Introduction to representation theory

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"Average" on cube problem

Example

Pick a die. Each step we replace the number on each side with average of neighbor sides. What would be the numbers written after long time?

What is a representation?

Outline

- Main definitions and some theorems
- Characters
- Frobenius duality
- Solution

Formal definition

Let V be a vector space over some field k. Let $GL(V)$ be the group of invertible homomorphisms $V \to V$.
Let G be a finite group.

Definition

A linear representation of G in V is a pair (V, ρ) where $\rho : G \to GL(V)$ is a group homomorphism

For convenience we will sometime denote the pair (V, ρ) with ρ_V.

Example

Let $\rho_V : V \to GL(V) : v \mapsto 1$. Clearly that is a representation. We will call it trivial.

Definition

Homomorphism of representations (V, ρ) and (W, τ) is a vector space homomoprhism $V \to W$ making the following diagram commute:

$$\begin{tikzcd}
V \arrow[r, "\rho(g)"] \arrow[d, "T"'] & V \arrow[d, "T"] \\
W \arrow[r, "\tau(g)"'] & W
\end{tikzcd}$$

If T is isomorphism of vector spaces, then induced (by the property above) representation homorphism is *representation isomophism*.

Subrepresentations

Definition

Let V be a representation. We call W a subrepresentation of representation V if it's a subspace $W \subset V$ as a vector space and invariant under G. That is, for all $g \in G$ and for all $w \in W$ we have $g.w \in W$.

Lemma

Let W, V be two representation and let f be a representation homomoprhism W to V. Then $\mathrm{Ker\ f} \subset \mathrm{V}$ and $\mathrm{Im\ f} \subset \mathrm{W}$ are subrepresentations.

Reducability

Definition

The representation of G is called irreducible if the space V is not empty and no nontrivial subspace is invariant under G. Or equivalently irreducible representation has no nontrivial subrepresentations.

Lemma

If $k = \mathbb{C}$ any representation of a finite group is a direct sum of irreducible representations.

Schur's lemma

Lemma (Schur's lemma)

Let (V, ρ_v) and (W, ρ_w) be irreducible representations of the same group G.

1. *If $T : (V, \rho_v) \to (W, \rho_w)$ is a representation homomoprhism, then $T = 0$ or T is a representation isomophism.*
2. *If G is finite and $T : (V, \rho_v) \to (V, \rho_v)$ is and endomorphism (homomorphism of representation to itself) then $T = \lambda I$ for some field element λ.*

Induced representation

Let G be a group, that acts on finite set X. Let $L(X)$ be a space of functions on X. Then we have representation T on $L(X)$ given as:

$$[T(g)f](x) = f(xg^{-1})$$

Now fix a point x_0 and pick a stationary subgroup H of that point. Then

$$F(g) = f(x_0 g^{-1})$$

will be a subrepresentation corresponding to a subspace invariant under H.

Induced representation

Now let's generalize this observation. Let U be a representation of some subgroup H in finite-dimensional V. Consider a space of functions F, s.t. $F \ni f : G \to V$, s.t.

$$F(hg) = U(h) \cdot F(g)$$

where $h \in H, g \in G$.

Definition

F defined as above called induced representation of G by subgroup H.

Characters

Let T be representation of group G, then the character χ:

Definition

$$\chi_T(g) = \mathrm{Tr}\ \mathrm{T(g)}$$

Theorem

Irreducible of G in finite dimensional space if fully defined by character function.

What is $\chi(e)$ in terms of the representation space?

Characters

To avoid using measure theory, we will define

Definition

$$(f, y) := \frac{1}{|G|} \sum_{g \in G} f(g)\overline{y(g)}$$

Theorem

Characters of irreducible representations are orthogonal with respect to scalar product above.

Frobenius duality

Let $H < G$ be groups, and let χ_H be a character defining some representation of H, χ_G a character of a representation of G in the same space, then:

Theorem

$$(Ind_G \chi_H, \chi_G) = (\chi_H, \chi_G \mid_H)$$

"Average" on cube problem

Example

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Cube transformation group

Lemma

Group G of orientation-preserving symmetries of cube is isomorphic to S_4.

Now let's fix one face on the cube. The subgroup $Z/4Z$ of S_4 that preserve particular side.
Now define V_r to be the space of all functions defined on the faces of our cube, which is the $6-$ dimensional real space. Now pick the trivial representation of $Z/4Z$ in V_r and induce a representation T on G.

Solution via representations

Consider an operator

$$[Lf](x) = \frac{1}{4} \sum_{\substack{y \in CF \\ (x,y)}} f(y),$$

where $x \in CF$ is particular face in the set CF of cube faces, f is some function on the set of spaces and L is an operator on the space of functions.

Lemma

Action of L and action of G commute.

Let's pay closer attention to the representation T. It's induced from representation of $Z/4Z$, thus decomposes into 3 irreducibles: Let's denote them as $T = T_1 \oplus T_2 \oplus T_3$.

For each subrepresentation there should be a corresponding invariant subspace.

We can pick invariant subspaces of V_r in the following way:

V_1: constant functions,

V_2: even functions which sum (over cube) to 0

V_3: odd functions.

On constant function space one can see that eigenvalue is $\lambda_1 = 1$.
For an operator from V_2 we can choose a function that is 1 on the first pair of cube's faces, 0 on the other, and -1 on the last one.
Thus $\lambda_2 = \frac{-1}{2}$.
For a representative of function V_3: fix a face and assign 1 to it, -1 to opposite face, and 0 to all other faces.
Thus $\lambda_3 = 0$.
Clearly in the limit we fall into V_1.
Now since projection of initial function $f = (1, 2, 3, 4, 5, 6)$ on V_1 is constant function $\frac{21}{6}$, those are desired numbers.