$$(f \times g)\ (a, b) = (f\ a, g\ b)$$

$$(f = \perp)$$

Instance of $\rightarrow^{+}$

$$
\begin{array}{l}
\textbf{newtype}\ a \rightarrow^{+} b = \mathit{AddFun}\ (a \rightarrow b) \\
\textbf{instance}\ \mathit{Category}\ (\rightarrow^{+})\ \textbf{where} \\
\quad \textbf{type}\ \mathit{Obj}\ (\rightarrow^{+}) = \mathit{Additive} \\
\quad \mathit{id} = \mathit{AddFun}\ \mathit{id} \\
\quad \mathit{AddFun}\ g \circ \mathit{AddFun}\ f = \mathit{AddFun}\ (g \circ f) \\
\textbf{instance}\ \mathit{Monoidal}\ (\rightarrow^{+})\ \textbf{where} \\
\quad \mathit{AddFun}\ f \times \mathit{AddFun}\ g = \mathit{AddFun}\ (f \times g) \\
\textbf{instance}\ \mathit{Cartesian}\ (\rightarrow^{+})\ \textbf{where} \\
\quad \mathit{exl} = \mathit{AddFun}\ \mathit{exl} \\
\quad \mathit{exr} = \mathit{AddFun}\ \mathit{exr} \\
\quad \mathit{dup} = \mathit{AddFun}\ \mathit{dup}
\end{array}
$$

Fork and Join

- $\nabla :: \mathit{Cartesian}\ k \Rightarrow (a\ \texttt{'k'}\ c) \rightarrow (a\ \texttt{'k'}\ d) \rightarrow (a\ \texttt{'k'}\ (c \times d))$
- $\Delta :: \mathit{Cartesian}\ k \Rightarrow (c\ \texttt{'k'}\ a) \rightarrow (d\ \texttt{'k'}\ a) \rightarrow ((c + d)\ \texttt{'k'}\ a)$

...Machine Learning...

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Universidade do Minho

26 de Abril

Indice

A short introdution

- We want to calculate $\mathcal{D}$.
- However, $\mathcal{D}$ is not computable.
- Solution: reimplement corollaries using category teory

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A short introdution

Corollary 1.1

NOTA: adicionar definição do corolário 1.1 aqui

Corollary 2.1

NOTA: adicionar definição do corolário 2.1 aqui

Corollary 3.1

NOTA: adicionar definição do corolário 3.1 aqui

A category is a collection of objects(sets and types) e morphisms(operation between objects), with 2 basic operations(identity and composition) of morfisms, and 2 laws:

- (C.1) $id \circ f = id \circ f = f$
- (C.2) $f \circ (g \circ h) = (f \circ g) \circ h$

N

ote: for this paper, objects are data types and morfisms are functions

$$\begin{array}{l} \textbf{class}\ \mathit{Category}\ k\ \textbf{where} \\ \quad id :: (a'k'a) \\ \quad (\circ) :: (b'k'c) \to (a'k'b) \to (a'k'c) \end{array}$$

$$\begin{array}{l} \textbf{instance}\ \mathit{Category}\ (\to)\ \textbf{where} \\ \quad id = \lambda a \to a \\ \quad g \circ f = \lambda a \to g\ (f\ a) \end{array}$$

Functors

A functor F between 2 categories $\mathcal{U}$ *and* $\mathcal{V}$ is such that:

- given any object $t \lambda \textbf{in}\ \mathcal{U}$ there exists a object $F\ t \lambda \textbf{in}\ \mathcal{V}$
- given any morphism $m :: a \to b \lambda \textbf{in}\ \mathcal{U}$ there exists a morphism $F\ m :: F\ a \to F\ b \lambda \textbf{in}\ \mathcal{V}$
- $F\ id\ (\lambda \textbf{in}\ \mathcal{U}) = id\ (\lambda \textbf{in}\ \mathcal{V})$
- $F\ (f\ \$\backslash\ circ\ \$\ g) = F\ f\ \$\backslash\ circ\ \$\ F\ g$

