陰的および陽的スパース同定のためのラグランジアンとニュートン力学の同時同定による非線形力学の発見

Discovering Nonlinear Dynamics by Simultaneous Lagrangian and Newtonian Identification for Implicit and Explicit Sparse Identification

Tohoku University - School of engineering - NeuroRobotics Laboratory

Eymeric CHAUCHAT C4TM1417 - 18th November 2025

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White box system

$$\ddot{x} = -c_1\dot{x} - k_1 y$$

$$\ddot{y} = -c_2\dot{y} - k_2 x$$

Black box system

$$h_1 = \sigma(W_{11}^{(1)} x + W_{12}^{(1)} y + b_1^{(1)})$$

$$h_2 = \sigma(W_{21}^{(2)} h_1 + W_{22}^{(2)} \dot{x} + b_2^{(2)})$$

$$h_3 = \sigma(W_{31}^{(3)} h_2 + W_{32}^{(3)} \dot{y} + b_3^{(3)})$$

$$\vdots$$

$$h_n = \sigma(W^{(n)} h_{n-1} + b^{(n)})$$

$$\ddot{x} = W_1^{(out)} h_n + b_1^{(out)}$$

$$\ddot{y} = W_2^{(out)} h_n + b_2^{(out)}$$

Opening

White box, Black box system identification

Pros:
- Physically interpretable
- Generalizes well
- Requires less data

Cons:
- Scales with expert knowledge
- Difficult to model complex systems

White box system

$$\ddot{x} = -c_1 \dot{x} - k_1 y$$

$$\ddot{y} = -c_2 \dot{y} - k_2 x$$

Pros:
- Scales to complex systems
- No expert knowledge required

Cons:
- Not physically interpretable
- Poor generalization
- Requires large amount of data

Black box system

$$h_1 = \sigma(W_{11}^{(1)} x + W_{12}^{(1)} y + b_1^{(1)})$$

$$h_2 = \sigma(W_{21}^{(2)} h_1 + W_{22}^{(2)} \dot{x} + b_2^{(2)})$$

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1 What is SINDy

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Sparse
Identification of
Nonlinear
Dynamics

1 What is SINDy

Sparse Identification of Nonlinear Dynamics

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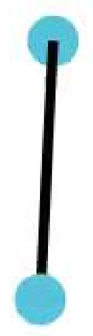
Sparse Identification of Nonlinear Dynamics



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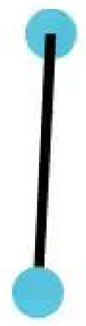

$$\sum F\left(\theta, \dot{\theta}\right) = m\ddot{\theta}$$

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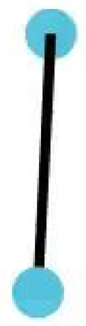
Sparse Identification of Nonlinear Dynamics



$$\ddot{\theta} + \frac{g}{L}\sin(\theta) = 0$$

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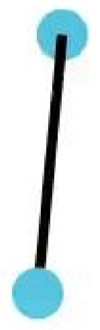

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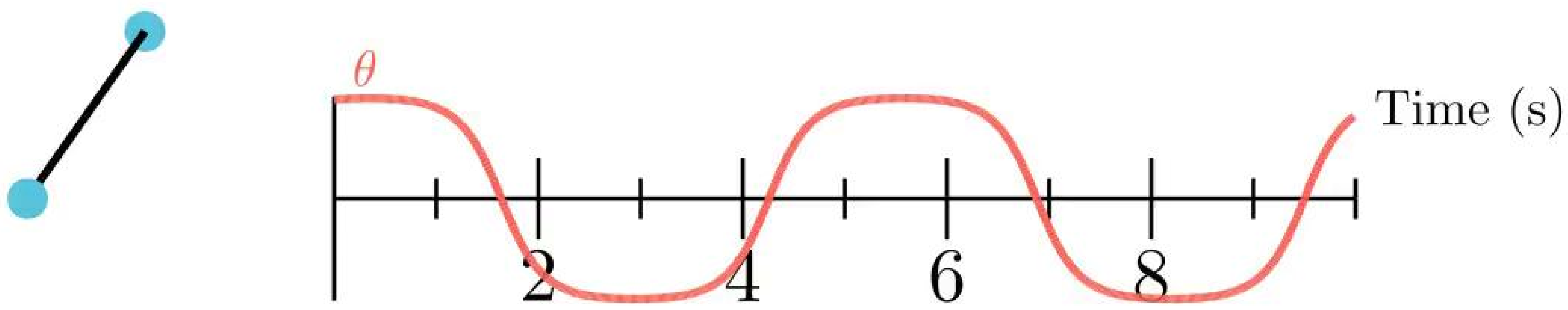

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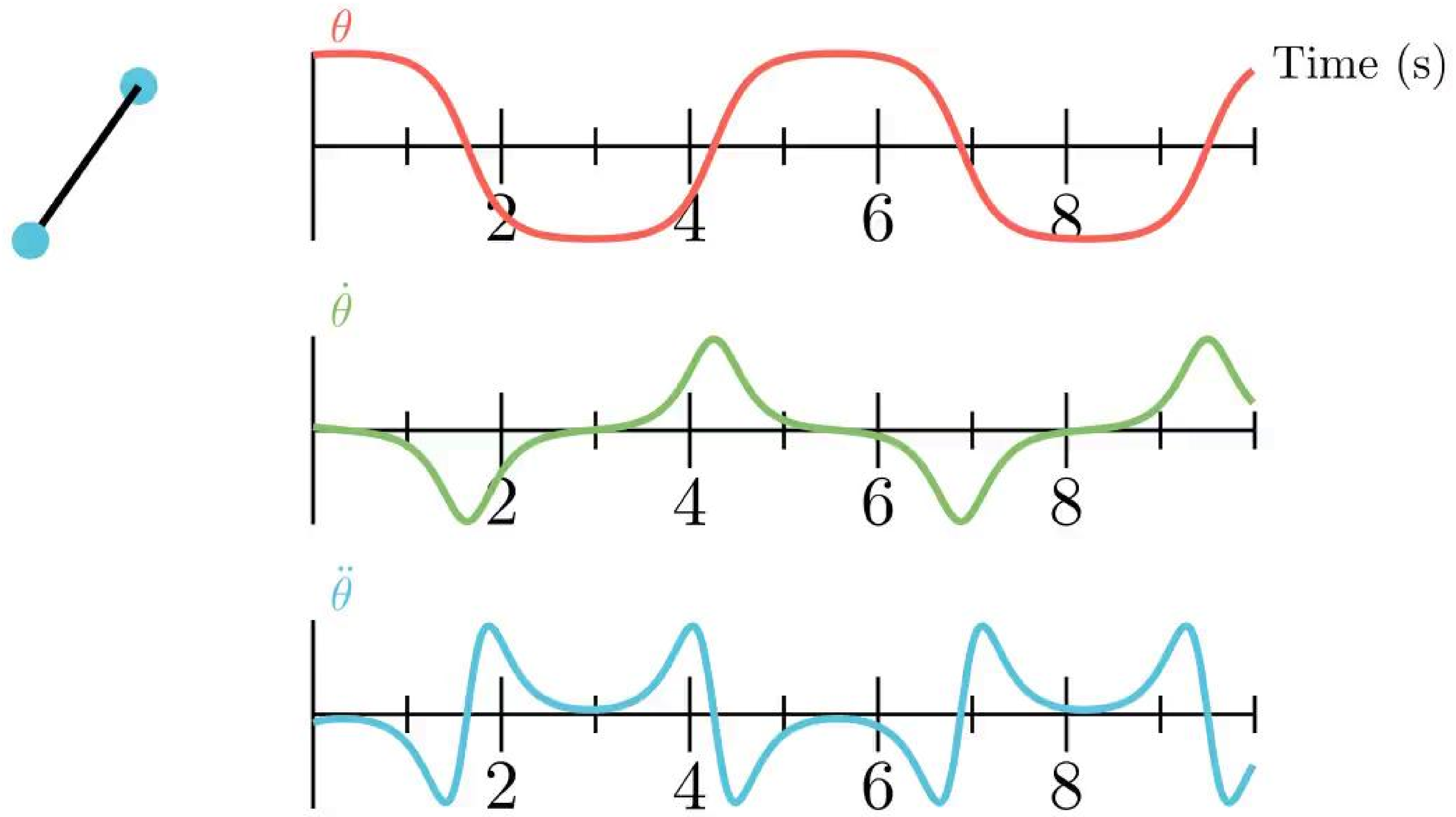

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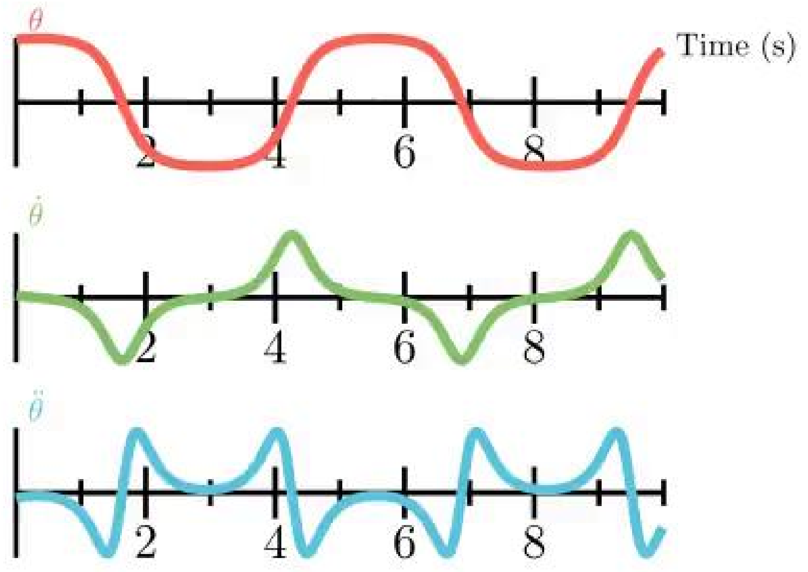

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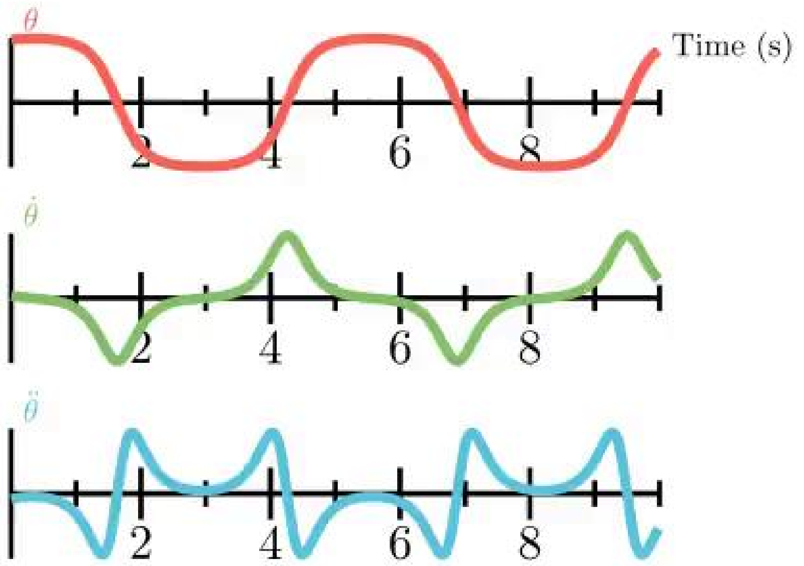

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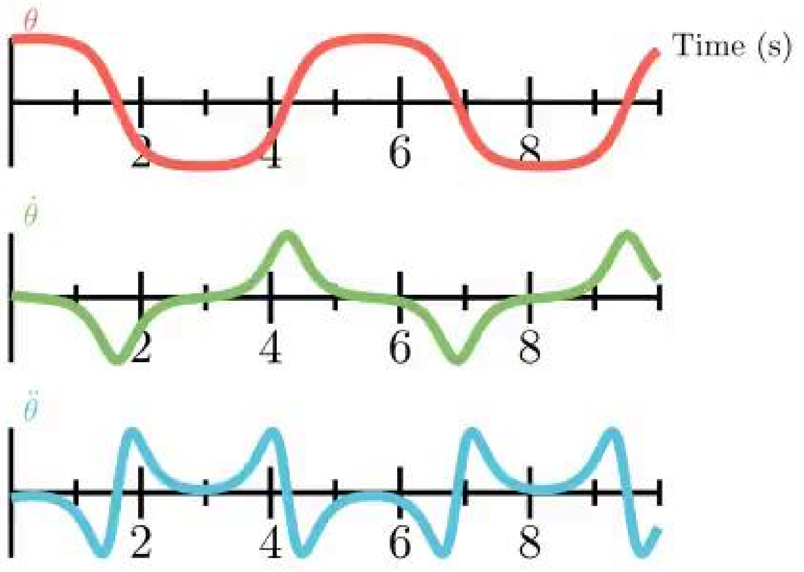

$$\boxed{\sum F\left(\theta, \dot{\theta}\right)} = m\ddot{\theta}$$

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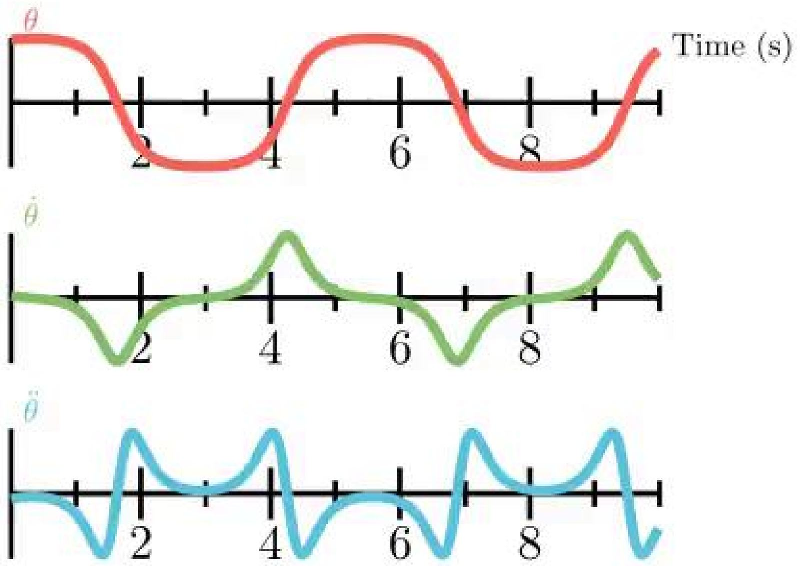
Sparse Identification of Nonlinear Dynamics



