Limits

Definitions

Precise Definition : We say $\lim\limits_{x\to a} f(x) = L$ if for every $\varepsilon > 0$ there is a $\delta > 0$ such that whenever $0 < |x-a| < \delta$ then $|f(x) - L| < \varepsilon$.

"Working" Definition : We say $\lim\limits_{x\to a} f(x) = L$ if we can make $f(x)$ as close to L as we want by taking x sufficiently close to a (on either side of a) without letting $x = a$.

Right hand limit : $\lim\limits_{x\to a^+} f(x) = L$. This has the same definition as the limit except it requires $x > a$.

Left hand limit : $\lim\limits_{x\to a^-} f(x) = L$. This has the same definition as the limit except it requires $x < a$.

Limit at Infinity : We say $\lim\limits_{x\to\infty} f(x) = L$ if we can make $f(x)$ as close to L as we want by taking x large enough and positive.

There is a similar definition for $\lim\limits_{x\to-\infty} f(x) = L$ except we require x large and negative.

Infinite Limit : We say $\lim\limits_{x\to a} f(x) = \infty$ if we can make $f(x)$ arbitrarily large (and positive) by taking x sufficiently close to a (on either side of a) without letting $x = a$.

There is a similar definition for $\lim\limits_{x\to a} f(x) = -\infty$ except we make $f(x)$ arbitrarily large and negative.

Relationship between the limit and one-sided limits

$\lim\limits_{x\to a} f(x) = L \quad\Rightarrow\quad \lim\limits_{x\to a^+} f(x) = \lim\limits_{x\to a^-} f(x) = L$

$\lim\limits_{x\to a^+} f(x) = \lim\limits_{x\to a^-} f(x) = L \quad\Rightarrow\quad \lim\limits_{x\to a} f(x) = L$

$\lim\limits_{x\to a^+} f(x) \neq \lim\limits_{x\to a^-} f(x) \quad\Rightarrow\quad \lim\limits_{x\to a} f(x)$ Does Not Exist

Properties

Assume $\lim\limits_{x\to a} f(x)$ and $\lim\limits_{x\to a} g(x)$ both exist and c is any number then,

1. $\lim\limits_{x\to a} [cf(x)] = c\lim\limits_{x\to a} f(x)$
2. $\lim\limits_{x\to a} [f(x) \pm g(x)] = \lim\limits_{x\to a} f(x) \pm \lim\limits_{x\to a} g(x)$
3. $\lim\limits_{x\to a} [f(x)g(x)] = \lim\limits_{x\to a} f(x) \; \lim\limits_{x\to a} g(x)$
4. $\lim\limits_{x\to a} \left[\dfrac{f(x)}{g(x)}\right] = \dfrac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)}$ provided $\lim\limits_{x\to a} g(x) \neq 0$
5. $\lim\limits_{x\to a} [f(x)]^n = \left[\lim\limits_{x\to a} f(x)\right]^n$
6. $\lim\limits_{x\to a} \left[\sqrt[n]{f(x)}\right] = \sqrt[n]{\lim\limits_{x\to a} f(x)}$

Basic Limit Evaluations at $\pm\infty$

1. $\lim\limits_{x\to\infty} \mathbf{e}^x = \infty$ & $\lim\limits_{x\to-\infty} \mathbf{e}^x = 0$
2. $\lim\limits_{x\to\infty} \ln(x) = \infty$ & $\lim\limits_{x\to 0^+} \ln(x) = -\infty$
3. If $r > 0$ then $\lim\limits_{x\to\infty} \dfrac{b}{x^r} = 0$
4. If $r > 0$ and x^r is real for negative x then $\lim\limits_{x\to-\infty} \dfrac{b}{x^r} = 0$
5. n even : $\lim\limits_{x\to\pm\infty} x^n = \infty$
6. n odd : $\lim\limits_{x\to\infty} x^n = \infty$ & $\lim\limits_{x\to-\infty} x^n = -\infty$
7. n even : $\lim\limits_{x\to\pm\infty} a\,x^n + \cdots + b\,x + c = \operatorname{sgn}(a)\infty$
8. n odd : $\lim\limits_{x\to\infty} a\,x^n + \cdots + b\,x + c = \operatorname{sgn}(a)\infty$
9. n odd : $\lim\limits_{x\to-\infty} a\,x^n + \cdots + c\,x + d = -\operatorname{sgn}(a)\infty$

Note : $\operatorname{sgn}(a) = 1$ if $a > 0$ and $\operatorname{sgn}(a) = -1$ if $a < 0$.

Evaluation Techniques

Continuous Functions

If $f(x)$is continuous at a then $\lim\limits_{x\to a} f(x) = f(a)$

Continuous Functions and Composition

$f(x)$ is continuous at b and $\lim\limits_{x\to a} g(x) = b$ then

$\lim\limits_{x\to a} f(g(x)) = f\left(\lim\limits_{x\to a} g(x)\right) = f(b)$

Factor and Cancel

$$\lim_{x\to 2} \frac{x^2+4x-12}{x^2-2x} = \lim_{x\to 2} \frac{(x-2)(x+6)}{x(x-2)}$$

$$= \lim_{x\to 2} \frac{x+6}{x} = \frac{8}{2} = 4$$

Rationalize Numerator/Denominator

$$\lim_{x\to 9} \frac{3-\sqrt{x}}{x^2-81} = \lim_{x\to 9} \frac{3-\sqrt{x}}{x^2-81}\,\frac{3+\sqrt{x}}{3+\sqrt{x}}$$

$$= \lim_{x\to 9} \frac{9-x}{(x^2-81)(3+\sqrt{x})} = \lim_{x\to 9} \frac{-1}{(x+9)(3+\sqrt{x})}$$

$$= \frac{-1}{(18)(6)} = -\frac{1}{108}$$

Combine Rational Expressions

$$\lim_{h\to 0} \frac{1}{h}\left(\frac{1}{x+h} - \frac{1}{x}\right) = \lim_{h\to 0} \frac{1}{h}\left(\frac{x-(x+h)}{x(x+h)}\right)$$

$$= \lim_{h\to 0} \frac{1}{h}\left(\frac{-h}{x(x+h)}\right) = \lim_{h\to 0} \frac{-1}{x(x+h)} = -\frac{1}{x^2}$$

