CHAPTER 6

6. Random Variable, Probability Distributions and Their Application

6.1 Definition of Random Variable

At the beginning of Chapter 1 we talked about the idea of a random experiment. When a random experiment is performed, the interest may not be in all of the details of the experimental results (outcomes) but only in the value of some numerical quantity determined by the result.

For instance, in conducting the random experiment, tossing dice, one might often be interested in the sum of the two dice rather than he/she is really concerned about the sample points (outcomes) of the individual dice. That is, one might want to know whether or not the sum is seven , for instance, instead of he/she is concerned over whether the actual outcome was (1,6) or (2,5) or (3,4) or (4,3) or (5,2) or (6,1). Also, in another situation, one might also concern with number of head of the outcomes in flipping a coin thrice rather than concerned with the categorical outcomes, HHH, HHT,…, TTT; or with whether lifetime of a piece of equipment exceeds 100 hours; the number of aces in a bridge hand, or the time it takes for a light bulb to burn out, etc. Those variables, such as sum, difference, time and so on are quantities which depend on those random actions. Such quantities change from one experiment to another depending on which outcomes resulted from an experiment. Such assignments used to measure or describe numerically the different aspects or characteristics of the outcome of our random experiment are known as random variables.

$$S = \{\underset{\substack{\uparrow \\ \text{outcome}}}{\alpha}\} \qquad \underset{\substack{\uparrow \\ \text{r.v}}}{X}(\alpha)$$

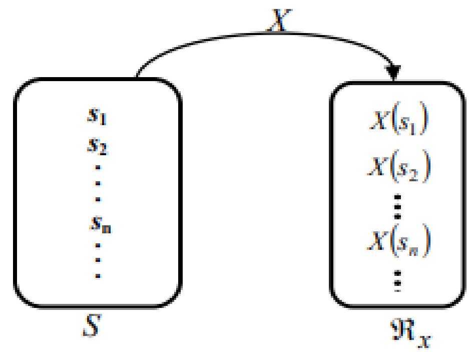
The numerical values of random variables are unknown and depend on the outcome of particular experiment. Since the value of a random variable is determined by the outcome S ={α} X (α) outcome of the experiment, we may assign probabilities to its possible values. More precisely, we have the following definition.

Definition:

Let S be the sample space associated with some random experiment, E. A random variable X is a function or a transformation that assigns a real number $X(s)$ to each outcome or element $s \in S$, *i.e.*, a random variable X is a real-valued function

$$X : S \to \Re_X$$

which associates a number, $X(s)$, with each outcome s in S. The domain of X is a *sample space* of a random experiment and its range is *the real line* as shown in the figure below.



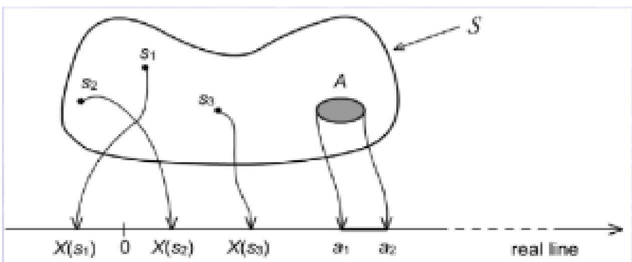


Fig. 2.2: A mapping $X(\ )$ from S to the real line.

The figure shows the transformation of a few individual sample points as well as the transformation of the event A, which falls on the real line segment $[a_1, a_2]$.

Remark 3.1:

- $X(s) = x$ means that the value of the random variable X associated with outcome $s \in S$ is x.
- $\Re x$ is a subset of real number consisting the image of $s \in S$ and is called the image or range space (set).

Example 6.1: If X is a random variable, then it is a function from the elements of the sample space to the set of real numbers. i.e. X is a function $X: S \to R$.

Flip a coin three times, let X be the number of heads in three tosses.

S = (HHH, THH, HTH, HHT, TTH, THT, HTT, TTT)

X (HHH) =3

X (HHT) =X (HTH) =X (THH) =2

X (HTT) =X (THT) =X (TTH) =1

X (TTT) =0

X= {0, 1, 2,}

❖ ***Types of Random Variables***

1. **Discrete random variable**: are variables which can assume only a specific number of values which are clearly separated and they can be counted.

- A random variable X is called discrete (or discrete type), if X takes on a finite or countably infinite number of values; that is, either finitely many values such as $x_1, \ldots, x_n$, or countably infinite many values such as $x_0, x_1, x_2, \ldots$.

Or discrete random variable can be described as follows:

- Take whole numbers (like 0, 1, 2, 3 etc.)
- Take finite or countably infinite number of values
- Jump from one value to the next and cannot take any values in between.

Example 6.2:

Experiment	Random Variable (X)	Variable values
Children of one gender in a family	Number of girls	0, 1, 2, …
Answer 23 questions of an exam	Number of correct	0, 1, 2, …, 23

2. **Continuous random variable**: are variables that can assume any values on proper interval $I \subseteq \Re$.

Or continuous random variables can be described as follow:

- Take value from the set of real numbers.
- Usually obtained by measuring.
- Take infinite number of values in an interval.
- Too many to list, like discrete variable.

Example 6.3: The following examples are some examples of continuous r.v's

Experiment	Random Variable X	Variable values
Weigh 100 People	Weight	45.1, 78, …
Measure Time Between Arrivals	Inter-Arrival time	0, 1.3, 2.78, …

6.2 Definition of Probability Distribution

Definition: A **probability distribution** is a complete list of all possible values of a random variable and their corresponding probabilities.

6.2.1 Probability Distribution of Discrete Random Variables

- If X is a discrete random variable, the function given by *f(x) = P (X = x)* for each *x* within the range of ***X*** is called the ***probability distribution*** or ***probability mass function (pmf)*** of ***X***.

Example 6.4: Consider the possible outcomes for the exp't of tossing three coins together.

