Work, Energy and Momentum

❖ Outlines

❖ Work *done by constant force*
❖ Work *done by variable force*
❖ Work-Kinetic Energy Theorem
❖ Impulse and Momentum
❖ Linear Momentum and Impulses
❖ Conservation of Momentum
❖ Collision
❖ Center of mass

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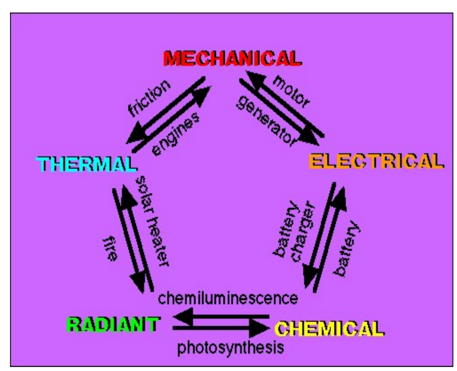
Energy

❖ Energy

- Energy is a property of the state of a system, not a property of individual objects.
- Energy is expressed in joules (J)
- 4.19 J = 1 calorie
- Energy is conserved. It can be transferred from one object to another or change in form, but cannot be created or destroyed
- Energy can be expressed more specifically by using the term **work(W)**



Work = ***The Scalar Dot Product between Force and Displacement.*** If you apply a force on an object and it covers a displacement in the direction of the force you have supplied energy to, or done work on, that object.

Energy

❖ Kinetic Energy

- Kinetic Energy is energy associated with the state of motion of an object
- For an object moving with a speed of ***v***

$$KE = \frac{1}{2} m v^2$$

Special case: Constant Acceleration

Remember result eliminating t: $\boxed{v^2 - v_0^2 = 2a(x - x_0)}$

Multiply by ½ m: $\frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 = ma(x - x_0)$

$$= ma\Delta x$$

But $F=ma!$

$$\Delta(\frac{1}{2}mv^2) = F\Delta x$$

Work

❖ Work *W*

- Start with $\frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 = F_x \Delta x$ → Work "W"
- Work provides a link between force and energy
- Work done on an object is transferred to/from it
- If W > 0, energy added: "transferred to the object"
- If W < 0, energy taken away: "transferred from the object"

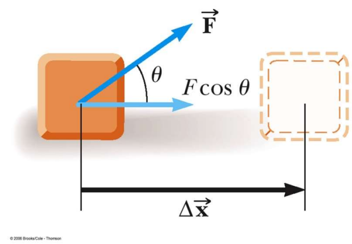
Work

❖ Work *done by constant force*

- The work, W, done by a constant force on an object is defined as the product of the component of the force along the direction of displacement and the magnitude of the displacement.

$$W = (F cos\theta)\Delta x$$

  - F is the magnitude of the force
  - Δx is the magnitude of the object's displacement
  - θ is the angle between F and Δx

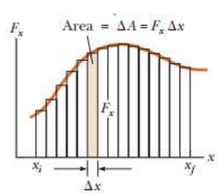


- **In this case it means that F and Δx must be** **parallel.** To ensure that they are parallel we add the cosine on the end.

Work

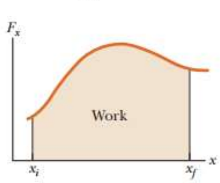
❖ Work *done by variable force*

❖ Suppose an object is displaced along the x-axis under the action of a force F_x that acts in the x-direction and varies with position.



$$W_1 \cong F_x \Delta x$$

$$W \cong F_1 \Delta x_1 + F_2 \Delta x_2 + F_3 \Delta x_3 + \cdots$$



- The work done by a variable force acting on an object that undergoes a displacement is equal to the area under the graph of F_x versus x.

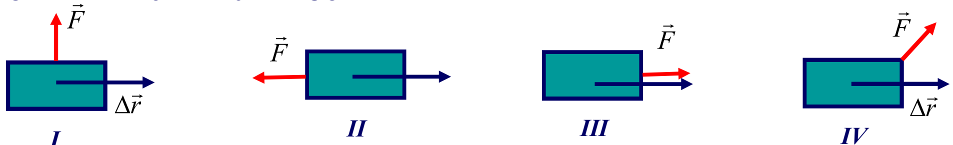
Work

❖ Work: + or -?

