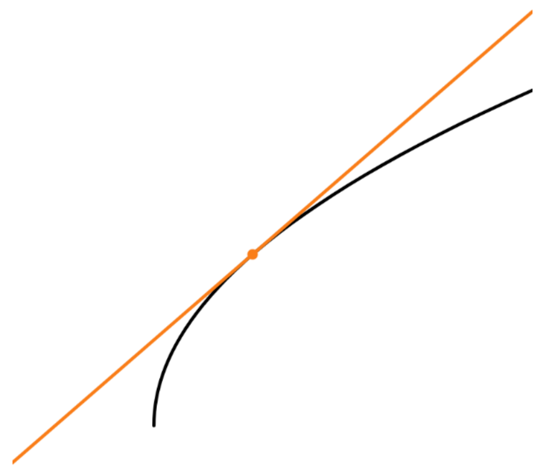
Chapter 1, Lesson 1

Linearization & Linear Approximation

In calculus, we often encounter functions that are too complex to evaluate or analyze exactly at every point. The process of linearization (also called linear approximation) provides a powerful method to simplify these functions near a specific point by approximating them with a linear function, or tangent line. As we've learned in previous lessons/courses, the *tangent* line to a curve is a line that, within a small interval around a specific point, models the behaviour of the curve so closely that the two are nearly indistinguishable. In this lesson, we will explore how to use the derivative to construct linear approximations of functions and interpret their accuracy.



Linearization

$f(x) \approx L(x)$ at $x = a$

$L(x) - f(a) = f'(a)(x-a)$

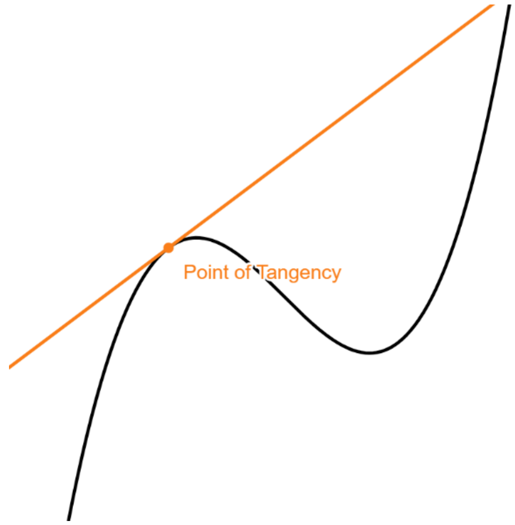
Note: $L(x)$ can also be notated as $T(x)$

LEARNING GOALS

- Describe the linear approximation to a function at a point
- Write the linearization of a given function
- Use linearization to approximate the values of radical and/or reciprocal calculations
- Determine if the value obtained by a linear approximation is an overestimate or underestimate of the actual values of the curve

What Is Linearization?

Linearization is the process of approximating a nonlinear function with a linear function near a specific point. This approximation uses the tangent line to the function at that point to represent the function's behavior in a small neighborhood. The basic premise of linearization is that in a small region around the point of tangency, the tangent line approximates the values of the function so closely that we can use it to *estimate* values around the point of tangency.



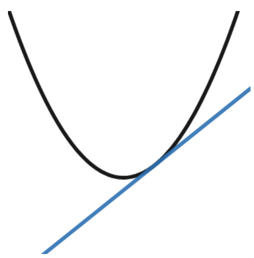
Example 1 *– Introduction to Linearization*

For some differentiable function $g(x)$, it is known that $g(2)=7$ and $g'(2)=-3$. Use the tangent line approximation at $x=2$ to estimate $g(2.1)$.

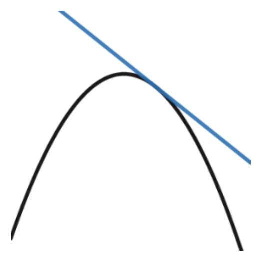
Overestimates vs Underestimates

Since linearization only provides an *estimate* of the values on a curve around the point of tangency, our results will always have a degree of inaccuracy. When performing a linear approximation, we always must be aware of if our results are *overestimates* or *underestimates* of the actual values of the function. This determination is based on the *concavity* of the curve.

When $f(x)$ is concave up, the tangents are *below* the graph, so the linearization *underestimates* the actual values of the function



When $f(x)$ is concave down, the tangents are *above* the graph, so the linearization *overestimates* the actual values of the function



Example 2 *– Overestimates vs Underestimates*

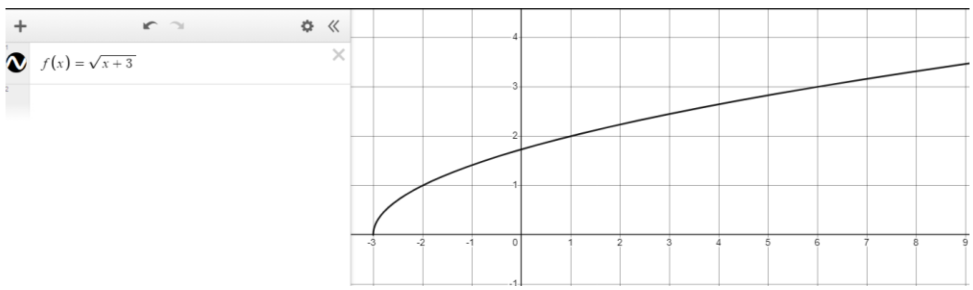
Find the linearization of the function $f(x)=\sqrt{x+3}$ at $x=1$ and use it to approximate $\sqrt{3.98}$ and $\sqrt{4.05}$. Would this be an overestimate or an underestimate?

Desmos Connection

We can use Desmos to help us visualize and calculate linearization! Visualizing a linear approximation is one of most powerful ways that we can understand this topic.

1. Preface your equation with $f(x) =$

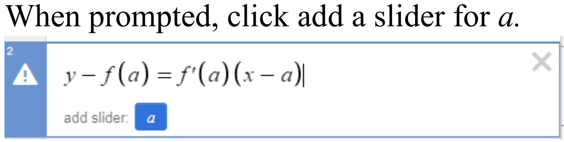
   Using the example from above, let's enter $f(x) = \sqrt{x+3}$ into Desmos



2. Set up your linearization

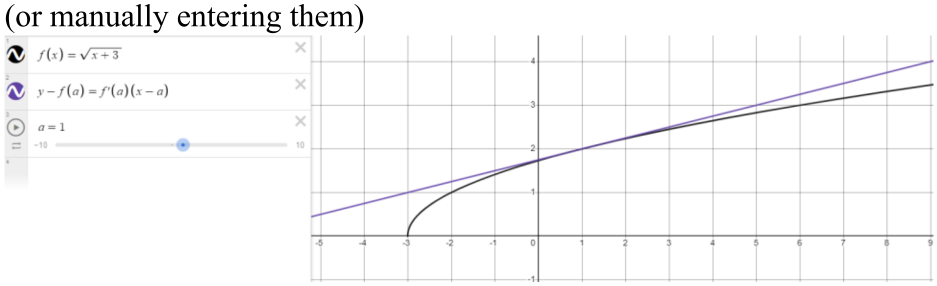
   Now that $f(x)$ is defined, you can now reference it within Desmos. On the next line, type in $y - f(a) = f'(a)(x-a)$. This equation may look familiar to you…it is simply the point-slope equation for a line, where $(a, f(a))$ is a sample point (in this case, the center of linearization) and $f'(a)$ is the slope of the tangent!

