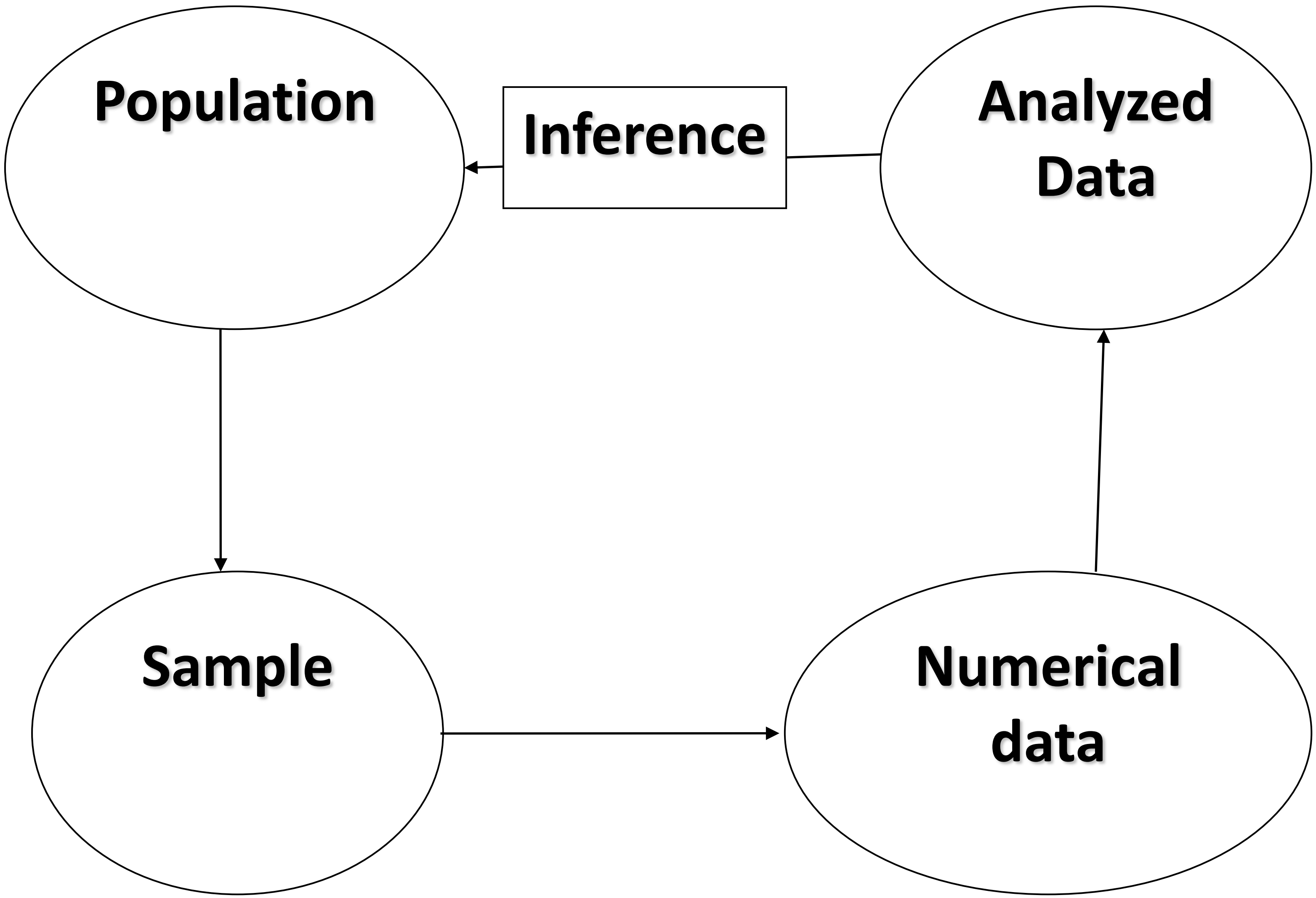
CHAPTER 8
Estimation and Hypothesis Testing

March, 2023

ESTIMATION AND HYPOTHESIS TESTING

- **Inference:** is the process of making interpretations or conclusions from sample data for the totality of the population.
- It is only the sample data that is ready for inference.
- In statistics there are two ways though which inference can be made.
    - Statistical estimation
    - Statistical hypothesis testing.


Population
Inference
Analyzed Data
Sample
Numerical data

Statistical Estimation

Definition: The assignment of value(s) to a population parameter based on a value of the corresponding sample statistic is called ***estimation*****.**

Estimate and Estimator

The value(s) assigned to a population parameter based on the value of a sample statistic is called an **estimate**. The sample statistic used to estimate a population parameter is called an **estimator**.

Example 1**:**

An example of population parameters and their corresponding estimator;

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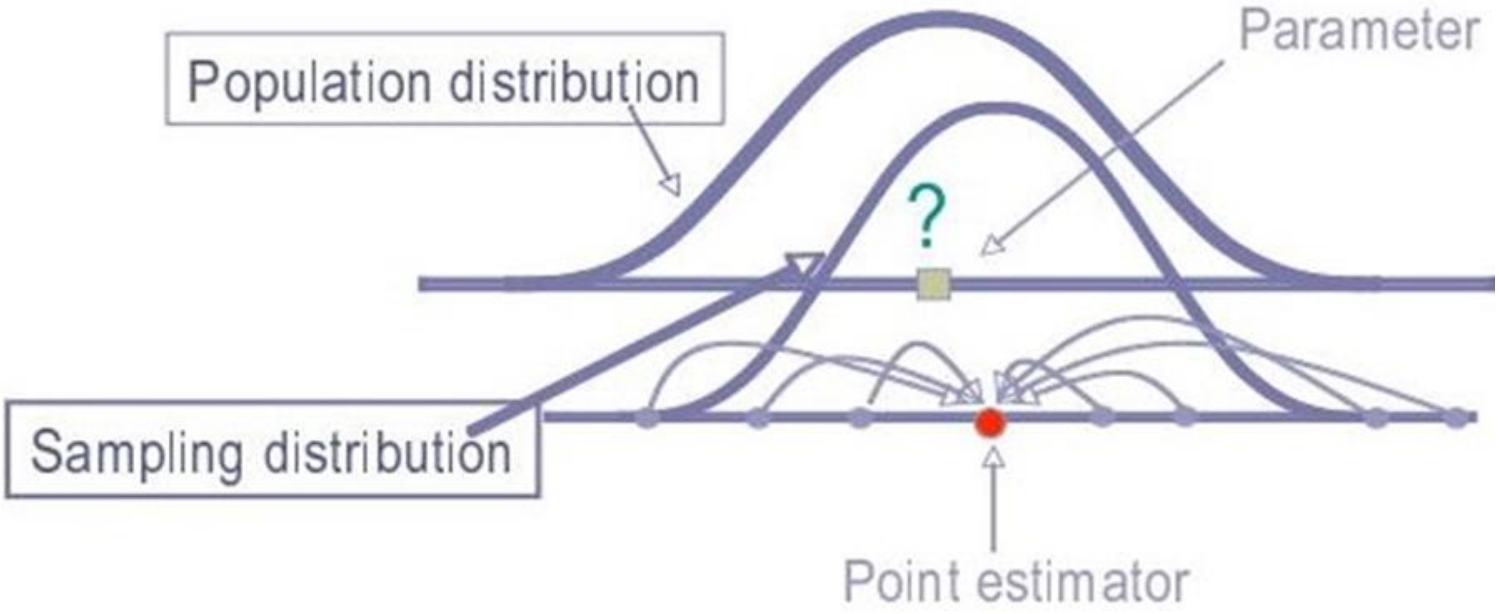
Parameter	Estimator
Population Mean, μ	$\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}$
Population variance, σ^2	$s^2 = \frac{(x_i - \bar{x})^2}{n-1}$
Population S.D, σ	$s = \sqrt{\frac{(x_i - \bar{x})^2}{n-1}}$
Population proportion, p	$\hat{p} = \frac{x}{n}$

