integers. Describe

a. $\{x \in \mathbf{R} \mid -2 < x < 5\}$
b. $\{x \in \mathbf{Z} \mid -2 < x < 5\}$
c. $\{x \in \mathbf{Z}^+ \mid -2 < x < 5\}$.

Solution a. $\{x \in \mathbf{R} \mid -2 < x < 5\}$ is the open interval of real numbers strictly between -2 and 5. It is pictured as follows:



b. $\{x \in \mathbf{Z} \mid -2 < x < 5\}$ is the set of all integers between -2 and 5. It is equal to the set $\{-1, 0, 1, 2, 3, 4\}$.

c. Since all the integers in $\mathbf{Z}^+$ are positive, $\{x \in \mathbf{Z}^+ \mid -2 < x < 5\} = \{1, 2, 3, 4\}$.

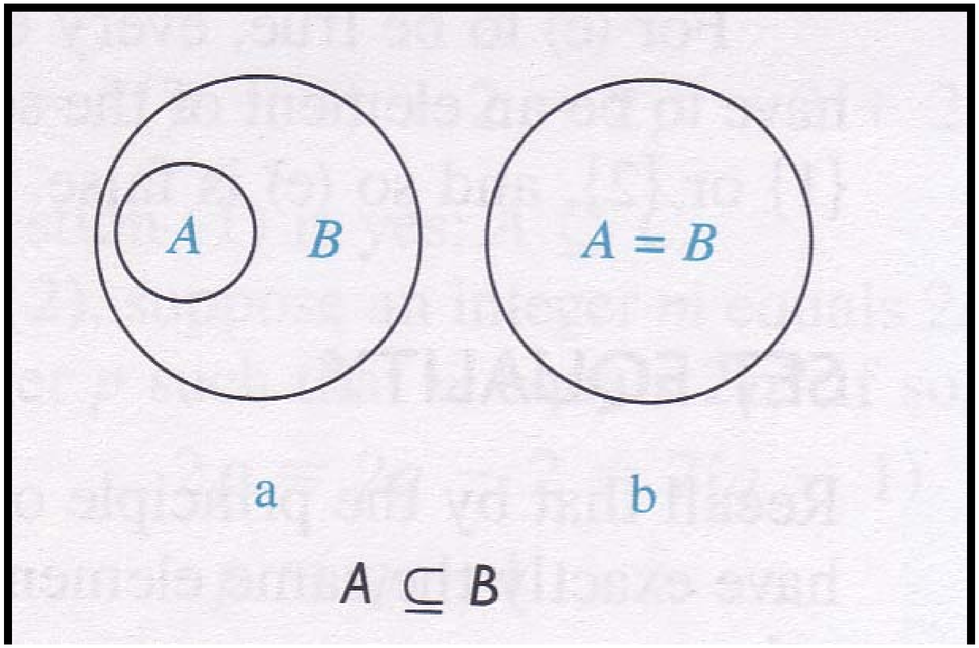
Subsets

If A and B are sets, A is called a **subset** of B, written $\boldsymbol{A \subseteq B}$, if, and only if, every element of A is also an element of B.

Symbolically:

$$A \subseteq B \quad \Leftrightarrow \quad \forall x, \text{ if } x \in A \text{ then } x \in B.$$

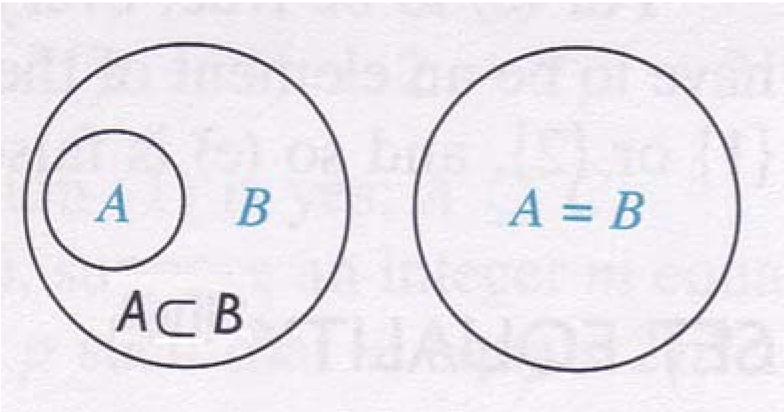
The phrases A *is contained in* B and B *contains* A are alternative ways of saying that A is a subset of B.



Proper Subset ⊂

DEFINITION

Let A and B be sets. A is a **proper subset** of B if, and only if, every element of A is in B but there is at least one element of B that is not in A.



Proper Subset

- Some authors prefer to use the symbol $\subset$ to indicate *proper* subset . This usage makes $\subseteq$ and $\subset$ analogous to $\leq$ and $<$.
- If $x \leq y$ then x may be equal to y, or maybe not, but if $x < y$, then x is strictly less than y. Similarly, if $A \subseteq B$, then A may or may not be equal to B, but if $A \subset B$, then A is definitely not equal to B.

Not a Subset

It follows from the definition of subset that a set A is not a subset of a set B, written $A \not\subseteq B$, if, and only if, there is at least one element of A that is not an element of B. Symbolically:

$$\boxed{A \not\subseteq B \quad \Leftrightarrow \quad \exists x \text{ such that } x \in A \text{ and } x \notin B.}$$

Example

Suppose B76, XR3, D54, ES2, and XL5 are the model numbers of certain pieces of equipment. Let $A = \{B76, XR3, D54, XL5\}$, $B = \{B76, D54\}$, and $C = \{ES2, XL5\}$.