Sample space, S = (HHH, THH, HTH, HHT, TTH, THT, HTT, TTT)

Let the r.v. X be the № of heads that will turn up when three coins tossed

X = {0, 1, 2, 3}

P (X = 0) = P (TTT) = 1/8,

P(X=1) = P (HTT) +P (THT) + P (TTH) =1/8+1/8+1/8 = 3/8

P(X=2) = P (HHT) +P (HTH) +P (THH) = 1/8+1/8+1/8 = 3/8,

P(X=3) = P (HHH) = 1/8

Then, the probability distribution for the random discrete variable X will be:

X = x	0	1	2	3
P(X=x)	1/8	3/8	3/8	1/8

Exercise! Let a pair of fair dice be tossed and let X denote the sum of the points obtained. Then find the probability distribution of X

Remark: - The probability distribution function or probability mass function *f(x)*, of a discrete random variable X should satisfy the following two conditions:

1. $f(x) \geq 0$
2. $\sum_x f(x) = 1$ (the summation is taken over all possible values of *x*)

Exercise! Could ***f(x) be*** a *p.m.f?* Justify your answer. $f(x) = \frac{x+2}{25}$ For *x* = 1, 2, 3, 4, 5

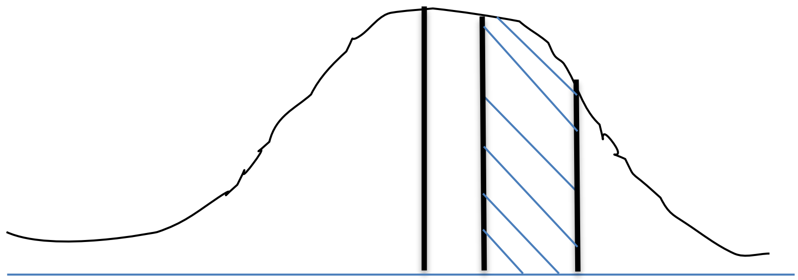
6.2.2 Probability Distribution of Continuous Random Variables

- A function ***f(x)***, defined over the set of real numbers, is called ***probability density function*** of a continuous random variable X if and only if

$P(a \leq X \leq b) = \int_a^b f(x)dx$ for any real constant $a \leq b$.

- ***Probability density function:*** is a probability distribution whose random variable is continuous. Probability of a single value is zero and probability of an interval is the area bounded by curve of probability density function and interval on x-axis. Let **a** and **b** be any two values; a <b. The prob. that X assumes a value that lies b/n a and b is equal to the area under the curve a and b.

- I.e. P (a$\le x \le b$) area under curve b/n a and b



$\bar{x}$ a b

Fig. probability density functions of X

Since X must assume some values, it follows that.

p (a $\le x \le b$) area of shaded region

$$P(a \le X \le b) = P(a < X \le b) = P(a \le X < b) = P(a \le X < b) = P(a \le X < b) = \int_a^b f(x)\,dx$$

Note:

- ✓ P (X = a) = 0 for any point a
- ✓ The total area under the density curve must equal 1.

□ ***Probability density*** function also called ***probability densities (p.d.f.)***, or simply ***densities***.

Remark:

- ❖ The probability density function ***f (x)*** of a continuous random variable X, has the following properties (always satisfy the following conditions):
  1. $f(x) \ge 0$ for all x, or for $-\infty < x < \infty$
  2. $\int_{-\infty}^{\infty} f(x)\,dx = 1$

□ If X is a continuous random variable and a and b are real constants with $a \le b$, then

$$P(a \le X \le b) = P(a < X \le b) = P(a \le X < b) = P(a < X < b)$$

Example 6.5: If X is a random variable with probability density function *f(x),*

$$f(x) = \begin{cases} k.e^{-3x} & for x > 0 \\ 0 & elsewhere \end{cases}$$

Find the constant k and compute *P (0.5 ≤ X ≤ 1)*?

Exercise! Show that $f(x) = 3x^2$ for 0 < x < 1 can represent density function?

Example 6.6: If the function *f(x)* is the density function of a r.v. *X*, where *f(x)* is given by

$f(x) = \begin{cases} Cx^2 & 0 < x < 3 \\ 0 & otherwise \end{cases}$. Then find

(a) The constant C

(b) Compute *P*(1 < x < 2)?

6.3 Expectation And Variance of Random Variable

Every discrete random variable X has a point associated with it. The points collectively are known as a ***probability mass function (pmf)*** which can be used to obtain probabilities associated with the random variable. Similarly for continuous random variable we can say ***probability density function (pdf).***

✓ **Expected Value (Mean) of Random Variable**

- Let X be a discrete random variable X whose possible values are X_1, X_2 …., X_n with the probabilities P(X1), P(X2),P(X3),…….P(Xn) respectively.

  Then the expected value of X, E(X) is defined as:

  $E(X) = X_1P(X_1) + X_2P(X_2) + \text{---} + X_n P(X_n)$

$$\mathbf{E(X)} = \sum_{i=1}^{n} X_i P(X = x_i)$$

Example 6.7: what is the expected value for the r.v from the previous (Tossing a fair coin three times) example?

- ***Solution***

$X = \{0,1,2,3,\} = X_1 = 0, \quad X_2 = 1, \quad X_3 = 2, \quad and X_4 = 3$

$P(X=x_1) = 1/8$, $P(X= X_2) = 3/8$, $P(X= X_3) = 3/8$ and $P(X= x_4) = 1/8$

$$E(X) = \sum_{i=1} X_i P(X = x_i)$$

$= 0(1/8) \ +1(3/8) + 2(3/8) +3(1/8) = 12/8 \ = 1.5$

- Let the random variable *X* is continuous with *p.d.f. f(x)*, it`s expected value is defined by:

$$\mu = E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx$$, provided that the integral exists

Example 6.8: ***Suppose*** that you are expecting a message at some time past 5 P.M. From experience it is know that X, the number of hours after 5 P.M. until the message arrives, is a random variable with the following probability density function:

$$f(x) = \begin{cases} \frac{1}{1.5} & \text{if } 0 < x < 1.5 \\ 0 & \text{otherwise} \end{cases}$$

What will be the expected amount of time past 5 P.M. until the message arrives?