- There are two types estimation.

- **Point Estimation**

- It is a procedure that results in a single value as an estimate for a parameter.



Interval estimation

- It is the procedure that results in the interval of values as an estimate for a parameter, which is interval that contains the likely values of a parameter.
- It deals with identifying the upper and lower limits of a parameter. The limits by themselves are random variable.

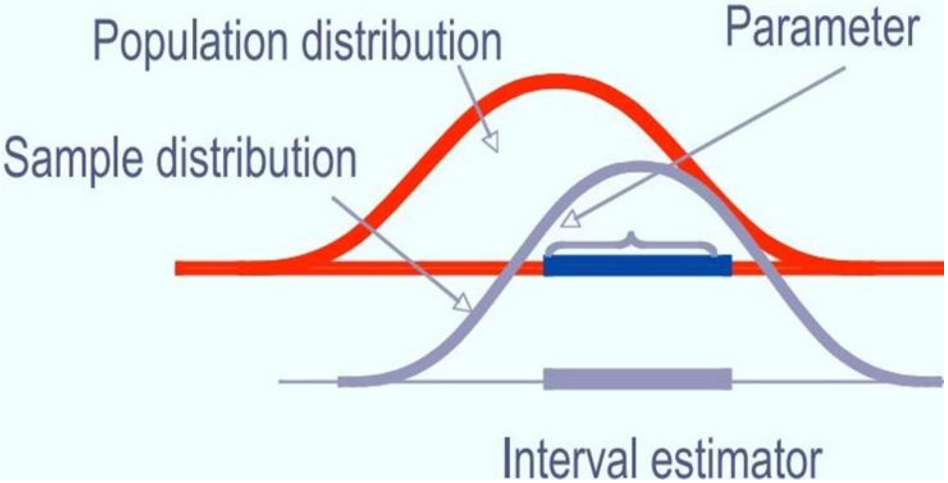


Figure 8.2: interval estimator for unknown population parameter

Definition of terms

- **Confidence Interval**: An interval estimate with a specific level of confidence.
- **Confidence Level**: The percent of the time the true value will lie in the interval estimate given.
- **Consistent Estimator**: An estimator which gets closer to the value of the parameter as the sample size increases.
- **Degrees of Freedom**: The number of data values which are allowed to vary once a statistic has been determined.

- **Estimator:** A sample statistic which is used to estimate a population parameter. It must be unbiased, consistent, and relatively efficient.
- **Estimate:** Is the different possible values which an estimator can assumes.
- **Interval Estimate:** A range of values used to estimate a parameter.
- **Point Estimate:** A single value used to estimate a parameter.
- **Relatively Efficient Estimator:** The estimator for a parameter with the smallest variance.
- **Unbiased Estimator:** An estimator whose expected value is the value of the parameter being estimated.

❖ Properties of best estimator

- An estimator or statistic is considered to be **best estimator** if it satisfies the following properties.

1. Unbiasedness

- An estimator is said to be unbiased if its expected value is equal to its parameter value.

i.e. E(estimator) = Parameter.

For Instance, the sample mean ($\bar{x}$) is an unbiased estimator of the population mean μ.

Example

A population of five units has the observations 7, 6, 8, 4 and 10. Random samples of size 3 are drawn, and the possible number of samples is $\binom{5}{3} = 10$ without replacement, as given below.

Sample number	**Samples**	**Means**
1	7,6,8	$\bar{x}_1 = 7$
2	7,6,4	$\bar{x}_2 = {}^{17}/_3$
3	7,6,10	$\bar{x}_3 = {}^{23}/_3$
4	7,8,4,	$\bar{x}_4 = {}^{19}/_3$
5	7,8,10	$\bar{x}_5 = {}^{25}/_3$
6	7,4,10	$\bar{x}_6 = 7$
7	6,8,4	$\bar{x}_7 = 6$
8	6,8,10	$\bar{x}_8 = 8$
9	6,4,10	$\bar{x}_9 = {}^{20}/_3$
10	8,4,10	$\bar{x}_{10} = {}^{22}/_3$
	Mean	$\boldsymbol{\bar{x}} = \mathbf{7}$

$$Population\ Mean = \mu = \frac{4+6+7+8+10}{5} = 7 = E[\boldsymbol{\bar{x}}]$$

$$\Rightarrow thesample\ mean\ \boldsymbol{\bar{x}}\ is\ unbiased\ estimatore\ of\ population\ mean\ \mu$$

2. Consistency

- A statistic (estimator) is said to be consistent estimator of the population parameter if it is more and more closer to the value of the parameter as the sample size increases.

- For any distribution, sample mean $\bar{x}$ is a consistent estimator of the population mean, μ

3. Efficiency

- An estimator is said to be efficient if its value remains the same or stable as the sample size changes. This property is called **sampling stability**.

Example, Sample mean ($\bar{x}$) has good sampling stability. That is why it is most widely used average in business data.

