UNIT-THREE

3. APPLICATION OF DERIVATIVES

3.1 The extreme/ absolute and local/ of a continuous function

Many of our applications in this unit will revolve around minimum and maximum values of a function. So, in this sub unit we will try to concentrate ourselves on the theorems, definition, maximum (absolute), minimum (absolute), local maximum and local minimum values and critical points of a continuous function f on closed interval [a, b].

Let f be a function on closed interval [a, b] and suppose the graph of f be the following one.

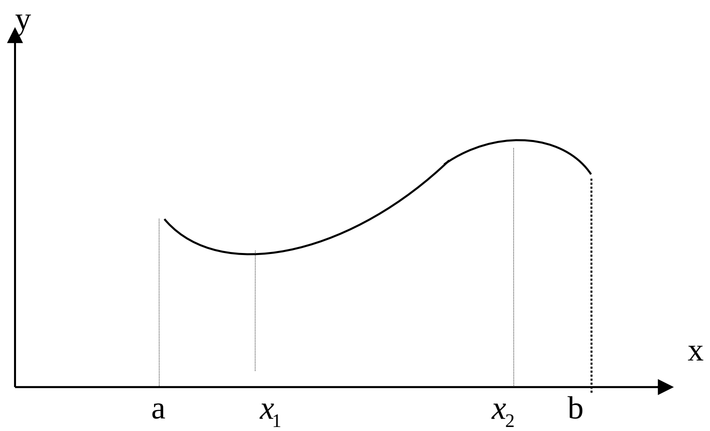

fig 3.1

As it is shown in the fig3.1 above, f attains its maximum value at $x_2 \in$ [a, b] and minimum value at $x_1 \in$ [a, b]

Definition 3.1 Let f be a continuous function on [a, b]. Then

i) f is said to have maximum value on [a, b], if and only if there exist $x_0 \in$ [a, b] such that f(x) $\leq$ f(x_0) $\forall$ x $\in$ [a, b] and f(x_0) is called the absolute maximum value of f on [a, b]

ii) f is said to have minimum value on [a, b], if and only if there exist $x_0 \in$ [a, b] such that f(x_0) $\leq$ f(x) $\forall$ x $\in$ [a, b] and f(x_0) is called the absolute minimum value of f on [a, b].

iii) a maximum or minimum value of f on [a, b] is called extremum value of f on [a, b].

Example-1

Let f(x) = x, then find the extremum value(s) of f (if possible) on the interval [0, 3].

Solution:

Since the function f(x) = x is continuous function on the interval [0, 3] by definition 3.1, there exist $x_0 \in [0, 3]$ such that $f(x) \leq f(x_0)$ $\forall x \in [0, 3]$

$\Rightarrow$ **f (x_0) is absolute maximum value of f on [0, 3] and**

there exist $x_0 \in [0, 3]$ such that $f(x_0) \leq f(x)$ $\forall x \in [0, 3]$

$\Rightarrow$ **f (x_0) is absolute minimum value of f on [0, 3].**

But as it is shown in the fig3.2 below, $3 \in [0, 3]$ and $f(x) \leq f(3)$ $\forall x \in [0, 3]$

$\Rightarrow$ f (3) =3 is absolute maximum value of f on [0, 3] and

since $0 \in [0, 3]$ and $f(0) \leq f(x)$ $\forall x \in [0, 3]$

$\Rightarrow$ f (0) =0 is absolute minimum value of f on [0, 3].

Therefore, f (3) =3 and f (0) =0 are the extremum values of f on [0, 3]

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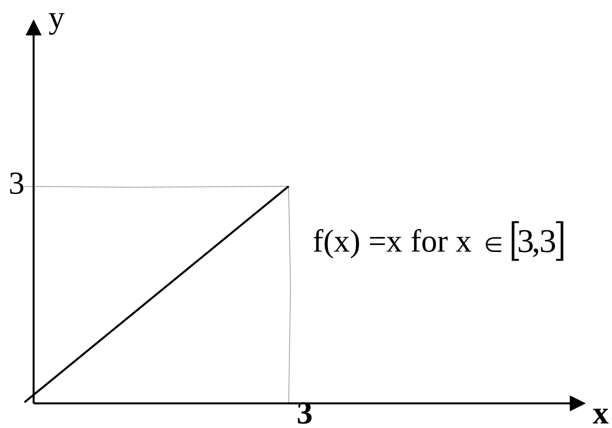


Fig3.2

Example-2

Let f(x) =x^2, find the extremum value of f on [-2, 2].

Solution:

The following fig3.3 will support our solution.

Clearly f is continuous on [-2, 2] and since $0 \in [-2, 2]$, $f(0) \leq f(x)$ $\forall x \in [-2, 2]$ which implies that f (0) =0 is absolute minimum value of f on [-2, 2] and

$f(-2) = f(2) = 4 \geq f(x)$ $\forall x \in [-2, 2]$

Hence f (-2) =f (2) =4 is the absolute maximum value of f on [-2, 2].

Therefore, f (0) =0 and f (-2) = f (2) = 4 are the extremum values of f on [-2, 2] as shown in the fig4.3 below.

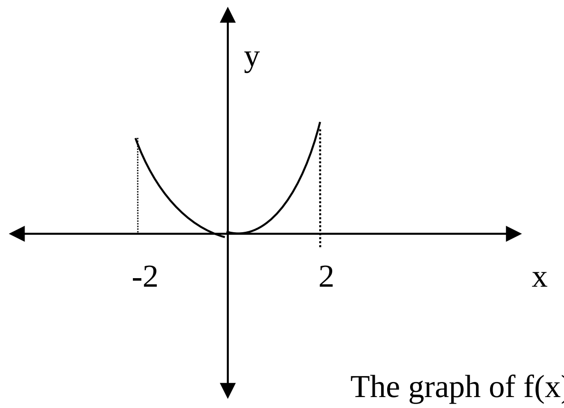


Fig 3.3 The graph of $f(x) = x^2$ for $x \in [-2, 2]$

<u>Theorem</u> 3.1 (Extreme Value Theorem)

If f is continuous function on closed interval [a, b], then f has a maximum and minimum value on [a, b].

<u>Remark</u>

Theorem4.1 works if the given function satisfies the following two pre-conditions

i.e. the function must be: i) Continuous on [a, b] ii) defined on a closed interval [a, b]

Example-1

Let f(x) = x, for $x \in [0\ 2)$. Since f (0) =0 < f(x) $\forall x \in [0\ 2) \Rightarrow 0$ is minimum value of f on [0, 2) but, even though f is continuous on [0, 2), f has no maximum value on [0, 2). Because f is not defined on closed interval or we don't know the upper boundary of the interval [0, 2) as shown in the fig4.4 below.

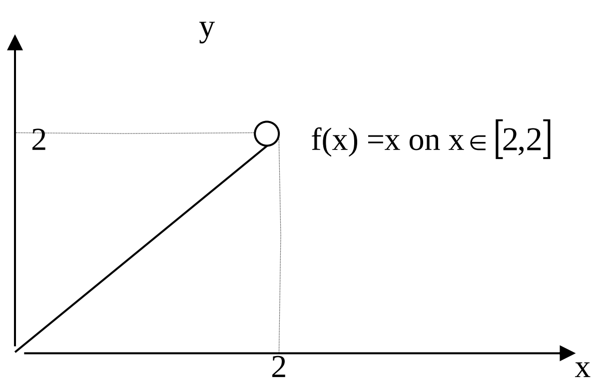


fig3.4

Note that: The same function with the same defining rule may have different extreme value depending on defined closed interval d of f.

Example-2 Let f(x) = $\begin{cases} \frac{1}{x} & if \quad x \neq 0 \\ x \, if & x = 0 \end{cases}$ on the interval [-3, 3]

Solution

Obviously, the function f is defined on the closed interval [-3, 3] satisfy the pre-condition (ii) above. But, f is not continuous at x = 0.

Therefore, f has neither maximum nor minimum value on [-3, 3] as it is shown in the fig 3.5 bellow.

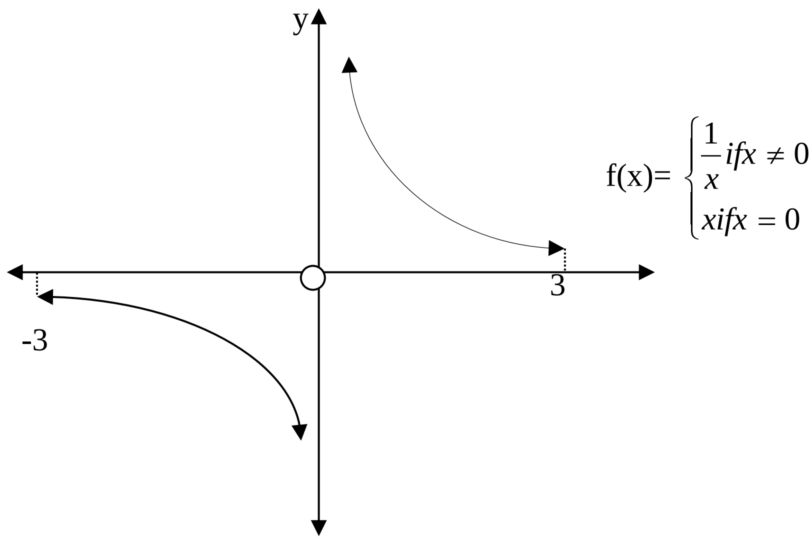


Fig3.5.

