UNIT II

2.1 Derivatives

2.1.1 Geometric significance of the Derivative

Secant and Tangent Lines

The concept of derivative arose in connection with the geometric tangent to a plane curve and also in connection with the physical quantity velocity. We shall treat these in turn (tangent in this section and velocity in the application).

Consider two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ on the graph of $y=f(x)$. The line joining these two points is called a **secant line** and has a slope given by

$$m_{PQ} = \frac{y_2 - y_1}{x_2 - x_1}$$

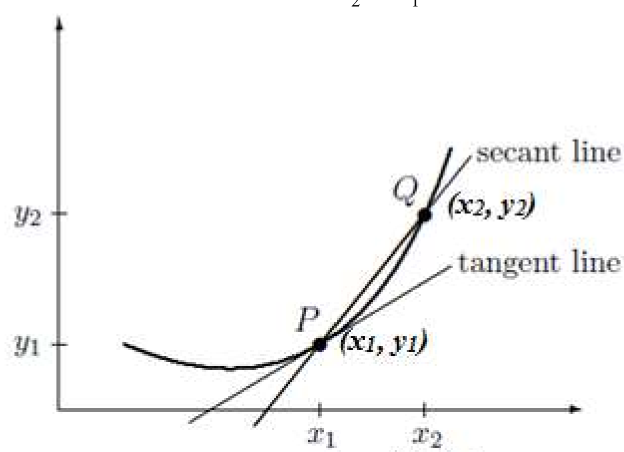


Fig 3.1

If we let $h=x_2 - x_1$, *then*

$x_2 = x_1+h$ and $y_2=f(x_2)=f(x_1+h)$

The **slope** of the **secant line joining** *P* and *Q* is then

$$m_{PQ} = \frac{y2 - y1}{x2 - x1} = \frac{f(x2) - f(x1)}{x2 - x1} = \frac{f(x_1 + h) - f(x_1)}{h}$$

Let's now imagine that *Q* slides along the curve towards point *P*. As it does so, the slope of the secant line joining *P* and *Q* will more closely approximate the slope of a tangent line to the curve at *P*. We can in fact define the slope of the tangent line at point *P* as the **limiting value of** m_{PQ} **as point *Q* approaches *P*.**

As point ***Q* approaches *P*, the value of *h* = x_2 - x_1 approaches zero**. The slope of the tangent line at *P* is then

$$m_{\tan} = \lim_{h \to 0} \frac{f(x_1 + h) - f(x_1)}{h} \text{ if the limit exists.}$$

<u>Note that:-</u>

1. The variable h is not exactly equal to 0 in the above limit, but it is simply approaching to 0 from the left and from the right side.
2. The tangent line to the graph of a function f at the point $(x_1, f(x_1))$ is:

i. the line through *P* with slope $m_{\tan}$, if $\lim_{h \to 0} \frac{f(x_1 + h) - f(x_1)}{h}$ exists and

its equation is $y - y_1 = m_{\tan}(x - x_1)$

ii. the vertical tangent line x =x_1, if $\lim_{x \to x_1} \frac{f(x) - f(x_1)}{x - x_1} = \pm\infty$.

Note that: -

1. The special limit which is very useful in calculus.

$$\boxed{\lim_{h \to 0} \frac{a^h - 1}{h} = \ln a \quad \text{where } a>0,\ a \neq 1}$$

2. Another important limit is finding the limit of the expression of the form

$$\frac{f(x) - f(a)}{x - a} \quad \underline{\text{[Difference quotient]}}$$

If we let h = x-a, then the difference quotient becomes

$$\frac{f(a+h) - f(a)}{h}$$

3. Difference quotient and its limit play an important role in calculus.

Example1. What is the equation of a tangent line to the graph of $f(x) = x^2 + 2x$ at the point (-3, 3)?

Solution: Let $f(x) = x^2 + 2x$ and x_1 = -3, then if it exists the slope of the tangent line becomes

$$m_{\tan}\ (-3) = \lim_{x\to -3}\frac{f(x)-f(-3)}{x-(-3)} = \lim_{x\to -3}\frac{(x^2+2x)-(3))}{x+3} = \lim_{x\to -3}\frac{x^2+2x-3}{x+3}$$

$$= \lim_{x\to -3}\frac{(x-1)(x+3)}{x+3} = \lim_{x\to -3}(x-1) = -3 - 1 = -4$$

$\Rightarrow$ The slope of the tangent line at (-3, 3) is -4

The equation of a tangent line is

$$\frac{y-3}{x-(-3)} = -4 \Rightarrow \frac{y-3}{x+3} = -4 \Rightarrow y - 3 = -4(x+3) \Rightarrow y-3 = -4x-12 \Rightarrow y = -4x-9$$

$\therefore$ The equation of the tangent line to the graph of $f(x) = x^2 + 2x$ at the point (-3, 3) is

y = -4x-9

Another method

Let h = x-(-3), when x $\to$-3 then h $\to$ 0 so

$$m_{\tan}\ (-3) = \lim_{x\to -3}\frac{f(x)-f(-3)}{x-(-3)} = \lim_{h\to 0}\frac{f(h+-3)-f(-3)}{h} = \lim_{h\to 0}\frac{f(h-3)-f(-3)}{h}$$

$$= \lim_{h\to 0}\frac{f[(h-3)^2+2(h-3)]-3}{h} = \lim_{h\to 0}\frac{h^2-6h+9+2h-6-3}{h}$$

$$= \lim_{h\to 0}\frac{h^2-4h}{h} = \lim_{h\to 0}\frac{h(h-4)}{h} = \lim_{h\to 0} h - 4 = 0 - 4 = -4$$

Example 2.Find the equation of the tangent line to the graph of $f(x) = \sin x$ at the point (0, 0).

Solution:

Let f(x) = sin x and x_1 = 0, then if exists the slope of the required tangent line becomes

$$m_{\tan}\ (0) = \lim_{x\to 0}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0}\frac{\sin x - \sin 0}{x} = \lim_{x\to 0}\frac{\sin x - 0}{x}$$

$$= \lim_{x\to 0}\frac{\sin x}{x} = 1$$

So the equation of the tangent line is

$$y - y_1 = m_{\tan}(0)(x - x_1)$$
$$\Rightarrow y - 0 = 1(x-0)$$
$$\Rightarrow y = x$$

$\therefore$ The equation of the tangent line to the graph of f(x) = sin x at the point (0, 0) is y=x

Example 3: Let g(x) = $\begin{cases} 2x+1, if x>1 \\ x^2+2, if x \le 1 \end{cases}$ is the graph of g(x) have a line tangent at the point (1, 3). Then find its equation.