- Work can be positive, negative, or zero. The sign of the work depends on the direction of the force relative to the displacement

$$W = (F\cos\theta)\Delta x = F.\Delta x$$

  - Work positive: W > 0 if 90° > θ > 0°
  - Work negative: W < 0 if 180° > θ > 90°
  - Work zero: W = 0 if θ = 90°
  - Work maximum if θ = 0°
  - Work minimum if θ = 180°



$W_I = 0$ $\quad W_{II} = -F\Delta r$ $\quad W_{III} = F\Delta r$ $\quad W_{IV} = F\Delta r\cos\theta$

Work

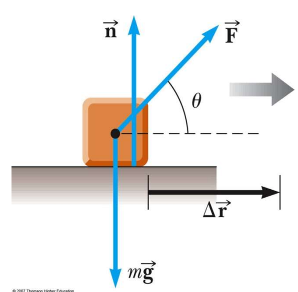
❖ Work Done by Multiple Forces

- If more than one force acts on an object, then the total work is equal to the algebraic sum of the work done by the individual forces

$$W_{\text{net}} = \sum W_{\text{by individual forces}}$$

  - Remember work is a scalar, so this is the algebraic sum

$$W_{net} = W_g + W_N + W_F = (F\cos\theta)\Delta r$$

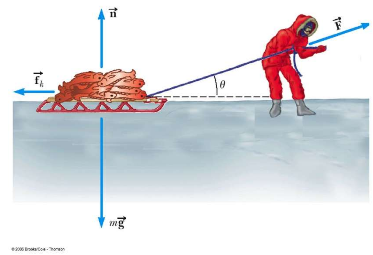


Work

❖ Example

- Suppose $\mu_k = 0.200$, How much work done on the sled by friction, and the net work if $\theta = 30^\circ$ and he pulls the sled 5.0 m?

$$F_{net,y} = N - mg + F\sin\theta = 0$$
$$N = mg - F\sin\theta$$



$$W_{fric} = (f_k \cos 180^\circ)\Delta x = -f_k \Delta x$$
$$= -\mu_k N \Delta x = -\mu_k (mg - F\sin\theta)\Delta x$$
$$= -(0.200)(50.0kg \cdot 9.8m/s^2$$
$$-1.2\times 10^2 N \sin 30^\circ)(5.0m)$$
$$= -4.3\times 10^2 J$$

$$W_{net} = W_F + W_{fric} + W_N + W_g$$
$$= 5.2\times 10^2 J - 4.3\times 10^2 J + 0 + 0$$
$$= 90.0J$$

Work –Kinetic Energy Theorem

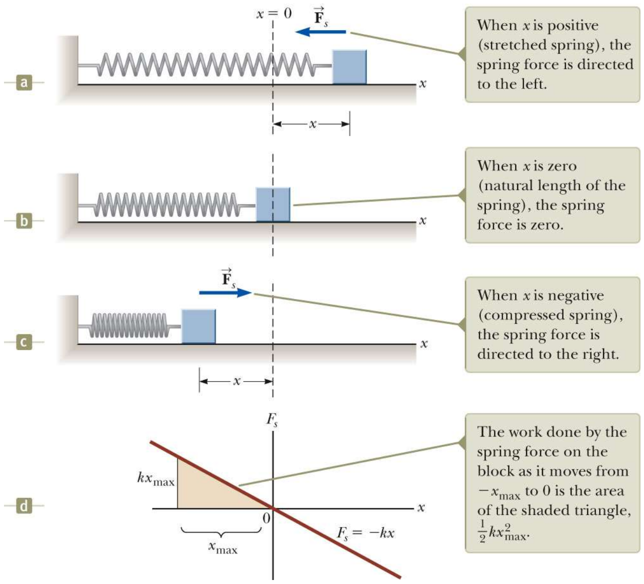
❖ Work-Kinetic Energy Theorem

- When work is done by a net force on an object and the only change in the object is its speed, the work done is equal to the change in the object's kinetic energy
  - Speed will increase if work is positive
  - Speed will decrease if work is negative

