CHAPTER-3

GRAPHY THEORY

3.0 Introduction

Graph theory is a branch of mathematics that deals with arrangements of certain objects and relationships between these objects. Graph theory is broadly classified into two: **undirected** graphs and **directed** graphs (**digraphs**). In this chapter we will discuss a number of important cases of undirected and directed graphs, as well as a few important properties that may be possessed by graphs, such as planarity and colorability.

3.1 Basic Definitions and examples

Definition: Undirected graph G is a pair of sets (V,E) consisting of two things

i) A set V= V(G) whose elements are called vertices, Points or node of G

ii) A set E= E(G) called edge list of G

- V is called a vertex set and E is called an edge list.
- Vertices u and v are said to be adjacent if there is an edge e= {u,v}between them.

In such a case u and v are called the end points of e.

- The edge e is said to be incident on each of its end points u and v.

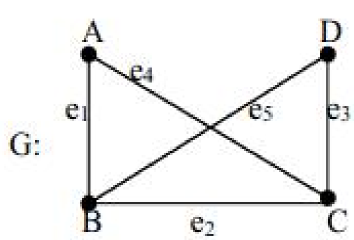
If G is a finite:

- $|V(G)|$ denotes the number of vertices in G.
- $|E(G)|$ denotes the number of edges in G.

Note: An edge should not pass through more than two vertices.

Undirected graphs are pictured by diagrams in a natural way, specifically, each vertex v in V is represented by a dot (small circle), and edge e = $\{v_1, v_2\}$ is represented by a curve which connects its end points v_1 and v_2.

Example 1: consider the graph G given below.



G is a graph with G= (V,E) where

i. V = {A,B,C,D}

ii. E= $\{e_1,e_2,e_3,e_4,e_5\}$ with

$e_1 = \{A,B\}$, $e_2 = \{B,C\}$, $e_3= \{C,D\}$, $e_4= \{A,C\}$ and $e_5= \{B,D\}$

Definition: Two or more edges joining the same pair of vertices are called multiple (parallel) edges. An edge joining a vertex to itself is called a Loop.

Depending on loops (self loops) and parallel edges we state the following types of graphs.

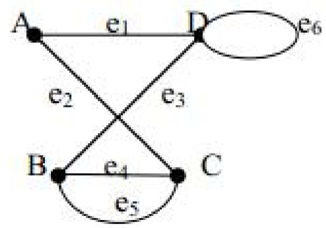
Simple graph: A graph with no loops and parallel edges is called simple graph.

Multigraph: A graph which consists of parallel (multiple edges) is called a multigraph.

Pseudo graph: A graph which consists of loops and parallel edges is called pseudo graph.

Example 2: State the nature of following graph G'.

G':



- G' is a graph with set of pairs (V,E) where
  i. V= {A,B,C,D} is a vertex set
  ii. E= $\{e_1,e_2,e_3,e_4,e_5,e_6\}$ is an edge list

- e_6 is a loop where as e_4 and e_5 are multiple edges
- The graph is not a multigraph but it is a pseudo graph

Graphs have proven to be an extremely useful tool for analyzing situations involving a set of elements in which various pairs of elements are related by some property. The following are some examples dealing with real life situations.

Examples

1. Electrical network
Transistors: - vertices
Wire: - edges.

2. Telephone communication
Telephones and switching center:-vertices
Telephone lines: -edges

3. Computer flow chart
Instructions:-vertices
Logical flow:-edge

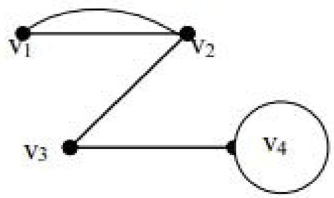
4. Organization chart
People:-vertices
Link between the people:- edge

Degree of a vertex: The degree of a vertex V in a graph G is equal to:

1. The number of edges in G which contain V if it has no loops.
2. The number of edges in G which contain V plus twice the number of loop(s) if it has loop(s).

Note: For a graph with loops, each loop contributes 2 to the degree of the corresponding vertex.

Example1: Determine the degree of each vertex of the graph given below.