   When prompted, click add a slider for *a*.



3. Graph different tangent lines

   You should now see the tangent line to your graph at the value $x = a$. You can now dynamically change your tangent line by choosing different *a* values by using your slider (or manually entering them)



Use Linearization to approximate $\sqrt{65}$

a. Decide on a logical function/place to center the approximation

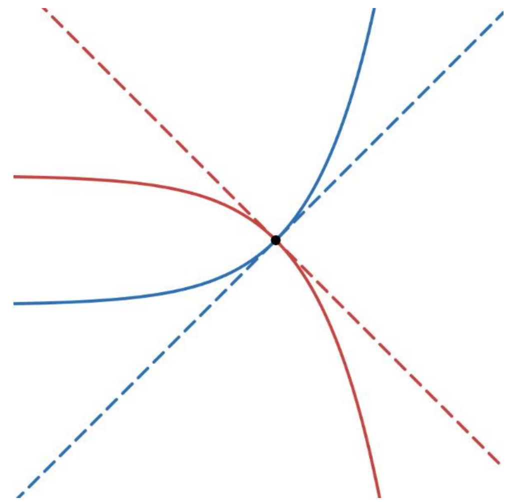
b. State the point of tangency

c. Determine the equation of the tangent

d. Use the equation to calculate the linear approximation.

Advanced Limit Techniques

Let's revisit limits! Throughout Calculus, there were many different strategies that we learned for taking limits, including direct substitution, multiplying by a conjugate, factoring, etc. However, some limits can result in *indeterminate* forms, which are challenging to evaluate directly. In these cases, we can use L'Hôpital's Rule to help. L'Hôpital's Rule provides a systematic method for resolving these indeterminate forms by leveraging derivatives. Building on our previous knowledge of limits and derivatives, this lesson will introduce L'Hôpital's Rule and demonstrate how it simplifies the evaluation of limits of indeterminate forms.



L'Hôpital's Rule

Suppose *f* and *g* are differentiable functions over an open interval containing *a,* except possibly at *a*. If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$, *or* if $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = \pm\infty$, then

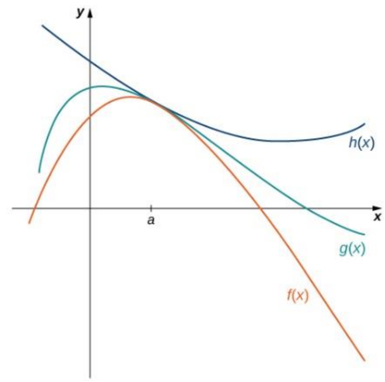
$$\boxed{\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}}$$

LEARNING GOALS

- Identify indeterminate quotients
- Recognize when to use L'Hôpital's Rule.
- Use L'Hôpital's Rule to solve the limit of an indeterminate quotient
- Compare using L'Hôpital's rule to other limit evaluation strategies
- Evaluate the limit of a function by using the Squeeze Theorem
- Evaluate the limit of a composite function

The Squeeze Theorem

The Squeeze Theorem is a unique tool that allows us to examine the behaviour of a function which is too complicated (or impossible) to examine directly. The Squeeze Theorem works by "trapping" a function between two simpler ones whose limits are easier to find. If both of these bounding functions approach the same value at a certain point, then the function in the middle must "squeeze" through the other two functions.

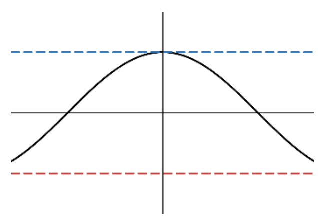


Let $f(x), g(x)$ and $h(x)$ be defined over an open interval containing *a*. If $f(x) \le g(x) \le h(x)$ for all $x \ne a$ in an open interval containing *a*, and if $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L,\ L \in \mathbb{R},$ then $\lim_{x \to a} g(x) = L.$

Let's see how we can use the Squeeze Theorem to evaluate $\lim_{x \to 0} x \cos x$

1. Start by stating the range of the trigonometric ratio

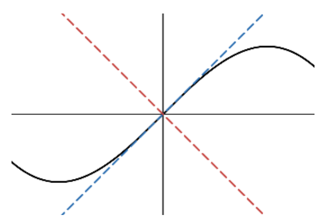
$$-1 \le \cos x \le 1$$



2. Manipulate the inequality so that the middle portion is the original question

$$(-1) \cdot x \le (\cos x) \cdot x \le (1) \cdot x$$

$$-x \le x \cos x \le x$$



3. Evaluate the limits of the outer boundaries

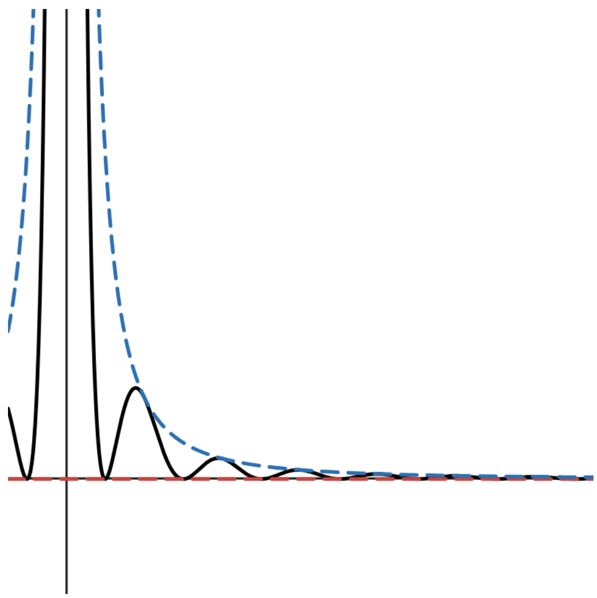
$$\lim_{x \to 0}(-x) \le \lim_{x \to 0} x \cos x \le \lim_{x \to 0}(x)$$

$$\lim_{x \to 0}(-x) = 0 \qquad \lim_{x \to 0}(x) = 0$$

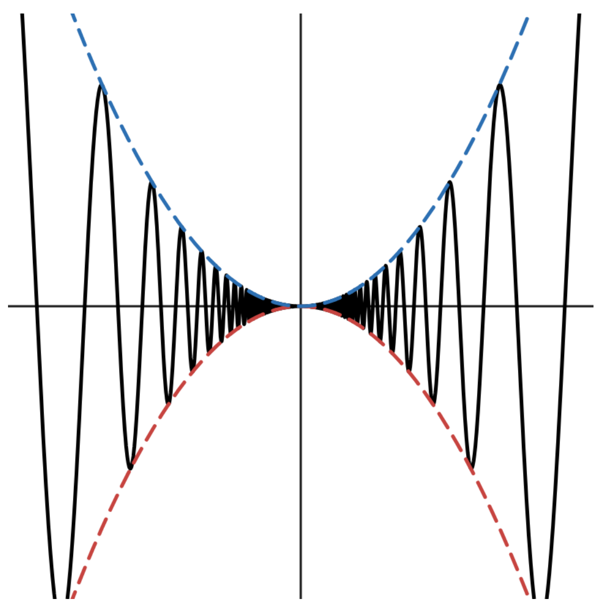
Therefore, by the Squeeze Theorem, $\lim_{x \to 0} x \cos x = 0.$

Use the Squeeze Theorem to evaluate each limit.