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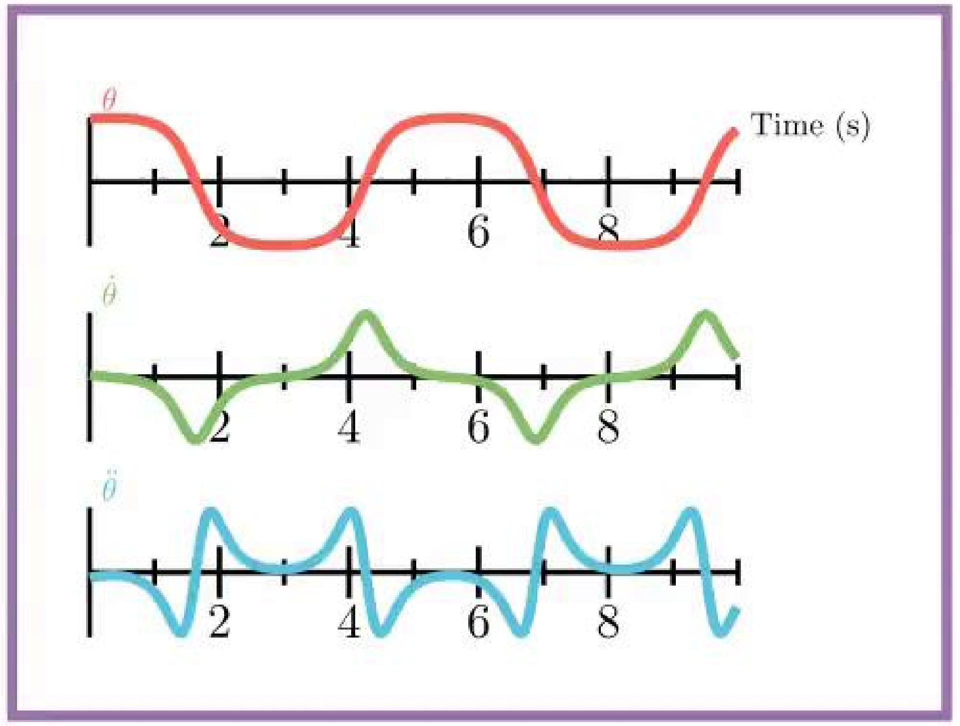
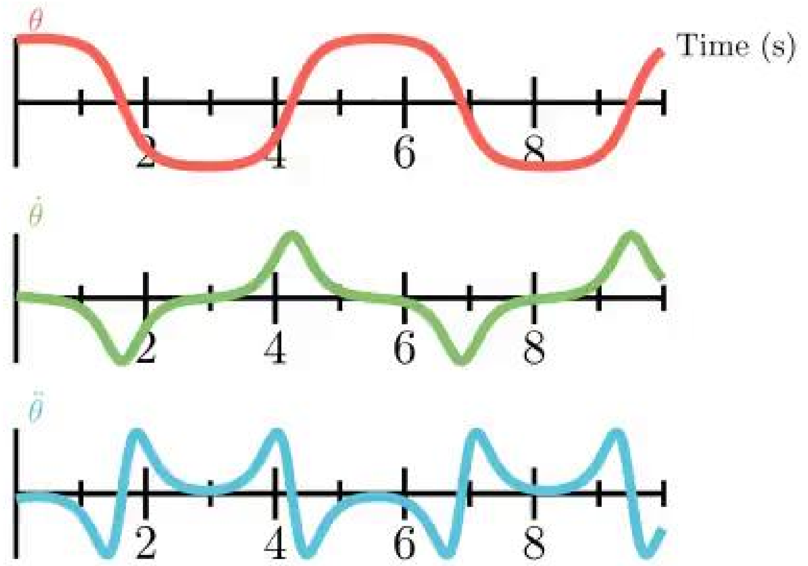
Sparse Identification of Nonlinear Dynamics



SINDy generates system from data

1 What is SINDy



Sparse Identification of Nonlinear Dynamics

Catalog of candidate functions

$$\sin(\theta)$$

$$\cos(\theta)$$

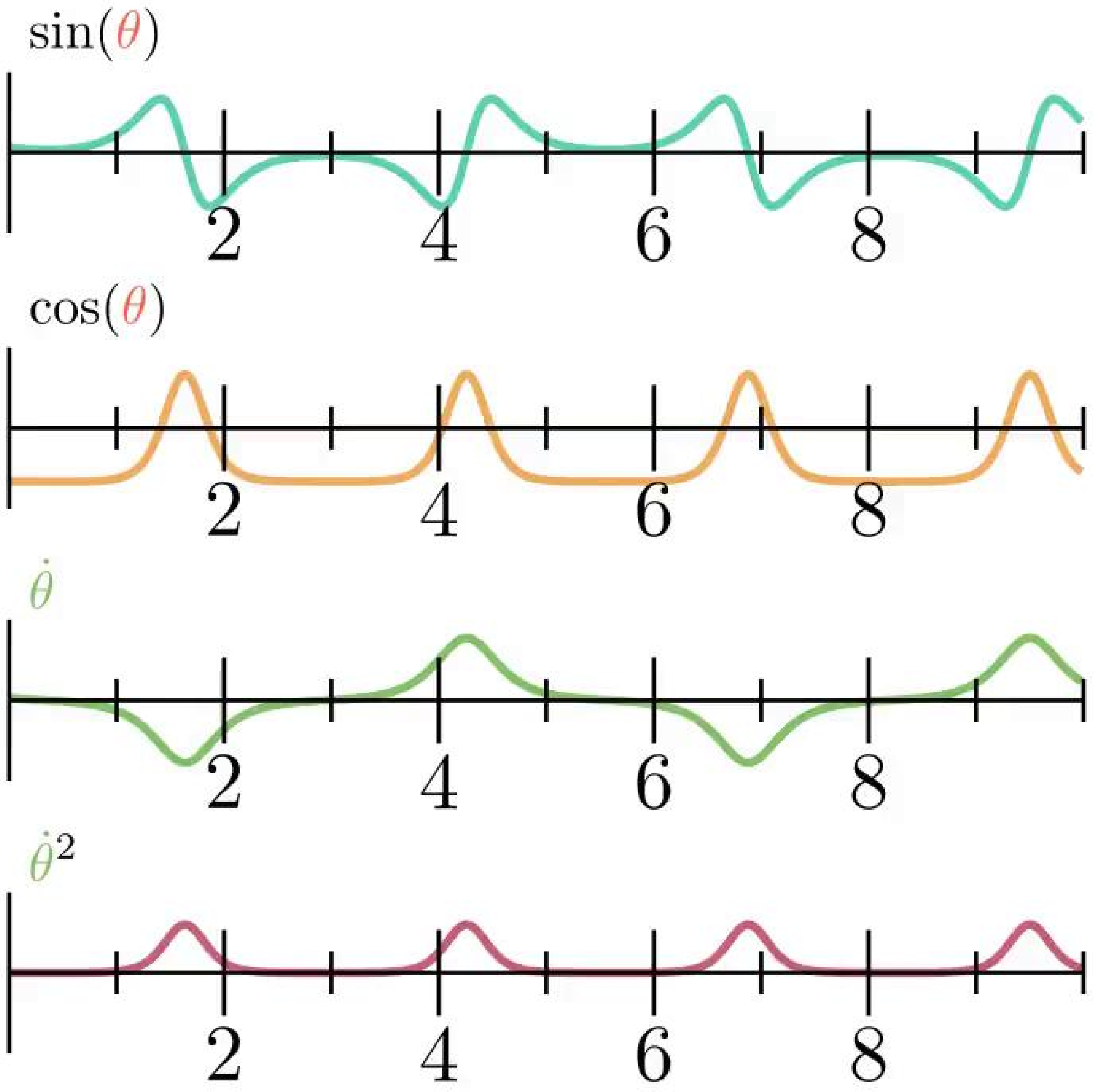
$$\dot{\theta}$$

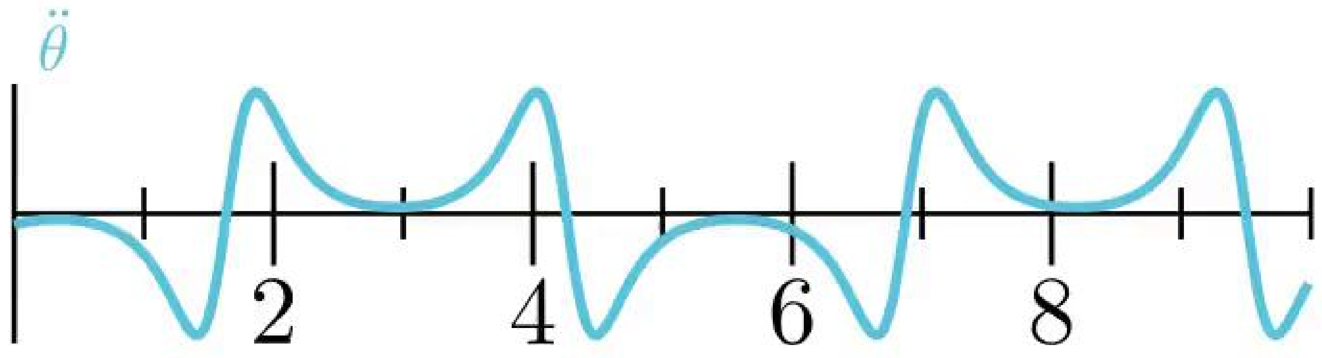
$$\dot{\theta}^2$$

1 What is SINDy

Sparse Identification of Nonlinear Dynamics

Catalog of candidate functions





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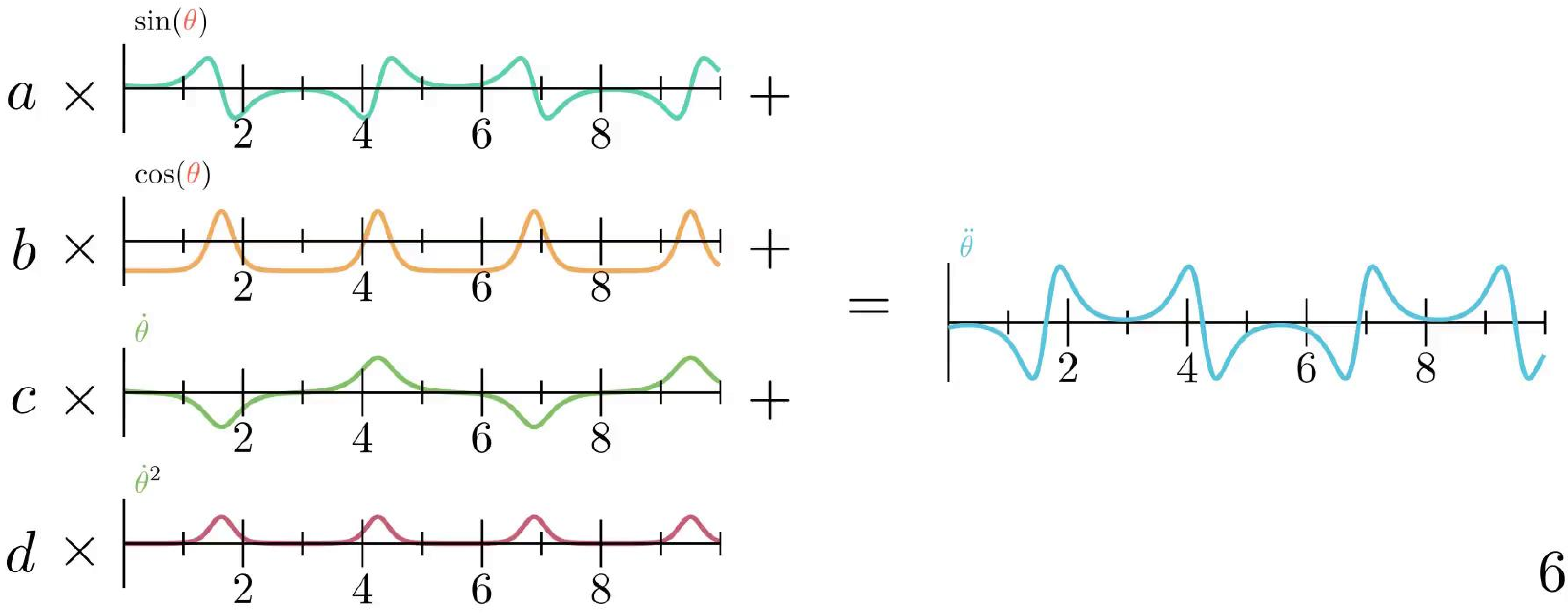
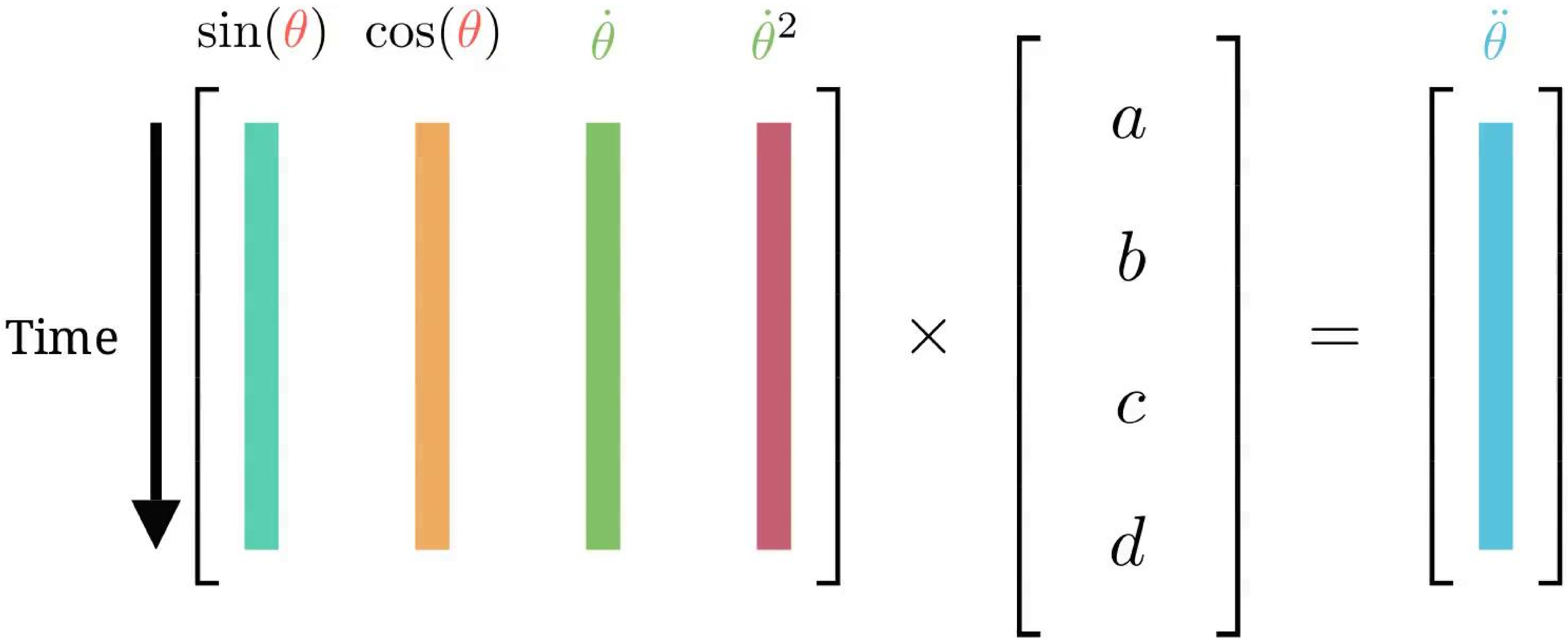
Catalog of candidate functions