Definition4.2: (Local extreme values)

Let f be a continuous function on an interval I. Then,

i) f is said to have a local maximum value at x_1, if there exist an open interval (c, d) $\subseteq$ I containing x_1 such that f attains its maximum on (c, d) at $x_1 \Rightarrow$ f (x) $\leq$ f(x_1) $\forall$ x $\in$(c, d).

ii) f is said to have a local minimum value at x_2, if there exist an open interval (c, d) $\subseteq$ I containing x_2 such that f attains its minimum value of f on (c, d) at x_2

$\Rightarrow$ f(x_1) $\leq$ f (x) $\forall$ x $\in$(c, d).

iii) the value that is either local maximum or local minimum value is called **local extreme values.**

iv) a point at which the function has local maximum or local minimum value is called **local extremum point.**

Consider the following figure

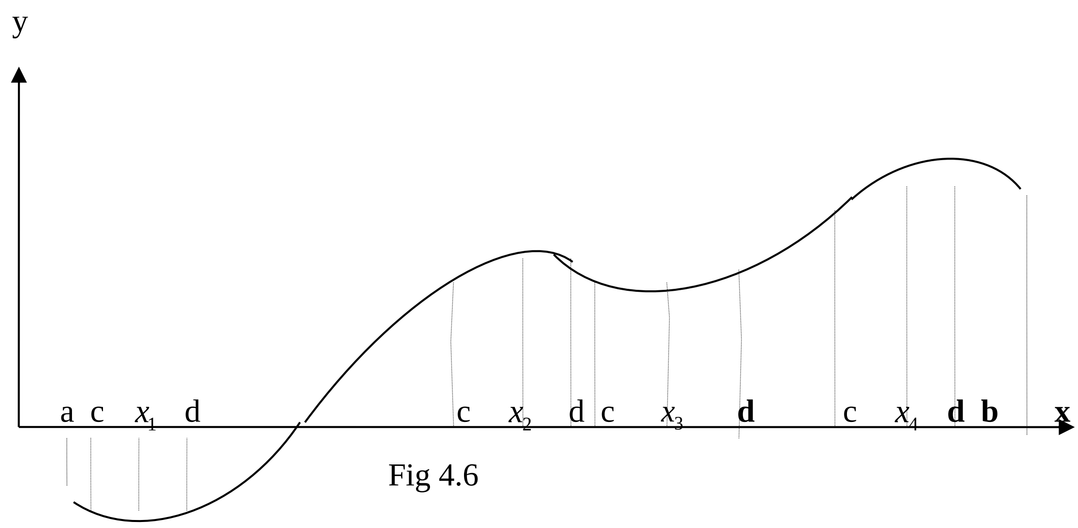


Fig 4.6

f has local maximum value at x_2 and x_4 &

f has local minimum value at x_1 and x_3

f attains absolute maximum value on [a, b] at x_4 and

f attains absolute minimum value on [a, b] at x_1

Note that: For a continuous function f on [a, b]

- ❖ x_0 is local extremum point of f on [a, b] if and only if $x_0 \in (a, b)$. In other word, the end points a and b can't be local extremum points of f on [a, b] and hence f(a) and f(b) are neither local max nor local min of f on [a, b].
- ❖ If f attains its max or min in the interior of [a, b], then the points are local extreme points.
- ❖ Sometimes the maximum & minimum value of a function f are called **absolute maximum and absolute minimum values** of f respectively in order to emphasize the difference from local max and local min of the function f.

Theorem 3.2 (**Fundamental theorem of local extreme value**)

Suppose f is continuous function on the interval I and differentiable in the interior of I. If x_o is a local extreme point of f, then $f'(x_o) = 0$

Proof

To proof the fundamental theorem of local extreme value theorem we will consider two cases. i) When f has a local maximum value at x_o and ii) f has local minimum value at x_o.

Since the proof of (i) and (ii) are identical, we will proof only (i).

Suppose f is continuous function and f has a local maximum value at $x_o \in I$, then there exist an open interval $(c, d) \subseteq I$ containing x_o such that $f(x) \leq f(x_o)$ for all $x \in (c, d)$.

But for every $x \in (c, d)$ we have $\frac{f(x) - f(x_0)}{x - x_0} \geq 0$ if $x < x_o$ and $\frac{f(x) - f(x_0)}{x - x_0} \leq 0$ if $x > x_o$

Let $h = x - x_o$, then $x = x_o + h$

$$\lim_{x \to x_0^-} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{h \to 0^-} \frac{f(x_0 + h) - f(x_0)}{h} \geq 0.$$

Because $f(x_0 + h) \leq f(x_0)$ and $h = x - x_0 > 0$

Similarly, $\lim_{x \to x_0^+} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{h \to 0^+} \frac{f(x_0 + h) - f(x_0)}{h} \leq 0$

Because $f(x_0 + h) \leq f(x_0)$ and $h = x - x_0 > 0$

Since f is differentiable at x_0 we have

$$0 \leq \lim_{h \to 0^-} \frac{f(x_0 + h) - f(x_0)}{h} = \lim_{h \to 0^+} \frac{f(x_0 + h) - f(x_0)}{h} \leq 0$$

This implies that, $0 \leq f'(x_0) \leq 0$.

It follows that, $f'(x_0) = 0$

Note that: The reverse of theorem4.2 may not be true.

i.e. $f'(x_o) = 0$ doesn't necessary implies that x_o is extreme points of f.

For instance, let $f(x) = x^3$, thus $f'(x) = 3x^2$, $f'(x) = 0$ if and only if $3x^2 = 0$

Which implies that $x = 0$ here $f'(0) = 0$ but f neither has local maximum nor local minimum value as it shown in the figure4.7 below.

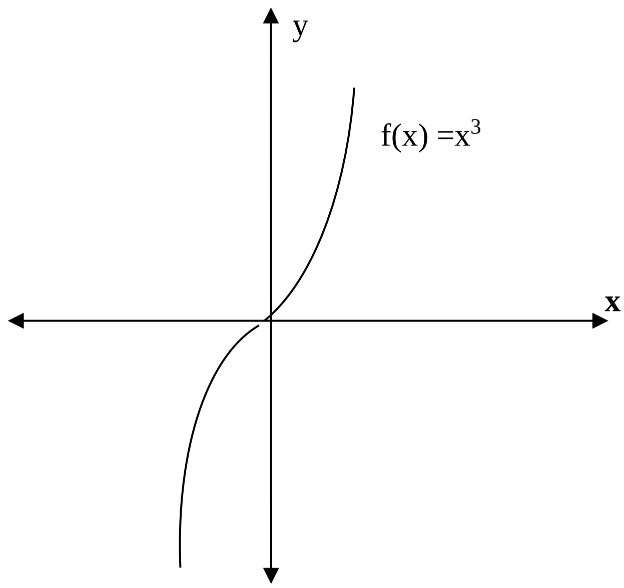


Figure3.7

Definition 3.3:

If c is a number in the domain of a function f and if either $f'(c)=0$ or $f'(x)$ doesn't exist at x=c, then the number c is called a critical number of f. And the point (c, f(c)) is called the critical point of f.

Example-1: Let $f(x)=x^2-4x+4$, then find the critical number(s) and points of f

Solution

$f(x)=x^2-4x+4 \Rightarrow f'(x)=2x-4$

$f'(x)=0$ if and only if $2x-4=0$

$\Rightarrow x=2$

Therefore x=2 is the only critical number of f and (2, 0) is a critical point.

Example-2: Let $f(x)=x^3-9x^2+15x$ .then, find the critical number(s) of f

Solution

$f(x)=x^3-9x^2+15x \Rightarrow f'(x)=3x^2-18x+15$

$f'(x)=0 \Leftrightarrow 3x^2-18x+15=0 \Rightarrow x=1$ or $x=5$

Therefore, x=1 and x=5 are the critical numbers of f

Example-3: Let $f(x)=\sqrt{1-x^2}$ for $\in x$ [-1, 1]. Then find the critical number(s) of f

Solution

f(x) = $\sqrt{1-x^2}$ $\Rightarrow$ $f'(x)$ = $\frac{-x}{\sqrt{1-x^2}}$ (by using quotient rule of derivatives)

$\Rightarrow$ $f'(x)$ =0 if and only if $\frac{-x}{\sqrt{1-x^2}}$ =0

$\Rightarrow$x=0 and 0$\in$ $[-1,1]$

$\Rightarrow$x=0 the critical number of f and -1&1 are elements of the domain of f and since $f'(x)$ doesn't exit at x=1&x=-1 implies that 1, -1 and 0 are the critical numbers of a function f(x) = $\sqrt{1-x^2}$ on[-1, 1].

Activity3.1

Find the critical numbers and points of a function:-

i) f(x) = $\frac{1}{x}$, x‡ 0   ii) f(x) = $\sqrt{1-x}$, x$\in$[-∞, 1]

iii)f(x) = x^2e^x iv) f(x) =x+ $\frac{1}{x}$, x$\neq$ 0

3.3 Role's theorem and Mean Value Theorem (MVT)

Suppose a function f is continuous on the interval [a, b] and differentiable in the interior points of [a, b] then, the only domain of the extreme value of f on [a, b] is either at the end points of [a, b] or at the critical points of the given function f.

Therefore, to find the extreme values finding the critical point is very important one.

The following theorem tells us some criteria about critical points of f on [a, b].

Theorem3.3 Role's theorem

Let f be continuous function on [a, b] and differentiable on (a, b). If f(a) = f(b), then there is a number c in (a, b) such that $f'(c)$ =0

Proof

Let f be continuous function on [a, b] and differentiable on (a, b) and let f (a) =f (b). Here we see two cases.