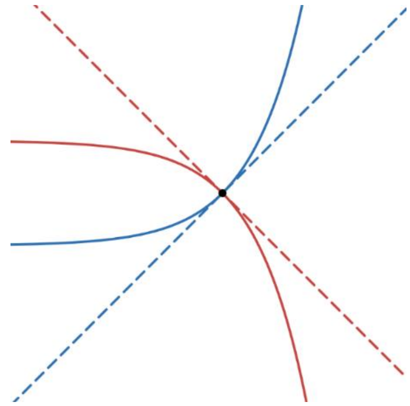
a. $\lim\limits_{x \to \infty} \dfrac{1+\cos x}{x^2}$



b. $\lim\limits_{x \to 0} x^2 \cos\left(\dfrac{\pi}{20x}\right)$



L'Hôpital's Rule



When we were first learning limits, we learned to evaluate limits using a variety of strategies, including direct substitution, multiplying by a conjugate, factoring, etc. However, some limits can result in *indeterminate forms*, which are challenging to evaluate directly. In these cases, we can use L'Hôpital's Rule to help. L'Hôpital's Rule provides a systematic method for resolving these indeterminate forms by using derivatives.

Suppose *f* and *g* are differentiable functions over an open interval containing *a*. If $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{0}{0}$ *or* if $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\pm\infty}{\pm\infty}$, then L'Hôpital's Rule tells us to evaluate this limit by differentiating the numerator and denominator *independently* (i.e. *not* by using the Quotient Rule). In other words:

$$\boxed{\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}}$$

Example 2 *– Verifying Special Limits*

Use L'Hôpital's Rule to verify the "special limits" we have previously learned.

a. $\lim\limits_{x \to 0} \dfrac{\sin x}{x}$

b. $\lim\limits_{x \to 0} \dfrac{1 - \cos x}{x}$

__Example 3__ – Revisiting Some Limits

The following limits are ones that we have solved before using a variety of strategies. Evaluate each limit using a previously learned strategy, *then* use L'Hôpital's Rule to verify the result.

	Previous Technique	L'Hôpital's Rule
a. $\lim_{x \to 3} \frac{x^2 - 9}{x - 3}$		
b. $\lim_{x \to 0} \frac{\sqrt{4+x} - 2}{x}$		
c. $\lim_{x \to 3} \frac{\frac{1}{x} - \frac{1}{3}}{x - 3}$		

***Example 4** – Applying L'Hôpital's Rule*

Determine if the following limit produces the *indeterminate forms* $\frac{0}{0}$ or $\frac{\infty}{\infty}$ and if so, apply L'Hôpital's Rule to evaluate the limit.

a. $\lim_{x \to 0} \frac{1-\cos(2x)}{5x}$

b. $\lim_{x \to 0} \frac{\sin x}{2x^2 - 1}$

c. $\lim_{x \to 2} \frac{\sin x}{x^2 - 4}$

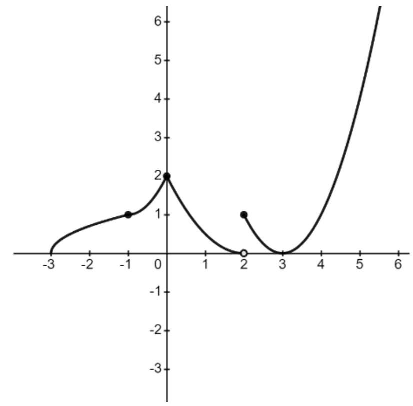
d. $\lim_{x \to \infty} \frac{\ln(e^{3x} + x)}{x}$

***Example 5** – L'Hôpital's Rule with Arbitrary Functions*

The functions *f* and *h* are twice differentiable with $\lim_{x \to 2} h(x) = 4$. The function *h* satisfies $h(x) = \frac{x^2 - 4}{1 - \left(f(x)\right)^3}$. It is known that $\lim_{x \to 2} h(x)$ can be evaluated using L'Hôpital's Rule. Use $\lim_{x \to 2} h(x)$ to find $f(2)$ and $f'(2)$

Limits with Composite Functions

When determining the limit of a *composite* function, the final answer can sometimes be deceptive! This is particularly true when the x value being approached is a point of discontinuity on one of the functions in the composition. In these cases, it is best to approach the question using *sided limits* and verify that the limit approaching from both sides are congruent with each other.



In the picture shown above, let's find $\lim_{x\to 0} f\left(f(x)\right)$. A common mistake is to automatically assume that the limit does not exist (DNE). The reason for this misconception is that one might take the limit of the inner function and see that it is 2, then note that the limit as x approaches 2 is nonexistent. Since this is a composite function, though, a more nuanced approach is needed!

First, we take the *sided limits* of the inner function. As we do this, though, we should also notate the direction that the *outputs* are approaching from using the "+" and "–" superscripts, respectively.

$$\lim_{x\to 0^-} f(x) = 2^-$$

$$\lim_{x\to 0^+} f(x) = 2^-$$

In both cases, we can see that as the inputs approach 0 from either side, the *outputs* are approaching 2 *from below* (or, in other words, from values that are *less than* 2).

As is always the case with composite functions, the outputs of the inner function become the inputs of the outer function. So now we take the limit as the inputs approach 2 *from values less than 2* (or, in this case, from the left).

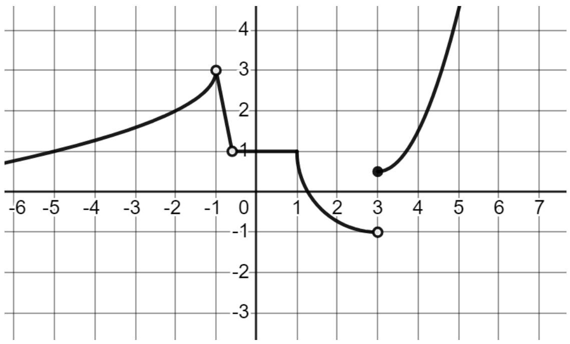
$$\lim_{x\to 2^-} f(x) = 0$$

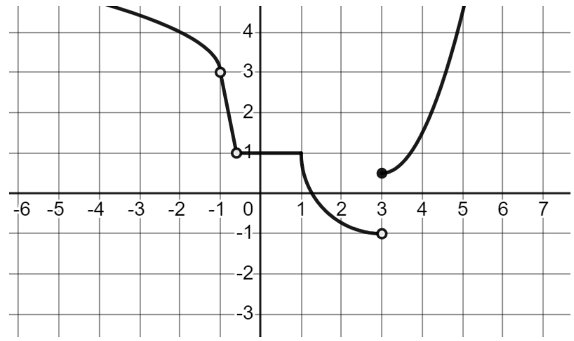
$$\lim_{x\to 2^-} f(x) = 0$$

Therefore, $\lim_{x\to 0} f\left(f(x)\right) = 0.$

***Example 6** – Limits with Composite Functions (Graphically)*

Two different graphs of the function f are shown. What is the difference in computing $\lim_{x\to -1} f\left(f(x)\right)$ for both functions?





