Chapter 6

- Random Variables and Probability Distributions

6.1 Concept of a Random Variable

Statistics is concerned with making inferences about populations and population characteristics. Experiments are conducted with results that are subject to chance. The testing of a number of electronic components is an example of a **statistical experiment**, a term that is used to describe any process by which several chance observations are generated. It is often important to allocate a numerical description to the outcome.

For example, the sample space giving a detailed description of each possible outcome when three electronic components are tested may be written

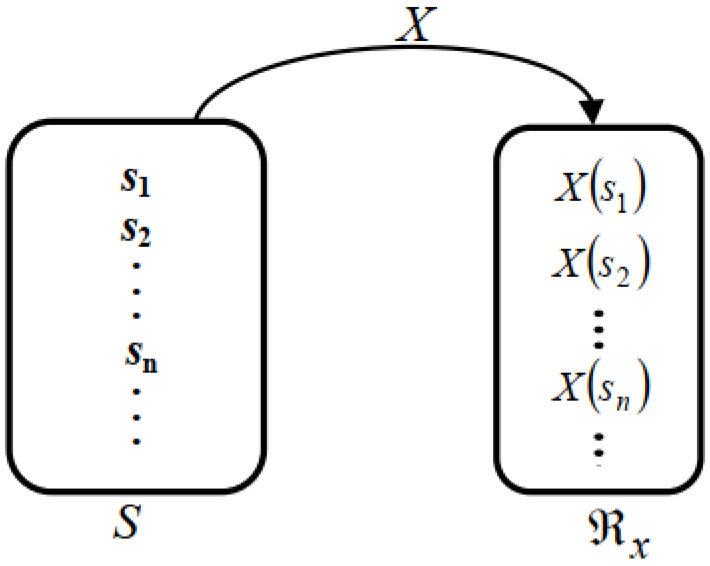
$$S = \{NNN, NND, NDN, DNN, NDD, DND, DDN, DDD\},$$

where ***N*** denotes ‘**non-defective**’ and ***D*** denotes ‘**defective**’. **One is naturally concerned with the number of defectives that occur.** Thus, each point in the sample space will be *assigned a numerical value* of 0, 1, 2, or 3. These values are, of course, random quantities *determined by the outcome of the experiment*.

They may be viewed as values assumed by the *random variable* X, the number of defective items when three electronic components are tested.

Definition 6.1 : Let S be the sample space associated with some random experiment, E . A random variable X is a function or a transformation that assigns a real number $X(s)$ to each outcome or element $s \in S$, *i.e.*, a random variable X is a real-valued function which maps a sample point to a real value.

$$X : S \to \Re_x$$



- A random variable, usually shortened to r.v. (rv),

- r.v. denoted by capital letters, such as X, Y, Z.

- The value of the r.v. X at the sample point s is X(s),

Example 6.1:

- ❖ Assume tossing of three distinct coins once
- ❖ The sample space is

S = {HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}.

X(s) = the number of heads (H's) in S.

S	HHH	HHT	HTH	THH	HTT	THT	TTH	TTT
X(s)	3	2	2	2	1	1	1	0

Thus, X(HHH) = 3, X(HHT) = X(HTH) = X(THH) = 2,

X(TTT) = 0 and X(HTT) = X(THT) = X(TTH) = 1,

X(S) = {0, 1, 2, 3}.

Discrete Random Variables

Definition 6.2:

- ❖ A random variable X is called discrete (or of the discrete type)
  - ➢ If X takes on a finite or countably infinite number of values (that is, either finitely many values such as $x_1, \dots, x_n$, or countably infinite many values such as $x_0, x_1, x_2, \dots$.)

OR

- ❖ we can describe discrete random variable as, it
  - ✓ Take whole numbers (like 0, 1, 2, 3 etc.)
  - ✓ Take finite or countably infinite number of values
  - ✓ Jump from one value to the next and cannot take any values in between.

Example 6.2:

Experiment	Random Variable (X)	Variable values
Children of one gender in a family	Number of girls	0, 1, 2, …
Answer 23 questions of an exam	Number of correct	0, 1, 2, ..., 23
Count cars at toll between 11:00 am & 1:00 pm	Number of cars arriving	0, 1, 2, ..., n

Probability Distributions

- Simple probabilities can be computed from elementary consideration or either of the methods described in the previous chapter.
- However, in dealing with probabilities of whole classes of events, we have to consider more efficient ways of analysis of probability.
- For this purpose we should know the concept of a probability distribution.
- Moreover probability distribution is defined for random variables (r.v’s).

Probability Distribution of Discrete Random Variables

Definition 6.3:

- If X is a discrete random variable,
- The function given by

$$f(x) = P(X = x) \text{ for each } x \text{ within the range of } \boldsymbol{X}$$

is called the *probability distribution* or *probability mass function* *(p.m.f)* of ***X***.

Example 6.3:

Find the probability mass function corresponding to the random variable X of ***example 6.1***. That is the r.v X = {0, 1, 2, 3}.

$$f(0) = P(X = 0) = \frac{1}{8} \qquad f(1) = P(X = 1) = \frac{3}{8}$$

$$f(2) = P(X = 2) = \frac{3}{8} \qquad f(3) = P(X = 3) = \frac{1}{8}$$

Remark:

- The probability distribution (mass) function *f(x)*, of a discrete random variable X, satisfy the following two conditions

1. $f(x) \geq 0$
2. $\sum_{x} f(x) = 1,$ the summation is taken over all possible values of *x*.

