# Algebra General Exam - January 2024

## Your UVa ID Number:

- There is a total of 80 points in 8 problems.
- Please provide *complete proofs* and justify *every* statement that you make.
- Make sure that the solution to every problem is a *continuous text* written legibly and in the correct order.
- If you refer to a *standard theorem*, please, *state* this theorem clearly and fully, and *justify* that this theorem applies in the situation at hand.
- Vague statements and hand-waving arguments will not be appreciated.
- You may assume the validity of one part of the problem to do another part even if you unable to prove the first part (in which case you, of course, will not get credit for it).

DO EACH PROBLEM ON A SEPARATE SHEET OF PAPER, AND STAPLE THEM TOGETHER IN THE CORRECT ORDER BEFORE TURNING THE EXAM IN.

**Sign below the pledge:**

"On my honor, I pledge that I have neither given nor received help on this assignment."

1. (10 points) Let $R \subset \mathbb{Q}$ be a subring with 1 of the field of rational numbers.
(a) (3 points) Show that if $\frac{a}{b} \in R$ where $a, b \in \mathbb{Z}$ $(b \neq 0)$ with g.c.d.$(a, b) = 1$ then $\frac{1}{b} \in R$.
(b) (4 points) Show that $R$ coincides with the localization $\mathbb{Z}_S$ of $\mathbb{Z}$ with respect to some multiplicative set $S \subset \mathbb{Z}$ which you should describe explicitly.
(c) (3 points) For a prime $p$, we define
$\mathbb{Z}_{(p)} := \left\{ \frac{a}{b} \mid a, b \in \mathbb{Z}, \ \ p \nmid b \right\}$.
Show that every $\mathbb{Z}_{(p)}$ is a maximal proper subring of $\mathbb{Q}$, and conversely, every maximal proper subring $R \subset \mathbb{Q}$ coincides with one of the $\mathbb{Z}_{(p)}$'s.

2. (12 points) Let $K$ be a field.
(a) (5 points) Let $K[x]$, $K[y]$ and $K[x, y]$ be the rings of polynomials in $x$, $y$ and $x, y$, respectively. Show that the natural inclusions $K[x] \hookrightarrow K[x, y]$ and $K[y] \hookrightarrow K[x, y]$ give rise to a ring (actually, $K$-algebra) homomorphism $\varphi\colon K[x] \otimes_K K[y] \to K[x, y]$, and that this homomorphism is actually a ring isomorphism.
(b) (7 points) Let $K(x)$, $K(y)$ and $K(x, y)$ be the fields of rational functions in $x$, $y$ and $x, y$, respectively, (i.e., the fields of fractions of $K[x]$, $K[y]$ and $K[x, y]$, respectively). Show that the natural inclusions $K(x) \hookrightarrow K(x, y)$ and $K(y) \hookrightarrow K(x, y)$ give rise to a ring (actually, $K$-algebra) homomorphism $\psi\colon K(x) \otimes_K K(y) \to K(x, y)$. Derive, using $\psi$, that $K(x) \otimes_K K(y)$ is not a field.
Hint: You may show that $\frac{1}{x+y}$ is not in the image of $\psi$.

3. (10 points) Let $G$ be a finite group of order $n$. Set $X = \{(x, y) \in G \times G \mid xy = yx\}$. Show that $X$ has size $nh$ where $h$ is the number of conjugacy classes of $G$. (*Hint.* Let $C$ be a fixed conjugacy class in $G$. Show that the number of elements $(x, y) \in X$ with $x \in C$ equals $n$.)

4. (10 points)Let $T\colon V \to V$ be a linear transformation of an $n$-dimensional vector space $V$ over a field $K$, and let $\chi_T(x)$ and $\mu_T(x)$ be the characteristic and minimal polynomials of $T$ in $K[x]$.
(a) (3 points) Prove that if $\deg \mu_T(x) < n$ then $V$ has a $T$-invariant subspace different from $\{0\}$ and $V$.
(b) (7 points) Prove that $V$ does not have any $T$-invariant subspaces different from $\{0\}$ and $V$ if and only if $\chi_T(x)$ is irreducible in $K[x]$.

5. (10 points) Let $f(x) \in K[x]$ be a nonconstant monic polynomial in one variable over a field $K$, and let $f = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ be its factorization into a product of powers of distinct irreducible monic polynomials. Find, with proof, necessary and sufficient conditions in terms of $r$ and $\alpha_1, \ldots, \alpha_r$ for the quotient ring $R = K[x]/(f)$ to be:
(a) (2 points) a field;
(b) (4 points) a local ring which is not a field;
(c) (4 points) a direct product of fields.

6. (8 points) Let $L/K$ be a finite Galois extension of degree $n$. Show that for every prime $p$ dividing $n$ there exists an intermediate subfield $K \subset M \subset L$ such that the degree $[M : K]$ is prime to $p$.

7. (10 points) Consider the following matrix $A \in M_4(\mathbb{Q})$:

$$A = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 2 & 3 & 0 & 1 \\ -2 & 2 & 5 & -1 \\ 0 & 0 & 0 & 5 \end{pmatrix}$$

Determine the characteristic polynomial $\chi_A(x)$, the minimal polynomial $\mu_A(x)$, the rational canonical form of $A$ and, if applicable, the Jordan canonical form of $A$.

8. (10 points) If $K = \mathbb{F}_q$ is a finite field, prove that any $\alpha \in K$ can be written in the form $\alpha = \beta^2 + \gamma^2$ with $\beta, \gamma \in K$.
Hint: How many elements in $K$ are squares?