***Case i*:** Suppose f is constant function, then $f'(x)$ =0 for all x $\in$ (a, b).

Therefore, there is a number c $\in$(a, b) such that $f'(c)$ =0

Case ii: Suppose f is not constant function, since f is continuous function on [a, b], by extremum value theorem f attains its maximum and minimum value on [a, b] and since

f (a) =f (b), at least one of these values must occur at a point c in (a, b) and by hypothesis f is differentiable at c and hence $f'(c) = 0$

Example-1 Verify Role's theorem for function $f(x) = x^2 - 3x + 2$

Solution:

$f(x) = x^2 - 3x + 2 = (x-1)(x+2)$

Here f (1) =f (2) =0 and since f is a polynomial function, it is continuous on [1, 2] and differentiable on (1, 2) by Rolle's Theorem there is a number $c \in (1, 2)$ such that $f'(c) = 0$.

$f(x) = x^2 - 3x + 2 \Rightarrow f'(x) = 2x - 3$ but we have $f'(c) = 0$.

$\Rightarrow 2c - 3 = 0 \Rightarrow c = \dfrac{3}{2} \in (1, 2)$.

Therefore, $f(x) = x^2 - 3x + 2$ satisfy the Role's theorem.

Note that:

We can deduce from Rolle's Theorem that, to apply the theorem the given function f should satisfy the following pre-conditions.

The function should be:

i) Continuous on [a, b]   ii) differentiable on (a, b) &   iii) f (a) = f (b)

Example-2: Let f(x) be a continuous function on [-1, 1] defined by f (x) =/x / then, find the critical numbers of f on the given domain.

Solution

Even though f(x) =/x / is continuous function on [-1, 1] and f (-1) =f (1) =1, f is not differentiable at x =0 which is $\in [-1, 1]$ or the pre condition (ii) is not satisfied.

Therefore, we can't get a number $c \in [-1, 1]$ such that $f'(c) = 0$

What do you expect if the above two (i and ii) conditions are satisfied but f (a) ‡ f (b)?

Do you have another option for such case?

The answer will be given by the following theorem.

Theorem 3.4 (Mean Value Theorem)

Let f be continuous function on [a, b] and differentiable on (a, b). Then, there is a number c in (a, b) such that $f'(c) = \dfrac{f(b)-f(a)}{b-a}$.

Proof

The proof of the mean value theorem can be illustrated as follows.

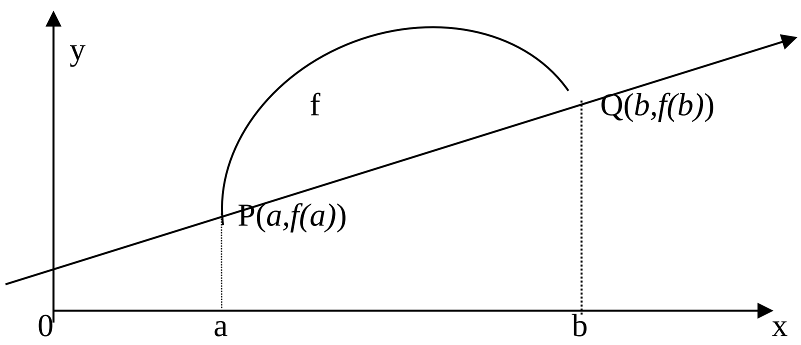


Fig3.8

The first thing that we need is the equation of the secant line that goes through the two points P and Q by using to point form of equation of a line as shown above. That is

$$y = f(a) + \frac{f(b)-f(a)}{b-a}(x-a)$$

Let's now define a new function *g(x)* as to the difference between *f(x)* and the equation of the secant line or,

$$g(x) = f(x) - \left(f(a) + \frac{f(b)-f(a)}{b-a}(x-a) \right) = f(x) - f(a) - \frac{f(b)-f(a)}{b-a}(x-a)$$

Next, let's notice that because $g(x)$ is the sum of $f(x)$, which is assumed to be continuous on $[a, b]$, and a linear polynomial, which we know to be continuous everywhere, we know that $g(x)$ must also be continuous on $[a, b]$.

Since g is continuous function on [a, b], differentiable on (a, b) and g (a) =g (b) =0

Thus by Rolle's Theorem there is $c \in (a, b)$ such that $g'(c) = 0$.

But $g'(c) = f'(c) - \left[\dfrac{f(b)-f(a)}{b-a} \right]$

Hence $0 = f'(c) - \left[\dfrac{f(b)-f(a)}{b-a}\right]$

This implies that $f'(c) = \dfrac{f(b)-f(a)}{b-a}$ which completes the proof.

Example-1: Verify the mean value theorem holds for the function g(x) = x^2-6x on [1, 4]

Solution

Since g(x) = x^2-6x is a polynomial function, g(x) is continuous on [1, 4] and differentiable on (1, 4), by mean value theorem there exists a number c $\in$(1, 4) such that $g'(c) = \dfrac{g(4)-g(1)}{4-1}$ =

$\dfrac{[(4)^2-6(4)]-[(1)^2-6(1)]}{4-1}$ = -1

And g(x) = x^2-6x $\Rightarrow$ $g'(x)$ = 2x-6 $\Rightarrow$ $g'(c)$ = 2c-6

But $g'(c)$ = -1 $\Rightarrow$ 2c-6=-1

$\Rightarrow$ c = $\dfrac{5}{2}$ $\in$(1, 4)

Thus the number c is $\dfrac{5}{2}$

Example-2: Verify the mean value theorem holds for the function f(x) = $\dfrac{x-1}{x}$ on [1, 3].

Solution

Clearly, f(x) = $\dfrac{x-1}{x}$ is continuous function on the interval [1, 3] and differentiable on (1, 3) and hence, by mean value theorem there exists c $\in$ (1, 3) such that

$f'(c) = \dfrac{f(b)-f(a)}{b-a} = \dfrac{f(3)-f(1)}{3-1} = \dfrac{1}{3}$.

But, $f'(x) = \dfrac{(x-1)'(x)-(x)'(x-1)}{(x)^2} = \dfrac{1}{x^2}$ since $f'(c) = \dfrac{1}{3}$

$\Rightarrow f'(c) = \dfrac{1}{(c)^2} = \dfrac{1}{3}$ $\Rightarrow$ c = $\pm\sqrt{3}$ however, only $\sqrt{3} \in (1, 3)$

Therefore, the number c = $\sqrt{3}$ only satisfy mean value theorem.

Note that, f(x) = $\frac{x-1}{x}$ is not continuous and differentiable function at zero if the given interval contains zero for instance on [-1, 3].

Activity3.2

1) Verify the **Rolle's Theorem** by finding the value of x for which f(x) and f $'$(x) vanish.

i) f(x) = 1-x^2 ii) f(x) = $\frac{x^2-9}{x^2+1}$ iii) let f(x) = $2-|x|$, then f (-2) =f (2), is there some number $x_0 \in$ (-2, 2) such that f $'$(x_0) =0? Does this result contradict with Rolle's Theorem?

2) Verify the mean value theorem hold true or not for the following functions.

i) f(x) = x^2 +3x on [1, 3]   ii) f(x) = $\frac{x}{x^2-1}$ on [-2, 0]   iii) f(x) = $|x-2|$ on [1, 4]

iv) f(x) = $\begin{cases} 1+x if x \geq 0 \\ 1-x if x < 0 \end{cases}$ on [-2, 3]   v) f(x)= cosx on [0, 6π]

vi) f(x) =x^3-x on [-1, 2]   vii) f(x) = x^4-2x^2+1 on [-2, 2]

3.4 Monotonic functions.

Definition 3.4:

A function f on the interval I contained in the domain of f said to be:

i) Increasing function on the interval I, if f(x_1) $\leq$ f(x_2) for all $x_1 < x_2 \in I$

ii) Decreasing function on the interval I, if f(x_1) $\geq$ f(x_2) for all $x_1 < x_2 \in I$

iii) Strictly increasing function on the interval I, if f(x_1) < f(x_2) for all $x_1 < x_2 \in I$

iv) Strictly decreasing function on the interval I, if f(x_1) > f(x_2) for all $x_1 < x_2 \in I$

v) Monotonic on the interval I, if it is either increasing or decreasing on the interval I

vi) Strictly monotonic on the interval I, if it is either strictly increasing or strictly decreasing on the interval I.

- ❖ Can you give an example for a function which is monotonic function but it is not strictly monotonic? Yes!!

Constant functions are monotonic but they are not strictly monotonic functions for instance, f(x) = 3 is monotonic but it is not strictly monotonic function.

Because f(x) = 3 for all $x \in$ R, $f(x_1) \leq f(x_2)$ or $f(x_1) \geq f(x_2$ for all $x_1 < x_2 \in$/R. In other saying

$3 \leq 3$ or $3 \geq 3$ is true for any $x_1 < x_2 \in$ R.

$\Rightarrow$ f is increasing or decreasing on its domain but it is not strictly increasing or strictly decreasing on its domain.

Therefore, f is monotonic but it is not strictly monotonic

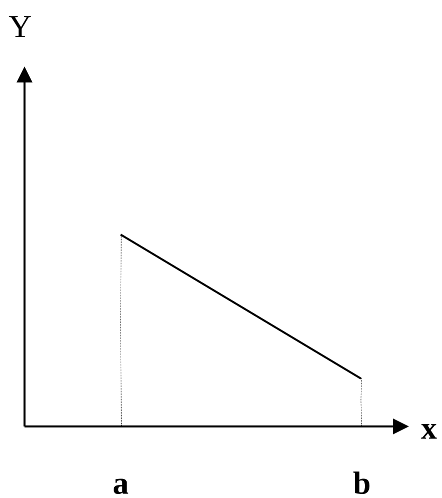
Note that:

Graphical interpretation of strictly increasing and strictly decreasing function f

- ❖ If f is strictly increasing on I then the graph of f is rising as we move from left to right of x-axis.
- ❖ If f is strictly decreasing on I then the graph of f is falling dawn as we move from left to right of x-axis.