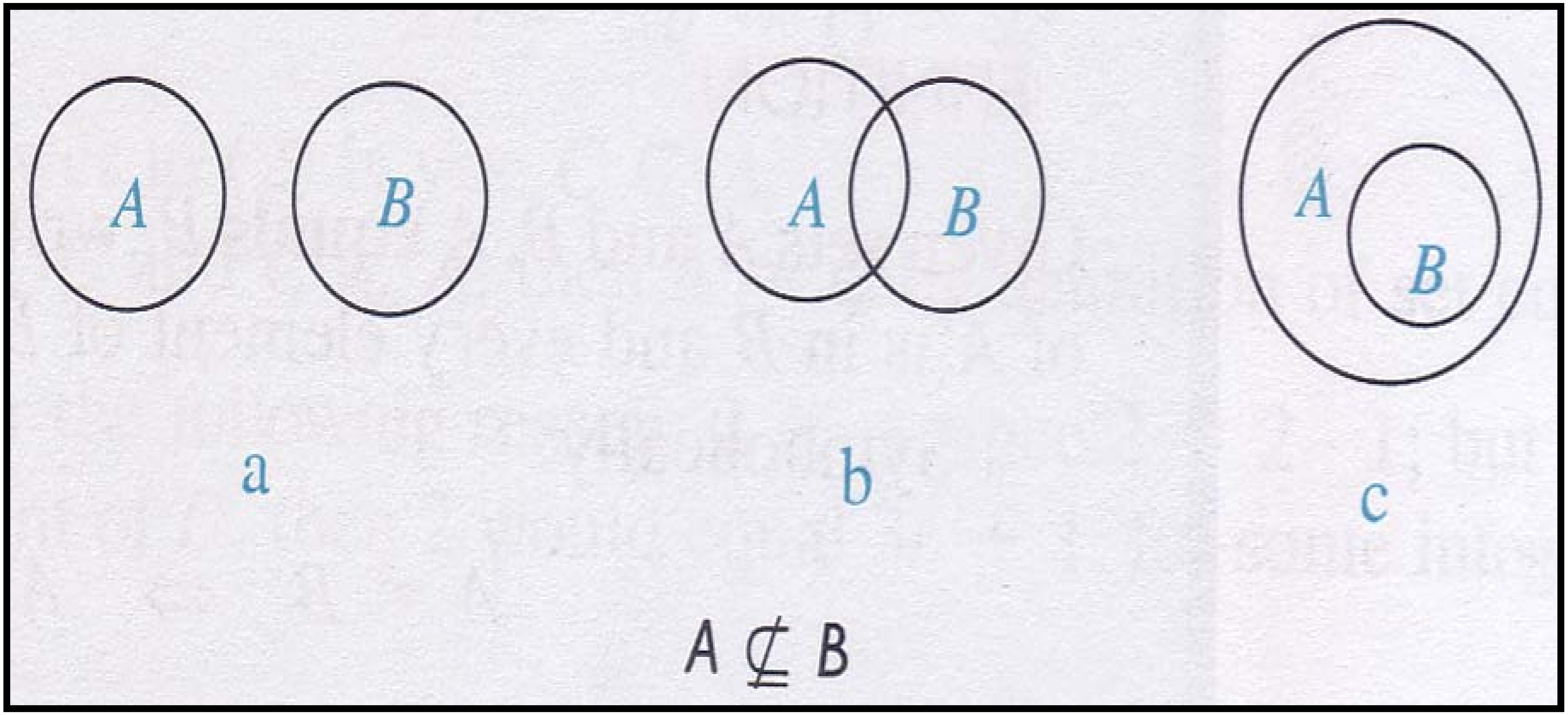
a. Is $B \subseteq A$? b. Is $C \subseteq A$? c. Is $B \subseteq B$?

Solution

a. Yes. Both elements of B are in A.

b. No; ES2 is in C but not in A.

c. Yes. Both elements of B are in B. (The definition of subset implies that any set is a subset of itself.)


A
B
a
A
B
b
A
B
c
A ⊄ B

Example

Which of the following are true statements?

a. $2 \in \{1, 2, 3\}$ b. $\{2\} \in \{1, 2, 3\}$ c. $2 \subseteq \{1, 2, 3\}$
d. $\{2\} \subseteq \{1, 2, 3\}$ e. $\{2\} \subseteq \{\{1\}, \{2\}\}$ f. $\{2\} \in \{\{1\}, \{2\}\}$

Solution Only (a), (d), and (f) are true.

For (b) to be true, the set $\{1, 2, 3\}$ would have to contain the element $\{2\}$. But the only elements of $\{1, 2, 3\}$ are 1, 2, and 3, and 2 is not equal to $\{2\}$. Hence (b) is false

For (c) to be true, the number 2 would have to be a set and every element in the set 2 would have to be an element of $\{1, 2, 3\}$. This is not the case, so (c) is false

For (e) to be true, every element in the set containing only the number 2 would have to be an element of the set whose elements are $\{1\}$ and $\{2\}$. But 2 is not equal to $\{1\}$ or $\{2\}$, and so (e) is false.

Set Equality

DEFINITION

Given sets A and B, A **equals** B, written $\boldsymbol{A = B}$, if, and only if, every element of A is in B and every element of B is in A.

Symbolically:

$$A = B \quad \Leftrightarrow \quad A \subseteq B \quad \text{and} \quad B \subseteq A.$$

This version of the definition of equality implies the following:

> To know that a set A equals a set B, you must know that $A \subseteq B$ and you must also know that $B \subseteq A$.