L'Hospital's/L'Hôpital's Rule

If $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{0}{0}$ or $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\pm\infty}{\pm\infty}$ then,

$\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \lim\limits_{x\to a} \dfrac{f'(x)}{g'(x)}$, a is a number, ∞ or $-\infty$

Polynomials at Infinity

$p(x)$ and $q(x)$ are polynomials. To compute $\lim\limits_{x\to\pm\infty} \dfrac{p(x)}{q(x)}$ factor largest power of x in $q(x)$ out of both $p(x)$ and $q(x)$ then compute limit.

$$\lim_{x\to -\infty} \frac{3x^2-4}{5x-2x^2} = \lim_{x\to -\infty} \frac{x^2\left(3-\frac{4}{x^2}\right)}{x^2\left(\frac{5}{x}-2\right)}$$

$$= \lim_{x\to -\infty} \frac{3-\frac{4}{x^2}}{\frac{5}{x}-2} = -\frac{3}{2}$$

Piecewise Function

$\lim\limits_{x\to -2} g(x)$ where $g(x) = \begin{cases} x^2+5 & \text{if } x < -2 \\ 1-3x & \text{if } x \geq -2 \end{cases}$

Compute two one sided limits,

$$\lim_{x\to -2^-} g(x) = \lim_{x\to -2^-} x^2+5 = 9$$
$$\lim_{x\to -2^+} g(x) = \lim_{x\to -2^+} 1-3x = 7$$

One sided limits are different so $\lim\limits_{x\to -2} g(x)$ doesn't exist. If the two one sided limits had been equal then $\lim\limits_{x\to -2} g(x)$ would have existed and had the same value.

Some Continuous Functions

Partial list of continuous functions and the values of x for which they are continuous.

1. Polynomials for all x.
2. Rational function, except for x's that give division by zero.
3. $\sqrt[n]{x}$ (n odd) for all x.
4. $\sqrt[n]{x}$ (n even) for all $x \geq 0$.
5. $\mathbf{e}^x$ for all x.
6. $\ln(x)$ for $x > 0$.
7. $\cos(x)$ and $\sin(x)$ for all x.
8. $\tan(x)$ and $\sec(x)$ provided $x \neq \cdots, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \cdots$
9. $\cot(x)$ and $\csc(x)$ provided $x \neq \cdots, -2\pi, -\pi, 0, \pi, 2\pi, \cdots$

Intermediate Value Theorem

Suppose that $f(x)$ is continuous on $[a,b]$ and let M be any number between $f(a)$ and $f(b)$. Then there exists a number c such that $a < c < b$ and $f(c) = M$.

Derivatives

Definition and Notation

If $y = f(x)$ then the derivative is defined to be $f'(x) = \lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$.

If $y = f(x)$ then all of the following are equivalent notations for the derivative.

$$f'(x) = y' = \frac{df}{dx} = \frac{dy}{dx} = \frac{d}{dx}\left(f(x)\right) = Df(x)$$

If $y = f(x)$ all of the following are equivalent notations for derivative evaluated at $x = a$.

$$f'(a) = y'\big|_{x=a} = \left.\frac{df}{dx}\right|_{x=a} = \left.\frac{dy}{dx}\right|_{x=a} = Df(a)$$

Interpretation of the Derivative

If $y = f(x)$ then,

1. $m = f'(a)$ is the slope of the tangent line to $y = f(x)$ at $x = a$ and the equation of the tangent line at $x = a$ is given by $y = f(a) + f'(a)(x-a)$.
2. $f'(a)$ is the instantaneous rate of change of $f(x)$ at $x = a$.
3. If $f(t)$ is the position of an object at time t then $f'(a)$ is the velocity of the object at $t = a$.

Basic Properties and Formulas

If $f(x)$ and $g(x)$ are differentiable functions (the derivative exists), c and n are any real numbers,

1. $\dfrac{d}{dx}\left(c\right) = 0$
2. $\left(c\,f(x)\right)' = c\,f'(x)$
3. $\dfrac{d}{dx}\left(x^n\right) = n\,x^{n-1}$ – **Power Rule**
4. $\left(f(x) \pm g(x)\right)' = f'(x) \pm g'(x)$
5. $\left(f(x)\,g(x)\right)' = f'(x)\,g(x) + f(x)\,g'(x)$ – **Product Rule**
6. $\left(\dfrac{f(x)}{g(x)}\right)' = \dfrac{f'(x)\,g(x) - f(x)\,g'(x)}{\left(g(x)\right)^2}$ – **Quotient Rule**
7. $\dfrac{d}{dx}\left(f\left(g(x)\right)\right) = f'\left(g(x)\right)\,g'(x)$ – **Chain Rule**