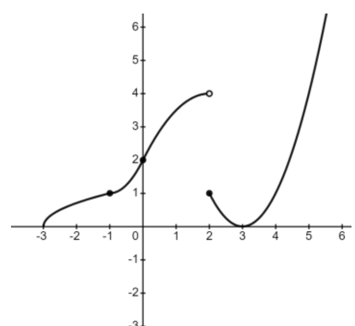
a. 3 b. 0.5 a. 3 b. 0.5

c. –1 d. DNE c. –1 d. DNE

***Example 7** – Limits with Composite Functions*

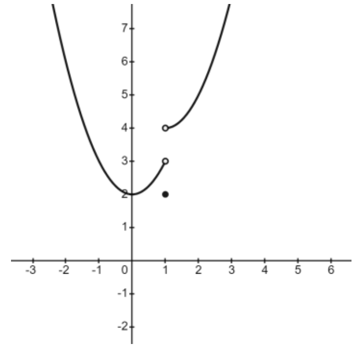
Evaluate each limit using the accompanying graphs.

a. $\lim_{x\to 0} f\left(f(x)\right)$



Graph of f

b. $\lim_{x\to 0} f(1-x^2)$

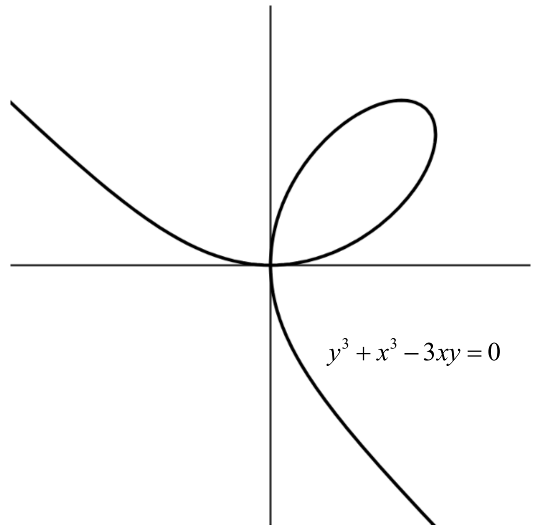


Graph of f

Chapter 1, Lesson 3

Implicit Differentiation

So far, we have only taken derivatives of *explicitly* defined functions, or, in other words, functions that are written in the form $y = f(x)$. But what happens when we try to differentiate a function that is not explicitly defined*?* These types of functions are said to be *implicitly* defined, where the relationship between the variables is tangled together in an equation that we can't (or simply don't want to) rewrite in the form $y = f(x)$. In such cases, we have to use *implicit differentiation.* In this lesson, we will learn about the process of implicit differentiation to take the derivatives of equations/expressions that involve the differentiation of entire functions.



Implicitly Defined Functions

An implicitly defined function is a function where we *can't* (or don't want to) solve explicitly for the dependent variable.

LEARNING GOALS

- Identify and describe the difference between an explicitly defined function and an implicitly defined function
- Find the derivative of a complicated function by using implicit differentiation
- Use implicit differentiation to determine the equation of a tangent line

Which of these can be written explicitly as a function of x and which can not? For those that can, solve explicitly for y, then differentiate the function.

a. $x^2 + y^3 = 8$

b. $xy = 25$

c. $3x^4 - xy + y^3 = 12$

d. $\cos y = 3x + 3y$

e. $\log_3 y = x^2 - 3x$

f. $xy - 3y = 12x$

Implicit Differentiation

While it might sound silly, implicit differentiation can be summed up by saying "the derivative of y is **the derivative of *y*.**" For instance, the derivative of $x + y = 0$ is $1 + \frac{dy}{dx} = 0$ since the derivative of x is 1 and the derivative of y is $\frac{dy}{dx}$. We can then apply chain rule, product rule, quotient rule etc to differentiate more sophisticated results.

Let's develop our understanding of this principle by taking a derivative using familiar methods, then by using implicit differentiation.

Consider the function $f(x) = xy^2$, where $y = \sin x$. Find $f'(x)$ two different ways, as outlined below, simultaneously, and make comparisons between the two methods.

Substituting $y = \sin x$ then using derivative rules	Using implicit differentiation (and not substituting $y = \sin x$).

Conclusion:

***Example 2** – Using Implicit Differentiation*

Find $\frac{dy}{dx}$ given $x^3 + y^3 = 18xy$

***Example 3** – Second Derivatives using Implicit Differentiation*

An object's motion can be described by $y^2 + 2y = 2x + 1$. Determine if the object is speeding up or slowing down when $x = 1$.

***Example 4** – Tangent Lines with Implicit Differentiation*

Find the equation of the tangent line to the curve $y^3 + x^3 - 3xy = 0$ at the point $P\left(\frac{3}{2}, \frac{3}{2}\right)$.

***Example 5** – Differentiating with x as a Function*

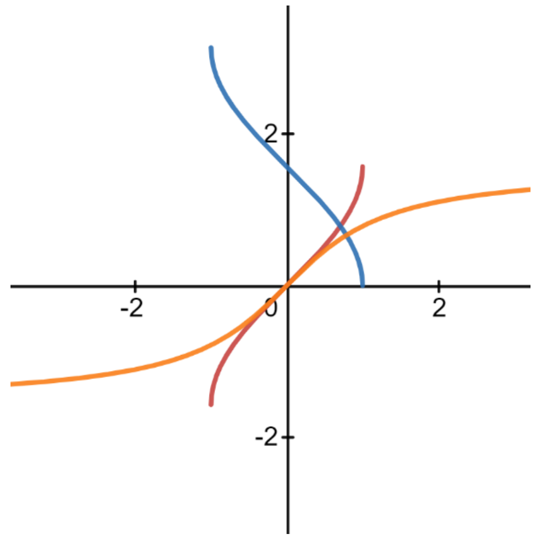
For the following function, first find $\frac{dy}{dx}$. Then, find $\frac{dx}{dy}$

$3x^2y - y^2 = x^3 + 3y$

Chapter 1, Lesson 4

Derivatives of Inverse Functions

In previous lessons, we have spent some time time taking the derivatives of the primary and secondary trigonometric functions. We have also worked with the *inverse* trigonometric functions $\arcsin x$, $\arccos x$, and $\arctan x$. Up until now, though, we have not taken the derivatives of these functions. That will now change! In this lesson, we will learn about the derivatives of the inverse trigonometric functions, and learn about how these derivatives are derived by using implicit differentiation (which we have also recently been introduced to). We will also broaden and generalize this understanding by learning a formula for taking the derivative of *any* inverse function.



Derivative of an Inverse Function

If some function $f(x)$ is both invertible and differentiable, then its inverse function is also differentiable at any point $\left(a, f^{-1}(a)\right)$ using the formula:

$$(f^{-1})'(a) = \frac{1}{f'\left(f^{-1}(a)\right)}$$

LEARNING GOALS

- Calculate the derivative of an inverse function at a specified point
- Determine the derivatives of the inverse trigonometric functions

Example 1 *– Differentiating an Inverse Function*

The function *h* is given by $h(x) = x^5 + 3x - 2$ and $h(1) = 2$. If h^{-1} is the inverse of *h,* what is the value of $\left(h^{-1}\right)'(2)$?