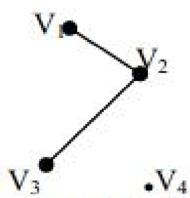


Solution:
deg (v_1)= 2
deg (v_2)= 3
deg (v_3)= 2
deg (v_4)= 3

Definition: - i) A vertex is said to be even if its degree is even number and odd if its degree is odd number.
ii) A vertex of degree zero is called **isolated** vertex.
iii) A vertex of degree one is called **pendant** vertex.

Example 2: Consider the following graph G.

G:



G = (V, E), where i) V= $\{v_1, v_2, v_3, v_4\}$
ii) E = $\{\{v_1, v_2\}, \{v_2, v_3\}\}$

deg (v_1) = 1 = deg (v_3), deg (v_2)=2, deg (v_4) = 0
V_2 and V_4 are even vertices
V_1 and V_3 are odd vertices

Moreover,
- V_3 and V_4 are not adjacent
- V_2 and V_3 are adjacent

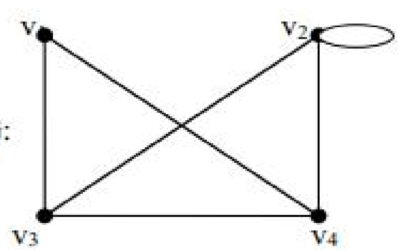
Minimum and maximum degree

Let G be a graph. The minimum and maximum degrees of G are denoted by δ (G) and $\Delta(G)$ respectively and given by:

$$\delta(G) = \min \{\deg(v); v \in V(G)\} \;\&\; \Delta(G) = \max \{\deg v; v \in V(G)\}$$

Example: Find the minimum and maximum degree of G given below.

G:



Solution: δ (G) =2, i.e., deg(v_1)
$\Delta(G)$ = 4, i.e., deg(v_2)

3.2 Directed Graphs

Directed graphs are graphs in which the edges are one way. Such graphs are frequently more useful in various dynamical systems such as:

- Digital computer
- Flow system
- Communication system
- Transportation system

Definition: a digraph D is a graph consisting of two things:

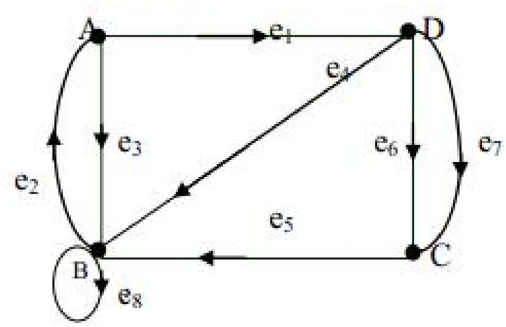
i) A set V whose elements are called vertices, Points or node of D

ii) A set E whose elements are order pairs (u,v) of distinct vertices called arcs or directed edges of D.

Suppose e = (u,v) is a directed edge in a digraph D. Then the following terminologies are used.

- e begins at u and ends at v.
- u is the origin or initial point of e where as v is destination or terminal point of e.
- **v** is the successor of **u** and **u** is the predecessor of **v**.
- u is adjacent to v where as v is adjacent from u
- If u = v then e is a loop.
- The set of all successor of a vertex u is : $suss(u) = \{v \in V : (u,v) \in E\}$

Example 1: Let D be the directed graph shown in the following figure.



- $e_4 = (D,B) \neq (B,D)$
- e_8 is a loop.
- Succ(A) = {D}. That is D is adjacent from A.
- E_6 and e_7 are parallel arcs.

3.3 Matrix representation of Graphs

The essential features of a graph are:

i. The adjacency relationships between vertices and
ii. The incidence relationships between vertices and edges

Thus graphs can be represented by any one of the following types of matrices.

a) Adjacency matrix
   - a matrix that describes the adjacency relationships between vertices of a graph.
b) Incidence Matrix
   - A matrix that describes the incidence relationships between vertices and edges of a graph.

