Algebra General Exam - January 2025

Your UVa ID Number:

- There is a total of 80 points in 8 problems.
- Please provide *complete proofs* and justify *every* statement that you make.
- Make sure that the solution to every problem is a *continuous text* written legibly and in the correct order.
- If you refer to a *standard theorem*, please, *state* this theorem clearly and fully, and *justify* that this theorem applies in the situation at hand.
- Vague statements and hand-waving arguments will not be appreciated.
- You may assume the validity of one part of the problem to do another part even if you unable to prove the first part (in which case you, of course, will not get credit for it).

DO EACH PROBLEM ON A SEPARATE SHEET OF PAPER, AND STAPLE THEM TOGETHER IN THE CORRECT ORDER BEFORE TURNING THE EXAM IN.

Sign below the pledge:

"On my honor, I pledge that I have neither given nor received help on this assignment."

Signature (Use ID number instead): ______________________

(1) (11pt) Let $G = \mathrm{GL}_n(\mathbb{C}) = \{g \in M_n(\mathbb{C}) \mid g \text{ is invertible}\}$ be the group of all invertible $n \times n$ complex matrices. Consider the set $U = \{X \in M_n(\mathbb{C}) \mid (X - I_n)^n = O\}$, where I_n denotes the identity matrix.
   (a) (3pt) Show that the correspondence $(g, X) \mapsto gXg^{-1}$ defined an action of G on U.
   (b) (4pt) Show that the number of orbits of G on U is finite.
   (c) (4pt) Provide a complete list of representatives for orbits, for $n = 4$.

(2) (8pt) Consider a group action $\cdot : G \times X \to X$.
   (a) (3pt) Show that G naturally extends to an action on X^k away from the multi-diagonal, i.e., on the set $\{(x_1, \ldots, x_n) \in X^k | x_i \neq x_j \text{ if } i \neq j\}$.
   (b) (5pt) The group action of G on X is said to be *k-transitive* if the above action on X^k away from the multi-diagonal is transtive. Prove that the natural action of the alternating group A_n on $\{1, \ldots, n\}$ is $(n-2)$-transitive.

(3) (9pt) For an odd number n and an orthogonal matrix $A \in \mathrm{O}_n(\mathbb{R}) \backslash \mathrm{SO}_n(\mathbb{R})$, where

$$\mathrm{O}_n(\mathbb{R}) = \left\{A \in \mathrm{GL}_n(\mathbb{R}) | AA^t = I_n\right\}$$

$$\mathrm{SO}_n(\mathbb{R}) = \left\{A \in \mathrm{GL}_n(\mathbb{R}) | AA^t = I_n, \quad \det(A) > 0\right\},$$

show that -1 is an eigenvalue of A.

(4) (10pt) Let $\phi : \mathbb{Z}^r \to \mathbb{Z}^r$ be an homomorphism of free abelian groups given by multiplication by a square matrix $A \in \mathrm{M}_r(\mathbb{Z})$.
   (a) (5pt) Show that $\mathrm{Im}(\phi)$ is of finite index in the target if and only if $\det(A) \neq 0$.
   (b) (5pt) Show that when the index is finite it is equal to $|\det(A)|$.

(5) (11pt) Let $R = C[-1, 1]$ be the ring of continuous real-valued functions on the interval $[-1, 1]$ with the standard addition and multiplication of functions.
   (a) (3pt) Give an example of zero divisors in R.
   (b) (4pt) Give an example of non-invertible elements that are not zero divisors.

(c) (4pt) Let $I \subset R$ be the set of functions $f \in R$ that vanish at 0, i.e., $f(0) = 0$. Show that I is a maximal ideal of R which is not principal.

(6) (8pt) Let A be a finitely generated abelian group such that $A \otimes_{\mathbb{Z}} \mathbb{Z}/(25) \cong A$. Classify all such As up to isomorphism.

(7) (12pt) A finite group G has the following partially complete character table (of its complex irreducible representations), where c_i is the cardinality of the conjugacy class of C_i, and C_1 is the conjugacy class of the group unit

C_i	C_1	C_2	C_3	C_4	C_5
c_i	1	1	2	2	2
χ_1	1	1	1	1	1
χ_2	1	1	-1	1	-1
χ_3	1	1	1	-1	
χ_4	1	1	-1	-1	
χ_5					

(a) (4pt) Complete the character table;

(b) (4pt) Determine the isomorphism type of the center $Z(G)$;

(c) (4pt) Determine the isomorphism type of the abelianization $G/[G, G]$.

(8) (11pt) Let F be a field of characteristic $p > 0$ and consider the polynomial $f(x) := x^p - x - c \in F[x]$ for some $c \in F$. Let α be a root of $f(x)$ in an algebraic closure $\overline{F}$ of F.

(a) (3pt) Prove that $f(x)$ has roots $\alpha, \alpha + 1, ..., \alpha + p - 1$.

(b) (4pt) Prove that $F(\alpha)/F$ is a Galois extension.

(c) (4pt) Prove that $f(x)$ is either irreducible or splits into linear factors in $F[x]$.