1 What is SINDy

Sparse Identification of Nonlinear Dynamics

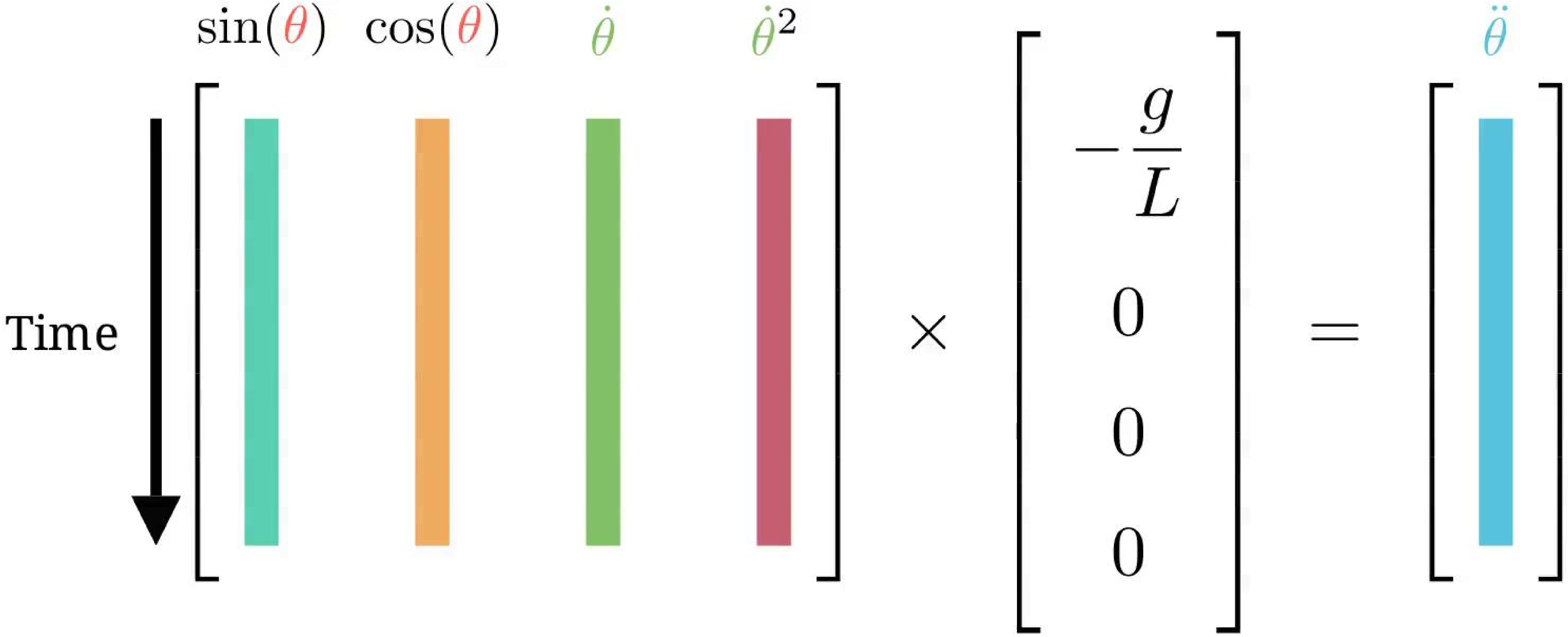
We can define the SINDy linear system



1 What is SINDy

Sparse Identification of Nonlinear Dynamics

We can define the SINDy linear system



2.1 Sindy type : classic SINDy

$$\begin{bmatrix} f_1 & f_2 & f_3 & f_4 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} F_{ext} \end{bmatrix}$$

Brunton, S. L., Proctor, J. L. & Kutz, J. N. Discovering governing equations from data: Sparse identification of nonlinear dynamical systems. (2016).

2.2 Sindy type : SINDy-PI

$$\begin{bmatrix} f_1 & f_2 & f_3 & f_4 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$

Kaheman, K., Kutz, J. N. & Brunton, S. L. SINDy-PI: A Robust Algorithm for Parallel Implicit Sparse Identification of Nonlinear Dynamics. (2020).

2.2 Sindy type : SINDy-PI

Null space optimization is HARD

$$\begin{bmatrix} f_1 & f_2 & f_3 & f_4 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$

Kaheman, K., Kutz, J. N. & Brunton, S. L. SINDy-PI: A Robust Algorithm for Parallel Implicit Sparse Identification of Nonlinear Dynamics. (2020).

2.2 Sindy type : SINDy-PI

$$\begin{bmatrix} f_1 & f_2 & f_3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} f_4 \end{bmatrix}$$

Kaheman, K., Kutz, J. N. & Brunton, S. L. SINDy-PI: A Robust Algorithm for Parallel Implicit Sparse Identification of Nonlinear Dynamics. (2020).

2.2 Sindy type : SINDy-PI

$$\begin{bmatrix} f_1 & f_2 & f_3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} f_4 \end{bmatrix}$$

What if the right term is not in our catalog ?

Kaheman, K., Kutz, J. N. & Brunton, S. L. SINDy-PI: A Robust Algorithm for Parallel Implicit Sparse Identification of Nonlinear Dynamics. (2020).

2.2 Sindy type : SINDy-PI

$$\begin{bmatrix} f_4 & f_2 & f_3 \end{bmatrix} \begin{bmatrix} d \\ b \\ c \end{bmatrix} = \begin{bmatrix} f_1 \end{bmatrix} \qquad \begin{bmatrix} f_1 & f_4 & f_3 \end{bmatrix} \begin{bmatrix} a \\ d \\ c \end{bmatrix} = \begin{bmatrix} f_2 \end{bmatrix}$$

$$\begin{bmatrix} f_1 & f_2 & f_4 \end{bmatrix} \begin{bmatrix} a \\ b \\ d \end{bmatrix} = \begin{bmatrix} f_3 \end{bmatrix} \qquad \begin{bmatrix} f_1 & f_2 & f_3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} f_4 \end{bmatrix}$$

Kaheman, K., Kutz, J. N. & Brunton, S. L. SINDy-PI: A Robust Algorithm for Parallel Implicit Sparse Identification of Nonlinear Dynamics. (2020).

2.2 Sindy type : SINDy-PI

$$\begin{bmatrix} f_1 & f_2 & f_3 & f_4 \end{bmatrix} \begin{bmatrix} 0 & a_2 & a_3 & a_4 \\ b_1 & 0 & b_3 & b_4 \\ c_1 & c_2 & 0 & c_4 \\ d_1 & d_2 & d_3 & 0 \end{bmatrix} = \begin{bmatrix} f_1 & f_2 & f_3 & f_4 \end{bmatrix}$$

The SINDy-Parallel-Implicit formulation

Kaheman, K., Kutz, J. N. & Brunton, S. L. SINDy-PI: A Robust Algorithm for Parallel Implicit Sparse Identification of Nonlinear Dynamics. (2020).

2.2 Sindy type : SINDy-PI

$$\begin{bmatrix} f_1 & f_2 & f_3 & f_4 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0.3 & 0.0 \\ 0.5 & 0 & 0.3 & 2.0 \\ 0.1 & 0.0 & 0 & 0.0 \\ 0.3 & 0.5 & 0.7 & 0 \end{bmatrix} = \begin{bmatrix} f_1 & f_2 & f_3 & f_4 \end{bmatrix}$$

The SINDy-Parallel-Implicit formulation

Kaheman, K., Kutz, J. N. & Brunton, S. L. SINDy-PI: A Robust Algorithm for Parallel Implicit Sparse Identification of Nonlinear Dynamics. (2020).

2.2 Sindy type : SINDy-PI

$$
\begin{bmatrix} f_1 & f_2 & f_3 & f_4 \end{bmatrix}
\begin{bmatrix}
0 & 0 & 0.3 & 0.0 \\
0.5 & 0 & 0.3 & 2.0 \\
0.1 & 0.0 & 0 & 0.0 \\
0.3 & 0.5 & 0.7 & 0
\end{bmatrix}
=
\begin{bmatrix} f_1 & f_2 & f_3 & f_4 \end{bmatrix}
$$

The SINDy-Parallel-Implicit formulation

Kaheman, K., Kutz, J. N. & Brunton, S. L. SINDy-PI: A Robust Algorithm for Parallel Implicit Sparse Identification of Nonlinear Dynamics. (2020).

3.1 SINDy limitation

Until now, I showed one coordinate system

But in Newton formulation we have one equation per coordinate

$$\sin(q), \cos(q), \ddot{q}, \dot{q}$$

3.1 SINDy limitation

Until now, I showed one coordinate system

But in Newton formulation we have one equation per coordinate

$$\sin(q_1), \cos(q_1), \ddot{q}_1, \dot{q}_1, \sin(q_2), \cos(q_2), \ddot{q}_2, \dot{q}_2$$

3.1 SINDy limitation

Until now, I showed one coordinate system

But in Newton formulation we have one equation per coordinate

$$
\begin{array}{cccccc}
\sin(q_1)^2 & \sin(q_1)\cos(q_1) & \sin(q_1)\ddot{q}_1 & \sin(q_1)\dot{q}_1 & \sin(q_1)\sin(q_2) & \sin(q_1)\cos(q_2) \\
\sin(q_1)\ddot{q}_2 & \sin(q_1)\dot{q}_2 & \cos(q_1)^2 & \cos(q_1)\ddot{q}_1 & \cos(q_1)\dot{q}_1 & \cos(q_1)\sin(q_2) \\
\cos(q_1)\cos(q_2) & \cos(q_1)\ddot{q}_2 & \cos(q_1)\dot{q}_2 & \ddot{q}_1^2 & \ddot{q}_1\dot{q}_1 & \ddot{q}_1\sin(q_2) \\
\ddot{q}_1\cos(q_2) & \ddot{q}_1\ddot{q}_2 & \ddot{q}_1\dot{q}_2 & \dot{q}_1^2 & \dot{q}_1\sin(q_2) & \dot{q}_1\cos(q_2) \\
\dot{q}_1\ddot{q}_2 & \dot{q}_1\dot{q}_2 & \sin(q_2)^2 & \sin(q_2)\cos(q_2) & \sin(q_2)\ddot{q}_2 & \sin(q_2)\dot{q}_2 \\
\cos(q_2)^2 & \cos(q_2)\ddot{q}_2 & \cos(q_2)\dot{q}_2 & \ddot{q}_2^2 & \ddot{q}_2\dot{q}_2 & \dot{q}_2^2
\end{array}
$$

This lead to 36 terms to consider for only 8 bases function

3.1 SINDy limitation

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$\sin(q_1)^2$	$\sin(q_1)\cos(q_1)$	$\sin(q_1)\ddot{q}_1$	$\sin(q_1)\dot{q}_1$	$\sin(q_1)\sin(q_2)$	$\sin(q_1)\cos(q_2)$
$\sin(q_1)\ddot{q}_2$	$\sin(q_1)\dot{q}_2$	$\cos(q_1)^2$	$\cos(q_1)\ddot{q}_1$	$\cos(q_1)\dot{q}_1$	$\cos(q_1)\sin(q_2)$
$\cos(q_1)\cos(q_2)$	$\cos(q_1)\ddot{q}_2$	$\cos(q_1)\dot{q}_2$	$\ddot{q}_1^2$	$\ddot{q}_1\dot{q}_1$	$\ddot{q}_1\sin(q_2)$
$\ddot{q}_1\cos(q_2)$	$\ddot{q}_1\ddot{q}_2$	$\ddot{q}_1\dot{q}_2$	$\dot{q}_1^2$	$\dot{q}_1\sin(q_2)$	$\dot{q}_1\cos(q_2)$
$\dot{q}_1\ddot{q}_2$	$\dot{q}_1\dot{q}_2$	$\sin(q_2)^2$	$\sin(q_2)\cos(q_2)$	$\sin(q_2)\ddot{q}_2$	$\sin(q_2)\dot{q}_2$
$\cos(q_2)^2$	$\cos(q_2)\ddot{q}_2$	$\cos(q_2)\dot{q}_2$	$\ddot{q}_2^2$	$\ddot{q}_2\dot{q}_2$	$\dot{q}_2^2$

$\sin(q_1)^2$	$\sin(q_1)\cos(q_1)$	$\sin(q_1)\ddot{q}_1$	$\sin(q_1)\dot{q}_1$	$\sin(q_1)\sin(q_2)$	$\sin(q_1)\cos(q_2)$
$\sin(q_1)\ddot{q}_2$	$\sin(q_1)\dot{q}_2$	$\cos(q_1)^2$	$\cos(q_1)\ddot{q}_1$	$\cos(q_1)\dot{q}_1$	$\cos(q_1)\sin(q_2)$
$\cos(q_1)\cos(q_2)$	$\cos(q_1)\ddot{q}_2$	$\cos(q_1)\dot{q}_2$	$\ddot{q}_1^2$	$\ddot{q}_1\dot{q}_1$	$\ddot{q}_1\sin(q_2)$
$\ddot{q}_1\cos(q_2)$	$\ddot{q}_1\ddot{q}_2$	$\ddot{q}_1\dot{q}_2$	$\dot{q}_1^2$	$\dot{q}_1\sin(q_2)$	$\dot{q}_1\cos(q_2)$
$\dot{q}_1\ddot{q}_2$	$\dot{q}_1\dot{q}_2$	$\sin(q_2)^2$	$\sin(q_2)\cos(q_2)$	$\sin(q_2)\ddot{q}_2$	$\sin(q_2)\dot{q}_2$
$\cos(q_2)^2$	$\cos(q_2)\ddot{q}_2$	$\cos(q_2)\dot{q}_2$	$\ddot{q}_2^2$	$\ddot{q}_2\dot{q}_2$	$\dot{q}_2^2$

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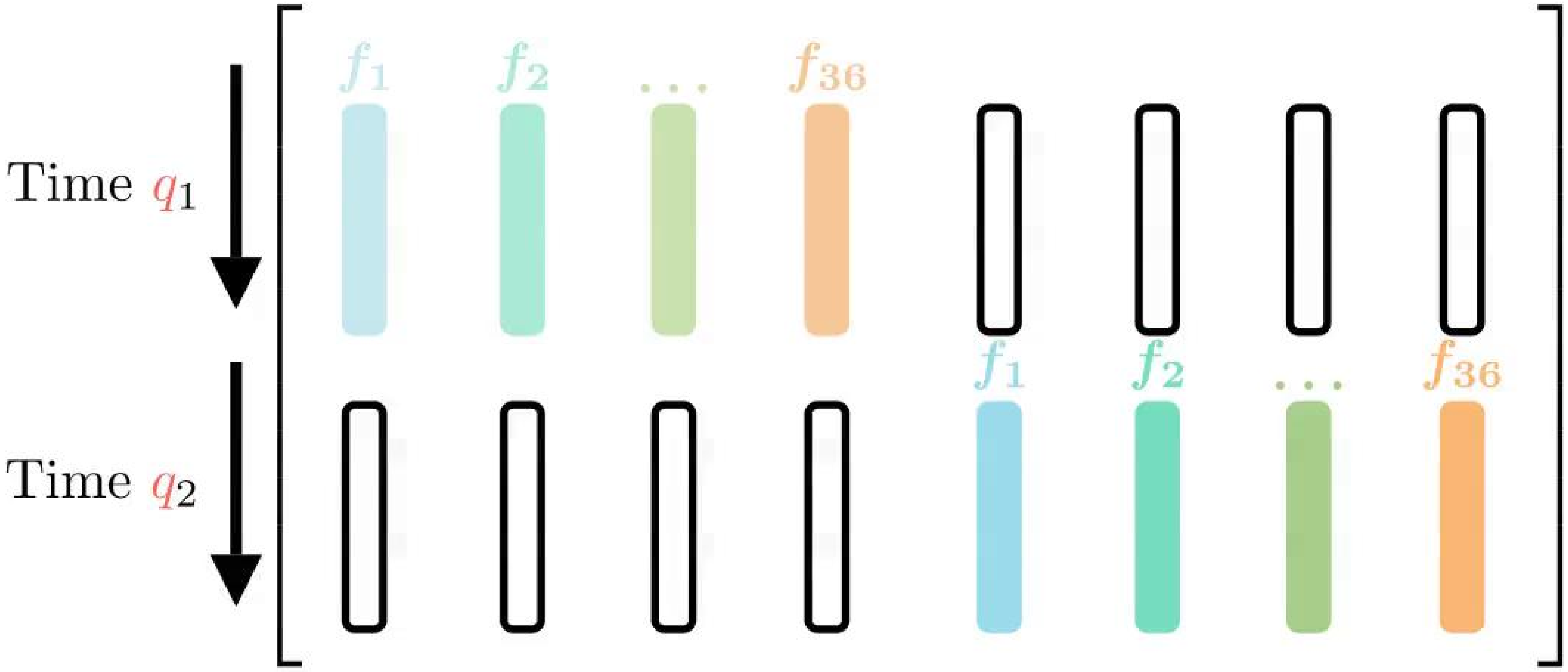