Example 6.4:

Check whether the function given by

$$f(x) = \frac{x+2}{25} \quad \text{for } x = 1, 2, 3, 4, 5$$

is a p.m.f or not.

<u>***Distribution Functions for Discrete Random Variables***</u>

Definition 6.4:

- If X is a discrete random variable, the function given by

$$F(x) = P(X \leq x) = \sum_{t \leq x} f(t)$$ for all x in *R* and t ∈ *X*.

- Where *f(t)* is the value of probability distribution or *p.m.f* of X at *t*,

- is called the ***distribution function***, or ***the cumulative distribution function of X***.

Distribution Functions (Continued)

- If X takes on only a finite number of values $x_1, x_2, \ldots, x_n$, then the distribution function is given by

$$F(x) = \begin{cases} 0 & -\infty < x < x_1 \\ f(x_1) & x_1 \le x < x_2 \\ f(x_1) + f(x_2) & x_1 \le x_2 \le x < x_3 \\ \vdots & \vdots \\ f(x_1) + \cdots + f(x_n) & x_n \le x < \infty \end{cases}$$

Example 6.5:

- Find the distribution function F of the total number of heads obtained in four tosses of a balanced coin?

X(S) = {0, 1, 2, 3, 4}

- The cumulative distribution function *F(X)* will be the following;

$$f(x=0) = \frac{1}{16} \quad f(x=1) = \frac{4}{16}$$

$$f(x=2) = \frac{6}{16} \quad f(x=3) = \frac{4}{16}$$

$$f(x=4) = \frac{1}{16}$$

$$F(X) = \begin{cases} 0 & for \quad x < 0 \\ \frac{1}{16} & for\, 0 \le x < 1 \\ \frac{5}{16} & for\, 1 \le x < 2 \\ \frac{11}{16} & for\, 2 \le x < 3 \\ \frac{15}{16} & for\, 3 \le x < 4 \\ 1 & for \quad x \ge 4 \end{cases}$$

Continuous Random Variables

Definition 6.7:

- A r.v X is called continuous (or of the continuous type) if X takes all values in a proper interval $I \subseteq \mathfrak{R}$.

OR

- we can describe continuous random variables as follows:
  - ✓ Take whole or fractional number.
  - ✓ Obtained by measuring.
  - ✓ Take infinite number of values in an interval.
  - ✓ Too many values to list like discrete random variables

Example 6.6:

- The following examples are continuous r.v.s

Experiment	Random Variable X	Variable values
Weigh 100 People	Weight	45.1, 78, ...
Measure Part Life	Hours	900, 875.9, …
Ask Food Spending	Spending	54.12, 42, ...
Measure Time Between Arrivals	Inter-Arrival time	0, 1.3, 2.78, ...

Probability Density Function of Continuous Random Variables

Definition 6.6:

- A function with values *f(x)*, defined over the set of all real numbers,
- Is called a ***probability density function*** (*p.d.f.*) of the continuous random variable X if and only if $P(a \le x \le b) = \int_a^b f(x)\,dx$ for any real constant $a \le b$
- Probability density function also referred as ***probability densities***, ***probability function***, or simply ***densities***.

Remark:

- The probability density function $f\,(x)$ of the continuous random variable X, has the following properties (satisfy the conditions)

  1. $f(x) \geq 0$ for all x, or for $-\infty < x < \infty$
  2. $$f\,(x) = \int_{-\infty}^{\infty} f\,(x)\,dx = 1$$

- If X is a continuous random variable and a and b are real constants with $a \leq b$, then

$$P\,(a \leq x \leq b) = P\,(a < x \leq b) = P\,(a \leq x < b) = P\,(a < x < b)$$

Example 6.7:

- If X is a continuous random variable with probability density *f(x)*

$$f(x) = \begin{cases} k.e^{-3x} & for\ x > 0 \\ 0 & elsewhere \end{cases}$$

Find (a) The constant k and (b) *P* $(0.5 \leq X \leq 1)$?

Example 6.8:

- Show that

$$f(x) = 3x^2$$

for 0 < x < 1 represent the density function?

Distribution Functions of Continuous Random Variables

Definition 6.7:

- If X is a continuous random variable and the value of its probability density is *f (t)*, then function given by

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)\,dt$$

- is called the ***distribution function,*** or
- The ***cumulative distribution*** of the continuous r.v. X.

Distribution Function (continued)

Theorem 6.1:

- If $f\,(x)$ and $F(x)$ are the values of the probability density and the distribution function of X at x, then

$$P\,(a \leq x \leq b) = F(b) - F(a)$$

- For any real constant a and b with a ≤ b, and

$$f(x) = \frac{d\,F(x)}{dx}$$

where the derivative exist.