Solution:

The tangent line at x=1 exists if $\lim_{x\to 1} \frac{g(x)-g(1)}{x-1}$ exists or $\pm\infty$.

$$\lim_{x\to 1^+} \frac{g(x)-g(1)}{x-1} = \lim_{x\to 1^+} \frac{(2x+1)-3}{x-1} = \lim_{x\to 1^+} \frac{2x-2}{x-1} = \lim_{x\to 1^+} \frac{2(x-1)}{x-1} = \lim_{x\to 1^+} 2 = 2.$$

And

$$\lim_{x\to 1^-} \frac{g(x)-g(1)}{x-1} = \lim_{x\to 1^-} \frac{(x^2+2)-3}{x-1} = \lim_{x\to 1^-} \frac{x^2-1}{x-1} = \lim_{x\to 1^-} \frac{(x-1)(x+1)}{x-1} = \lim_{x\to 1^-} (x+1) = 2$$

Since $\lim_{x\to 1^+} \frac{g(x)-g(1)}{x-1} = \lim_{x\to 1^-} \frac{g(x)-g(1)}{x-1} = 2$, so $\lim_{x\to 1} \frac{g(x)-g(1)}{x-1}$ exists and equal to 2.

$\Rightarrow$ The slope ($m_{\tan}$ (1)) of the tangent line is 2.

Therefore, the equation of the tangent line is

$$\frac{y-3}{x-1} = 2 \quad \Rightarrow \quad y-3=2(x-1) \quad \Rightarrow \quad y-3=2x-2 \quad \Rightarrow \quad \underline{\underline{y=2x+1}}$$

Note: If $\lim_{x\to x_1} \frac{f(x)-f(x_1)}{x-x_1} = \pm\infty$, then the graph of f has a vertical tangent line, i.e. the line x = x_1.

Example 4: Let f(x) = $x^{1/3}$. Then at (0, 0) the graph has a vertical line $\underline{\underline{x = 0}}$ as a tangent line because:-

$$\lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{x^{1/3}-0}{x} = \lim_{x\to 0} \frac{1}{x^{2/3}} = \infty$$

Example 5: Let f(x) = x^2 and g(x) = x^3. For what value (s) of x is (are) tangent lines to the graphs of these functions parallel?

Solution:

Let the required value of x be a, and the slope of tangent line for f(x) at x = a is m_1 and g(x) at x = a is m_2; then the two tangent are parallel if $m_1 = m_2$.

But $m_1 = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{x\to a} \frac{x^2-a^2}{x-a} = \lim_{x\to a} \frac{(x-a)(x+a)}{x-a} \lim_{x\to a} x+a = \underline{\underline{2a}}$ and

$m_2 = \lim_{x\to a} \frac{g(x)-g(a)}{x-a} \quad \lim_{x\to a} \frac{x^3-a^3}{x-a} \quad \lim_{x\to a} \frac{(x-a)(x^2+ax+a^2)}{x-a}$

$= \lim_{x\to a} (x^2+ax+a^2) = \underline{\underline{3a^2}}$

Since $m_1=m_2 \Rightarrow 2a=3a^2 \Rightarrow 3a^2-2a=0 \Rightarrow a(3a-2)=0$

$\Rightarrow a = 0$ or $3a - 2 = 0 \quad \Rightarrow a = 0$ or $a = \frac{2}{3}$

$\therefore$ The values of x are 0 and $\frac{2}{3}$

Exercise 3.1

i. Find the equation of the tangent line for the following functions graphs at the given point.

a/ $f(x) = x^2$, if $x=1$ at the point (0,1) b/ $g(x) = x^2$ at the point (x, y)

c/ $h(x) \begin{cases} x^3 \text{ if } x \geq 0 \text{ at the point } (0,0) \\ x^2 \text{ if } x<0 \end{cases}$

ii. Do the following functions have tangent line to their graphs at the given point?

a. $f(x) = |x|$ at the point (1, 1)? at the point (0, 0) ?

b. $g(x) = \begin{cases} x \text{ for } x<0 \text{ at the point } (0,0) \text{ ?} \\ x^2 \text{ for } x \geq 0 \end{cases}$

iii. Let $f(x) = x$ and $g(x) = x^2$. For what value (s) of x is (are) tangent lines to the graphs of these functions parallel or perpendicular

2.2 Definition of the Derivative

In determining the slope $m_{\tan}(x_1)$ to the curve $y = f(x)$ at point $x=x_1$, we were led to the formula.

$$m_{\tan}(x_1) = \lim_{x\to x_1} \frac{f(x)-f(x_1)}{x-x_1}$$

If *f(x)* is a function defined at x_1, we can let $h = x - x_1$ and then $x = x_1+h$. From this we can see that as x gets closer to x_1, $x - x_1 = h$ gets closer to 0 ($x \to x_1 \Rightarrow h\to 0$). Then the above slope of the tangent formula becomes.

$$m_{\tan}(x_1) = \lim_{x\to x_1} \frac{f(x)-f(x_1)}{x-x_1} = \lim_{h\to 0} \frac{f(x_1+h)-f(x_1)}{h}$$

Using these, we are now on the position to define the derivative of a function and its derivative at a point.

Definition 3.1

Let a function f be defined at x_0. If $\lim_{x \to x_0} \frac{f(x)-f(x_o)}{x-x_0}$ or $\lim_{h \to 0} \frac{f(x_0+h)-f(x_0)}{h}$

exists, then the value of the limit is called **the derivative of f at x_0** and is denoted by

$f'(x_0)$ or $D_x f(x_o)$.

If the limit exists, then $f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h)-f(x_0)}{h}$ or $f'(x_0) = \lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0}$

In this case, we say f has derivative at x_0 or the derivative of f exists at x_0.

In general for any arbitrary element x in the domain of the function f,

if $\lim_{h \to o} \frac{f(x+h)-f(x)}{h}$ exists ,then the **derivative of** f , denoted by $f'(x)$ or $\frac{d}{dx}$ f(x) or

$D_x f(x)$ is defined as $f'(x) = \lim_{h \to 0_{11}} \frac{f(x+h)-f(x)}{h}$. The process of determining $f'(x)$ is

called **differentiation.**

Note that: the derivative is the limit of $\frac{f(x)-f(x_o)}{x-x_o}$ as $(x-x_o) \to 0$ $[x \to x_o]$ if this limit exists.

However, if the limit may fail to exist, and in this case the derivative is not defined.

Definition 3.2

1. A function f is said to be **differentiable at a point x_0** in its domain, if and only if f has derivative at x_o.
2. A function f is said to be **differentiable in its domain**, if and only if f is differentiable at every point in its domain.

