UNIT V

5. INTEGRATION

5.1 Introduction

In this unit, we will be looking at integrals. Integrals are the third and final major topic that will be covered in this section. As with derivatives this unit will be devoted almost exclusively to finding and computing integrals. Applications will be given in the following unit. There are really two types of integrals that we'll be looking at in this unit: Indefinite Integrals and Definite Integrals. The first half of this unit is devoted to indefinite integrals and the last half is devoted to definite integrals. As we will see in the last half of the unit if we don't know indefinite integrals we will not be able to do definite integrals.

Unit Objectives

At the end of this unit you will be able to:

- Understand the general concept of integral calculus.
- Identify different techniques of integration.
- Integrate numerous functions.
- Relate integral with derivative of function.
- Perform the calculation of area bounded by some simple curves.
- Derive the volume formula of sphere.
- Apply the knowledge of integral calculus in their life.
- Understand the fundamental theorem of calculus I and II.
- Compute the area bounded by simple curves.

5.2 Antiderivative and Indefinite Integral

The word "integral" may also refer to antiderivative in a mild abuse of language. (The antiderivative of a function is that function whose derivative is equal to the first function.) Though they are closely related through the fundamental theorem of calculus
(which is to be discussed later), -the two notions are conceptually distinct. When one wants to clarify this distinction, an antiderivative integral is referred to as an **indefinite integral** (a function), while the integrals discussed in this article are termed **definite integrals.**

Definition 5.1

Let f be a continuous function on its domain.

A function $F(x)$ is called antiderivative of $f(x)$ if and only if for all x in the domain of f

$F'(x) = f(x)$.

Example 1.Consider f(x) = x, domain of f is the set of real numbers

Then $F(x) = \frac{x^2}{2} + c$, where $c \in \Re$ is an antiderivative of f.

i.e., $F'(x) = x$.

Observe that the antiderivative of a given function is not unique; for instance

x^2, $x^2 + 5$, $x^2 - 4$ and $x^2 + c$, for c is arbitrary constant are all antiderivatives of

$f(x) = 2x$.

$$\text{Since } \frac{d}{dx}(x^2) = \frac{d}{dx}(x^2+5) = \frac{d}{dx}(x^2-4) = 2x.$$

Definition 5.2

If $F(x)$ is any anti-derivative of $f(x)$, then the most general anti-derivative of $f(x)$ is called an **indefinite integral** and denoted, $\int f(x)dx = F(x) + c$, c is any constant

In this definition, the $\int$ is called the **integral symbol**, $f(x)$ is called the **integrand**, x is called the **integration variable** and the "c" is called the **constant of integration**.

Thus, we write $\int 2x dx = x^2 + c$

Example 2: For $x \neq 0$, the function $F(x) = \frac{1}{x^2} + c$ is an antiderivative of $f(x) = \frac{-2}{x^3}$ for an arbitrary constant number c. Thus $\int f(x)dx = \int \frac{-2}{x^3}dx = \frac{1}{x^2} + c$

Example 3:

For an arbitrary constant number c the function F(x) = Sin x +c is an antiderivative of the function f(x) = Cos x , for all $x \in \Re$.

Example 4:

Since $\left(\frac{x^{n+1}}{n+1}\right)' = (n+1)\frac{x^n}{n+1} = x^n$ *, the function*

F(x) $= \frac{x^{n+1}}{n+1} + c$ is an antiderivative of f(x) $=x^n$ for $n \neq -1$ and an arbitrary constant c moreover,

F(x) $= \ln|x| + c$ is an antiderivative of f(x) $= \frac{1}{x}$.

> **<u>Theorem 5.1</u>**
>
> Let f be continuous on [a, b]and F and G are antiderivative f on [a, b],then there exist $c \in \Re$ such that,
>
> $$F(x) = G(x) + c \quad \forall x \in [a,b].$$

Proof:

We assume that F(x) and G(x) are not identically equal to each other.

Thus, $F'(x) = f(x) = G'(x) \ \forall x \in (a,b)$ *and*

$F'(a) = f(a) = G'(a)$,

$F'(b) = f(b) = G'(b)$

F and G are continuous on [a, b] and differentiable in (a, b)

$\therefore (F-G)'(x) = F'(x) - G'(x) = 0, \forall x \in (a,b)$

Thus, by corollary 1of mean value theorem, F – G is a constant function

$\Rightarrow \exists c \in \Re \, such that (F-G)(x) = c$

$\therefore F(x) = G(x) + c \quad \forall x \in [a,b].$

By the above theorem, one can conclude that every continuous function has an infinite number of antiderivative and any two of them only differing by a constant.

Theorem 5.2

If F and G are antiderivative of the continuous functions f and g respectively, and c is any constant then

i) cF is an antiderivative of cf

ii) F+G is antiderivative of f + g

iii) F – G is antiderivative of f – g

Proof of (iii)

Since $(F - G)' = F' - G'$ = f – g it follows that F – G is antiderivative of (f – g).

Similarly you can verify the remaining.

Example: Find the antiderivative of $f(x) = 3x^2 - 8x^3 - 6x + 2$

Solution:

$$\int f(x)dx = \int (3x^2 - 8x^3 - 6x + 2)dx = \int 3x^2 dx - \int 8x^3 dx - \int 6x dx + \int 2 dx$$

$$= 3\int x^2 dx - 8\int x^3 dx - 6\int x dx + 2\int dx = \frac{3x}{2+1}^{2+1} - \frac{8x}{3+1}^{3+1} - \frac{6x}{2}^{1+1} + 2x + c$$

$$= x^3 - 2x^4 - 3x^2 + 2x + c.$$

$\therefore F(x) = x^3 - 2x^4 - 3x^2 + 2x + c$ is antiderivative of $3x^2 - 8x^3 - 6x + 2$.

Example: Show that the polynomial; Q(x) = $a_0 x + \frac{a_1 x^2}{2} + \frac{a_2 x^3}{3} \dots + \frac{a_n x^{n+1}}{n+1} + c$ is an antiderivative of the polynomial P(x) where P(x) = $a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$.

Since, $\frac{a_j x^{j+1}}{j+1} + c$ is an antiderivative of the function $a_j x^j$ for j = 0,1,2,3, . . .,n. It follows that Q(x) is an antiderivative of P(x).

Definition 5.3

The set of all antiderivative of a continuous function f is called the Indefinite Integral of f and is denoted symbolically by $\int f(x)dx$. The function f is called the integrand and x is called the variable of integration.

Thus , if f(x) is an antiderivative of the continuous function f and c is any constant ,then the relation $\int f(x)dx =$ F(x) +c holds and previous theorem can now expressed as follows;

$i)\ \int k\, f(x)\, dx = k\int f(x)dx. = kF(x) + c$.

ii) $\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx = F(x) + G(x) + c$.

iii) $\int [f(x) - g(x)]dx = \int f(x)dx - \int g(x)dx = F(x) - G(x) + c.$

The process of finding the set of antiderivative of the continuous function f is called Integration of the function f with respect to the variable of integration x.

5 .3 Integration of Simple Trigonometric and Exponential Functions

I. Integration Formulas

A number of formulas below follow immediately from the standard differentiation formulas of earlier unit.

1) $\int Sinxdx = -Cosx + c$

2) $\int Cosxdx = Sinx + c$

3) $\int Sec^2 xdx = \tan x + c$

4) $\int e^x dx = e^x + c$

5) $\int x^n dx = \dfrac{x^{n+1}}{n+1} + c \ ,n \neq 1$

6) $\int x^n dx = \ln|x| + c; \quad If\ n = -1$

7) $\int a^x dx = \dfrac{a^x}{\ln a} + c\ ; \quad a > 0 \,\&\, a \neq 1$

<u>Exercises 5.1</u>

Integrate the followings

1) $\int \dfrac{x^3}{2} dx$

2) $\int 9x^8\, dx$

3) $\int 2Sinx\, dx$

4) $\int \dfrac{Cos\, x}{2} dx$

5) $\int \dfrac{1}{Cos^2 x} dx$

6) $\int \dfrac{4}{x}\, dx$

7) $\int 3^x\, dx$

8) $\int 2\, e^x\, dx$

9) $\int (x + 2Cos\, x)dx$

10) $\int \left(2x^3 + \dfrac{9}{7}x^8 + 2x \right)dx$

11) $\int \frac{x^3+4x}{\sqrt{x}}\,dx$ 12) $\int \left(\frac{1}{2}e^x + 3Sinx\right)dx$

13) $\int Sin\,5x\,dx$ 14) $\int \frac{1}{x-1}dx$

15) $\int (x-4)^{23}\,dx$ 16) $\int Tan^2\,x\,dx$

17) $\int \frac{1}{3x}dx$ 18) $\int \left(\frac{1}{x^4} - \sqrt[3]{x^2}\right)dx$

19) $\int \frac{x-2}{\sqrt{x^3}}dx$ 20) $\int \frac{(x^2-1)^2}{x^3}\,dx$

5.4 Techniques of Integration

There are some important methods to evaluate indefinite integrals which we call techniques of integration.

I. Integration by substitution

Let's see integration by substitution by examples.

Example 1: Suppose we want to find the antiderivative of $\int (x+2)^7\,dx$.

One can find the antiderivative of $(x+2)^7$ by expanding in to a polynomial of degree 7 and then integrate term wise using the formula

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + c; \ \ where\ n \neq 1 \ ------------------ \ \otimes$$

However, this method for large degree is cumbersome and tedious; thus using integration by substitution method is preferable.