4. Sufficiency

A statistic is said to be a sufficient estimator of the parameter if it contains all the information in the sample regarding the parameter. In other words, a sufficient statistic utilizes all the information that a given sample can furnish about the parameter.

Example: Sample mean is the most sufficient estimator because it takes in to account all the information in the population.

I. Point and Interval estimation of the population mean: μ

Point Estimation about population mean μ:

- Another term for *statistic* is *point estimate*.
- For instance, sum of x_i over n is the point estimator used to compute the estimate of the population mean μ. That is $\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}$, is a point estimator of the population mean.

II. Confidence interval estimation of the population mean μ.

- The interval estimate of a confidence interval is defined by;

the sample statistic $\pm$ margin of error.

- A confidence interval consists of three parts.
  - A confidence level.
  - A statistic.
  - A margin of error.

A confidence level

The **confidence level** is the *probability* that the value of the parameter falls within the range specified by the confidence interval surrounding the statistic.

A statistic

A statistic means here, it means that any point estimator of a population parameter. *For example* $\bar{x}$.

Margin of Error

In a confidence interval, the range of values above and below the sample statistic is called the **margin of error**.

$$ME = Z_{\alpha/2}\sigma_{\bar{x}} = Z_{\alpha/2}\sqrt{\frac{\sigma^2}{n}} = Z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \text{ , since } \sigma^2_{\bar{x}} = \frac{\sigma^2}{n}$$

To gain further insight into μ, we surround the point estimate with a **margin of error**:

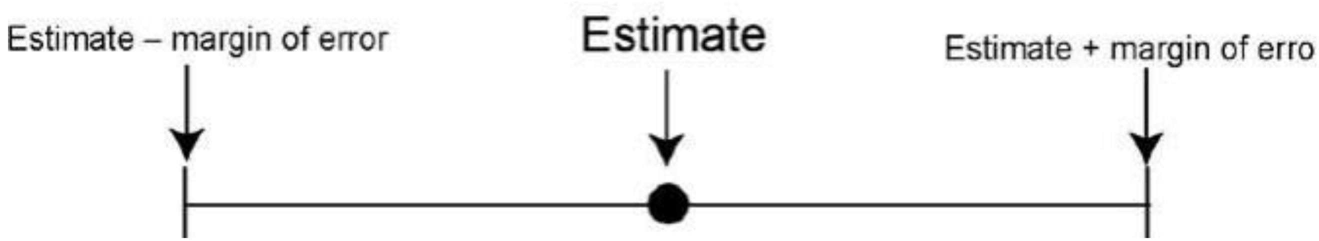


Figure 8.3: margin of an error

A 100 (1-α) % CI for μ is given by:

$$\bar{x} \pm ME$$

$$or\ \bar{x} \pm Z\alpha/_2 \frac{\sigma}{\sqrt{n}} = \bar{x} - Z\alpha/_2 \frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{x} + Z\alpha/_2 \frac{\sigma}{\sqrt{n}}$$

Table 8.1: The value of $Z_{\alpha/2}$ for different level of significance (α) and different confidence level

100 (1−α) %	***α***	**$Z_{\alpha/2}$**
90%	.10	1.64
95%	.05	1.96
99%	.01	2.58

❖ Confidence Interval Estimation for population mean (μ)

Case I: *Sampling from a normally distributed population with known variance σ^2 (for large or small sample)*

- It is equivalent to say a "Confidence Interval Estimation" when the sample data is *normally distributed.*

- Recall that z_α denotes the value of z for which the area under the standard normal curve to its right equal to α . Analogously $z_{\alpha/2}$ denotes the value of z for which the area to its right is $\alpha/2$, and $-z_{\alpha/2}$ denotes the value of z for which the area to its left is $\alpha/2$.
- Consider the following figure,

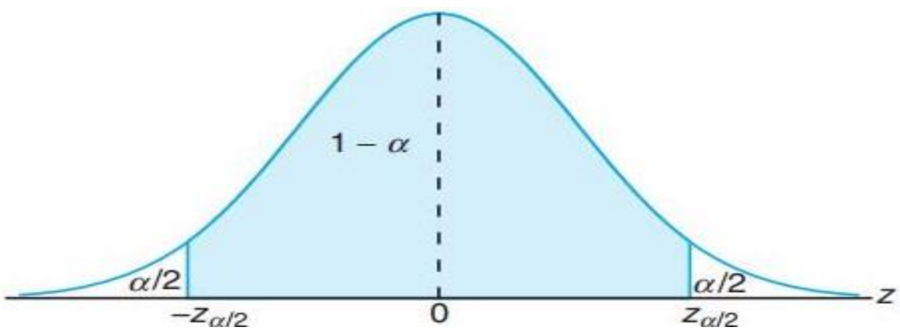


Figure 8.4: P (-zα/2 < Z < zα/2) = 1 - α.

From the above figure we have:

$$P\left[-z_{\alpha/2} < Z < z_{\alpha/2}\right] = 1-\alpha$$

$$\Rightarrow P\left[-z_{\alpha/2} < \frac{\bar{x}-\mu}{\sigma/\sqrt{n}} < z_{\alpha/2}\right] = 1-\alpha$$

$$\Rightarrow P\left[-z_{\alpha/2}\,{}^{\sigma}/_{\sqrt{n}} < \bar{x}-\mu < z_{\alpha/2}\,{}^{\sigma}/_{\sqrt{n}}\right] = 1-\alpha$$

$$\Rightarrow P\left[\bar{x}-z_{\alpha/2}\,{}^{\sigma}/_{\sqrt{n}} < \mu < \bar{x}+z_{\alpha/2}\,{}^{\sigma}/_{\sqrt{n}}\right] = 1-\alpha$$

Thus a 100 $(1-\alpha)$% confidence interval for the population mean μ is given by:

$$\bar{x} \pm z_{\alpha/2}\,{}^{\sigma}/_{\sqrt{n}} \;or\; \bar{x} \pm ME \text{----------------} (8.1)$$

Example

The average zinc concentration recovered from a sample of measurements taken in 36 different locations in a river is found to be 2.6 grams per milliliter. Find the 95% and 99% confidence intervals for the mean zinc concentration in the river. Assume that the population is normally distributed with standard deviation is 0.3 gram per milliliter.