- ***Properties of Expectation***

1. If X = K is a constant, then E(X) = K
2. Suppose K is a constant and X is random variable Then E(KX) = K E(X)
3. If X is random variable and suppose a and b are constants. Then E (aX + b) = a E(X) + b
4. Let X and Y are any two random variables. Then E (X + Y) = E(X) + E(Y). This can be generalized to *n* random variables, That is, if $X_1, X_2, X_3, \ldots, X_n$ are random variables then,

$$E(X_1 + X_2 + X_3 + \ldots + X_n) = E(X_1) + E(X_2) + E(X_3) + \ldots + E(X_n)$$

5. Let X and Y are any two random variables. If X and Y are independent. Then

$$E(XY) = E(X)\, E(Y)$$

Exercise!

If the random variables X and Y have density functions $f(x) = \frac{1}{8}x$ and $f(y) = \frac{1}{12}y$ respectively for $0 < x < 4$ and $1 < y < 5$. Find

A. E (X)

B. E(Y)

C. E (X + Y)

D. E (2X + 3Y)

E. E (XY) if X and Y are independent?

✓ **Variance of Random Variable**

- If X is a discrete random variable with expected value μ (i.e. E(X) = μ), then the variance of X, denoted by Var (X), is defined by

Var (X) = $E(X-\mu)^2$

$= E(X^2) - \mu^2$ or $\mathbf{E(X^2) - [E(X)]^2}$

$= \sum_{i=1}^{n}(x_i)^2 P(x_i) - \mu^2$ where, $E(X^2) = \sum_{i=1}^{n}(x_i)^2 P(x_i)$

Examples 6.9: Consider a random variable X that takes a value either 1 or 0 with respective probabilities P and 1-P. find the expected value as well as the variance of the r.v X.

Solution $X_1 = \{0, 1\}$

P(X=1) = P and P(X=0) = 1-P

E (X) = $\sum xiP(X = x_i) = 0.P(X=0) + 1.P(X=1)$

= 0(1-P) + 1(P) = P

Var (X) $= E(X^2) - \mu^2$

$= \sum xi^2 P(X = x_i) - \mu^2$

$= [0^2 P(x=0) + 1^2 P(x=1)] - P^2$

$= [0(1-p)+1(p)] - P^2$

$= P - P^2 = P(1-P)$

Example 6.10: Two fair coin are tossed. Determine Var (X) where X is the number of heads that appear.

Solution

X = No of heads = $\{0,1,2,\}$ $\{HH, TH, HT, TT\}$

P (X = 0) =¼ , P (X = 1) = ½, P(X=2) = ¼

E (X) = 0.P(X=0) +1.P (X=1) +2P(X=2) = 0 (1/4) + 1(1/2) +2(1/4) = 1=E(X)

$E(X^2) = 0^2P(X=1) + 1^2.P(X=1) + 2^2P(X=2) = 0(1/4) + 1(1/2) + 4(1/4) = 3/2$

Var (X) = E(X^2) - μ^2 = 3/2-1=1/2

Exercise! Find the variance and standard deviation of the random variable X. X is number of heads come up in tossing of three distinct coins once.

❖ **Properties of Variance**

1. Let X be a random variable and K is a constant then, V (X + K) = V(X)
2. For a constant K and a random variable X then, V(KX) = K^2 V(X)
3. Let $X_1, X_2, X_3, \ldots, X_n$ be *n* independent random variable, then

   $$V(X_1 + X_2 + X_3 + \ldots + X_n) = V(X_1) + V(X_2) + V(X_3) + \ldots + V(X_n)$$
4. Let X be a random variable with finite variance. Then for any real number a,

   $$V(X) = E[(X - a)^2] - [E(X) - a]^2.$$

 If X is a continuous random variable, then the variance of X, denoted by Var (X), is defined by $\boldsymbol{Var(X) = \delta^2 = \int_{-\infty}^{\infty} X^2 f(X) - \mu^2}$, Provided that the integral exists

Examples 6.11: Let a continuous random variable X has probability density function given by $f(x) = \begin{cases} 2e^{-2x} & x > 0 \\ 0 & x \le 0 \end{cases}$ for a constant K.

Find

(a) The variance of X (c) *Var (KX)*

(b) The standard deviation of X (d) *Var (K + X)*

Example 6.12: Two fair coins are tossed. Determine Var (X) where X is the number of heads that appear.

Solution A) X = No of heads = $\{0,1,2,\}$ $\{HH, TH\ ,\ HT\ ,TT\}$

P (X = 0) =¼ , P (X = 1) = ½, P(X=2) = ¼

E (X) = 0.P(X=0) +1. P (X=1) +2. P(X=2) = 0 (1/4) + 1(1/2) +2(1/4) = 1=E(X)

E(X^2) = 0^2P(X=1) +1^2.P(X=1) +2^2P(X=2) = 0(1/4) + 1(1/2) +4(1/4) = 3/2

Var (X) = E(X^2) - μ^2 = 3/2-1=1/2

6.4 COMMON PROBABILITY DISTRIBUTIONS

In practice most of the times a detailed list as given in the examples above will not be presented. Instead, a model or formula will be used. These models are named like ***uniform distribution, normal distribution, binomial distribution, negative binomial distribution, exponential distribution, Poisson distribution, geometric distribution, hyper-geometric distribution etc.*** If the random variable is discrete/ continuous, the model is known as discrete/continuous probability distribution respectively.

6.4.1 Common Discrete Probability Distributions

A. ***Binomial Distribution***

It is used to represent the probability distribution of discrete random variables. Binomial means two categories. The successive repetition of an observation (trial) may result in an outcome which possesses or which does not possess a specified character. Our primary interest will be either of these possibilities. Conventionally, the outcome of primary interest is termed as ***success.*** The alternative outcome is termed as ***failure.*** These terminologies are used irrespective of the nature of the outcome. For example, non-germination of a seed may be termed as success.