Therefore, to find the result substitute u for (x+2)

Now if u = (x+2)

$\Rightarrow$ du = dx,

Then the variable of integration **x** will be replaced by **u.**

Hence, the given indefinite integral has the form;

$$\int (x+2)^7\,dx = \int u^7\,du$$

Now we can use the above ($\otimes$) to get the antiderivative of the given function

$$\therefore \int (x+2)^7 \quad = \int u^7 du \;=\; \frac{u^8}{8} + c$$

And substituting back the value for u, we have the result

$$\therefore \qquad \int (x+2)^7 dx = \frac{(x+2)^8}{8} + c$$

Example 2: Find $\int \cos 5x \, dx$

Solution: Let u = 5x

$\Rightarrow$ du = 5dx

$\Rightarrow \dfrac{du}{5} = dx$

Then, $\int \cos 5x dx \;=\; \int \cos u \frac{du}{5} \;= \frac{1}{5}\int \cos u du$

$$= \frac{1}{5}(\sin u) + c$$

$$= \frac{1}{5}(\sin 5x) + c$$

<u>Theorem 5.3</u>

If for all x $\in$[a ,b], the function $g'(x)$ is continuous and f(u) is continuous at g(x) ,then

$$\int f(u)du \quad = \int f(g(x).g'(x)dx, \;\; where \;\; g(x) = u$$

Proof :

If F(u) is an antiderivative of f(u), then we have

$$[F(g(x))]' = F'(g(x)).g'(x) \;=\; f(g(x)).g'(x).$$

i.e., converting the chain rule of differentiation in to method of integration by substitution

$\therefore$ F (g(x)) is an antiderivative of f (g(x)).g'(x)

Thus we have;

$$\int [F(g(x))]' \, dx \;=\; \int F'(g(x).g'(x)dx \;= \int f(g(x)).g'(x)dx$$

Now since u = g(x) , then $\int f(u)\,du = \int f(g(x)).g'(x)dx$

Example 1: Integrate $\int \left[(x^5+6)^5 - 2\right]5x^4 dx$

Solution:

Let g(x) = x^5 + 6 = u

$\Rightarrow$ $g'(x) = 5x^4dx$ = du

Now, $\int \left[(x^5+6)^5 - 2\right]5x^4 dx \quad = \int (u^5 - 2)du$

$$= \frac{u^6}{6} - 2u + c \text{, but } u = x^5 + 6$$

$$= \frac{(x^5+6)^6}{6} - 2(x^5+6) + c$$

Therefore, $\int \left[(x^5+6)^5 - 2\right]5x^4 dx \quad = \quad \frac{(x^5+6)^6}{6} - 2(x^5+6) + c$

Example 2:

Integrate $\int \frac{x+2}{\sqrt{4x+x^2}}dx$

Solution: Let u = 4x + x^2

$\Rightarrow$ du = 2(2+x)dx

$\Rightarrow$ du/2 = (x+2) dx

$$\int \frac{x+2}{\sqrt{4x+x^2}}dx = \int \frac{du}{2\sqrt{u}} = \frac{1}{2}\int u^{-\frac{1}{2}}du = \frac{1}{2}\frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c = \frac{1}{2}\frac{1}{\frac{3}{2}}u^{\frac{3}{2}} + c$$

$$= \frac{1}{2}\times\frac{2}{3}u^{\frac{3}{2}} + c = \frac{1}{3}u\frac{3}{2} + c = \frac{1}{3}(4x+x^2)^{\frac{3}{2}} + c$$

$$\therefore, \int \frac{x+2}{\sqrt{4x+x^2}}\,dx = \frac{\sqrt{(4x+x^2)^3}}{3} + c$$

Example 3: $\int \tan x dx$

Solution: $\int \tan x\, dx = \int \frac{\sin x}{\cos x} dx$ $\quad$ *Let* $u = \cos x \Rightarrow du = -\sin x dx$

$$\Rightarrow \quad -du = \sin x dx$$

$$\therefore \quad \int \frac{\sin x}{\cos x} dx = \int -\frac{du}{u} = -\int \frac{du}{u} = -\ln|u| + c$$

$$= -\ln|\cos x| + c = \ln(|\cos x|)^{-1} + c$$

$$= \ln\left|\frac{1}{\cos x}\right| + c = \underline{\underline{\ln|\sec x| + c}}$$

Example 4: Integrate $\int \frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}} dx$

Solution: Let u $=1 + \sqrt{x}$

$$\text{du} = \frac{1}{2\sqrt{x}} dx \quad \Rightarrow \quad 2du = \frac{1}{\sqrt{x}} dx$$

Therefore, $\int \frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}} dx = \int 2\sqrt{u}\ du = 2\int \sqrt{u}\ du = 2\frac{u^{\frac{3}{2}}}{3/2} + c$

$$= \underline{\underline{\frac{4}{3}\left(1+\sqrt{x}\right)^{\frac{3}{2}} + c}}$$

<u>Exercises 5.2</u>

Integrate the followings

1. $\int \sin 2x dx$ 2. $\int \sin^2 x \cos x dx$ 3. $\int e^{\frac{x}{3}} dx$

4. $\int \frac{dx}{2+2x}$ 5. $\int x\sqrt{2x+1} dx$ 6. $\int \cos(3-x) dx$

7. $\int \frac{4\sqrt{x}}{1+x^{\frac{5}{4}}} dx$ 8 . $\int e^x \cos(e^x) dx$ 9. $\int \frac{\ln x}{x} dx$

10. $\int \frac{x}{x^2+1} dx$ 11. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$ 12. $\int \frac{e^x}{1+e^x} dx$

13. $\int \frac{\sin x}{1+3\cos x} dx$ 14. $\int x^2 \sqrt{5-2x}\, \mathrm{dx}$ 15. $\int \frac{dx}{(x+\ln x)}$

16. $\int \sec^2 x \sqrt{4\tan x} dx$ 17. $\int \frac{x^2+2x}{\sqrt{1+x}} dx$

II. Integration by parts

As we converted the chain rule of differentiation in to the method of integration by substitution, in this section we will use the product theorem for differentiation in terms of integral.

The resulting integration theorem will allow us to evaluate integrals of the form such as;

$\int \ln x dx$, $\int e^{-x} \cos x dx$, *and* $\int x \cos x dx$

Theorem 5.4 (Integration by parts)

Let f and g be functions with continuous derivative on their domains;

$$\int f(x)\, g'(x)\, dx = f(x)\, g(x) - \int f'(x) g(x)\, dx$$

Proof:

Using the product rule for differentiation,

i.e., $\left[f(x)\,g(x)\right]' = f'(x)\,g(x) \; + f(x)g'(x)$ ---------------- Θ

Since f, g, f' and g' are continuous the indefinite integrals

$\int \left[f(x).g(x)\right]'dx = \int f'(x)\,g(x)dx \; + \int f(x)\,g'(x)dx$, which is reduced to;

$\Rightarrow f(x)\; g(x) = \int f'(x)g(x) + \int f(x)g'(x)dx$

$\Rightarrow \int f(x)\,g'(x)dx$ = f(x) g(x) - $\int f'(x)\,g(x)dx.$

Example 1: Integrate $\int x\cos xdx$

Solution: Let f(x) =x and $g'(x)$ =cosxdx

$f'(x)$ =1.dx, g(x) = sinx and using the above theorem consequently,

$\int xosxdx$ = xsinx - $\int \sin xdx$

= xsinx – (-cosx) +c

= <u>xsinx +cosx +c.</u>

Example 2: Find $\int x^2 e^x dx$

Solution:

Let f(x) = $x^2 \Rightarrow \quad f'(x) = 2xdx \quad and \quad g'(x) = e^x dx \quad \Rightarrow g(x) = e^x$

Thus $\int x^2 e^x dx = x^2 e^x - \int 2xe^x dx$

$= x^2 e^x - 2\int x\,e^x dx$

Again let $f(x) = x \quad \Rightarrow f'(x) = dx$

$g'(x) = e^x dx \quad \Rightarrow g(x) = e^x$

$\therefore\; x^2 e^x - 2\int xe^x dx = x^2 e^x - 2(xe^x - \int e^x dx$

$$= x^2e^x \ - 2xe^x \ + 2e^x \ + c$$

$$\therefore \ \int x^2 e^x dx \ = \ e^x(x^2 - 2x + 2) + c$$

Example 3: Evaluate $\int \ln x dx$

Solution:

Let f(x) =lnx $\Rightarrow f'(x) \ \frac{1}{x} dx$ $\qquad g' = \text{dx} \ \Rightarrow g(x) = x$

$$\therefore \quad \int \ln x dx = \text{xlnx} - \int \frac{1}{x} x dx$$

$$= \ x \ln x - \int dx$$

$$= \underline{\underline{x \ln x - x \ + c}}$$

Example 4: Find $\int x \ln x dx$

Solution: Let f(x) = ln x $\qquad$ g '(x) = xdx

$$\Rightarrow f'(x) = \frac{1}{x} dx \qquad g(x) = \frac{x^2}{2}$$

$$\therefore \quad \int x \ln x dx \qquad = \frac{x^2}{2} \ln x - \int \frac{1}{x} \frac{x^2}{2} dx$$

$$\frac{x^2}{2} \ln x - \frac{1}{2} \int x dx \qquad = \frac{x^2}{2} \ln x - \frac{1}{2}\left(\frac{x^2}{2}\right) + c$$

$$= \underline{\underline{\frac{x^2}{2} \ln x - \tfrac{1}{4} x^2 + c}}$$

Example 5: Find $\int e^x \cos x dx$

<u>**Solution**</u>: Let $f(x) = \ e^x$ $\qquad$ $and \ g'(x) \ = \cos x dx$

$$\Rightarrow f'(x) = e^x dx \qquad and \quad g(x) = \sin x$$

$$\int e^x \cos x dx = e^x \sin x - \int e^x \sin x dx$$

Again let $h(x) = e^x$ and $k^{'}(x) = \sin x dx$

$\Rightarrow h^{'}(x) = e^x$ and $k(x) = -\cos x$

As a result, $\int e^x \cos x dx = e^x \sin x - \int e^x \sin dx$

$$= e^x \sin x - [h(x)k(x) - \int h^{'}(x)k(x)dx] = e^x \sin x + e^x \cos x - \int e^x \cos x dx$$

$$\Rightarrow \int e^x \cos x dx + \int e^x \cos x dx = e^x \sin x + e^x \cos + c$$

$$\Rightarrow 2\int e^x \cos dx = e^x \sin x + e^x \cos x + c \qquad \Rightarrow \int e^x \cos x dx = \underline{\underline{\frac{1}{2} e^x (\sin x + \cos x) + c}}$$

Exercises 5.3

Find the indefinite integrals of the following by using integration by parts method

1) $\int x \sin x dx$ 2) $\int xe^{-x} dx$ 3) $\int (x+1)e^x dx$

4) $\int (\ln x)^2 dx$ 5) $\int x^2 \sin x \, dx$ 6) $\int (2x+3) \cos x dx$

7) $\int \frac{\ln x}{\sqrt{x}} dx$ 8) $\int x^2 \ln x \, dx$ 9) $\int e^x (x+3)^2 dx$ 10) $\int \frac{\ln x}{x^2} dx$

11) $\int (\ln x)^3 dx$

III. Integration by partial fraction

Consider a rational function R(x) which is a quotient of two polynomials. That is

$R(x) = \dfrac{P(x)}{Q(x)}$ *assuming that the degree of* $P(x)$ *is lessthan the degree of* $Q(x)$.