Adjacency Matrix

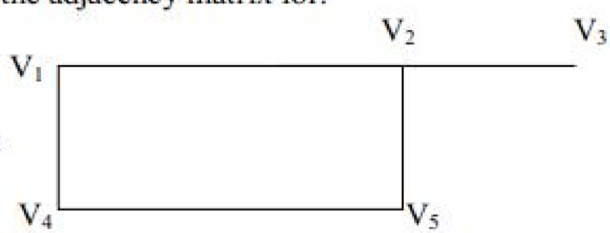
Definition: Suppose G is a graph with m vertices and suppose the vertices have been ordered, say $v_1, v_2, - - - , v_m$. Then the adjacency matrix $A = [a_{ii}]$ of the graph G is the $m \times n$ matrix defined by:

$$a_{ij} = \begin{cases} n, & \text{if there are n edges joining } v_i \text{ and } v_j \\ 0, & \text{other wise.} \end{cases}$$

Examples

1. Draw the adjacency matrix for:

G:



Solution: Let the vertex set be ordered and labeled as $V = \{v_1, v_2, v_3, v_4, v_5\}$

Then the adjacency matrix A is:

$$A = \begin{array}{c|ccccc} & v_1 & v_2 & v_3 & v_4 & v_5 \\ \hline v_1 & 0 & 1 & 0 & 1 & 0 \\ v_2 & 1 & 0 & 1 & 0 & 1 \\ v_3 & 0 & 1 & 0 & 0 & 0 \\ v_4 & 1 & 0 & 0 & 0 & 1 \\ v_5 & 0 & 1 & 0 & 1 & 0 \end{array}$$

Incidence Matrix

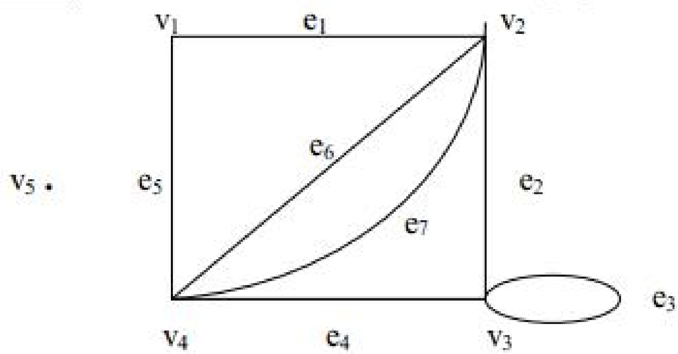
Definition: Suppose G is a graph with vertices $v_1, v_2, \ldots, v_m$. The incidence matrix $I = [b_{ij}]$ of the graph G is given by:

$$b_{ij} = \begin{cases} 1, \text{if } e_j \text{ is incident on } v_i \\ 0, \text{otherwise} \end{cases}$$

Note: 1. The incident matrix of a graph is not necessarily a square matrix.

2 An incident matrix I has a row for each vertex and a column for each edge.

Example: Find the incidence matrix I for the graph shown below.



Solution: I is a 5X7 matrix.

$$\therefore I = \begin{array}{c|ccccccc} & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \\ \hline v_1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ v_2 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ v_3 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ v_4 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ v_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}$$

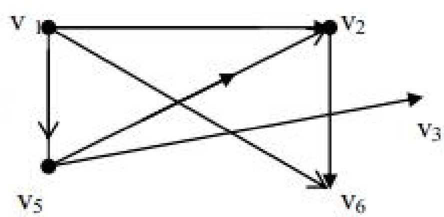
3.4 Matrix Representation of diagraphs

Adjacency matrix: The adjacency matrix A= $[a_{ij}]$ of a diagraph is defined as a matrix with:

$$a_{ij} = \begin{cases} n \text{ if number of } (v_i, v_j) \in E \text{ is } n \\ 0 \text{ otherwise} \end{cases}$$

Example 1: Write the adjacency matrix for the following diagraph.