In binomial distribution the experiment consisting the following criteria/assumptions

- There is only two outcomes in Bernoulli trials (success or failure)
- Fixed number of trials (n) i.e. n should be discrete
- At each trial the probability of success (p) remains the same
- n trials are independent.

The variable X which represents the count of the number of successes in Bernoulli trials will be a discrete random variable. The probability distribution of such discrete random variable X is called the binomial distribution. The binomial distribution is given by the probability mass function (pmf):

P[X=x] = $\binom{n}{x} p^x q^{n-x}$ for all possible values of X.

Where, n= number of trials

x= number of successes in n trials

n-x = number of failures in n trials

p = probability of success ($P = \frac{x}{N}$)

q = 1 - p = probability of failure

$\binom{n}{x}$ = the possible number of ways in which x successes can occur.

Note:

The binomial distribution is determined by two parameters n and p.

The expected value of the binomial distribution is np and the variance is npq.

Example 6.13: A given mid-exam contains 10 multiple choice questions, and each question has four alternatives with one exact answer. Find the probability that the student exactly answered

i. 3 questions

ii. 8 questions

iii. At least 3 questions

Using binomial distribution, we can get the probability value easily. That is n = 10,

p = ¼ (the chance of getting answer from 4 alternatives)

q = 1- p = 1- ¼ = ¾

The possible marks for a student from 10 questions are X = 0, 1, 2, 3. . . 10.

$P(X = x) = P(x: n, p) = \binom{n}{x} p^x q^{n-x}$

i. $P(X = 3) = \binom{10}{3} (0.25)^3 (0.75)^7 = 0.250$

ii. $P(X = 8) = \binom{10}{8} (0.25)^8 (0.75)^2 = 0.00386$

iii. P (X >= x) = 1 - P (X < x). Hence P (X >= 3) = 1 – P (X < 3)

= 1 – {P (X = 0) + P (X = 1) + P (X = 2)}

$P(X = 0) = \binom{10}{0} (0.25)^0 (0.75)^{10} = 0.0563$

$P(X = 1) = \binom{10}{1} (0.25)^1 (0.75)^9 = 0.1877$

$P(X = 2) = \binom{10}{2} (0.25)^2 (0.75)^8 = 0.2816$

∴ P (X >= 3) = 1 – (0.0563 + 0.1877 + 0.2816) = 0.4744

The mean = np = 2.5. The variance = n.p.q = 1.875

Example 6.14: Suppose that a population of size N = 500 consists of 300 dominants and 200 recessives. For a sample of size n = 10, calculate the probabilities: -

a) Exactly 2 individuals will be recessive.
b) At least 2 individuals will be recessive.
c) At most 1 individual will be recessive.
d) At most 5 individuals will be recessive**. (Exercise!!!)**

B. *Poisson Distribution*

The Poisson distribution is also used to represent the probability distribution of a discrete random variable. It is employed in describing random events that occur rarely over a continuum of time or space. The Poisson distribution bears a close similarity to the binomial distribution. Suppose that we are interested in the number of occurrences of an event E in a time period of length t. This time period can be split into n equal intervals, each of length t/n. These n intervals can be treated as n trials by Bernoulli process. But there is difficult. Since the event occurs at various points of time, it can occur twice or more in one of the trials of length t/n.

In case of binomial distribution, the event is **dichotomous**, and hence there is no possibility of such multiple occurrences within a single trial. In order to overcome this difficulty, we make n larger and larger. When n is large, the trials are shorter in terms of length of time. As a result, the probability of occurrence of an event in a single trial would be smaller. It is equivalent of saying that it is a ***rare event***. The binomial distribution can still be used to represent the distribution of such random events. However, the computations become tedious since n is very large. This can be explained by example.

Suppose that the number of insects caught in a trap is being studied and that the data are collected on the number of insects caught per hour. Assume that the probability that an insect will be caught in any single minute is 0.06. Assume further that the events of insects being trapped are mutually independent and the probability p = 0.06 remains same for all the minutes. We may use the binomial distribution to calculate the number of insects caught per hour by considering each minute as a separate Bernoulli trial. If x is the number of insects caught in a minute, then we have:

$$P[X=x] = \binom{60}{x}(0.06)^x(0.94)^{60-x}$$

Instead of dividing the hour into minutes the seconds may be used as basic units. Then the value of p would be reduced to, p=0.06/60=0.001. Considering each second as a Bernoulli trial, we would have a sample size 60×60=3600 for a period of one hour. The binomial distribution would now be:

$$P[X=x] = \binom{3600}{x}(0.001)^x(0.999)^{3600-x}$$

Thus, when n becomes larger and larger the computations using binomial become tedious. Fortunately, it has been shown by Poisson that the value of $\binom{n}{x}p^x q^{n-x}$ approaches the value of $\frac{(np)^X e^{-np}}{x!}$, when n becomes large and p becomes small in such a way that the equality, np = λ is maintained.

The Poisson distribution is given by the pmf,

P[X=x] = $\frac{e^{-\lambda}\lambda^x}{x!}$**.**

Where, λ = np, x = the number of times an event occurs.

e= Naperian base = 2.7182…

In Short, the probability of *X* occurrences in an interval of time, volume, area, etc., for a variable where λ (Greek letter lambda) is the mean number of occurrences per unit (time, volume, area, etc.) is $P(X;\lambda) = \frac{e^{-\lambda}\lambda^x}{x!}$ where *X* _ 0, 1, 2, . . .

The value of $e^{-\lambda}$ can be obtained directly from mathematical tables. In case of Poisson distribution, the counts of alternative events, i.e., failures are not of interest. This is a contrast between binomial and Poisson distributions. For Poisson distribution all that we need is np, the mean number of successes. We need not know about n and p individually.

Thus, the Poisson distribution is determined by the parameter λ..**The special property of Poisson distribution is that its mean and variance are same to** λ **.**

i.e. mean = variance = λ.

Example 6.15: In Black Lion Hospital, the average new born female baby in every 24 hour is 7. What is the probability that

i. No female babies are born in a day?

ii. Only three female babies are born per day?

iii. Two female babies are born in 12 hours?