Common Derivatives

$\dfrac{d}{dx}\left(x\right) = 1$

$\dfrac{d}{dx}\left(\sin(x)\right) = \cos(x)$

$\dfrac{d}{dx}\left(\cos(x)\right) = -\sin(x)$

$\dfrac{d}{dx}\left(\tan(x)\right) = \sec^2(x)$

$\dfrac{d}{dx}\left(\sec(x)\right) = \sec(x)\tan(x)$

$\dfrac{d}{dx}\left(\csc(x)\right) = -\csc(x)\cot(x)$

$\dfrac{d}{dx}\left(\cot(x)\right) = -\csc^2(x)$

$\dfrac{d}{dx}\left(\sin^{-1}(x)\right) = \dfrac{1}{\sqrt{1-x^2}}$

$\dfrac{d}{dx}\left(\cos^{-1}(x)\right) = -\dfrac{1}{\sqrt{1-x^2}}$

$\dfrac{d}{dx}\left(\tan^{-1}(x)\right) = \dfrac{1}{1+x^2}$

$\dfrac{d}{dx}\left(a^x\right) = a^x \ln(a)$

$\dfrac{d}{dx}\left(\mathbf{e}^x\right) = \mathbf{e}^x$

$\dfrac{d}{dx}\left(\ln(x)\right) = \dfrac{1}{x}, \quad x > 0$

$\dfrac{d}{dx}\left(\ln|x|\right) = \dfrac{1}{x}, \quad x \neq 0$

$\dfrac{d}{dx}\left(\log_a(x)\right) = \dfrac{1}{x\ln(a)}, \quad x > 0$

Chain Rule Variants

The chain rule applied to some specific functions.

1. $\frac{d}{dx}\left(\left[f(x)\right]^n\right) = n\left[f(x)\right]^{n-1} f'(x)$
2. $\frac{d}{dx}\left(\mathbf{e}^{f(x)}\right) = f'(x)\,\mathbf{e}^{f(x)}$
3. $\frac{d}{dx}\left(\ln\left[f(x)\right]\right) = \frac{f'(x)}{f(x)}$
4. $\frac{d}{dx}\left(\sin\left[f(x)\right]\right) = f'(x)\cos\left[f(x)\right]$
5. $\frac{d}{dx}\left(\cos\left[f(x)\right]\right) = -f'(x)\sin\left[f(x)\right]$
6. $\frac{d}{dx}\left(\tan\left[f(x)\right]\right) = f'(x)\sec^2\left[f(x)\right]$
7. $\frac{d}{dx}\left(\sec\left[f(x)\right]\right) = f'(x)\sec\left[f(x)\right]\tan\left[f(x)\right]$
8. $\frac{d}{dx}\left(\tan^{-1}\left[f(x)\right]\right) = \frac{f'(x)}{1+\left[f(x)\right]^2}$

Higher Order Derivatives

The 2^{nd} Derivative is denoted as $f''(x) = f^{(2)}(x) = \frac{d^2 f}{dx^2}$ and is defined as $f''(x) = \left(f'(x)\right)'$, *i.e.* the derivative of the first derivative, $f'(x)$.

The n^{th} Derivative is denoted as $f^{(n)}(x) = \frac{d^n f}{dx^n}$ and is defined as $f^{(n)}(x) = \left(f^{(n-1)}(x)\right)'$, *i.e.* the derivative of the $(n-1)^{st}$ derivative, $f^{(n-1)}(x)$.

Implicit Differentiation

Find y' if $\mathbf{e}^{2x-9y} + x^3 y = \sin(y) + 11x$. Remember $y = y(x)$ here, so products/quotients of x and y will use the product/quotient rule and derivatives of y will use the chain rule. The "trick" is to differentiate as normal and every time you differentiate a y you tack on a y' (from the chain rule). Then solve for y'.

$$\mathbf{e}^{2x-9y}(2-9y') + 3x^2y^2 + 2x^3y\,y' = \cos(y)y' + 11$$
$$2\mathbf{e}^{2x-9y} - 9y'\mathbf{e}^{2x-9y} + 3x^2y^2 + 2x^3y\,y' = \cos(y)y' + 11 \qquad \Rightarrow \qquad y' = \frac{11 - 2\mathbf{e}^{2x-9y} - 3x^2y^2}{2x^3y - 9\mathbf{e}^{2x-9y} - \cos(y)}$$
$$\left(2x^3y - 9\mathbf{e}^{2x-9y} - \cos(y)\right)y' = 11 - 2\mathbf{e}^{2x-9y} - 3x^2y^2$$

Increasing/Decreasing – Concave Up/Concave Down

Critical Points

$x = c$ is a critical point of $f(x)$ provided either

1. $f'(c) = 0$ or,
2. $f'(c)$ doesn't exist.

Increasing/Decreasing

1. If $f'(x) > 0$ for all x in an interval I then $f(x)$ is increasing on the interval I.
2. If $f'(x) < 0$ for all x in an interval I then $f(x)$ is decreasing on the interval I.
3. If $f'(x) = 0$ for all x in an interval I then $f(x)$ is constant on the interval I.

Concave Up/Concave Down

1. If $f''(x) > 0$ for all x in an interval I then $f(x)$ is concave up on the interval I.
2. If $f''(x) < 0$ for all x in an interval I then $f(x)$ is concave down on the interval I.

Inflection Points

$x = c$ is a inflection point of $f(x)$ if the concavity changes at $x = c$.

Extrema

Absolute Extrema

1. $x = c$ is an absolute maximum of $f(x)$ if $f(c) \ge f(x)$ for all x in the domain.
2. $x = c$ is an absolute minimum of $f(x)$ if $f(c) \le f(x)$ for all x in the domain.