If it is not expressed in that way, we must change the given expression in to the desired property by using long division

Example1: $\dfrac{x^4+3x+2}{x^2-1} = (x^2+1) + \dfrac{3x+3}{x^2-1}$

Therefore, the rational function R(x) is the sum of the polynomials (x^2+1) and the rational function $\dfrac{3x+3}{x^2-1}$. Where the degree of the numerator is less than the degree of the denominator.

In this case, we shall give a systematic method, known as a partial fraction of integrating rational functions.

For the method we need the support of the following theorem which we shall accept without proof.

Theorem 5.5

If two polynomials in the same variable x are identical, then the corresponding powers of x have the same coefficients.

That is, if $4x^3 + 3x + 9 = Ax^4 + Bx^3 + cx^2 + Dx + E$ for all x, then

A= 0, B = 4 , C = 0 D= 3 E = 9

We engage this method only for the following cases:

i) Q(x) factors into distinct linear factors
ii) Q(x) factors in to repeated linear factors.

Integration of rational function of $R(x) = \dfrac{P(x)}{Q(x)}$ by the method of partial fraction consists essentially of two steps, namely:

1) Factorizing Q(x) and
2) Expressing the rational function as a sum of simple rational functions whose denominators are linear factors.

Example 1: Integrate $\int \dfrac{5}{x^2-9}dx$

Solution :

The integrand is $\dfrac{P(x)}{Q(x)} = \dfrac{5}{x^2-9}$

Thus, factoring (x^2 - 9) and we obtain $\frac{5}{x^2-9} = \frac{5}{(x-3)(x+3)}$

Therefore, we express as a sum of simple rational functions to get

$\frac{5}{(x-3)(x+3} = \frac{A}{x-3} + \frac{B}{x+3}$, *where A and B are* constants to be determined

This gives, $\frac{5}{(x-3)(x+3)} = \frac{A(x+3) + B(x-3)}{(x-3)(x+3)}$.

$$\Rightarrow 5 = (A+B)x + (A-B)3$$
$$\Rightarrow A+B = 0 \quad and \quad 3(A-B) = 5$$
$$\begin{cases} A-B = \frac{5}{3} \\ A+B = 0 \end{cases} \quad solving\ simul\tan eously$$

$$\Rightarrow A = -B \quad and\ B = -\frac{5}{6} \quad thus, A = \frac{5}{6}$$
$$\therefore \quad \frac{5}{x^2-9} = \frac{5}{6(x-3)} - \frac{5}{6(x+3)}$$
$$\therefore \int \frac{5}{x^2-9}dx = \int \frac{5}{6(x-3)}dx - \int \frac{5}{6(x+3)}$$
$$= \frac{5}{6}\int \frac{dx}{(x-3)} - \frac{5}{6}\int \frac{dx}{(x+3)}$$
$$= \frac{5}{6}\ln|x-3| - \frac{5}{6}\ln|x+3| + c$$
$$= \frac{5}{6}\ln|x-3| + \frac{5}{6}\ln(|x+3|)^{-1} + c$$
$$= \frac{5}{6}\ln|x-3| + \frac{5}{6}\ln\left|\frac{1}{x+3}\right| + c$$
$$= \frac{5}{6}\ln\left|\frac{x-3}{x+3}\right| + c$$
$$\therefore \int \frac{5}{x^2-9}dx = \underline{\underline{\frac{5}{6}\ln\left|\frac{x-3}{x+3}\right| + c}}$$

Example 2: Evaluate; $\int \frac{(2x^3 + 11x^2 + 17x + 7)}{(x^2 + 5x + 6)}dx$

Solution:

By long division method we obtain; $\dfrac{2x^3 + 11x^2 + 17x + 7}{x^2 + 5x + 6} = (2x+1) + \dfrac{1}{(x+2)(x+3)}$

$$\therefore \quad \int \frac{(2x^3 + 11x^2 + 17x + 7)}{(x^2 + 5x + 6)} dx = \int (2x+1) dx + \int \frac{dx}{(x+2)(x+3)}$$

Now first treat the rational part thus;

Let $\dfrac{A}{x+2} + \dfrac{B}{x+3} = \dfrac{1}{(x+2)(x+3)}$

$$\Rightarrow \frac{A(x+3) + B(x+2)}{(x+2)(x+3)} = \frac{1}{(x+2)(x+3)} \qquad \Rightarrow \frac{Ax + Bx + 3A + 2B}{(x+2)(x+3)} = \frac{1}{(x+2)(x+3)}$$

$$\Rightarrow (A+B)x + 3A + 2B = 1 \qquad \Rightarrow A + B = 0 \;\; and \;\; 3A + 2B = 1$$

$$\Rightarrow A = -B. \quad then,\ 3A + 2B = 1$$

$$\Rightarrow 3A - 2A = 1$$

$$\Rightarrow A = 1 \; and \;\; B = -1$$

$$\int \frac{(2x^3 + 11x^2 + 17x + 7)}{(x^2 + 5x + 6)} dx = \int (2x+1) dx + \int \frac{dx}{(x+2)(x+3)}$$

$$= \int (2x+1) dx + \int \frac{dx}{x+2} - \int \frac{dx}{x+3}$$

$$= x^2 + x + \ln|x+2| - \ln|x-3| + c$$

$$= x^2 + x + \ln|x+2| + \ln\left|\frac{1}{x+3}\right| + c = x^2 + x + \ln\left|\frac{x+2}{x+3}\right| + c$$

$$\therefore \quad \int \frac{(2x^3 + 11x^2 + 17x + 7)}{(x^2 + 5x + 6)} dx = \underline{\underline{x^2 + x + \ln\left|\frac{x+2}{x+3}\right| + c}}$$

Example 3: Evaluate $\displaystyle\int \frac{x}{x^2 - 2x + 1} dx$

Solution:

$$\text{Let } \frac{x}{x^2 - 2x + 1} = \frac{A}{x-1} + \frac{B}{(x-1)^2} \qquad \Rightarrow Ax - A + B = x$$

$$\Rightarrow A = 1 \quad and \; -A + B = 0 \;\; \Rightarrow A = B \;\; \therefore,\ B = 1$$

$$\int \frac{x}{x^2 - 2x + 1} dx = \int \frac{A}{x-1} dx + \int \frac{B}{(x-1)^2} dx$$

$$= \int \frac{1}{x-1} dx + \int \frac{1}{(x-1)^2} dx = \ln|x-1| - \frac{1}{x-1} + c$$

$$\int \frac{x}{x^2 - 2x + 1}dx = \ln|x-1| - \frac{1}{x-1} + c$$

Example 4: Evaluate $\int \left(\frac{6x^4 - 6x^3 + 4x^2 + 3x - 2}{x^3 - x^2} \right) dx$

Solution:

For R(x) $= \frac{P(x)}{Q(x)}$ the degree of P(x) >the degree of Q(x), so we use the long division method to express the given integrand

$$\therefore \frac{6x^4 - 6x^3 + 4x^2 + 3x - 2}{x^3 - x^2} = 6x + \frac{4x^2 + 3x - 2}{x^3 - x^2}$$

$$\int \left[\frac{6x^4 - 6x^3 + 4x^2 + 3x - 2}{x^3 - x^2} \right] dx = \int 6xdx + \int \left[\frac{4x^2 + 3x - 2}{x^3 - x^2} \right] dx$$

$$= 3x^2 + \int \frac{4x^2 + 3x - 2}{x^2(x-1)} dx$$

$$Let \quad \frac{4x^2 + 3x - 2}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1}$$

$$\Rightarrow 4x^2 + 3x - 2 = Ax(x-1) + B(x-1) + Cx^2$$

$$= Ax^2 - Ax + Bx - B + Cx^2 \quad = (A+C)x^2 + (B-A)x - B$$

$$\therefore, \begin{cases} A + C = 4 \\ B - A = 3 \quad and\ B = 2 \end{cases}$$

$$\Rightarrow B - A = 3, \qquad \Rightarrow 2 - A = 3, \qquad \Rightarrow A = -1$$

Also A+C = 4 $\quad \Rightarrow -1 + C = 4 \quad \Rightarrow C = 5$

$$\int \frac{4x^2 + 3x - 2}{x^2(x-1)} dx = \int \frac{A}{x} dx + \int \frac{B}{x^2} dx + \int \frac{C}{x-1} dx = -\int \frac{1}{x} dx + 2\int \frac{1}{x^2} dx + 5\int \frac{1}{x-1} dx$$

$$= -\ln|x| - \frac{2}{x} + 5\ln|x-1| + c$$

$$= 5\ln|x-1| - \ln|x| - \frac{2}{x} + c = \ln\left|\frac{(x-1)^5}{x}\right| - 2\frac{1}{x} + c$$