In this case λ = 7 per day

No female baby born in a day $\Rightarrow$ P (X = 0) = $\frac{e^{-7}7^{0}}{0!}$ = e^{-7} = 0.0138189

Only three female babies are born $\Rightarrow$ P (X = 3) = $\frac{e^{-7}7^{3}}{3!}$ = 0.78998

Two female babies are born in 12 hours → in this case λ = 7/2 = 3.5

$$P(X = 2) = \frac{e^{-3.5}(3.5)^2}{2!} = 0.184959$$

Example 6.16: In some experiments it was observed that the incidence of stem fly in black gram was 6 percent. Suppose we examine 50 black gram plants in a field at random. What is the probability that at most 3 plants will be found to be affected by stem fly? **Exercise!!!**

6.4.2 *Common Continuous Probability Distribution*

A. *Normal Distribution:*

The most important and widely used probability distribution is normal distribution. It is also known as ***Gaussian distribution***. Most of the distributions occurring in practice, for instance, binomial, Poisson, etc., can be approximated by normal distribution. Further, many of the sampling distributions like Student's t-distribution, F-distribution, & χ^2 - distributions tend to normality for ***large samples***. Therefore, the normal distribution finds an important place in statistical inference.

The normal distribution is used to represent the probability distribution of a continuous random variable like life expectancies of some product, the volume of shipping container etc. Its probability density function is expressed by the relation,

$$\boldsymbol{f(x) = \frac{1}{\delta\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\delta}\right)^2}}$$

In the above formula, π = a constant equaling 22/7 = 3.14

e = Naperian base equaling 2.7182…

μ = population mean.

δ = population standard deviation.

x = a given value of the rv in the range $-\infty \leq x \leq \infty$.

For a normal distribution the frequency curve will be symmetrical or bell shaped. However, not all symmetrical curves are normal.

- **Properties of Normal Distribution**:

1. A normal distribution curve is bell-shaped.
2. The mean, median, and mode are equal and are located at the center of the distribution.
3. A normal distribution curve is unimodal (i.e., it has only one mode).
4. The curve is symmetric about the mean, which is equivalent to saying that its shape is the same on both sides of a vertical line passing through the center.
5. The curve is continuous; that is, there are no gaps or holes. For each value of *X*, there is a corresponding value of *Y.*
6. The curve never touches the *x* axis. Theoretically, no matter how far in direction the curve extends, it never meets the *x* axis—but it gets increasingly closer.
7. The total area under a normal distribution curve is equal to 1.00, or 100%. This fact may seem unusual, since the curve never touches the *x* axis, but one can prove it mathematically by using calculus. i.e. $\int_{-\infty}^{\infty} \boldsymbol{f(x)dx} = \int_{-\infty}^{\infty} \frac{\mathbf{1}}{\boldsymbol{\delta\sqrt{2\pi}}} \boldsymbol{e}^{-\frac{1}{2}\left(\frac{x-\mu}{\delta}\right)^2} \boldsymbol{dx}$ **=1.**
8. The area under the part of a normal curve that lies within 1 standard deviation of the mean is approximately 0.68, or 68%; within 2 standard deviations, about 0.95, or 95%; and within 3 standard deviations, about 0.997, or 99.7%.
9. The Probability that a random variable will have a value between any two points is equal to the area under the curve between those points.

The shape of the normal curve is completely determined by two parameters μ & δ. For any given δ, there can be a number of normal curves each with a different μ. Likewise, for any given μ, there can be a number of normal curves each with a different δ. In order to make such all distributions readily comparable with each other, their individuality as expressed by their mean and standard deviation has to be suppressed. This is done by transforming the normal variable into standard normal variable.

- **Standard normal Distribution**

It is a normal distribution with mean 0 and variance 1. Normal distribution can be converted to standard normal distribution as follows. If X has normal distribution with mean μ_X and standard deviation, then the standard normal distribution variant Z is given by Z=$\frac{x-\mu}{\delta}$

$$P(Z) = \frac{1}{\sqrt{2\pi}} e^{-Z^2/2}$$

Properties of the standard normal distribution:

- The same as normal distribution, but the mean is zero and the variance is one.
- Areas under the standard normal distribution curve have been tabulated in various ways. The most common ones are the areas between Z = 0 and a positive value of Z and the area less than any negative/positive value of Z.

Given a normal distributed random variable X with mean μ and standard deviation σ:

P (b<X<a) $= P(\frac{b-\mu}{\delta} < \frac{x-\mu}{\delta} < \frac{a-\mu}{\delta})$

$P(X < a) = P\left(\frac{x-\mu}{\delta} < \frac{a-\mu}{\delta}\right)$ But, $\frac{x-\mu}{\delta} = Z$ Standard normal r.v.

⇔ P$\left(Z < \frac{a-\mu}{\delta}\right)$

Note: i) P (a<x<b) = P (a<=X<b)

= P (a<X<=b)

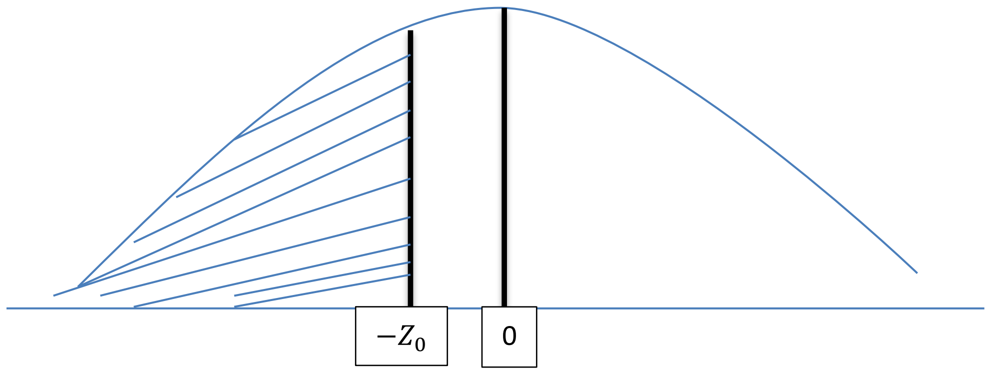
=P (a<=X<=b)