Example 6.9:

- Find the distribution function of the random variable X and evaluate $P\ (0.5 \le X \le 1)$,

- if the probability density of X is $f\ (x)$

$$f(x) = \begin{cases} 3e^{-3x} & for\ x > 0 \\ 0 & elsewhere \end{cases}$$

Solution:

- For x > 0

$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_{0}^{x} f(t)\,dt$$

$$= -e^{-3t}\Big|_{0}^{x}$$

$$= 1 - e^{-3x}$$

- Since *F(X) ≤ 0*, we can rewrite *F(X)* as

$$F(X) = \begin{cases} 0 & for\ x \le 0 \\ 1 - e^{-3x} & for\ x > 0 \end{cases}$$

- Hence

$$P\,(0.50 \le x \le 1) = F(1) - F(0.50)$$

$$= \left(1 - e^{-3}\right) - \left(1 - e^{-1.5}\right) = 0.173$$

Example 6.10:

(a) Find the constant C such that the function *f(x)* is the density function of a r.v. *X*, where *f(x)* is given by

$$f(x) = \begin{cases} Cx^2 & 0 < x < 3 \\ 0 & otherwise \end{cases}$$

(b) Compute $P(1 < x < 2)$?

(c) Find the distribution function for the random variable X?

(d) Use the result of (c) to find $P(1 < x \leq 2)$?

Example 6.11:

- A r.v. X has distribution function, *F(x)* given by:

$$F(x) = \begin{cases} 0, & x \leq 0 \\ 2c\left(x^2 - \frac{1}{3}x^3\right), & 0 < x \leq 2 \\ 1, & x > 2. \end{cases}$$

(i) Determine the corresponding *p.d.f.* (*f(x)*).

(ii) Determine the constant *c*?

Properties of distribution functions, F(x) :

1. *$0 \leq F(x) \leq 1$ for all x in R*

2.. *F(x)* is nondecreasing [i.e., *$F(x) \leq F(y)$ if $x \leq y$*]

3.F(x) is continuous from the right [i.e., for all x]

$$\lim_{h \to 0^{+}} F(x+h) = F(x)$$

4. $\lim_{x \to -\infty} F(x) = 0$ and $\lim_{x \to \infty} F(x) = 1$

6.2 Expectation and Variance of a Random Variable

- ✓ **Expectation of a Random Variable**

❖ Introduction to expectation: mean and variance of r.v

- The ***expectation*** of a random variable X is very often called ***the mean*** of X and
- Denoted by ***E(X)***.
- The mean, or expectation, of the random variable X gives a single value that acts as a representative or average of the values of the r.v X,
- For this reason it is often called a ***measure of central tendency***.

Expectation of a Random Variable

Definition 6.9:

- Let X be a discrete random variable which takes values x_i ($x_1, \ldots, x_n$) with corresponding probabilities

$$P(X = x_i) = f(x_i),\ i = 1, \ldots, n.$$

- Then the *expectation* of X (or *mathematical expectation* or *mean value* of X) is denoted by $E(X)$ and is defined as:

$$E(X) = \sum_{i=1}^{n} x_i f(x_i)$$

$$E(X) = x_1 f(x_1) + \ldots + x_n f(x_n)$$

The last summation is taken over all appropriate values of x.

. . . *(Continued)*

- If the random variable X takes on (countably) infinitely many values x_i with corresponding probabilities

$$f(x_i), \qquad i = 1, 2, \ldots,$$

- Then the expectation of X is defined by:

$$E(X) = \sum_{i=1}^{\infty} x_i f(x_i) \qquad \text{provided that} \qquad \sum_{i=1}^{\infty} |x_i| f(x_{ii}) < \infty$$

. . . *(Continued)*

- As a special case where the probabilities are all equal, we have

$$E(x) = \frac{x_1 + x_2 + . . . + x_n}{n}$$

which is called the arithmetic mean, or simply the mean, of $x_1, x_2, . . . , x_n$.

Example 6.12:

Find the expectation of the sum of points in tossing of a fair die?

Definition 6.10:

- Let the random variable *X* is continuous with *p.d.f. f(x),* its expectation is defined by:

$$E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx \quad \text{provided this integral exists}$$

Example 6.13:

$$f(x) = \begin{cases} \frac{1}{2}x & 0 < x < 2 \\ 0 & otherwise \end{cases}$$

- The density function of a random variable X is given by

Find the expected value of X?

Properties of Expectation

1. If $X = K$, where K is a constant, then

$$E(X) = K$$

2. Suppose K is a constant and X is random variable, then

$$E(KX) = KE(X)$$

3. If X is random variable and suppose a and b are constants, then

$$E(aX + b) = a\,E(X) + b$$

3. Let X and Y are any two random variables, then

$$E(X + Y) = E(X) + E(Y).$$

- This can be generalized to *n* random variables; *i.e*, if $X_1, X_2, X_3, \ldots, X_n$ are random variables then,

$$E(X_1 + X_2 + X_3 + \ldots + X_n) = E(X_1) + E(X_2) + E(X_3) + \ldots + E(X_n)$$

Properties of Expectation . . . *(Continued)*

- Let X and Y are any two random variables.
- If X and Y are independent, then

$$E(XY) = E(X)E(Y)$$

Examples 6.14:

If the random variables X and Y have density functions

f(x) = 1/8x and f(y) = 1/12y respectively

for 0 < x < 4 and 1 < y <5.

Find

a. E (X) *b. E(Y)* *C. E(X + Y)* *d. E(2X + 3Y)*

e. E(XY) if X and Y are independent?

Variance of a Random Variable and Its Properties

- The *variance* (or the *standard deviation*) is a measure of the *dispersion*, or *scatter*, of the values of the random variable about the mean.