D_1:



Solution: Let the vertex set V of D_1 is labeled and ordered as V= $\{v_1, v_2, v_3 v_4 v_5\}$. Thus, the adjacency matrix for $D_1$is:

$$A(D_1) = \begin{array}{c|ccccc} & V_1 & V_2 & V_3 & V_4 & V_5 \\ \hline V_1 & 0 & 1 & 0 & 1 & 1 \\ V_2 & 0 & 0 & 0 & 1 & 0 \\ V_3 & 0 & 0 & 0 & 0 & 0 \\ V_4 & 0 & 0 & 0 & 0 & 0 \\ V_5 & 0 & 1 & 1 & 0 & 0 \end{array}$$

Incidence Matrix of a diagraph

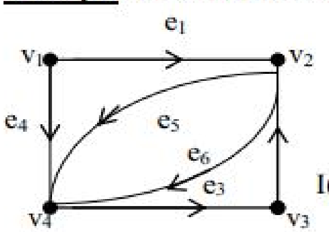
The incidence matrix I(D) of a loop-free diagraph is the nxm matrix in which:

$$a_{ij} = \begin{cases} 1 \text{ if arc } e_j \text{ is incident from } v_i \\ -1 \text{ if arc } e_j \text{ is incident to } v_i \\ 0 \text{ otherwise.} \end{cases}$$

Note: An incidence matrix has a row for each vertex and a column for each arc.

Example*:* Find the incidence matrix I (D) of the diagraph in the figure.

D_3:



Solution: The incidence matrix for D is given as:

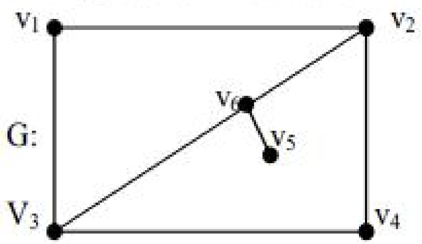
$$I(D_3) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & -1 \end{pmatrix} \end{array}$$

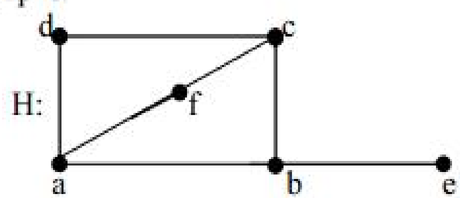
3.5 Isomorphism of Graphs

It is possible for two graph diagrams two look different but to represent the same graph. On the other hand, it is possible to look similar but to represent different graphs.

Definition: Graphs G(V,E) and G*(V*,E*) are said to be isomorphic if there exists a one-to – one correspondence $f : V \to V^*$ such that {u,v} is an edge of G iff $\{f(u), f(v)\}$ is an edge of G*.

Example: Show that G and H are isomorphic graphs.





Solution: - The correspondence is

$f(v_1) = d$ $\quad$ $f(v_3) = a$ $\quad$ $f(v_5) = e$

$f(v_2) = c$ $\quad$ $f(v_4) = f$ $\quad$ $f(v_6) = b$

That is,

$\{v_1, v_2\} \in \mathrm{E}(G) \Leftrightarrow \{f(v_1), f(v_2)\} = \{d, c\} \in \mathrm{E}(H)$

$\{v_3, v_4\} \in \mathrm{E}(G) \Leftrightarrow \{f(v_3), f(v_4)\} = \{a, f\} \in \mathrm{E}(H)$

$\vdots$

$\{v_5, v_6\} \in \mathrm{E} \subset G) \Leftrightarrow \{f(v_5), f(v_6)\} = \{e, b\} \in \mathrm{E}(H)$

$\therefore G \cong H$.

3.5 paths and connectivity

Paths:

Let G be a graph and e be an edge in G with end points u and v. Then the ordered triple (u,e,v) is a step (walk) in G.

Definition: A path in a graph G consists of an alternating finite sequence of vertices and edges of the form:

$v_0, e_1, v_1, e_2, v_2, \ldots e_{n-1}, v_{n-1}, e_n, v_n$

Where each edge e_i contains the vertices v_{i-1} and v_i (which appears on the sides of e_i in the sequence).

Sometimes, with the understanding that consecutive vertices are adjacent, the path can be written as:

$(v_0, v_1, \ldots v_n)$ with the idea that consecutive vertices are adjacent.