Therefore, $\int \left(\frac{6x^4 - 6x^3 + 4x^2 + 3x - 2}{x^3 - x^2} \right) dx = \int 6xdx + \int \frac{A}{x} dx + \int \frac{B}{x^2} + \int \frac{C}{x-1} dx$

$$= 6\int xdx + \int \frac{-1}{x} dx + \int \frac{2}{x^2} + \int \frac{1}{x-1} dx$$

$$= 6(\frac{x^2}{2}) - \ln|x| - \frac{2}{x} + 5\ln|x - 1 + c|$$

$$= 3x^2 + \ln\left|\frac{(x-1)^5}{x}\right| - \frac{2}{x} + c$$

Exercises 5.5

Integrate the following

1. $\int \frac{x+1}{x-1}dx$
2. $\int \frac{dx}{x^2+2x}$
3. $\int \frac{2x-3}{2-x}dx$
4. $\int \frac{4x-3}{x^2+6x+8}dx$
5. $\int \frac{dx}{x^2-1}$
6. $\int \frac{x}{(x+1)^2}dx$
7. $\int \frac{dx}{4-x^2}$
8. $\int \frac{x^2}{x+2}dx$
9. $\int \frac{3x}{(x-2)^2}dx$
10. $\int \frac{dx}{(x-1)(x-2)}$
11. $\int \frac{x^3}{2-x}dx$
12. $\int \frac{(x+1)^3}{x^2-x}dx$
13. $\int \frac{dx}{2+x-x^2}$
14. $\int \frac{x^3+x-2}{(x-2)^2}dx$
15. $\int \left(\frac{2x^4+3x^3-x^2+x-1}{x^3-x}\right)dx$
16. $\int \frac{(x-1)}{x(x-2)(x+1)}dx$
17. $\int \frac{2x+3}{x(x+1)^2}dx$
18. $\int \frac{x+1}{x-x^2}dx$
19. $\int \frac{3}{2x^2-x^3}dx$
20. $\int \frac{x^3+!}{x^3-4x}dx$

IV. Integration of trigonometric functions

There are integrals involving trigonometric functions which cannot be evaluated easily using the methods already developed.

However, such integral can be evaluated by transforming the integrand by means of trigonometric identities such as;

$Sin^2 x + Cos^2 x = 1$

$\Rightarrow \; Cos^2 x = 1 - Sin^2 x$

$$= \frac{1}{2}(1 + \cos 2x)$$

Similarly, $\sin^2 x = \frac{1}{2}(1-\cos 2x)$

Example1: Evaluate $\int \sin^4 x\,dx$

Solution: Using the above identity

$$\int \sin^4 x\,dx = \int (\sin^2 x)^2 dx = \int\left[\frac{[1-\cos 2x]}{2}\right]^2 dx = \frac{1}{4}\int\left(1-2\cos 2x+\cos^2 2x\right)dx$$

$$= \frac{x}{4}-\frac{2}{4}\int \cos 2x + \frac{1}{4}\int \cos^2 2x\,dx$$

$$= \frac{x}{4}-\frac{1}{4}\sin 2x + \frac{1}{4}\int\left[\frac{1+\cos 4x}{2}\right]dx$$

$$= \frac{x}{4}-\frac{1}{4}\sin 2x + \frac{x}{8} + \frac{1}{32}\sin 4x + c$$

$$\int \sin^4 x\,dx = \frac{3x}{8} - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + c\,.$$

Example 2: Find $\int \cos^2 x\,dx$

Solution: By using trigonometric identity $Cos^2 x = \frac{1+\cos 2x}{2}$, we have

$$\int \cos^2 x\,dx = \int\left(\frac{(1+\cos 2x)}{2}\right)dx = \frac{1}{2}\int(1+\cos 2x)dx$$

$$= \frac{1}{2}\int dx + \frac{1}{2}\int \cos 2x\,dx = \frac{x}{2}+\frac{1}{2}\left(\frac{\sin 2x}{2}\right)+c$$

$$= \frac{x}{2} + \frac{1}{4}\sin 2x + c$$

$$\therefore \int \cos^2 x\,dx = \underline{\frac{x}{2} + \frac{1}{4}\sin 2x + c}$$

<u>Example 3</u>: Evaluate $\int \cos^5 x dx$

<u>Solution:</u>

$$\int \cos^5 x dx = \int \cos^4 x \cos x dx = \int (\cos^2 x)^2 \cos x dx = \int\left[1-\sin^2 x\right]^2 \cos x\,dx$$

Now let u= sinx, then du = cosxdx (by making substitution)

$$\int \cos^5 x dx = \int\left[1-\sin^2 x\right]^2 \cos x dx = \int\left(1-u^2\right)^2 du = \int\left(1-u^2+u^4\right)du$$

$$= u-\frac{2}{3}u^3+\frac{u^5}{5}+c$$

$$= \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + c$$

Therefore, $\int \cos^5 x dx \quad = \quad \underline{\underline{\sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + c}}$

Example 4: Evaluate $\int \frac{\sin^3 x}{\cos^4 x} dx$

Solution:

$\int \frac{\sin^3 x}{\cos^4 x} dx \quad = \quad \int \frac{(1-\cos^2 x)\sin x dx}{\cos^4 x}$. Now we let u =cosx , then du = -sinxdx

$$\int \frac{\sin^3 x}{\cos^4 x} dx \quad = \quad -\int \frac{\sin x(1-\cos^2 x)}{\cos^4 x} dx \quad = \quad -\int \frac{(1-\cos^2 x)\sin x}{\cos^4 x} dx \quad = -\int \frac{(1-u^2)}{u^4} du$$

$$-\int \left(\frac{1}{u^4} - \frac{1}{u^2}\right) du \quad = \frac{1}{3u^3} \quad - \frac{1}{u} \quad + c$$

$$= \frac{1}{3\cos^3 x} - \frac{1}{\cos x} \quad + c$$

$$\int \frac{\sin^3 x}{\cos^4 x} dx \quad = \quad \underline{\underline{\frac{1}{3\cos^3 x} - \frac{1}{\cos x} + c}}$$

Example 5: Evaluate $\int \sin^2 x \cos^3 x \, dx$

Solution: We factor out cos x and write the rest

$$\int \sin^2 x \cos^3 x dx \quad = \quad \int \sin^2 x \cos^2 x \cos x dx$$

$$= \int \sin^2 x(1 - \sin^2 x)\cos x dx$$

Now we let u = sinx so that du =cosxdx. Thus, we obtain

$$\int \sin^2 x \cos^3 x \, dx \quad = \quad \int u^2(1-u^2) du$$

$$= \quad \int (u^2 - u^4) du \; = \; \frac{1}{3}u^3 - \frac{1}{5}u^5 + c \; = \; \frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x \; + c$$

$$\int \sin^2 x \cos^3 x\,dx = \underline{\underline{\frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x + c}}$$

Exercises 5.5

Integrate the following

1) $\int \cos^3 x dx$ 2) $\int \sin^2 x dx$ 3) $\int \tan x dx$

4) $\int \cos 3x dx$ 5) $\int \cot^2 x dx$ 6) $\int \sin x \cos 2x dx$

7) $\int \frac{\sin x}{\sqrt{\cos x}} dx$ 8) $\int \sec^2 8x dx$ 9) $\int \tan^3 x \sec^4 x dx$

Miscellaneous substitutions

Sometimes we need a combination of the above discussed techniques to evaluate a given functions.

To show this we have some examples

Example 1: Find $\int e^{\sqrt{x}} dx$

Solution

Using substitution method; Let $t = \sqrt{x}$ $\quad thus,\ dt = \frac{1}{2\sqrt{x}} dx \Rightarrow 2\sqrt{x}dt = dx$

$$\Rightarrow 2tdt = dx$$

Thus, $\int e^{\sqrt{x}} dx = \int e^t 2t\,dt = 2\int e^t t\,dt.$ *Next using integration by parts* ;

Let f(x) = t , $f'(x)$ = dt and $g'(x)$ = e^tdt then g(x) = e^t .

Thus $2\int e^t dt = 2\left[te^t - \int e^t dt\right] = 2[te^t - e^t] + c$

$$= \boxed{\underline{\underline{2\left(\sqrt{x}\ e^{\sqrt{x}} - e^{\sqrt{x}}\right)}} + c}$$

Example 2: Find $\int x^3 e^{x^2} dx$

Solution:

Now let us apply integration by parts first;

$\int x^3 e^{x^2} dx = \int x^2 e^{x^2} x dx$, then let $\left[f(x) = x^2 \quad and\ g'(x) = x e^{x^2} dx\right]$

$$\left[\Rightarrow f'(x) = 2xdx \quad and\ g(x) = \frac{e^{x^2}}{2}\right]$$

$$\int x^3 e^{x^2} dx = \left[\frac{x^2 e^{x^2}}{2} - \int ex^2 xdx\right];$$

Next we use substitution method to evaluate $\int e^{x^2} xdx$, If u = x^2 then du =2xdx

$$\Rightarrow \frac{du}{2} = xdx \quad Thus \int e^{x^2} dx = \int \frac{e^u du}{2} = \frac{1}{2}\int e^u du = \frac{1}{2} e^{x^2} + c$$

$$\int x^3 e^{x^2} dx \quad \left[\underline{\underline{\frac{x^2 e^{x^2}}{2} - \frac{e^{x^2}}{2} + c}}\right]$$

Example 3: Find $\int \frac{dx}{e^x + 1}$

Solution:

To find the solution, first we use substitution method

Now we let $\left[t = e^x + 1 \quad \Rightarrow dt = e^x dx \quad \Rightarrow dx = \frac{dt}{e^x} = \frac{dt}{t-1}\right]$