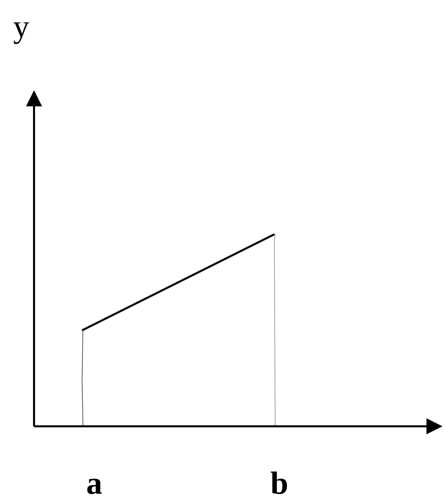
Take a look at the following figures

(a)



f is strictly decreasing on[a, b]

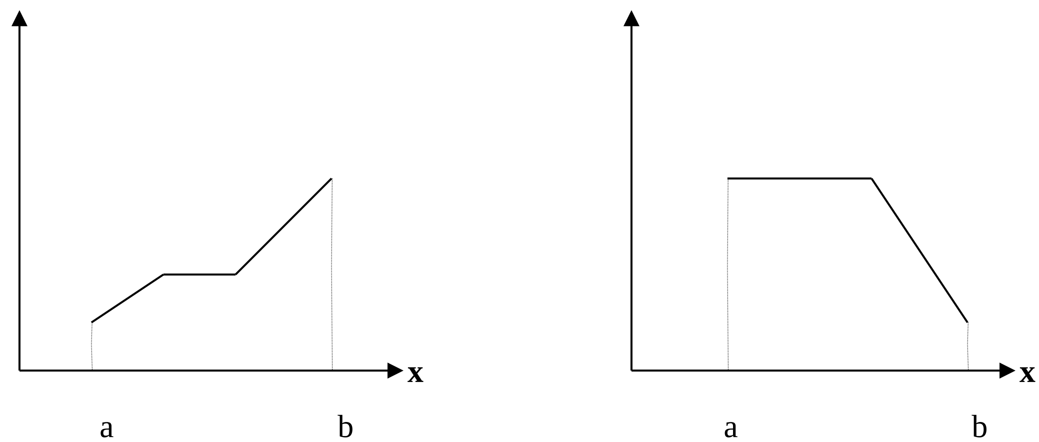
(b)



f is strictly increasing on[a, b]

y **(c)**

y **(d)**



f is increasing on[a, b] **f is decreasing on[a, b]**

fig4.9

Example-1: Let f(x) =x+2, then for any real number $x_1 < x_2 \Rightarrow x_1+2 < x_2+2$

$\Rightarrow f(x_1) < f(x_2) \ \forall \ x_1 < x_2 \in R$

Hence ,f is strictly increasing on R

Example-2: Let f(x) = x^2, then determine the interval at which f strictly increasing and strictly decreasing.

Solution

Let $x_1, x_2 \in [0, \infty)$ and let $0 \leq x_1 < x_2$

$\Rightarrow (x_1)^2 < (x_2)^2 \Rightarrow x_1^2 < x_2^2$

$\Rightarrow f(x_1) < f(x_2)$ for all $0 \leq x_1 < x_2 \in [0, \infty)$

Hence, f is strictly increasing on $[0, \infty)$

To find the intervals at which f strictly decreasing for any real number

x_1 and $x_2 \in [-\infty, 0]$,

Let $x_1 < x_2 \leq 0 \Rightarrow (x_1)^2 > (x_2)^2 \Rightarrow x_1^2 > x_2^2$

$\Rightarrow f(x_1) > f(x_2) \ \forall \ x_1 < x_2 \leq 0 \in (-\infty, 0] \Rightarrow$ f is strictly decreasing on $(\infty, 0]$ as it is shown in the figure below.

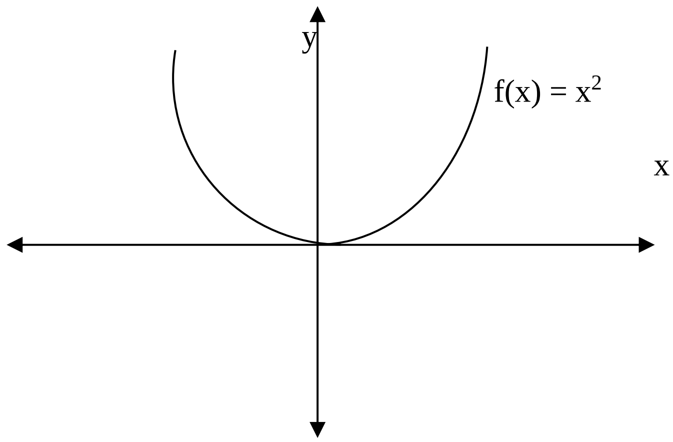


Fig3.10

Activity 3.3

Determine the intervals at which the following function is strictly increasing and strictly decreasing.

a) $f(x) = x-1$ b) $p(x) = -x+3$ c) $g(x) = -2+x-x^2$ d) $h(x) = x^3$

e) $f(x) = x^2+\frac{1}{x^2}$ f) $f(x) = \frac{4}{x^2+2}$ g) $f(x) = xe^2$

Now, we are in a position to discuss the very important theorem which used to find the intervals in which the given function increasing and decreasing. In addition to this, it is very important to sketch the graph of a function.

Theorem 3.5

Let f be continuous function on interval I & differentiable at each interior point of I, then

i) if $f'(x) > 0$ for all $x \in I$ then f is strictly increasing on I.

ii) if $f'(x) \geq 0$ for all $x \in I$ then f is increasing on I.

iii) if $f'(x) < 0$ for all $x \in I$ then f is strictly decreasing on I.

iv) if $f'(x) \leq 0$ for all $x \in I$ then f is decreasing on I.

Proof:

Suppose $f^{'}(x) > 0$ for each interior point of I and let x and z be any two points on I such that x < z, then by Mean Value Theorem there is $c \in (x, z)$ such that $f^{'}(c) = \dfrac{f(z)-f(x)}{z-x}$

But, $f^{'}(c) > 0$, then it follows that $\dfrac{f(z)-f(x)}{z-x} > 0$

$\Rightarrow f(z) > f(x)$ (since z-x >0).

Since this is true for every pair number x and z on I with x< z, it follows that f is strictly increasing on I.

The poof of ii), iii & iv are analogous to this. Therefore, it is left as an exercise

Hint: Start with suppose $f^{'}(x) \geq 0$ for each interior of x in I and repeat the procedure in the proof (i).

Example-1: Let $f(x) = x^2-6x+9$ be a function. Find the interval at which f(x) is strictly-increasing and strictly decreasing

Solution:

Clearly, $f(x) = x^2-6x+9$ is continuous and differentiable on its entire domain.

Thus, $f^{'}(x) = 2x-6=2(x-3)$.

But $f^{'}(x)>0 \Leftrightarrow 2(x-3) >0 \Rightarrow x >3$ and $f^{'}(x) <0 \Leftrightarrow 2(x-3) <0 \Rightarrow x<3$

Hence, $f^{'}(x)>0 \ \forall x \in (3, \infty)$ and $f(x) <0 \ \forall x \in (-\infty, 3)$

Thus, by above definition f(x) strictly increasing on $(3,\infty)$ and f(x) is strictly decreasing on $(-\infty, 3)$.

Note that:

- Since $f^{'}(x) \geq 0$ on $[3, \infty]$ and $f^{'}(x) \leq 0$ on $[-\infty, 3]$ implies that f is increasing on$[3,\infty]$ and decreasing on $[-\infty, 3]$
- To find the interval at which $f^{'}(x) >0$ & $f^{'}(x) <0$ we can solve it analytically or by using sign chart. But it is recommended to use sign chart for inequalities which contain products (quotients) of two or more than two factors.

Example-2: Let $f(x) = x^3-3x+7$. At which interval f is strictly increasing and strictly decreasing.

Solution:

$f(x) = x^3-3x+7 \Rightarrow f^{'}(x) =3x^2-3 = 3(x^2-1)$

Let us use sign chart to find the intervals at which the graph of f is strictly increasing on and strictly decreasing.