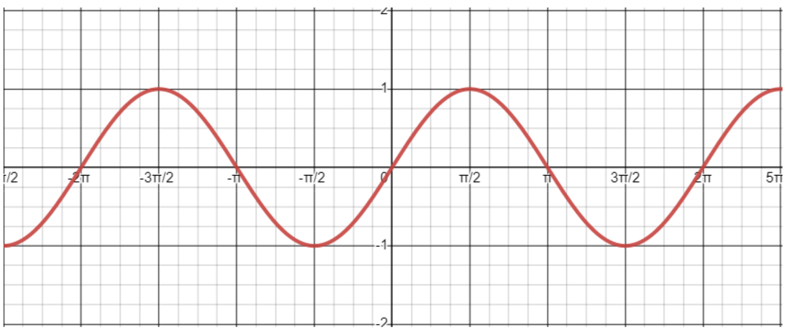
Example 2 *– Differentiating an Inverse*

Let *f* be a differentiable function such that $f(5) = 20, f(7) = 5, f'(5) = -4,\ f'(7) = -\frac{5}{2}$. The function *g* is differentiable and $g(x) = f^{-1}(x)$ for all *x*. What is the value of $g'(5)$?

Building the Inverse Trigonometric Functions

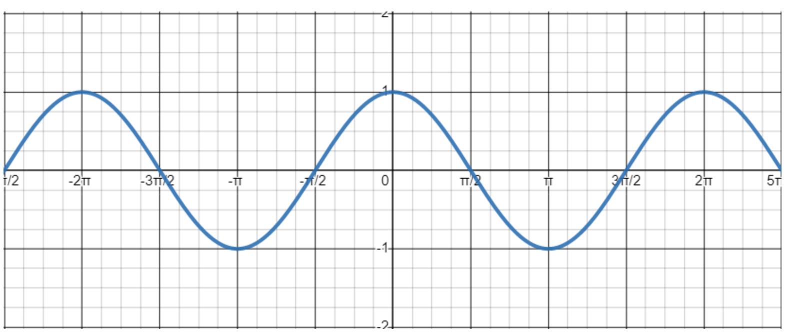
In *Precalculus II,* we learned how to construct the inverse trigonometric functions by taking invertible portions of the primary trigonometric functions. Let's review how this is done before we learn how to differentiate them.

Building $y = \arcsin(x)$



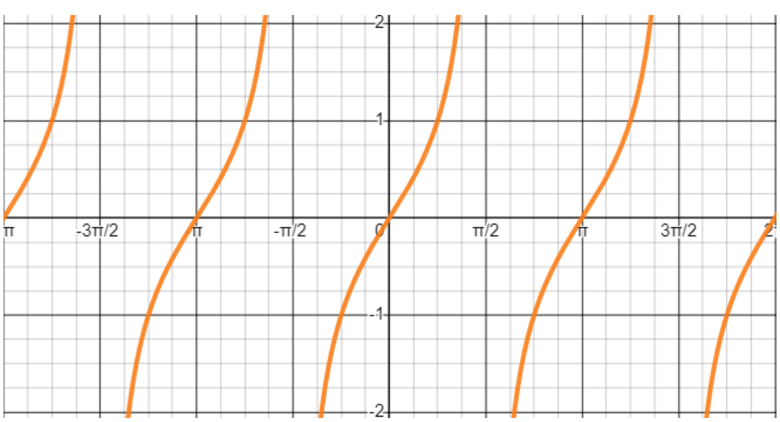
	$y = \sin x$	$y = \arcsin x$
Domain		
Range		

Building $y = \arccos(x)$



	$y = \cos x$	$y = \arccos x$
Domain		
Range		

Building $y = \arctan(x)$



	$y = \tan x$	$y = \arctan x$
Domain		
Range		

Derivatives of Inverse Trigonometric Functions

The derivatives of the inverse trigonometric functions can be derived using implicit differentiation. Let's see how this works by deriving the derivative formula for $y = \arcsin x$.

First, we start with with the inverse function $\qquad y = \arcsin x$

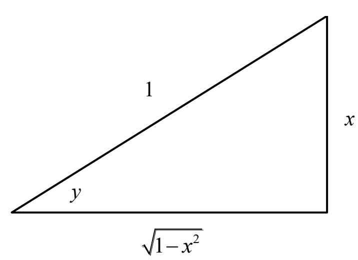
Now, invert this function by swapping the input and output to obtain a more familiar funciton $\qquad \sin y = x$

$$\frac{d}{dx}(\sin y) = \frac{d}{dx}(x)$$

Now, differentiate both sides implicitly $\qquad \cos y \cdot \frac{dy}{dx} = 1$

$$\frac{dy}{dx} = \frac{1}{\cos y}$$

This equation is still not solved for explicitly. Therefore, let's go back to our original ratio that $\sin y = x$. While it's a bit unintuitive, this means that from some angle y in a right triangle, the opposite side length has length x and the hypotenuse has length 1. Using the Pythagorean Theorem, we can then determine that the adjacent side has a length of $\sqrt{1-x^2}$.



From this triangle, we can see that $\cos y = \sqrt{1-x^2}$

Therefore, We can conclude that $\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$

***Example 3** – Deriving the Derivatives with Implicit Differentiation*

Derive the derivatives of the remaining two inverse trigonometric functions

a. $y = \arccos x$

b. $y = \arctan x$

Summary of Inverse Trigonometric Derivatives

$\frac{d}{dx}\arcsin x = \boxed{}$

$\frac{d}{dx}\arccos x = \boxed{}$

$\frac{d}{dx}\arctan x = \boxed{}$

Find each derivative

a. $y = \arcsin(x^4)$

b. $y = \arccos(10x^2)$

c. $y = \arcsin\left(\frac{1}{x}\right)$

d. $y = \arctan(8x^3)$

Example 5 – *Revisiting L'Hôpital's Rule*

Determine if the following limit produces the *indeterminate forms* $\frac{0}{0}$ or $\frac{\infty}{\infty}$ and if so, apply L'Hôpital's Rule to evaluate the limit.

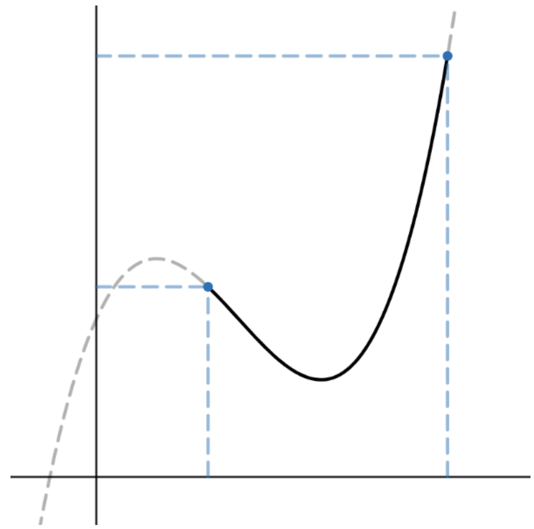
a. $\lim_{x \to 0} \frac{\ln(1+x^2)}{\arctan(x^2)}$

b. $\lim_{x \to 1} \frac{\arcsin x - \arccos x}{\ln x}$

CHAPTER 1, LESSON 5

THE INTERMEDIATE VALUE THEOREM

In our study of calculus, we have previously learned about the concept of *continuity at a point*. Building on this knowledge, we can now begin to study one of the most important consequences of function continuity. Many times in real-life applications, we don't necessarily need to always know *when* a certain result will occur, but only *that* it occurs. This is the main principle behind the Intermediate Value Theorem, which claims that a continuous function that travels from one endpoint to another must hit all possible values between those endpoints. In this lesson, we will learn about the Intermediate Value Theorem and use it to prove a variety of results in real-life and theoretical contexts.