Note that:

i. The derivative of a function at a given point x_0 is the slope of the tangent line to the graph of f at (x_0, f (x_0)).

ii. To differentiate a function means to find its derivative.

iii. We can use the formals $f'(xo) = \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}$ or

$f'(x_0) = \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0}$ to find the derivative of f at x_0.

Example 6: Find the derivative of a function at the given number if exists.

a/ $f(x) = 4x-x^2+5$; at x = -1

b/ $g(x) = \left(\frac{1}{2}\right)^x$; at x = -1

c/ $h(x) = \sqrt{x}$; at x = 0

Solution:

a. For $f(x) = 4x- x^2+5$, find $f'(-1)$

$$f'(-1) = \lim_{x\to -1} \frac{f(x)-f(-1)}{x-(-1)} = \lim_{x\to -1} \frac{(4x-x^2+5)-0}{x+1} = \lim_{x\to -1} \frac{-x^2+4x+5}{x+1}$$

$$= \lim_{x\to -1} \frac{(5-x)(x+1)}{x+1}$$

$$= \lim_{x\to -1} (5-x) = 5-(-1) = \underline{\underline{6}}$$

;2b. $g(x) = \left(\frac{1}{2}\right)^x$ find $g'(-1)$

$$g'(-1) = \lim_{h\to 0} \frac{g(-1+h)-g(-1)}{h} = \lim_{h\to 0} \frac{\left(\frac{1}{2}\right)^{-1+h} - \left(\frac{1}{2}\right)^{-1}}{h} = \lim_{h\to 0} \frac{\left(\frac{1}{2}\right)^{-1} . \left(\frac{1}{2}\right)^{h} - \left(\frac{1}{2}\right)^{-1}}{h}$$

$$= \lim_{h\to 0} \frac{\left(\frac{1}{2}\right)^{-1}\left[\left(\frac{1}{2}\right)^{h} - 1\right]}{h} = \lim_{h\to 0} \frac{2\left[\left(\frac{1}{2}\right)^{h} - 1\right]}{h} = 2\lim_{h\to 0} \frac{\left(\frac{1}{2}\right)^{h} - 1}{h} = 2\ln\left(\frac{1}{2}\right) = \underline{\underline{-2\ln 2}}$$

c. $h(x)=\sqrt{x}$, find $h'(0)$

$$h'(0) = \lim_{x\to 0}\frac{h(x)-h(0)}{x-0} \quad \lim_{x\to 0}\frac{\sqrt{x}-\sqrt{0}}{x} = \lim_{x\to 0}\frac{\sqrt{x}}{x} = \lim_{x\to 0}\frac{1}{\sqrt{x}}$$

$\therefore$ The derivative doesn't exist at x = 0. (**Why?**)

<u>Example:7</u> Find the derivative of the following functions by stating the domain where it is differentiable .

a/ f(x) = 2 b/ g(x) = $\sqrt{x+1}$ c/ f(x)= cos x

Solution:

a. $f(x) = \lim_{x\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{x\to 0}\frac{2-2}{h} = \lim_{x\to 0}\frac{0}{h} = \lim_{x\to 0} 0 = \underline{\underline{0}}$

$\therefore$ For <u>all real numbers</u> $f'(x) = 0$

b. $g'(x) = \lim_{x\to 0}\frac{g(x+h)-g(x)}{h} = \lim_{x\to 0}\frac{\sqrt{x+h+1}-\sqrt{x+1}}{h}\cdot\frac{\sqrt{x+h+1}+\sqrt{x+1}}{\sqrt{x+h+1}+\sqrt{x+1}}$

$$= \lim_{x\to 0}\frac{(x+h+1)-(x+1)}{h(\sqrt{x+h+1}+\sqrt{x+1}} = \lim_{x\to 0}\frac{h}{h(\sqrt{x+h+1}+\sqrt{x+1}}$$

$$= \lim_{x\to 0}\frac{1}{\sqrt{x+h+1}+\sqrt{x+1}}$$

$$= \frac{1}{\sqrt{x+1}+\sqrt{x+1}} = \underline{\underline{\frac{1}{2\sqrt{x+1}}}}$$

$\therefore$ For $x \in (-1,\infty)$ $f'(x) = \frac{1}{2\sqrt{x+1}}$

c. For f(x) = cosx,

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h}$$

$$= \lim_{h\to 0}\frac{\cos x\cos h-\sin x\sin h-\cos x}{h} = \lim_{h\to 0}\frac{\cos x(\cos h-1)-\sin x\sin h}{h}$$

$$= \lim_{h\to 0}\frac{\cos x(\cos h-1)}{h} - \lim_{h\to 0}\frac{\sin x\sin h}{h} = \cos x\lim_{h\to 0}\frac{\cos h-1}{h} - \sin x\lim_{h\to 0}\frac{\sin h}{h}$$

$$= \cos x\lim_{h\to 0}\frac{\cos h-1}{h}\cdot\frac{\cos h+1}{\cos h+1} - \sin x\lim_{h\to 0}\frac{\sin h}{h}$$

$$= \cos x\lim_{h\to 0}\frac{\cos^2 h-1}{h(\cos h+1)} - \sin x\lim_{h\to 0}\frac{\sin h}{h}$$

$$= -\cos x \lim_{h\to 0} \frac{\sin h . \sin h}{h(\cos h + 1)} - \sin h \lim_{h\to 0} \frac{\sin h}{h}$$

$$= -\cos \lim_{h\to 0} \frac{\sin h}{h} . \lim_{h\to 0} \frac{\sin h}{\cos h + 1} - \sin x \lim_{h\to 0} \frac{\sin h}{h}$$

$= -\cos x.1.0 - \sin x .1 = 0 - \sin x = \underline{\underline{- \sin x}}$

$\therefore$ For all real number $f'(x) = -\sin x$

Example 8: Let the function f is given by $f(x) = 3x^2+mx-2$ for some constant m. If $f'(1) = 4$, then what is the value (s) of m ?