Definition 6.11:

- Let X is a random variable.
- The variance of X, denoted by V(X) or Var(X) or , defined as: δ_x^2

$$V(X) = E\,(X - E(X))^2$$

- The positive square root of V(X) is called the standard deviation of X and denoted by σ_x

Theorem 6.2:

- Let X be a random variable,
- Then the variance of X is equivalently given by

$$V(X) = E(x^2) - [E(X)]^2$$

Examples 6.15:

- Find the variance and the standard deviation of the number come up on the top on a single toss of a fair die?

Properties of Variance

- Let X be a random variable and K is a constant then,

$$V(X + K) = V(X)$$

- For a constant K and a random variable X then,

$$V(KX) = K^2\, V(X)$$

- Let $X_1, X_2, X_3, \ldots, X_n$ be *n* independent random variable, then

$$V(X_1 + X_2 + X_3 + \ldots + X_n) = V(X_1) + V(X_2) + V(X_3) + \ldots + V(X_n)$$

- Let X be a random variable with finite variance. Then for any real number a,

$$V(X) = E[(X - a)^2] - [E(X) - a]^2.$$

Examples 6.16:

A continuous random variable X has probability density given by

$$f(x) = \begin{cases} 2e^{-2x} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

and for a constant K. Find

(a) The variance of X (b) The standard deviation of X (c) Var(KX) (d) Var(K + X)

Exercise!

- For the random variable X
- If $E[(X - 1)^2] = 10, \quad E[(X - 2)^2] = 6.$
- Find

  (a) *E(X)*?

  (b) *Var(X)*?

  (c) Standard deviation of *X*?

6.3. Common Probability Distributions:

❖ Common Discrete Distributions: Binomial and Poisson

Bernoulli Distribution

- If an experiment has two possible outcomes
  - ➢“success” with probability Θ and
  - ➢“Failure” with probability $(1 - \Theta)$
- Then the number of success (0 or 1) has a Bernoulli distribution.

Definition 6.12:

- A random variable X has Bernoulli distribution and it referred to as a Bernoulli random variable if and only if its probability distribution given by:

$$f(x;\theta) = \theta^{x}(1-\theta)^{1-x} \quad \text{for } x = 0, 1$$

Properties of Bernoulli Random Variable

- Le X be a Bernoulli random variable having probability of success then the:

✓ ***Mean***

$$E(X) = \sum_{x=0}^{1} x\, f(X=x) \quad = \sum_{x=0}^{1} x\,\theta^{x}(1-\theta)^{1-x} = \theta$$

✓ ***Variance***

$$\mathrm{Var}(X) = E(X - E(x))^2 = \theta\,(1-\theta)$$

Example 6.17:

- Tossing a coin and considering heads as success and tails as failure.
- Checking items from a production line: success = not defective, failure = defective.
- Phoning a call centre: success = operator free; failure = no operator free.
- Success of medical treatment.
- Student passes exam.
- A fair die is tossed. Let X = 1 only if the first toss shows a "4" or "5". Then

$$X \sim B(1, \frac{1}{3})$$

Binomial Distribution

- Repeated trials play an important role in probability and statistics,
- Especially when the number of trial is fixed,
- The parameter (the probability of success) is same for each trial, and
- The trial are all independent.
- Several random variables are a rise in connection with repeated trials.
- The one we shall study here concerns the total number of success.

Definition 6.13:

- A random variable X has ***Binomial distribution*** and

- It referred to as a ***Binomial random variable*** if and only if its probability distribution given by

$$f(x; n, \theta) = \binom{n}{x} \theta^x (1 - \theta)^{n-x} \quad \text{for } x = 0, 1, \ldots, n$$

- In general binomial distribution has the following characteristics

  - It consists of *n* trials
  - Each trials results in either of out puts (“success” or “failure”)
  - The *n* trials are independent
  - The probability of success is ϴ and the probability of failure is (1-ϴ)
  - The probability of success or ϴ is the same for each trials

Properties of Binomial Distribution:

- Let X be a Binomial distribution with *n* number of trials and probability of success then the: θ

- ***Mean***

$$E(X) = \mu = \sum_{x=1}^{n} x\, f(X=x) = \sum_{x=1}^{n} x \binom{n}{x} \theta^{x} (1-\theta)^{n-x} = n\theta$$

- ***Variance***

$$Var(X) = E(X - E(x))^2 = n\theta\,(1-\theta)$$

Example 6.18:

1. Find the probability of getting 5 heads and seven tails in 12 flips of a balanced coin?

Solution Number of ways of selecting 5 heads cut of total 12 flips $= {}^{12}C_5$.

Probability of getting one head in a coin $= \frac{1}{2}$

Also, probability of getting one tail in a coin $= \frac{1}{2}$

Probability of getting 5 head $= \left(\frac{1}{2}\right)^5$

Probability of getting 7 tails $= \left(\frac{1}{2}\right)^7$

So, required probability

$$= {}^{12}C_5\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^7 = {}^{12}C_5\left(\frac{1}{2}\right)^{12} = \frac{{}^{12}C_5}{2^{12}}$$

2. The *Statistical Bulletin* published by Metropolitan Life Insurance Co. reported that 2% of all American births result in twins. If a random sample of 8000 births is taken, find the mean, variance, and standard deviation of the number of births that would result in twins.

