CHAPTER-TWO

RECURRENCE RELATIONS

2.1. Introduction

Recursion involves recursive definition of an algorithm, a set or a sequence in terms of itself. When a recursive definition is used as a tool for solving combinatorial problems, an equation called recurrence relation is applied to represent present and future values on the basis of earlier or prior terms. It is clear that in counting problems we analyze a given situation and then express the result in terms of the results for certain smaller non-negative integers. Once the recurrence relation is determined, one can solve the equation at any $n \in W$- non negative integer n. With access to a computer, such relation is highly valuable, especially if it cannot be solved explicitly.

Recurrence relations are the discrete counterparts to the continuous ideas of the ordinary differential equations. Recurrence relations are also called **difference equations** or **recurrence equations**.

Thus, this chapter is devoted to the study of recursively defined discrete functions and the solutions of recurrence relations associated to these recursively defined functions.

2.2 The Notation of Sequences

Counting, enumeration and discrete problems are some typical areas where sequences are exploited most. In realty, they are ranges of special kinds of functions. The distinguishing feature of sequences is their domain. In most cases, they assume the set of natural numbers as a domain. Under our study of recursive definitions and specifically recurrences relations, the set of whole numbers will be regarded as the domain of sequences.

Definition: If f is a function that maps the set of whole numbers W or any non-empty finite subset of W in to a certain number set, then the range of f is called a sequence. By convention, the elements of the range of the function f, i.e. the elements of the sequence are known as terms.

Note: A sequence is often called a discrete function or a numeric function

Notations:

(1) Subscripted symbols of the form a_n, b_n, f_n, etc are used to denote the image of a whole number n under the sequence f. If we select, say a_n to denote the image of $n \in W$ under f, then $a_n = f(n)$, n= 0,1,2, …

(2) We use the notation $\{f_n\}_{n=0}^{\infty}$ or simply $\{f_n\}$ to denote a sequence f.

In other words,

$$\{a_n\}_{n=0}^{\infty} = \{f_0, f_1, f_2, ---\} = \text{the range of f (which is the sequence f).}$$

If $n \in W$ begins at some non negative integer $k \geq 1$, then the notation $\{a_n\}_{n=k}^{\infty}$ is applied for the sequence.

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Note: (1) If the domain of a sequence $\{a_n\}$ is the set of all non negative integers W, then $\{a_n\}$ is called an infinite sequence and if the domain is any finite non-empty subset of w, it is called a finite sequence.

(2) If the sequence $\{a_n\} = \{a_0, a_1, a_2, \ldots\}$ is:

a. the set of integers, then $\{a_n\}$ is called a sequence of integers.
b. the set of real numbers, then $\{a_n)$ is termed as a real sequence.
c. the set of complex numbers, then $\{a_n)$ is known as a complex sequence.

2.2.1 Methods of Describing Sequences

There are several methods of representing a sequence and the most commonly used techniques are the following.

a. Enumerating the first few terms of the sequence. Note that we list terms of the sequence till a rule for telling the present and future values is observed.
b. Supplying a rule that defines a sequence f as an explicit function a(n), preferably written as $a_n = f(n)$, $n \geq 0$. It should be noted that $a_n = f(n)$ depends upon n and only n.
c. A technique called recursive definition stated in terms of recurrence relations and initial conditions may be used to describe the terms of a sequence.

Examples

1. The numbers: 0,1,1,2,3,5,8,13, … form a sequence that begins with the two terms 0 and 1. Each new term, there after, is a sum of the previous two terms. As we shall see later, the numbers in this sequence are called the Fibonacci number.
   This description is called enumeration method.

2. The sequence $\{a_n\}_{n=0}^{\infty}$ where $a_n = 5(3^n)$ or simply $\{5(3^n)\}_{n=0}^{\infty}$ represents the geometric sequence 5,15,45,135,…
   This description is called explicit method.

3. If we denote the $(n+1)^{th}$ Fibonacci number by f_n we have:
   $f_n = f_{n-1} + f_{n-2,}\ n \geq 2$, with $f_0 = 0$ and $f_1=1$.
   This is called a recursive definition for the sequence of Fibonacci numbers given in example1

The recursive definition for the Fibonacci sequence is composed of two parts, namely, the equation: $f_n = f_{n-1} + f_{n-2}$, $n \geq 0$ and the values $f_0=0$ and $f_1=1$. These properties are enjoyed by all other recursively defined sequences.