Fermat's Theorem

If $f(x)$ has a relative (or local) extrema at $x = c$, then $x = c$ is a critical point of $f(x)$.

Extreme Value Theorem

If $f(x)$ is continuous on the closed interval $[a, b]$ then there exist numbers c and d so that,

1. $a \le c, d \le b$,
2. $f(c)$ is the absolute maximum in $[a, b]$,
3. $f(d)$ is the absolute minimum in $[a, b]$.

Finding Absolute Extrema

To find the absolute extrema of the continuous function $f(x)$ on the interval $[a, b]$ use the following process.

1. Find all critical points of $f(x)$ in $[a, b]$.
2. Evaluate $f(x)$ at all points found in Step 1.
3. Evaluate $f(a)$ and $f(b)$.
4. Identify the absolute maximum (largest function value) and the absolute minimum (smallest function value) from the evaluations in Steps 2 & 3.

Relative (local) Extrema

1. $x = c$ is a relative (or local) maximum of $f(x)$ if $f(c) \ge f(x)$ for all x near c.
2. $x = c$ is a relative (or local) minimum of $f(x)$ if $f(c) \le f(x)$ for all x near c.

1^{st} Derivative Test

If $x = c$ is a critical point of $f(x)$ then $x = c$ is

1. a relative maximum of $f(x)$ if $f'(x) > 0$ to the left of $x = c$ and $f'(x) < 0$ to the right of $x = c$.
2. a relative minimum of $f(x)$ if $f'(x) < 0$ to the left of $x = c$ and $f'(x) > 0$ to the right of $x = c$.
3. not a relative extrema of $f(x)$ if $f'(x$ is the same sign on both sides of $x = c$.

2^{nd} Derivative Test

If $x = c$ is a critical point of $f(x)$ such that $f'(c) = 0$ then $x = c$

1. is a relative maximum of $f(x)$ if $f''(c) < 0$.
2. is a relative minimum of $f(x)$ if $f''(c) > 0$.
3. may be a relative maximum, relative minimum, or neither if $f''(c) = 0$.

Finding Relative Extrema and/or Classify Critical Points

1. Find all critical points of $f(x)$.
2. Use the 1^{st} derivative test or the 2^{nd} derivative test on each critical point.

Mean Value Theorem

If $f(x)$ is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b) then there is a number $a < c < b$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

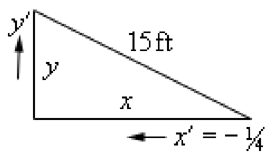
Newton's Method

If x_n is the n^{th} guess for the root/solution of $f(x) = 0$ then $(n+1)^{st}$ guess is $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$ provided $f'(x_n)$ exists.

Related Rates

Sketch picture and identify known/unknown quantities. Write down equation relating quantities and differentiate with respect to t using implicit differentiation (*i.e.* add on a derivative every time you differentiate a function of t). Plug in known quantities and solve for the unknown quantity.

Example A 15 foot ladder is resting against a wall. The bottom is initially 10 ft away and is being pushed towards the wall at $\frac{1}{4}$ ft/sec. How fast is the top moving after 12 sec?



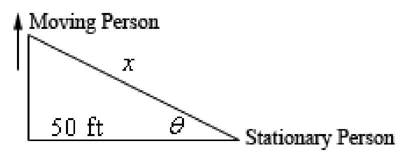
x' is negative because x is decreasing. Using Pythagorean Theorem and differentiating,

$x^2 + y^2 = 15^2 \quad \Rightarrow \quad 2x\,x' + 2y\,y' = 0$

After 12 sec we have $x = 10 - 12\left(\frac{1}{4}\right) = 7$ and so $y = \sqrt{15^2 - 7^2} = \sqrt{176}$. Plug in and solve for y'.

$7\left(-\frac{1}{4}\right) + \sqrt{176}\;y' = 0 \quad \Rightarrow \quad y' = \frac{7}{4\sqrt{176}}$ ft/sec

Example Two people are 50 ft apart when one starts walking north. The angle θ changes at 0.01 rad/min. At what rate is the distance between them changing when $\theta = 0.5$ rad?



We have $\theta' = 0.01$ rad/min. and want to find x'. We can use various trig functions but easiest is,

$\sec(\theta) = \frac{x}{50} \quad \Rightarrow \quad \sec(\theta)\tan(\theta)\;\theta' = \frac{x'}{50}$

We know $\theta = 0.5$ so plug in θ' and solve.

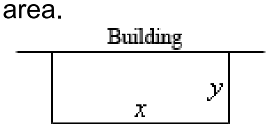
$$\sec(0.5)\tan(0.5)\,(0.01) = \frac{x'}{50}$$
$$x' = 0.3112 \text{ ft/min}$$

Remember to have calculator in radians!

Optimization

Sketch picture if needed, write down equation to be optimized and constraint. Solve constraint for one of the two variables and plug into first equation. Find critical points of equation in range of variables and verify that they are min/max as needed.

Example We're enclosing a rectangular field with 500 ft of fence material and one side of the field is a building. Determine dimensions that will maximize the enclosed area.



Maximize $A = xy$ subject to constraint $x + 2y = 500$. Solve constraint for x and plug into area.

$$x = 500 - 2y \quad \Rightarrow \quad \begin{aligned} A &= y(500 - 2y) \\ &= 500y - 2y^2 \end{aligned}$$

Differentiate and find critical point(s).

