Introduction to representation theory

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"Average" on cube problem

Example

Pick a die. Each step we replace the number on each side with average of neighbor sides. What would be the numbers written after long time?

What is a representation?

Outline

- Main definitions and some theorems
- Characters
- Frobenius duality
- Solution

Formal definition

Let V be a vector space over some field k. Let GL(V) be the group of invertible homomorphisms $V \to V$. Let G be a finite group.

Definition

A linear representation of G in V is a pair (V, ρ) where $\rho: G \to GL(V)$ is a group homomorphism

For convenience we will sometime denote the pair (V, ρ) with ρ_V .

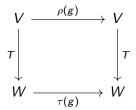
Example

Let $\rho_V: V \to GL(V): v \mapsto 1$. Clearly that is a representation. We will call it trivial.



Definition

Homomorphism of representations (V, ρ) and (W, τ) is a vector space homomorphism $V \to W$ making the following diagram commute:



If T is isomorphism of vector spaces, then induced (by the property above) representation homorphism is *representation isomorphism*.

Subrepresentations

Definition

Let V be a representation. We call W a subrepresentation of representation V if it's a subspace $W \subset V$ as a vector space and invariant under G. That is, for all $g \in G$ and for all $w \in W$ we have $g.w \in W$.

Lemma

Let W,V be two representation and let f be a representation homomorphism W to V. Then $\operatorname{Ker} f \subset V$ and $\operatorname{Im} f \subset W$ are subrepresentations.

Reducability

Definition

The representation of G is called irreducible if the space V is not empty and no nontrivial subspace is invariant under G. Or equivalently irreducible representation has no nontrivial subrepresentations.

Lemma

If $k = \mathbb{C}$ any representation of a finite group is a direct sum of irreducible representations.

Schur's lemma

Lemma (Schur's lemma)

Let (V, ρ_v) and (W, ρ_w) be irreducible representations of the same group G.

- If $T: (V, \rho_v) \to (W, \rho_w)$ is a representation homomorphism, then T = 0 or T is a representation isomorphism.
- 2 If G is finite and $T:(V,\rho_v)\to (V,\rho_v)$ is and endomorphism (homomorphism of representation to itself) then $T=\lambda I$ for some field element λ .

Induced representation

Let G be a group, that acts on finite set X. Let L(X) be a space of functions on X. Then we have representation T on L(X) given as:

$$[T(g)f](x) = f(xg^{-1})$$

Now fix a point x_0 and pick a stationary subgroup H of that point. Then

$$F(g) = f(x_0 g^{-1})$$

will be a subrepresentation corresponding to a subspace invariant under H



Induced representation

Now let's generalize this observation. Let U be a representation of some subgroup H in finite-dimensional V. Consider a space of functions F, s.t. $F \ni f : G \to V$, s.t.

$$F(hg) = U(h) \cdot F(g)$$

where $h \in H, g \in G$.

Definition

F defined as above called induced representation of G by subgroup H.



Characters

Let T be representation of group G, then the character χ :

Definition

$$\chi_{\mathcal{T}}(g) = \operatorname{Tr} \, \mathrm{T}(g)$$

Theorem

Irreducible of G in finite dimensional space if fully defined by character function.

What is $\chi(e)$ in terms of the representation space?



Characters

To avoid using measure theory, we will define

Definition

$$(f,y) := \frac{1}{|G|} \sum_{g \in G} f(g) \overline{y(g)}$$

Theorem

Characters of irreducible representations are orthogonal with respect to scalar product above.

Frobenius duality

Let H < G be groups, and let χ_H be a character defining some representation of H, χ_G a character of a representation of G in the same space, then:

Theorem

$$(Ind_G\chi_H,\chi_G)=(\chi_H,\chi_G\mid_H)$$

"Average" on cube problem

Example

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Cube transformation group

Lemma

Group G of orientation-preserving symmetries of cube is isomorphic to S_4 .

Now let's fix one face on the cube. The subgroup Z/4Z of S_4 that preserve particular side.

Now define V_r to be the space of all functions defined on the faces of our cube, which is the 6- dimensional real space. Now pick the trivial representation of Z/4Z in V_r and induce a representation T on G.

Solution via representations

Consider an operator

$$[Lf](x) = \frac{1}{4} \sum_{\substack{y \in CF \\ (x,y)}} f(y),$$

where $x \in CF$ is particular face in the set CF of cube faces, f is some function on the set of spaces and L is an operator on the space of functions.

Lemma

Action of L and action of G commute.

Let's pay closer attention to the representation \mathcal{T} . It's induced from representation of $\mathbb{Z}/4\mathbb{Z}$, thus decomposes into 3 irreducibles:

Let's denote them as $T = T_1 \oplus T_2 \oplus T_3$.



For each subrepresentation there should be a corresponding invariant subspace.

We can pick invariant subspaces of V_r in the following way:

 V_1 : constant functions,

 V_2 : even functions which sum (over cube) to 0

 V_3 : odd functions.

On constant function space one can see that eigenvalue is $\lambda_1=1$. For an operator from V_2 we can choose a function that is 1 on the first pair of cube's faces, 0 on the other, and -1 on the last one.

Thus $\lambda_2 = \frac{-1}{2}$.

For a representative of function V_3 : fix a face and assign 1 to it, -1 to opposite face, and 0 to all other faces.

Thus $\lambda_3 = 0$.

Clearly in the limit we fall into V_1 .

Now since projection of initial function f = (1, 2, 3, 4, 5, 6) on V_1 is constant function $\frac{21}{6}$, those are desired numbers.