ii) $P(-\infty < Z < \infty) = 1$

- Consider the situations under the standard normal curve. It is clear that

$P(0<Z)=0.5=P(Z<0)$

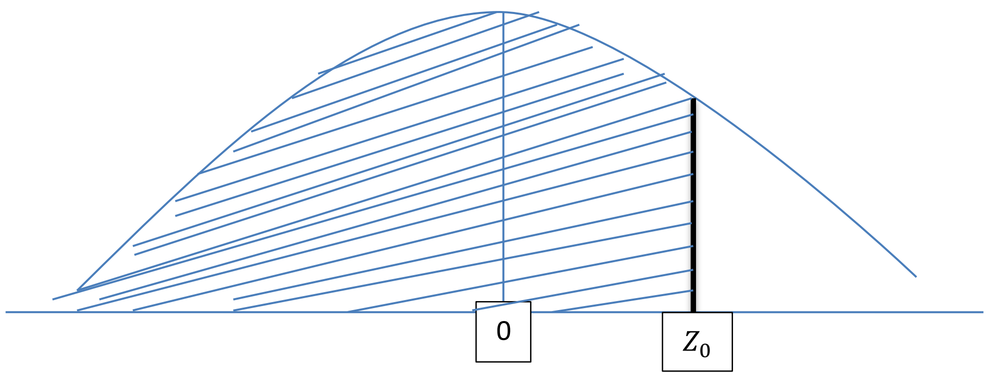
i) Let Z_0 be negative number then,



- $To\ compute\ the\ value\ of\ P(Z < -Z_0)$

  $Use\ the$ standard $normal\ distribition\ Table\ for\ the\ negative\ value\ of\ Z_0.$

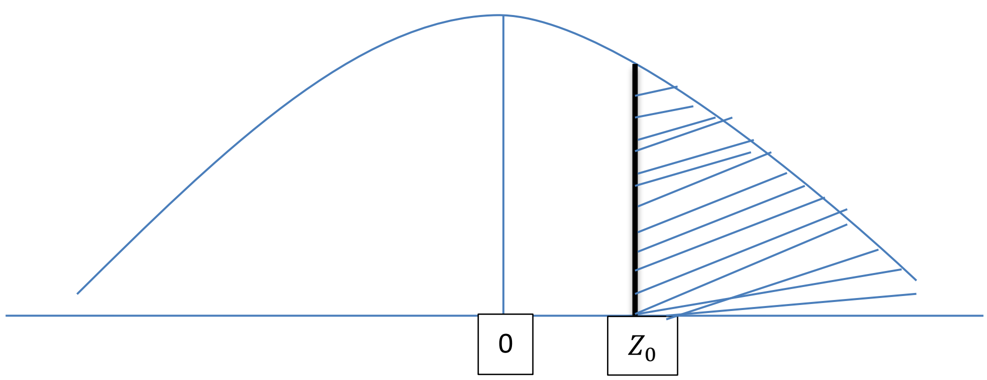
ii) If Z_0 is positive real number, then



- To compute $the\ value\ of\ P(Z < Z_0),$

$Here\ also\ you\ can\ use$ standard $normal\ distribition\ Table\ for\ the\ positive\ value\ of\ Z_0.$

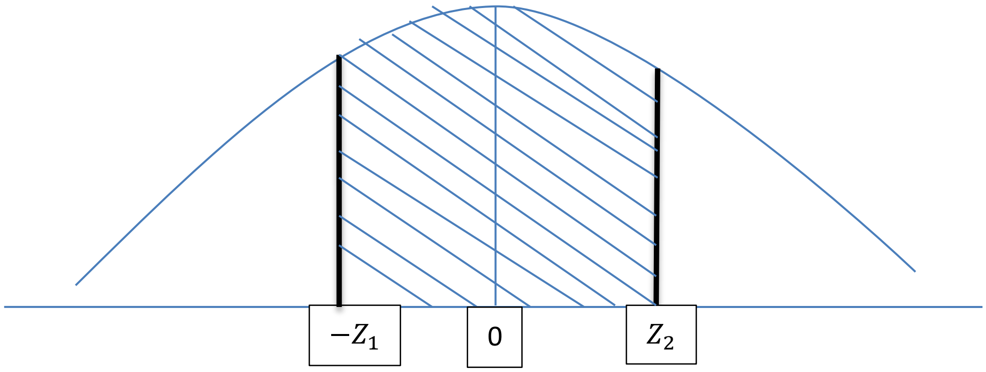
iii) If Z_0 is positive real number, then



- To compute $the\ value\ of\ \ P(Z > Z_0),$

  $First\ read\ from\ the\ table\ for\ P(Z < Z_0)$, and then subtract that value from 1. It will give you the area of the shaded region; which mean that $P(Z > Z_0)$

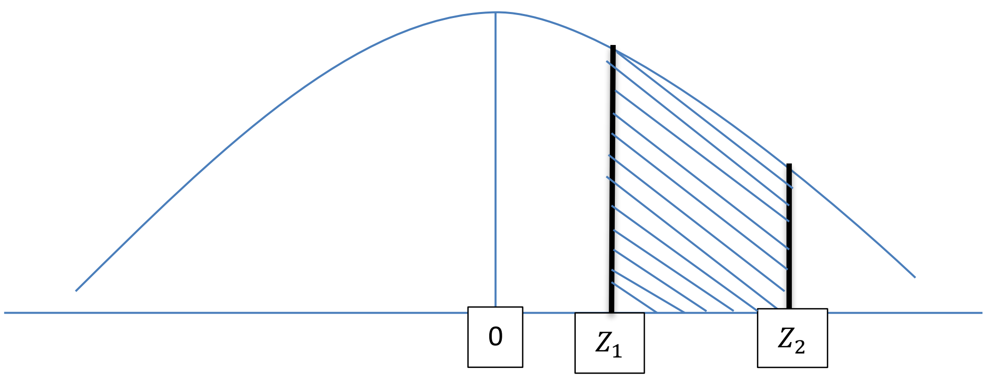
iv) Let Z_1 be a negative number and Z_2 be positive real number, then



- To compute *the value of* $P(-Z_1 < Z < Z_2)$, *you can subtract* $P(Z < -Z_1)$ *from* $P(Z < Z_2)$.