Intermediate Value Theorem

Let f be continuous over a closed interval [*a, b*]. If $f(a) \leq f(c) \leq f(b)$, then there is a number c such that $c \in [a,b]$. In other words, if we have a *continuous* portion of a function that achieves 2 output values, then the function *must* achieve all real output values in between those two values.

LEARNING GOALS

- Describe the Intermediate Value Theorem and use it to prove a variety of results in real-life and theoretical contexts

***Example 1** – The IVT – Basic Application*

Use the Intermediate Value Theorem to show that the function $f(x) = x^3$ achieves an output value of 10.

***Example 2** – Existence of Roots*

Use the Intermediate Value Theorem to show that the function $f(x) = x^3 + x - 1$ has at least one real root somewhere between 0 and 1.

***Example 3** – Applicable Context*

Suppose a thermometer shows 12°C at 8 AM and 23°C at noon. Assuming the temperature changes continuously, use the IVT to explain why there must have been a time between 8 AM and noon when the temperature was exactly 20°C.

***Example 4** – The IVT and Abstract Thinking*

Consider the function $f(x) = -30\cos\left(\frac{\pi x}{7}\right) + 50$. Prove that there is at least one value $c \in (20, 40)$ such that $f(c) = 70$.

Example 5 *– The IVT and Abstract Thinking*

Two friends, Daniel and Jason, are riding a luge down a steep hill for 10 minutes. Daniel's velocity, in miles per hour, at time t hours is given by a differentiable function D for $0 \le t \le 10$. Values for $D(t)$ at select values of t are given in the table below. Jason's velocity, in miles per hour, at time t hours is given by the piecewise function J defined below.

t	0	2	5	8	10
$D(t)$	1	8	1.5	5	10

$$J(t) = \begin{cases} 5e^{5-t} & 0 \le t \le 5 \\ 12 - 3t - t^2 & 5 < t \le 10 \end{cases}$$

Is there a time in the first 5 minutes when Daniel's velocity is equal to Jason's velocity?

Example 6 *– Applying the Intermediate Value Theorem*

Describe why each of the following situations is an *incorrect* interpretation of the Intermediate Value Theorem.

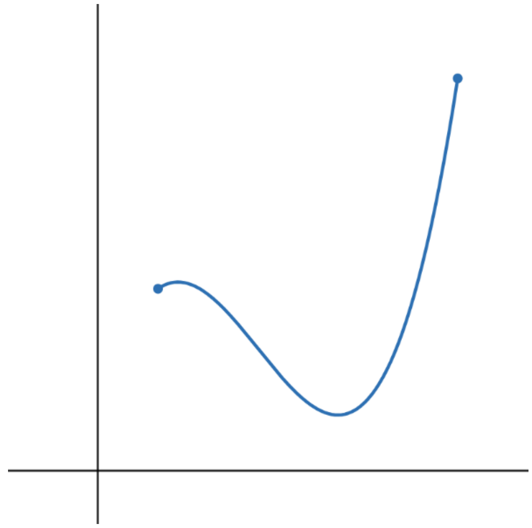
a. For the function $f(x) = \frac{1}{x}$, we know that $f(-1) < 0$ and $f(1) > 0$. Therefore, $f(x)$ has a root on the interval [–1, 1] by the IVT.

b. For some continuous function $f(x)$, whose domain is all real numbers, it is known that $f(1) = 3$ and $f(10) = 17$. Therefore, by the IVT, this function achieves no real roots on $x \in (1,10)$

Chapter 1, Lesson 6

The Extreme Value Theorem

Understanding the behavior of functions at their highest and lowest points (referred to as extreme values) is a fundamental concept in calculus. By now, we should understand the similarities and differences between the concepts of a *local* extreme value and an *absolute/global* extreme value. While we have practiced finding local extreme values using 1st derivatives, we have yet to precisely identify absolute maxima and minima. Building on our prior knowledge of continuity and derivatives, we will now explore how to identify extreme values and apply the *Extreme Value Theorem*. In this lesson, we will learn about the circumstances under which we are *guaranteed* to find an extreme value and how to find it.



Extreme Value Theorem

If a function contains no points of discontinuity on a closed interval [*a*, *b*], then the function *must* have an absolute maximum and an absolute minimum on that interval

LEARNING GOALS

- Define absolute extrema
- Define local extrema
- Explain how to find the critical points of a function over a closed interval
- Explore the Extreme Value Theorem (EVT) and use it to find the absolute maximum / absolute minimum value of a function on a closed interval

Finding Absolute Extrema

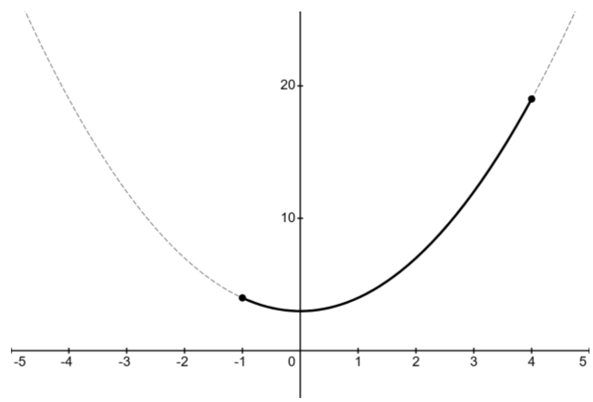
If a function satisfies the Extreme Value Theorem (i.e. if it contains no points of discontinuity on an interval), then the absolute extrema can be found by doing the following:

- Evaluate the function at the given endpoints (i.e. the start and end of the interval)
- Evaluate the function at all *critical numbers*
- Compare the values. The highest value is the *absolute maximum* and the lowest value is the *absolute minimum*

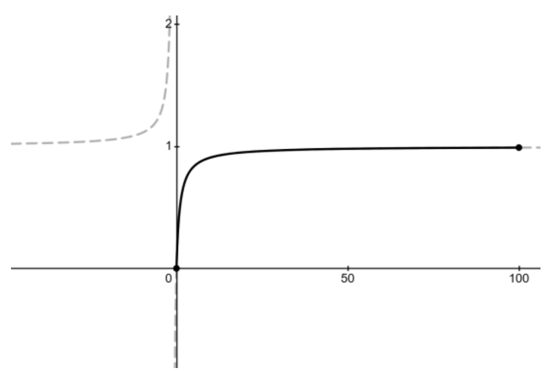
Example 1 *– Finding Absolute Extrema*

For each of the following, find the Absolute Extrema.

a. $f(x) = x^2 + 3, \quad x \in [-1, 4]$



b. $y = \dfrac{x}{1+x}, \quad x \in [0, 100]$



***Example 2** – The Extreme Value Theorem*

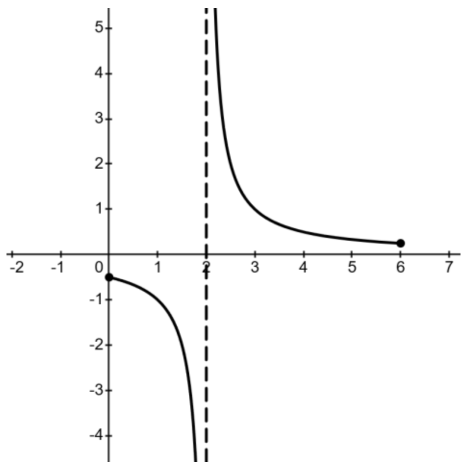
Consider the function $f(x)=\sqrt{4-x^2}$. State the interval over which the EVT applies, then find the absolute maximum and absolute minimum values of *f* over that interval.