		-1	1
(x+1)	- - - - - -	+ + + + + +	+ + + + + +
(x-1)	- - - - - -	- - - - - -	+ + + + + +
f'(x) =3(x+1)(x-1)	+ + + + + +	- - - - - -	+ + + + + +

The above sign chart shows us $f^{'}(x) > 0$ on $(-\infty, -1) \cup (1, \infty)$.This implies that, f(x) is strictly increasing on $(-\infty, -1) \cup (1, \infty)$ and $f^{'}(x) < 0$ on the interval (-1, 1) this implies that f is strictly decreasing on (-1, 1). Moreover, since $f^{'}(x) \geq 0$ on $(-\infty, -1] \cup [1, \infty) \Rightarrow$

f is increasing on $(-\infty, -1] \cup [1, \infty)$ and $f^{'}(x) \leq 0$ on $[-1, 1] \Rightarrow$ f is increasing on [-1, 1]

Activity: 3.4

Find the intervals at which each of the following functions increasing and decreasing

a) $f(x) = x^2-2x-8$ b) $f(x) = x+\frac{1}{x}$ c) $f(x) = \frac{x^2}{2} - \frac{2}{x}$

d) $f(x) = \begin{cases} -x+1, if x > 1 \\ 0, if x \in (-1,1) \\ x+1, if x < -1 \end{cases}$ e) f(x) = cosx f) f(x) = sinx

3.5 First and second Derivative Tests

Let $f(x) = x^2-x-2$, hopefully you are familiar with the graph of f is called parabola.

a) Is the parabola up or dawn ward? b) What about $g(x) = -x^2-x-2$?

c) Which function has local maximum value and which has local minimum value?

d) Clearly the local maximum & local minimum values f and g are occurred at turning points (vertices) of the quadratic equation graphs and we compute the local maximum or local minimum values of f(x) & g(x) by using the formula $\left(-\frac{b}{2a}, \frac{4ac-b^2}{4a}\right)$.

The value of $\frac{4ac-b^2}{4a}$ is local max if the parabola is dawn ward or local min if the parabola is up ward & the value occurs at point $\frac{-b}{2a}$ of the quadratic function.

But this formula doesn't work for every function.

Therefore, the following theorems (first & second derivative test) are very important to test the local extreme values of a given function.

Theorem 4.6 (first derivative test)

Let f be continuous function on I and c be an interior point of I. For some h >0 assume (c-h , c + h) $\in$ I, then

i)f has a local maximum at c , if $f'(x) > 0$ for all $x \in$ (c-h , c) and

$f'(x) < 0$ for all $x \in$ (c , c + h)

ii)f has a local minimum at c , if $f'(x) < 0$ for all $x \in$ (c-h , c) and

$f'(x) > 0$ for all $x \in$ (c , c + h)

Proof:

Suppose a function f is continuous function on I, then for some h >0, (c-h, c+ h) $\subset$ I by hypotheses $f'(x) > 0$ for all $x \in$ (c-h, c) by theorem 4.5 f is strictly increasing on (c-h, c). Thus, by f(c) >f(x) for all $x \in$ (c-h, c) ------ (*).

Again by hypotheses $f'(x) < 0$ for all $x \in$ (c, c+h). Hence, by theorem 4.5 f is strictly decreasing on [c, c+h].-----------(**)

Thus, from (*) and (**) we have f(c) >f(x) for all $x \in$ (c-h, c+h).

Hence, f(c) is a local maximum. That means f has a local maximum at c.

Since the proof of (ii) is quit similar to (i) the proof of (ii) left as an exercise.

Example-1: Let $f(x) = x^2-x-2$. Find local extreme values of a function.

Solution:

Since $f(x) = x^2-x-2$ is a polynomial function, it is continuous and differentiable all over its domain and $f'(x) = 2x-1$.

$f'(x) > 0$ if and only if $2x-1 > 0$ and $f'(x) < 0$ if and only if $2x-1<0$

$\Rightarrow$ $f^{'}(x) > 0$ for $x \in (-\infty, \frac{1}{2})$ and $f^{'}(x) < 0$ for $\in$ $x\ (-\infty, \frac{1}{2})$.

Hence, by first derivative test f has local minimum value of $f(\frac{1}{2}) = \frac{-9}{4}$ at a point $x = \frac{1}{2}$ in its domain. Therefore, the extreme value of f is $\frac{-9}{4}$.

Example -2: Find the extreme value of a function $f(x) = x^3+3x^2-9x$.

Solution:

Since $f(x) = x^3+3x^2-9x$ is polynomial function, it is continuous and differentiable all over its domain, $\Rightarrow f^{'}(x) = 3x^2+6x-9 = 3(x-1)(x+3)$.

Now let's use sign chart to identify the intervals at which $f^{'}(x) > 0$ and $f^{'}(x) < 0$.

		-3	1
(x-1)	- - - - - -	- - - - - -	+ + + + + +
(x+3)	- - - - - -	+ + + + + +	+ + + + + +
f'(x) =3(x-1)(x+3)	+ + + + + +	- - - - -	+ + + + + +

The sign chart indicates us $f^{'}(x) > 0$ on $(-\infty, -3)$ and $f^{'}(x) < 0$ on $(-3, 1)$

$\Rightarrow$ by first derivative theorem f has local maximum value at point $x = -3$

i.e. $f(-3) = 27$ is local maximum value of f, similarly $f^{'}(x) < 0$ on $(-3, 1)$ and

$f^{'}(x) > 0$ on $(1, \infty)$ $\Rightarrow$ f has local minimum value at point $x = 1$.

i.e. $f(1) = -5$ is local minimum value of f.

Example-3: Find the local extremum value(s) of a function $f(x) = |x|$.

Solution

Since $f(x) = |x| \Rightarrow f(x) = x$ for every $x > 0 \Rightarrow f'(x) = 1$ for $\forall\, x \in (0, \infty)$ and

$f(x) = |x| \Rightarrow f(x) = -x$ for every $x < 0 \Rightarrow f^{'}(x) = -1$ $\forall\, x \in (-\infty, 0)$ and $f^{'}(x)$ doen't exist at point $x = 0$ (because $\lim_{x\to 0^+} f(x) = 1 \neq \lim_{x\to 0^-} f(x) = -1$).

However, $f^{'}(x) < 0$ on $(-\infty, 0)$ and $f^{'}(x) > 0$ on $(0, \infty)$.

Thus, by first derivative test theorem f has local minimum value at point $x = 0$.

Hence, f(0) =0 is local minimum value of f as it shown in the fig 3.11below.

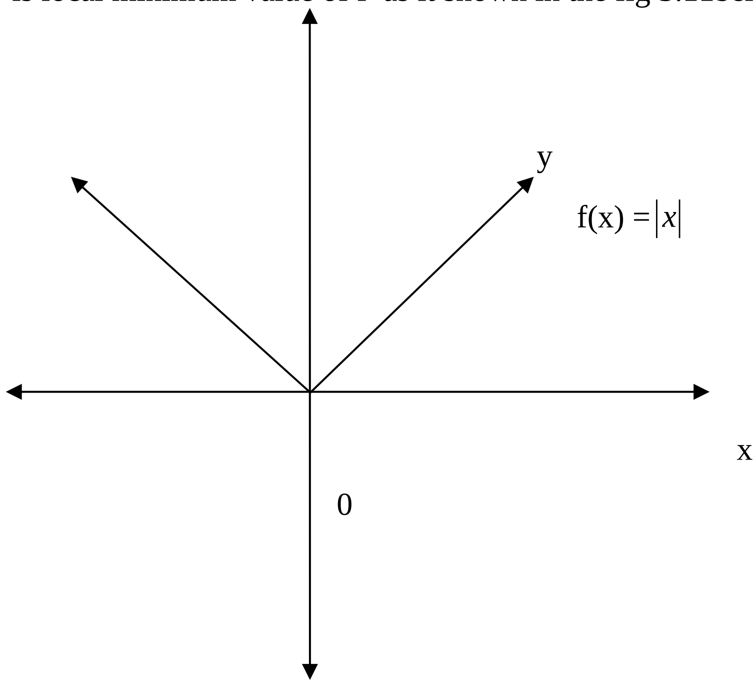


Fig3.11

Theorem 4.7 (Second Derivative Test)

Suppose f is a continuous function on interval I and let c be any interior points of I such that $f'(c) = 0$ and $f''(c)$ exist. Then

i) if $f''(c) < 0$, then f has a local maximum value at point x = c

ii) if $f''(c) > 0$, then f has a local minimum value at point x = c

iii) if $f''(c) = 0$, then we cannot draw any conclusion about extreme values

Proof:

To prove (i) let c be an interior point of I such that $f'(x) = 0$ and $f''(c) < 0$.

But $f''(c) = \lim_{x\to c} \frac{f'(x) - f'(c)}{x - c} = \lim_{x\to c} \frac{f'(x)}{x - c}$

Since $f''(c) < 0$, it follows that $\lim_{x\to c} \frac{f'(x)}{x - c} < 0$

Thus there exits h>0 with (c-h, c+h)$\subset$ I such that $\frac{f'(x)}{x-c} < 0$ for all x $\in$(c-h, c+h)

and x$\neq$ c. Now we will consider two cases.

Case I: for x $\in$(c-h, c), since x-c<0 $\frac{f'(x)}{x-c} < 0$ implies that f'(x) > 0.

Case II: for x $\in$(c, c+h), since x-c >0, $\frac{f'(x)}{x-c} < 0$ implies that f'(x) < 0.

Thus, by first derivative test theorem, f has a local maximum value at c.

The proof of (ii) is analoous to(i) threfore, it is left as an exercise.

Example-1: Find the local extremum point(s) and value(s) of a function f(x) = x^3+3x^2+4.

Solution:

f(x) = $x^3+3x^2+4 \Rightarrow$ f'(x) = $3x^2+6x$ =3x(x+2)

$f'(x)$=0 if and only if 3x(x+2) = 0 $\Rightarrow$3x =0 or x+2 = 0

$\Rightarrow$x = 0 or x = -2. hence $f'(0) = 0$ and $f'(-2) = 0$.

Then the second derivative of $f''(x) = 6x+6 = 6(x+1)$

and hence $f''(0) = 6 > 0$ and $f''(-2) = -6 < 0$

Thus, by second derivative test theorem, f has local maximum value at x= -2 i.e.f(-2) = 8 and f has local minimum value at point x = 0 i.e f(0) = 4.