2.3 Recursive Definition and Recurrence Relations

2.3.1 Recursive Definition

A technique of defining an algorithm, a set or a function in terms of itself by:

(i) Giving a rule for finding present and future values from earlier or prior values.

(ii) Specifying one or more starting values to activate the rule mentioned in (i)

An algorithm, a set or a function stated in terms of these two conditions is called a **recursively** defined *algorithm*, *set* or *function*, respectively.

Note that a recursive definition is *well defined* if and only if it satisfies conditions (i) and (ii) above. If any one of these two properties is lacking, the recursive definition may not describe the required phenomenon in a unique manner.

Example:

Let us use a geometric sequence to illustrate that both the rule and the starting values are really essential in recursive definitions.

Recall that a geometric sequence is an infinite array of numbers, such as 5,15,45,135, …, where the division of any term ,other than the first, by its immediate predecessor is a constant, called the common ratio r. For our sequence this common ratio r is 3 since $\frac{15}{5}=\frac{45}{15}=\frac{135}{45}=---=3$. If a_o, a_1, a_2, ---,are in a geometric sequence, then $\frac{a_1}{a_0}=\frac{a_2}{a_1}=\frac{a_3}{a_2}=\ldots=\frac{a_{n+1}}{a_n}=\ldots=r$, the common ratio. In this particular geometric sequence $a_{n+1}= 3a_n$, $n \geq 0$ is the rule for finding present and future values from earlier or prior terms.

The equation $a_{n+1}= 3a_n$, $n \geq 0$, does not, however, define a unique geometric sequence. The sequence 7, 21, 63, 189, … also satisfies the relation. To pinpoint the particular sequence described by $a_{n+1}= 3a_n$, we need to know one of the terms of that sequence as a starting value. Hence,

$a_{n+1}= 3a_n$, $n \geq 0$ and $a_0=5$

uniquely defines the sequence 5,15,45,135,…, whereas

$a_{n+1} = 3a_n$ $n \geq 0$ and $a_1=21$ identifies 7,21,63,189, … as the geometric sequence under study.

2.3.2 Recurrence Relations and Initial Conditions

The expressions for permutations and combinations are one of the most fundamental tools for counting the elements of finite sets. They often prove to be inadequate and many problems of computer science require a different approach. Hence, recurrence relation emerges in this section as another tool for solving combinatorial problems.

Recurrence relations are often called difference equations or recurrence equations.

The salient characteristic of a recurrence relation is the specification of the term f_n as a function of the prior terms f_0, f_1 , f_2 , … , f_{n-1}. However, a recurrence relation by itself is not sufficient to define a unique sequence; we must also specify the values of some initial terms. This is why in our definition of the Fibonacci sequence, we set $f_0 = 0$ and $f_1 = 1$ as initial conditions.

Recall that a recursive definition of a discrete function specifies one or more initial values and a rule for determining subsequent terms from those that precede them. When recursive definitions are applied to solve combinatorial problems, the equation involved in these definitions, which is employed for finding present terms from the preceding once is called a recurrence relation.

Recurrence relation:
A recurrence relation for a sequence $\{a_n\}$ and a non negative integer n_o, is a formula that expresses a_n in terms of one or more of the previous values $a_0, a_1, \ldots, a_{n-1}$ of the sequence for all integers $n \geq n_0$.

Initial conditions: Initial conditions, which are also called boundary conditions of the recurrence relation, are the values of one or more starting terms of the sequence specified in the form

$a_o = k$, $a_1 = r$, etc.

for some constants $k, r \in R$. Note that the computation of terms of a sequence from the recurrence relation is initiated by the boundary conditions.

Explicit sequence: A function $a_n = f(n)$ that defines the term a_n of a sequence $\{a_n\}$ on the basis of a non-negative integer n alone is called an explicit sequence of n.

Solution of a Relation:
If each term of an explicit sequence $a_n = f(n)$, $\forall\ n \in w$ satisfies a given recurrence relation, then the explicit sequence (i.e., the explicitly defined sequence) $\{a_n\}$ is called the solution of the difference equation. The procedure followed to find the explicit sequence $\{a_n\}$ that solves a recurrence relation is called solving.