3.2 The Lab SINDy introduction

One solution, the lagrangian formulation

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One solution, the lagrangian formulation

One equation per system

$$\mathcal{L} = T - V$$

with T the kinetic energy and V the potential energy

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One solution, the lagrangian formulation

One equation per system

$$\mathcal{L} = T - V$$

with T the kinetic energy and V the potential energy

Euler-Lagrange equation $\downarrow$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_1}\right) - \frac{\partial \mathcal{L}}{\partial q_1} = 0$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_2}\right) - \frac{\partial \mathcal{L}}{\partial q_2} = 0$$

3.2 The Lab SINDy introduction

One solution, the lagrangian formulation

One equation per system

Lagrange space

$$\mathcal{L} = T - V$$

with T the kinetic energy and V the potential energy

Euler-Lagrange equation

Newton space

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_1}\right) - \frac{\partial \mathcal{L}}{\partial q_1} = 0$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_2}\right) - \frac{\partial \mathcal{L}}{\partial q_2} = 0$$

3.3 The Lab SINDy formulation

Let's take our catalog function again

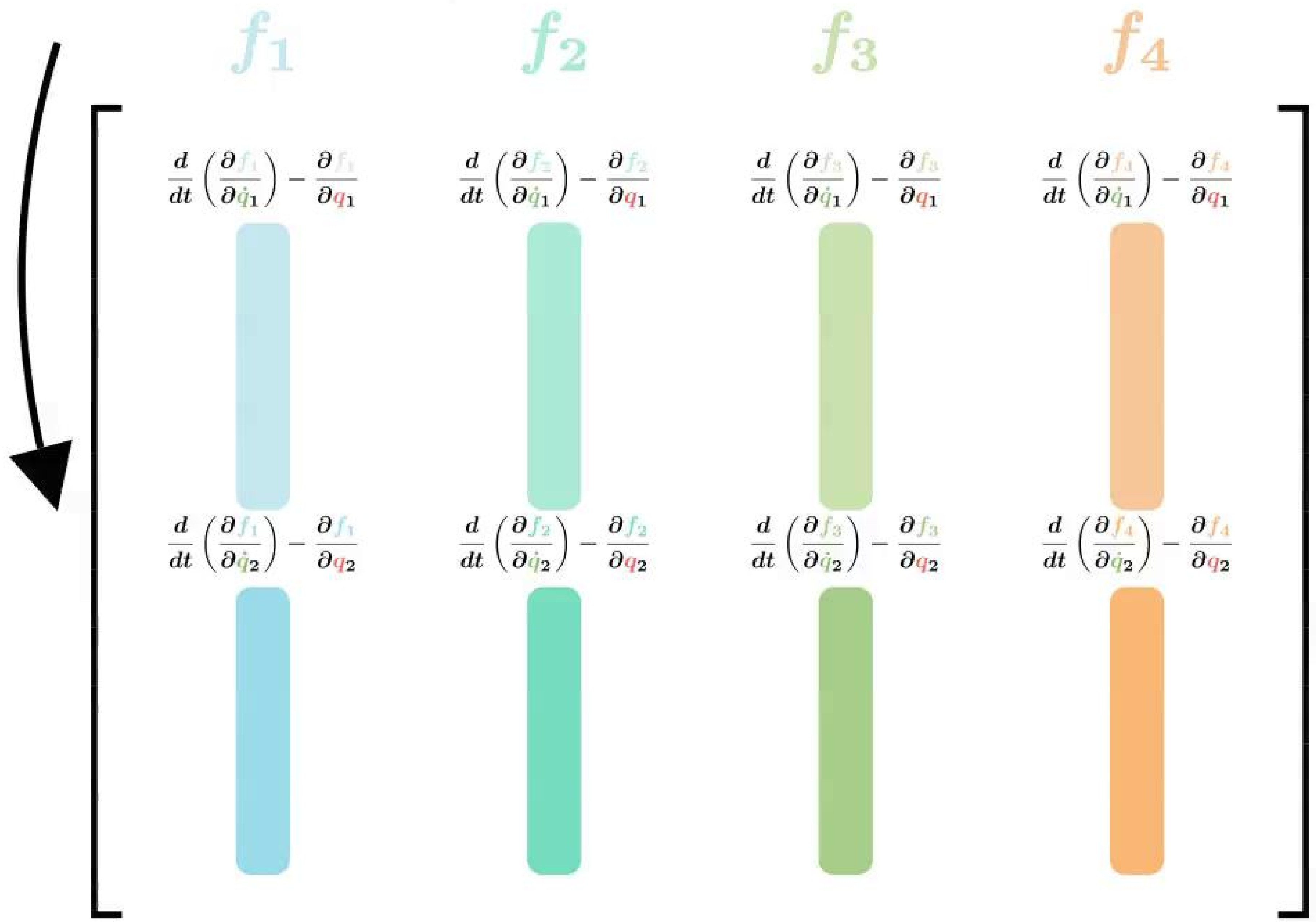
f_1 f_2 f_3 f_4

Purnomo, A. & Hayashibe, M. Sparse identification of Lagrangian for nonlinear dynamical systems via proximal gradient method. (2023).

3.3 The Lab SINDy formulation

Let's take our catalog function again

Euler-Lagrange

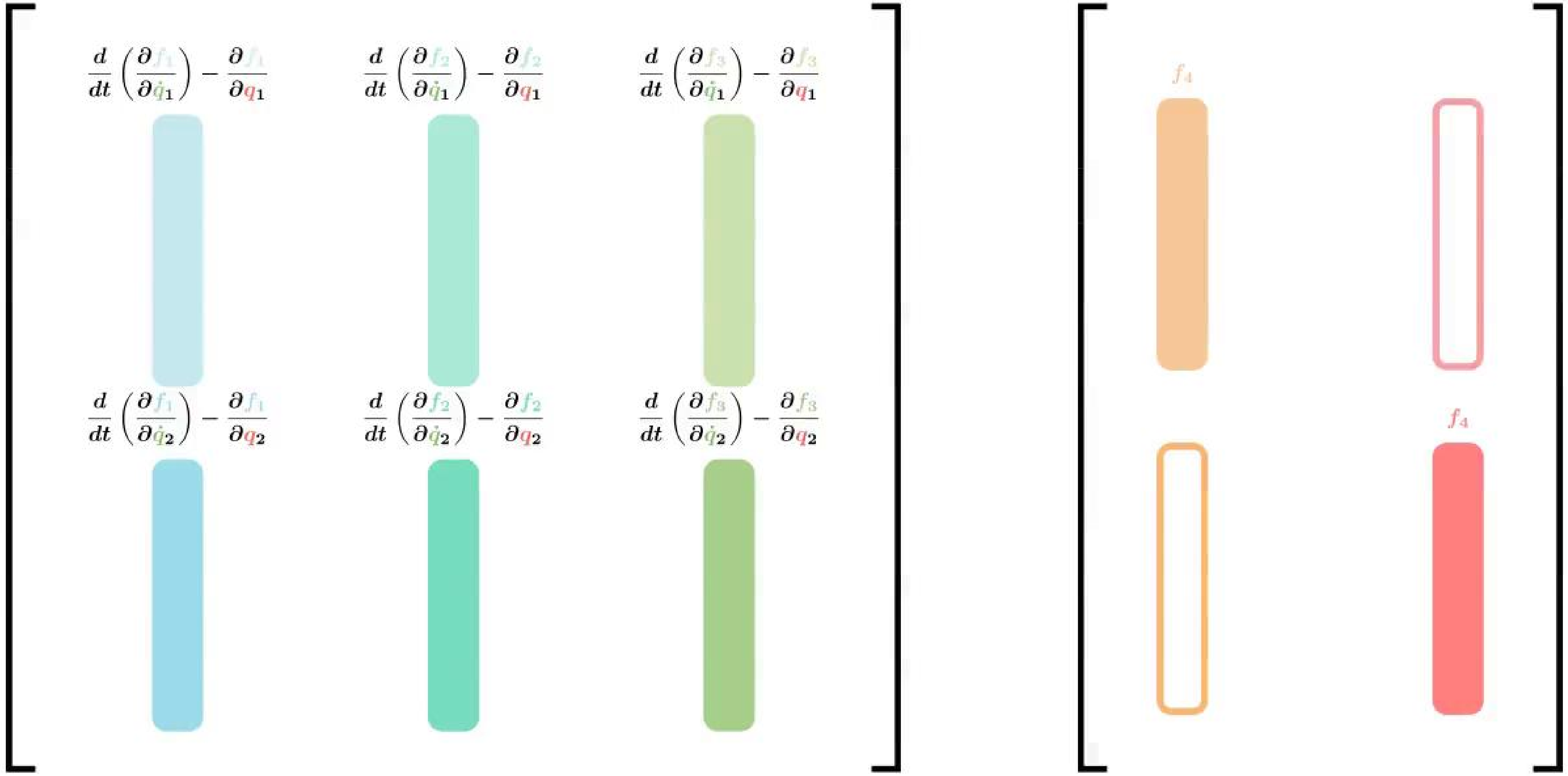


XL-SINDy formulation, more compact !

Purnomo, A. & Hayashibe, M. Sparse identification of Lagrangian for nonlinear dynamical systems via proximal gradient method. (2023).

4.1 First addition

How could we have a compact catalog with the advantages of newton ?



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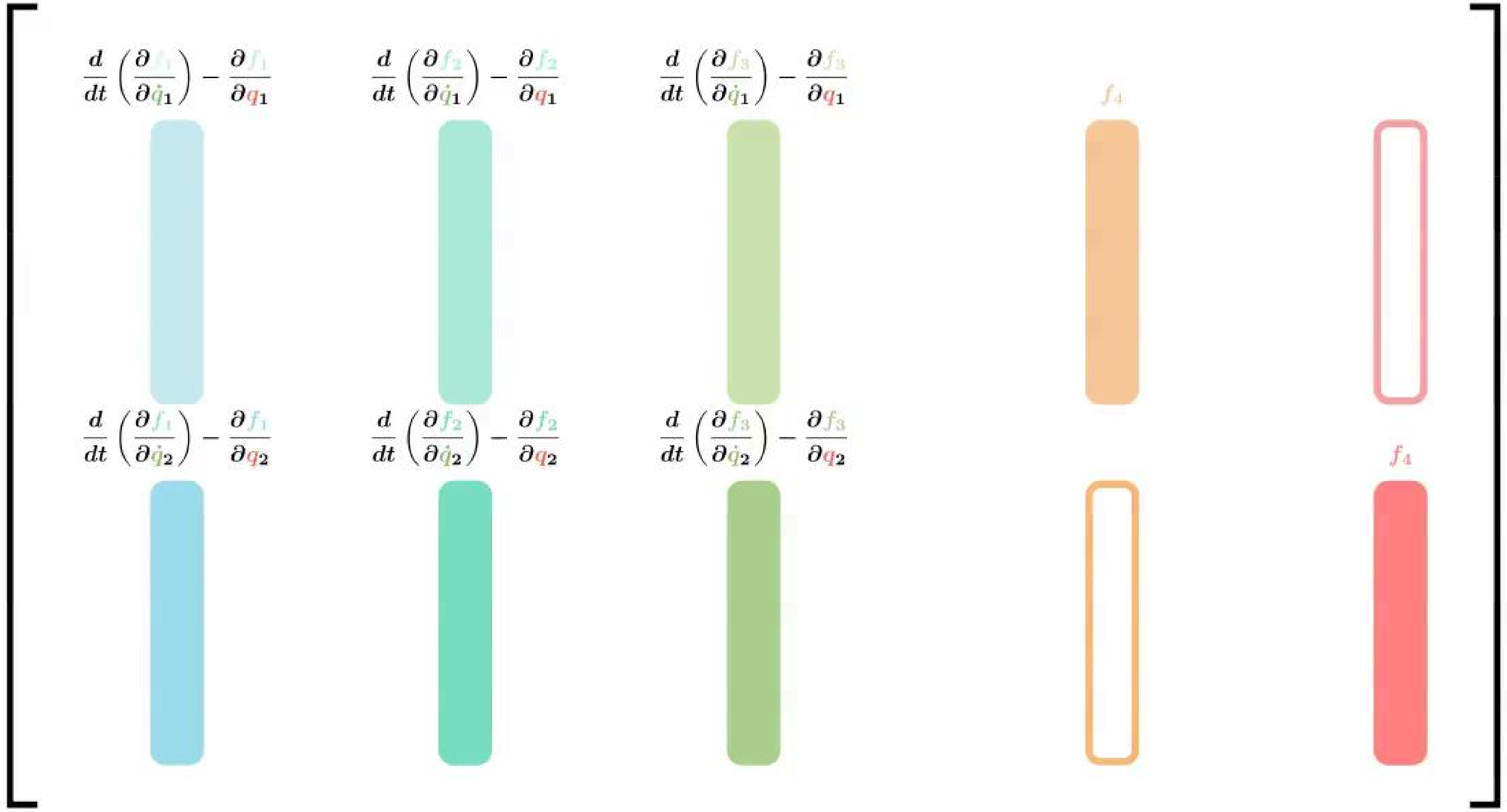
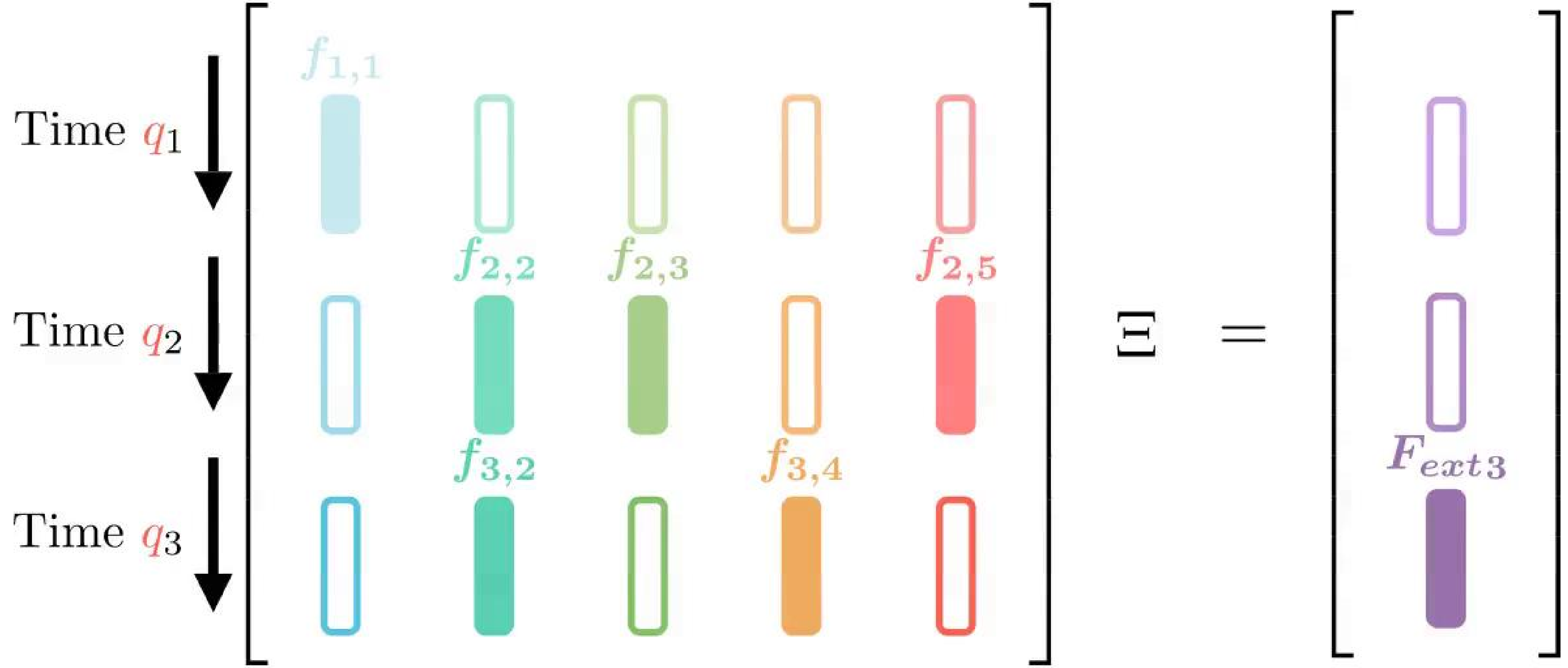


An unified Lagrange-Newton SINDy matrix

4.2 Second addition

How can we guess implicit and explicit in complex system ?