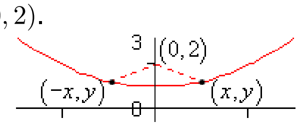
$A' = 500 - 4y \Rightarrow \quad y = 125$

By 2^{nd} derivative test this is a relative maximum and so is the answer we're after. Finally, find x.

$x = 500 - 2(125) = 250$

The dimensions are then 250 x 125.

Example Determine point(s) on $y = x^2 + 1$ that are closest to $(0,2)$.



Minimize $f = d^2 = (x-0)^2 + (y-2)^2$ and the constraint is $y = x^2 + 1$. Solve constraint for x^2 and plug into the function.

$$\begin{aligned} x^2 = y - 1 \Rightarrow f &= x^2 + (y-2)^2 \\ &= y - 1 + (y-2)^2 = y^2 - 3y + 3 \end{aligned}$$

Differentiate and find critical point(s).

$f' = 2y - 3 \quad \Rightarrow \quad y = \frac{3}{2}$

By the 2^{nd} derivative test this is a relative minimum and so all we need to do is find x value(s).

$x^2 = \frac{3}{2} - 1 = \frac{1}{2} \quad \Rightarrow \quad x = \pm\frac{1}{\sqrt{2}}$

The 2 points are then $\left(\frac{1}{\sqrt{2}}, \frac{3}{2}\right)$ and $\left(-\frac{1}{\sqrt{2}}, \frac{3}{2}\right)$.

Integrals

Definitions

Definite Integral : Suppose $f(x)$ is continuous on $[a,b]$. Divide $[a,b]$ into n subintervals of width Δx and choose x_i^* from each interval. Then $\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f\left(x_i^*\right)\Delta x$.

Anti-Derivative : An anti-derivative of $f(x)$ is a function, $F(x)$, such that $F'(x) = f(x)$.

Indefinite Integral : $\int f(x)\,dx = F(x) + c$ where $F(x)$ is an anti-derivative of $f(x)$.

Fundamental Theorem of Calculus

Part I : If $f(x)$ is continuous on $[a,b]$ then $g(x) = \int_a^x f(t)\,dt$ is also continuous on $[a,b]$ and $g'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x)$.

Part II : $f(x)$ is continuous on $[a,b]$, $F(x)$ is an anti-derivative of $f(x)$, *i.e.* $F(x) = \int f(x)\,dx$, then $\int_a^b f(x)\,dx = F(b) - F(a)$.

Variants of Part I :

$$\frac{d}{dx}\int_a^{u(x)} f(t)\,dt = u'(x)f\big[u(x)\big]$$

$$\frac{d}{dx}\int_{v(x)}^{b} f(t)\,dt = -v'(x)f\big[v(x)\big]$$

$$\frac{d}{dx}\int_{v(x)}^{u(x)} f(t)\,dt = u'(x)f\big[u(x)\big] - v'(x)f\big[v(x)\big]$$

Properties

$\int f(x) \pm g(x)\,dx = \int f(x)\,dx \pm \int g(x)\,dx$

$\int c f(x)\,dx = c\int f(x)\,dx$, c is a constant

$\int_a^b f(x) \pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

$\int_a^b c f(x)\,dx = c\int_a^b f(x)\,dx$, c is a constant

$\int_a^a f(x)\,dx = 0$

$\int_a^b c\,dx = c(b-a)$, c is a constant

$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

$\left|\int_a^b f(x)\,dx\right| \le \int_a^b \left|f(x)\right| dx$

$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ for any value c.

If $f(x) \ge g(x)$ on $a \le x \le b$ then $\int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$

If $f(x) \ge 0$ on $a \le x \le b$ then $\int_a^b f(x)\,dx \ge 0$

If $m \le f(x) \le M$ on $a \le x \le b$ then $m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$

Common Integrals

$\int k\,dx = k\,x + c$

$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + c,\ n \neq -1$

$\int x^{-1}\,dx = \int \frac{1}{x}\,dx = \ln|x| + c$

$\int \mathbf{e}^u\,du = \mathbf{e}^u + c$

$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + c$

$\int \ln(u)\,du = u\ln(u) - u + c$

$\int \cos(u)\,du = \sin(u) + c$

$\int \sec(u)\tan(u)\,du = \sec(u) + c$

$\int \tan(u)\,du = \ln|\sec(u)| + c$

$\int \sin(u)\,du = -\cos(u) + c$

$\int \csc(u)\cot(u)du = -\csc(u) + c$

$\int \tan(u)\,du = -\ln|\cos(u)| + c$

$\int \sec^2(u)\,du = \tan(u) + c$

$\int \sec(u)du = \ln\left|\sec(u) + \tan(u)\right| + c$

$\int \frac{1}{a^2+u^2}\,du = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + c$

$\int \csc^2(u)\,du = -\cot(u) + c$

$\int \csc(u)du = -\ln\left|\csc(u) + \cot(u)\right| + c$

$\int \frac{1}{\sqrt{a^2-u^2}}\,du = \sin^{-1}\left(\frac{u}{a}\right) + c$

Standard Integration Techniques

u Substitution : $\int_a^b f(g(x))\,g'(x)\,dx$ will convert the integral into $\int_a^b f(g(x))\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$ using the substitution $u = g(x)$ where $du = g'(x)dx$. For indefinite integrals drop the limits of integration.