***Example 3** – Logical Reasoning with the EVT*

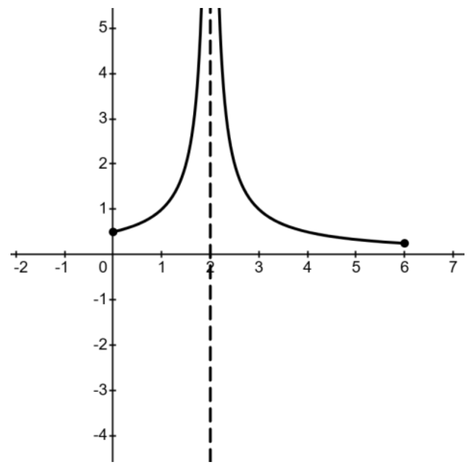
Suppose $h(t)$ represents the height above the loading point of a roller coaster car (in meters) at time *t* (in seconds) during a ride. The model is valid for $0 \le t \le 120$, and $h(t)$ is continuous. Explain, using the Extreme Value Theorem, why there must be both a highest point and a lowest point during the ride. Then, discuss why you do **not** need to find the actual values to make this conclusion.

Example 4 – *Graphical Representations*

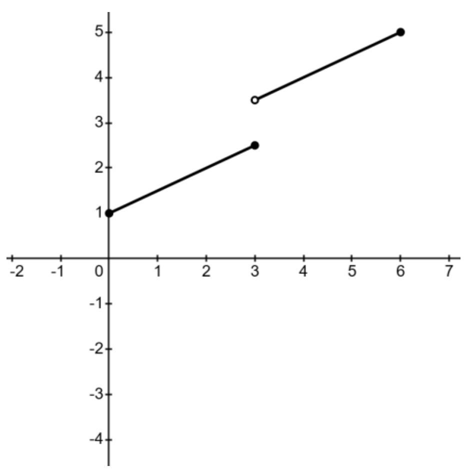
Consider each of these functions on the closed interval [0, 6]. Determine if they have an Absolute Maximum and/or Absolute Minimum



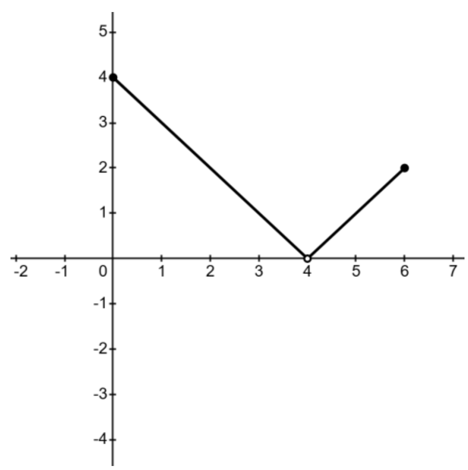
- Absolute Maximum
- Absolute Minimum



- Absolute Maximum
- Absolute Minimum



- Absolute Maximum
- Absolute Minimum



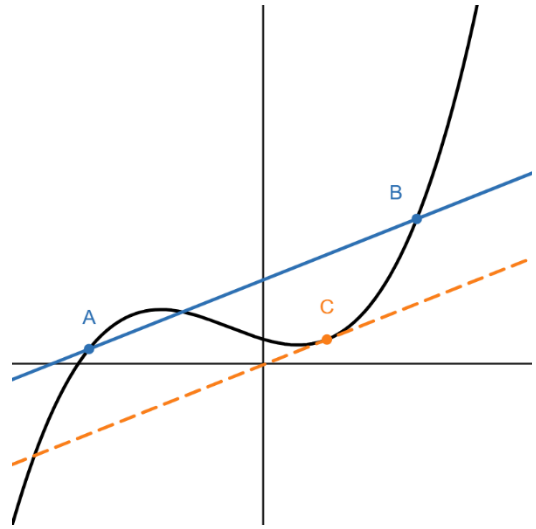
- Absolute Maximum
- Absolute Minimum

Which of the above functions does the EVT *not* guarantee an absolute maximum / minimum? Explain why.

Chapter 1, Lesson 7

The Mean Value Theorem

Over the past couple of years, we have been continually building on our understanding of rates of change. Today, we will learn about the *Mean Value Theorem,* which is a concept that proposes a powerful connection between the average rate of change of a function over an interval and the instantaneous rate of change at a specific point within that interval. Building on our understanding of derivatives as rates of change and continuity from previous lessons, this theorem establishes a relationship between the global behavior of a function and its local behavior. In this lesson, we will explore the formal statement of the Mean Value Theorem and how to apply it to solve problems in a variety of contexts



Mean Value Theorem

If $y = f(x)$ is continuous at every point of the closed interval [*a, b*] and differentiable at every point of its interior (*a, b*), then there is at least one point $c \in (a,b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

LEARNING GOALS

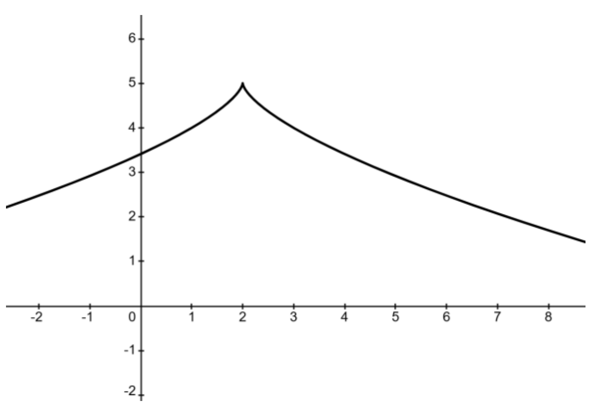
- Describe the significance of the Mean Value Theorem (MVT)
- Explain the meaning of Rolle's Theorem
- Apply the Mean Value Theorem to find when the value of an instantaneous rate was equal to the value of the average rate of change over an interval
- State the importance of the Mean Value Theorem in real-life applications

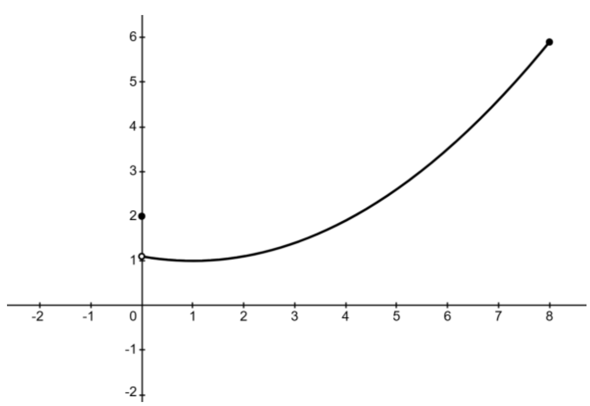
***Example 1** – Conceptualizing the Mean Value Theorem*

Vicki lives 150 miles from the beach. She leaves her home at noon and arrives at the beach at 3:00 p.m. What is the average rate of change of her distance travelled in miles with respect to time in hours? What does the MVT tell us about this trip?