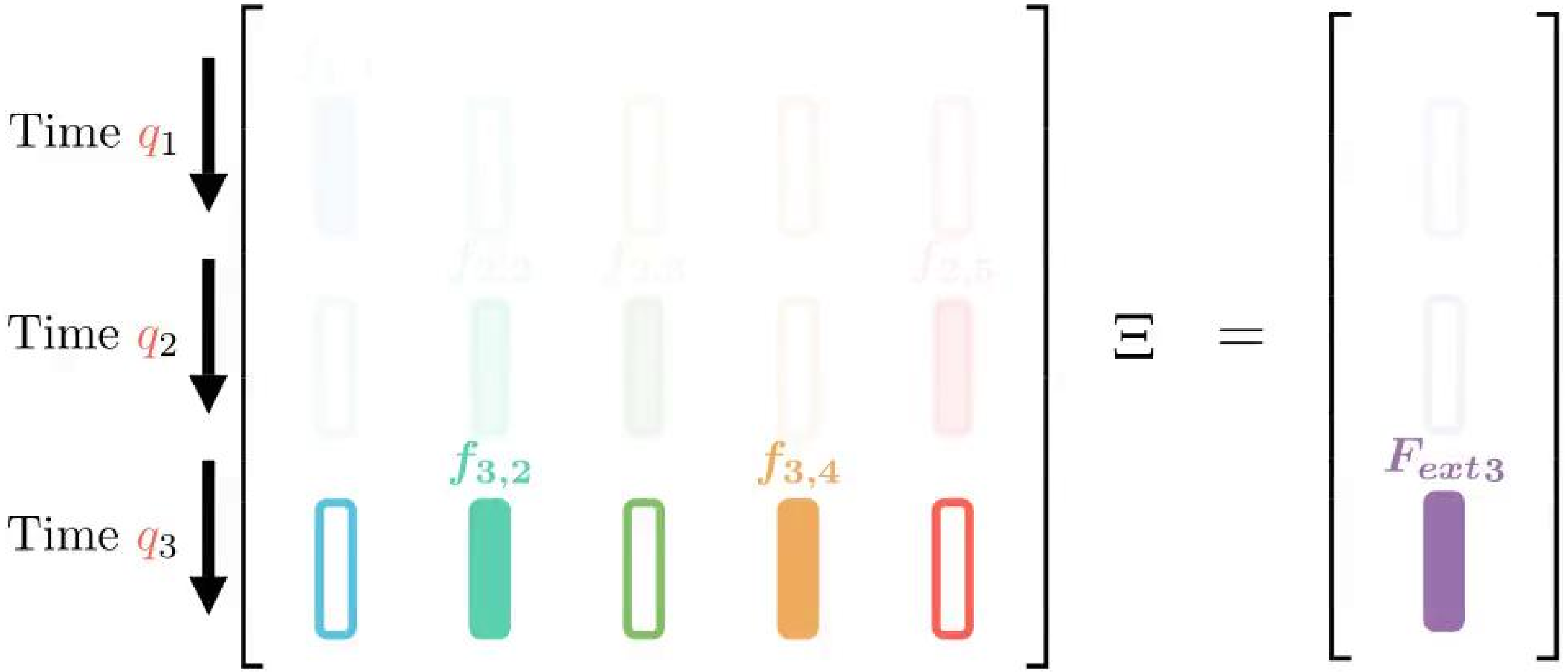
4.2 Second addition

How can we guess implicit and explicit in complex system ?

1. Where are external forces ?

4.2 Second addition

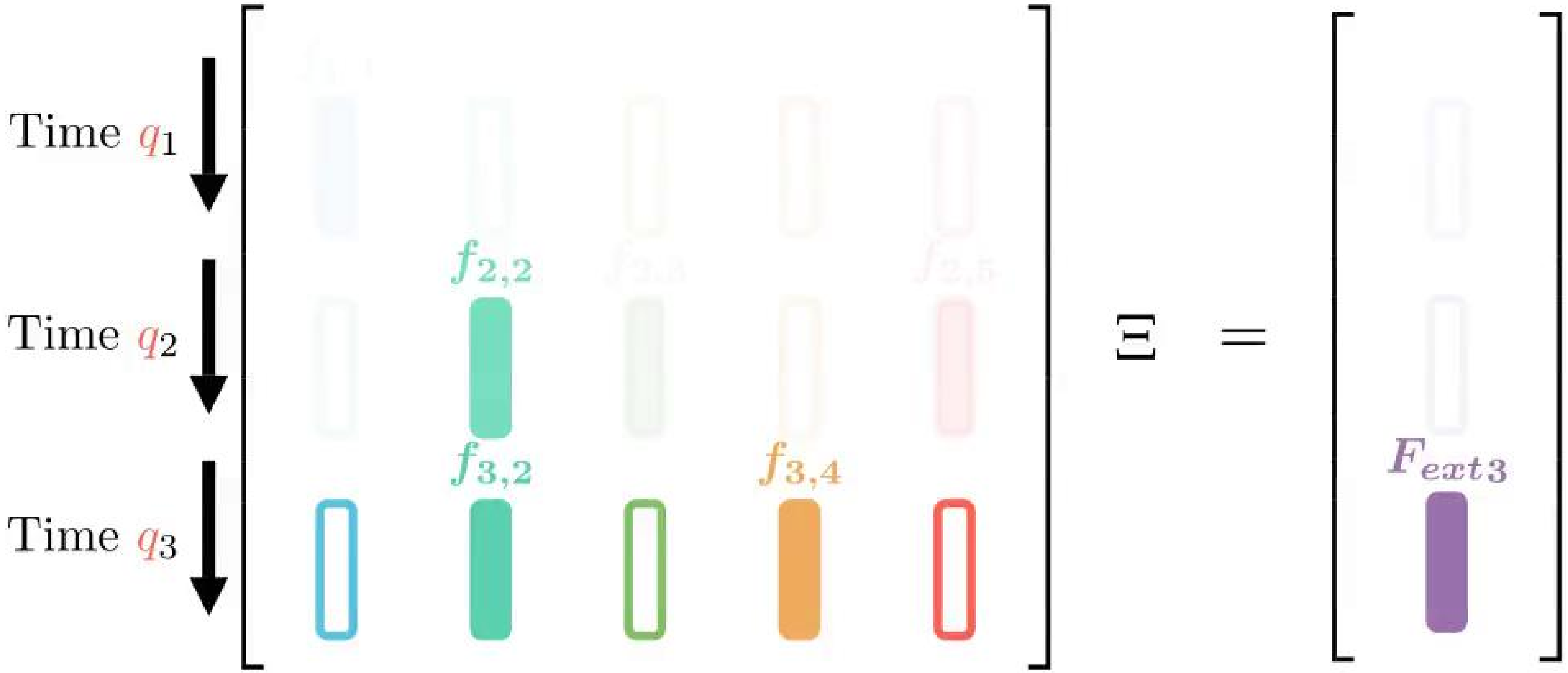
How can we guess implicit and explicit in complex system ?



1. Where are external forces ?
2. What functions are activated ?

4.2 Second addition

How can we guess implicit and explicit in complex system ?



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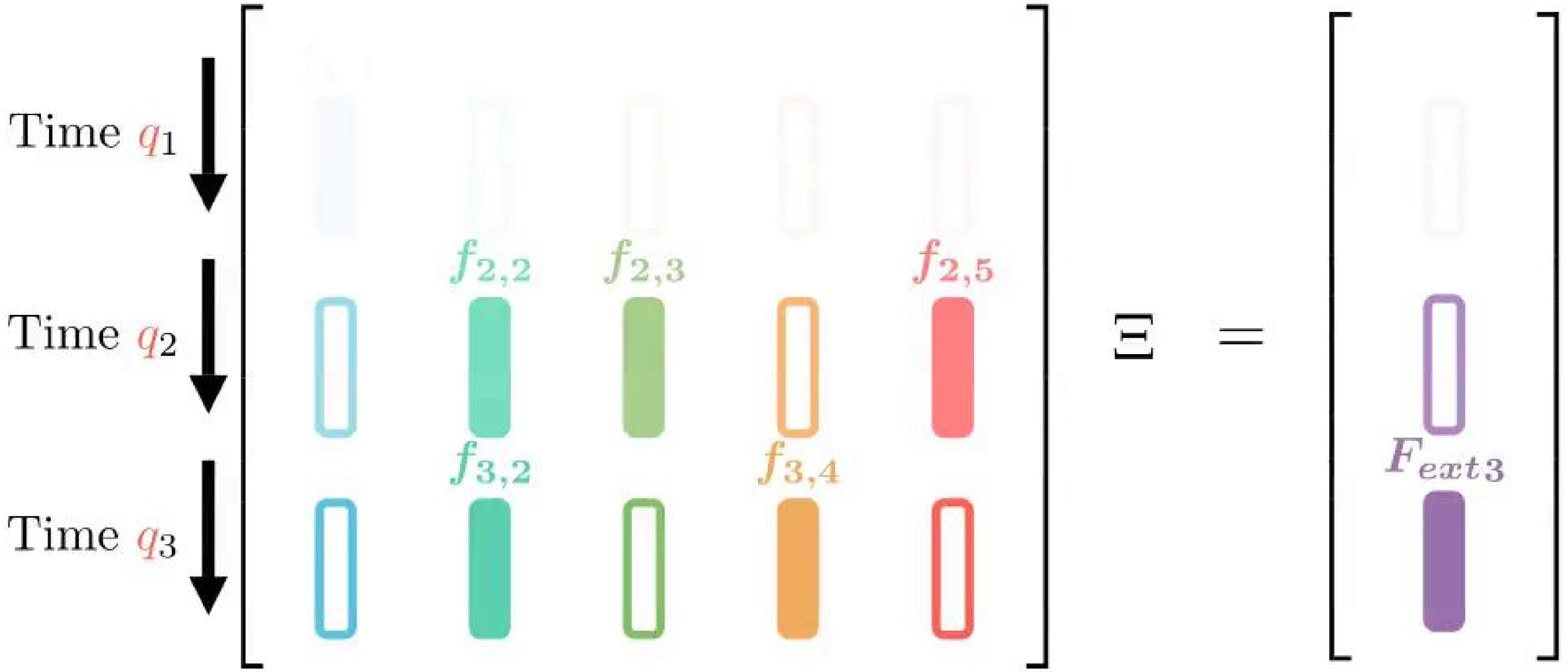

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2. What functions are activated ?
3. What coordinate activate these functions ?

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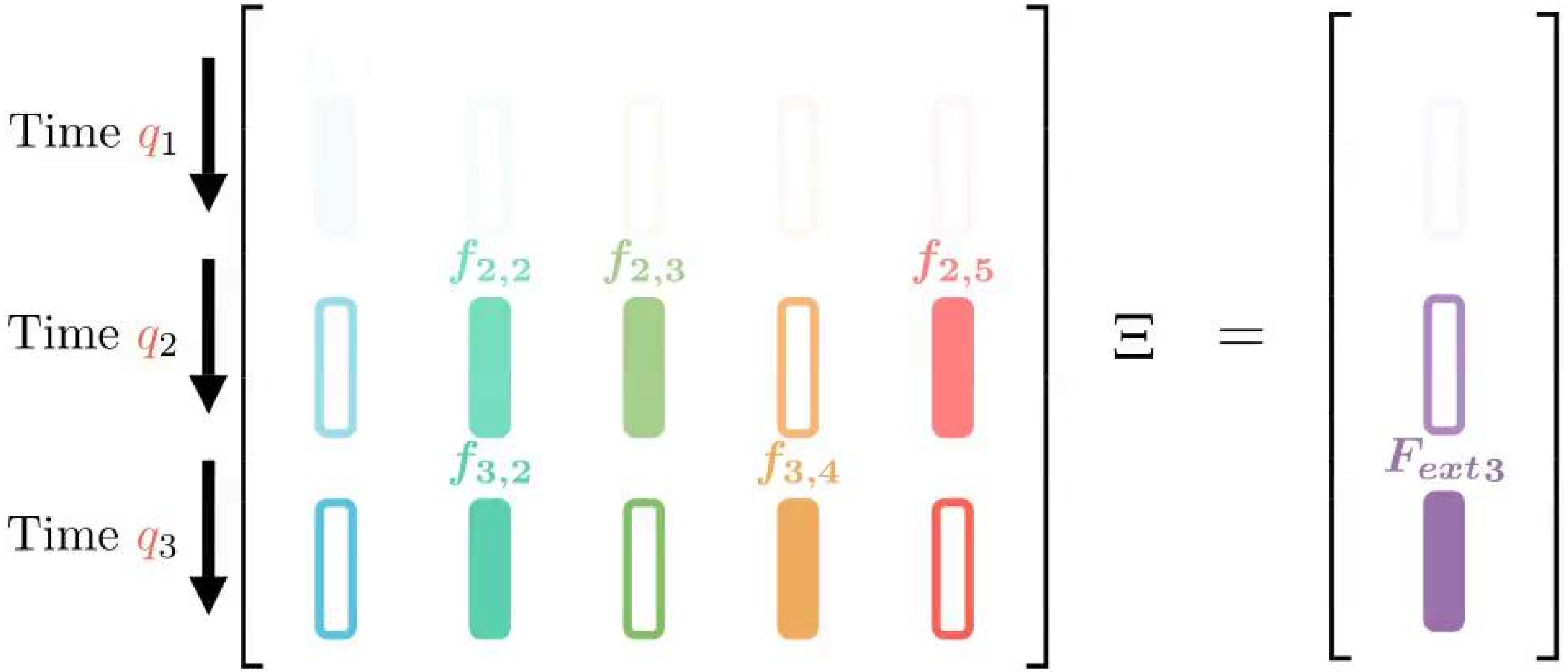


