## **Convexity Theorem and Integrable Systems**

Mikhail Tikhonov

November 30, 2020

University of Virginia

Introduction and required definitions

## **Exponential map**

Let G be a Lie group. By a one-parameter subgroup of G we mean homorphism  $\mathbb{R} \to G$ .

$$\mathsf{hom}_{\mathsf{Lie}}(\mathbb{R}, G) \ni \theta \mapsto \left. \frac{d}{dt} \right|_{0} \theta(t) \in T_{e}G$$

We define the Lie group exponential map to be:

$$T_eG \ni X \mapsto \exp X = \theta(1) \in G$$
,

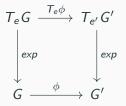
where  $\theta$  is the one-parameter subgroup of G corresponding to X.

1

## **Exponential map**

#### Proposition

The exponential map is smooth and natural, i.e.



where  $\phi \in \text{hom}(G, G')$ .

Moreover if  $X \in T_eG$  then  $\exp(t+s)X = (\exp tX) \cdot (\exp sX)$  for all  $t, s \in \mathbb{R}$ , where  $\cdot$  stands for Lie group product.

2

## Riemannian analogue of exponential map

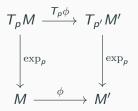
Let (M, g) be a riemannian manifold and let  $p \in M$  be a point of M. The riemannian exponential map starting at p is defined by:

$$T_pM \ni X \mapsto \exp_p X = \gamma_X(1) \in M$$

where  $\gamma_X$  is the unique geodesic starting at p with tangent vector X

### **Proposition**

The exponential map is smooth and natural, i.e.



#### **Normal coordinates**

#### **Proposition**

There exist U and V open neighborhoods respectively of  $o \in T_pM$  and  $p \in M$ , s.t.  $\exp_p : U \to V$  is a diffeomorphism. Futhermore if we fix an orthonormal basis of  $T_pM$  we obtain an isomorphism F with  $\mathbb{R}^n$ , combined with  $\exp^{-1}$  gives normal cooridnate chart.

### **Proposition**

Let G be a Lie group equipped with bi-invariant riemannian metric. Then the Lie group exponential map is precisely the reimannian exponential map starting at the identity.

## Morse theory

Let  $f:M\to\mathbb{R}$  be a Morse-Bott function on a compact riemannian manifold whose critical submanifolds have all index and coindex different from one. The the level sets of f are connected.

# Convexity theorem

## Convexity theorem

Let (M, w) be a compact connected symplectic manifold, and let  $\mathbb{T}^m$  be an m-torus. Suppose that  $\psi: \mathbb{T}^m \to \operatorname{Sympl}(M, w)$  is a hamiltonian action with moment map  $\mu: M \to \mathbb{R}^m$ . Then:

- 1. the levels of  $\mu$  are connected
- 2. the image of  $\mu$  is convex
- 3. the image of  $\mu$  is convex hull of the images of the fixed points of the action