$$W_{net} = KE_f - KE_i = \Delta KE$$

$$W_{net} = \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2$$

Elastic Potential Energy



$$\lim_{\Delta x \to 0} \sum_{x_i}^{x_f} F_x \Delta x = \int_{x_i}^{x_f} F_x dx$$

$$W = \int_{x_i}^{x_f} F_x dx = \int_{x_i}^{x_f} -kx\, dx$$

$$= \int_{-x_{max}}^{0} -kx\, dx = \tfrac{1}{2}kx^2$$

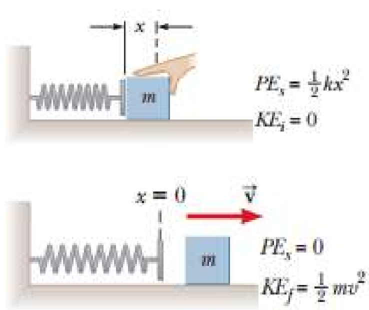
$$W = \int_{x_i}^{x_f} -kx\, dx = \tfrac{1}{2}kx_i^2 - \tfrac{1}{2}kx_f^2$$

Work done by spring on block

Elastic Potential Energy

❖ If $x_i = 0$ and $x_f = x$, elastic potential energy (EPE) can be written as;

$$W = EPE = \frac{1}{2}kx^2$$



❖From conservation of energy;

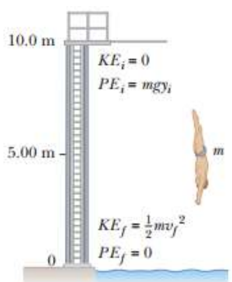
$$(EPE + KE)_i = (EPE + KE)_f$$

$$\frac{1}{2}kx_i^2 + \frac{1}{2}mv_i^2 = \frac{1}{2}kx_f^2 + \frac{1}{2}mv_f^2$$

Gravitational Potential Energy

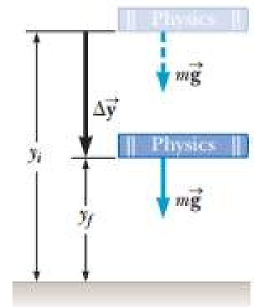
❖ Potential energy is a property of a **system,** rather than of a single object, because it's due to the relative positions of interacting objects in the system.

$$W = F_g \Delta y cos\theta = F_g(y_i - y_f)cos0 = -F_g(y_f - y_i)$$



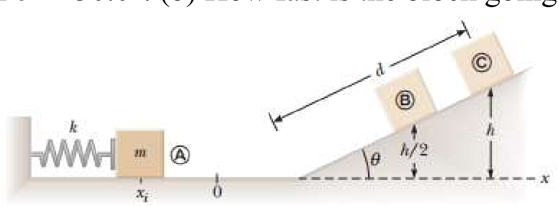
$$KE_i + PE_i = KE_f + PE_f$$

$$\tfrac{1}{2}mv_i^2 + mgy_i = \tfrac{1}{2}mv_f^2 + mgy_f$$



Example

A 0.5kg block rests on a horizontal, frictionless surface. The block is pressed back against a spring having a constant of k = 625 N/m, compressing the spring by 10 cm to point . Then the block is released. (a) Find the maximum distance d the block travels up the frictionless incline if $\theta = 30.0°$. (b) How fast is the block going when halfway to its maximum height?



$$\tfrac{1}{2}mv_i^2 + mgy_i + \tfrac{1}{2}kx_i^2 = \tfrac{1}{2}mv_f^2 + mgy_f + \tfrac{1}{2}kx_f^2$$

$$\tfrac{1}{2}kx_i^2 = mgh = mgd\sin\theta$$

$$d = \frac{\frac{1}{2}kx_i^2}{mg\sin\theta} = \frac{\frac{1}{2}(625\text{ N/m})(-0.100\text{ m})^2}{(0.500\text{ kg})(9.80\text{ m/s}^2)\sin(30.0°)}$$