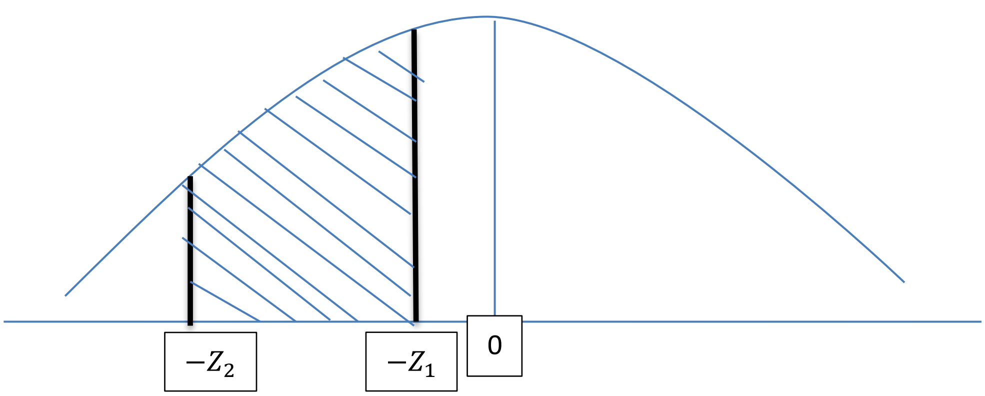
  i.e, $P(-Z_1 < Z < Z_2) = P(Z < Z_2) - P(Z < -Z_1)$

v) If Z_1 and Z_2 are both positive real numbers with $Z_1 < Z_2$



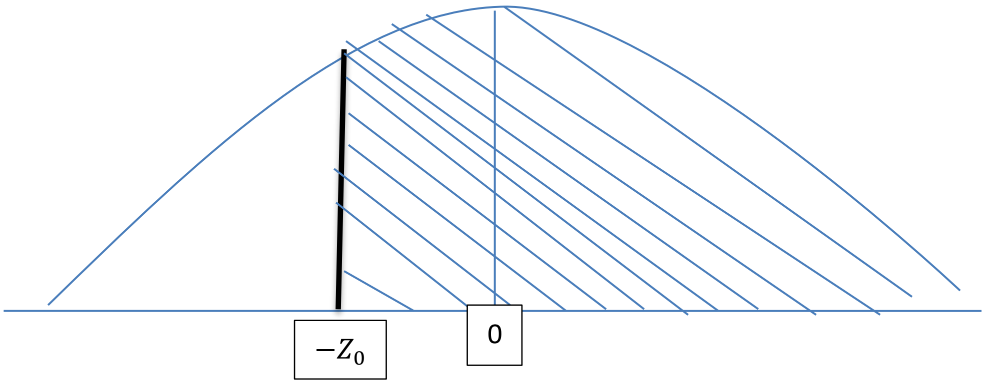
- To compute *the value of* $P(Z_2 > Z < Z_1)$, simply you can subtract $P(Z < Z_1)$from $P(Z < Z_2)$using a standard normal distribution Table.

vi) If Z_1 and Z_2 are both negative real numbers with $-Z_2 < -Z_1$



- To compute $the\ value\ of\ \ P(-Z_2 < Z < -Z_1)$, simply you can subtract $P(Z < -Z_2)$from $P(Z < -Z_1)$using a standard normal distribution Table with the negative standardized values.

vii) If Z_0 is negative real number,



- To compute $the\ value\ of\ \ P(-Z_0 < Z)$, simply you can read the value of $P(Z < -Z_0)$ from standard normal distribution Table and then subtract this value from 1.

  i.e., $P(-Z_0 < Z) = 1 - P(Z < -Z_0)$

NOTE: As the value of δ increases, the curve becomes more and more flat and vice versa.

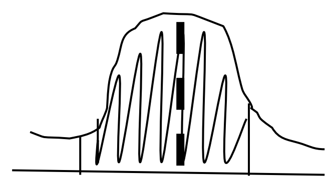
<u>Examples 6.17:</u> - For a standard normal variable Z find

a) P (-2.2 <Z < 1.2) **c)** P (0 < Z < 0.96)

b) P (Z > 1.05) d) P (-1.45 < Z < 0)

Solution:

a)



a)

P (-2.2 < Z < 1.2) = P (Z < 1.2) - P (Z < -2.2)

= 0.846136 – 0.021692

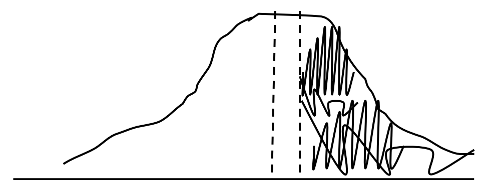
= <u>0.82444</u>

b) = P (Z > 1.05) = 1 - P (Z < 1.05)

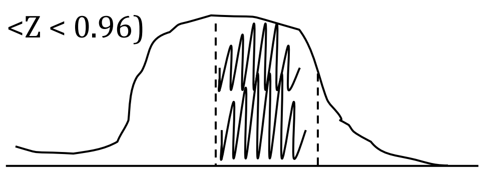
= 1- 0.853141

= <u>0.14686</u>

b)



c) P (0 <Z < 0.96)

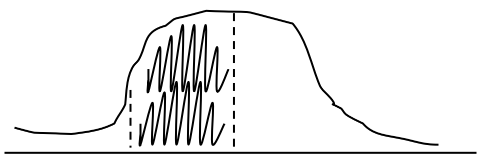


$\Rightarrow$ P (0 <Z < 0.96) = P (Z < 0.96) - P (Z < 0)

= 0.831472 – 0.5

= <u>0.331472</u>

d) P (-1.45 <Z<0) =?