$\int \frac{dx}{e^e + 1} = \int \frac{dt}{t(t-1)}$; Now using partial fraction method on $\frac{1}{t(t-1)}$

$$\frac{1}{t(t-1)} = \frac{A}{t} + \frac{B}{t-1} \quad \Rightarrow A(t-1) + Bt = 1 \quad \Rightarrow (A+B)t - A = 1$$

$$But\ A + B = 0 \Rightarrow A = -B$$
$$and\ -A = 1 \Rightarrow A = -1 \quad Thus\ B = 1$$

$$\therefore \int \frac{dt}{t(t-1} = \int \frac{-1}{t} dt + \int \frac{1}{t-1} dt = -\ln|t| + \ln|t+1| + c = \ln\left|\frac{t-1}{t}\right| + c$$

$$= \ln\left|\frac{e^{x+1-1}}{e^x + 1}\right| + c = \ln\left|\frac{e^x}{e^x + 1}\right| + c$$

$$\therefore \int \frac{dx}{e^e + 1} = \underline{\underline{\ln\left|\frac{e^x}{e^e + 1}\right| + c}}$$

For the case $\sqrt[n]{ax+b}$, $Substitute \quad ax + b = u^n$

Example4: *Find* $\int \frac{dx}{x\sqrt{1-x}}$

Solution: *Let* $1-x=u^n \Rightarrow u=\sqrt{1-x}$ *And* $1-u^n = x \Rightarrow dx=-2udu$

$$\therefore \int \frac{dx}{x\sqrt{1-x}} = \int \frac{-2udu}{(1-u^2)u} = -2\int \frac{du}{1-u^2}$$

Now apply partial fraction

i.e, $\frac{1}{1-u^2} = \frac{A}{1-u} + \frac{B}{1+u} \quad \Rightarrow A+Au+B-Bu=1 \Rightarrow A+B=1$ *and* $(A-B)u=0 \Rightarrow A=B$

$\therefore$ 2A = 1 $\Rightarrow A=\frac{1}{2}$ *and* $B=\frac{1}{2}$

$$\therefore \quad -2\int \frac{du}{1-u^2} = -2\left[\int \frac{\frac{1}{2}du}{1-u} + \int \frac{\frac{1}{2}du}{1+u}\right]$$

$$= -\int \frac{1}{u-1} - \int \frac{du}{1+u} = \ln|1-u| - \ln|u+1| + c = \ln\left|\frac{1-u}{1+u}\right| + c$$

$$= \ln\left|\frac{1-\sqrt{1-x}}{1+\sqrt{1+x}}\right| + c$$

Example 5: Evaluate $\int \frac{dx}{(x-2)\sqrt{x+2}}$

Solution: Now, let x+2 = u^2 $\Rightarrow u=\sqrt{x+2} \quad \Rightarrow x=u^2-2 \Rightarrow dx=2udu$

$$\int \frac{dx}{(x-2)\sqrt{x+2}} = \int \frac{2udu}{(u^2-2-2)u} = 2\int \frac{du}{u^2-4} = 2\int \frac{du}{(u-2)(u+2)}$$

Next apply partial fraction method Let $\frac{1}{(u-2)(u+2)} = \frac{A}{u-2} + \frac{B}{u+2}$

$$\Rightarrow Au+2A+Bu-2B=1 \qquad \Rightarrow (A+B)u=0 \Rightarrow A=-B$$

$$\textit{and } 2(A-B)=1 \Rightarrow A-B=\frac{1}{2} \textit{ and } \sin ce\, A=-B\, then -B-B=\frac{1}{2}$$

$$\Rightarrow B=-\frac{1}{4} \quad \textit{and} \quad A=\frac{1}{4}$$

$$\therefore \int \frac{dx}{(x-2)\sqrt{x+2}} = 2\int \frac{du}{(u-2)(u+2)} = 2\left[\frac{1}{4}\int \frac{du}{(u-2)} - \frac{1}{4}\int \frac{du}{u+2}\right]$$

$$= \frac{1}{2}\ln|u-2| - \frac{1}{2}\ln|u+2| + c$$

$$= \frac{1}{2}\left[\ln\left|\frac{u-2}{u+2}\right|\right] + c = \frac{1}{2}\ln\left|\frac{\sqrt{x+2}\ -2}{\sqrt{x+2}\ +2}\right| + c$$

Exercises 5.6

Evaluate the following integrals

1) $\int \frac{x^3}{e^{2x}}dx$ 2) $\int x\cos 3x dx$ 3) $\int \sec^3 x dx$

4) $\int x\sec^2 x dx$ 5) $\int x\sqrt{2x+5}\, dx$ 6) $\int \frac{\sqrt{x+4}}{x}dx$

7) $\int 3x^3\sqrt{x^2+2}\ dx$ 8) $\int \frac{dx}{1+3e^x+2e^{2x}}$ 9) $\int \cos^2 x \tan^5 x dx$

10) $\int \left(\sqrt{1+e^{2x}}\right)dx$ 11) $\int \sec^5 x dx$ 12) $\int \frac{x^3}{e^{x^2}}dx$

;8

5.5 The Definite Integral and fundamental theorem of calculus

Our goal in this section is to show how area under a curve, such as the area A of a rectangle, of triangle so on can be expressed as a limit of a sum of terms which is called a definite integral.

We will then introduce a result called the fundamental theorem of calculus that allows to compute definite integrals and thus find area and other quantities by using indefinite integration (antiderivative) methods of the (previous section).

Now, consider the region S in fig5.1 which is bounded above by f(x) = x^2, below by the x-axis from x = 1 to x=3.

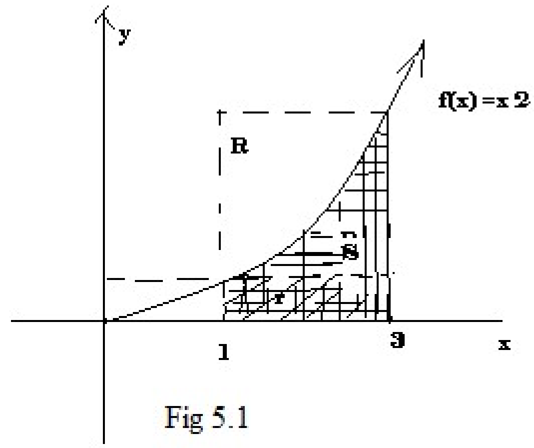


Fig 5.1

Let R and r be rectangles with the same base on the x-axis of closed interval [1, 3] enclosing S and enclosed by R respectively, then

A(r)≤ A(S)≤ A(R), where A(r),A(S) and A(R) are the area r, S, and R respectively .If we substitute the given interval [1, 3] in to two equal sub-intervals [1, 2] and [2, 3]and if we denote by R_1 and R_2 the rectangles with bases [1,2] and [2,3] enclosing S and by r_1 and r_2 the rectangles on bases [1,2] and [2,3] enclosed by the region S as shown figure

5.2 below we have A (r_1) + A (r_2) ≤A(S) ≤ A (R_1) +A (R_2).

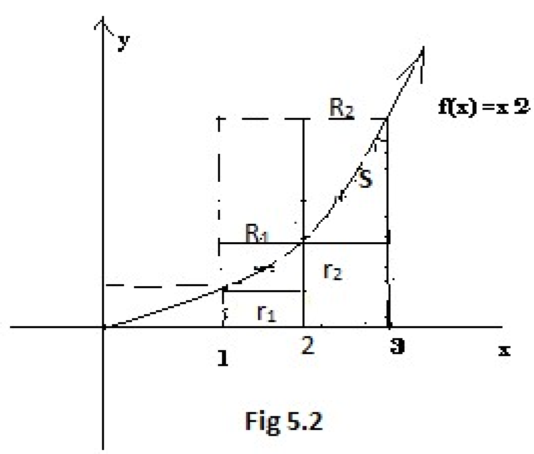


Fig 5.2

If we subdivide the interval [1, 3] in to four equal sub divisions [1, 3/2], [3/2, 2], [2,5/2] and [5/2, 3]and if we denote by R_1, R_2, R_3, and R_4 the rectangles enclosing S and by r_1, r_2, r_3, and r_4 the rectangles enclosed by S we have

$A(r_1) + A(r_2) + A(r_3) + A(r_4) \leq A(S) \leq A(R_1) + A(R_2) + A(R_3) + A(R_4)$

$$\Rightarrow \sum_{i=1}^{4} A(r_i) \leq A(S) \leq \sum_{i=1}^{4} A(R_i)$$

If we continue sub-dividing the interval [1, 3] in to n equal subintervals

$1 \quad x_1 \quad x_2 \quad x_3 \; . \; . \; . \; x_{i-1} \quad x_i \quad x_{i+1} \; . \; . \; . \; x_{n-1} \quad x_n = 3$

and denoting the enclosing and the enclosed rectangles on each subintervals by R_i and r_i respectively we find $\sum_{i=1}^{n} A(r_i) \leq A(S) \leq \sum_{i=1}^{n} A(R_i)$

Therefore, we can observe that as the number of equal sub intervals increase the length of each subinterval decreases or tend to zero.

This small sub-interval is denoted Δx and $\Delta x = x_j - x_{j-1}$ and hence the difference

$\sum_{j=1}^{n} A(R_j) - \sum_{j=1}^{n} A(r_j)$ *gets smaller and smaller* and eventually approaches zero.