$$= \boxed{1.28\text{ m}}$$

$$\tfrac{1}{2}mv_i^2 + mgy_i + \tfrac{1}{2}kx_i^2 = \tfrac{1}{2}mv_f^2 + mgy_f + \tfrac{1}{2}kx_f^2$$

$$\tfrac{1}{2}kx_i^2 = \tfrac{1}{2}mv_f^2 + mg\left(\tfrac{1}{2}h\right)$$

$$\frac{k}{m}x_i^2 = v_f^2 + gh$$

$$v_f = \sqrt{\frac{k}{m}x_i^2 - gh}$$

$$= \sqrt{\left(\frac{625\text{ N/m}}{0.500\text{ kg}}\right)(-0.100\text{ m})^2 - (9.80\text{ m/s}^2)(0.640\text{ m})}$$

$$v_f = \boxed{2.50\text{ m/s}}$$

Power

❖Power (P)

- Power, the rate at which energy is transferred.
- One useful application of Energy is to determine the rate at which we store or use it.

SI. Unit: watt ($w = J/s$)

$$P = \frac{W}{t} \rightarrow \frac{Fx}{t} \rightarrow Fv$$

$$P = \frac{mgh}{t}$$

$$P = \frac{\frac{1}{2}mv^2}{t}$$

Linear Momentum and Impulses

❖ Linear Momentum

- Linear momentum is a measure of how hard it is to stop a moving object.
- There are two factors that make hard to stop, its mass (m) and its velocity ($\vec{v}$).
- The greater the mass the harder it is to stop, the faster an object is moving the harder it is to stop.
- *Linear momentum ($\vec{p}$) is defined as the product of the mass (m) and velocity ($\vec{v}$) of a particle.*

$$\vec{p} = m\vec{v}$$

$$\vec{p} = \vec{p}_x + \vec{p}_y + \vec{p}_z \qquad |p| = \sqrt{{p_x}^2 + {p_y}^2 + {p_z}^2}$$

$$\vec{p}_x = m\vec{v}_x \qquad \vec{p}_y = m\vec{v}_y \qquad \vec{p}_z = m\vec{v}_z$$

Linear Momentum and Impulses

- If a resultant force acts on a body, it will cause that body's momentum to change. The momentum change occurs in the direction of the force, at a rate proportional to the magnitude of that force.

- The net force can be expressed in terms of change in momentum divided by time, or the rate of change of momentum.

$$F_{net} = m\vec{a} = m\frac{\Delta\vec{v}}{\Delta t} = \frac{\Delta\vec{p}}{\Delta t}$$

  - The change in an object's momentum $\Delta\vec{p}$ divided by the elapsed time Δt equals the constant net force F_{net} acting on the object:
  - The magnitude of the momentum p of an object of mass m can be related to its kinetic energy KE:

$$KE = \frac{p^2}{2m}$$

Linear Momentum and Impulses

❖Impulse

❖ Impulse is the magnitude of a force multiplied by the time for which it acts.

- The linear momentum of an object is conserved when $F_{net} = 0$.

- If a constant force $\vec{F}$ acts on an object, the **impulse** $\vec{I}$ delivered to the object over a time interval Δt is given by;

$$\vec{I} = \vec{F}\Delta t = \Delta\vec{p} = m\vec{v}_f - m\vec{v}_i$$

Impulse-momentum theorem

❖ *States that the impulse of the force acting on an object equals the change in momentum of the object*

Linear Momentum and Impulses

❖Example

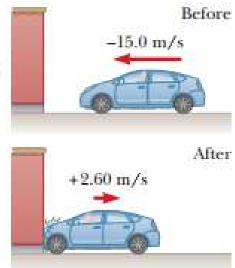
A car of mass $1.5 \times 10^3 kg$ collides with a wall and rebounds. The initial and final velocities of the car are $\vec{v}_i = -15\ m/s$ and $\vec{v}_f = 2.5\ m/s$, respectively. If the collision lasts for $0.15\ s$, find **(a)** the impulse delivered to the car due to the collision and **(b)** the size and direction of the average force exerted on the car.