<u>Soln.</u>

The point estimate of μ is $\bar{x}$ = 2.6. The z-value leaving an area of 0.025 to the right, and therefore an area of 0.975 to the left, is $z_{0.025}$ = 1.96 (from table). Hence, the 95% confidence interval is

-
- $[\bar{x} - z_{\alpha/2}\, {}^{\sigma}/_{\sqrt{n}} < \mu < \bar{x} + z_{\alpha/2}\, {}^{\sigma}/_{\sqrt{n}}]$

  - $= [2.6 - 1.96\, {}^{(.3)}/_{\sqrt{36}} < \mu < 2.6 + 1.96\, {}^{(.3)}/_{\sqrt{36}}]$

    - $[2.50 < \mu < 2.70]$

  - $\Rightarrow$ 95%CI for population mean μ will be $[2.50, 2.70]$

Case II: *Large sample confidence interval for the population mean μ.*

If the parent distribution of the population from which the sample is drawn is unknown or is not normal but the sample size is large; i.e., the sample size is ≥ 30, then by applying the concept of "**Central Limit Theorem**", a 100(1 −α)% confidence interval for the population mean μ will be given by:

$$\bar{x} \pm z_{\alpha/2}\, {}^{\sigma}\!/_{\sqrt{n}} \ if\ \sigma\ is\ known\ and ---------(8.2)$$

$$\bar{x} \pm z_{\alpha/2}\, {}^{s}\!/_{\sqrt{n}} \ if\ \sigma\ is\ unknown ----------(8.3)$$

Example

A forester wishes to estimate the average number of "count trees" per acre (trees larger than a specified size) on a 2,000-acre plantation. She can then use this information to determine the total timber volume for trees in the plantation. A random sample of $n = 50$ one-acre plots is selected and examined. The average (mean) number of count trees per acre is found to be 27.3, with a standard deviation of 12.1. Use the described information and construct 99% confidence interval estimation for μ, (the mean number of count trees per acre for the entire plantation).

Soln.

Given: n = 50, sample mean = 27.3, s = 12.1 confidence level = 99%

We use the general confidence interval with confidence coefficient equal to .99 and a $z_{\alpha/2}$-value equal to 2.58 (see Table 8.1). Substituting into the formula $\bar{x} \pm 2.58\,{}^{\sigma}/_{\sqrt{n}}$ and replacing σ with s, we have;

$$\bar{x} \pm 2.58\,{}^{\sigma}/_{\sqrt{n}} = 27.3 \pm 2.58\,{}^{12.1}/_{\sqrt{50}}$$

This corresponds to the confidence interval 27.3 $\pm$ 4.41 that is, the interval from 22.89 to 31.71.

∴ A 100 (1- α) % CI = 99%CI for a population mean μ=[22.89 , 31.71]

Thus, we are 99% sure that the average number of count trees per acre is between 22.89 and 31.71.

Case III. *Small Sample Confidence interval for the Population Mean: Sampling from a normally distributed population with σ^2 unknown and n < 30.*

Suppose that the population of interest has a normal distribution with unknown mean $\boldsymbol{\mu}$ and unknown variance $\boldsymbol{\sigma^2}$**.** Assume that a random sample of size n, say $x_1, x_2, \ldots x_n$ is available, and let $\bar{\boldsymbol{x}}$ and $\boldsymbol{s^2}$ be the sample mean and variance, respectively. We wish to construct a two-sided CI on $\boldsymbol{\mu}$. If the variance $\boldsymbol{\sigma^2}$ is known, we know that $z = {}^{(\bar{x}-\mu)}/_{({}^{\sigma}/_{\sqrt{n}})}$ has a standard normal distribution. When $\boldsymbol{\sigma^2}$ is unknown, a logical procedure is to replace $\boldsymbol{\sigma}$ with the sample standard deviation s. The random variable Z now becomes

$$t = {}^{(\bar{x}-\mu)}/_{({}^{s}/_{\sqrt{n}})}.$$

The *t*-Distribution

Let $x_1, x_2,\ldots, x_n$ be a random sample from a normal distribution with unknown mean $\boldsymbol{\mu}$ and unknown variance $\boldsymbol{\sigma^2}$. The random variable

$$t = \frac{(\bar{x} - \mu)}{\left({}^{s}\!/\!{}_{\sqrt{n}}\right)} -------------------(8.4)$$

has a *t* distribution with *n* - 1 degrees of freedom.

When:

1. the population from which the sample is selected is (approximately normally distributed),

2. the sample size is small (that is, $n < 30$), and

3. the population standard deviation σ is not known, the normal distribution is replaced by the student *t*- distribution to construct confidence intervals about μ.

The t Confidence Interval on μ.

It is easy to find a 100 $(1-\alpha)$ % confidence interval on the mean of a normal distribution with unknown variance by proceeding essentially as we did in in the above section (Case I). We know that the distribution of $t = \frac{\bar{x}-\mu_{\bar{x}}}{s/\sqrt{n}}$ is t with $n-1$ degrees of freedom. Letting $\left(t_{\frac{\alpha}{2}}, n-1\right)$be the upper 100 $\frac{\alpha}{2}$ percentage point of the t distribution with $n-1$ degrees of freedom, we may write:

$$P\left(-t_{\frac{\alpha}{2}}, n-1 \leq t \leq t_{\frac{\alpha}{2}}, n-1\right) = 1-\alpha$$

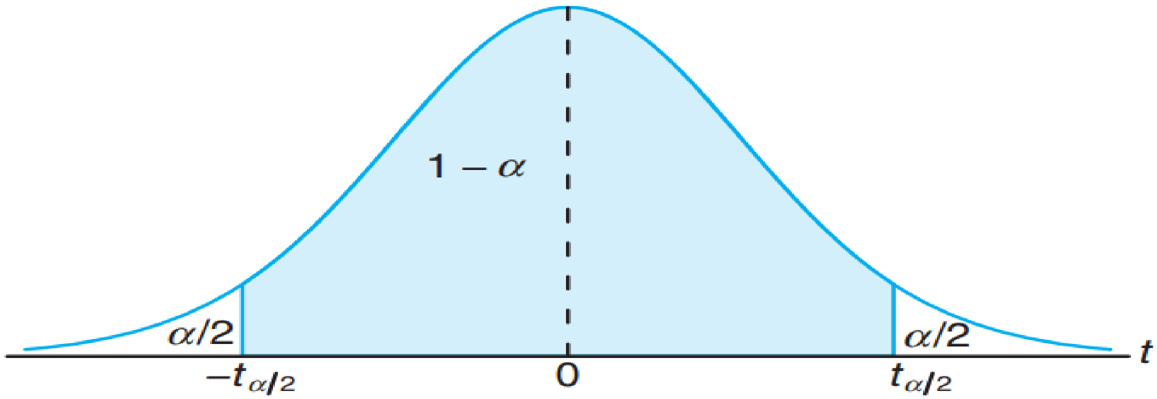


Figure 8.5: P $(-t_{\alpha/2} < t < t_{\alpha/2}) = 1 - \alpha$.