Example

Let sets A, B, C, and D be defined as follows:

$$A = \{n \in \mathbf{Z} \mid n = 2p, \text{ for some integer } p\},$$
$$B = \text{the set of all even integers},$$
$$C = \{m \in \mathbf{Z} \mid m = 2q - 2, \text{ for some integer } q\},$$
$$D = \{k \in \mathbf{Z} \mid k = 3r + 1, \text{ for some integer } r\}.$$

a. Is $A = B$? b. Is $A = C$? c. Is $A = D$?

Solution a. Yes. $A = B$ because every integer of the form $2p$, for some integer p, is even (so $A \subseteq B$), and every integer that is even can be written in the form $2p$, for some integer p (so $B \subseteq A$).

Example (Cont.)

b. Yes. $A = C$ if, and only if, every element of A is in C and every element of C is in A. Considering the definitions of A and C, deciding whether $A = C$ involves deciding whether both of the following questions can be answered yes:

1. Can any integer that can be written in the form $2p$, for some integer p, also be written in the form $2q - 2$, for some integer q?
2. Can any integer that can be written in the form $2q - 2$, for some integer q, also be written in the form $2p$, for some integer p?

To answer question (1), suppose an integer n equals $2p$, for some integer p. Can you find an integer q so that n equals $2q - 2$? If so, then

$$2q - 2 = 2p$$

$$2q = 2p + 2 = 2(p + 1)$$

and thus $$q = p + 1.$$

Example (Cont.)

So, if $n = 2p$, where p is an integer, let $q = p + 1$. Then q is an integer (since it is a sum of integers) and

$$2q - 2 = 2(p + 1) - 2 = 2p - 2 + 2 = 2p.$$

Hence the answer to question (1) is yes: $A \subseteq C$.

To answer question (2), suppose an integer m equals $2q - 2$, for some integer q. Can you find an integer p such that m equals $2p$? If so, then

$$2p = 2q - 2 = 2(q - 1)$$

and thus $$p = q - 1.$$

So, if $m = 2q - 2$, where q is an integer, let $p = q - 1$. Then p is an integer (since it is a difference of two integers), and

$$2p = 2(q - 1) = 2q - 2.$$

Hence the answer to (2) is yes: $C \subseteq A$.

Since $A \subseteq C$ and $C \subseteq A$, then $A = C$ by definition of set equality.

Example (Cont.)

c. No. $A \neq D$ for the following reason: $2 \in A$ since $2 = 2 \cdot 1$; but $2 \notin D$. For if 2 were an element of D, then 2 would equal $3r + 1$, for some integer r. Solving for r would give

$$3r + 1 = 2$$

$$3r = 2 - 1$$

$$3r = 1$$

$$r = \frac{1}{3}.$$

Operations on sets

DEFINITION

Let A and B be subsets of a universal set U.

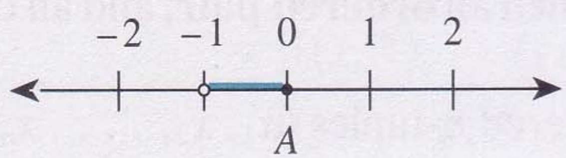
1. The **union** of A and B, denoted $\boldsymbol{A \cup B}$, is the set of all elements x in U such that x is in A or x is in B.
2. The **intersection** of A and B, denoted $\boldsymbol{A \cap B}$, is the set of all elements x in U such that x is in A and x is in B.
3. The **difference** of B minus A (or **relative complement** of A in B), denoted $B - A$, is the set of all elements x in U such that x is in B and x is not in A.
4. The **complement** of A, denoted $\boldsymbol{A^c}$, is the set of all elements x in U such that x is not in A.

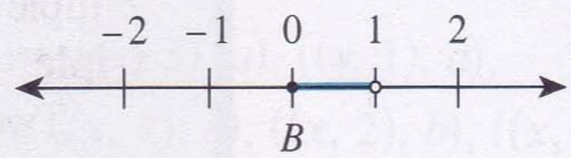
Symbolically:

$$A \cup B = \{x \in U \mid x \in A \text{ or } x \in B\},$$
$$A \cap B = \{x \in U \mid x \in A \text{ and } x \in B\},$$
$$B - A = \{x \in U \mid x \in B \text{ and } x \notin A\},$$
$$A^c = \{x \in U \mid x \notin A\}.$$

Example

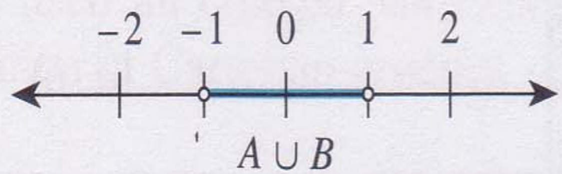
Let the universal set be the set $\mathbf{R}$ of all real numbers and let $A = \{x \in \mathbf{R} \mid -1 < x \leq 0\}$ and $B = \{x \in \mathbf{R} \mid 0 \leq x < 1\}$. These sets are shown on the number lines below.





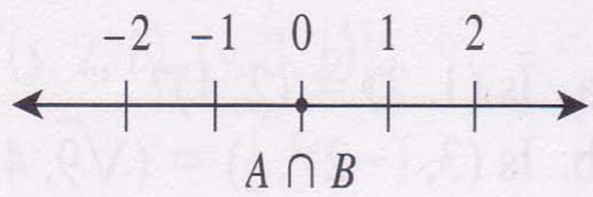
Find $A \cup B$, $A \cap B$, and A^c.