$$p_i = mv_i = (1.50 \times 10^3\ \text{kg})(-15.0\ \text{m/s}) = -2.25 \times 10^4\ \text{kg}\cdot\text{m/s}$$

$$p_f = mv_f = (1.50 \times 10^3\ \text{kg})(+2.60\ \text{m/s}) = +0.390 \times 10^4\ \text{kg}\cdot\text{m/s}$$

$$I = p_f - p_i = +0.390 \times 10^4\ \text{kg}\cdot\text{m/s} - (-2.25 \times 10^4\ \text{kg}\cdot\text{m/s})$$

$$I = \boxed{2.64 \times 10^4\ \text{kg}\cdot\text{m/s}}$$



$$F_{av} = \frac{\Delta p}{\Delta t} = \frac{2.64 \times 10^4\ \text{kg}\cdot\text{m/s}}{0.150\ \text{s}} = \boxed{+1.76 \times 10^5\ \text{N}}$$

Conservation of Momentum

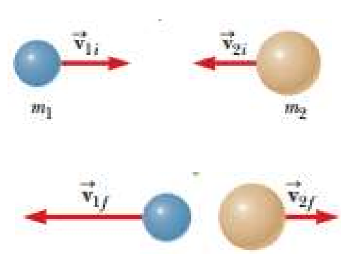
❖ When a collision occurs in an isolated system, the total momentum of the system doesn't change with the passage of time.

❖ Instead, it remains constant both in magnitude and in direction. The momenta of the individual objects in the system may change, but the vector sum of *all* the momenta will not change.

$$\vec{F}_{21}\Delta t = m_1\vec{v}_{1f} - m_1\vec{v}_{1i} \qquad \vec{F}_{12}\Delta t = m_2\vec{v}_{2f} - m_2\vec{v}_{2i}$$

$$\vec{F}_{21}\Delta t = -\vec{F}_{12}\Delta t \qquad m_1\vec{v}_{1f} - m_1\vec{v}_{1i} = -(m_2\vec{v}_{2f} - m_2\vec{v}_{2i})$$

$$\boxed{m_1\vec{v}_{1i} + m_2\vec{v}_{2i} = m_1\vec{v}_{1f} + m_2\vec{v}_{2f}}$$



❖ When no net external force acts on a system, the total momentum of the system remains constant in time.

Collision in 1D

❖ The total momentum of the system just before the collision equals the total momentum just after the collision as long as the system may be considered isolated.

❖ We define an inelastic collision as a collision in which momentum is conserved, but kinetic energy is not.

- When two objects collide and stick together, the collision is called *perfectly inelastic*.

❖ An elastic collision is defined as one in which both momentum and kinetic energy are conserved.

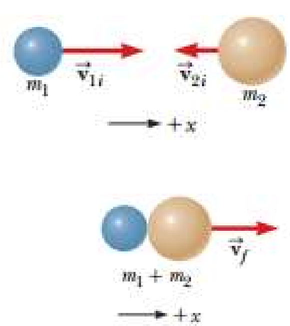
Collision in 1D

❖ *Perfectly inelastic collisions*

- Before a perfectly inelastic collision the objects move independently.
- After the collision the objects remain in contact. System momentum ***is*** conserved, but system energy is ***not*** conserved.

$$m_1 v_{1_i} + m_2 v_{2_i} = (m_1 + m_2) v_f$$

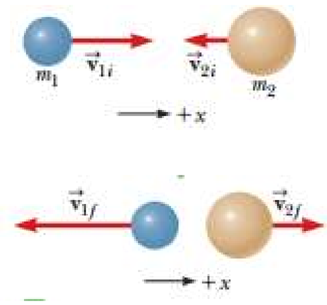
$$\boxed{v_f = \frac{m_1 v_{1i} + m_2 v_{2i}}{m_1 + m_2}}$$



Collision in 1D

❖ *Elastic collisions*

- Before a perfectly inelastic collision the objects move independently.
- After the collision the object velocities change, but ***both*** the energy and momentum of the system are conserved.