Solution:

If $f(x) = 3x^2+mx-2$ is given, then its derivative at x =1 is

$$f'(1) = \lim_{x\to 1} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1} \frac{(3x^2+mx-2)-(m+1)}{x-1} = \lim_{x\to 1} \frac{3x^2+mx-m-3}{x-1}$$

$$= \lim_{x\to 1} \frac{3x^2-3+mx-m}{x-1} = \lim_{x\to 1} \frac{3(x^2-1)+m(x-1)}{x-1} = \lim_{x\to 1} \frac{3(x-1)(x+1)+m(x-1)}{x-1}$$

$$= \lim_{x\to 1} \frac{(x-1)[3(x+1)+m]}{x-1}$$

$$= \lim_{x\to 1} [3(x+1)+m] = \underline{\underline{6+m}}$$

But $f'(1) = 4 \Rightarrow 6+m = 4 \Rightarrow \underline{\underline{m = -2}}$

Exercise 3.2

i. Let $f(x) = x^2-x+1$. Find $f'(0)$ and $f'(2)$ and draw the lines tangent to the graph of f at the corresponding points .

ii. Find the derivative of each of the following functions by stating the domain in which the function is differentiable.

a/ $f(x) = 4x+3$

b/ $g(x) = x^2$

c/ $h(x) = \sin x$

d/ $f(x) = \frac{1}{x}$

e/ $g(x) = \sqrt{x}$

iii. For the given functions find the derivative at the given points if exists.

a/ $f(x) = \begin{cases} x \text{ if } x \le 0 \\ 1+x \text{ if } x > 0 \end{cases}$ at $x = 0$

b/ $g(x) = \begin{cases} -2x^2+2 \text{ if } x \ge 0 \\ -\frac{1}{2}x^2 \text{ if } x < 0 \end{cases}$ at $x = 0$

c/ $h(x) = |x+1|$ at $x = -1$

iv. Let the function $f(x) = x^2$ is given. Then find

a/ the equation of the tangent line and

b/ the equation of the line perpendicular to the tangent at the point (0,0) .

2.4 Continuity and Differentiability

In the previous unit (unit II: Limit and Continuity), we have already discussed about the continuity of a function at a point and on the interval. So, in this sub-unit, we will consider the derivative of a function at all numbers in the domain of a function at which the function is differentiable.

For fixed *x* and *t* approaches *x*, We have the following definition

Definition 3.3:

Let *f* be a function and *S* be a subset of the domain of *f*. If $\lim_{t\to x} \frac{f(t)-f(x)}{t-x}$ exists for each x in *S*, then f is said to be differentiable on *S* and its derivative is a function on S denotes by f' (*S*), so that $f'(x) = \lim_{t\to x} \frac{f(t)-f(x)}{t-x}$

Note that: In the above formula of $f'(x)$, x represents any number at which f is differentiable. However, where the limit on the right side of the formula is evaluated, t is the variable and x is regarded as a constant.

Definition 3.4

If a function f is differentiable at each number in its domain, then f is said to be a differentiable function.

Example 9: For constants a and b, let the function f is given by f(x) = ax+ b. Find the derivative of f in $(-\infty, \infty)$ or IR .

Solution: Follow the following steps:

Step1: f(x) = a x + b

Step2: f(t) = at + b

step3 : $\frac{f(t)-f(x)}{t-x} = \frac{at+b-(ax+b)}{t-x} = \frac{at-ax}{t-x} = \frac{a(t-x)}{t-x} = a$ for $t \neq x$.

step4: $\lim_{t \to x} \frac{f(t)-f(x)}{t-x} = \lim_{t \to x} a = \underline{\underline{a}}$

:. For any constants a and b, the derivative of f (x) = ax + b in IR is a.

Activity 3.2

1. Let f(x) = c for $x \in$ IR, where c is a constant, find $f'(x)$
2. Let f(x) = x for $x \in$ IR, find $f'(x)$
3. Let f(x) = ax for $x \in$ IR, where a is a constant, find $f'(x)$,

Example10: Let $f(x) = x^2$, find $f'(x)$ for all $x \in (-\infty,\infty)$

Solution:

$$f'(x) = \lim_{t \to x} \frac{f(t)-f(x)}{t-x} = \lim_{t \to x} \frac{t^2-x^2}{t-x} = \lim_{t \to x} \frac{(t-x)(t+x)}{t-x} = \lim_{t \to x} (t+x) = x+x = \underline{\underline{2x}}$$

Activity 3.3:

1. Let $f(x) = ax^2$ where a is any constant, then find $f'(x)$ for $x \in (-\infty, \infty)$.
2. Let $f(x) = ax^2 + b$ where a, b constants, find $f'(x)$ in $(-\infty, \infty)$.
3. Let $f(x) = ax^2 + bx + c$ where a, b, c constants, then find $f'(x)$ in $(-\infty,\infty)$
4. Let $f(x) = \sqrt{x}$ for all $x \in (0, \infty)$, find the derivative of f(x) on $(0, \infty)$

Note that:

The above functions are differentiable in their domain. However, there are functions that are not differentiable in their domain.

Example 11:-Let $f(x) = |x-4|$, then show that f is not differentiable at x = 4.

Solution: Clearly the domain of f(x) is IR.

Now let's check whether $f'(4) = \lim_{x\to 4} \frac{f(x)-f(4)}{x-4}$ exists or not.

But form absolute definition we have:

$$f(x) = |x-4| = \begin{cases} x-4 \text{ if } x \geq 4 \\ 4-x \text{ if } x < 4 \end{cases}$$

$$\lim_{x\to 4^-} \frac{f(x)-f(4)}{x-4} = \lim_{x\to 4^-} \frac{4-x-0}{x-4} = \lim_{x\to 4^-} \frac{x-4}{x-4} = \lim_{x\to 4^-} \frac{-(x-4)}{x-4} = \lim_{x\to 4^-} -1 = -1$$

And $\lim_{x\to 4^+} \frac{f(x)-f(4)}{x-4} = \lim_{x\to 4^+} \frac{x-4-0}{x-4} = \lim_{x\to 4^+} \frac{x-4}{x-4} = \lim_{x\to 4^+} 1 = 1$

Since the right hand and the left hand limits at 4 are <u>not</u> equal, it follows

$$\lim_{x\to 4} \frac{f(x)-f(4)}{x-4} = \lim_{x\to 4} \frac{|x-4|-0}{x-4} \quad \text{does not exist.}$$

$\therefore$ f doesn't have a derivative at 4.

$\Rightarrow$ f is not differentiable function on IR. But f is differentiable for all x $\in$ IR/{4}

2.5 Differentiation Formulas (Techniques of Differentiation)

I. Properties of Derivative

From the previous section, we have seen that:

- For any constant c and $x \in (-\infty, \infty)$ the derivative of f(x)=c is 0 .
- For any constants a and b, the derivative of f(x) = ax + b in $(-\infty, \infty)$ is a.
- If $f(x) = x^2$, then $f'(x) = 2x$ for all x $\in$ IR.
- If $f(x) = \sqrt{x}$, then $f'(x) = \frac{1}{2\sqrt{x}}$ for x $\in (0,\infty)$

Properties of derivative are some of the techniques of differentiation. These properties are given on the basis of the derivative of combined functions.

<u>Theorem 3.1</u>

If f is differentiable at x_o, then f is continuous at x_o.

[We will see the proof of the theorems in Math 261 Course]

Note that: The converse of the above theorem (Theorem2.1) is false

i.e., if f is continuous at x_o, then f may or may not be differentiable at x_o.

