# Convexity Theorem and Integrable Systems

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# Introduction and required definitions

# Exponential map

Let $G$ be a Lie group. By a one-parameter subgroup of $G$ we mean homorphism $\mathbb{R} \to G$.

$$\mathrm{hom}_{\mathrm{Lie}}(\mathbb{R}, G) \ni \theta \mapsto \left.\frac{d}{dt}\right|_0 \theta(t) \in T_e G$$

We define the Lie group exponential map to be:

$$T_e G \ni X \mapsto \exp X = \theta(1) \in G,$$

where $\theta$ is the one-parameter subgroup of $G$ corresponding to $X$.

# Exponential map

**Proposition**
*The exponential map is smooth and natural, i.e.*

$$\begin{tikzcd} T_e G \arrow[r, "T_e\phi"] \arrow[d, "exp"] & T_{e'} G' \arrow[d, "exp"] \\ G \arrow[r, "\phi"] & G' \end{tikzcd}$$

*where* $\phi \in \mathrm{hom}(G, G')$*.*

*Moreover if* $X \in T_e G$ *then* $\exp\,(t+s)X = (\exp\, tX) \cdot (\exp\, sX)$ *for all* $t, s \in \mathbb{R}$*, where* $\cdot$ *stands for Lie group product.*

# Riemannian analogue of exponential map

Let $(M, g)$ be a riemannian manifold and let $p \in M$ be a point of $M$. The riemannian exponential map starting at $p$ is defined by:

$$T_pM \ni X \mapsto \exp_p X = \gamma_X(1) \in M$$

where $\gamma_X$ is the unique geodesic starting at $p$ with tangent vector $X$

**Proposition**
*The exponential map is smooth and natural, i.e.*

$$\begin{tikzcd}
T_pM \arrow[r, "T_p\phi"] \arrow[d, "\exp_p"] & T_{p'}M' \arrow[d, "\exp_p"] \\
M \arrow[r, "\phi"] & M'
\end{tikzcd}$$

# Normal coordinates

**Proposition**
*There exist $U$ and $V$ open neighborhoods respectively of $o \in T_pM$ and $p \in M$, s.t. $\exp_p : U \to V$ is a diffeomorphism. Futhermore if we fix an orthonormal basis of $T_pM$ we obtain an isomorphism $F$ with $\mathbb{R}^n$, combined with $\exp^{-1}$ gives normal cooridnate chart.*

**Proposition**
*Let $G$ be a Lie group equipped with bi-invariant riemannian metric. Then the Lie group exponential map is precisely the reimannian exponential map starting at the identity.*

# Morse theory

Let $f : M \to \mathbb{R}$ be a Morse-Bott function on a compact riemannian manifold whose critical submanifolds have all index and coindex different from one. The the level sets of $f$ are connected.

# Convexity theorem

## Convexity theorem

Let $(M, w)$ be a compact connected symplectic manifold, and let $\mathbb{T}^m$ be an m-torus. Suppose that $\psi : \mathbb{T}^m \to \mathrm{Sympl(M, w)}$ is a hamiltonian action with moment map $\mu : M \to \mathbb{R}^m$. Then:

1. the levels of $\mu$ are connected
2. the image of $\mu$ is convex
3. the image of $\mu$ is convex hull of the images of the fixed points of the action