That is, $\lim_{\Delta x \to 0} \sum_{j=1}^{n} A(R_j) = \lim_{\Delta x \to o} \sum_{j=1}^{n} A(r_j) = A(S)$

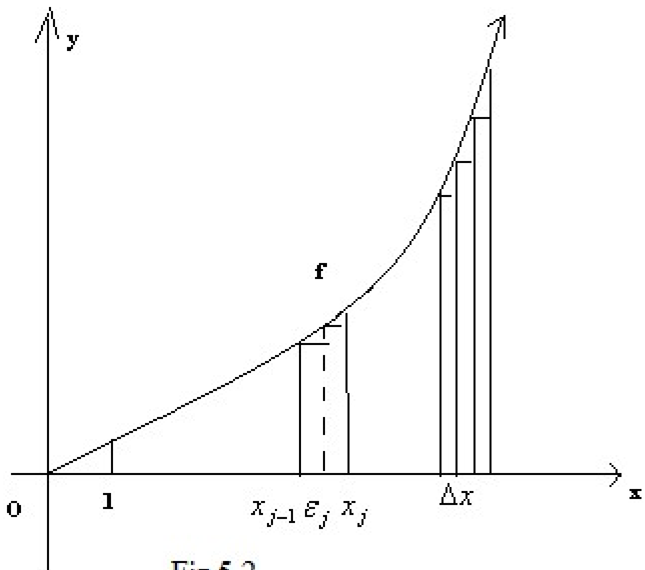


Fig 5.2

Since, $A(R_j) = f(x_j)\Delta x$ and

$A(r_j) = f(x_{j-1})$ *are close to each other* (*where* $\Delta x = x_j - x_{j-1}$) *as* $\Delta x \to 0$, we can approximate A (R_j) and A(r_j) by the expression $f(\varepsilon_j)\Delta x$ where ε is any point in the subinterval $\lfloor x_{j-1}, x_j \rfloor$ and thus write

$$A(S) = \lim_{\Delta x \to 0} \sum_{j=1}^{n} f(\varepsilon_j)\Delta x = \lim_{\Delta x \to 0} \sum_{j=1}^{n} (\varepsilon_j)^2 \Delta x \quad Since\ f(x) = x^2$$

- • Using the above illustration of the notion of area under a curve, we associate the idea with the concept of definite integral as follow;

I. The Definite Integral

Let f(x) be a function that is continuous on the interval $a \le x \le b$. Subdivide the interval $a \le x \le b$ in to n equal parts using the points a= x_0 , x_1, x_2, . . . x_{j-1}, x_j, x_{j+1}, . . .,x_n =b and the subdivided interval [a, b] using the points;

[a, x_1] ,[x_1, x_2] [x_2 ,x_3] . . . [x_j, x_{j+1}] . . . [x_{n-2}, x_{n-1}], [x_{n-1} , x_n] each of width $\Delta x = \dfrac{b-a}{n}$

and choose a number x_j from the j^{th} subinterval for j = 1, 2, 3, . . . n.

For the sum, [f(x_1) + f(x_2) + . . . f(x_n)] Δx called an integral sum or Riemann sum of f in the interval [a, b].

Then the definite integral of f on the interval $a \leq x \leq b$ *denoted by* $\int_a^b f(x)dx.$ is the limit of the Riemann sum as $n \to \infty$; That is:

$$\int_a^b f(x)\,dx = \lim_{n \to +\infty} \sum_{j=1}^{n} [f(x_1) + f(x_2) + \ . \ \ . \ \ . + f(x_n)]\Delta x$$

$$= \lim_{n \to \infty} \sum_{j=1}^{n} f(x_j)\Delta x$$

The function f(x) is called the integrand, and the numbers ***a*** and ***b*** are called the lower and the upper limits of integration respectively.

The process of finding the definite integral is called definite integration.

II. Area as a Definite Integral

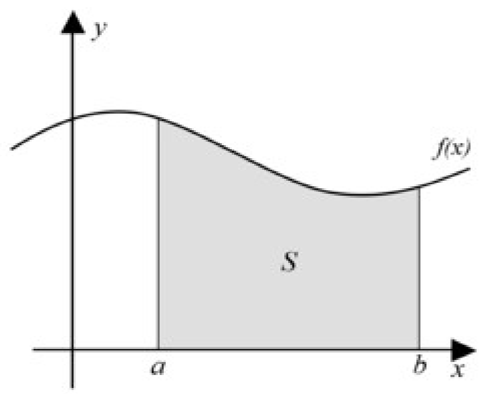


Fig 5.3

Suppose f is continuous and f(x) ≥ 0 *On* $[a,b]$

If the limit of the integral sum $\lim_{\Delta x \to 0} \sum_{j=1}^{n} f(\varepsilon_j)\Delta x$ *exists* and equal to the real number S, then the number S is called the definite integral of f over the interval [a, b] and is denoted by; S = $\int_a^b f(x)dx$ is read as "the integral of f with respect to x from a to b.'' and we say that f is integrable.

The symbol $\int_a^b f(x)\,dx$ used for definite integral is essentially is the same as the symbol $\int f(x)\,dx$ for the indefinite integral, even though the definite integral is a specific number

While the indefinite integral is a family of functions, the antiderivative of f.

It is not quite evident from the definition of the definite integral whether the limit of integral sum $\lim_{\Delta x \to o} \sum_{j=1}^{n} f(\varepsilon_j)\Delta x$ exists for all functions.

We shall now state a theorem without proof which assures us when a function is integrable.

> **Theorem 5.6 (Existence of definite integral)**
>
> If the function f is continuous on [a, b] ,then $\lim_{\Delta x \to 0} \sum_{j=1}^{n} f(\varepsilon_j)\Delta x$ *exists.*
>
> That is, f is integrable over [a, b].

Example 1: Let f(x) = cos(x) then find the integral f over $\left[0, \dfrac{\pi}{2}\right]$

Solution:

Since cos(x) is continuous in $\left[0, \dfrac{\pi}{2}\right]$, then the integral of $\int_0^{\frac{\pi}{2}} \cos x \; dx$ exists.

$$\therefore \int_0^{\frac{\pi}{2}} \cos x \; dx = \sin x \Bigg|_{x=0}^{x=\frac{\pi}{2}} = \sin\left(\frac{\pi}{2}\right) - \sin 0 = 1$$

Example 2: Let p(x) be any polynomial function, then find the integral of p (x) over $(-\infty, \infty)$.

Solution: Since any polynomial p(x) is continuous in the finite interval [a, b], the integral $\int_a^b p(x)dx.$ *exists.*

Example 3: Let $f(x) = \dfrac{1}{x}$, then find definite integral of f over $[-1,1]$

Solution: Since the function f(x) $= \dfrac{1}{x}$ is not defined at x = 0;and $0 \in$ [-1,1] the integral $\int_{-1}^{1} \left(\dfrac{1}{x}\right) dx$ does not exists.

Example 4: Use the definition of definite integral to show that $\int_a^b k\,dx = k(b-a)$ where k is any constant.

Solution: Subdivide the given interval [a, b] in to n equal subintervals each of length

$$\Delta x = \frac{b-a}{n}$$

That is $[a, x_1]$, $[x_1, x_2]$, $[x_2, x_3]$ . . . $[x_{j-1}, x_j]$ . . . $[x_n, b]$

Since f(x) = k we have $f(\varepsilon_j) = k \;\; for \;\; any\, \varepsilon \in \lfloor x_{j-1}, x_j \rfloor$ consequently we have

$$\sum_{j=1}^{n} f(\varepsilon_j)\Delta x = \sum_{j=1}^{n} k\Delta x = k\sum_{j=1}^{n} \Delta x = k\sum_{j=1}^{n} \frac{b-a}{n} = \frac{kn(b-a)}{n} = k(b-a)$$

$$\therefore \int_a^b k\,dx = k(b-a) \; .$$

Exercise 5.7

1. Let S be the region bounded by f(x) = x^3, the line x =2,the x axis and x = 4 and ;

Estimate A(S) by subdivided the interval [2, 4] in to

a) 2 equal subintervals b) 4 equal subintervals

2. Let S be the region bounded by f(x) = sinx, x = 0, the x axis and x = π

Estimate A(S) by subdividing the interval [0, π]

a) 2 equal subintervals b) 4 equal subintervals.

3. Determine whether the following functions are integrable or not in the given interval

a) cosx , [-π, π] b) sin3x , [-π/2 , π/2]

c) tanx , [0 , π/2]   d) tanx [0 , π/4]

e) sinx +cosx , [0, π/2]   f) 1/x , [1/2 , 100]

g) $3x^2 + 7x - 2$, [-1 ,1]   h) $\frac{x^2+2}{x-1}$, [0,2]

The following properties of definite integrals are immediate consequences of its definition and can be used to simplify the computation of definite integrals.

Theorem 5.7

If f and g are integrable over [a, b] and k is any constant then kf, f +g, f – g are integrable on $a \le x \le b$. And

1) $\int_a^b k\, f(x)dx = k\int_a^b f(x)\,dx$; *Constant muliple rule.*

2) $\int_a^b (f(x) \pm g(x))dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$; *sum or difference rule.*

3) $\int_a^a f(x)dx = 0$ ($\therefore$ *a line segment has no area.*)

4) $\int_a^b f(x)dx = -\int_b^a f(x)dx$

5) If f is integrable on $a \le x \le b$ and c$\in$[a, b] then

$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$; Subdivision rule

6) Let f is integrable over [a , b]

If m$\le$ f(x) $\le$ M for all x $\in [a,b]$,then m(b-a) $\le \int_a^b f(x)dx \le M(b-a)$

Proof of 1:

Since $\lim_{\Delta x\to 0}\sum_{j=1}^{n} kf(\varepsilon_j)\Delta x = k\lim_{\Delta x\to 0}\sum_{j=1}^{n} f(\varepsilon_j)\Delta x$ it follows that kf is integrable over [a,b] and $\int_a^b kf(x)dx = k\int_a^b f(x)dx$

Proof of 2:

Since f and g are integrable over [a, b] $\lim_{\Delta x_1\to 0}\sum_{j=1}^{n} f(\varepsilon_j)\Delta x_1$ *and* $\lim_{\Delta x_2\to 0}\sum_{j=1}^{n} g(\eta_j)\Delta x_2$ exists where Δx_1 and Δx_2 are the length of the subintervals of [a, b]corresponding to the functions f and g and ε_j *and* η_j are any two elements in the j[th] subinterval corresponding to the functions f and g. Let $\Delta x_1 = \Delta x_2 = \Delta x$ *and* $\varepsilon_j = \eta_j$ *we have*

$\lim_{\Delta x\to 0}\sum_{j=1}^{n}[f(\eta_j)\pm g(\eta_j)]\Delta x = \lim_{\Delta x\to 0}\sum_{j=1}^{n} f(\eta_j)\Delta x \pm \lim_{\Delta x\to 0}\sum_{j=1}^{n} g(\eta_j)\Delta x$ and hence f$\pm$ g is integrable in [a ,b]

Hence $\int_a^b (f(x)\pm g(x))dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$

Proof of 5:

We assume that we can always subdivide the interval [a, b] in to n equal subintervals with c as one of the end points of a subinterval [a, b] = [a, c] and [c, b] respectively then

$$\left(\sum_{j=1}^{n} f(\varepsilon_j)\Delta x \ ; \sum_{j=1}^{k} f(\varepsilon_j)\Delta x \ \ and \ \sum_{j=k+1}^{n} f(\varepsilon_j)\Delta x_j \right) \ \text{-------}\otimes$$

$where\ k\ is the number\ of\ sub\text{int}erval\ of [a,c].$

Since f is continuous on [a, b] and hence on [a, c] and [c, b].