$\Rightarrow$ There are functions which are continuous but not differentiable at a point.

Example 11. f (x) = $|x-1|$ is continuous at x = 1, but f is not differentiable at x =1(**why**?)

Theorem 3.2

Let f and g be two differentiable functions at the point x, then.

i. f + g and f - g are differentiable at x and

$$(f+g)'(x) = f'(x) + g'(x) \text{ and}$$

$$(f-g)'(x) = f'(x) - g'(x).$$

ii. f.g is differentiable at x and

$$(f.g)'(x) = f'(x)g(x) + f(x)g'(x)$$

iii. If g (x) ≠ 0, $\frac{f}{g}$ is differentiable at x and

$$\left(\frac{f}{g}\right)'(x) = \left(\frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}\right)$$

Note that: The above theorem is applicable for a finite number of differentiable functions.

E.g. $(f+g+h+k)'(x) = f'(x) + g'(x) + h'(x) + k'(x)$

Example 12: Find the derivative of f(x) = x –cos x

Solution: Let f(x) = x-cosx then

$f'(x) = (x - \cos x)' = (x)' - (\cos x)'$

But $(x)' = 1$ and $(\cos x)' = -\sin x$

$\Rightarrow f'(x) = 1-(-\sin x) = \underline{\underline{1+\sin x}}$

Theorem 3.3

If f is differentiable at x, then for any number c the function cf is differentiable at x and $(cf)'(x) = cf'(x)$.

Example 13: Use the properties of derivative to find the derivative of each of the following functions at a point x in its domain.

a. $f(x) = ax^2+bx+c$ where a, b, c constants

b. $g(x) =2x^2-3x-\sqrt{x}$ for $x > 0$

c. $h(x) = x^4$

d. $f(x) = \dfrac{x+1}{x^2-2x}$ for $x\in IR/\{2,0\}$

Solution:

a. If $f(x) =ax^2+x+c$, then since *f* is a combination of differentiable functions then by the above theorem we have;

$$f'(x) = (ax^2+bx+c)' = (ax^2)'+(bx)'+(c)' = a(2x)+b(1)+0 = 2ax+b$$

b. If $g(x) =2x^2-3x-\sqrt{x}$, then

$$g'(x) = (2x^2-3x-(\sqrt{x}))' = (2x^2)'-(3x)'-(\sqrt{x})'$$

$$= 2(x^2)'-3(x)'-(\sqrt{x})'$$

$$= 2(2x)-3(1)-\left(\frac{1}{2\sqrt{x}}\right) = 4x-3-\frac{1}{2\sqrt{x}}$$

c. If $h(x) =x^4$, then h(x) can be written as $h(x) =x^2.x^2$

$$h'(x)= (x^4)' = (x^2.x^2)' = (x^2)' = (x^2)'.x^2+x^2(x^2)' =(2x).x^2+x^2(2x)$$

$$= 2x^3+2x^3= \underline{\underline{4x^3}}$$

d. If $f(x) = \dfrac{x+1}{x^2-2x}$, then

$$f'(x) = \left(\frac{x+1}{x^2-2x}\right)' = \frac{(x+1)'(x^2-2x)-(x+1)(x-2x)}{(x^2-2x)^2}$$

$$= \frac{[(x')+(1')](x^2-2x)-(x+1)[(x^2)'-(x^2)']}{(x^2-2x)^2} = \frac{(1+0)(x^2-2x)-(x+1)[2x-2(x)']}{(x^2-2x)^2}$$

$$= \frac{1(x^2-2x)-(x+1)[2x-2(1)}{(x^2-2x)^2} = \frac{x^2-2x-(x+1)(2x-2)}{(x^2-2x)^2}$$

$$= \frac{x^2-2x-2x^2+2}{(x^2-2x)^2} \quad = \quad \frac{-x^2-2x+2}{(x^2-2x)^2} \quad = \quad \frac{-x^2-2x+2}{\underline{\underline{x^4-4x^3+4x^2}}}$$

Exercise 3.3

i. Using properties of derivative find the derivative of the following functions.

a/ $f(x)= x^3+4x^2+3$

b/ $g(x) = \frac{1}{x^2}$ for $x\neq 0$

c/ $h(x)=3x^2-2\sqrt{x}$ for $x>0$

d/ $l(x) = \frac{1}{x^2+4}$

ii. Let $f(x) =3x+x^2$. If $f'(a) =7$, find a

iii. If $f(x) = \tan x$, then find $f'(x)$ in its domain

II. Derivatives of power functions

Proposition 1 $(x^n)' = nx^{n-1}$ for $n\in N$.

Note that: this proposition and quotient derivative property gives us

$$\left(\frac{1}{x^n}\right)' = \frac{-n}{x^{n+1}} \qquad \text{for } x \neq 0 \text{ and } n\in N$$

In general: Let r be any fixed non- zero real number. If $f(x) =x^r$,then $f'(x) = rx^{r-1}$ in the domain of x^{r-1}

Example 14: Find the derivatives of the functions if

a/ $f(x)\ 2x^7-3x^5+x^21$

b/ $g(x) = 3x^{\frac{2}{3}} +2x^5 + \frac{2}{x^4}$ for $x\neq 0$

c/ $h(x)= \frac{x^2+1}{\sqrt{x}}$ for $x>0$

Solution: a. $f(x) = 2x^7 -3x^5+x^2+1$, then

$$f'(x) =(2x^7-3x^5+x^2+1)' = (2x^7)' -(3x^5)' + (x^2)' + (1)' = 2(x^7)' -3 (x^5)' +(x^2)' +(1)'$$

$$= 2(7x^6)-3(5x^4)+(2x)+0 = \underline{14x^6-15x^4+2x}$$

b) If g(x) $= 3x^{\frac{2}{3}} + 2x^5 + \frac{2}{x^4}, then$

$$g'(x) = (3x^{\frac{2}{3}} + 2x^5 + \frac{2}{x^4})'$$

$$= \left(3x^{\frac{2}{3}}\right)' + \left(2x^5\right)' + \left(\frac{2}{x^4}\right)' = 3\left(x^{\frac{2}{3}}\right)' + 2\left(x^5\right)' + 2\left(\frac{1}{x^4}\right)'$$

$$= 3\left(\frac{2}{3}x^{\frac{2}{3}-1}\right) + 2\left(5x^4\right) + 2\left(\frac{-4}{x^5}\right) = 2x^{\frac{-1}{3}} + 10x^4 - \frac{8}{x^5} = \frac{2}{x^{\frac{1}{3}}} + 10x^4 - \frac{8}{x^5}$$

c) If h(x) = $\frac{x^2+1}{\sqrt{x}}$, then

$$h'(x) = \left(\frac{x^2+1}{\sqrt{x}}\right)' = \frac{(x^2+1)'.\sqrt{x} - (x^2+1)(\sqrt{x})'}{\left(\sqrt{x}\right)^2} = \frac{2x\sqrt{x} - \frac{x^2+1}{2\sqrt{x}}}{x} forx > 0$$

$$= \frac{4x^2 - x^2 - 1}{2x\sqrt{x}} \qquad = \underline{\underline{\frac{3x^2-1}{2x\sqrt{x}}}}$$

III. Derivatives of the Basic Trigonometric functions

From the previous section, we have already seen that

(sinx)' = cos x and

(cos x)' = -Sin x for x $\in$ IR

Since the basic trigonometric functions such as tan x, sec x cosec x and cot x are combinations of sinx and /or cosx, by using the division property of derivative we can find the derivatives of these trigonometric functions.