Note

Given this papers category properties(objects are data types) we have that functors map types to themselfs

Objective

Let's start by defining a new data type:
newtype $\mathcal{D}\ a\ b = \mathcal{D}\ (a \to b\ x\ (a\,\$\backslash\ multimap\,\$\ b))$
and adapting $\mathcal{D}^+$ to use it:

Adapted definition

$$\hat{\mathcal{D}} :: (a \to b) \to \mathcal{D}\ a\ b$$
$$\hat{\mathcal{D}}\ f = \mathcal{D}\ (\mathcal{D}^+\ f)$$

Our objective is to deduce an instance of Category for $\mathcal{D}$ **where** $\hat{\mathcal{D}}$ is a functor.

Instance deduction

Using corollaries 3.1 and 1.1 we deduce that

- (DP.1) -- bigDplus id = λ a -> (id a,id)
- (DP.2) —- $\mathcal{D}^+ \ (g \,\$\backslash\ circ \,\$\ f) = \$\backslash lambda \,\$\ a \rightarrow$
  $\textbf{let}\ \{(b, f') = \mathcal{D}^+\ f\ a; (c, g') = \mathcal{D}^+\ g\ b\}\ \textbf{in}\ (c, g'\ \$\backslash\ circ\,\$\ f')$

saying that $\hat{\mathcal{D}}$ *a functor is equivalent to, for all f e g functions* **of** *corre*

$id = \hat{\mathcal{D}}\ id = \mathcal{D}\ (\mathcal{D}^+\ id)$

$\hat{\mathcal{D}}\ g\ \$\backslash\ circ\,\$\ \hat{\mathcal{D}}\ f = \hat{\mathcal{D}}\ (g\ \$\backslash\ circ\,\$\ f) = \mathcal{D}\ (\hat{\mathcal{D}}\ (g\ \$\backslash\ circ\,\$\ f))$

Instance deduction

Based on (DP.1) e (DP.2) we'll rewrite the above into the following defenition:

$id = \mathcal{D}\ (\$\backslash lambda\,\$\ a \rightarrow (id\ \ a, id))$

$bidDhat\ g\,\$\backslash\ circ\,\$\ \hat{\mathcal{D}}\ f = \mathcal{D}\ (\$\backslash lambda\,\$\ a \rightarrow \textbf{let}\ \{(b, f') = \mathcal{D}^+\ f\ a; (c, g') = \mathcal{D}^+\ g\ b\}\ \textbf{in}\ (c, g'\ \$\backslash\ circ\,\$\ f'))$

The first equasion has a trivial solution(define id of instance as $\mathcal{D}\ (\$\backslash lambda\,\$\ a \rightarrow (id\ \ a, id)))$

To solve the secound we'll first solve a more general one: $\mathcal{D}\ g\,\$\backslash\ circ\,\$\ \mathcal{D}\ f = \mathcal{D}\ (\$\backslash lambda\,\$\ a \rightarrow \textbf{let}\ \{(b, f') = f\ \ a; (c, g') = g\ b\lambda\}\ \textbf{in}\ (c, g'\ \$\backslash\ circ\,\$\ f'))$, and this has an equivalently trivial solution in our instance.

Instance deduction

$\hat{\mathcal{D}}$ definition for linear functions

$$linearD :: (a \to b) \to \mathcal{D}\ a\ b$$
$$linearD\ f = \mathcal{D}\ (\lambda a \to (f\ a, f))$$

s

Categorical instance we've deduced

$$\textbf{instance}\ Category\ \mathcal{D}\ \textbf{where}$$
$$id = linearD\ id$$
$$\mathcal{D}\ g \circ \mathcal{D}\ f = \mathcal{D}\ (\lambda a \to \textbf{let}\ \{(b, f') = f\ a; (c, g') = g\ b\}\ \textbf{in}\ (c, g' \circ$$

Instance proof

In order to prove that the instance is correct we must observe if it follows laws (C.1) and (C.2).
First we must make a concession: that we only use morfisms arising from $\mathcal{D}^+$(we can force this by transforming $\mathcal{D}$ into an abstract type). If we do, then $\mathcal{D}^+$ is a functor.