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Poisson Distribution

- When the number of trial *n* is large,
- The calculation of binomial probabilities of *Definition 6.13* will usually involve a prohibitive amount of work.
- Poisson distribution shall present as a probability distribution that can be used to approximate binomial probabilities of this kind.
- Especially when $n \to \infty$, $\theta \to 0$ and , $n\theta$ remains constant.
- By letting the constant ($n\theta$) to be λ ,
- *That is* , $n\theta = \lambda$ *and hence,* $\theta = \frac{\lambda}{n}$
- The binomial distribution will be write

$$f(x; n, \quad \theta) = \binom{n}{x}\left(\frac{\lambda}{n}\right)^x\left(1 - \frac{\lambda}{n}\right)^{n-x} \qquad \textit{for } x = 0, 1, \; . \; . \; . \; , n$$

- After some mathematical computation we would the following distribution indicated in Definition 6.14

Definition 6.13:

- A random variable X has Poisson distribution and
- It referred to as a Poisson random variable
- *If and only if* its probability distribution given by

$$f(x;\ \lambda) = \frac{\lambda^x\ e^{-\lambda}}{x!} \quad \text{for } x = 0, 1, 2, \ . \ . \ .$$

- *Thus when* $n \to \infty$, $\theta \to 0$ and $n\theta$ remains constant
- The number of success is a random variable having a poison distribution with the parameter λ.

Properties of Poisson distribution

- Le X is a Poisson distribution with an average number of time an event occur λ then:

Mean

$$E(X) = \mu = \sum_{x=0}^{\infty} x \frac{\lambda^x e^{-\lambda}}{x!} = \lambda$$

Variance

$$Var(X) = E(X - E(x))^2 = \lambda$$

Example 6.19:

- ➢ Number bacteria per small volume of fluid
- ➢ Number of car accident per a given period of time
- ➢ Number of customers arrived during a given period of time.

Example 6.20:

- If 2% of the books bound at a certain bindery have defective bindings,
- Use the poison approximation to the binomial distribution to determine
- The probability that 5 of 400 books bound by this bindery will have defective binding?

Common continuous distributions: Normal, t, and chi-square distribution

6.4. Common continuous distributions: Normal, t, and chi-square Distribution

Normal Distribution

- The normal distribution is in many ways is the cornerstone of modern statistical theory.

Definition 6.15:

- A continuous random variable X has a normal distribution and
- It referred to as a normal random variable
- If and only if its probability distribution is given by

$$N(x;\mu,\sigma)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

for $-\infty < x < \infty$, *where* $\sigma > 0$

Properties of Normal Distribution

- Le X be a normal random variable with parameters then:

Mean

$$E(X) = \int_{-\infty}^{\infty} x \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-1}{2}(\frac{x-\mu}{\sigma})^2} dx = \mu$$

Variance

$$= \delta^2$$

$$Var(X) = E(X - E(x))^2$$

Example 6.21:

- Suppose we are looking at a national examination
- Whose scores would be approximately normal with

  $\mu = 500$ and $\sigma = 100$.

- What is the probability that a score falls between 600 and 750?

Definition 6.16: (Standard Normal distribution)

- The normal distribution with μ = 0 and $\sigma = 1$
- It referred to as standard normal distribution which is given by:

$$N(x; 0, \quad 1) = \frac{1}{\sqrt{2\pi}} e^{\frac{-1}{2}x^2} \qquad \text{for } -\infty < x < \infty$$

Properties of Standard Normal Distribution

- Le X be a **Standard normal** random variable with parameters then

Mean

$$E(X) = \int_{-\infty}^{\infty} x \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = 0$$

Variance

$$\boldsymbol{Var\ (X) = E(X - E(x))^2 = 1}$$

Transformation to Standard Normal

- We can transform any normal random variable X with mean μ and standard deviation σ into a standard normal random variable Z With mean 0 and standard deviation 1.

- The linear transformation is:

$$Z = \frac{X - \mu}{\sigma}$$

Example 6.22:

- On a statistics examination the mean was 78 and the standard deviation was 10.

  (a) Determine the standard scores of two students whose grades were

  i. 93 and

  ii. 62 respectively,

  (b) Determine the grades of two students whose standard scores were

  i. -0.6 and

  ii. 1.2 respectively

Additional Properties of Normal Distribution

1. It is bell shaped and is symmetrical about its mean and it is mesokurtic. The maximum ordinate is at $x = \mu$ and is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}$$

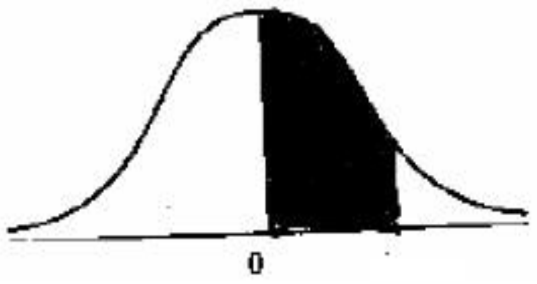
2. It is asymptotic to the axis, i.e., it extends indefinitely in either direction from the mean.