$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f}$$

$$m_1(v_{1i} - v_{1f}) = m_2(v_{2f} - v_{2i})$$

$$\tfrac{1}{2} m_1 v_{1i}^2 + \tfrac{1}{2} m_2 v_{2i}^2 = \tfrac{1}{2} m_1 v_{1f}^2 + \tfrac{1}{2} m_2 v_{2f}^2$$

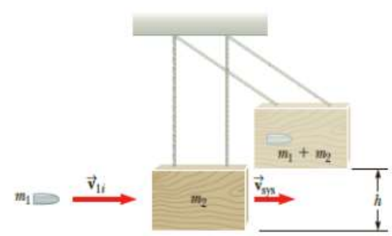
$$m_1(v_{1i} - v_{1f})\,(v_{1i} + v_{1f}) = m_2(v_{2f} - v_{2i})\,(v_{2f} + v_{2i})$$

$$v_{1i} - v_{2i} = -(v_{1f} - v_{2f})$$

Collision in 1D

❖ Example

The bullet is fired into a large block of wood suspended from some light wires. The bullet is stopped by the block, and the entire system swings up to a height h. It is possible to obtain the initial speed of the bullet by measuring h and the two masses. Assume $m_1 = 5\ g, m_2 = 1\ kg\ and\ h = 5\ cm.$ **(a)** Find the velocity of the system after the bullet embeds in the block. **(b)** Calculate the initial speed of the bullet.



$$(KE + PE)_{\text{after collision}} = (KE + PE)_{\text{top}}$$

$$\tfrac{1}{2}(m_1 + m_2)v_{sys}^2 + 0 = 0 + (m_1 + m_2)gh$$

$$v_{sys}^2 = 2gh$$

$$v_{sys} = \sqrt{2gh} = \sqrt{2(9.80\ \text{m/s}^2)(5.00 \times 10^{-2}\ \text{m})}$$

$$v_{sys} = \boxed{0.990\ \text{m/s}}$$

$$p_i = p_f$$

$$m_1 v_{1i} + m_2 v_{2i} = (m_1 + m_2)v_{sys}$$

$$v_{1i} = \frac{(m_1 + m_2)v_{sys}}{m_1}$$

$$v_{1i} = \frac{(1.005\ \text{kg})(0.990\ \text{m/s})}{5.00 \times 10^{-3}\ \text{kg}} = \boxed{199\ \text{m/s}}$$

Collision in 2D

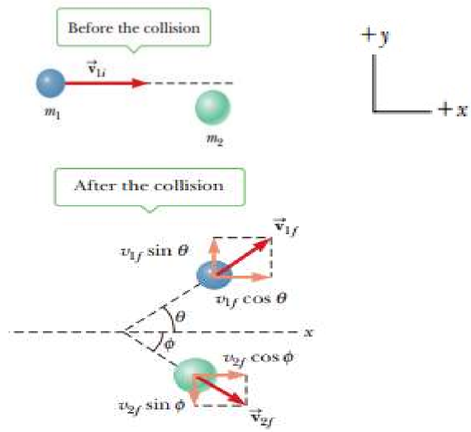
❑ The general equation of 2D collision

$$m_1v_{1ix} + m_2v_{2ix} = m_1v_{1fx} + m_2v_{2fx}$$

$$m_1v_{1iy} + m_2v_{2iy} = m_1v_{1fy} + m_2v_{2fy}$$

x-component: $m_1v_{1i} + 0 = m_1v_{1f}\cos\theta + m_2v_{2f}\cos\phi$

y-component: $0 + 0 = m_1v_{1f}\sin\theta + m_2v_{2f}\sin\phi$



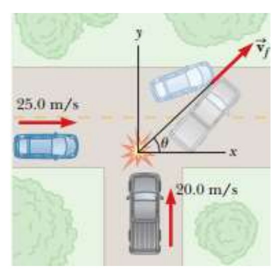
❑ If the collision is elastic, we can write a third equation, for conservation of energy, in the form