Solution $A \cup B = \{x \in \mathbf{R} \mid -1 < x \leq 0 \text{ or } 0 \leq x < 1\} = \{x \in \mathbf{R} \mid -1 < x < 1\}$.

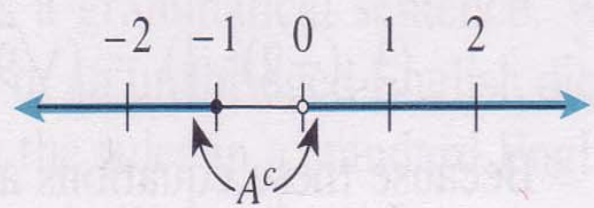


Example (Cont.)

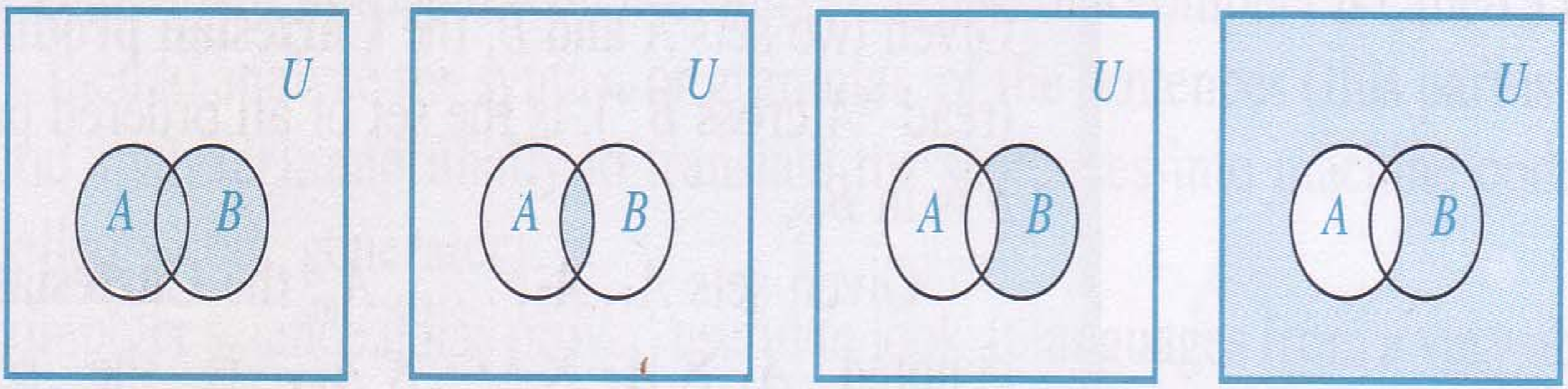
$A \cap B = \{x \in \mathbf{R} \mid -1 < x \leq 0 \text{ and } 0 \leq x < 1\} = \{0\}.$



$$\begin{aligned} A^c &= \{x \in \mathbf{R} \mid \text{it is not the case that } -1 < x \leq 0\} \\ &= \{x \in \mathbf{R} \mid \text{it is not the case that } (-1 < x \text{ and } x \leq 0)\} && \text{by definition of the double inequality} \\ &= \{x \in \mathbf{R} \mid x \leq -1 \text{ or } x > 0\} && \text{by De Morgan's law} \end{aligned}$$



Example (Cont.)



Shaded region represents $A \cup B$.

Shaded region represents $A \cap B$.

Shaded region represents $B - A$.

Shaded region represents A^c.

Ordered n-tuple

DEFINITION

Let n be a positive integer and let $x_1, x_2, \ldots, x_n$ be (not necessarily distinct) elements. The **ordered n-tuple, $(x_1, x_2, \ldots, x_n)$,** consists of $x_1, x_2, \ldots, x_n$ together with the ordering: first x_1, then x_2, and so forth up to x_n. An ordered 2-tuple is called an **ordered pair**, and an ordered 3-tuple is called an **ordered triple**.

Two ordered n-tuples $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ are **equal** if, and only if, $x_1 = y_1, x_2 = y_2, \ldots, x_n = y_n$.

Symbolically:

$$(x_1, x_2, \ldots, x_n) = (y_1, y_2, \ldots, y_n) \quad \Leftrightarrow \quad x_1 = y_1, x_2 = y_2, \ldots, x_n = y_n.$$

In particular:

$$(a, b) = (c, d) \quad \Leftrightarrow \quad a = c \text{ and } b = d.$$

Example

a. Is $(1, 2) = (2, 1)$?

b. Is $(3, (-2)^2, \frac{1}{2}) = (\sqrt{9}, 4, \frac{3}{6})$?

Solution a. No. By definition of equality of ordered pairs,

$$(1, 2) = (2, 1) \quad \Leftrightarrow \quad 1 = 2 \text{ and } 2 = 1.$$

But $1 \neq 2$, and so the ordered pairs are not equal.

b. Yes. By definition of equality of ordered triples,

$$(3, (-2)^2, \tfrac{1}{2}) = (\sqrt{9}, 4, \tfrac{3}{6}) \quad \Leftrightarrow \quad 3 = \sqrt{9} \text{ and } (-2)^2 = 4 \text{ and } \tfrac{1}{2} = \tfrac{3}{6}.$$

Because these equations are all true, the two ordered triples are equal.

Cartesian product

DEFINITION

Given two sets A and B, the **Cartesian product** of A and B, denoted $\mathbf{A \times B}$ (read "A cross B"), is the set of all ordered pairs (a, b), where a is in A and b is in B.