Example 15 : Find the derivatives of

a/ f(x) = sec x for x $\neq \frac{n\pi}{2}$; n$\in$z

b/ g(x) = cosec x for x $\neq n\pi$; n$\in$z

Solution: a. f'(x) = (sec x)' $= \left(\frac{1}{\cos x}\right)' = \frac{(1)'.\cos x - 1(\cos x)'}{(\cos x)^2} = \frac{0.\cos x - 1(-\sin x)}{\cos^2 x}$

$$= \frac{0+\sin x}{\cos^2 x} = \frac{\sin x}{\cos^2 x} = \frac{\sin x}{\cos x}.\frac{1}{\cos x} = \tan x.\sec x$$

b. $g'(x) = (\text{cosec } x)' = \left(\frac{1}{\sin x}\right)' = \frac{(1)'.\sin x - 1(\sin x)'}{(\sin x)2} = \frac{0.\sin x - \cos x}{\sin^2 x}$

$$= \frac{0-\cos x}{\sin^2 x} = \frac{-\cos x}{\sin^2 x} = \frac{-\cos x}{\sin x}.\frac{1}{\sin x} = -\cot x.\cos ecx$$

$\therefore$ **For example 15**, we can find

- ❖ $(\sec x)' = \tan x.\sec x$ for $x \neq \frac{n\pi}{2}$; $n\in z$. and
- ❖ $(\text{cosec} x)' = -\cos x.\ \text{cosec} x$ for $x \neq n\pi$; $n\in z$

Example 16:- Find the derivatives of

a/ $f(x) = \tan x$ for $x \neq \frac{n\pi}{2}$; $n \in z$

b/ $g(x) = \cot x$ for $x \neq n\pi$; $n\in z$

c/ $h(x) = \cos x + x\sin x$. or $x \in IR$

Solution: a. $f'(x) = (\tan x)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{(\sin x)'.\cos x - \sin x.(\cos x)'}{(\cos x)^2}$

$$= \frac{\cos x.\cos x - \sin x.(-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

b. $g'(x) = (\cot x)' = \left(\frac{\cos x}{\sin x}\right)' = \frac{(\cos x)'.\sin x - \cos x.(\sin x)'}{(\sin x)^2}$

$$= \frac{(-\sin x).\sin x - \cos x(\cos x)}{\sin^2 x}$$

$$= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = \frac{-(\sin^2 x + \cos^2 x)}{\sin^2 x}$$

$$= \frac{-1}{\sin^2 x} = -\underline{\underline{\text{cosec}^2 x}}$$

c. $h'(x) = (\cos x + x\sin x)' = (\cos x)' + (x\sin x)' = (\cos x)' + [(x)'.\sin x + x(\sin x)']$

$$= -\sin x + \sin x + x\cos x = \underline{\underline{x\cos x}}$$

$\therefore$ **From example 16 (a and b), we can find**

- ❖ $(\tan x)' = \sec^2 x$ for $x \neq \frac{n\pi}{2}$; $n\in z$. and
- ❖ $(\cot x)' = -\text{cosec}^2 x$ for $x \neq \frac{n\pi}{2}$, $n\pi$; $n\in z$

2.5 The Chain Rule

In the previous section, we have tried to find the derivatives of combinations of differentiable functions. Now we are going to find the derivatives of compositions of function. If f and g are any two differentiable functions, then we can find the derivative of the composition of functions [i.e., (fog) (x) or (gof)(x)] in their domains by using the following theorem.

Remember that: If $f(x) = x^3$ and $g(x) = x+2$, then the composition functions are :-

$$(fog)(x) = f(g(x)) = f(x+2) = (x+2)^3 \text{ and}$$

$$(gof)(x) = g(f(x)) = g(x^3) = x^3+2$$

Theorem 3.4 (Chain Rule)

If f and g are two continuous functions such that g is differentiable at x and f is differentiable at g(x) , then

$$[(fog)(x)]' = [f(g(x)]' = f'(g(x)).g'(x)$$

Example 17: Differentiate the following functions by using the chain rule.

a. $P(x)= (x^4-2x^3-x^2-1)^3$

b. $k(x)= \sqrt{5x-8}$ for $x > \frac{8}{5}$

c. $r(x) = \cos^3 (x^2+1)$

Solution:

a. let $g(x) = x^4-2x^3- x^2-1$ and $f(x) =x^3$, then $p(x) = (fog)(x)$

Therefore, $p'(x) = f'(g(x)).\, g'(x)$. But $f'(x) =(x^3)' =3x^2$ and

$g'(x)=(x^4-2x^3+x^2-1)' = 4x^3-6x^2+2x$

So, $f'(g(x)) = f'(x^4-2x^3+x^2-1) = 3(x^4-2x^3+x^2-1)^2$

$$\therefore p'(x) =3(x^4-2x^3+x^2-1)^2.(4x^3-6x^2+2x) =\underline{6x(x^4-2x^3+x^2-1)^2 (2x^2-3x+1)}$$

b. Let $g(x) = 5x - 8$ and f(x) = $\sqrt{x}$, then k(x) = (fog) (x). But $g'(x) = 5$ and $f'(x) = \frac{1}{2\sqrt{x}}$

So, using the chain rule we get,

$$k'(x) = f'(g(x)).g'(x)$$

$$= f'(5x-8).g'(x)$$

$$= \frac{1}{2}(5x-8)^{\frac{-1}{2}}(5)$$

$$= \frac{1}{2\sqrt{5x-8}}(5)$$

$$= \frac{5}{2\sqrt{5x-8}}$$

d. Let h(x) =x^2+1 , g(x) =cos x and f(x) =x^3 then r(x) = [fo(goh] (x) = f(g(h(x)))

r'(x) = [f(g(h(x)))]' = f'[g(h(x))]'. [g(h(x))] =f'[g(h(x))]. g'(h(x)).h'(x).

but f' x)= $(x^3)' = 3x^2$, g'(x) = (cosx)' = -sinx and h'(x) = (x) = $(x^2+1)'$ =2x

So, g(h(x)) = g(x^2+1) = -sin(x^2+1)

$\therefore$ r'(x) = 3 $\cos^2(x^2+1)$. [-sin(x^2+1)].(2x) = - $\underline{\underline{6x \cos^2 (x^2+1) \sin (x^2)}}$.