3. Total area under the curve sums to 1, i.e., the area of the distribution on each side of the mean is 0.5. $\Rightarrow \int_{-\infty}^{\infty} f(x)dx = 1$

4. It is unimodal

5. $Mean = Median = \text{mod}\,e$

6. The probability that a random variable will have a value between any two points is equal to the area under the curve between those points.



t - Distribution

- In probability and statistics, Student's t-distribution (or simply the t-distribution) is a probability distribution that
- arises in the problem of estimating the mean of a normally distributed population when the sample size is small.
- Student's t-tests used for the statistical significance test of the difference between two sample means, and for confidence intervals for the difference between two population means.
- The Student's t-distribution is a special case of the generalized hyperbolic distribution.
- The parameter ν is called the number of degrees of freedom.
- The distribution depends on ν, but not μ or σ;
- Lack of dependence on μ and σ is what makes the t-distribution important in both theory and practice.

t – Distribution *(Continued)*

Definition 6.17:

- A random variable X has a t distribution (students t–distribution)

- If its probability distribution given by

$$f(x,v) = \frac{\Gamma\left(\frac{v+1}{2}\right)}{\sqrt{v\pi}\,\Gamma\left(\frac{v}{2}\right)} \left(1+\frac{x^2}{v}\right)^{-(v+1)/2} \quad \textit{for} - \infty < x < \infty$$

- With v degrees of freedom.

- If v is large ($v \geq 30$),

- The graph of $f(x, v)$ closely approximates the standard normal curve.

Properties of t Distribution

- Le X be a t-distribution random variable with parameter *v* then

Mean

$$E(X) = \int_{-\infty}^{\infty} x \frac{\Gamma(\frac{v+1}{2})}{\sqrt{v\pi}\,\Gamma(\frac{v}{2})} \left(1 + \frac{x^2}{v}\right)^{-(v+1)/2} dx = 0 \qquad \textbf{\textit{for } } \boldsymbol{v > 2}$$

Variance

$$Var(X) = E(X - E(x))^2 = \frac{v}{v-2}, \qquad \textit{for } v > 2$$

Chi - Square Distribution

- Let $X_1, X_2, \ . \ . \ . \ , X_v$ is *v* independent normally distributed random variables with mean zero and variance 1.
- Consider the random variable χ^2

$$\chi^2 = X_1^2 + X_2^2 + . \ . \ . + X_v^2$$

- Where the notation used for a r.v. *X* having the Chi-Square distribution with *v* degree of freedom (*d.f*) is

$$X \sim \chi_v^{\ 2}$$

- It is possible to show that for $x \geq 0$

$$P(\chi^2 \leq x) = \frac{1}{2^{v/2}\,\Gamma(\frac{v}{2})} \int_0^x u^{(\frac{v}{2})-1}\, e^{-x/2}\, du \quad \text{and}$$

$$P(\chi^2 \leq x) = 0 \quad \text{for } x < 0$$

- This distribution is called the Chi- square distribution and
- *v* is the degree of freedom,
- Which will formally defined as follow:

Definition 6.18:

- A random variable X has a Chi - square distribution and
- It referred to as a Chi - square random variable
- If and only if its probability distribution given by

$$f\left(x;\frac{v}{2},2\right)=\begin{cases}\dfrac{1}{2^{\frac{v}{2}}\,\Gamma(\frac{v}{2})}\,x^{\frac{v-2}{2}}\,e^{\frac{-x}{2}} & for\ x>0, \quad v>0\ and\ integer,\\ 0 & elsewhere\end{cases}$$

- On the other way round χ^2 distribution is
- a special case of the gamma distribution for $\alpha=\frac{v}{2}$ and $\beta=2$.
- This distribution is used in certain statistical inference problems involving
    - ✓ Confidence intervals for variances and
    - ✓ Testing hypotheses about variances.

Properties of Chi - square Distribution

- Let X be an Chi - square random variable with
- parameters α and β then

Mean

$$E(X) = \int_{-\infty}^{\infty} x \frac{1}{2^{\frac{v}{2}}\,\Gamma(\frac{v}{2})} x^{\frac{v-2}{2}} e^{\frac{-x}{2}} dx = v$$

Variance

$$Var(X) = E(X - E(x))^2 = 2v$$

$$\Phi(z) = P(Z \le z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2}\, du$$

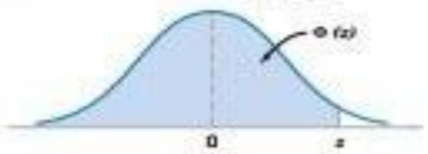


Table II Cumulative Standard Normal Distribution (continued)