$$P\left(-t_{\frac{\alpha}{2}}, n-1 \leq \frac{\bar{x}-\mu_{\bar{x}}}{s/\sqrt{n}} \leq t_{\frac{\alpha}{2}}, n-1\right) = 1-\alpha$$

Rearranging this last equation yields

$$P\left(\bar{x} - t_{\frac{\alpha}{2}}, n-1\frac{s}{\sqrt{n}} \leq \mu \leq \bar{x} + t_{\frac{\alpha}{2}}, n-1\frac{s}{\sqrt{n}}\right) = 1-\alpha \text{-------------} (8.5)$$

If $\bar{x}$ and s are the mean and standard deviation of a random sample from a normal distribution with unknown variance $\boldsymbol{\sigma^2}$, a **100 $(1-\alpha)$ % confidence interval on $\boldsymbol{\mu}$** is given by:

$$\bar{x} - t_{\frac{\alpha}{2}}, (n-1)\frac{s}{\sqrt{n}} \le \mu \le \bar{x} + t_{\frac{\alpha}{2}}, (n-1)\frac{s}{\sqrt{n}} ----(8.6)$$

where $t_{\frac{\alpha}{2}}, n-1$ is the upper $100\frac{\alpha}{2}$ percentage point of the t distribution with $n-1$ degrees of freedom.

Example

The contents of seven similar containers of sulfuric acid are 9.8, 10.2, 10.4, 9.8, 10.0, 10.2, and 9.6 liters. Find a 95% confidence interval for the mean contents of all such containers, assuming an approximately normal distribution.

<u>Soln.</u>

The sample mean and standard deviation for the given data are $\bar{x} = 10.0$ and $s = 0.283$

$$100(1-\alpha)\%CI = 95\%$$

$$\Rightarrow \alpha = 0.05$$

$$\Rightarrow \frac{\alpha}{2} = 0.025$$

We use student's t-distribution since we are given small sample size and population variance is unknown.

$$\Rightarrow t\alpha_{/2}, n-1 = t_{0.025}, 6$$

$$= 2.447 \; from\; t - distribution\; table$$

Hence,

95% confidence interval for μ is

$$10.0 - (2.447)\left(\frac{0.283}{\sqrt{7}}\right) < \mu < 10.0 + (2.447)\left(\frac{0.283}{\sqrt{7}}\right).$$

which reduces to $9.74 < \mu < 10.26$.

Problems:

1. From a normal sample of size 25 a mean of 32 was found .Given that the population standard deviation is 4.2. Find A 95% confidence interval for the population mean.

2. A drug company is testing a new drug which is supposed to reduce blood pressure. From the six people who are used as subjects, it is found that the average drop in blood pressure is 2.28 points, with a standard deviation of 0.95 points. What is the 95% confidence interval for the mean change in pressure?

Answer

1. Use case I

$$\bar{X} = 32, \qquad \sigma = 4.2, \qquad 1-\alpha = 0.95 \Rightarrow \alpha = 0.05, \alpha/2 = 0.025$$

$$\Rightarrow Z_{\alpha/2} = 1.96 \quad \textit{from} \quad \textit{table.}$$

$$\Rightarrow \textit{The required interval will be } \bar{X} \pm Z_{\alpha/2}\, \sigma/\sqrt{n}$$

$$= 32 \pm 1.96 * 4.2/\sqrt{25}$$

$$= 32 \pm 1.65$$

$$= \underline{\underline{(30.35,\ 33.65)}}$$

2. Use case III

$$\bar{X} = 2.28, \quad S = 0.95, \quad 1-\alpha = 0.95 \Rightarrow \alpha = 0.05, \ \alpha/2 = 0.025$$

$$\Rightarrow t_{\alpha/2} = 2.571 \ \ with\ df = 5 \quad from\ table.$$

$$\Rightarrow The \ \ required\ interval \ \ will\ be\ \bar{X} \pm t_{\alpha/2}\, S/\sqrt{n}$$

$$= 2.28 \pm 2.571 * 0.95/\sqrt{6}$$

$$= 2.28 \pm 1.008$$

$$= \underline{\underline{(1.28, \ 3.28)}}$$

Hypothesis testing

- **Hypothesis**; is statement about a population developed for the purpose of testing.
- **Hypothesis testing:** is a procedure based on sample evidence and probability to determine theory whether the hypothesis is a reasonable statement.
- **Test statistic**: is a value determined from a sample information, used to determine whether to reject the null hypothesis or not.
- **Decision rule**: is a statement of the condition under which the null hypothesis is rejected and the conditions under which it is not rejected.
- **Critical region:** A region where the null hypothesis is rejected

There are two types of hypothesis:

- **Null hypothesis**:

  It is the hypothesis to be tested.

  It is the hypothesis of equality or the hypothesis of no difference.

  It is denoted by H_0.

- **Alternative hypothesis**:

  It is the hypothesis available when the null hypothesis has to be rejected.

  It is the hypothesis of difference.

  It is denoted by H_1 or H_a.

- **Types of errors:**

- Testing hypothesis is based on sample data which may involve sampling errors.

- The following table gives a summary of possible results of any hypothesis test:

Null hypothesis	Decision	
	Reject H_0	Don't reject H_0
H_0 is true	**Type I Error**	Right Decision
H_0 is false	Right Decision	**Type II Error**

Type I error Rejecting the null hypothesis when it is true. It is denoted by α

Type II error: Accepting the null hypothesis when it is false. It is denoted by β

There are three different methods which used to test hypotheses. These are:

1. The Critical Value Method (**The Traditional Approach**)
2. The P-value Method (**The Modern Approach**)
3. The Confidence Interval Method

❖ **Type of Tests**

Based on the form of the null and alternative hypotheses, we have two types of tests:

- **One-sided**(one tailed) tests and

Where the alternative hypotheses are inequalities type are called one-sided (one-tailed) tests. μ_o is the hypothesized (assumed) mean.

$$\textbf{\textit{Form:}}\ a)\ \begin{cases} H_0\colon \mu = \mu_0 \\ H_0\colon \mu < \mu_0 \end{cases} or \begin{cases} H_0\colon \mu \geq \mu_0 \\ H_0\colon \mu < \mu_0 \end{cases} \Rightarrow \boldsymbol{Left\ tailed\ test}$$

$$b)\ \begin{cases} H_0\colon \mu = \mu_0 \\ H_0\colon \mu > \mu_0 \end{cases} or \begin{cases} H_0\colon \mu \leq \mu_0 \\ H_0\colon \mu > \mu_0 \end{cases} \Rightarrow \boldsymbol{Right\ tailed\ test}$$

- **Two-sided** (two tailed) test.