Derivatives of Logarithmic and Exponential functions

We know the number e is an irrational number approximately e $\cong$ 2.718 and is defined as:

$$e = \lim_{t \to 0} (1+t)^{\frac{1}{t}}$$

$$e = \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x$$

Definition 3.5:

The logarithm of x, x >0, to the base *e* is called the **natural logarithm** and denoted by *lnx*.

Note: lnx = $\log^{x}{}_{e}$, where x>0

Theorem 3.5

Let a >0 and a ≠1. then

$$(\log_a^x)' = \frac{1}{x} \log_a^e$$

- Using change of base for the logarithm of $\log_a^e$, then we have $\frac{1}{\log_e^a} = \frac{1}{\log e^a} = \frac{1}{\ln a}$

So the above formula can be written as:-

$$(\log_a^x)' = \frac{1}{x\ln a} \quad \text{for } x>0,\ a>0\ a\neq 1.$$

Note that: $(\ln x)' = (\log_e^x)' = \dfrac{1}{x\ln e} = \dfrac{1}{x}$

$$\therefore (\ln x)' = \frac{1}{x}$$

Example18: Find the derivatives of the following functions.

a. $f(x) = \log_7^x$

b. $h(x) = \log_2(x^2+1)$

c. $p(x) = \ln(\cos x)$

Solution:

a. if $f(x) = \log_7^x$, then $f'(x) = \left(\log_7^x\right)' = \underline{\underline{\dfrac{1}{x\ln 7}}}$

b. If $h(x) = \log_2(x^2+1)$, let $p(x) = x^2+1$ and $g(x) = \log_2^x$, then $h(x) = g(p(x))$.

Thus, $h'(x) = [g(p(x))]' = g'(p(x)).p'(x)$

But $g'(x) = (\log_2^x)' = \dfrac{1}{x\ln 2}$ and $p'(x) = (x^2+1)^1 = 2x$

So, $g(p(x)) = \dfrac{1}{p(x)\ln 2} = \dfrac{1}{(x^2+1)\ln 2}$

$$\therefore h'(x) = \frac{1}{(x^2+1)\ln 2}.(2x) = \frac{2x}{(x^2+1)\ln 2}$$

c. If $p(x) = \ln(\cos x)$, let $g(x) = \cos x$ and $f(x) = \ln x$, then $p(x) = (fog)(x)$

Thus, $p'(x) = [(fog)(x)]' = f'(g(x)).\ g'(x)$

But $f^{'}(x) = \frac{1}{x}$ and $g^{'}(x) = -\sin x$

So $f^{'}(g(x)) = \frac{1}{g(x)}\, g'(x)$

$\therefore\ p^{'}(x) = \frac{1}{\cos x}.(-\sin x) = \frac{-\sin x}{\cos x} = -\underline{\underline{\tan x}}$

By using the derivative of the logarithm function and some facts about exponential and logarithmic functions we can find the derivative of the exponential function.

What is exponential function?

Definition 3.6:

Let a>0 and $a \neq 1$. The exponential function of a^x is defined as $y = a^x$ if and only if $\log_a{}^y = x$.

Since $y = a^x$, it becomes $\log a^y = x$

How we derivate the exponential function?

Let the exponential function is given by $f(x) = a^x$ for $a>0, a\neq 1$

Since $x = \log_a a^x$, we have $x = \log_a f(x)$

$\Rightarrow (x)^1 = (\log_a f(x))^1 \quad \Rightarrow \quad 1 = \frac{1}{f(x)\ln a}.f^{'}(x)$ [Why?]

$\Rightarrow 1 = \frac{f^{'}(x)}{f(x).\ln a} \quad \Rightarrow f^{'}(x) = f(x).\ln a \quad \Rightarrow \quad (a^x)^{'} = a^x \ln a.$

Theorem 3.6

Let $f(x) = a^x$ where $a > 0$ and $a \neq 1$.then

$$f^{'}(x) = (a^x)^{'} = \frac{a^x}{\log a^e} = a^x \ln a$$

<u>**Note that**</u> : $(e^x)^{'} = e^x.\ln e = e^x$

Example 17: Find the derivatives of the following

a/ $f(x)=3^x$

b/ $g(x)=e^{3x}$

c/ $p(x)= 3^{3x-1}$

d/ $g(x) = e^{\cos x}$

Solution

a. If $f(x) =3^x$ then $f'(x) = (3^x)'=3^x.\ln 3$

b. If $g(x) =e^{3x}$

let $h(x) =3x$ and $f(x) = e^x$, then $g(x) = (fog)(x)$

So, $g'(x)= [(foh)(x)]' = f'(h(x)).h'(x)$

But $f'(x) = (e^x)' = e^x$ and $h'(x) = (3x)'=3$

So, $f'(h(x)= f'(3x) =e^{3x}$

$\therefore\ g'(x) =(e^{3x}) =e^{3x}(3) = \underline{\underline{3e^{3x}}}$

c. If $(p(x) =3^{3x-1}$,

let $g(x) = 3x-1$ and $f(x) = 3^x$, then $p(x) =(fog)(x)$.

So, $p'(x) = [(fog)(x)]' = f'(g(x)) .g'(x)$

But $f'(x) = (3^x)' =3^x \ln 3$ and $g'(x) = (3x-1)' = 3$

So, $f'(g(x)) = f'(3x-1) = 3^{3x-1}.\ln 3$

$\therefore p'(x) = (3^{3x-1})' =3^{3x-1} \ln 3\ (3) = \underline{\underline{3^{3x}\ .\ln 3}}$

d. If $g(x) =e^{\cos x}$, let $g(x) =\cos x$ and $f(x) =e^x$, then $g(x) = (fog)(x)$

So, $g'(x) = [(fog)(x)]'= f'(g(x)) .g'(x)$

But $f'(x) =(e^x)' = e^x$ and $g'(x) = (\cos x)'= -1 \sin x$.

So, $f'(g(x)) = f'(\cos x) =e^{\cos x}$

$\therefore g'(x) =(e^{\cos x})'= e^{\cos x} . (-\sin x) = -e^{\cos x} \sin x$

Derivatives of Hyperbolic Functions

The last set of functions that we're going to be looking in this chapter at are the hyperbolic functions. In many physical situations combinations **of e^x and e^{-x}** arise fairly often. Because of this these combinations are given names.