Example $\int_1^2 5x^2\cos\left(x^3\right)\,dx$

$u = x^3 \quad \Rightarrow \quad du = 3x^2dx \quad \Rightarrow \quad x^2dx = \frac{1}{3}du$

$x = 1 \quad \Rightarrow \quad u = 1^3 = 1 \quad :: \quad x = 2 \quad \Rightarrow \quad u = 2^3 = 8$

$$\int_1^2 5x^2\cos\left(x^3\right)\,dx = \int_1^8 \frac{5}{3}\cos(u)\,du = \frac{5}{3}\sin(u)\Big|_1^8 = \frac{5}{3}\Big(\sin(8) - \sin(1)\Big)$$

Products and (some) Quotients of Trig Functions

For $\int \sin^n(x)\cos^m(x)\,dx$ we have the following :

1. **n odd.** Strip 1 sine out and convert rest to cosines using $\sin^2(x) = 1 - \cos^2(x)$, then use the substitution $u = \cos(x)$.
2. **m odd.** Strip 1 cosine out and convert rest to sines using $\cos^2(x) = 1 - \sin^2(x)$, then use the substitution $u = \sin(x)$.
3. **n and m both odd.** Use either 1. or 2.
4. **n and m both even.** Use double angle and/or half angle formulas to reduce the integral into a form that can be integrated.

For $\int \tan^n(x)\sec^m(x)\,dx$ we have the following :

1. **n odd.** Strip 1 tangent and 1 secant out and convert the rest to secants using $\tan^2(x) = \sec^2(x) - 1$, then use the substitution $u = \sec(x)$.
2. **m even.** Strip 2 secants out and convert rest to tangents using $\sec^2(x) = 1 + \tan^2(x)$, then use the substitution $u = \tan(x)$.
3. **n odd and m even.** Use either 1. or 2.
4. **n even and m odd**. Each integral will be dealt with differently.

Trig Formulas : $\sin(2x) = 2\sin(x)\cos(x)$, $\cos^2(x) = \frac{1}{2}(1+\cos(2x))$, $\sin^2(x) = \frac{1}{2}(1-\cos(2x))$

Example $\int \tan^3(x)\sec^5(x)\,dx$

$$\begin{aligned}\int \tan^3 x\,\sec^5 x\,dx &= \int \tan^2 x\,\sec^4 x\,\tan x\,\sec x\,dx \\ &= \int \left(\sec^2(x) - 1\right)\sec^4(x)\,\tan(x)\sec(x)dx \\ &= \int \left(u^2 - 1\right)u^4du \qquad \left[u = \sec(x)\right] \\ &= \tfrac{1}{7}\sec^7(x) - \tfrac{1}{5}\sec^5(x) + c\end{aligned}$$

Example $\int \frac{\sin^5(x)}{\cos^3(x)}\,dx$

$$\begin{aligned}\int \frac{\sin^5 x}{\cos^3 x}\,dx &= \int \frac{\sin^4 x\sin x}{\cos^3 x}\,dx = \int \frac{(\sin^2 x)^2\sin x}{\cos^3 x}\,dx \\ &= \int \frac{(1-\cos^2(x))^2\sin(x)}{\cos^3(x)}\,dx \qquad \left[u = \cos(x)\right] \\ &= -\int \frac{(1-u^2)^2}{u^3}du = -\int \frac{1-2u^2+u^4}{u^3}du \\ &= \tfrac{1}{2}\sec^2(x) + 2\ln\left|\cos(x)\right| - \tfrac{1}{2}\cos^2(x) + c\end{aligned}$$

Integration by Parts : $\int u\,dv = uv - \int v\,du$ and $\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$. Choose u and dv from integral and compute du by differentiating u and compute v using $v = \int dv$.

Example $\int x\mathbf{e}^{-x}\,dx$

$u = x \quad dv = \mathbf{e}^{-x} \quad \Rightarrow \quad du = dx \quad v = -\mathbf{e}^{-x}$

$$\int x\mathbf{e}^{-x}\,dx = -x\mathbf{e}^{-x} + \int \mathbf{e}^{-x}\,dx = -x\mathbf{e}^{-x} - \mathbf{e}^{-x} + c$$

Example $\int_3^5 \ln(x)\,dx$

$u = \ln(x) \quad dv = dx \quad \Rightarrow \quad du = \frac{1}{x}dx \quad v = x$

$$\int_3^5 \ln(x)\,dx = x\ln(x)\Big|_3^5 - \int_3^5 dx = (x\ln(x) - x)\Big|_3^5 = 5\ln(5) - 3\ln(3) - 2$$

Trig Substitutions : If the integral contains the following root use the given substitution and formula to convert into an integral involving trig functions.

$\sqrt{a^2 - b^2x^2} \Rightarrow x = \frac{a}{b}\sin(\theta)$	$\sqrt{b^2x^2 - a^2} \Rightarrow x = \frac{a}{b}\sec(\theta)$	$\sqrt{a^2 + b^2x^2} \Rightarrow x = \frac{a}{b}\tan\theta$
$\cos^2(\theta) = 1 - \sin^2(\theta)$	$\tan^2(\theta) = \sec^2(\theta) - 1$	$\sec^2(\theta) = 1 + \tan^2(\theta)$

Example $\int \frac{16}{x^2\sqrt{4-9x^2}}\,dx$

$x = \frac{2}{3}\sin(\theta) \quad \Rightarrow \quad dx = \frac{2}{3}\cos(\theta)\,d\theta$

$$\sqrt{4-9x^2} = \sqrt{4 - 4\sin^2(\theta)} = \sqrt{4\cos^2(\theta)} = 2\left|\cos(\theta)\right|$$