$$\tfrac{1}{2}m_1v_{1i}^2 = \tfrac{1}{2}m_1v_{1f}^2 + \tfrac{1}{2}m_2v_{2f}^2$$

Collision in 2D

❖ Example

A car with mass $1.5 \times 10^3 kg$ traveling east at a speed of 25 m/s collides at an intersection with a $2.5 \times 10^3 kg$ van traveling north at a speed of 20 m/s. Find the magnitude and direction of the velocity of the wreckage after the collision, assuming that the vehicles undergo a perfectly inelastic collision



$$\sum p_{xi} = m_{car}v_{car} = (1.50 \times 10^3\text{ kg})(25.0\text{ m/s}) = 3.75 \times 10^4\text{ kg}\cdot\text{m/s}$$

$$\sum p_{xf} = (m_{car} + m_{van})v_f\cos\theta = (4.00 \times 10^3\text{ kg})v_f\cos\theta$$

$$3.75 \times 10^4\text{ kg}\cdot\text{m/s} = (4.00 \times 10^3\text{ kg})v_f\cos\theta$$

$$\sum p_{yi} = m_{van}v_{van} = (2.50 \times 10^3\text{ kg})(20.0\text{ m/s}) = 5.00 \times 10^4\text{ kg}\cdot\text{m/s}$$

$$\sum p_{yf} = (m_{car} + m_{van})v_f\sin\theta = (4.00 \times 10^3\text{ kg})v_f\sin\theta$$

$$5.00 \times 10^4\text{ kg}\cdot\text{m/s} = (4.00 \times 10^3\text{ kg})v_f\sin\theta$$

$$\tan\theta = \frac{5.00 \times 10^4\text{ kg}\cdot\text{m/s}}{3.75 \times 10^4\text{ kg}\cdot\text{m}} = 1.33$$

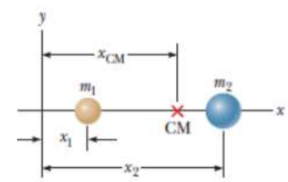
$$\theta = 53.1°$$

$$v_f = \frac{5.00 \times 10^4\text{ kg}\cdot\text{m/s}}{(4.00 \times 10^3\text{ kg})\sin 53.1°} = 15.6\text{ m/s}$$

$$|v_f| = \sqrt{(v_f\cos\theta)^2 + (v_f\sin\theta)^2} = 15.6m/s$$

Center of mass

❖ The center of mass of the system is located somewhere on the line joining the two particles and is closer to the particle having the larger mass.



$$x_{cm} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2}$$

If we have more than two particles

$$x_{cm} = \frac{m_1x_1 + m_2x_2 + m_3x_3 + \cdots m_nx_n}{m_1 + m_2 + m_3 + \cdots m_n} = \frac{\sum_1^n m_nx_n}{\sum_1^n m_n} = \frac{1}{M}\sum_1^n m_nx_n$$

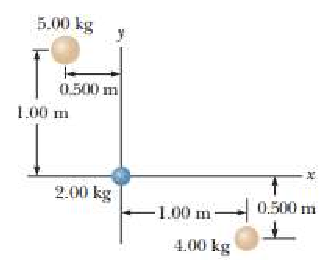
❖ In general

$$r_{cm} = x_{cm} + y_{cm} + z_{cm} = \frac{1}{M}\left(\sum_1^n m_nx_n + \sum_1^n m_ny_n + \sum_1^n m_nz_n\right)$$

Center of mass

❖ Example

Three objects are located in a coordinate system. Find the center of mass.



$$M = (5 + 2 + 4)kg = 11kg$$

$$\sum m_nx_n = \big((5 \times -0.5) + (2 \times 0) + (4 \times 1)\big)kg.m = 1.5kg.m$$

$$x_{cm} = \frac{\sum m_nx_n}{M} = \frac{1.5}{11}m = 0.136m$$

$$\sum m_ny_n = \big((5 \times 1) + (2 \times 0) + (4 \times -0.5)\big)kg.m = 3kg.m$$

$$y_{cm} = \frac{\sum m_ny_n}{M} = \frac{3}{11}m = 0.273m$$

$$r_{cm} = x_{cm} + y_{cm} = (0.136, 0.273)m$$

Equilibrium of Particles

❖ *Torque (moment of force) (τ) the force multiplied by the perpendicular distance from the point about which the moment is being measured.*