Examples

1. Show that the explicit sequence $\{a_n\}$ where $a_n = 2^{n+1}-1$ for $n \geq 1$ is a solution of the recurrence relation:

$a_n = 3a_{n-1} - 2a_{n-2}$, $n \geq 3$.

Solution:

To show that $a_n = 2^{n+1}-1$ $\forall n \in N$ is a solution of the recurrence relation:

$a_n = 3a_{n-1} - 2a_{n-2},$

first observe that the terms of the explicit sequence $\{a_n\}$ at n, n-1, and n-2, respectively, are:

$a_n = 2^{n+1} - 1 \ldots$ [i]

$a_{n-1} = 2^{n-1+1} - 1 = 2^n - 1 \ldots$ [ii]

$a_{n-2} = 2^{n-2+1} - 1 = 2^{n-1} . 1$ --- [iii]

Substituting these formulas into the right-hand side of the recurrence relation, we get:

$$
\begin{aligned}
3a_{n-1} - 2a_{n-2} &= 3\,[2^n - 1] - 2\,[2^{n-1} - 1] \ \ldots \text{ by [ii] and [iii]} \\
&= 3\,(2^n) - 3 - 2\,(2^{n-1}) + 2 \\
&= 3\,(2^n) - 2^n - 1 \\
&= 2\,(2^n) - 1. \\
&= 2^{n+1} - 1 \\
&= a_n \quad \ldots\ldots \text{ by [i].}
\end{aligned}
$$

Thus, we conclude that: $a_n = 3a_{n-2} - 2a_{n-2}$ whenever $a_n = 2^{n+1}-1$. Consequently the explicit sequence $\{a_n\}_{n=1}^{\infty}$ where $a_n = 2^{n+1}-1\ \forall n \geq 1$ is a solution of the given recurrence relation.

2. Show that the sequence $\{f_n\}$ defined explicitly by $f_n = 2(-4)^n + 3$ is a solution of the recurrence relation $f_n = -3\, f_{n-1} + 4\, f_{n-2}$.

Solution: Given the explicitly defined sequence $f_n = 2(-4)^n + 3$, to show that it is a solution of the relation: : $f_n = -3f_{n-1} + 4\, f_{n-2}$, we begin with RHS.

$$
\begin{aligned}
-3\, f_{n-1} + 4f_{n-2} &= -3\,[2(-4)^{n-1} + 3] + 4\,[2(-4)^{n-2} + 3] \\
&= -6\,(-4)^{n-1} - 9 + 8\,(-4)^{n-2} + 12 \\
&= -6\,(-4)^{n-1} + (-2)\,(-4)\,(-4)^{n-2} + 3 \\
&= -6\,(-4)^{n-1} - 2\,(-4)^{n-1} + 3 \\
&= -8\,(-4)^{n-1} + 3 \\
&= 2\,(-4)\,(-4)^{n-1} + 3 \\
&= 2\,(-4)^n + 3. \\
&= f_n
\end{aligned}
$$

Therefore, the sequence $\{f_n\}$ where $f_n = 2(-4)^n + 3$ is a solution of the recurrence relation.

3. Suppose that the discrete function f is defined recursively by:

$$f(0) = 2 \text{ and}$$
$$f(n+1) = 2f(n) + 3$$

Then find f(1), f(2), f(3), f(4) and f(5)

Solution: From the recursive definition, it follows that:

$$
\begin{aligned}
f(1) &= 2f\,(0) + 3 \\
&= 2(2)+3 = 7\ /// \\
f(2) &= 2f(1) + 3 = 2\,(7) + 3 = 17\ /// \\
f(3) &= 2f(2) + 3 = 2(17) + 3 = 37\ /// \\
f(4) &= 2f(3) + 3 == 2(37)+3 = 77\ /// \\
f(5) &= 2f(4) + 3 = 2(77) + 3 = 157\ ///
\end{aligned}
$$

4. *Let a and b be positive integers, and suppose Q is defined recursively as follows.*

$$
Q\,(a, b) = \begin{cases} 0, \text{ if } a < b \\ Q\,(a-b, b) + 1, \text{ if } b \leq a. \end{cases}
$$

(a) find (i) Q (2,5) (ii) Q(12 ,5)
(b) What does this function Q do?
(c) Find Q (5861, 7).