Given sets $A_1, A_2, \ldots, A_n$, the **Cartesian product** of $A_1, A_2, \ldots, A_n$, denoted $\mathbf{A_1 \times A_2 \times \cdots \times A_n}$, is the set of all ordered n-tuples $(a_1, a_2, \ldots, a_n)$ where $a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n$.

Symbolically:

$$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\},$$

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n\}.$$

Example

Let $A = \{x, y\}$, $B = \{1, 2, 3\}$, and $C = \{a, b\}$.

a. Find $A \times B$. b. Find $(A \times B) \times C$. c. Find $A \times B \times C$.

Solution a. $A \times B = \{(x, 1), (x, 2), (x, 3), (y, 1), (y, 2), (y, 3)\}$

b. The Cartesian product of A and B is a set, so it may be used as one of the sets making up another Cartesian product. This is the case for $(A \times B) \times C$:

$$(A \times B) \times C = \{(u, v) \mid u \in A \times B \text{ and } v \in C\}$$ by definition of Cartesian product

$$= \{((x, 1), a), ((x, 2), a), ((x, 3), a), ((y, 1), a), ((y, 2), a), ((y, 3), a), ((x, 1), b), ((x, 2), b), ((x, 3), b), ((y, 1), b), ((y, 2), b), ((y, 3), b)\}$$

Example (Cont.)

c. The Cartesian product $A \times B \times C$ is superficially similar to, but is not quite the same mathematical object as, $(A \times B) \times C$. $(A \times B) \times C$ is a set of ordered pairs of which one element is itself an ordered pair, whereas $A \times B \times C$ is a set of ordered triples. By definition of Cartesian product,

$$\begin{aligned} A \times B \times C &= \{(u, v, w) \mid u \in A, v \in B, \text{ and } w \in C\} \\ &= \{(x, 1, a), (x, 2, a), (x, 3, a), (y, 1, a), (y, 2, a), \\ &\quad (y, 3, a), (x, 1, b), (x, 2, b), (x, 3, b), (y, 1, b), \\ &\quad (y, 2, b), (y, 3, b)\}. \end{aligned}$$

SECTION 5.2

Properties of Sets

Theorem 5.2.1

THEOREM 5.2.1 Some Subset Relations

1. Inclusion of Intersection: For all sets A and B,

$$\text{(a) } A \cap B \subseteq A \quad \text{and} \quad \text{(b) } A \cap B \subseteq B.$$

2. Inclusion in Union: For all sets A and B,

$$\text{(a) } A \subseteq A \cup B \quad \text{and} \quad \text{(b) } B \subseteq A \cup B.$$

3. Transitive Property of Subsets: For all sets A, B, and C,

$$\text{if } A \subseteq B \quad \text{and} \quad B \subseteq C, \text{ then } A \subseteq C.$$

Procedural Versions of Set Definitions

Let X and Y be subsets of a universal set U and suppose x and y are elements of U.

1. $x \in X \cup Y \Leftrightarrow x \in X$ or $x \in Y$
2. $x \in X \cap Y \Leftrightarrow x \in X$ and $x \in Y$
3. $x \in X - Y \Leftrightarrow x \in X$ and $x \notin Y$
4. $x \in X^c \Leftrightarrow x \notin X$
5. $(x, y) \in X \times Y \Leftrightarrow x \in X$ and $y \in Y$

Theorem 5.2.2

THEOREM 5.2.2 Set Identities

Let all sets referred to below be subsets of a universal set U.

1. Commutative Laws: For all sets A and B,
$$\text{(a)}\quad A \cap B = B \cap A \quad\text{and}\quad \text{(b)}\quad A \cup B = B \cup A.$$
2. Associative Laws: For all sets A, B, and C,
$$\text{(a)}\quad (A \cap B) \cap C = A \cap (B \cap C) \quad\text{and}$$
$$\text{(b)}\quad (A \cup B) \cup C = A \cup (B \cup C).$$
3. Distributive Laws: For all sets, A, B, and C,
$$\text{(a)}\quad A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \quad\text{and}$$
$$\text{(b)}\quad A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$
4. Intersection with U (U Acts as an Identity for $\cap$): For all sets A,
$$A \cap U = A.$$
5. Double Complement Law: For all sets A,
$$(A^c)^c = A.$$

Theorem 5.2.2

6. Idempotent Laws: For all sets A,

$$\text{(a)}\ A \cap A = A \quad \text{and} \quad \text{(b)}\ A \cup A = A.$$

7. De Morgan's Laws: For all sets A and B,

$$\text{(a)}\ (A \cup B)^c = A^c \cap B^c \quad \text{and} \quad \text{(b)}\ (A \cap B)^c = A^c \cup B^c.$$

8. Union with U (U Acts as a Universal Bound for $\cup$):

$$A \cup U = U.$$

9. Absorption Laws: For all sets A and B,

$$\text{(a)}\ A \cup (A \cap B) = A \quad \text{and} \quad \text{(b)}\ A \cap (A \cup B) = A.$$

10. Alternate Representation for Set Difference: For all sets A and B,

$$A - B = A \cap B^c.$$

Theorem 5.2.3

THEOREM 5.2.3 Intersection and Union with a Subset

For any sets A and B, if $A \subseteq B$, then

$$\text{(a) } A \cap B = A \quad \text{and} \quad \text{(b) } A \cup B = B.$$

Proof of part (a):