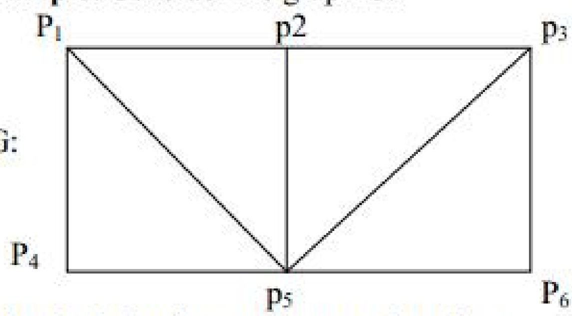
Given the path $p = (v_0,e_1,v_1,e_2,v_2,\ldots,v_{n-1},e_n,v_n)$. Then:

- P is said to **traverse** the edges $e_1,e_2,---,e_n$ and **visit** the vertices $v_0,v_1,\ldots,v_{n-1},v_n$
- v_0 is called the initial vertex and v_n is the terminal vertex of p.
- The number n of edges is called the length of the path.
- A path is said to be closed if $v_0 = v_n$.
- P is said to be open if $v_0 \neq v_n$.
- P is called a simple path if all the vertices are distinct.
- P is called a cycle if it is a closed simple path (i.e., all the vertices are distinct except v_0 and v_n .
- P is called a trial if all the edges are distinct (i.e., a path that does not traverse the same edge more than once)
- A loop is a cycle of length 1
- A simple path of length ≥ 1 with no repeated edges and whose end points are equal is called a circuit.

Note: A closed path in which all the edges are distinct is called a closed trial.

Example: Consider the graph G.

G:



Take the following sequences of vertices.

$$\alpha = (P_4, P_1, P_2, P_5, P_1, P_2, P_3, P_6)$$

$$\beta = (p_4, p_1, p_5, p_2, p_6)$$

$$\wp = (p_4, p_1, p_5, p_2, p_3, p_5, p_6)$$

$$\delta = (P_4, P_1, P_5, P_3, P_6)$$

$$p = (P_4, P_1, P_2, P_3, P_5, P_4)$$

α is an open path form p_4 to p_6 but it is not a cycle and not a trial as well .

β is not a path since there is no edge $\{p_2,p_6\}$ in G.

$\wp$ is a trial (no edge is used twice) but not a cycle.

The sequence δ is a simple path as well as a trial between p_4 and p_6 but not the shortest path between p_4 and p_6.

The shortest path between p_4 and p_6 is (p_4,p_5,p_6) which has length 2.

The sequence p is a circuit of length 5.

3.5.2 Special classes of graphs

There are a number of interesting special classes of graphs such as complete, regular, bipartite, cycle, path and wheel graphs. The first three will be discussed in section 3.6.

Cycle graph:

- A cycle graph of order n is a connected graph whose edges form a cycle of length n.
- Cycle graphs are denoted by C_n.

Path graph:

- A path graph of length n is a graph obtained by removing an edge from a cycle graph C_n.
- Path graph of order n is denoted by P_n.

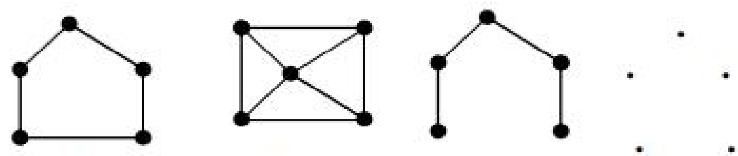
Wheel Graph:

- A wheel of order n is a graph obtained by joining a single vertex (the 'hub') to each vertex of a cycle graph.
- Wheel graph is denoted by W_n.

Null Graph:

- A null graph of order n is a graph with n vertices and no edges and is denoted by N_n.

Example: graphs of classes C_5, W_5, P_5, and N_5 are shown in the figure below from a through d respectively.



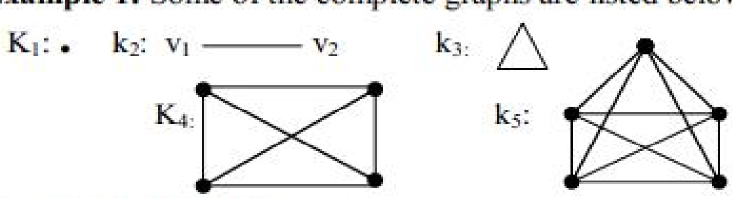
(a) (b) (c) (d)

3.6 Complete, regular and Bipartite Graphs

Complete graph: A graph G is said to be complete if every vertex in G is connected to every other vertex in G. Thus a complete graph G must be connected.