***Example 2** – Applying the Mean Value Theorem*

Which of the two functions below can the Mean Value Theorem be applied to on the interval [0, 8]? For either that does, find the value of *c* which satisfies the Mean Value Theorem. For either that does not, explain why.





Example 3 *– Can MVT Be Applied?*

For each of the following functions described below, determine whether the Mean Value Theorem can be applied on the interval $[-3,3]$. If it can be applied, explain how you know. If it cannot be applied, explain why not.

a. $f(x)$ is continuous for all real numbers

b. $f(x)$ is differentiable on $(-3,3)$.

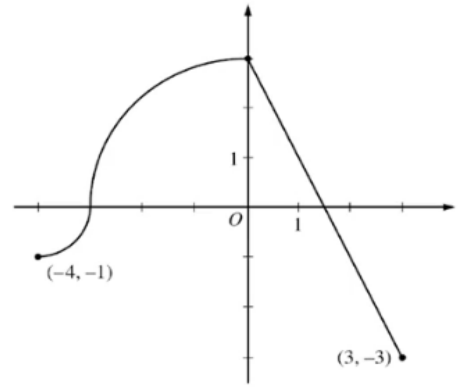
c. $f(x) = |x| + 4$

d. $f(x) = \sin\left(\frac{\pi x}{3}\right)$

e. $f(x) = \tan\left(\frac{\pi x}{3}\right)$

***Example 4** – Understanding the Theorem*

The continuous function f is defined on the interval $-4 \le x \le 3$. The graph of f is shown to the right.



Graph of f

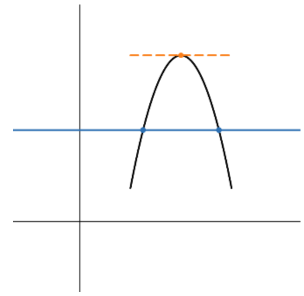
Find the average rate of change of f on the interval $-4 \le x \le 3$. It is known that there is *NO* point $c \in (-4,3)$ for which $f'(c) = \dfrac{f(3)-f(-4)}{3+4}$.

Explain why this statement does *NOT* contradict the Mean Value Theorem

***Example 5** – Applying the MVT*

For $f(x) = x^3 - 3x$ on the interval [−2, 2], verify that the function satisfies the conditions of the Mean Value Theorem. Then, find the point(s) c in the interval that satisfies the conclusion of the Mean Value Theorem.

Rolle's Theorem



If $y = f(x)$ is continuous at every point of the closed interval [*a, b*] and differentiable at every point of its interior (*a, b*), and if $f(a) = f(b)$, then there is at least one point $c \in (a,b)$ such that $f'(c) = 0$

***Example 6** – Rolle's Theorem*

Let $f(x) = x^3 - 3x^2 + 2x + 1$ on $[0,2]$. Use Rolle's Theorem to confirm that $f(x)$ has at least one critical point in [0, 2]

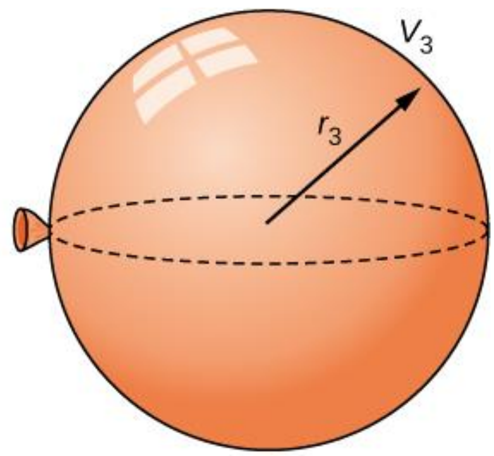
***Example 7** – Applying the MVT (Rolle's Theorem)*

Show that the function $f(x) = x^3 + e^x$ has <u>*exactly*</u> one real root

Chapter 1, Lesson 8

Related Rates Problems

We have already learned that derivatives give us an indication of the *rate of change* of a quantity with respect to another quantity. Many real-world problems involve quantities that change over time and are interconnected. In other words, these rates of change are *related* by their relationship to each other and to time. *Related Rates* is a technique that helps us analyze how these quantities change with respect to one another by using derivatives. This method relies on understanding implicit differentiation, a concept we have previously studied, and applying it to dynamic situations. In this lesson, we will explore how to solve related rates problems by identifying relationships between variables, differentiating implicitly with respect to time, and interpreting the results.



Steps For Related Rates

1. Draw a diagram of the situation and clearly define your variables
2. Write down your *known* and *needed* rates (with respect to time), as well as your "**when**"
3. Write an appropriate equation/equations that relate your given/needed variables together
4. Differentiate implicitly with respect to time
5. Substitute your given rate(s) and your "**when**" and solve for your needed rate.
6. *Directly* answer the question (pay close attention to what it is asking)

LEARNING GOALS

- Express changing quantities in terms of derivatives
- Find relationships among the derivatives in a given problem
- Use the chain rule and implicit differentiation to find the rate of change of one quantity that depends on the rate of change of other quantities
- Identify and describe situations where the rate of change of a quantity is not uniform

***Example 1** – Introduction to Related Rates (Implicit Differentiation Review)*

Differentiate each of the following functions implicitly with respect to *time.*

a. $A = \pi r^2$

b. $V = \pi r^2 h$

***Example 2** – Related Rates (Perimeter / Area)*

The length of a rectangle is increasing by 3 centimeters per second and the width is decreasing at 5 centimeters per second. How fast is the perimeter of the rectangle changing at the instant that the length is 10cm and the width is 8cm?

***Example 3** – Related Rates (Non-Constant Dimension)*

A spherical balloon is being filled with air at a constant rate of 2 cm^3/sec. How fast is the radius increasing when the radius is 3cm?

***Example 4** – Related Rates (Constant Dimension)*

A large cylindrical water tank, whose diameter is 4 metres, is being filled with water at a constant rate of 3 cubic metres per second. Find the rate at which the water level is rising when the water is 5 metres deep.

Example 5 *– Similar Triangles – Cross Section*

A water tank has the shape of an inverted cone with a base radius of 5 metres and a height of 10 metres. Water is leaking out of the tank at a rate of 2 cubic metres per minute. Find the rate at which the water level is dropping when the water is 6 metres deep.

Example 6 *– Similar Triangles (Movement)*

A 6-ft tall person walks away from a 10-ft lamppost at a constant rate of 3 ft/sec. What is the rate that the tip of the shadow moves away from the pole when the person if 10 ft away from the pole?