Therefore, each of the integral sums in $\otimes$ tends respectively to

$$\int_a^b f(x)dx, \quad \int_a^c f(x)dx, \ and \int_c^b f(x)dx$$

Since $\sum_{j=1}^{n} f(\varepsilon_j)\Delta x = \sum_{j=1}^{k} f(\varepsilon_j)\Delta x + \sum_{j=k+1}^{n} f(\varepsilon_j)\Delta x$ By the limit theorem we have

$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$. Therefore, property (5) has geometrical interpretation when f(x) is positive as seen in the figure below.

Therefore, clearly the area of the region ABDF is the sum of the region of ACEF and CBDF.

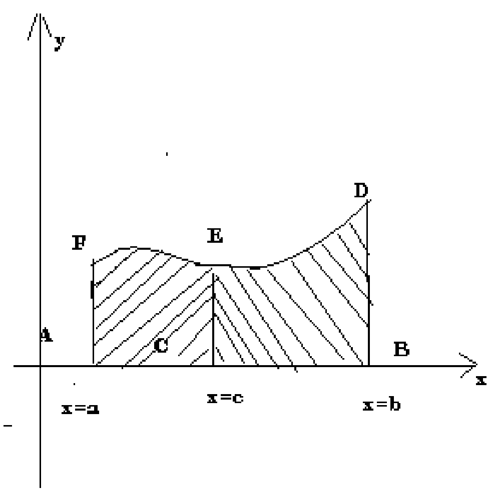


Fig5.4

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx \text{ for } a \le b$$

Proof of property 6:

Since $M - f \geq 0 \quad and \quad f - m \geq 0$ we have by property (4)

$$\int_a^b (M-f)(x)dx \geq 0 \; and \qquad \int_a^b (m-f)(x)dx \geq 0$$ and by property (3)

$$\int_a^b (M-f)(x)dx \; = \int_a^b Mdx - \int_a^b f(x)dx \; \geq 0$$

$$M(b-a) - \int_a^b f(x)dx \geq 0 \qquad Thus, \; M(b-a) \geq \int_a^b f(x)\,dx$$

$$\int_b^a f(x)\;dx \leq M(b-a) \;------------\otimes$$

Similarly, $\int_b^a (f-m)(x)\,dx \geq 0 \quad \Rightarrow \int_b^a f(x)\,dx \; - \; \int_b^a m\,dx \; \geq o \; \Rightarrow \int_a^b f(x)\,dx \geq \int_a^b m\,dx$

$$\Rightarrow m(b-a) \leq \int_a^b f(x)\,dx \quad -------------(\otimes\otimes)$$

Combining

$\otimes \; and \otimes\otimes \; we \; obtain \quad \underline{\underline{m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)}}$

Theorem 5.8 (Mean Value Theorem for Integrals)

If f is continuous on the interval [a, b] , then there is a number x_0 between a and b such that $\int_a^b f(x)\,dx \; = f(x_0)(b-a)$.

Proof: Since f is continuous on [a, b] there are numbers x_1 and x_2 in [a, b] such that

$f(x_1) = m$, $f(x_2) = M$ and $\forall x \in [a,b]$; $\quad m \leq f(x) \leq M$.

By property (5) of theorem 5.7 we have:

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a) \qquad \Rightarrow m < \frac{1}{b-a}\int_a^b f(x)\,dx \le M$$

Thus, there is a number $y_0 \in (m, M)$ such that:

$\frac{1}{b-a}\int_a^b f(x)\,dx = y_0$. And by Intermediate Value Theorem, there is a number

$x_0 \in [x_1, x_2]$ *such that* $f(x_0) = y_0$ and hence; $\int_a^b f(x)\,dx = \underline{\underline{f(x_0)(b-a)}}$

5.6 The Fundamental Theorem of Calculus

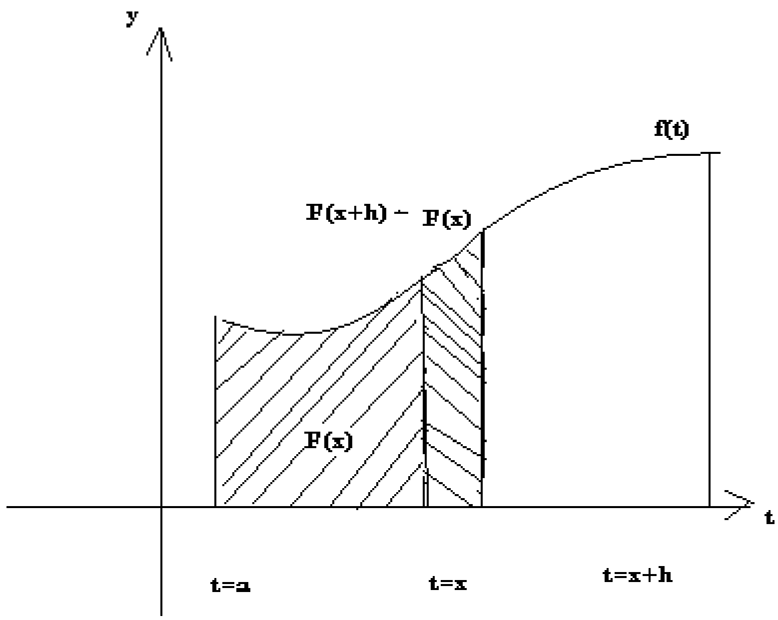
The purpose of this section is to develop a general method for evaluating $\int_a^b f(x)\,dx$ that does not require computing various sums.

Thus, the method will allow us to evaluate many (but not all) of the integrals that arise in applications.

Let us consider the definite integral $F(x) = \int_a^x f(t)\,dt$ for a continuous function f(t) defined on [a, b], where the lower limit of integration is fixed, that is constant, and the upper limit varies within the interval [a, b].

Since for each $x \in [a, b]$ the definite integral $\int_a^x f(t)\,dt$ is a fixed number, it is clear that F(x) is a function.

For positive function f(t) the function F(x) represents the area of the region bounded by the graph of f(t) , t = a , the x-axis, and t = x Fig5.5



Theorem 5.9 (Fundamental Theorem of Calculus I (FTCI))

If the function f(t) is continuous in [a, b] and for $x \in [a,b]$; $F(x) = \int_a^x f(t)\,dt$ then F(x) is differentiable on [a, b] and $F'(x) = f(x)$.

Proof:

Suppose x and x+h are in (a, b). We then have:

$$F(x+h) = \int_a^{x+h} f(t)\,dt$$

$$F(x+h) = \int_a^{x+h} f(t)\,dt = \int_a^{x} f(t)\,dt + \int_x^{x+h} f(t)\,dt$$

$$\Rightarrow F(x+h) - F(x) = \int_x^{x+h} f(t)\,dt \,.$$

Finally, assume that $h \neq 0$ and we get

$$\frac{F(x+h)-F(x)}{h} = \frac{1}{h}\int_x^{x+h} f(t)dt \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Let's now assume that $h > 0$ and since we are still assuming that x and x+h are in (a, b) we know that f(x) is continuous on $[x, x+h]$ and so by the extreme value theorem we know that there are numbers c and d in $[x, x+h]$ so that f(c) = m is the absolute minimum of f(x) in $[x, x+h]$ and that f(d) = M is the absolute maximum of f(x) in $[x, x+h]$.

So, by the integral properties we then know that we have,

$$mh \leq \int_x^{x+h} f(t)dt \leq Mh \text{ or } f(c)h \leq \int_x^{x+h} f(t)dt \leq f(d)h$$

Now divide both sides of this by h to get

$$f(c) \leq \frac{1}{h}\int_x^{x+h} f(t)dt \leq f(d)$$

and then use (1) to get, $f(c) \leq \dfrac{F(x+h)-F(x)}{h} \leq f(d)$ ……….(2)

Next, if $h < 0$ we can go through the same argument above except we will be working on [x+h, x] to arrive at exactly the same inequality above. In other words, (2) is true provided $h \neq 0$.

Now, if we take $h \to 0$ we also have $c \to x$ and $d \to x$ because both c and d are between x and x+h. this means that we have the following two limits.

$\lim_{h\to 0} f(c) = \lim_{c\to x} f(c) = f(x)$ and $\lim_{h\to 0} f(d) = \lim_{d\to x} f(d) = f(x)$

The Squeeze Theorem then tell us that,

$$\lim_{h\to 0}\frac{F(x+h)-F(x)}{h} = f(x) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

But, the left side of this is exactly the definition of the derivatives of F(x) and so we get that $F^{'}(x) = f(x)$.

$\Rightarrow$ F(x) is differentiable on the interval (a, b).

Fundamental theorem of Calculus (I) enables us to find a function F(x) whose derivative is the given function f(x) .In other words, the Theorem asserts that every continuous function on [a, b] is a derivative of some function.

Theorem 5.10: (Fundamental Theorem of Calculus(II)(FTCII))
Let f(t) be continuous on [a, b]. If G(t) is an antiderivative of f(t) on [a, b], then $\int_a^b f(t)\,dt = G(b) - G(a)$.