Suppose A and B are sets with $A \subseteq B$. To show part (a) we must show that $A \cap B \subseteq A$ and $A \subseteq A \cap B$. We already know that $A \cap B \subseteq A$ by the inclusion of intersection property. To show that $A \subseteq A \cap B$, let $x \in A$. *[We must show that $x \in A \cap B$.]* Since $A \subseteq B$, then $x \in B$ also. Hence

$$x \in A \quad \text{and} \quad x \in B,$$

and thus

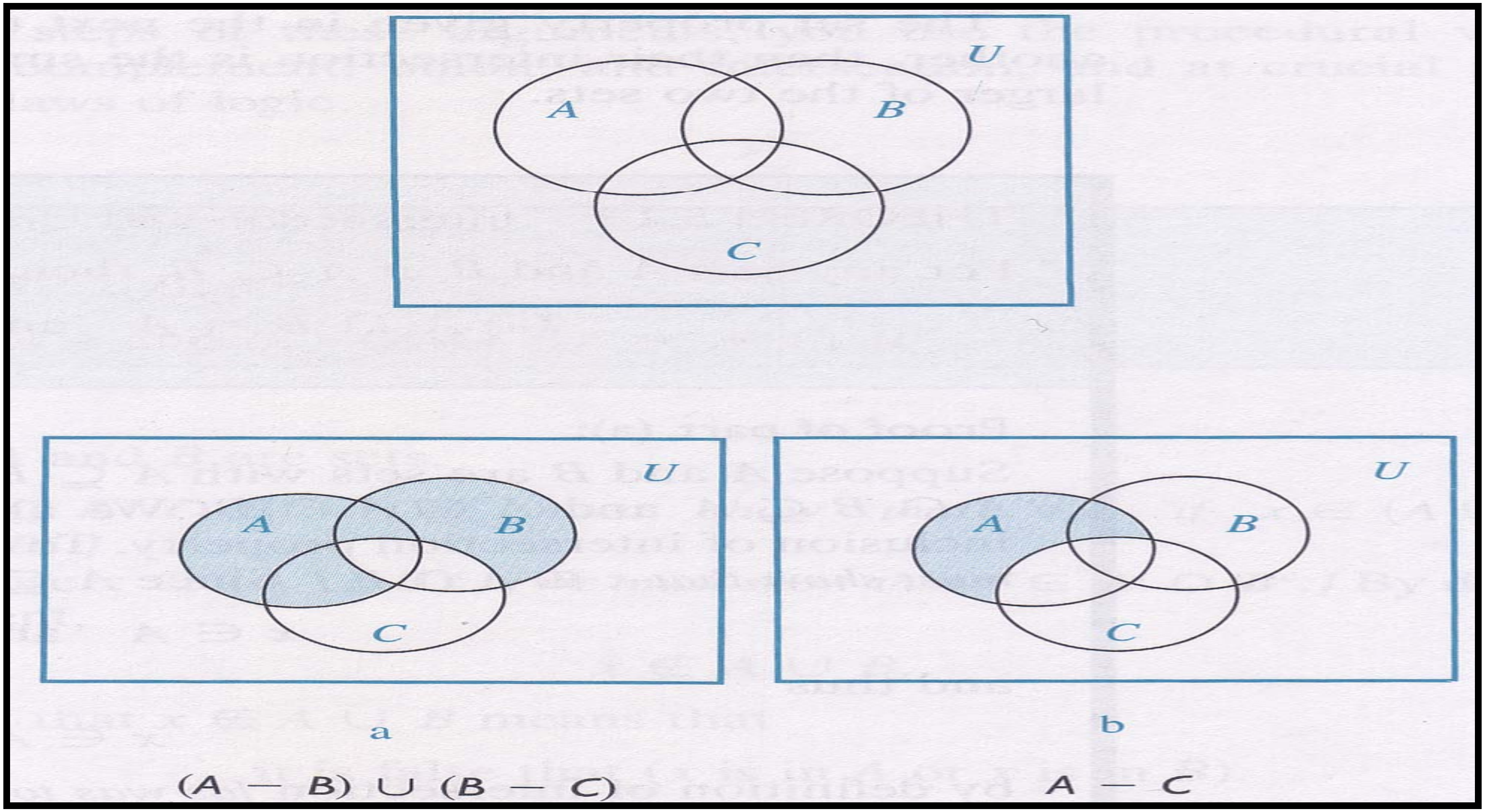
$$x \in A \cap B$$

by definition of intersection *[as was to be shown]*.

Example

- Is the following set property true?

For all sets A, B, and C, $(A - B) \cup (B - C) = A - C$



Example (Cont.)

- This property is true if and only if , the given equality holds for all sets A, B and C.
  - So it is false if and only if , there are sets A, B, and C for which the equality does not hold.

Example

Prove that for all sets, A, B, and C

$$(A \cup B) - C = (A - C) \cup (B - C).$$

Solution Let sets A, B, and C be given. Then

$(A \cup B) - C = (A \cup B) \cap C^c$	by the alternate representation of set difference law
$= C^c \cap (A \cup B)$	by the commutative law for $\cap$
$= (C^c \cap A) \cup (C^c \cap B)$	by the distributive law
$= (A \cap C^c) \cup (B \cap C^c)$	by the commutative law for $\cap$
$= (A - C) \cup (B - C)$	by the alternate representation of set difference law.

SECTION 5.3

The Empty Set, Partitions, Power Sets, and Boolean Algebras

Theorem 5.3.1

THEOREM 5.3.1 A Set with No Elements Is a Subset of Every Set

If $\emptyset$ is a set with no elements and A is any set, then $\emptyset \subseteq A$.