$\Rightarrow$ P (-1.45 < Z < 0) = P (Z < 0) - P (Z < -1.45)

= 0.5 – 0.073529

= <u>0.4265</u>

<u>Example 6.18:</u> In a normally distributed population of 1000 wage earners with a mean income of 750 birr per month and variance of the income 576, what will be the number of persons who earn between 600 and 1100 birr?

<u>Givens</u> <u>Required</u>

Number of wage earners who earn b/n 600 and 1100

N=1000

$\mu = 750$

$\delta^2 = 576$

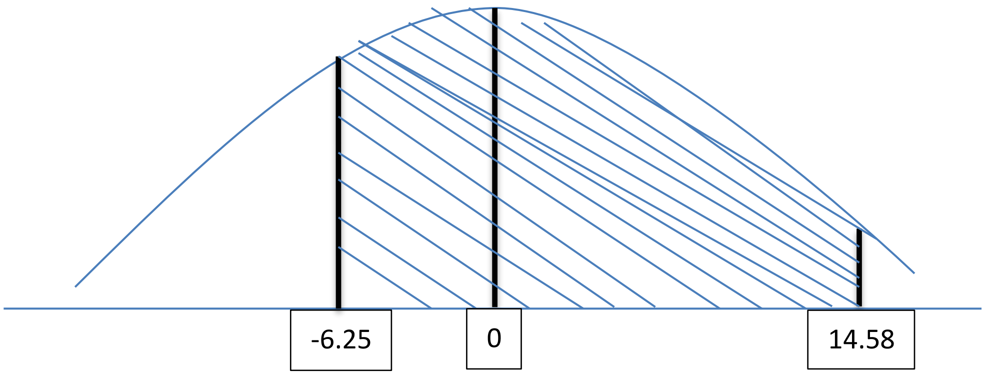
<u>Sol.</u> $\mathrm{P}(600 < \mathrm{X} < 1100) = ?$

$$\Rightarrow \mathrm{P}\left(\frac{600-750}{24} \leq \frac{X-\mu}{\delta} \leq \frac{1100-750}{24}\right) = \mathrm{P}(-6.25 \leq \mathrm{Z} \leq 14.58)$$

$$= P(Z < 14.58) - P(Z < -6.25)$$

$$= 0.999967 - 0.000033$$

$\underline{= 0.999934 \sim 1}$ this means that almost all of the population earns the wage in between 600birr and 1100birr.

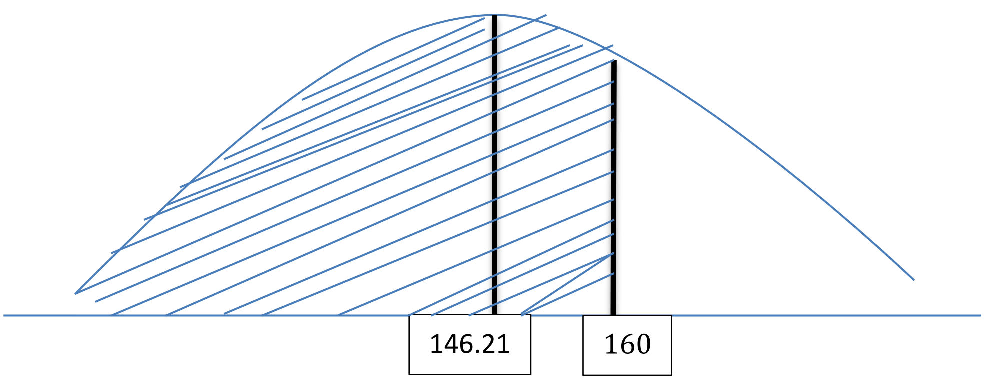


Example 6.19: Find the z value such that the area under the standard normal distribution curves between 0 and the z value is 0.2123.

Example 6.20: A survey found that women spend on average \$146.21 on beauty products during the summer months. Assume the standard deviation is \$29.44. Find the percentage of women who spend less than \$160.00. Assume the variable is normally distributed.

Solution

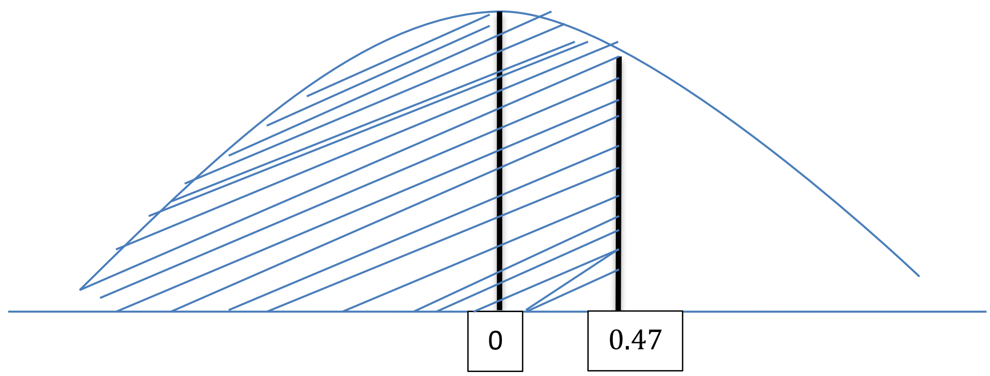
Step 1 Draw the figure and represent the area as shown below



Step 2 Find the z value corresponding to \$160.00.

$$Z = \frac{X - \mu}{\delta} = \frac{\$160 - \$146.21}{29.44} = 0.47$$

Hence \$160.00 is 0.47 of a standard deviation above the mean of \$146.21, as shown in the z distribution.



Step 3 Find the area, using Standard Normal Distribution Table. The area under the curve to the left of z = 0.47 is **0.6808.**

Therefore **0.6808, or 68.08%,** of the women spend less than $160.00 on beauty products during the summer months.