(C.1) proof

$id\ \$\backslash\ circ\ \$\ \hat{\mathcal{D}}$
$= \hat{\mathcal{D}}\ id\ \$\backslash\ circ\ \$\ \hat{\mathcal{D}}\ f - functor\ law\ for\ id\ (specification$ **of** $\hat{\mathcal{D}})$
$= \hat{\mathcal{D}}\ (id\ \$\backslash\ circ\ \$\ f) - functor\ law\ for\ (\$\backslash circ\$)$
$= \hat{\mathcal{D}}\ f - cathegorical\ law$

Instance proof

(C.2) proof

$$
\begin{aligned}
&\hat{\mathcal{D}}\ h \,\$\backslash\ circ\,\$\ (\hat{\mathcal{D}}\ g \,\$\backslash\ circ\,\$\ \hat{\mathcal{D}}\ f)\\
&= \hat{\mathcal{D}}\ h \,\$\backslash\ circ\,\$\ \hat{\mathcal{D}}\ (g \,\$\backslash\ circ\,\$\ f) - \textit{functor law for } (\$\backslash circ\$)\\
&= \hat{\mathcal{D}}\ (h \,\$\backslash\ circ\,\$\ (g \,\$\backslash\ circ\,\$\ f)) - \textit{functor law for } (\$\backslash circ\$)\\
&= \hat{\mathcal{D}}\ ((h \,\$\backslash\ circ\,\$\ g) \,\$\backslash\ circ\,\$\ f) - \textit{categorical law}\\
&= \hat{\mathcal{D}}\ (h \,\$\backslash\ circ\,\$\ g) \,\$\backslash\ circ\,\$\ \hat{\mathcal{D}}\ f - \textit{functor law for } (\$\backslash circ\$)\\
&= (\hat{\mathcal{D}}\ h \,\$\backslash\ circ\,\$\ \hat{\mathcal{D}}\ g) \,\$\backslash\ circ\,\$\ \hat{\mathcal{D}}\ f - \textit{functor law for } (\$\backslash circ\$)
\end{aligned}
$$

Note

This proofs don't require anything from $\mathcal{D}$ *and* $\hat{\mathcal{D}}$ aside from functor laws. As such, all other instances of categories created from a functor won't require further proofs.

Generalized parallel composition will be defined using a monoidal category:

class *Category k* $\Rightarrow$ *Monoidal k* **where**
  $(x) :: (a \text{ 'k' } c) \rightarrow (b \text{ 'k' } d) \rightarrow ((a \times b) \text{ 'k' } (c \times d))$

instance *Monoidal* $(\rightarrow)$ **wh**
  $f \text{ 'x' } g = \lambda(a, b) \rightarrow (f\ a, g$

Monoidal Functor definition

A monoidal functor F between categories $\mathcal{U}$ *and* $\mathcal{V}$ is such that:

- F is a functor
- F (f $\ *times*$ g) = F f $\ *times*$ F g

Instance deduction

From corollary 2.1 we can deduce that:
$\mathcal{D}^+ \ (f \ \$\backslash \ times \ \$ \ g) = \$\backslash lambda \ \$ \ (a, b) \rightarrow \textbf{let} \ \{(c, f') = \mathcal{D}^+ \ f \ a; (d, g') = \mathcal{D}^+ \ g \ b\} \ \textbf{in} \ ((c, d), f' \ \$\backslash \ times \ \$ \ g')$
Defining F from $\hat{\mathcal{D}}$ leaves us with the following definition:
$\mathcal{D} \ (\mathcal{D}^+ \ f) \ \$\backslash \ times \ \$ \ \mathcal{D} \ (\mathcal{D}^+ \ g) = \mathcal{D} \ (\mathcal{D}^+ \ (f \ \$\backslash \ times \ \$ \ g))$
Using the same method as before, we replace $\mathcal{D}^+$ with it's definition and generalize the condition:
$\mathcal{D} \ f \ \$\backslash \ times \ \$ \ \mathcal{D} \ g = \mathcal{D} \ (\$\backslash lambda \ \$ \ (a, b) \rightarrow \textbf{let} \ \{(c, f') = f \ a; (d, g') = g \ b\} \ \textbf{in} \ ((c, d), f' \ \$\backslash \ times \ \$ \ g'))$
and this is enouth for our new instance.

Categorical instance we've deduced

instance *Monoidal* $\mathcal{D}$ **where**
$\mathcal{D}\ f\ x\ \mathcal{D}\ g = \mathcal{D}\ (\lambda(a,b) \rightarrow$ **let** $\{(c, f') = f\ a; (d, g') = g\ b\}$ **in** $(($

Cartesian categories and functors

$\textbf{class } \mathit{Monoidal}\ k \Rightarrow \mathit{Cartesean}\ k \textbf{ where}$
$\quad \mathit{exl} :: (a, b)\ \text{'k'}\ a$
$\quad \mathit{exr} :: (a, b)\ \text{'k'}\ b$
$\quad \mathit{dup} :: a'k'\ (a, a)$

$\textbf{instance } \mathit{Cartesean}\ (\rightarrow) \textbf{ w}$
$\quad \mathit{exl} = \lambda(a, b) \rightarrow a$
$\quad \mathit{exr} = \lambda(a, b) \rightarrow b$
$\quad \mathit{dup} = \lambda a \rightarrow (a, a)$

A cartesian functor F between categories $\mathcal{U}$ *and* $\mathcal{V}$ is such that:

- F is a monoidal functor
- F exl = exl
- F exp = exp
- F dup = dup

Instance deduction

From corollary 3.1 and from exl,exr and dup beeing linear function we can deduce that:
$\mathcal{D}^+$ $exl \lambda p \rightarrow (exl\ p, exl)$ $\mathcal{D}^+$ $exr \lambda p \rightarrow (exr\ p, exr)$
$\mathcal{D}^+$ $dup \lambda p \rightarrow (dup\ a, dup)$
With this in mind we'll deduce the instance: exl = $\mathcal{D}$ ($\mathcal{D}^+$ exl) exr = $\mathcal{D}$ ($\mathcal{D}^+$ exr) dup = $\mathcal{D}$ ($\mathcal{D}^+$ dup)

Instance deduction

Replacing $\mathcal{D}^+$ with it's definition and remembering linearD we can obtain:
exl = linearD exl exr = linearD exr dup = linearD dup
and we can directly convert this into a new instance:

Categorical instance we've deduced

instance *Cartesian D* **where**
$\quad exl = linearD\ exl$
$\quad exr = linearD\ exr$
$\quad dup = linearD\ dup$

Cocartesian category

This type of categories are the dual of the cartesian categories.

Note

In this paper coproducts are categorical products, i.e., biproducts

Definition

$$
\begin{array}{l}
\textbf{class } \mathit{Category}\ k \Rightarrow \mathit{Cocartesian}\ k \textbf{ where}: \\
\quad \mathit{inl} :: a' k' \ (a, b) \\
\quad \mathit{inr} :: b' k' \ (a, b) \\
\quad \mathit{jam} :: (a, a) \ '\mathtt{k}' \ \ a
\end{array}
$$

Cocartesian functors

Cocartesian functor definition

A cocartesian functor F between categories $\mathcal{U}$ *and* $\mathcal{V}$ is such that:

- F is a functor
- F inl = inl
- F inr = inr
- F jam = jam