In which the critical region (rejection region) includes both large and small values of the test statistic.

$$\textbf{\textit{Form:}}\ \begin{cases} H_0\colon \mu = \mu_0 \\ H_0\colon \mu \neq \mu_0 \end{cases} \Rightarrow \boldsymbol{Two\ tailed\ test}$$

❖ General steps in hypothesis testing:

- ✓ Use of the following sequence of steps in applying hypotheses-testing problem is recommended.

Step 1: State the hypotheses and identify the claim.

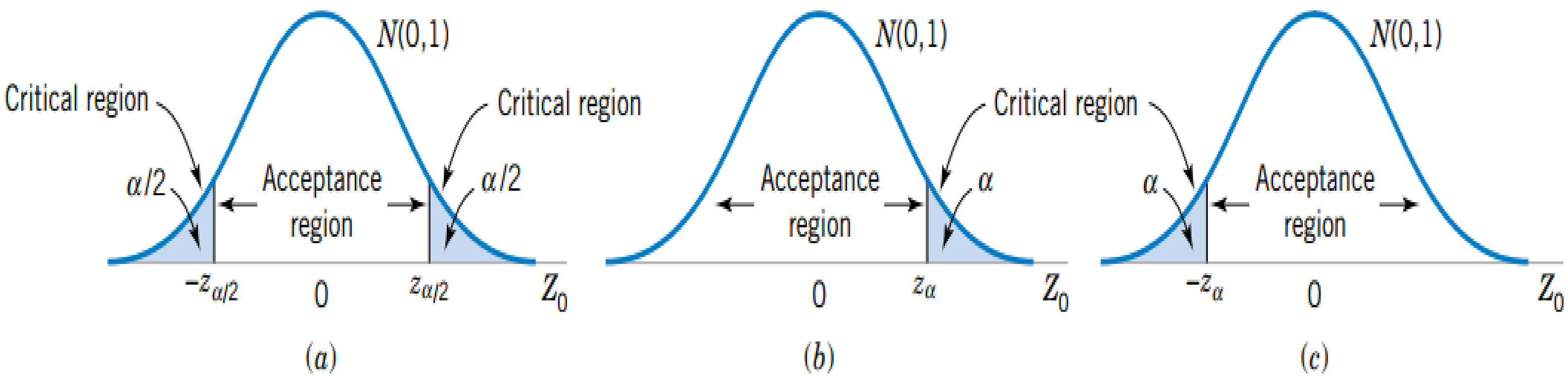
Step 2: Find the critical value(s).

Step 3: Compute the test value.

Step 4: Make the decision to reject or not reject the null hypothesis.

Step 5: Summarize the results.

☞ **Acceptance and Rejection Region**



❖ **One sample Z-test Rejection Rule**

✓ Null hypothesis $H_0 \; : \mu = \mu_0$

✓ Test statistic $Z_{call} = Z_0 = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}}$

Alternative Hypothesis	**Rejection Criteria** Reject H_0 if:
$H_a \; : \mu \neq \mu_0$	$Z_0 > Z_{\alpha/2} \; or \; Z_0 < -Z_{\alpha/2}$
$H_a \; : \mu > \mu_0$	$Z > Z_\alpha$
$H_a \; : \mu < \mu_0$	$Z_0 < -Z_\alpha$

Hypothesis testing about one population mean :

- Suppose the assumed or hypothesized value of population mean is denoted by μ_0, then one can formulate two sided (1) and one sided (2 and 3) hypothesis as follows:

$$H_0 : \mu = \mu_0 \quad vs \quad H_1 : \mu \neq \mu_0$$

$$H_0 : \mu = \mu_0 \quad vs \quad H_1 : \mu > \mu_0$$

$$H_0 : \mu = \mu_0 \quad vs \quad H_1 : \mu < \mu_0$$

Case I: When sampling is from a normal distribution with population variance is known

- ✓ The relevant test statistic is $$Z_{cal} = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}$$

Example 18

A producer of an electric bulbs claims that the average life length of its product is 1800 hrs. A sample of 400 bulbs gave mean life length of 1780 hrs. Suppose it's known that life lengths are normally distributed with standard deviation of 200 hrs. Would you support the producers claim at 5% level of significance? Test also that the average life length of the bulbs is less than the producer's claims at 1% level of significance.

Soln.

Step-1: formulate the hypothesis

$$H_0: \mu = \mu_0$$
$$H_a: \mu \neq \mu_0$$

Step-2: identify the significance level: $\alpha = 0.05$

Step-3: the parent Population is normal, with $\sigma = 200$ $(known)$, $n = 400$ Then the test statistic will be:

$$Z_{Call} = \left|\frac{\bar{x} - \mu}{\sigma/\sqrt{n}}\right|$$

Step 4: the critical region is

$$\left|\frac{\bar{x} - \mu}{\sigma/\sqrt{n}}\right| > Z_{\alpha/2}$$

i.e. reject H_0 if $\left|\frac{\bar{x}-\mu}{\sigma/\sqrt{n}}\right| > Z_{\alpha/2}$

Step 5: Evaluate the critical region and compare with tabulated value

$$Z_{call} = \left|\frac{1780-1800}{200/\sqrt{400}}\right| = |-2| = 2 \text{ and}$$

$$Z_{tab} = Z_{\alpha/2} = Z_{0.05/2} = Z_{0.025} = 1.96\ (From\ table\ 8.1)$$

Step-6: decide whether or not H_0 should be rejected

Since $Z_{call} > Z_{tab}$, *reject Ho* that is the claim of the producer is not correct.