Solution: (a) (i) Q (2,5) = 0 since 2 < 5. ///

(ii) Q (12,5) = Q (7,5) +1
= [Q(2,5)+1] +1.
= Q (2,5) +2
= 0+2
= 2

(b) Each time b is subtracted from a, the value of the function Q is increased by 1. Hence Q (a,b) finds the quotient when a is divided by b.

(c) When we divide 5861 by 7, the quotient will be 837. Thus, according to the conclusion drawn in part (b) above, we have:

Q (5861,7) = 837.

2.4 Linear Recurrence Relation with Constant Coefficient

A recurrence relation of the form:

$$c_0 f_n + c_1 f_{n-1} + c_2 f_{n-2} + \dots + c_k f_{n-k} = f(n) \dots [1].$$

Where $c_0, c_1, c_2, \dots, c_k$ are constants, is called a linear recurrence relation with constant coefficients (LRRWCC).

***Note*:** The relation in [1] is linear since each term f_n f_{n-1}, f_{n-2}, …, f_{n-k}. appear only in a power of degree one.

ORDER OF RECURRENCE RELATION

If the constants c_0 and c_k in [1] are none zero, then relation [1] is known as the k^{th}- order linear recurrence relation with constant coefficients.

Note: The phrase " k^{th} – order " mean that the present term f_n of the relation depends on k previous terms, f_{n-1}, f_{n-2}, …, f_{n-k}.

Examples

1. The Fibonacci sequence defined by the recurrence relation: $F_n = F_{n-1} + F_{n-2}$, $n \geq 2$ with the initial conditions $F_0 = 0$ and $F_1 = 1$ is linear second-order.

2. $y_n - 4y_{n-1} = 0$, $n > 1$ with the boundary condition $y_1 = 3$ is a first-order linear recurrence relation with constant coefficients (LRRWCC of order-one).

3. $a_{K+1} - 5a_k + 4a_{k-1} - 6a_{k-2} = Ak+10$ defined for $k \geq 3$, together with the initial conditions $a_0 = \frac{7}{3}$ and $a_1 = a_2 = 5$ is a third-order linear recurrence relation with constant coefficients (LRRWCC of order-three).

HOMOGENEOUS RELATION

The recurrence relation [1] is called a k^{th} – order linear **homogeneous** recurrence relation with constant coefficients if and only if $f(n)= 0$ for all $n \in W$.

NONHOMOGENEOUS RELATION

The recurrence relation [1] is called a k^{th} – order linear **nonhomogeneous** recurrence relation with constant coefficients if and only if $f(n) \neq 0$ for some $n \in W$. That is, the relation:

$c_of_n + c_1f_{n-1} + c_2f_{n-2} + \dots c_kf_{n-k} = f(n) \neq 0$ for some $n \in W$ is termed as nonhomogeneous recurrence relation with constant coefficients (LNRRWCC).

Note: Non homogeneous recurrence relation is also called **Inhomogeneous** RR.

Examples

- The relation: $a_k = 5a_{k-1}, - 8a_{k-2}$,$k \geq 2$ with $a_0 =5$ and $a_1= 2$ is a 2^{nd} –order LHRRWCC, while $a_k= 5a_{k-1} -a_{k-2} +6a_{k-3}+ 4k+10$, $k \geq 3$ with $a_0 = \frac{7}{3}$ and $a_1= a_2 =5$ is a 3^{rd} –order LNHRRWCC.

2. *Classify the following recurrence relations*

(a) $f_n =nf_{n-1}$ *(b)* $a_n = a_{n-1}+ a_{n-3}$

(c) $b_n = b_{n-1}+2$ *(d)* $S_n = S_{n-2} + S_{n-4}$.

Solution:

(a) $f_n = nf_{n-1}$ is a first-order linear homogeneous recurrence relation with variable coefficients.

(b) $a_n = a_{n-1} + a_{n-3}$ is a third-order linear homogeneous recurrence relation with constant coefficients.

(c) $b_n = b_{n-1}+ 2$ is a first-order linear non homogeneous recurrence relation with constant coefficients.

(d) $S_n = S_{n-2} + S_{n-4}$ is a fourth-order linear homogeneous recurrence relation with constant coefficients.