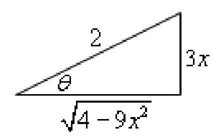
Recall $\sqrt{x^2} = |x|$. Because we have an indefinite integral we'll assume positive and drop absolute value bars. If we had a definite integral we'd need to compute θ's and remove absolute value bars based on that and,

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

In this case we have $\sqrt{4-9x^2} = 2\cos(\theta)$.

$$\int \frac{16}{\frac{4}{9}\sin^2(\theta)(2\cos\theta)}\left(\tfrac{2}{3}\cos\theta\right)d\theta = \int \frac{12}{\sin^2(\theta)}\,d\theta = \int 12\csc^2(\theta)d\theta = -12\cot(\theta) + c$$

Use Right Triangle Trig to go back to x's. From substitution we have $\sin(\theta) = \frac{3x}{2}$ so,



From this we see that $\cot(\theta) = \frac{\sqrt{4-9x^2}}{3x}$. So,

$$\int \frac{16}{x^2\sqrt{4-9x^2}}\,dx = -\frac{4\sqrt{4-9x^2}}{x} + c$$

Partial Fractions : If integrating a rational expression involving polynomials, $\int \frac{P(x)}{Q(x)}\,dx$, where the degree of $P(x)$ is smaller than the degree of $Q(x)$. Factor denominator as completely as possible and find the partial fraction decomposition of the rational expression. Integrate the partial fraction decomposition (P.F.D.). For each factor in the denominator we get term(s) in the decomposition according to the following table.

Factor of $Q(x)$	Term in P.F.D	Factor is $Q(x)$	Term in P.F.D
$ax + b$	$\frac{A}{ax+b}$	$(ax+b)^k$	$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_k}{(ax+b)^k}$
$ax^2 + bx + c$	$\frac{Ax+B}{ax^2+bx+c}$	$(ax^2+bx+c)^k$	$\frac{A_1x+B_1}{ax^2+bx+c} + \cdots + \frac{A_kx+B_k}{(ax^2+bx+c)^k}$

Example $\int \frac{7x^2+13x}{(x-1)(x^2+4)}\,dx$

$$\int \frac{7x^2+13x}{(x-1)(x^2+4)}\,dx = \int \frac{4}{x-1} + \frac{3x+16}{x^2+4}\,dx = \int \frac{4}{x-1} + \frac{3x}{x^2+4} + \frac{16}{x^2+4}\,dx$$

$$= 4\ln|x-1| + \tfrac{3}{2}\ln\left(x^2+4\right) + 8\tan^{-1}\left(\tfrac{x}{2}\right)$$

Here is partial fraction form and recombined.

$$\frac{7x^2+13x}{(x-1)(x^2+4)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+4} = \frac{A(x^2+4)+(Bx+C)(x-1)}{(x-1)(x^2+4)}$$

Set numerators equal and collect like terms.

$$7x^2 + 13x = (A+B)x^2 + (C-B)x + 4A - C$$

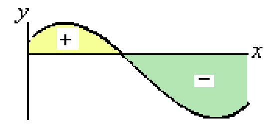
Set coefficients equal to get a system and solve to get constants.

$A + B = 7 \quad C - B = 13 \quad 4A - C = 0$

$A = 4 \quad B = 3 \quad C = 16$

Applications of Integrals

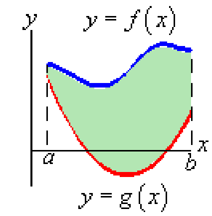
Net Area : $\int_a^b f(x)\,dx$ represents the net area between $f(x)$ and the x-axis with area above x-axis positive and area below x-axis negative.



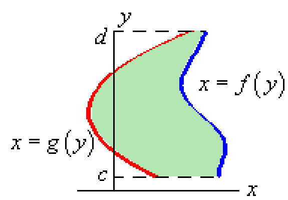
Area Between Curves : The general formulas for the two main cases for each are,

$y = f(x) \;\Rightarrow\; A = \int_a^b [\text{upper function}] - [\text{lower function}]\,dx$ & $x = f(y) \;\Rightarrow\; A = \int_c^d [\text{right function}] - [\text{left function}]\,dy$

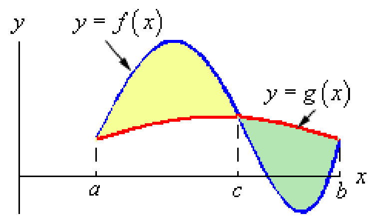
If the curves intersect then the area of each portion must be found individually. Here are some sketches of a couple possible situations and formulas for a couple of possible cases.



$$A = \int_a^b f(x) - g(x)\,dx$$



$$A = \int_c^d f(y) - g(y)\,dy$$



$$A = \int_a^c f(x) - g(x)\,dx + \int_c^b g(x) - f(x)\,dx$$

Volumes of Revolution : The two main formulas are $V = \int A(x)\,dx$ and $V = \int A(y)\,dy$. Here is some general information about each method of computing and some examples.

Rings

$A = \pi\left((\text{outer radius})^2 - (\text{inner radius})^2\right)$

Limits: x/y of right/bot ring to x/y of left/top ring

Horz. Axis use $f(x)$, $g(x)$, $A(x)$ and dx.

Vert. Axis use $f(y)$, $g(y)$, $A(y)$ and dy.

Cylinders/Shells

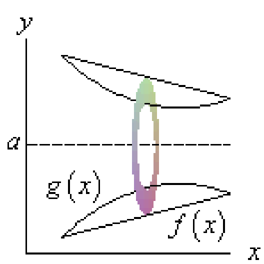
$A = 2\pi(\text{radius})(\text{width / height})$

Limits : x/y of inner cyl. to x/y of outer cyl.