Example

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<u>Soln.</u>

Step-1: formulate the hypothesis

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$$H_a: \mu \neq \mu_0$$

Step-2: identify the significance level: $\alpha = 0.05$

Step-3: the parent Population is normal, with $\sigma = 200$ (*known*), $n = 400$ Then the test statistic will be:

$$Z_{Call} = \left| \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} \right|$$

Step 4: the critical region is

$$\left| \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} \right| > Z_{\alpha/2}$$

i.e. reject H_0 *if* $\left| \frac{\bar{x}-\mu}{\sigma/\sqrt{n}} \right| > Z_{\alpha/2}$

Step 5: Evaluate the critical region and compare with tabulated value

$$Z_{call} = \left| \frac{1780 - 1800}{200/\sqrt{400}} \right| = |-2| = 2 \text{ and}$$

$$Z_{tab} = Z_{\alpha/2} = Z_{0.05/2} = Z_{0.025} = 1.96 \ (\text{From table } 8.1)$$

Step-6: decide whether or not H_0 should be rejected

Since $Z_{call} > Z_{tab}$, *reject Ho* that is the claim of the producer is not correct.

Example

A company has a computer system that can process at most 1200 bills per hour. A new system is tested which processes an average of 1260 bills per hour with a standard deviation of 215 bills in a sample of 40 hours. Test if the new system is significantly better than old one at the 5% level of significance.

Soln.

1. H_0: $\mu \leq 1200$

   H_a: $\mu > 1200$
2. $\alpha = 0.05$.
3. The population is non- normal, σ is unknown

   But, n = 40 ⇒ large sample size, $s = 215, and\ \bar{x} = 1260$

   Then by central limit theorem, the test statistic will be:

   $$Z = \frac{\bar{x}-\mu}{s/\sqrt{n}} \Rightarrow Z_{call} = \frac{1260-1200}{215/\sqrt{40}} = 1.76$$
4. Critical region: $Z_{call} > Z_{tab}$, where $Z_{tab} = Z_{\alpha} = Z_{0.05} = 1.645\ from\ table$
5. Decision Making and conclusion:

Since $Z_{call} > Z_{tab}$, reject H_0 that is we conclude that the new system represents an improvement over the old system at the $\alpha = 0.05$ level of significance.

Case II: When sampling from a normal distribution with unknown population variance and small sample size

- ✓ The relevant test statistic is $t_{cal} = \dfrac{\bar{X} - \mu_0}{S/\sqrt{n}}$

- **Two tailed test**

In this case we would use the t percentage points $-t_{\alpha/2},(n-1)$ and $t_{\alpha/2},(n-1)$ as the boundaries of the critical region so that we would reject $H_0: \mu = \mu_0$ if

$$t_0 > t_{\alpha/2},(n-1) \; or \; if \;\; t_0 < -t_{\alpha/2},(n-1)$$

Example

During a recent year the average cost of making a movie was \$54.8 million. This year, a random sample of 15 recent action movies had an average production cost of \$62.3 million with a variance of \$90.25 million. At the 0.05 level of significance, can it be concluded that it costs more than average to produce an action movie?

Source: New York Times Almanac

<u>Soln.</u>

Step 1: State the hypotheses and identify the claim.

H_0: $\mu = 54.8$ and

H_a: $\mu > 54.8$ (**Claim!**)

Step 2: Find the critical value(s).

The critical value is $t_{\alpha},(n-1)=t_{\alpha},0.05,(14) = 1.761 \rightarrow$ from Table

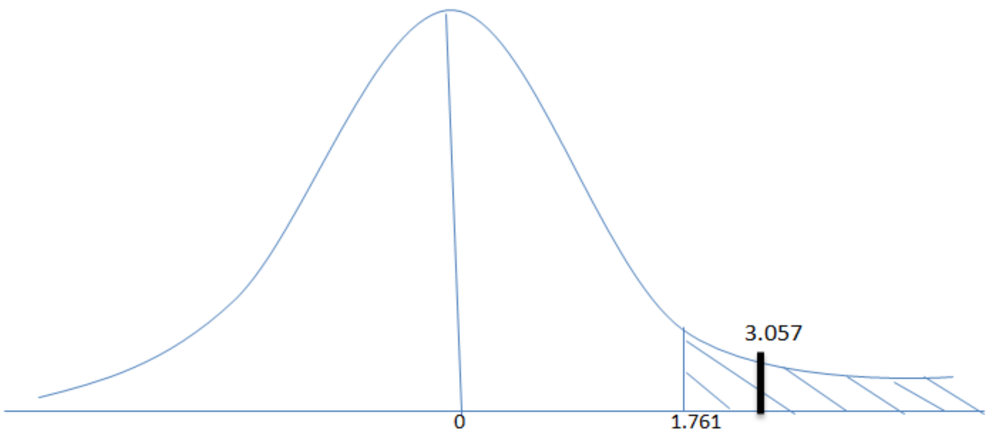
- **Step 3:** Compute the test value.

  Our test statistic here is student t-statistic which is equals to:

$$t_{call} = \frac{\bar{x} - \mu_0}{s/\sqrt{n}} = \frac{62.3 - 54.8}{9.5/\sqrt{15}} = \frac{7.5}{2.453} = 3.057$$

- **Step 4:** Make the decision to reject or not reject the null hypothesis.

  Since 3.057 falls in the critical region, see the **figure** bellow then we reject the null hypothesis.



- **Step 5:** Summarize the results.

  Based on the given α level of significance and the sample size, there is an enough evidence to support the claim that the production costs will more than the average cost.

Example

A medical investigation claims that the average number of infections per week at a hospital in southwestern Pennsylvania is 16.3. A random sample of 10 weeks had a mean number of 17.7 infections. The sample standard deviation is 1.8. Is there enough evidence to reject the investigator's claim at $\alpha = 0.05$? Assume the variable is normally distributed.

Source: Based on information obtained from Pennsylvania Health Care Cost Containment Council.

SOLUTION

Step 1 H_0: $\mu = 16.3$ (claim) and H_1: $\mu \neq 16.3$.

