Algebra General Exam - January 2023

Your UVa ID Number:

- Please Prove all your work and justify any statements that you make.
- State clearly and fully any theorem you use.
- Vague statements and hand-waving arguments will not be appreciated.
- You may assume the statement in an earlier part Proven in order to do a later part.

DO EACH PROBLEM ON A SEPARATE SHEET OF PAPER, AND STAPLE THEM TOGETHER IN THE CORRECT ORDER BEFORE TURNING THE EXAM IN.

Sign below the pledge:

"On my honor, I pledge that I have neither given nor received help on this assignment."

1. (a) (8 points) Let G be a finite group, and $H \subsetneq G$ be a proper subgroup. Prove that

$$G \neq \bigcup_{g \in G} gHg^{-1}.$$

(*Hint.* Estimate the number of elements in the right-hand side.)

(b) (8 points) Let G be a transitive subgroup of the symmetric group S_n. Prove that there exists $g \in G$ that has no fixed points on $I_n = \{1, 2, \ldots, n\}$ (i.e. there is no $i \in I_n$ such that $g(i) = i$).

2. Let $R = \mathbb{Z}\left[\sqrt[3]{2}\right] = \left\{a_0 + a_1\sqrt[3]{2} + a_2\left(\sqrt[3]{2}\right)^2 \mid a_0, a_1, a_2 \in \mathbb{Z}\right\}$.

(a) (5 points) Prove that R is a subring of $\mathbb{R}$.

(b) (6 points) Prove that the evaluation homomorphism $\mathbb{Z}[x] \to R$, $p(x) \mapsto p\left(\sqrt[3]{2}\right)$, yields a ring isomorphism

$$R \simeq \mathbb{Z}[x]/(x^3 - 2).$$

(c) (6 points) Using the isomorphism from part (b) and the 3rd Isomorphism Theorem, can you determine if the ideals $5R$ and $7R$ are prime or maximal?

3. (8 points) Let F be a field (possibly finite). Prove that the polynomial ring $F[x]$ has infinitely many prime ideals.
Note: You cannot assume without proof that $F[x]$ has infinitely many irreducible polynomials.

4. (10 points) Let R be a commutative ring with 1. Let $f\colon \mathbb{Z}^n \to \mathbb{Z}^m$ be a $\mathbb{Z}$-module homomorphism, and let $K = \ker f$. Prove that there exists a submodule $M \subset \mathbb{Z}^n$ such that

$\mathbb{Z}^n = K \oplus M$. (*Hint.* Argue that $P := \operatorname{im} f \subset \mathbb{Z}^m$ is projective.)
Reminder. An R-module P is called *projective* if, for any surjective module homomorphism $\varphi\colon M \to N$, every module homomorphism $\psi\colon P \to N$ admits a lifting $\tilde{\psi}\colon P \to M$ such that $\varphi \circ \tilde{\psi} = \psi$.

5. Let L/K be a finite Galois extension of degree d with Galois group G. In this problem, we think of the elements of G as linear transformations $\sigma : L \to L$ of the d-dimensional vector space L over K.

(a) (5 points) Let $\sigma \in G$ be an element of order n. Prove that the minimal polynomial of $\sigma : L \to L$ is $\mu(x) = x^n - 1$. (*Hint.* Use the standard theorem about linear independence of characters.)

(b) (10 points) Assume that p is a prime $\neq \operatorname{char} K$, and K contains a primitive p-th root of unity. Let L/K be a Galois extension of degree p, and let $\sigma \in \operatorname{Gal}(L/K)$ be a nontrivial automorphism. Prove that σ is diagonalizable. Identify its eigenvalues, the corresponding eigenvectors and derive from this analysis that $L = K\left(\sqrt[p]{a}\right)$ for some $a \in K^{\times}$. (*Note.* This gives an easier proof of Kummer theory in this special case).

6. (10 points) Let $K/\mathbb{Q}$ be a Galois extension with Galois group S_n (symmetric group) for some $n \geq 5$. Assume that K contains a primitive root of unity ζ_d. Prove that $d \leq 6$.

7. (a) (6 points) Let L/K be a field extension and $f(x) \in K[x]$. Prove that there is an isomorphism of L-algebras

$$L \bigotimes_K \frac{K[x]}{f(x)K[x]} \simeq \frac{L[x]}{f(x)L[x]}.$$

Note. The L-algebra structure on the ring $L \otimes_K \dfrac{K[x]}{f(x)K[x]}$ is given on simple tensors by $c \cdot (a \otimes \overline{g(x)}) := (ca) \otimes \overline{g(x)}$, where ca is the ring multiplication in L.

(b) (7 points) Suppose that L/K is a finite Galois extension of degree n. Prove that there is an isomorphism of L-algebras

$$L \otimes_K L \simeq \underbrace{L \times \cdots \times L}_{n}.$$

8. Let V be the $\mathbb{C}$-vector space of all polynomials $p(x, y)$ in two variables of total degree ≤ 2. Let $T : V \to V$ be the linear transformation $T(f) = \dfrac{\partial f}{\partial x}$.

(a) (2 points) Prove that T is nilpotent.

(b) (9 points) Construct a Jordan basis for T and compute its Jordan canonical form.