Example-2: Find the local extremum point(s) & value(s) of a rational function

f(x) = $\frac{x^2}{4}+\frac{4}{x}$, x $\neq 0$

Solution:

Clearly f(x) = $\frac{x^2}{4}+\frac{4}{x} = \frac{x^3+16}{4x} \Rightarrow f'(x) = \frac{3x^2(4x)-(4)(x^3+16)}{(4x)^2} = \frac{x^3-8}{2x^2}$

$f'(x) = 0$ if and only if $\frac{x^3-8}{2x^2}$ =0 $\Rightarrow$ $x^3-8=0 \Rightarrow (x-2)(x^2+2x+4)=0$

$\Rightarrow f'(x)=0$ for x = 2 only.

Then, $f''(x) = \frac{(3x^2)(2x^2)-(4x)(x^3-8)}{(2x^2)^2} = \frac{x^3+16}{2x^3}$

Since $f^{'}(2) = 0$ and $f^{''}(2) = \frac{3}{2} > 0$, by second derivative test theorem f has local mimimum value at point x =2 i.e. f(2) = 3.

3.6 Maximum and Minimum Values Of Continuous-Function on [a, b]

Theorem 3.8: **(Fundamental Theorem of Maximum and Minimum value)**

Suppose f is continuous function on a closed interval [a, b].

i) If f attains its maximum at x_0, then x_0 =a or x_0 =b or f attains its local maximum at x_0.

ii) If f attains its minimum at x_0, then x_0 =a or x_0 =b or f attains its local minimum at x_0.

Proof

i) Suppose f attains its maximum at x_0

If x_0 = a or x_0 =b then, obviously what the theorem state is true.

ii) Suppose $x_0 \neq a$ and $x_0 \neq b$ then $x_0 \in (a, b)$. But since x_0 is a point at which f attains its maximum, it follows that x_0 a local maximum.

The proof of (ii) is analogus to case(i) left as an exercise.

Steps to be followed when we are finding the points at which f attains its maximum and minimum values respectively on interval [a, b].

Finding the exterem value of *f(x)* on [*a, b*]

Step-1: Verify that the function is continuous on the interval [*a, b*].

Step-2: Find all critical points of *f(x)* that are in the interval [*a, b*].

Step-3: Evaluat the value of f at x = a, x =b and at the local extremum points

Step-4: Identify the maximum and minimum value of f and the points at which f attains its maximum and minimum value on [a, b] respectively in step-3.

Example-1: Find maximum and minimum value of f and at which point they attained respectively for a function $f(x) = 3-5x-x^2$ on [-3, 0**].**

Solution:

Step-1: Obviously $f(x) = 3-5x-x^2$ is continuous on [-3, 0]

Step-2: Find all critical points of *f(x)* that are in the interval [*a, b*].

$f^{'}(x) = -5-2x$ then $f^{'}(x) = 0$ if and only if $-5-2x = 0 \Rightarrow$ the only critical point is

$x = \frac{-5}{2} \in (-3, 0)$.

Since $f'(\frac{-5}{2}) = 0$ and $f''(\frac{-5}{2}) = -2 < 0$

Thus, by second derivative test theorem f has local maximum value at point $x = \frac{-5}{2}$.

But by fundamental theoem of maximum and minimum value theorem f attains its maximum and mimimum value at the point sellected from $\{-3, \frac{-5}{2}, 0\}$

Step-3: Clearly $f(-3) = 9$, $f(\frac{-5}{2}) = \frac{37}{4} = 9.25$, $f(0) = 3$

Step-4: Therefore, f attains its maximum value of $f(\frac{-5}{2}) = \frac{37}{4}$ and minimum vaue of $f(0) =3$ on [-3, 0] and they are attined at ponit $x = \frac{-5}{2}$ and $x = 0$ respectively .

Example-2: Find the maximum and minimum values of f and the points at which they attianed respectively for the function $f(x) = x^3-2x^2+x+4$ on [-1, 2]

Solution:

Step-1: Clearly $f(x) = x^3-2x^2+x+4$ is continuous on [-1, 2]

Step-2: Find all critical points of *f(x)* that are in the interval [*a*, *b*].

$f'(x) =3x^2-4x+1$ then $f'(x) = 0$ if and only if $3x^2-4x+1= 0$

$3x^2-4x+1= 0 \Rightarrow x =1$ or $x = \frac{1}{3}$ and moreover $1, \frac{1}{3} \in [-1, 2]$

$f'(x) =6x-4$ and hence $f''(1) = 2 >0$ & $f''(\frac{1}{3}) = -2 <0$

By second derivative test f attains its local maximum value at $x = \frac{1}{3}$ and local minimum value at $x =1$. But fundamental theorem of max & min theorem, f attains its maximum and/or minimum avlue at points sellected from $\{-1, \frac{1}{3}, 1, 2\}$.

Step-3: $f(-1) = -4$, $f(\frac{1}{3}) = \frac{4}{27}$, $f(1) = 0$, $f(2) = 4$

Step-4 f attains its maximum value of $f(2) = 4$ and minimum value of $f(-1) = -4$ on [-1, 2] and they attained at point x =2 and x= -1 respectively.

Now, every one of us surely in a position to tackle the problems in our-day-to-day activities by applying derivatives and fundamental theorem of maximum and minimum values.

Let us see some of practical applications of the theorem which is related to our day-to-day activities.

4.7 Optimization

In this section, we are going to look at optimization problems. In optimization problems we are looking for the largest value or the smallest value that a function can take. We saw how to one kind of optimization problem in the absolute extrema section where we found the largest and smallest value that a function would take on an interval.

This section is generally one of the more difficult for students taking a Calculus course. One of the main reasons for this is that a subtle change of wording can completely change the problem. There is also the problem of identifying the quantity that we'll be optimizing and the quantity that is the constraint and writing down equations for each.

The first step in all of these problems should be to very carefully read the problem. Once you've done that the next step is to identify the quantity to be optimized and the constraint.

Let's start the section with a simple problem to illustrate the kinds of issues will be dealing with here.

Example-1

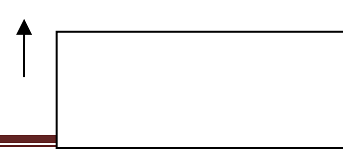
A mathematics teacher has given 5 meter long thin copper wire for his student to form a rectangle by using the wire as side of a rectangle. What is maximum area of a rectangle that the student can form from the wire and at which dimension?

Solution:

Let x and y be the wedth and length of the required rectangle's dimension.

Since the length of the wire is 6meter which implies that the perimetr (p)of a rectangle is 6meter.

$\Rightarrow$ p = 2(x+y) =6m



$\Rightarrow x+y = 3 \Rightarrow y=3-x$, in this case our domain is $\quad x$

$0 \leq x \leq 3$ since x and y are length $x \geq 0$ & $y \geq 0$ $\quad \longleftarrow y \longrightarrow$

But area of a rectangle (A) = (basea)(height) but base = y and height =x

$\Rightarrow A = xy = x(3-x)$ since $y = 3-x$

$\Rightarrow A(x) = 3x-x^2$

Now we want to find the largest value this will have on the interval [0, 3].

So, recall that the maximum value of a continuous function on a closed interval will occur at critical points and/or end points. As we've already pointed out the end points in this case will give zero area and so don't make any sense. That means our only option will be the critical points.

So let's get the derivative and find the critical points.

$A'(x) = 3-2x$.

$A'(x) =0$ if nad only if $3-2x =0 \Rightarrow x = \frac{3}{2}$ and $A''(x) = -2 \Rightarrow A''(\frac{3}{2}) = -2$

Since $A'(\frac{3}{2}) =0$ and $A''(\frac{3}{2}) =-2 < 0$ by second derivative test A(x) has local max at point $x = \frac{3}{2}$ on [0 , 3] and by fundamental theorem of maxium & minimum theorem,

A(x) attains its maximum and minimum avlue at point sellected from $\{0, 3, \frac{3}{2}\}$ and since, $A(x) = 3x-x^2 \Rightarrow A(0) = 0$, $A(3) = 0$ and $A(\frac{3}{2}) = \frac{9}{4}$.

Thus, the maximum area of a rectangle the student can form from 6m long wire is $\frac{9}{4}m^2$ and when the value of dimetions $x = \frac{3}{2}$ m and $y = \frac{3}{2}$ m. (by substituting $x = \frac{3}{2}$ in the equation $y=3-x$).

Example-2

A metal worker wants to make a water container metal box which is open on the top, from a rectangular metl sheet of 8m by 5m by cutting out congruent squares at each corner and then turning up the sides as shown in the fig below.

What should be the dimension of the metal box in order to have maximum value?