Proof (by contradiction):

Suppose not. *[We take the negation of the theorem and suppose it to be true.]* Suppose there exists a set $\emptyset$ with no elements and a set A such that $\emptyset \not\subseteq A$. *[We must deduce a contradiction.]* Then there would be an element of $\emptyset$ which is not an element of A *[by definition of subset]*. But there can be no such element since $\emptyset$ has no elements. This is a contradiction. *[Hence the supposition that there are sets $\emptyset$ and A, where $\emptyset$ has no elements and $\emptyset \not\subseteq A$, is false, and so the theorem is true.]*

Uniqueness of the Empty Set

COROLLARY 5.3.2 Uniqueness of the Empty Set

There is only one set with no elements.

Proof:

Suppose $\emptyset_1$ and $\emptyset_2$ are each sets with no elements. By Theorem 5.3.1 above, $\emptyset_1 \subseteq \emptyset_2$ since $\emptyset_1$ has no elements. Also $\emptyset_2 \subseteq \emptyset_1$ since $\emptyset_2$ has no elements. Thus $\emptyset_1 = \emptyset_2$ by definition of set equality.

Empty Set

DEFINITION

The unique set with no elements is called the **empty set**. It is denoted by the symbol $\emptyset$.

Element Method for Proving a Set Equals the Empty Set

To prove that a set X is equal to the empty set $\emptyset$, prove that X has no elements. To do this, suppose X has an element and derive a contradiction.

Theorem 5.3.3

THEOREM 5.3.3 Set Properties That Involve $\emptyset$

Let all sets referred to below be subsets of a universal set U.

1. Union with $\emptyset$ ($\emptyset$ Acts as an Identity for $\cup$): For all sets A,

$$A \cup \emptyset = A.$$

2. Intersection and Union with the Complement: For all sets A,

$$\text{(a) } A \cap A^c = \emptyset \quad \text{and} \quad \text{(b) } A \cup A^c = U.$$

3. Intersection with $\emptyset$ ($\emptyset$ Acts as a Universal Bound for $\cap$): For all sets A,

$$A \cap \emptyset = \emptyset.$$

4. Complements of U and $\emptyset$:

$$\text{(a) } U^c = \emptyset \quad \text{and} \quad \text{(b) } \emptyset^c = U.$$

Disjoint Sets

DEFINITION

Two sets are called **disjoint** if, and only if, they have no elements in common.

Symbolically:

$$A \text{ and } B \text{ are disjoint} \quad \Leftrightarrow \quad A \cap B = \emptyset.$$

Example

a. Let $A = \{1, 3, 5\}$ and $B = \{2, 4, 6\}$. Are A and B disjoint?
b. Given any sets A and B, are $A - B$ and B disjoint?

Solution a. Yes. By inspection A and B have no elements in common, or, in other words, $\{1, 3, 5\} \cap \{2, 4, 6\} = \emptyset$.

b. Yes. To prove that $A - B$ and B are disjoint, it is necessary to show that $(A - B) \cap B = \emptyset$. By the element method for proving a set equals the empty set, it suffices to show that $(A - B) \cap B$ has no elements. But by definition of intersection, any element in $(A - B) \cap B$ would be both in $A - B$ and in B. It follows by definition of difference that such an element would be both in B and not in B, which is impossible.

Example (Cont.)

Given any sets A and B, $(A - B)$ and B are disjoint.

Proof (by contradiction):

Suppose not. *[We take the negation of the theorem and suppose it to be true.]* Suppose there exist sets A and B such that $A - B$ and B are not disjoint. *[We must derive a contradiction.]* Then $(A - B) \cap B \neq \emptyset$, and so there is an element x in $(A - B) \cap B$. By definition of intersection, $x \in A - B$ and $x \in B$, and since $x \in A - B$, by definition of difference, $x \in A$ and $x \notin B$. Hence $x \in B$ and also $x \notin B$, which is a contradiction. *[Thus the supposition that there exist sets A and B such that A − B and B are not disjoint is false, and hence the proposition is true.]*

Mutually Disjoint Sets

DEFINITION

Sets $A_1, A_2, \ldots, A_n$ are **mutually disjoint** (or **pairwise disjoint or nonoverlapping**) if, and only if, no two sets A_i and A_j with distinct subscripts have any elements in common. More precisely, for all $i, j = 1, 2, \ldots, n$,

$$A_i \cap A_j = \emptyset \quad \text{whenever } i \neq j.$$

Example

a. Let $A_1 = \{3, 5\}$, $A_2 = \{1, 4, 6\}$, and $A_3 = \{2\}$. Are A_1, A_2, and A_3 mutually disjoint?

b. Let $B_1 = \{2, 4, 6\}$, $B_2 = \{3, 7\}$, and $B_3 = \{4, 5\}$. Are B_1, B_2, and B_3 mutually disjoint?

Solution a. Yes. A_1 and A_2 have no elements in common, A_1 and A_3 have no elements in common, and A_2 and A_3 have no elements in common.

b. No. B_1 and B_3 both contain 4.

Partitions of sets

DEFINITION

A collection of nonempty sets $\{A_1, A_2, \ldots, A_n\}$ is a **partition** of a set A if, and only if,

1. $A = A_1 \cup A_2 \cup \ldots \cup A_n$;

2. $A_1, A_2, \ldots, A_n$ are mutually disjoint.

Example

a. Let $A = \{1, 2, 3, 4, 5, 6\}$, $A_1 = \{1, 2\}$, $A_2 = \{3, 4\}$, and $A_3 = \{5, 6\}$. Is $\{A_1, A_2, A_3\}$ a partition of A?

b. Let $\mathbf{Z}$ be the set of all integers and let

$$T_0 = \{n \in \mathbf{Z} \mid n = 3k, \text{ for some integer } k\},$$
$$T_1 = \{n \in \mathbf{Z} \mid n = 3k + 1, \text{ for some integer } k\}, \text{ and}$$
$$T_2 = \{n \in \mathbf{Z} \mid n = 3k + 2, \text{ for some integer } k\}.$$

Is $\{T_0, T_1, T_2\}$ a partition of $\mathbf{Z}$?