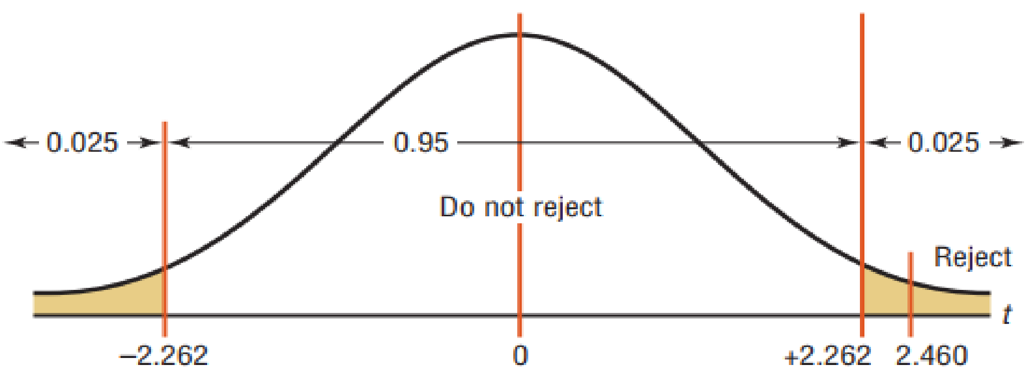
Step 2 The critical values are $+2.262$ and -2.262 for $\alpha = 0.05$ and d.f. $= 9$.

Step 3 The test value is

$$t = \frac{\overline{X} - \mu}{s/\sqrt{n}} = \frac{17.7 - 16.3}{1.8/\sqrt{10}} = 2.460$$

Step 4 Reject the null hypothesis since $2.460 > 2.262$. See Figure 8–20.

FIGURE 8–20
Summary of the *t* Test of Example 8–12



Step 5 There is enough evidence to reject the claim that the average number of infections is 16.3.

Example

Test the hypotheses that the average height content of containers of certain lubricant is 10 liters if the contents of a random sample of 10 containers are 10.2, 9.7, 10.1, 10.3, 10.1, 9.8, 9.9, 10.4, 10.3, and 9.8 liters. Use the 0.01 level of significance and assume that the distribution of contents is normal.

Given:

$$\bar{X} = 10.06, \; S = 0.25$$

$$t_{0.005}(9) = 3.2498$$

$$\mu = Population\ mean.\ \mu_0 = 10$$

$$H_0 : \mu = 10 \quad vs \quad H_1 : \mu \neq 10$$

$$\alpha = 0.01 (given)$$

Solution:

t- Statistic is appropriate because population variance is not known and the sample size is also small.

- Here we have two critical regions since we have two tailed hypothesis

$$The\ critical\ region\ is\ |t_{cal}| > t_{0.005}(9) = 3.2498$$

$$\Rightarrow (-3.2498,\ 3.2498)\ is\ acceptance\ region.$$

$$\Rightarrow t_{cal} = \frac{\bar{X} - \mu_0}{S/\sqrt{n}} = \frac{10.06 - 10}{0.25/\sqrt{10}} = 0.76$$

Decision: Don’t reject H_0 , since t_{cal} is in the acceptance region.

Conclusion

At 1% level of significance, we have no evidence to say that the average height content of containers of the given lubricant is different from 10 litters, based on the given sample data.

Case III: When sampling is from a non- normally distributed population or a population whose functional form is unknown but **large** sample size.

- If a sample size is large one can perform a test hypothesis about the mean by using:

- ***Test statistic is***

$$Z_{cal} = \frac{\bar{X} - \mu_0}{\sigma / \sqrt{n}}, \quad if\ \sigma^2\ is\ known.$$

$$= \frac{\bar{X} - \mu_0}{S / \sqrt{n}}, \quad if\ \sigma^2\ is\ unknown.$$

- The decision rule is similar to case I.

Example

Medical researchers have developed a new artificial heart constructed primarily of titanium and plastic. The heart will last and operate almost indefinitely once it is implanted in the patient's body, but the battery pack needs to be recharged about every four hours. A random sample of 50 battery packs is selected and subjected to a life test. The average life of these batteries is 4.05 hours with standard deviation 0.2 hour. Is there an evidence to support the claims that mean battery life is beneath 4 hours? Use $\alpha = 0.1$

Soln.

Step 1: State the hypotheses and identify the claim.

$$H_0: \mu = 4 hours \text{ and}$$
$$H_a: \mu < 4\ hours (Claim)$$

Step 2: Find the critical value(s).

Since $\alpha = 0.10$ and the test is a left-tailed test, the critical value is $-Z_{\alpha} = -Z_{0.10} = -1.28. \rightarrow \boldsymbol{from\,Table}$

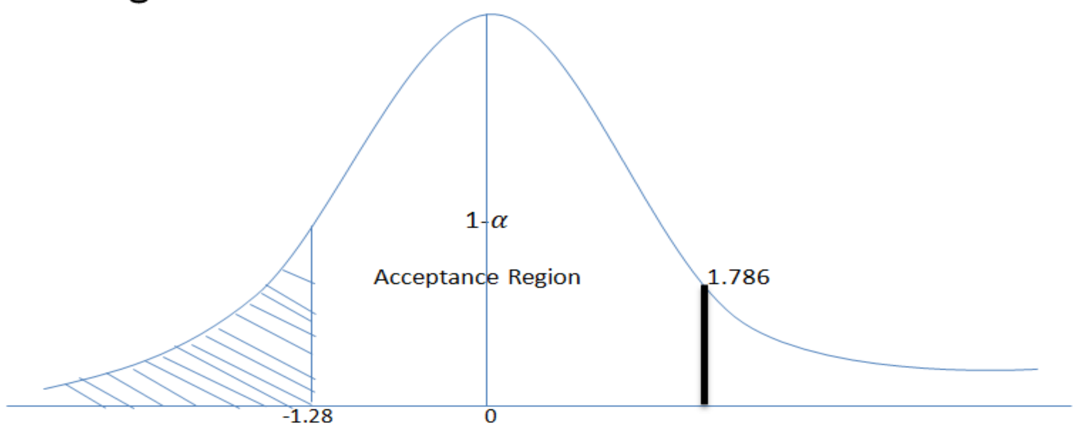
Step 3: Compute the test value.

Our test statistic here Z statistic which is equals to:

$$Z_{call} = Z_0 = \frac{\dot{x}-\mu_0}{s/\sqrt{n}} = \frac{4.05-4.00}{0.2/\sqrt{50}} = \frac{0.05}{0.028} = 1.786$$

Step 4: Make the decision to reject or not reject the null hypothesis.

Since the test value, 1.786, falls in a non-critical region, the decision is to not reject the null hypothesis. See the figure bellow.



Step 5: Summarize the results.

Based on the given α level of significance and the sample taken, there is an enough evidence to not support the claims that that the mean battery life is bellow 4 hours.

THANK YOU FOR ALL CONTACT HOUR!

Be Reader!!!