Solution:

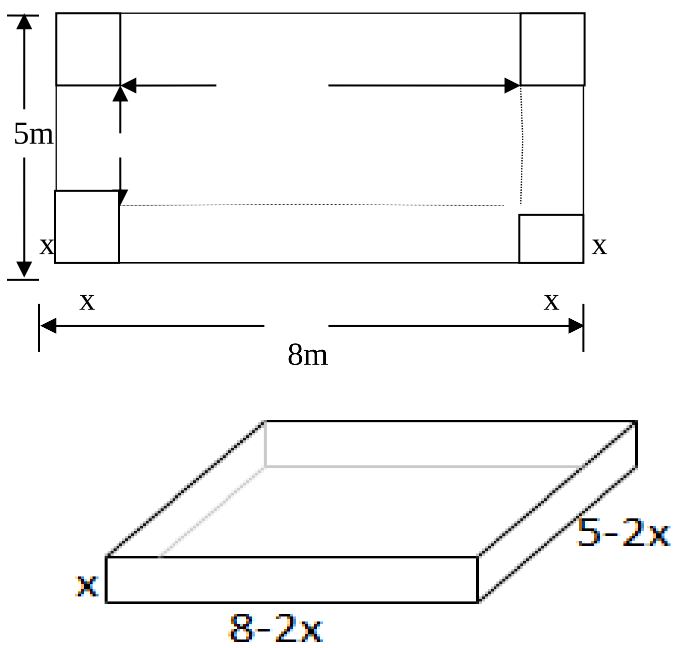


Fig4.12

Let x be the side of cut-out congruent squares, then the resulting box will has height=x, base area =(5-2x) by (8-2x) and volume V given by

$V(x) = x(5-2x)(8-2x)$, $x \in [0, \frac{5}{2}]$ **do you know why** $x \in [0, \frac{5}{2}]$?

$V(x) = 4x^3 - 26x^2 + 40x$, $x \in [0, \frac{5}{2}]$ $\Rightarrow V'(x) = 12x^2 - 52x + 40$

$V'(x) = 0$ if and only if $12x^2 - 52x + 40 = 0 \Rightarrow x = 1$ or $x = \frac{10}{3}$, but $\frac{10}{3} \notin [0, \frac{5}{2}]$

Hence, the only extremum point on $(0, \frac{5}{2})$ is 1 and $V''(x) = 24x - 52 \Rightarrow V''(1) = -28 < 0$

Thus, by second derivative theorem test V(x) attains its local maximum value at point x =1.

But, by fundamental theorem of maxium & minimum theorem, V(x) attains its maximum and minimum avlue at point sellected from {0, 1, $\frac{5}{2}$ } and Since

$V(x) = 4x^3 - 26x^2 + 40x \Rightarrow V(0) = 0$, $V(1) = 18$ and $V(\frac{5}{2}) = 0$.

Which implies that, the resulting metal box to have maximum volume, the cut-out congruent squares side length should be 1m. Which implies that, the resulting metal box dimensions should be height = x = 1m, bases (5-2x) and (8-2x). Since x=1, the resulting metal box will have maximum volume when it's height =1m, bases 3m and 6m.

Threfore,V-max= (1m)(3m)(6m) = 18m^3.

Activity 3.5

a) Find the maximum and minimum value(s) /if exist/ for a function

i) $f(x) = x^2$, on (-3, 2] ii) $g(x) = x^3+2x^2-4x+1$, on (-3, 2)

iii) $h(x) = \frac{x^2}{2-x}$, on [-3, 3].

b) The difference between two real numbers is18. sellect the numbers so that the product is as small as possible.

c) Determine the minimum area of a triangle formed by the x-axis, y-axis and a line through the point (2, 1).

d) A rectangle has perimeter 80meter. Find the length and wedth of a rectangle in order to has maximum area.

3.8 Rate of Change

It is real fact that things around us are on the process of change and it is natural to think of that change with respect to dependence of time, such as position, velocity, acceleration of moving object between two time interval t_1 and t_2. Not only those but also other change with respect to variable other than time can be treated in the same way and derivatives will have a great application here to determine, represent, and interpret and to draw conclusions about the rate at which things are changing around us at some instant t.

Let's begin by recalling some familiar ideas about motion. Recall that

$$\text{Distance} = \text{rate} \times \text{time or } rate = \frac{distance}{time}.$$

Definition 4.5:

The instantaneous rate of change of a function f(x) with respect to x over the interval x_0 to $x_0 + h$ is the derivatives of f (if exit).

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

Example-1: Suppose the area of a circle is changing with respect to its radius.

How fast is the area of a circle changing with respect to its radius when its radius is 8cm?

Solution:

Let the radius of circle be r and area of circle be A(r).

Then, $A(r) = \pi r^2 \Rightarrow A'(r) = 2\pi r$ but r = 8cm

$\Rightarrow A'(8cm) = 2\pi(8cm) = 16\pi$ cm.

Therefore, when the radius is 8cm, the area of the circle is changing its area at rate of $16\pi \frac{cm^2}{cm}$.

Remark: If any quantity Q (like distance, area, volume, speed, weight, etc) is a function of time, then the rate change of the quantity at any time t is given by the derivative of the function Q(t) with respect to time.

Example-2

1.8meter long man walks away from the point under 9 meter long light pole at rate of $4\frac{m}{s}$. How fast is his shadow lengthening at the moment when he is 12meter away from the base of the light pole?

Solution:

Let s (t) is the length of man's shadow after t seconds. Since a man is walking away at rate of 4 $\frac{m}{s}$, the distance traveled by man after t second is 4t meter as is shown in the figure below.

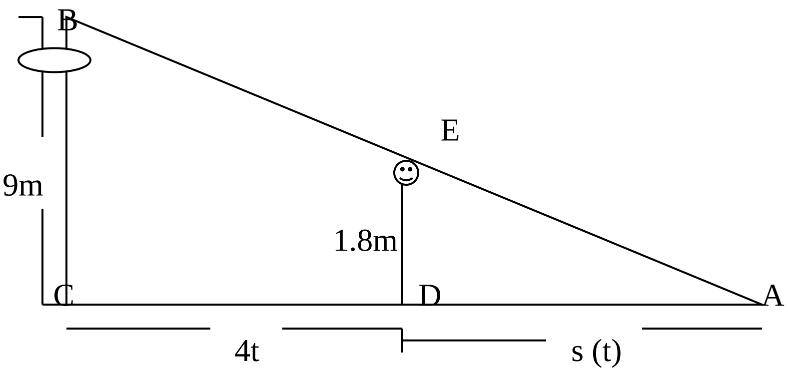


Fig3.13

Since the bigger triangle ABC is similar triangle with smaller triangle AED, then their corresponding sides are proportional.

$$\Rightarrow \frac{4t+s(t)}{s(t)} = \frac{9}{1.8} \quad \Leftrightarrow \quad 8(s\ (t)) = 1.8(4t + s\ (t))$$

$\Rightarrow$ s (t) = t $\Rightarrow$ $s'(t) = 1$

Thus, the rate of change of man's shadow is $1\frac{m}{s}$. But we are asked to find the rate of lengthening of the shadow after his travel 12m away from the base of pole.

The time taken to travel 12m is $\frac{12m}{4\frac{m}{s}}$ = 3 second.

Hence, the required rate of lengthening man's shadow after 3 second is given by $s'(t)$.

But $s'(t)$= 1 is constant function, $s'(3)= 1\frac{m}{s}$

$\Rightarrow$The man's shadow is lengthening at uniform rate of $1\frac{m}{s}$ all over the time.

Example-3

A boy wants to know the height of a tower and he release stone from the top of a tower and the stone lands on the top the ground after 4 second, then he compute the height.

a) Find the height of the tower what he has got.

b) How fast was the speed when the stone hits the ground?

c) How fast was the speed of the stone when t=3 second?

Solution:

a) Since the height of the tower varies with time t, let h (t) be the height of the tower.

But from our physics knowledge point of view,

$$h(t) = \frac{1}{2}gt^2 \text{, where g acceleration due to gravity} = 10\frac{kgm}{s^2}$$

$$h(t) = \frac{1}{2}(10)(4)^2 = 80\text{m}.$$

b) Let the speed of the stone after t second(s) be V (t), then since h (t) = $\frac{1}{2}gt^2$

$\Rightarrow$ V (t) = h $'$ (t) = gt (this is the known formula in physics)

$\Rightarrow$ The speed of the stone when it hits the ground will be V (4) = 10(4) = 40 $\frac{m}{s}$

c) The speed of the stone after 3 second will be V (3) = 10(3) = 30 $\frac{m}{s}$.

Activity 3.6

1) A water is falling in to a vertical cylindrical tank of radius 2m at the rate of $8\frac{m^3}{\min}$.

How fast is the water level rising?

2) A particle projected vertically up ward with the initial velocity 3.4m/s moves according to the law given by S(t) = 34.3t-4.9t^2. Where S(t) is the distance from the straight point at time t. Compute:

a) The velocity and acceleration of the particle when t= 5seconds and t= 8seconds

b) The maximum height where the particle reaches.

c) The time t when its height is 29.4m from the ground.

3.9 Curve Sketching

The other application of derivatives is in sketching the graph of functions

Hopefully, you will answer the questions from your pervious discussion of this course.

Question-1: What is the general meaning of asymptotes of a function?

Question-2: When do you say a function has vertical asymptote?

Question-3: What about horizontal and oblique asymptote?

Definition 3.6:

Let f be a function.

1) The line x =c is a vertical asymptote of the graph of f, if $\lim_{x\to c^+} f(x) = \pm\infty$ or $\lim_{x\to c^-} f(x) = \pm\infty$.
2) The line y = L is a horizontal asymptote of the graph of f, if $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$.

Procedures to be followed in sketching the graphs of functions:

Step-1: Determine i) x-intercept ii) y-intercept of the given function if it exists.

Step-2: Find all the asymptotes (vertical, horizontal or oblique) of a function and discuss the behaviors of given function near the asymptotes.

Step-3: Find the critical number(s) and determine the local extreme points and its values of a function.

Step-4: Find the interval at which the given function is increasing and decreasing.

Now it is a time to put the above collect information about the graph on x-y coordinate plane as follows.

i) Allocate the points where the graph of f intersects x-axis and y-axis.

ii) Draw the asymptotes of f (if exist) /note that the line should be broken/.

iii) Allocate all points whose x-coordinates are the critical points of f.

iv) If necessary plot other points

v) Sketch the complete graph of a function.

