Algebra General Exam - August 2024

Your UVa ID Number:

- There is a total of 80 points in 8 problems.
- Please provide *complete proofs* and justify *every* statement that you make.
- Make sure that the solution to every problem is a *continuous text* written legibly and in the correct order.
- If you refer to a *standard theorem*, please, *state* this theorem clearly and fully, and *justify* that this theorem applies in the situation at hand.
- Vague statements and hand-waving arguments will not be appreciated.
- You may assume the validity of one part of the problem to do another part even if you unable to prove the first part (in which case you, of course, will not get credit for it).

DO EACH PROBLEM ON A SEPARATE SINGLE-SIDED SHEET OF PAPER, AND STAPLE THEM TOGETHER IN THE CORRECT ORDER BEFORE TURNING THE EXAM IN.

Sign below the pledge:

"On my honor, I pledge that I have neither given nor received help on this assignment."

Signature: ___________________________

(1) (8pt) Let s, n be positivie integers, and I_n be the $n \times n$ identity matrix. Consider the following $sn \times sn$ matrix in $s \times s$ blocks

$$A_{s,n} = \begin{bmatrix} 0 & I_n & 0 & \dots & 0 & 0 \\ 0 & 0 & I_n & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & I_n \\ 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}.$$

Find the Jordan canonical form for $A_{s,n}$ over $\mathbb{C}$.

(2) (12 pt) Let $\mathbb{F}_q$ be a finite field of q elements. Denote by $\mathrm{GL}_n(\mathbb{F}_q)$ the group of invertible $n \times n$ matrices with coefficients in $\mathbb{F}_q$ and by $\mathrm{Gr}(k, n)$ the set of k-dimensional subspaces in the n-dimensional space $\mathbb{F}_q^n$ over $\mathbb{F}_q$, for $0 \leq k \leq n$.
   (a) (4pt) Find the cardinality of the group $\mathrm{GL}_4(\mathbb{F}_q)$.
   (b) (4pt) Show that the natural action of $\mathrm{GL}_n(\mathbb{F}_q)$ on $\mathbb{F}_q^n$ gives rise to a **transitive** action of $\mathrm{GL}_n(\mathbb{F}_q)$ on $\mathrm{Gr}(k, n)$.
   (c) (4 pt) Find the cardinality of the set $\mathrm{Gr}(3, 4)$.

(3) (8pt) Let S be a principal ideal domain (PID) and R be a subdomain of S which is also a PID. Let a, b be elements of R. Show that the greatest common divisors (gcd) of a and b in R and S differ only by a unit in S.

(4) (8pt) Fix a prime number p. Let $\mathbb{F}_p$ be the prime field of characteristic p and $\mathbb{F}_q$ be a finite field of $q = p^n$ elements. Show that, as commutative algebras, there is an isomorphism

$$\mathbb{F}_q \otimes_{\mathbb{F}_p} \mathbb{F}_q \cong \mathbb{F}_q^{\times n}.$$

(5) (10pt) If G is a group of order 483, show that $G \cong H \times K$, where H is a group of order 23, and K is a group of order 21.

(6) (12pt) Let D be the dihedral group of order 8. It can be described in terms of generators and relations as $D = \langle s, t \mid s^2 = 1 = t^4, st = t^{-1}s \rangle$.
  (a) (3pt) List all conjugacy classes of D.
  (b) (4pt) Describe all 1-dimension irreducible representations of D over $\mathbb{C}$.
  (c) (5pt) Compute the character table for D.

(7) (12pt) Consider the field extension $\mathbb{Q}(\sqrt{8 + 2\sqrt{15}})$ over $\mathbb{Q}$.
  (a) (4pt) Determine the degree of this extension.
  (b) (4pt) Show this is a Galois extension.
  (c) (4pt) Compute the Galois group for this extension.

(8) (10pt) Let $R = \mathbb{Q}[x]/(x^2 - 9)$, and M be an R-module. Show that there is a direct sum decompostion of R-modules $M \cong M_+ \oplus M_-$, where $x \in R$ acts on $M_\pm$ as ± 3.