Notation: The complete graph with n-vertices is denoted by k_n.

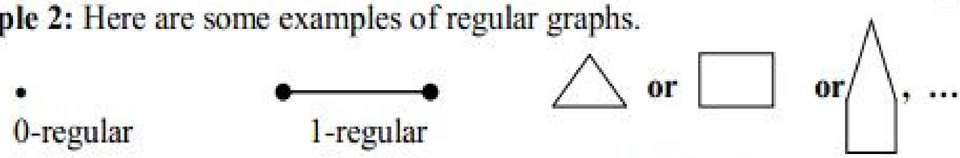
Example 1: Some of the complete graphs are listed below.



Remark: In a complete graph;

- There are $\frac{n(n-1)}{2}$ edges.
- All vertices are mutually adjacent.

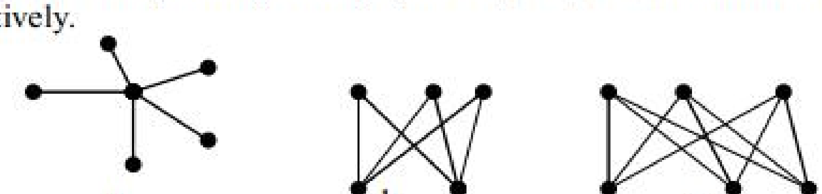
Regular graph: A graph G is said to be regular of degree k or K-regular if every vertex has degree k. In other words, a graph is regular if every vertex has the same degree.
Example 2: Here are some examples of regular graphs.



0-regular 1-regular

2-regular

Bipartite graphs: A graph G is said to be bipartite if its vertices V can be partitioned into two subsets M and N such that each edge of G connect a vertex of M to a vertex of N. (i.e. none of the edges in G connect vertices with in the same set M or N).
By a **complete Bipartite** graph we mean that each vertex of M is connected to each vertex of N; this graph is denoted by $K_{m,n}$ where m is the number of vertices in M and n is the number of vertices in N $(m \leq n)$.
Example 1: Complete bipartite graphs $K_{1,5}$, $K_{2,4}$, $K_{3,3}$ are shown below in a, b and c respectively.



a. b. c.

3.7 Euler and Hamilton Graphs

3.7.1 Eulerian Graphs

<u>**Eulerian Path**</u>:
Definition: An Eulerian path in a graph G(V,E) is a path which uses each edge in E exactly once.
An Euler path that begins and ends at the same vertex is called **Eulerian trial**.
A graph that contains an Eulerian (closed) trial is called an **Eulerian graph**.

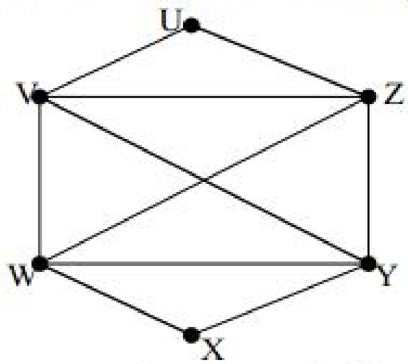
Examples

1. Consider the graph G and answer the following questions

a. Show that G has an Euler closed trial.

b. Find an Euler trial starting and ending at U.

G:



Solution a) G is a connected graph and degree of each vertex is even (i.e. deg(u)=deg(x)=2;deg (z)=deg(y)=deg(u) = deg(v)=4).

c) One Euler trial beginning and ending at U is UZWVZYXWYVU.

3.7.2 Hamilton Graph

Hamilton Paths:

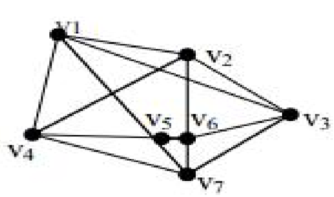
- a path that visits every vertex in a graph G exactly once is called a **Hamilton path**.
- A closed Hamilton path is called a **Hamilton cycle**.
- G is called a **Hamiltonian graph** if it admits a Hamiltonian cycle.