1. Where are external forces ?
2. What functions are activated ?
3. What coordinate activate these functions ?
4. Loop until no more discovery

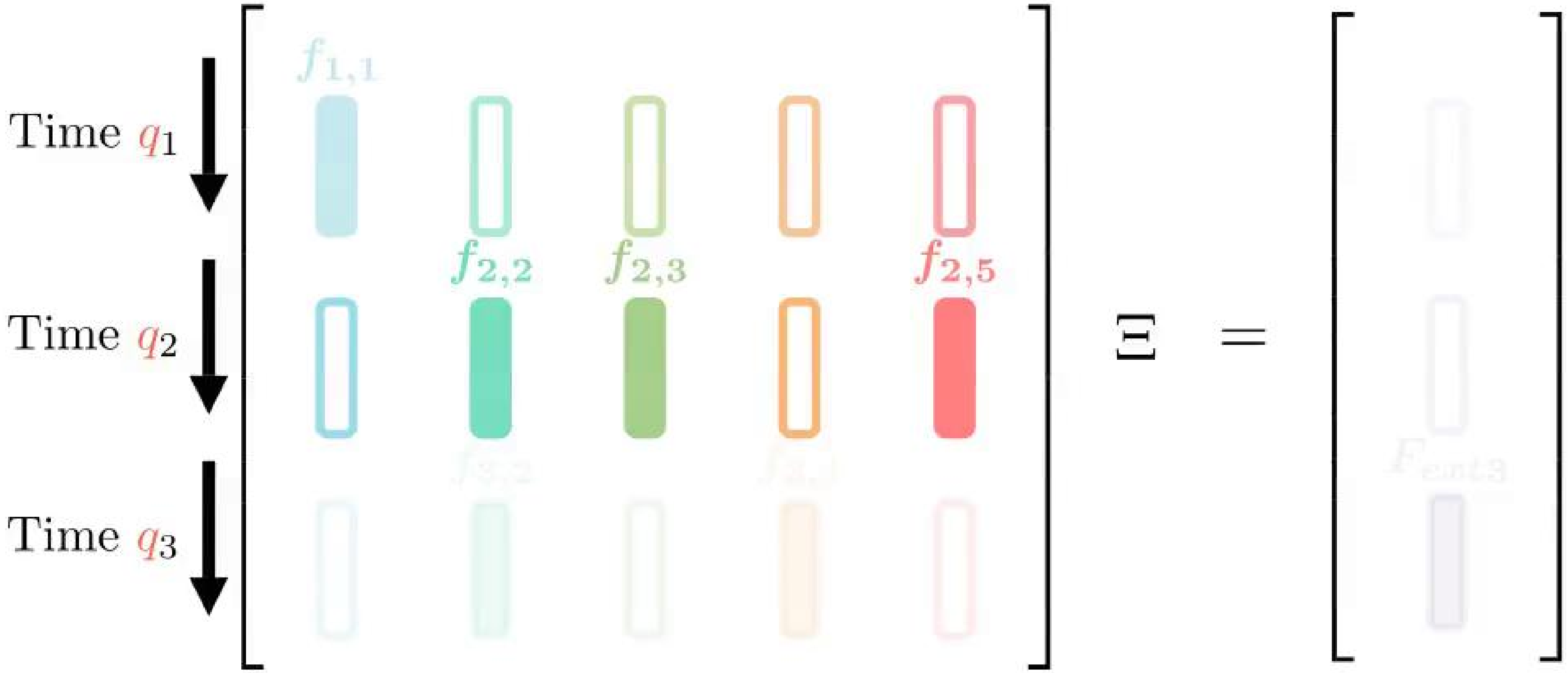
4.2 Second addition

How can we guess implicit and explicit in complex system ?

1. Where are external forces ?
2. What functions are activated ?
3. What coordinate activate these functions ?
4. Loop until no more discovery
5. Execute SINDy on this sub-system

4.2 Second addition

How can we guess implicit and explicit in complex system ?



1. Where are external forces ?
2. What functions are activated ?
3. What coordinate activate these functions ?
4. Loop until no more discovery
5. Execute SINDy on this sub-system
6. Execute SINDy-PI on non-discovered coordinate

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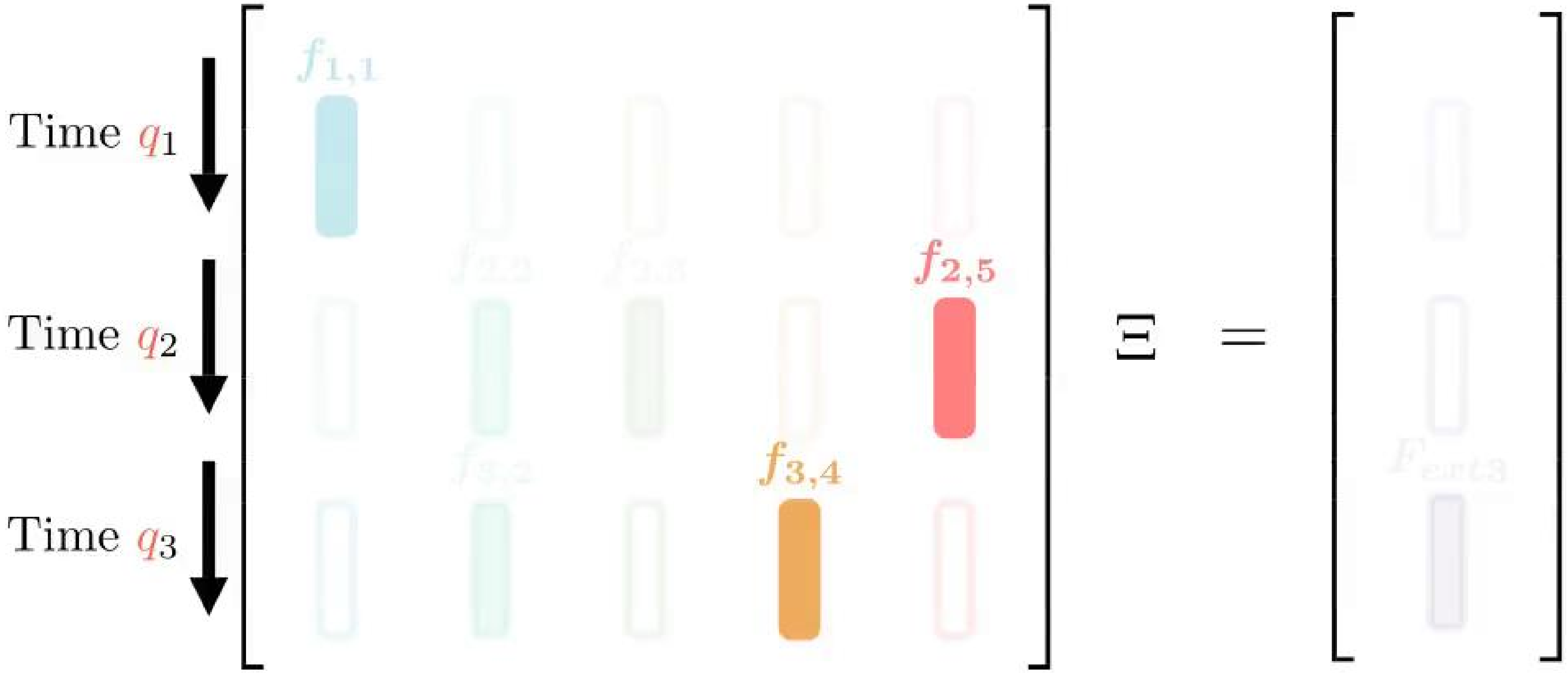

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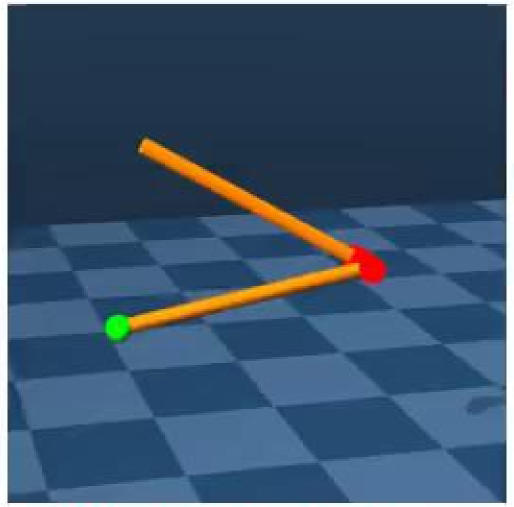
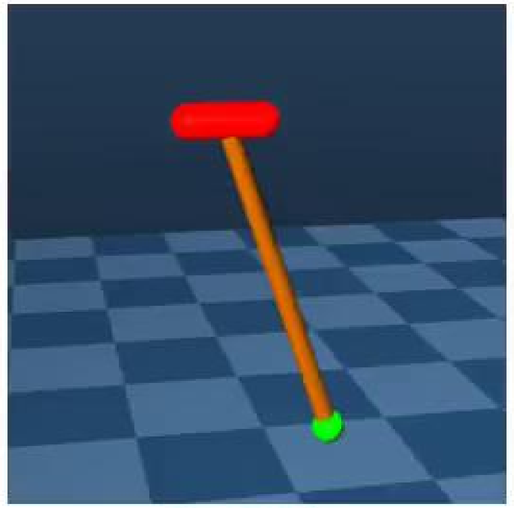
7. Combine everything

5.1 Result gathering

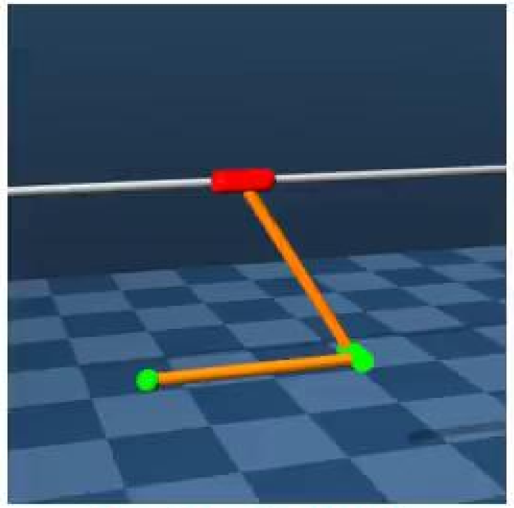
Test protocol has been established to evaluate the performance of the algorithm



Double Pendulum



Cartpole



Double Cartpole

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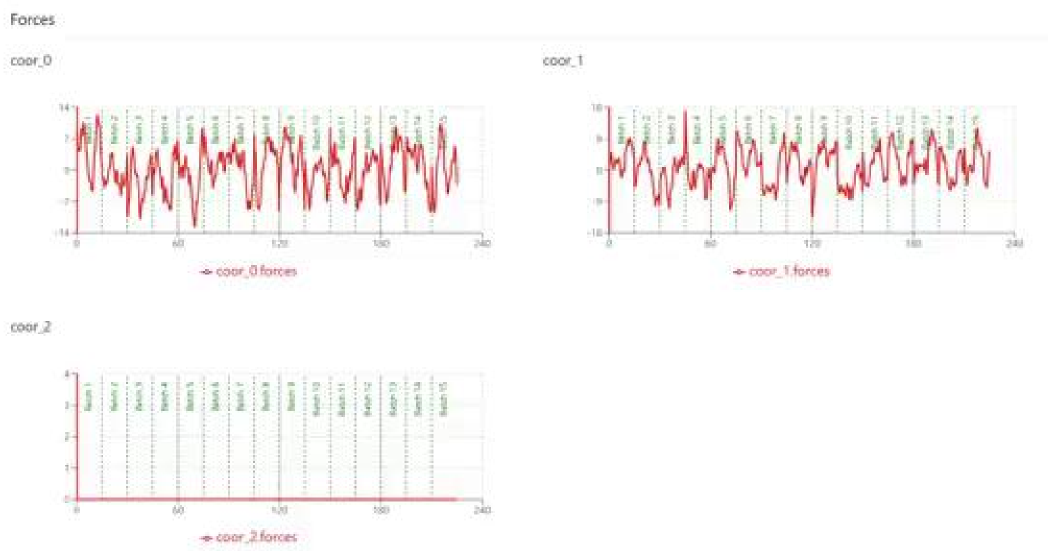
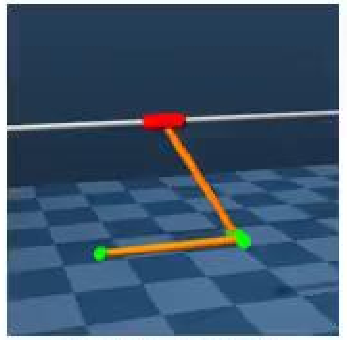


Force input



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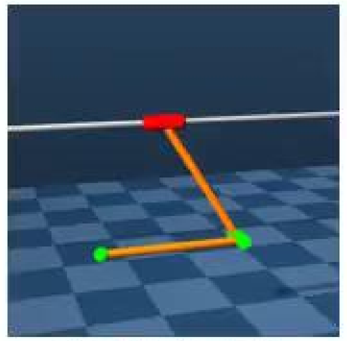

Double Cartpole

$$q, \dots, \dot{q}, \dots \longrightarrow \boxed{\text{System}} \longrightarrow \ddot{q}, \dots$$