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.500000	0.503989	0.507978	0.511967	0.515953	0.519939	0.532922	0.527903	0.531881	0.535856
0.1	0.539828	0.543795	0.547758	0.551717	0.555760	0.559618	0.563559	0.567495	0.571424	0.575345
0.2	0.579260	0.583166	0.587064	0.590954	0.594835	0.598706	0.602568	0.606420	0.610261	0.614092
0.3	0.617911	0.621719	0.625516	0.629300	0.633072	0.636831	0.640576	0.644309	0.648027	0.651732
0.4	0.655422	0.659097	0.662757	0.666402	0.670031	0.673645	0.677242	0.680822	0.684386	0.687933
0.5	0.691462	0.694974	0.698468	0.701944	0.705401	0.708840	0.712260	0.715661	0.719043	0.722405
0.6	0.725747	0.729069	0.732371	0.735653	0.738914	0.742154	0.745373	0.748571	0.751748	0.754903
0.7	0.758036	0.761148	0.764238	0.767305	0.770350	0.773373	0.776373	0.779350	0.782305	0.785236
0.8	0.788145	0.791030	0.793892	0.796731	0.799546	0.802338	0.805106	0.807850	0.810570	0.813267
0.9	0.815940	0.818589	0.821214	0.823815	0.826391	0.828944	0.831472	0.833977	0.836457	0.838913
1.0	0.841345	0.843752	0.846136	0.848495	0.850830	0.853141	0.855428	0.857690	0.859929	0.862143
1.1	0.864334	0.866500	0.868643	0.870762	0.872857	0.874928	0.876976	0.878999	0.881000	0.882977
1.2	0.884930	0.886860	0.888767	0.890651	0.892512	0.894350	0.896165	0.897958	0.899727	0.901475
1.3	0.903199	0.904902	0.906582	0.908241	0.909877	0.911492	0.913085	0.914657	0.916207	0.917736
1.4	0.919243	0.920730	0.922196	0.923641	0.925066	0.926471	0.927855	0.929219	0.930563	0.931888
1.5	0.933193	0.934478	0.935744	0.936992	0.938220	0.939429	0.940620	0.941792	0.942947	0.944083
1.6	0.945201	0.946301	0.947384	0.948449	0.949497	0.950529	0.951543	0.952540	0.953521	0.954486
1.7	0.955435	0.956367	0.957284	0.958185	0.959071	0.959941	0.960796	0.961636	0.962462	0.963273
1.8	0.964070	0.964852	0.965621	0.966375	0.967116	0.967843	0.968557	0.969258	0.969946	0.970621
1.9	0.971283	0.971933	0.972571	0.973197	0.973810	0.974412	0.975002	0.975581	0.976148	0.976705
2.0	0.977250	0.977784	0.978308	0.978822	0.979325	0.979818	0.980301	0.980774	0.981237	0.981691
2.1	0.982136	0.982571	0.982997	0.983414	0.983823	0.984222	0.984614	0.984997	0.985371	0.985738
2.2	0.986097	0.986447	0.986791	0.987126	0.987455	0.987776	0.988089	0.988396	0.988696	0.988989
2.3	0.989276	0.989556	0.989830	0.990097	0.990358	0.990613	0.990863	0.991106	0.991344	0.991576
2.4	0.991802	0.992024	0.992240	0.992451	0.992656	0.992857	0.993053	0.993244	0.993431	0.993613
2.5	0.993790	0.993963	0.994132	0.994297	0.994457	0.994614	0.994766	0.994915	0.995060	0.995201
2.6	0.995339	0.995473	0.995604	0.995731	0.995855	0.995975	0.996093	0.996207	0.996319	0.996427
2.7	0.996533	0.996636	0.996736	0.996833	0.996928	0.997020	0.997110	0.997197	0.997282	0.997365
2.8	0.997445	0.997523	0.997599	0.997673	0.997744	0.997814	0.997882	0.997948	0.998012	0.998074
2.9	0.998134	0.998193	0.998250	0.998305	0.998359	0.998411	0.998462	0.998511	0.998559	0.998605
3.0	0.998650	0.998694	0.998736	0.998777	0.998817	0.998856	0.998893	0.998930	0.998965	0.998999
3.1	0.999032	0.999065	0.999096	0.999126	0.999155	0.999184	0.999211	0.999238	0.999264	0.999289
3.2	0.999313	0.999336	0.999359	0.999381	0.999402	0.999423	0.999443	0.999462	0.999481	0.999499
3.3	0.999517	0.999533	0.999550	0.999566	0.999581	0.999596	0.999610	0.999624	0.999638	0.999650
3.4	0.999663	0.999675	0.999687	0.999698	0.999709	0.999720	0.999730	0.999740	0.999749	0.999758
3.5	0.999767	0.999776	0.999784	0.999792	0.999800	0.999807	0.999815	0.999821	0.999828	0.999835
3.6	0.999841	0.999847	0.999853	0.999858	0.999864	0.999869	0.999874	0.999879	0.999883	0.999888
3.7	0.999892	0.999896	0.999900	0.999904	0.999908	0.999912	0.999915	0.999918	0.999922	0.999925
3.8	0.999928	0.999931	0.999933	0.999936	0.999938	0.999941	0.999943	0.999946	0.999948	0.999950
3.9	0.999952	0.999954	0.999956	0.999958	0.999959	0.999961	0.999963	0.999964	0.999966	0.999967