Properties of Hamiltonian graph

- Only connected graphs can be Hamiltonian
- There is no simple criterion to identify a graph is Hamiltonian or not.

Theorem: If G is a simple graph with vertices $n \geq 3$ and if deg(u)+deg(v) $\geq n$ for all pairs of non-adjacent vertices u and v, then G is Hamiltonian. (The converse is not always true)

Example 1: show that a graph G given below is Hamilton.



Solution:

- $deg(v_1)+deg(v_6) = 8 \geq 7$
- $deg(v_1)+deg(v_7) = 8 \geq 7$
- $deg(v_2)+deg(v_5) = 8 \geq 7$
- $deg(v_2)+deg(v_7) = 8 \geq 7$
- $deg(v_4)+deg(v_6) = 8 \geq 7$
- $deg(v_1)+deg(v_5) = 8 \geq 7$, etc.

G satisfies the condition $deg(v_i)+deg(v_j) \geq 7$ for all Paris of non-adjacent vertices v_i and v_j.$i = j = 1, \ldots, 7$.

$\therefore G$ is Hamiltonian.

3.8 Planar Graphs and Graph Colors

3.8.1 *Planar Graphs*

Consider the complete graph on 4 vertices i.e., K_4. It has two common visual representations as shown in fig. 1 and fig. 2

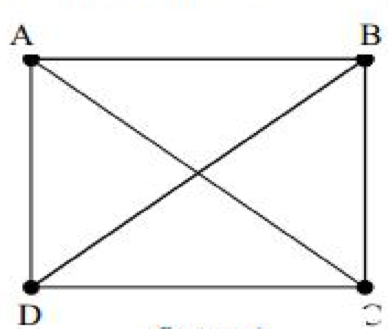


figure 1

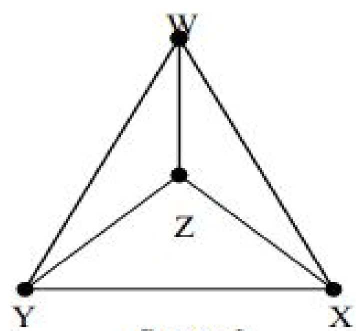


figure 2

In the representation of K_4 shown in fig. 1, the two edges AC and BD appear to cross at a point where there is no vertex. But in the second depiction, i.e., K_4 as represented in fig. 2, this crossing of edges doesn't happen (except at a common vertex).

This brings us to the topic of planarity. The graph in figure 2, the second depiction of K_4 is called a plane representation of K_4. Not every graph has a plane representation. A graph which has a plane representation is called a planner graph.

Definition (**planar Graph**)

A graph G which can be drawn in a plane with its edges intersecting only at (common) vertices is called a **planar graph**. A graph that has no such plane representation (or depiction) is called a **non-planner** graph.

Example 1: K_4 is a planar graph. The graph K_4 given in fig. 1 has planar depictions (or plane drawings) shown in fig. 2. a , b and c.

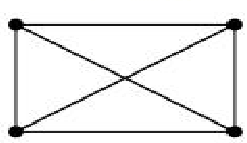
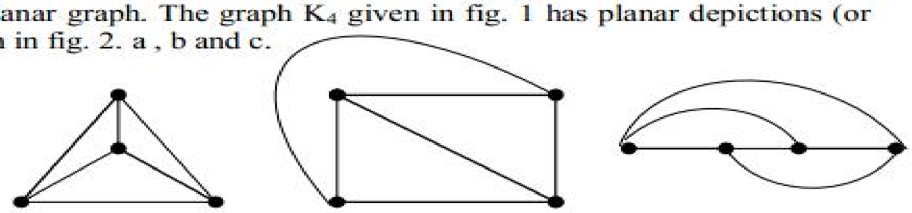


Figure 1



a) **b)** **c)**

Figure 2

Example 2: A complete graph k_5 and a complete bipartite graph $k_{3,3}$ are non planar.