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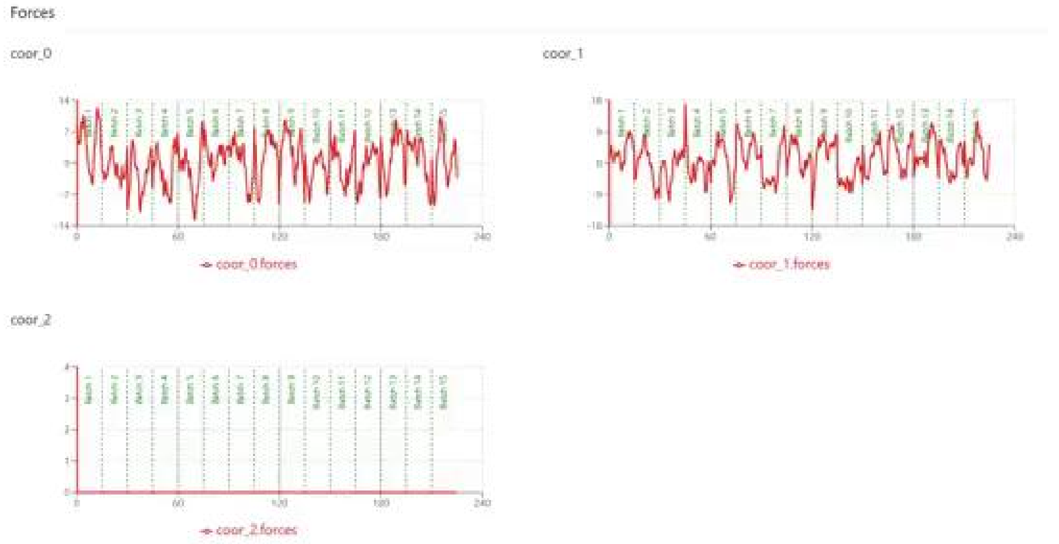
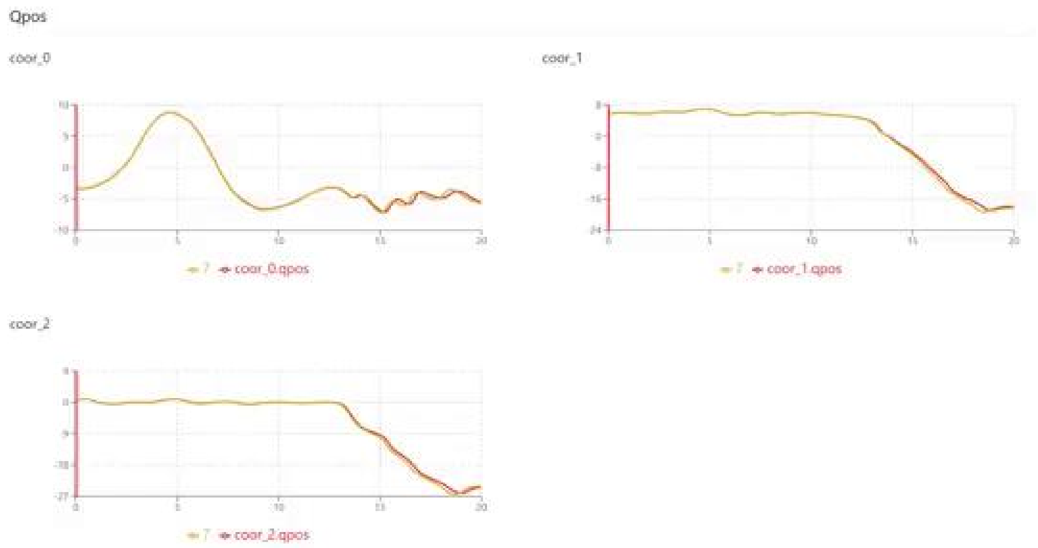
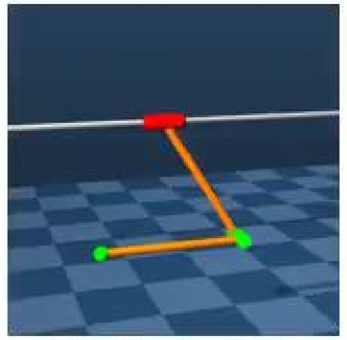
Test protocol has been established to evaluate the performance of the algorithm