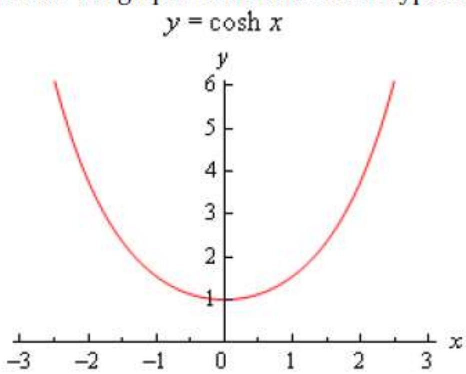
- There are six hyperbolic functions and they are defined as follows

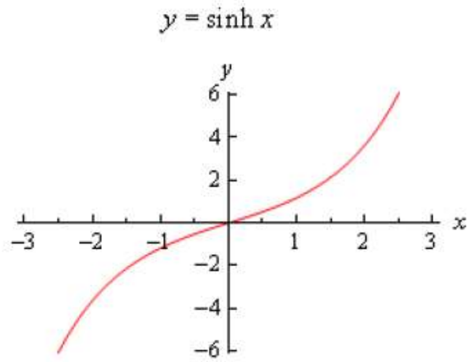
$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad\qquad \cosh x = \frac{e^x + e^{-x}}{2}$$

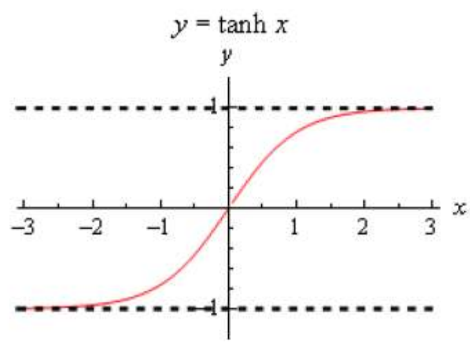
$tanhx = \frac{sinhx}{coshx}$ $\qquad$ $cothx = \frac{coshx}{sinhx} = \frac{1}{tanhx}$

$sechx = \frac{1}{coshx}$ $\qquad$ $cschx = \frac{1}{sinhx}$

Here are the graphs of the three main hyperbolic functions.







We also have the following fact about the hyperbolic functions

sinh(-x) = sinh(x) $\qquad$ cosh(-x) = cosh(x)

$\cosh^2(x) – \sinh^2(x) = 1$ $\qquad$ $1-\tanh^2(x) = \text{sech}^2(x)$

Note : $\frac{d}{dx}(e^{-x}) = -e^{-x}$

Example: The derivative for hyperbolic sine

$\frac{d}{dx}(sinhx) = \frac{d}{dx}(\frac{e^x - e^{-x}}{2}) = (\frac{e^x - (-e^{-x})}{2}) = (\frac{e^x + e^{-x}}{2})$ = **coshx**

Exercise

Show the following

Applied one

For the rest we can either use the definition of the hyperbolic function and/or the quotient rule. Here are all six derivatives.

$$\frac{d}{dx}(\sinh x) = \cosh x \qquad \frac{d}{dx}(\cosh x) = \sinh x$$
$$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x \qquad \frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$$
$$\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x \qquad \frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$$

Example Differentiate of the following functions

a. $f(x) = 2x^5 coshx$
b. $f(x) = \frac{sinhx}{x+1}$

Solution : a. $f'^{(x)} = 10x^4 coshx + (x) = 2x^5 sinhx$

b) $f'(x) = \frac{(x+1)coshx - sinhx}{(x+1)^2}$

2.6 Higher Order Derivatives

Example : let we consider $f(x) = 5x^3 - 3x^2 + 10x - 5$ then differentiate the function f(x)

We have $f'(x) = 15x^2 - 6x + 10$ this is first derivative

$f''(x) = [f'(x)]' = 30x - 6$ this is second derivative

$f'''(x) = [f''(x)]' = 30$ this is third derivative

Continuing, we can differentiate again

$f^{(iv)}(x) = [f'''(x)]' = 0$ this is fourth derivative

Note : if p(x) is a polynomial of degree n (i.e the largest exponent in the polynomial) then, $p^{(k)}(x) = 0$ for all $k \geq n + 1$

Notation for higher derivatives

$$f^{(3)}(x) = f'''(x)$$

$$f^{(iv)}(x) = f''''(x)$$

Collectively the second, third, fourth etc derivatives are called higher order derivatives

Alternate Notation

There is some alternate notation for higher order derivatives as well. Recall that there was a fractional notation for the first derivative

$$f'(x) = \frac{df}{dx}$$

And the higher order derivatives

$$f''(x) = \frac{d^2f}{dx^2} \qquad f'''(x) = \frac{d^3f}{dx^3} \qquad f^{(iv)}(x) = \frac{d^4f}{dx^4} \quad \text{etc}$$

2.6 Implicit Differentiation

Example : a) Find y' for xy =1

Solution : Let's rewrite the equation, we have

$$xy(x) = 1$$

Next differentiate both sides with respect to *x* as

$$\frac{d}{dx}(xy\,(x) = \frac{d}{dx}(1)$$

$$y(x) + x\frac{d}{dx}y\,(x) = 0$$

$$y + x\frac{dy}{dx} = 0$$

$$y + xy' = 0$$

$$y' = -\frac{y}{x}$$

$$y' = -\frac{1}{x^2}$$

b) Find *y'* for the function. $x^2 + y^2 = 9$

Solution: differentiate both sides with respect to *x* as

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}9$$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}y(x)^2) = \frac{d}{dx}9$$

$$2x + 2y\frac{dy}{dx} \quad = 0$$

$$2x + 2yy' = 0$$

$$y' = -\frac{2x}{2y} = -\frac{x}{y}$$

c. $x^3y^5 + 3x = 8y^3 + 1$

Solution : differentiate both side with respect to x and set y in the form of y(x). then we have

$$\frac{d}{dx}(x^3y^5 + 3x) = \frac{d}{dx}(8y^3 + 1)$$

$$\frac{d}{dx}(x^3y^5) + \frac{d}{dx}(3x) = \frac{d}{dx}(8y^3 + 1)$$

$$3x^2y^5 + x^35y^4\frac{dy}{dx} + 3 = 24y^2\frac{dy}{dx}$$

$$(-x^35y^4 + 24y^2)y' = \ 3x^2y^5 + 3$$

$$y' = \ \frac{3x^2y^5 + 3}{24y^2 \ -5x^3y^4}$$

Applied one