Algebra General Exam - August 2023

Your UVa ID Number:

- There is a total of 85 points in 8 problems
- Please provide *complete proofs* and justify *every* statement that you make
- Make sure that the solution to every problem is a *continuous text* written legibly and in the correct order
- If you refer to a *standard theorem*, please, *state* this theorem clearly and fully, and *justify* that this theorem applies in the situation at hand
- Vague statements and hand-waving arguments will not be appreciated
- You may assume the validity of one part of the problem to do another part even if you unable to prove the first part (in which case you, of course, will not get credit for it)

DO EACH PROBLEM ON A SEPARATE SHEET OF PAPER, AND STAPLE THEM TOGETHER IN THE CORRECT ORDER BEFORE TURNING THE EXAM IN

Sign below the pledge:

"On my honor, I pledge that I have neither given nor received help on this assignment."

1. Let G be a finite group, and let $g \in G$ be an element different from e (the identity) that lies in *every* subgroup $H \subset G$ such that $H \neq \{e\}$.

(1) (4 points) Show that G is a p-group for some prime p, and moreover, the order of g equals p.

(2) (3 points) Prove that if the group G is abelian then it must be cyclic.

(3) (3 points) Provide an example of a non-abelian (hence non-cyclic) finite group that contains a non-trivial element that lies in every nontrivial subgroup. (Please, indicate this element explicitly.)

2. Let G be a simple group of order 60. (Recall that a group is *simple* if it does not have any proper nontrivial normal subgroups.)

(1) (5 points) Use Sylow's theorems to show that G has a subgroup H of index 6.

(2) (5 points) Use the action of G on the coset space G/H by left multiplication to construct an embedding (= injective group homomorphism) ϕ of G into S_6 (symmetric group) and then show that the image $\phi(G)$ of this embedding is contained in $\Gamma := A_6$ (alternating group).

(3) (5 points) Thinking of G as a subgroup of Γ, use the action of G on the coset space Γ/G by left multiplication to embed G into A_5 and conclude that $G \simeq A_5$.

3. Let R be a commutative ring with 1, and let $I, J \subset R$ be two ideals.

(1) (3 points) Let us consider R, I and J as (left) R-modules. Show that the map $f\colon I \oplus J \to R$, $(x, y) \mapsto x + y$, is a homomorphism of R-modules.

(2) (7 points) Now assume that $I + J = R$ (in other words, I and J are *co-prime*). By analyzing the R-module homomorphism f from part (a), show that $I \oplus J \simeq R \oplus IJ$ as R-modules (here IJ denotes the product of I and J as ideals of R). (*Hint.* You may use the fact that R is a projective R-module.)

4. (10 points) Let $A = J_n(\lambda)$ be a Jordan block of size $n \geq 1$ with the eigenvalue λ over a field K. Find (with justification) all A-invariant subspaces of the n-dimensional space $V = K^n$. (*Hint.* Reduce to the case $\lambda = 0$ or use that V can be considered as a $K[t]$-module where the multiplication by t is given by the application of A.)

5. (10 points, 5 each) Identify the following tensor products:

(1) $\mathbb{Q}(x) \otimes_{\mathbb{Q}[x]} \mathbb{Q}(x)$ as $\mathbb{Q}[x]$-module;

(2) $\mathbb{Q}(\sqrt{2}) \otimes_{\mathbb{Q}} \mathbb{Q}(\sqrt{2})$ as $\mathbb{Q}$-algebra (your description of the resulting algebra should not contain any tensor products).

6. Let p be an odd prime, and consider the polynomial $f(x) = x^4 - x^2 + 1$ in $\mathbb{F}_p[x]$ (where $\mathbb{F}_p$ is the finite field with p elements) noting the identity $f(x) \cdot (x^2 + 1) = x^6 + 1$.

(1) (3 points) For which p is $f(x)$ separable?

(2) (4 points) Show that for $p > 3$ the polynomial $f(x)$ splits into linear factors over the extension $\mathbb{F}_{p^2}$ of $\mathbb{F}_p$. What happens for $p = 3$?

(3) (3 points) Find the minimal odd prime p for which $f(x)$ splits into linear factors already over $\mathbb{F}_p$.

7. (10 points) Use cyclotomic extensions to construct two *distinct* cyclic Galois extensions of $\mathbb{Q}$ of degree 7, and show that there exists a Galois extension of $\mathbb{Q}$ with Galois group isomorphic to $\mathbb{Z}/7\mathbb{Z} \times \mathbb{Z}/7\mathbb{Z}$.

8. Let K be a field of characteristic $p > 0$, let L/K be a cyclic Galois extension of degree p, and let σ be a generator of the Galois group $\mathrm{Gal}(L/K)$. In this problem, we consider σ as a linear transformation of L as a vector space over K.

(1) (3 points) Find the eigenvalues of σ.

(2) (3 points) Identify the Jordan canonical form of σ. (*Hint.* Find the number of Jordan blocks.)

(3) (4 points) Use your result from part (2) to show that there exists $\alpha \in L$ such that $\sigma(\alpha) = \alpha + 1$.