Horz. Axis use $f(y)$, $g(y)$, $A(y)$ and dy.

Vert. Axis use $f(x)$, $g(x)$, $A(x)$ and dx.

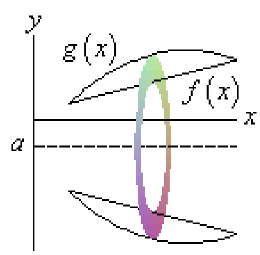
Ex. Axis : $y = a > 0$



outer radius : $a - f(x)$

inner radius : $a - g(x)$

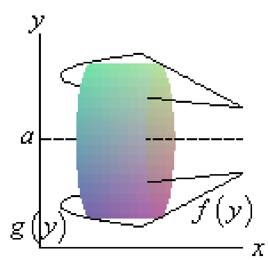
Ex. Axis : $y = a \le 0$



outer radius: $|a| + g(x)$

inner radius: $|a| + f(x)$

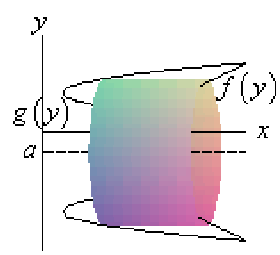
Ex. Axis : $y = a > 0$



radius : $a - y$

width : $f(y) - g(y)$

Ex. Axis : $y = a \le 0$



radius : $|a| + y$

width : $f(y) - g(y)$

These are only a few cases for horizontal axis of rotation. If the axis of rotation is the x-axis use the $y = a \le 0$ case with $a = 0$. For vertical axis of rotation ($x = a > 0$ and $x = a \le 0$) interchange x and y to get appropriate formulas.

Work : If a force of $F(x)$ moves an object in $a \le x \le b$, the work done is $W = \int_a^b F(x)\,dx$

Average Function Value : The average value of $f(x)$ on $a \le x \le b$ is $f_{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx$

Arc Length & Surface Area : The three basic formulas are,

$$L = \int_a^b ds \qquad SA = \int_a^b 2\pi y\, ds \text{ (rotate about } x\text{-axis)} \qquad SA = \int_a^b 2\pi x\, ds \text{ (rotate about } y\text{-axis)}$$

where ds is dependent upon the form of the function being worked with as follows.

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \text{ if } y = f(x),\ \ a \le x \le b \qquad ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \text{ if } x = f(t), y = g(t),\ \ a \le t \le b$$

$$ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy \text{ if } x = f(y),\ a \le y \le b \qquad ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \text{ if } r = f(\theta),\ \ a \le \theta \le b$$

With surface area you may have to substitute in for the x or y depending on your choice of ds to match the differential in the ds. With parametric and polar you will always need to substitute.

Improper Integral

An improper integral is an integral with one or more infinite limits and/or discontinuous integrands. Integral is called **convergent** if the limit exists and has a finite value and **divergent** if the limit doesn't exist or has infinite value.

Infinite Limit

1. $\displaystyle\int_a^\infty f(x)\, dx = \lim_{t\to\infty} \int_a^t f(x)\, dx$

2. $\displaystyle\int_{-\infty}^b f(x)\, dx = \lim_{t\to -\infty} \int_t^b f(x)\, dx$

3. $\displaystyle\int_{-\infty}^\infty f(x)\, dx = \int_{-\infty}^c f(x)\, dx + \int_c^\infty f(x)\, dx$ provided **both** integrals are convergent.

Discontinuous Integrand

1. Discontinuity at a : $\displaystyle\int_a^b f(x)\, dx = \lim_{t\to a^+} \int_t^b f(x)\, dx$

2. Discontinuity at b : $\displaystyle\int_a^b f(x)\, dx = \lim_{t\to b^-} \int_a^t f(x)\, dx$

3. Discontinuity at $a < c < b$: $\displaystyle\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx$ provided **both** are convergent.

Comparison Test for Improper Integrals : If $f(x) \ge g(x) \ge 0$ on $[a, \infty)$ then,

1. If $\displaystyle\int_a^\infty f(x)\, dx$ is convergent then $\displaystyle\int_a^\infty g(x)\, dx$ is convergent (if larger converges so does the smaller).
2. If $\displaystyle\int_a^\infty g(x)\, dx$ is divergent then $\displaystyle\int_a^\infty f(x)\, dx$ is divergent (if smaller diverges so does the larger).

Useful fact : If $a > 0$ then $\displaystyle\int_a^\infty \frac{1}{x^p}\, dx$ converges if $p > 1$ and diverges for $p \le 1$.

Approximating Definite Integrals

For given integral $\displaystyle\int_a^b f(x)\, dx$ and n (must be even for Simpson's Rule) define $\Delta x = \dfrac{b-a}{n}$ and divide $[a, b]$ into n subintervals $[x_0, x_1]$, $[x_1, x_2]$, …, $[x_{n-1}, x_n]$ with $x_0 = a$ and $x_n = b$ then,

Midpoint Rule : $\displaystyle\int_a^b f(x)\, dx \approx \Delta x \left[f(x_1^*) + f(x_2^*) + \cdots + f(x_n^*) \right]$, x_i^* is midpoint $[x_{i-1}, x_i]$

Trapezoid Rule : $\displaystyle\int_a^b f(x)\, dx \approx \frac{\Delta x}{2} \left[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n) \right]$

Simpson's Rule : $\displaystyle\int_a^b f(x)\, dx \approx \frac{\Delta x}{3} \left[f(x_0) + 4f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n) \right]$