Solution a. Yes. By inspection, $A = A_1 \cup A_2 \cup A_3$ and the sets A_1, A_2, and A_3 are mutually disjoint.

b. Yes. By the quotient-remainder theorem every integer n can be represented in exactly one of the three forms

$$n = 3k \quad \text{or} \quad n = 3k + 1 \quad \text{or} \quad n = 3k + 2,$$

for some integer k. This implies that no integer can be in any two of the sets T_0, T_1, or T_2. So T_0, T_1, and T_2 are mutually disjoint. It also implies that every integer is in one of the sets T_0, T_1, or T_2. So $\mathbf{Z} = T_1 \cup T_2 \cup T_3$.

Power Set

DEFINITION

Given a set A, the **power set** of A, denoted $\mathcal{P}(A)$, is the set of all subsets of A.

Example:

Find the power set of the set $\{x, y\}$. That is, find $\mathcal{P}(\{x, y\})$.

Solution $\mathcal{P}(\{x, y\})$ is the set of all subsets of $\{x, y\}$. Now since $\emptyset$ is a subset of every set $\emptyset \in \mathcal{P}(\{x, y\})$. Also any set is a subset of itself, so $\{x, y\} \in \mathcal{P}(\{x, y\})$. The only other subsets of $\{x, y\}$ are $\{x\}$ and $\{y\}$, so

$$\mathcal{P}(\{x, y\}) = \{\emptyset, \{x\}, \{y\}, \{x, y\}\}.$$

Theorem 5.3.5

THEOREM 5.3.5

For all sets A and B, if $A \subseteq B$ then $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

Proof:

Suppose A and B are sets such that $A \subseteq B$. *[We must show that $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.]*

Suppose $X \in \mathcal{P}(A)$. *[We must show that $X \in \mathcal{P}(B)$.]* Since $X \in \mathcal{P}(A)$, then $X \subseteq A$ by definition of power set. But $A \subseteq B$. Hence $X \subseteq B$ by the transitive property for subsets. It follows that $X \in \mathcal{P}(B)$ by definition of power set *[as was to be shown]*.

Thus $\mathcal{P}(A) \subseteq \mathcal{P}(B)$ by definition of subset *[as was to be shown]*.

THEOREM 5.3.6

For all integers $n \geq 0$, if a set X has n elements then $\mathcal{P}(X)$ has 2^n elements.

Proof (by mathematical induction):

Consider the property "Any set with n elements has 2^n subsets."

The property is true for $n = 0$: We must show that a set with zero elements has 2^0 subsets. But the only set with zero elements is the empty set, and the only subset of the empty set is itself. Thus a set with zero elements has one subset. Since $1 = 2^0$, the theorem is true for $n = 0$.

If the property is true for $n = k$, then it is true for $n = k + 1$: Let k be any integer with $k \geq 0$ and suppose that any set with k elements has 2^k subsets. *[This is the inductive hypothesis.]* We must show that any set with $k + 1$ elements has 2^{k+1} subsets. Let X be a set with $k + 1$ elements and pick an element z in X. Observe that any subset of X either contains z or it does not. Furthermore, any subset of X that does not contain z is a subset of $X - \{z\}$. And any subset A of $X - \{z\}$ can be matched up with a subset B, equal to $A \cup \{z\}$, of X that contains z. Consequently, there are as many subsets of X that contain z as do not, and thus there are twice as many subsets of X as there are subsets of $X - \{z\}$. But $X - \{z\}$ has k elements, and so

$$\text{the number of subsets of } X - \{z\} = 2^k \quad \text{by inductive hypothesis.}$$

Therefore,

$$\begin{aligned} \text{the number of subsets of } X &= 2 \cdot (\text{the number of subsets of } X - \{z\}) \\ &= 2 \cdot (2^k) \quad \text{by substitution} \\ &= 2^{k+1} \quad \text{by basic algebra.} \end{aligned}$$

Boolean Algebras

- A Boolean algebras is a set S together with two operations generally denoted (+)and (.) such that for all a and b in S, both a+b and a.b are in S and such that the following axioms hold:

1. For all a and b in S,

$$a + b = b + a$$ Commutative Law of +

$$a \cdot b = b \cdot a$$ Commutative Law of ·.

2. For all a, b, and c in S,

$$(a + b) + c = a + (b + c)$$ Associative Law of +

$$(a \cdot b) \cdot c = a \cdot (b \cdot c)$$ Associative Law of ·.

Boolean Algebras

3. For all a, b, and c in S,

$$a + (b \cdot c) = (a + b) \cdot (a + c) \qquad \text{Distributive Law of + over} \cdot$$

$$a \cdot (b + c) = (a \cdot b) + (a \cdot c) \qquad \text{Distributive Law of} \cdot \text{over +.}$$

4. There exist distinct elements 0 and 1 in S such that for all a in S,

$$a + 0 = a \qquad \text{0 Is an Identity for +}$$

$$a \cdot 1 = a \qquad \text{1 Is an Identity for} \cdot.$$

5. For each a in S, there exists an element denoted $\overline{a}$ and called the *complement* or *negation of a in S* such that

$$a + \overline{a} = 1 \quad \text{and} \quad a \cdot \overline{a} = 0. \qquad \text{Complement Laws.}$$