Example-1: Sketch the graph of a function $f(x) = x^3+x^2-x-1$

Solution

Step-1:

i) x-intercept can be obtained when y =0

$\Rightarrow x^3+x^2-x-1=0 \Leftrightarrow (x^3+x^2)-(x+1)=0$

$\Leftrightarrow x^2(x+1)-(x+1)=0 \Leftrightarrow (x^2-1)(x+1)= 0$

$\Rightarrow x =1$ or $x = -1$

Thus, the graph of f is intersects the x-axis at point (1, 0) and (-1, 0).

ii) y-intercept

$f(0) = (0)^3+(0)^2-(0)-1 = -1.$

Thus, (0, -1) is the point at which the graph of f intersects with y-axis.

Step-2: Asymptotes

i) Since $\lim\limits_{x\to c^+}$ $(x^3+x^2-x-1) = \lim\limits_{x\to c^-}$ $(x^3+x^2-x-1) = c^3+c^2-c-1 \neq \pm\infty \ \ \forall c \in \Re$.

This implies that the graph of f has no vertical asymptote.

ii) Since $\lim\limits_{x\to\pm\infty}$ f(x) = $\lim\limits_{x\to-\pm\infty}$ $(x^3+x^2-x-1) = \pm\infty$.

Thus, the graph of f has no horizontal asymptote.

Step-3: Critical numbers and local extreme

Since $f(x) = x^3+x^2-x-1$, then $f'(x) = 3x^2+2x-1$

$f'(x) = 0$ if and only if $3x^2+2x-1=0 \Rightarrow x = \frac{1}{3}$ or $x = -1$

Therefore, $= \frac{1}{3}$ and -1 are critical numbers of f.

$f''(x) = 6x+2$ and $f''(\frac{1}{3}) = 4 > 0$ and $f''(-1) = -4 < 0$

Hence, by second derivative test f has local maximum value of f (-1) = 0 and f has minimum value of $f(\frac{1}{3}) = \frac{-32}{27} \approx -1.185$.

Step-4: The intervals at which f increasing and decreasing.

Since $f'(x) = 3x^2+2x-1 = (x-\frac{1}{3})(x+1)$.

Let us use sign chart to identify at which point f '(x) changes its sign.

	-1		1/3	
(x-1/3)	- - - - -	- - - - -	0	+ + + + + +
(x+1)	- - - - -	0	+ + + + + +	+ + + + + +
(x-1/3) (x+1)	+ + + + + +	0	- - - - - 0	+ + + + + +

As it shown in the sign chart, $f'(x) > 0$ on the interval $(-\infty,-1)$ U $(\frac{1}{3},\infty)$

Thus, the graph of f is increasing (rising) on the interval $(-\infty,-1)$ U $(\frac{1}{3},\infty)$ and since

$f'(x) < 0$ on the interval $(-1,\frac{1}{3})$ which implies that the graph of f is decreasing (falling) on the interval $(-1, \frac{1}{3})$.

Finally, the graph of $f(x) = x^3+x^2-x-1$ looks like the following one.

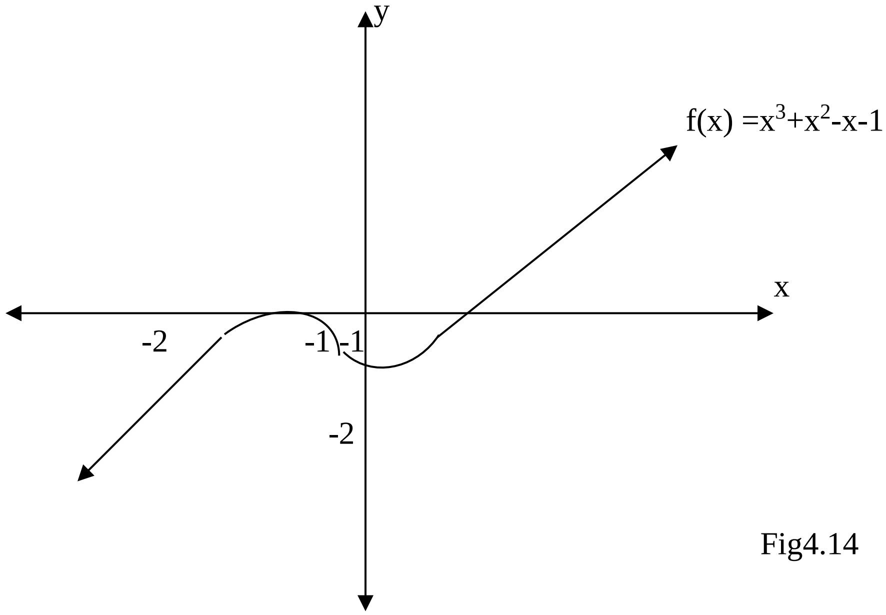


Fig4.14

Example-2: Sketch the graph of a function $f(x) = \dfrac{x-2}{x-1}$.

Solution

Step-1:

i) x-intercept can be obtained when y =0

Since $f(x) = \dfrac{x-2}{x-1} \Rightarrow \dfrac{x-2}{x-1} = 0 \Rightarrow x-2 = 0 \Rightarrow x = 2$

Thus, 2 is the x-intercept and (2, 0) is the intersection point of the graph of f and x-axis.

ii) y-intercept of a function obtained by computing f (0)

$f(0) = \dfrac{0-2}{0-1} = 2 \Rightarrow$ the graph of f intersects the y-axis at point (0, 2)

Step-2: Asymptotes

i) Since $\lim\limits_{x\to1^+}(\dfrac{x-2}{x-1}) = -\infty$ and $\lim\limits_{x\to1^-}(\dfrac{x-2}{x-1}) = \infty$

This implies that x=1 is the vertical asymptote of f.

ii) Since $\lim\limits_{x\to\pm\infty} f(x) = \lim\limits_{x\to-\pm\infty}(\dfrac{x-2}{x-1}) = \lim\limits_{x\to-\infty}\left(\dfrac{1-\frac{2}{x}}{1-\frac{1}{x}}\right) = \lim\limits_{x\to-\infty}\left(\dfrac{1-\frac{2}{x}}{1-\frac{1}{x}}\right) = 1$

Which implies that y=1 is the horizontal asymptote of f.

This implies that as /x/ increase indefinitely, the graph of f approach to the line y =1.

Note that: since the leading degree of numerator and denominator are equal f(x) has no oblique asymptote.

Step-3: Critical numbers and local extreme

f(x) = $\frac{x-2}{x-1}$ $\Rightarrow$ f' x) = $\frac{(x-2)'(x-1)-(x-1)'(x-2)}{(x-1)^2} = \frac{1}{(x-1)^2}$

$\Leftrightarrow$ f' (x) = $\frac{1}{(x-1)^2}$ $\Rightarrow$ f' (x)>0 for all x$\in$/R /{1}

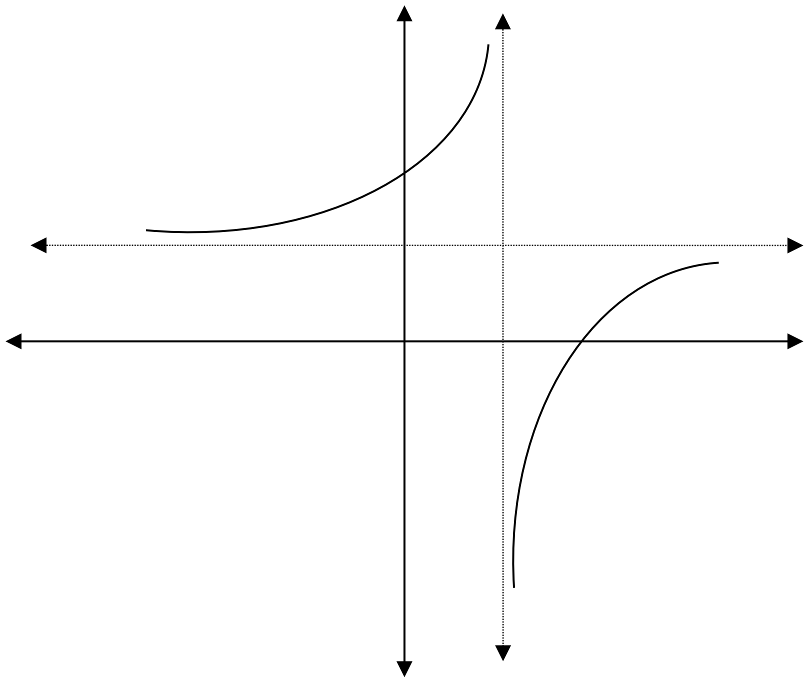
$\Rightarrow$ f' (x) doesn't exit at x = 1 but x=1 is not in the domain of f and hence f has no critical point or f has no local extreme number.

Step-4: The intervals at which f increasing and decreasing.

Since f' (x) = $\frac{1}{(x-1)^2}$ >0 $\forall x \in \Re/\{1\}$.

This implies that f is strictly increasing in its domain $(-\infty,1)u(1,\infty)$. The graph of

f(x) = $\frac{x-2}{x-1}$ looks like the following one.



y

$$f(x) = \frac{x-2}{x-1}.$$

2

1 y =1

2 x

x=1 Fig3.15

Activity 3.7

Sketch the graph of the following graphs.

a. $f(x) = x^3 - 6x + 1$

b. $f(x) = \dfrac{x^2}{x^2 - 9}$

c. $f(x) = x^4 - 2x^2 + 1$