$$\Phi(z) = P(Z \le z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} du$$

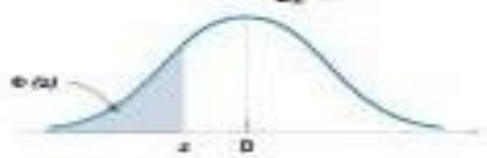


Table II Cumulative Standard Normal Distribution

z	−0.09	−0.08	−0.07	−0.06	−0.05	−0.04	−0.03	−0.02	−0.01	−0.00
−3.9	0.000033	0.000034	0.000036	0.000037	0.000039	0.000041	0.000042	0.000044	0.000046	0.000048
−3.8	0.000050	0.000052	0.000054	0.000057	0.000059	0.000062	0.000064	0.000067	0.000069	0.000072
−3.7	0.000075	0.000078	0.000082	0.000085	0.000088	0.000092	0.000096	0.000100	0.000104	0.000108
−3.6	0.000112	0.000117	0.000121	0.000126	0.000131	0.000136	0.000142	0.000147	0.000153	0.000159
−3.5	0.000165	0.000172	0.000179	0.000185	0.000193	0.000200	0.000208	0.000216	0.000224	0.000233
−3.4	0.000242	0.000251	0.000260	0.000270	0.000280	0.000291	0.000302	0.000313	0.000325	0.000337
−3.3	0.000350	0.000362	0.000376	0.000390	0.000404	0.000419	0.000434	0.000450	0.000467	0.000483
−3.2	0.000501	0.000519	0.000538	0.000557	0.000577	0.000598	0.000619	0.000641	0.000664	0.000687
−3.1	0.000711	0.000736	0.000762	0.000789	0.000816	0.000845	0.000874	0.000904	0.000935	0.000968
−3.0	0.001001	0.001035	0.001070	0.001107	0.001144	0.001183	0.001223	0.001264	0.001306	0.001350
−2.9	0.001395	0.001441	0.001489	0.001538	0.001589	0.001641	0.001695	0.001750	0.001807	0.001866
−2.8	0.001926	0.001988	0.002052	0.002118	0.002186	0.002256	0.002327	0.002401	0.002477	0.002555
−2.7	0.002635	0.002718	0.002803	0.002890	0.002980	0.003072	0.003167	0.003264	0.003364	0.003467
−2.6	0.003573	0.003681	0.003793	0.003907	0.004025	0.004145	0.004269	0.004396	0.004527	0.004661
−2.5	0.004799	0.004940	0.005085	0.005234	0.005386	0.005543	0.005703	0.005868	0.006037	0.006210
−2.4	0.006387	0.006569	0.006756	0.006947	0.007143	0.007344	0.007549	0.007760	0.007976	0.008198
−2.3	0.008424	0.008656	0.008894	0.009137	0.009387	0.009642	0.009903	0.010170	0.010444	0.010724
−2.2	0.011011	0.011304	0.011604	0.011911	0.012224	0.012545	0.012874	0.013209	0.013553	0.013903
−2.1	0.014262	0.014629	0.015003	0.015386	0.015778	0.016177	0.016586	0.017003	0.017429	0.017864
−2.0	0.018309	0.018763	0.019226	0.019699	0.020182	0.020675	0.021178	0.021692	0.022216	0.022750
−1.9	0.023295	0.023852	0.024419	0.024998	0.025588	0.026190	0.026803	0.027429	0.028067	0.028717
−1.8	0.029379	0.030054	0.030742	0.031443	0.032157	0.032884	0.033625	0.034379	0.035148	0.035930
−1.7	0.036727	0.037538	0.038364	0.039204	0.040059	0.040929	0.041815	0.042716	0.043633	0.044565
−1.6	0.045514	0.046479	0.047460	0.048457	0.049471	0.050503	0.051551	0.052616	0.053699	0.054799
−1.5	0.055917	0.057053	0.058208	0.059380	0.060571	0.061780	0.063008	0.064256	0.065522	0.066807
−1.4	0.068112	0.069437	0.070781	0.072145	0.073529	0.074934	0.076359	0.077804	0.079270	0.080757
−1.3	0.082264	0.083793	0.085343	0.086915	0.088508	0.090123	0.091759	0.093418	0.095098	0.096801
−1.2	0.098525	0.100273	0.102042	0.103835	0.105650	0.107488	0.109349	0.111233	0.113140	0.115070
−1.1	0.117023	0.119000	0.121001	0.123024	0.125072	0.127143	0.129238	0.131357	0.133500	0.135666
−1.0	0.137857	0.140071	0.142310	0.144572	0.146859	0.149170	0.151505	0.153864	0.156248	0.158655
−0.9	0.161087	0.163543	0.166023	0.168528	0.171056	0.173609	0.176185	0.178786	0.181411	0.184060
−0.8	0.186733	0.189430	0.192150	0.194894	0.197662	0.200454	0.203269	0.206108	0.208970	0.211855
−0.7	0.214764	0.217695	0.220650	0.223627	0.226627	0.229650	0.232695	0.235762	0.238852	0.241964
−0.6	0.245097	0.248252	0.251429	0.254627	0.257846	0.261086	0.264347	0.267629	0.270931	0.274253
−0.5	0.277595	0.280957	0.284339	0.287740	0.291160	0.294599	0.298056	0.301532	0.305026	0.308538
−0.4	0.312067	0.315614	0.319178	0.322758	0.326355	0.329969	0.333598	0.337243	0.340903	0.344578
−0.3	0.348268	0.351973	0.355691	0.359424	0.363169	0.366928	0.370700	0.374484	0.378281	0.382089
−0.2	0.385908	0.389739	0.393580	0.397432	0.401294	0.405165	0.409046	0.412936	0.416834	0.420740
−0.1	0.424655	0.428576	0.432505	0.436441	0.440382	0.444330	0.448283	0.452242	0.456205	0.460172
0.0	0.464144	0.468119	0.472097	0.476078	0.480061	0.484047	0.488033	0.492022	0.496011	0.500000

END