AP Calculus – Advanced Derivatives Practice (Free-Response)

Overview

The AP Calculus exam also features free-response questions in which you must provide detailed solutions and justifications. These questions often ask you to derive formulas, analyze graphs, and interpret the relationships among variables. The following set of 15 questions covers the key topics from your Advanced Derivatives unit (linearisation, L'Hôpital's rule, implicit differentiation, derivatives of inverse and inverse trigonometric functions, the Mean Value and Rolle's theorems, related rates, parametric derivatives, concavity, and graph analysis). You should plan to spend about **90 minutes** total (around six minutes per question) solving these questions.

Questions (provide full solutions and explanations)

1. **Linearisation and concavity.** Let $f(x) = \ln(x)$.
   a. Derive the linearisation $L(x)$ of f at $x = a > 0$.
   b. Use your expression to approximate $\ln(3.02)$ by linearising at $a = 3$.
   c. Determine, with justification, whether this approximation is an overestimate or an underestimate of the true value of $\ln(3.02)$.
2. **L'Hôpital's Rule.** Evaluate the limit

   $\lim_{x\to 0} \frac{\sin(3x)-3x}{x^3}$.

Provide a step-by-step solution that includes the application of L'Hôpital's rule and any necessary series expansions.

3. **Implicit differentiation.** The curve is defined implicitly by $x^2y + y^2 = 6$.
   a. Find $\dfrac{dy}{dx}$ in terms of x and y.
   b. Find the slope of the tangent line at the point $(2, 1)$.
   c. Write the equation of the tangent line at this point.
4. **Derivative of an inverse function.** Suppose f is differentiable and invertible. Given $f(2) = 5$, $f'(2) = 4$, $f(5) = 2$ and $f'(5) = 3$, compute $(f^{-1})'(5)$. Show all reasoning and state the formula you use.
5. **Mean Value Theorem.** Let $g(x) = x^3 - 3x$ on the interval $[-1, 2]$.
   a. Compute the average rate of change $\dfrac{g(2) - g(-1)}{2 - (-1)}$.
   b. State the Mean Value Theorem and verify that its hypotheses are satisfied for g on $[-1, 2]$.
   c. Find all numbers c in $(-1, 2)$ that satisfy the conclusion of the Mean Value Theorem for this function.
6. **Rolle's Theorem and roots.** Consider $h(x) = x^4 - 4x^2$ on the interval $[-2, 2]$.
   a. Show that the function satisfies the hypotheses of Rolle's Theorem on $[-2, 2]$.
   b. Find all values of c in $(-2, 2)$ guaranteed by Rolle's Theorem.
   c. Briefly interpret the geometric meaning of these points on the graph of h.
7. **Related rates (sphere).** A spherical balloon is being inflated so that its radius increases at a constant rate of 0.2 m/s.
   a. Derive a formula for the volume V of a sphere in terms of its radius r.
   b. Use implicit differentiation to relate $\dfrac{dV}{dt}$ and $\dfrac{dr}{dt}$.
   c. Find the rate at which the volume of the balloon is changing when the radius is 2 m. Express your answer in cubic metres per second.
8. **Second derivative and concavity.** Let $p(x) = \dfrac{x}{1+x}$ for $x \neq -1$.
   a. Compute $p'(x)$ and $p''(x)$.
   b. Determine all intervals where p is concave up and where it is concave down.
   c. Identify any inflection points and justify your answer.
9. **Antiderivative of $\frac{1}{1+x^2}$.**
   a. Evaluate $\displaystyle\int \frac{1}{1+x^2}\,dx$ and specify the family of antiderivatives.

b. Determine the particular antiderivative $F(x)$ that satisfies $F(0) = 0$.

c. Explain why this function is related to the derivative of an inverse trigonometric function.

10. **Parametric derivatives (chain rule).** A particle moves along a curve defined parametrically by $x(t) = t^2 - 2t$ and $y(t) = \ln(t+1)$ for $t > -1$.

    a. Derive a formula for $\dfrac{dy}{dx}$ in terms of t.

    b. Evaluate $\dfrac{dy}{dx}$ at $t = 1$.

    c. Interpret the meaning of your answer in the context of the particle's motion.

11. **Analyzing a graph of f'.** Suppose the graph of f' is positive on $-1 < x < 3$, negative on $3 < x < 5$, and $f'(x) = 0$ at $x = -1, 3, 5$. Additionally, $f''(x) > 0$ on $0 < x < 4$.

    a. Determine the intervals where f is increasing and decreasing.

    b. Identify the nature (local maximum, local minimum or neither) of the critical points $x = -1$, $x = 3$ and $x = 5$.

    c. Explain whether $x = 3$ is a point of inflection for f and justify your conclusion.

12. **Limit with exponential functions.** Evaluate

    $\lim_{x \to 0} \frac{e^{2x} - 1 - 2x}{x^2}$.

    Provide a detailed solution using either L'Hôpital's rule or a second-order Taylor expansion for e^{2x}.

13. **Implicit differentiation – tangent line.** The curve defined by $\cos(xy) = x$ passes through the point $\left(\dfrac{\pi}{2}, 1\right)$.

    a. Differentiate implicitly with respect to x to find an expression for $\dfrac{dy}{dx}$ in terms of x and y.

    b. Compute $\dfrac{dy}{dx}$ at $\left(\dfrac{\pi}{2}, 1\right)$.

    c. Write the equation of the tangent line to the curve at this point.

14. **Limit of logarithmic and power functions.** Prove that

    $\lim_{x \to \infty} \frac{\ln(x)}{\sqrt{x}} = 0$.

    Provide justification using an appropriate method (such as L'Hôpital's rule or a comparison test).

15. **Related rates – sliding ladder.** A 5-m ladder leans against a vertical wall. The bottom of the ladder is pulled away from the wall at a constant rate of 0.3 m/s. Let $x(t)$ be the distance of the bottom from the wall and $y(t)$ be the height of the top of the ladder above the ground.

a. Write the relationship between x and y using the Pythagorean theorem.

b. Differentiate implicitly with respect to time to relate $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

c. Compute $\frac{dy}{dt}$ when $x = 4$ m. State the sign of $\frac{dy}{dt}$ and interpret the result.