Force input

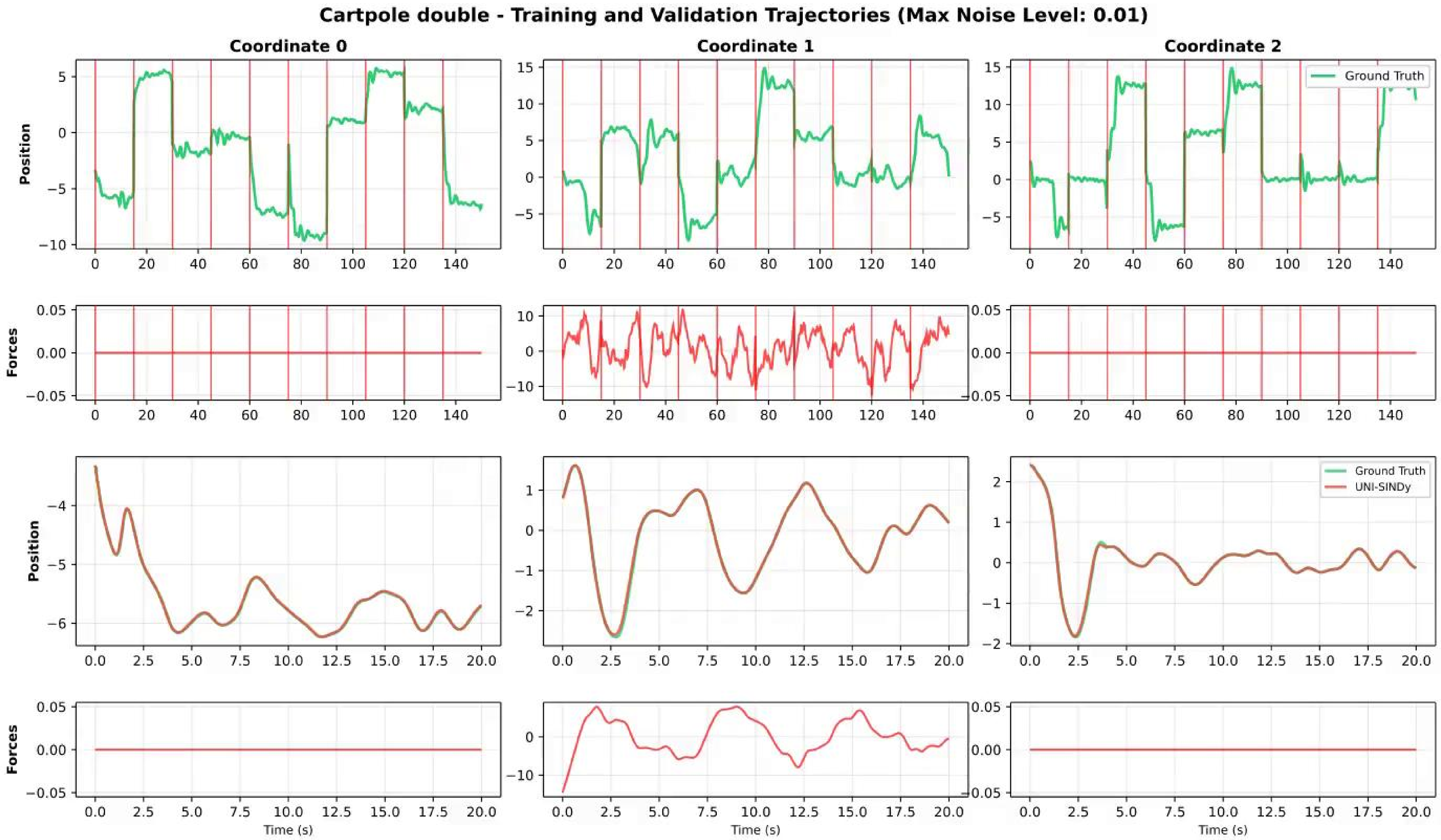


Comparison between real and discovered

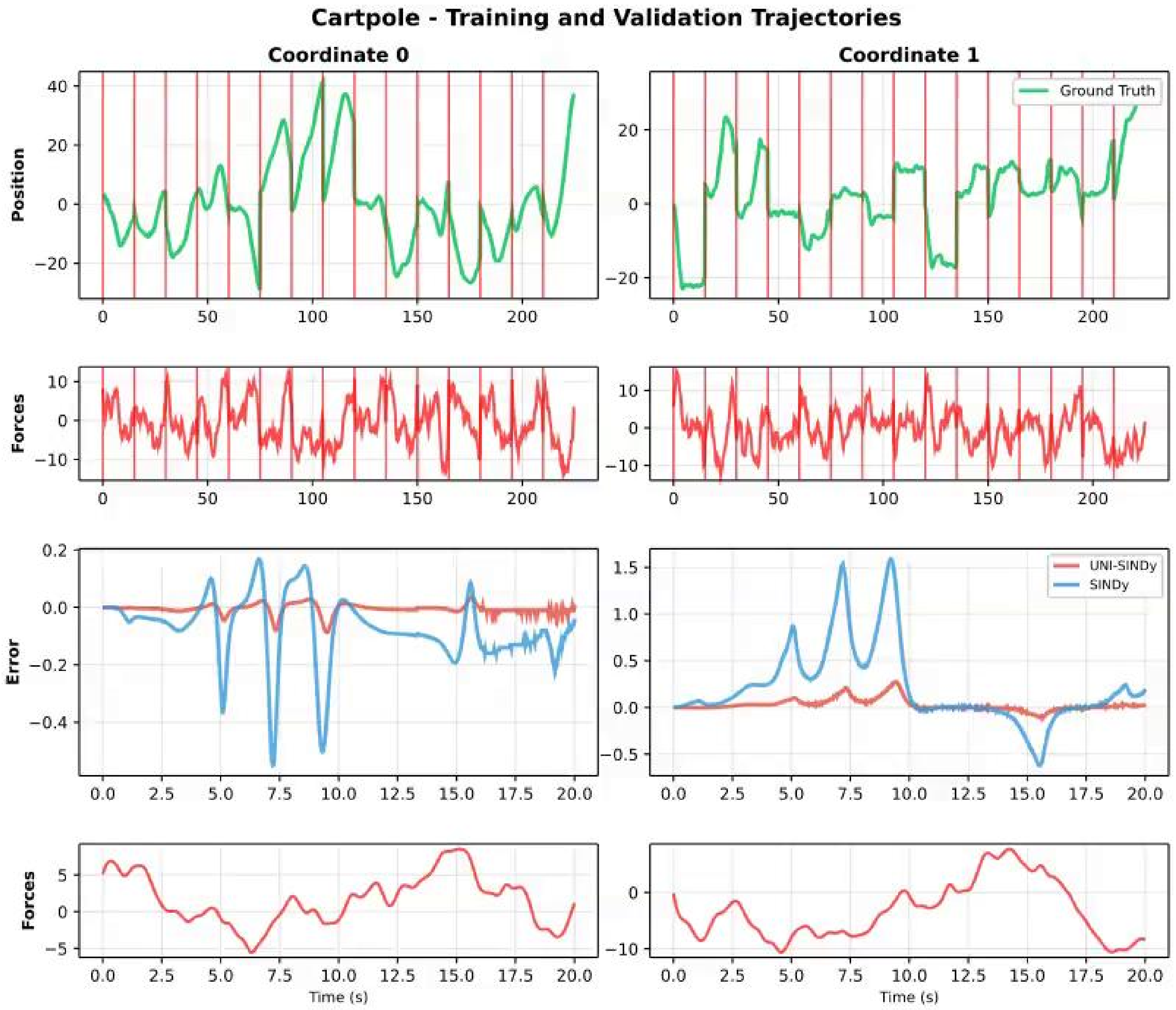


Double Cartpole

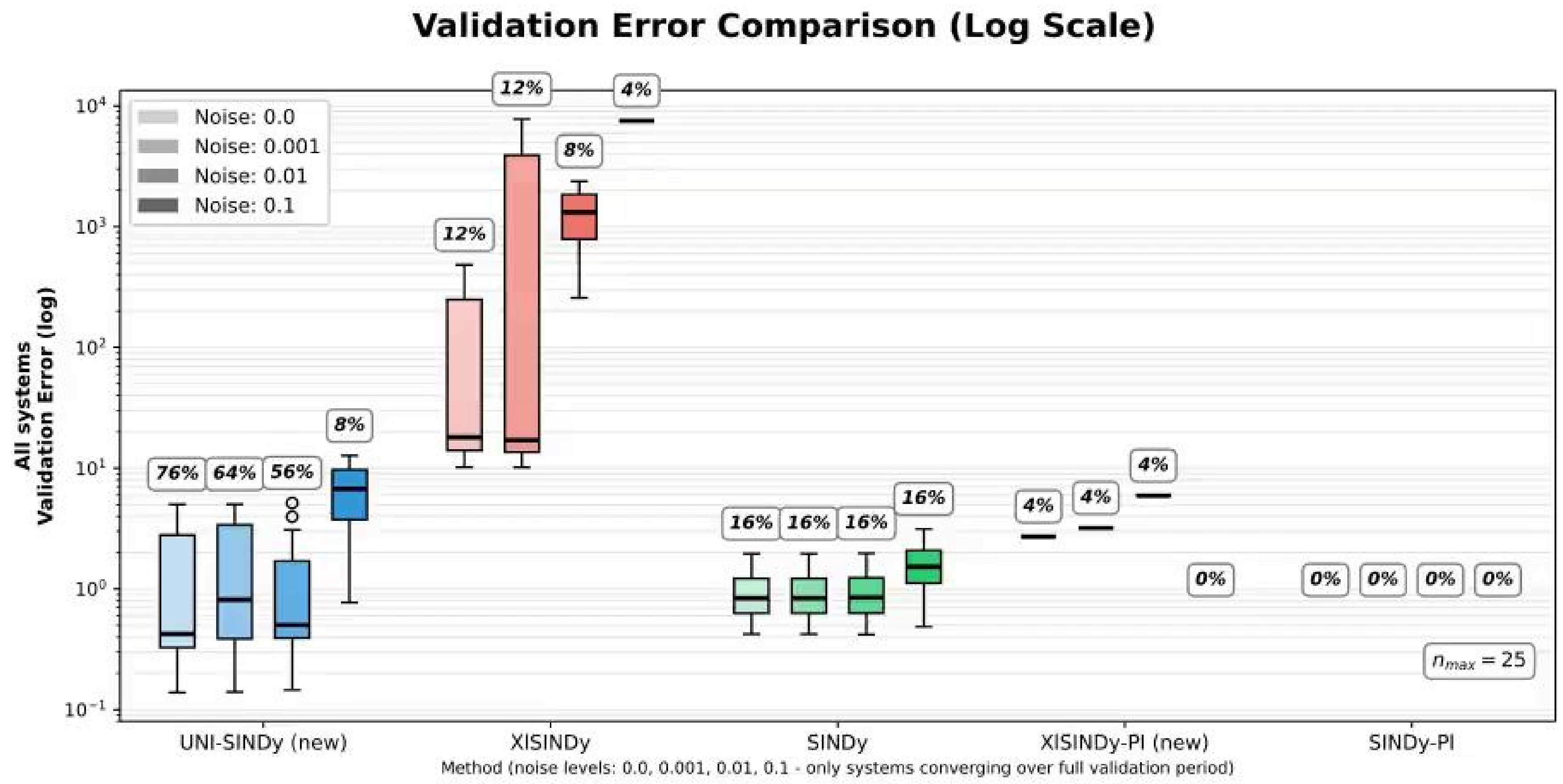
Example data 1

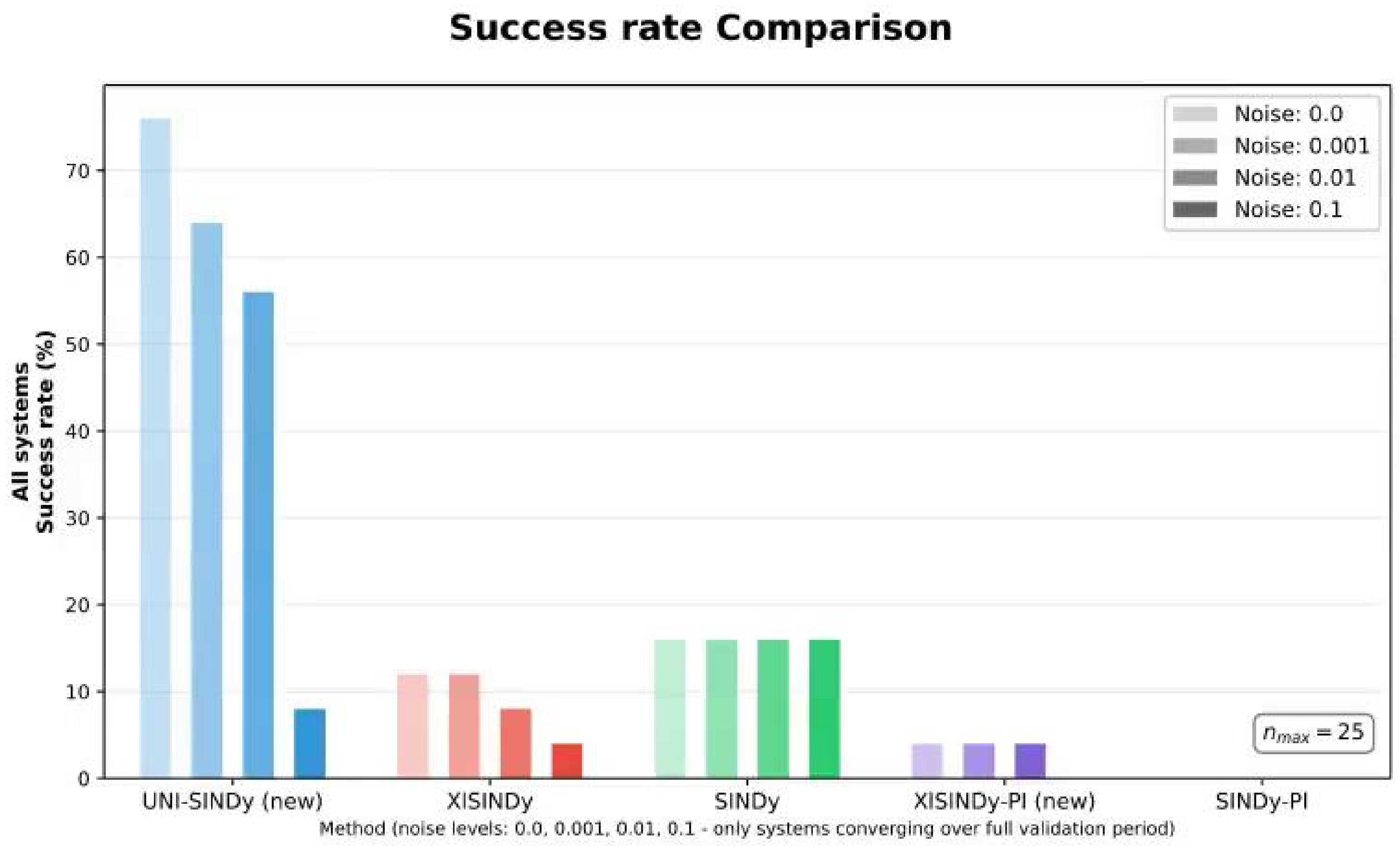


Example data 2

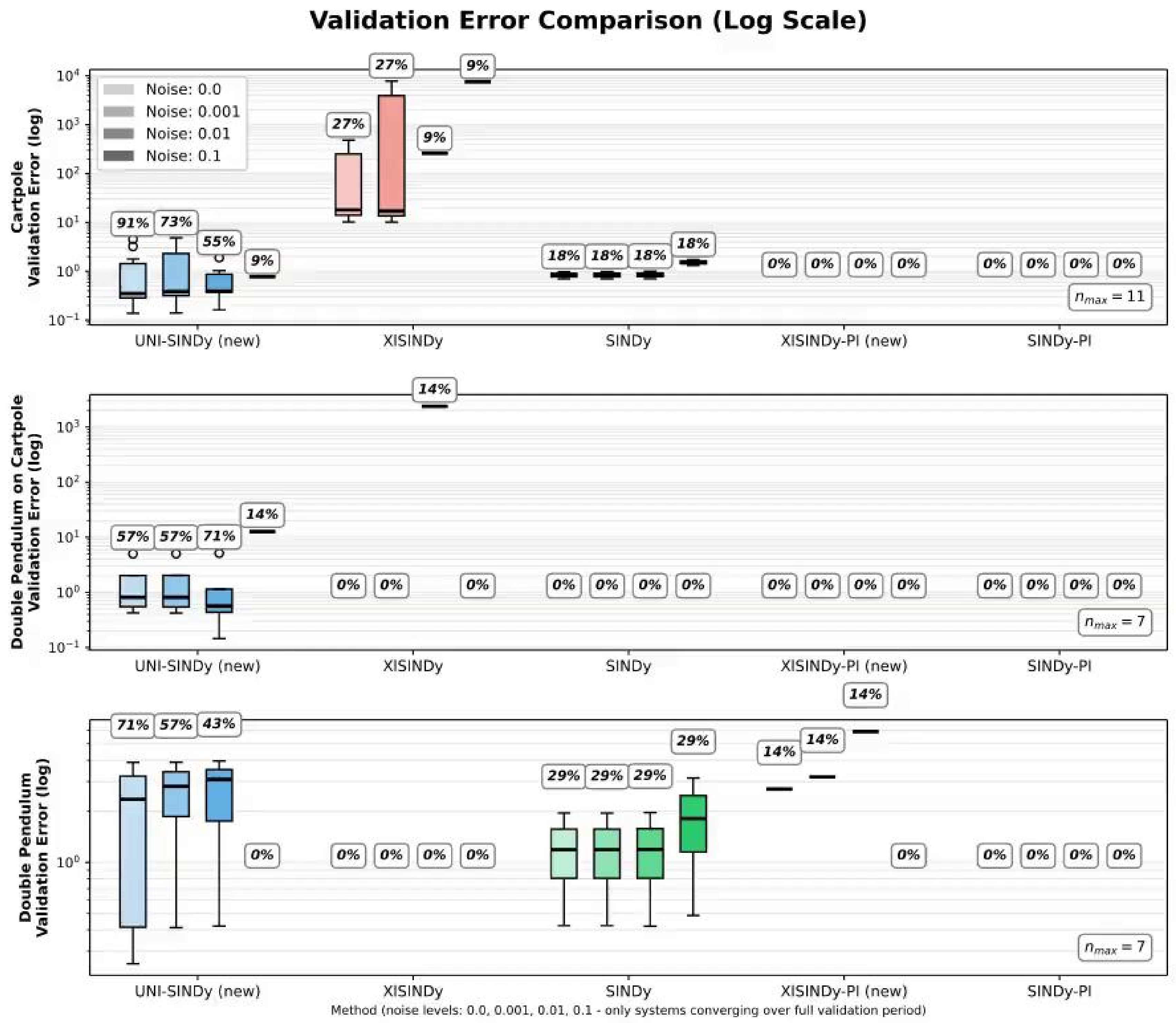


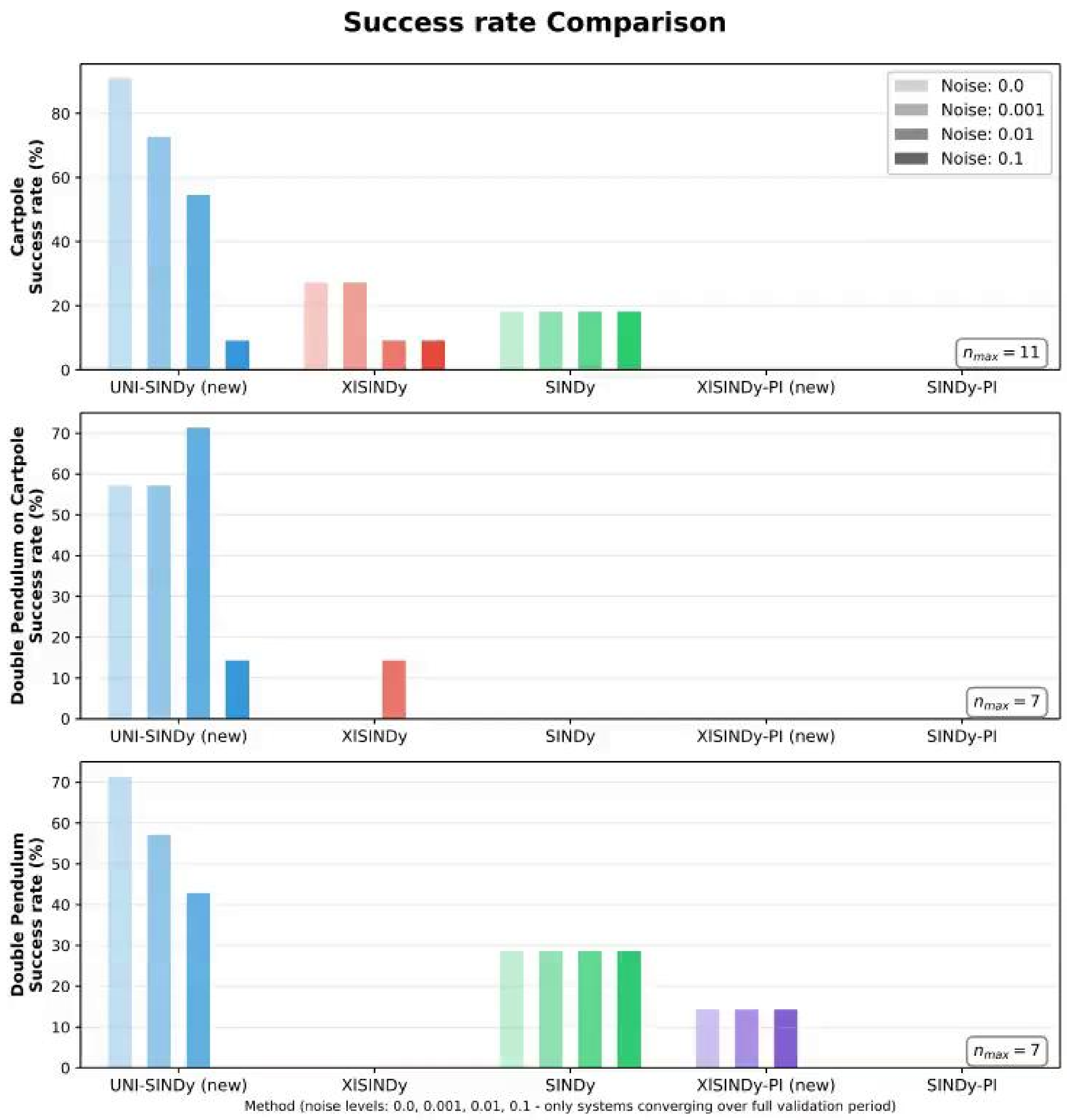
Noise analysis combined





Noise analysis per system





Discussion

Pros and cons, future works

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Pros:

- Compact catalogs
- Better discovery rate
- Handle complex system
- Unified framework

Cons:

- Complex implementation
- Still limited by catalog functions
- Ensemble-SINDy not yet implemented

Discussion

Pros and cons, future works

Pros:

- Compact catalogs
- Better discovery rate
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Cons:

- Complex implementation
- Still limited by catalog functions
- Ensemble-SINDy not yet implemented

Future works:

- Extend to more complex systems (ENSEMBLE-SINDy, catalog discovery)
- Improve catalog function selection
- Real world applications

6 Conclusion

Unified SINDy enables the use of more compact catalogs,

keep the advantages of both Newton and Lagrange formulation,

open the door to complex system identification.

Real world mixed implicit and explicit systems can also be identified

7 Questions and answers

Main library can be found here :
github.com/Eymeric65/py-xl-sindy

Presentation slide code, experiment data, batch generation code can be found here :
github.com/Eymeric65/py-xl-sindy-data-visualisation

Experiment visualisation website can be found here :
eymeric65.github.io/py-xl-sindy-data-visualisation/

5.2 Result table

SINDy-PI vs SINDy vs Mixed SINDy has been compared on
3 systems,3 levels of damping and every combinaison of implicit explicit

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SINDy-PI vs SINDy vs Mixed SINDy has been compared on
3 systems,3 levels of damping and every combinaison of implicit explicit

combo type	invalid	timeout	uncomplete	1st rank	2nd rank	2nd¿	Wins WC	winrate
mixed x explicit	90	5	20	40	6	0	30	0.294872
mixed x mixed	96	9	41	39	6	1	29	0.251366
sindy x explicit	127	84	9	8	8	0	2	0.105263
sindy x mixed	145	105	11	6	7	0	1	0.076923
mixed x implicit	14	12	1	5	4	0	4	0.375000
xlsindy x mixed	110	1	61	4	2	2	3	0.044693
xlsindy x implicit	8	8	1	3	0	0	1	0.250000
xlsindy x explicit	103	0	47	2	2	2	2	0.038462
sindy x implicit	31	31	0	0	0	0	0	0.000000

Math annexe

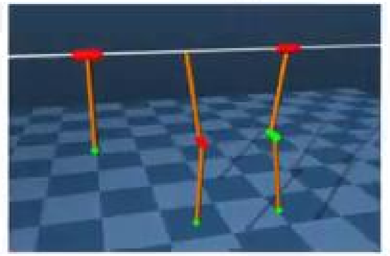
A

$$\begin{bmatrix} q^1_{t_i} & \cdots & q^n_{t_i} \\ \dot{q}^1_{t_i} & \cdots & \dot{q}^n_{t_i} \\ \ddot{q}^1_{t_i} & \cdots & \ddot{q}^n_{t_i} \end{bmatrix} = \mathbf{Q}_{t_i}$$

$q, \dot{q}, \ddot{q}$

output

Full-order system



input

u

$$\begin{bmatrix} u^1_{t_1} \\ \vdots \\ u^1_{t_r} \\ \vdots \\ u^n_{t_1} \\ \vdots \\ u^n_{t_r} \end{bmatrix} = \mathbf{U}$$

B

xL-SINDy $\left[\Theta_1, \ldots, \Theta_p\right]$

classical SINDy $\left[\Omega_1^{q^1}, \ldots, \Omega_1^{q^n}, \ldots, \Omega_m^{q^1}, \ldots, \Omega_m^{q^n}\right]$

$$\mathbf{E} = \begin{bmatrix} \mathcal{L}_1[\Theta_0](\mathbf{Q}_{t_1}) & \cdots & \mathcal{L}_1[\Theta_p](\mathbf{Q}_{t_1}) & \Omega_1^{q^1}(\mathbf{Q}_{t_1}) & \cdots & 0 & \cdots & \Omega_m^{q^1}(\mathbf{Q}_{t_1}) & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ \mathcal{L}_1[\Theta_0](\mathbf{Q}_{t_r}) & \cdots & \mathcal{L}_1[\Theta_p](\mathbf{Q}_{t_r}) & \Omega_1^{q^1}(\mathbf{Q}_{t_r}) & \cdots & 0 & \cdots & \Omega_m^{q^1}(\mathbf{Q}_{t_r}) & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ \mathcal{L}_n[\Theta_0](\mathbf{Q}_{t_1}) & \cdots & \mathcal{L}_n[\Theta_p](\mathbf{Q}_{t_1}) & 0 & \cdots & \Omega_1^{q^n}(\mathbf{Q}_{t_1}) & \cdots & 0 & \cdots & \Omega_m^{q^n}(\mathbf{Q}_{t_1}) \\ \vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ \mathcal{L}_n[\Theta_0](\mathbf{Q}_{t_r}) & \cdots & \mathcal{L}_n[\Theta_p](\mathbf{Q}_{t_r}) & 0 & \cdots & \Omega_1^{q^n}(\mathbf{Q}_{t_r}) & \cdots & 0 & \cdots & \Omega_m^{q^n}(\mathbf{Q}_{t_r}) \end{bmatrix}$$

Sparse regression : coordinate descent, hard threshold, proximal gradient

$$\mathbf{U} = \mathbf{E}\lambda$$

Retrieved lagrangian model : $L = [\Theta_1 \quad \ldots \quad \Theta_p] \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_p \end{bmatrix}$