Introduction to representation theory

M. Tikhonov

University of Virginia

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Outline

- Motivation
- Main definitions and theorems
- Characters
- Examples

What is representation?

Formal definition

Let V be a vector space over some field k. Denote invertible homomoprhism of V with $GL(V)$.
Let G be a finite group.

Definition

A linear representation of G in V is a pair (V, ρ) where $\rho : G \to GL(V)$ is a group homomoprhism

For convenience we will sometime denote the pair (V, ρ) with ρ_V.

Example

Let $\rho_V : V \to GL(V) : v \mapsto 1$. Clearly that is a representation. We will call it trivial.

Definition

Homomorphism of representations V, ρ and W, τ is a vector space homomoprhism $V \to W$ making following diagram commute:

$$\begin{tikzcd} V \arrow[r, "\rho(g)"] \arrow[d, "T"'] & V \arrow[d, "T"] \\ W \arrow[r, "\tau(g)"'] & W \end{tikzcd}$$

If T is isomorphism of vector spaces it is also isomorphism of representations.

Subrepresentations

Definition

Let V be a representation. We call W a subrepresentation of representation V if it's a subspace $W \subset V$ as a vector space and invariant under G. That is, for all $g \in G$ and for all $w \in W$ $g.w \in W$.

Lemma

Let W, V be two representation and consider f being representation homomoprhism. Then $\mathrm{Ker\ f \subset V}$ *and* $\mathrm{Im\ f \subset W}$ *are subrepresentations.*

Induced representation

Let G be a group, X be a G-homogenous space. Let $L(X)$ be a space of functions on X. Then we have representation T on $L(X)$ given as:

$$[T(g)f](x) = f(xg)$$

Now fix a point and pick a stationary subgroup H of that point. Then

$$F(g) = f(x_0 g)$$

will be a subrepresentation corresponding to a subspace invariant under H.

Now let's generalize this observation. Let U be a representation of some subgroup H in finite-dimensional V. Consider a space of functions F, s.t. $F \ni f : G \to V$, s.t.

$$F(hg) = U(h) \cdot F(g)$$

Reducability

Definition

The representation of G is called irreducible if the space V is not empty and no nontrivial subspace is invariant under G. Or equivalently irreducible representation has no nontrivial subrepresentations.

Lemma

Any reducible representation is a direct sum of irreducible representation.

Schur's lemma

Lemma (Schur's lemma)

Let (V, ρ_v) and (W, ρ_w) be irreducible representations of the same group G.

1. *If $T : (V, \rho_v) \to (W, \rho_w)$ is representation homomoprhism, then $T = 0$ or T is representation isomophism.*
2. *If G is finite and $T : (V, \rho_v) \to (V, \rho_v)$ is endomorphism (homorphism of presentation to itself) then $T = \lambda I$ for some colmplex number λ.*

"Average" on cube problem

Example

Pick a die. Each step we replace the number on each side with average of neighbor sides. What would be the numbers written after long time?

Cube transformation group

Lemma

Group of symmetries that preserve cube is isomorphic to S_4.

Now let's fix one face on the cube. Consider subgroup of S_4 that also preserve that side is Z/4Z.
Now define V_r to be a space of all functions defined on our cube, which is $6-$ dimensional real space. Now pick trivial representation of $Z/4Z$ in V_r and induce a representation T on G.

Solution via representations

Consider an operator

$$[Lf](x) = \frac{1}{4} \sum_{y \in CF} f(y)$$

, where $x \in CF$ is particular face in the set CF of cube faces, f is some functions on the set of spaces and L is an operator on the space of functions.

Lemma

Action of L and action of G commute.

Let's pay closer attention to representation T. Linking index $c(T, T)$ is 3, meaning that T is sum of 3 non-trivial representations.
Let's denote them as $T = T_1 \oplus T_2 \oplus T_3$.

To each subrepresentation there should be a corresponding invariant subspace.

We can pick invariant subspaces of V_r in the following way:

V_1: constant functions,

V_2: even functions which sums (over cube) to 0

V_3: odd functions.

On constant function space one can see that eigenvalue is $\lambda_1 = 1$.
For an operator from V_2 we can choose a function that is 1 on the first pair of cube's faces, 0 on the other, and -1 on the last one.
Thus $\lambda_2 = \frac{-1}{2}$.
For a representative of function V_3: fix a face and assign 1 to it, -1 to opposite face, and 0 to all other faces.
Thus $\lambda_3 = 0$.
Clearly in the limit we fall into V_1.
Now since projection of initial function $f = (1, 2, 3, 4, 5, 6)$ on V_1 is constant function $\frac{21}{6}$, those are desired numbers.