Proof: First let $F(x) = \int_a^x f(t)dt$ and then we know by FTC(I) that $F^{'}(x) = f(x)$ and so F(x) is an anti-derivative of f(x) on [a, b]. Further suppose that G(x) is any anti-derivative of f(x) on [a, b] that we want to choose. So, this means that we must have,

$$F^{'}(x) = G^{'}(x)$$

Then, by theorem 5.1 F and G differ by a constant on [a, b]. In other word for $a < x < b$ we have,

$$F(x) = G(x) + c$$

Now because F(x) and G(x) are continuous on [a, b], if we take the limit of this as $x \to a^{+}$ and $x \to b^{-}$ we can see that this also holds if x= a and x= b.

So, for $a \le x \le b$ we know that $F(x) = G(x) + c$. Let's use this and the definition of F(x) to do the following.

$$\begin{aligned} F(b) - F(a) &= (G(b) + c) - (G(a) \\ &= G(b) - G(a) \\ &= \int_a^b f(t)dt + \int_a^a f(t)dt \\ &= \int_a^b f(t)dt + 0 \\ &= \int_a^b f(x)dx \end{aligned}$$

Note that in the last step we used the fact that the variable used in the integral does not matter and so we could change the t's to x's.

This Theorem answers in full the problem of evaluating definite integrals.

$\int_a^b f(t)\,dt$ is to find another function whose derivative is f(x).

After that one only does a simple calculation

Notation $G(x)\Big|_a^b$ to denote the difference G(b) –G(a).

From the fundamental theorem, we can now conclude that if f(x) > 0, then the area of the region bounded by f(x), x-axis, x = a and x = b is

$$\int_a^b f(t)dt = F(x)\Big|_a^b = F(b) - F(a), \textit{ where } F \textit{ is an antiderivative of the function } f.$$

Example 1: Evaluate $\int_1^3 2x dx$

Solution: We know that F(x) = x^2 is antiderivative of f(x) = 2x

$$\int_1^3 2xdx = F(x)\Big|_1^3 = F(3) - F(1) = 3^2 - 1^2 = 8$$

Example 2: Evaluate $\int_2^5 \frac{-2}{x^3} dx$

Solution: we know that F(x) =1/x^2 is an antiderivative of $-\frac{2}{x^3}$,

$$\therefore \int_2^5 -\frac{2}{x^3}dx = \frac{1}{x^2}\Bigg|_2^5 = \frac{1}{5^2} - \frac{1}{2^2} = -\left(\frac{21}{100}\right)$$

Example 3: Evaluate $\int_0^{\frac{\pi}{2}} \sin x dx$

Solution: we know that F(x) = -cosx is an antiderivative of the function sinx

$$\therefore \int_0^{\frac{\pi}{2}} \sin x dx = F(x)\Big|_0^{\pi/2} = -\cos x\Big|_0^{\pi/2} = -\left[\cos\left(\frac{\pi}{2}\right) - \cos(0)\right] = \underline{\underline{1}}$$

Warning

We must be sure that the integral is continuous between the limits of integration in order to apply the fundamental Theorem of Calculus (II)

For example, we cannot use the FTC to evaluate $\int_{-1}^{1} \frac{-2}{x^2}dx$, since -2/x^3 is not continuous at x = 0 as $0 \in [-1, 1]$.

Example 4: Evaluate $\int_{-1}^{1} (2x^3 - 6x^2 + 2x + 1)dx$

Solution: Since $\frac{a_n x^{n+1}}{n+1}$ is an antiderivative of the function $a_n x^n$ for n$\neq -1$,and by the properties of integral we have;

$$\int_{-1}^{1} (2x^3 - 6x^2 + 2x + 1)dx = \left(\frac{2x^4}{4} - \frac{6x^3}{3} + x^2 + x\right)\Bigg|_{-1}^{1} = -2$$

Example 5: Find the area of the region bounded by f(x) = x^4, x =2, x = 3 and the x-axis

Solution: Since f(x) > o for $x \in [2,3]$, then the area of the region is

$$\int_{2}^{3} f(x)\,dx = \int_{2}^{3} x^4\,dx = \left(\frac{x^5}{5}\right)\Bigg|_{2}^{3} = \frac{211}{5}$$

Example 6: Find the area of the region bounded by ; f(x) = cosx,

$x = -\frac{\pi}{2}, \quad x = \frac{\pi}{2}$ *and the xaxis*

Solution: Since cosx ≥ 0 for $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ the area of the region is

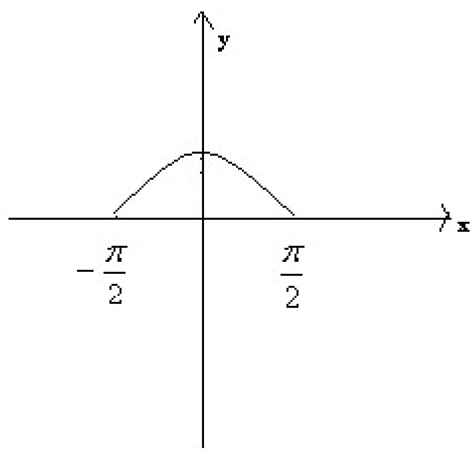


Fig5.6

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x dx = (\sin x)\Big|_{\pi/2}^{-\pi/2} = \sin\left(\frac{\pi}{2}\right) - \sin\left(-\frac{\pi}{2}\right) = 1-(-1) = 2$$

Exercises 5.8

I-In the following exercises, use the Fundamental Theorem of Calculus (I)to differentiate the given function of x.

1) f(x) = $\int_{0}^{x} \sqrt{\sin t}\,dt$ 2) g(x) = $\int_{\sqrt{x}}^{1} \frac{1}{t^3}dt$

3) $h(x) = \int_{0}^{x^3} \sqrt{1+t^2}\,dt$ 4) $k(x) = \int_{-1}^{x^2} 3^t\,dt$

II – In the following, evaluate the given integrals.

1) $\int_{1}^{2} 4x^2 dx$ 2) $\int_{1}^{2} \frac{dx}{x^2}$ 3) $\int_{-1}^{1} \sqrt{1-x}dx$

4) $\int_{0}^{\frac{\pi}{2}} \sin x\,dx$ 5) $\int_{0}^{1} \left(e^x + e^{-x}\right)dx$ 6) $\int_{0}^{2} \sqrt{2x+5}dx$

7) $\int_{2}^{3} \frac{dx}{x^2-1}$ 8) $\int_{1}^{2} \frac{dx}{x^2+2x}$ 9) $\int_{1}^{4} \frac{x^2-2}{x\sqrt{x}}dx$

10) $\int_{0}^{1} \frac{\ln(3x+1)}{3x+1}dx$ 11) $\int_{1}^{4} \frac{e^{2\sqrt{x}}}{\sqrt{x}}dx$ 12) $\int_{1}^{2} \frac{1}{x+1}dx$

5.7 Change of Variables

If for each t$\in [a,b]$, $g'(t)$ is continuous and the function f is continuous at g(t), then by the method of substitution we have:

$\int f(u)du = \int f(g(t))g'(t)dt$ -------------------- $\otimes$

If we integrate the right hand side from a to b ; Since u = g(t), the limit of integration of the left hand side of $\otimes$ above will be u = g(a) and u = g(b) and hence $\int_{g(a)}^{g(b)} f(u)du = \int_{a}^{b} f(g(t))g'(t)\,dt$.

Example 1 : Evaluate $\int_{0}^{\frac{\pi}{3}} \sqrt{\cos x}\sin x dx$

Solution: Let g(x) = cosx and f(u) =√u

$\Rightarrow g'(x) = -\sin x$ *thus the given integral takes the form*

$$\int_0^{\frac{\pi}{3}} \sqrt{\cos x}\,\sin x\,dx = \int_0^{\frac{\pi}{3}} f(g(x))g'(x)dx = \int_{g(0)}^{g(\frac{\pi}{3})} f(u)du = -\int_1^{\frac{1}{2}} f(u)du = -\int_1^{\frac{1}{2}} \sqrt{u}\;du = -\frac{2}{3}u^{\frac{3}{2}}\Big|_1^{1/2}$$

$$= \frac{2}{3} - \frac{\sqrt{2}}{6}.$$

Example 2: Evaluate the integral: $\int_1^3 \frac{\sqrt[3]{\ln x}}{3}dx$

Solution Let u = lnx; du = $\frac{1}{x}dx$ $\Rightarrow u(3) = \ln 3$ *and* $u(1) = \ln 1 = 0$

$$\therefore \int_1^3 \frac{\sqrt[3]{\ln x}}{x}dx = \int_0^{\ln 3} \sqrt[3]{u}\;du = \int_0^{\ln 3} u^{\frac{1}{3}}du = \frac{3}{4}u^{\frac{4}{3}}\Big|_0^{\ln 3} = \frac{3}{4}(\ln 3)^{\frac{4}{3}}$$

Exercises 5.9

Evaluate the following definite integrals.

1) $\int_0^{\frac{\pi}{2}} \cos(2x+1)dx$

2) $\int_0^{\left(\frac{\pi}{2}\right)} \cos\left(\sqrt{x}\right)dx$

3) $\int_0^{\frac{1}{2}} \frac{-2x}{1-x^2}dx$

4) $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin x \cos x\,dx$

5) $\int_0^1 \left(4x^3+1\right)\left(x^4+x\right)^6 dx$

6) $\int_3^4 \sqrt{x-3}dx$

7) $\int_{-2}^0 x\sqrt{2x^2+1}\;dx$

8) $\int_0^2 x^3\sqrt{(x^4+2)}\;dx$

9) $\int_0^{\frac{\pi}{3}} \sin x\,e^{\cos x}dx$

10) $\int_0^{\pi/4} \frac{\cos 3x}{(1-\sin 3x)^3}dx$

11) $\int_{-1}^2 xe^{2x^2}\,dx$

12) $\int_{1/2}^1 \frac{(\ln x)^3}{x}dx$

13) $\int_0^2 \frac{e^{2x}}{e^x+2}dx$

14) $\int_1^3 \frac{(y^{-1}+1)^3}{y^2}dy$

15) $\int_1^{\sqrt{e}} \frac{\ln(x)^3}{x}dx$

16) $\int_1^4 \frac{e^{-\sqrt{x}}}{\sqrt{x}}dx$