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} = rF\sin\theta$$

- An object in mechanical equilibrium must satisfy the following two conditions:

1. **The net external force must be zero:** $\sum \vec{F} = 0$
2. **The net external torque must be zero:** $\sum \vec{\tau} = 0$

- $\vec{a} = 0$
- $\vec{\alpha} = 0$

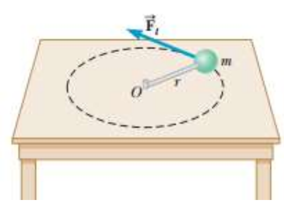
❑ Torque for clockwise moment is negative whereas for counterclockwise moment is positive.

Equilibrium of Particles

$$F_t = ma_t$$

$$F_t r = mra_t = mr^2\alpha = \tau$$

$$\tau = (mr^2)\alpha = \mathrm{I}\alpha$$

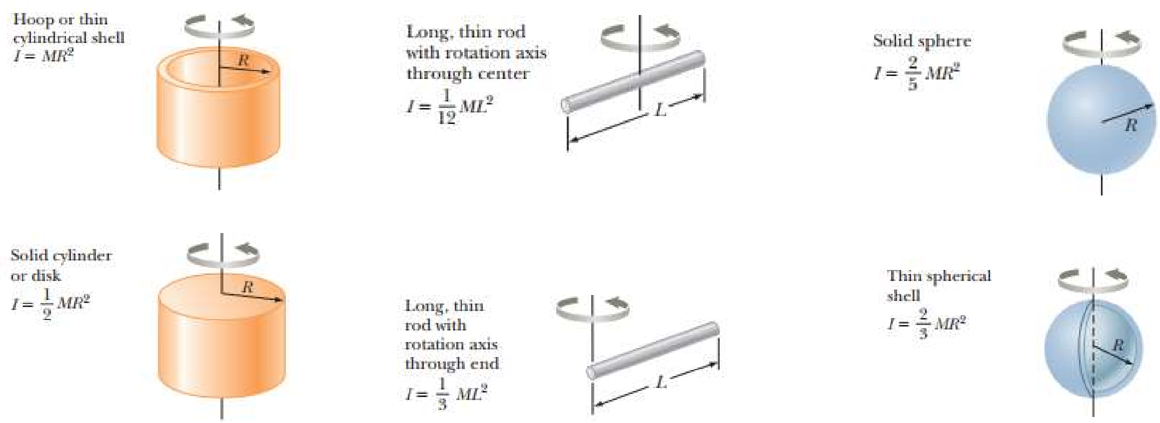


❖ $I = mr^2$ is called the **moment of inertia** of the object of mass *m*.

➢ The angular acceleration of an extended rigid object is proportional to the net torque acting on it.

Equilibrium of Particles

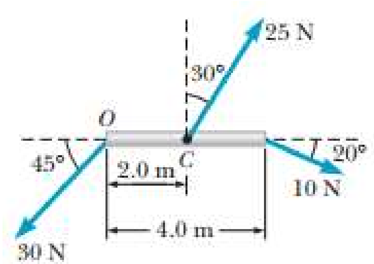
❑ Moments of Inertia for Various Rigid Objects



Equilibrium of Particles

❖ Example

Calculate the net torque (magnitude and direction) on the beam about (a) an axis through *O* and (b) an axis through *C*.



- $sin45 = 0.71$
- $sin60 = 0.87$
- $sin20 = 0.34$

- $30sin45 = 21.3N$
- $25sin60 = 21.75N$
- $10sin20 = 3.4N$

✓ About O

Clockwise

$\tau_1 = 3.4 \times 4 = 13.6Nm$

Counterclockwise

$\tau_2 = 21.75 \times 2 = 43.5Nm$

$\tau_{net} = \tau_2 - \tau_1 = 29.9Nm$

✓ About C

Clockwise

$\tau_1 = 3.4 \times 2 = 6.8Nm$

Counterclockwise

$\tau_2 = 21.3 \times 2 = 42.6Nm$

$\tau_{net} = \tau_2 - \tau_1 = 35.8Nm$

End

End of chapter two