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Dynamics of a predator-prey model with stage-structure on both species and anti-predator behavior

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ABSTRACT

In this paper, we have formulated and studied a stage-structure predator-prey model. Here, we consider stage-structure on both prey as well as predator population which means that the prey population is divided into two sub populations such as juvenile prey and adult prey, on the other hand, the predator population is also divided into two sub populations such as juvenile predator and adult predator. It is assumed that only adult predator have the ability to predation and they consume both juvenile prey as well as adult prey. Here, it is also considered that the growth rate of juvenile prey depends upon the adult prey population i.e., the juvenile prey has no reproduction capability. Also, Holling II and Holling IV response function have been used for the consumption of juvenile prey and adult prey by adult predator respectively. It is also considered two types of factors such as anti-predator behavior and group defense to formulate our proposed model. Mathematically, we have analyzed the positivity and boundedness of solutions, existence of equilibria, stability of the proposed system around these equilibrium points and Hopf bifurcation of interior equilibrium point. Finally, some numerical simulations have been presented to validate our theoretical results.

1. Introduction

Predator-prey model is an essential tool in mathematical ecology and specifically for our understanding of interacting populations in the natural environment. Population biology [1,2,7] is a subset study within ecology that evaluates factors that affect populations. A population is defined as a group of the same species living in a similar geographical area. In recent time, focus on general factors or components [5,10–12] of a population that researchers focus on when evaluating populations. When a population biologist begins to evaluate a population of species, they use many tools to gather information. Mathematical formulas and models are constructed based on the experiments and observations and then used to make predictions. Basically, the researchers need to look at factors that affect the population.

In anti-predator defense mechanism [9–12,34], predator recognition is an important component because most of the defenses necessitate prey first recognizing danger i.e., discriminating dangerous situations from harmless ones and then taking elusory action. There are several terms are used in the study of anti-predator defense mechanism such as: avoidance [33] i.e., prey limits its activities so as not be detected by predators, alarm signals or calls i.e., any alarm signal given in presence of predator, defense call i.e., prey calls loudly when predator near by, group defense i.e., prey prevents predation by attacking predators etc. Prey defenses can be a stabilizing factor in predator-prey interactions. Predation can be a strong agent of natural selection. Easily captured prey are eliminated and prey with effective defenses rapidly dominate the population. In this paper, we introduce the factor “group defence” which is shown by the prey species to their predator. In predator-prey dynamics, group defence is that type of terms which are used to describe the phenomenon whereby the predation is decreased or even prevented altogether due to increased ability of the prey to better defend themselves when their numbers are huge enough. In our proposed model, we incorporate the factor “group defence” to develop the model to show that preys are better able to defend themselves against predators when they are in groups.

A population is growing at a constant rate reaches a stable age distribution in which the proportion of individuals in each age class remain the same from year one to the next. Often, a single species cannot be properly modeled as one population, but instead is best treated as a

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structured population [19–24], where the individuals in the population are partitioned into classes or stages. In our proposed model, we considered stage-structure on both prey as well as predator population to make it more realistic. Here we, prescribed a four-tire predator-prey model which consists of juvenile prey, adult prey, juvenile predator and adult predator.

Functional responses describe the relationship between an individual's rate of consumption and food density. A consumer's functional response to changes in food density is an important component of population regulation. However, the interactions between the prey and predator follow the Lotka–Volterra model [3,4,17], which has the unsuitable assumption that the effect of the predation is to decrease the prey's per capita growth rate by a term which is proportional to the predator and prey populations. But in reality, in an ecosystem the interactions between predator and prey are often mosaic and complicated. In population dynamics, Holling proposed that there exist three different types of response function [13,17,18,25–29,31,32] such as Holling type I, Holling type II, Holling type III of the predator to the prey density which refer to the change in the density of prey attacked per unit time per predator as the prey density changes which are monotonic. But some field observations and experiments [8] indicate that the non-monotonic response occurs because when the prey abandance reaches a high level an inhibitory effect on the specific growth rate of population may occur. Considering such inhibitory effect, Andrews [15] proposed a functional response $f(x,y) = \frac{mx}{a+bx+\frac{x^2}{i}}$, which is called the Monod-Haldane function. Sokol and Howell [16] proposed a simplified Monod-Haldane function or Holling type-IV which is of the form: $f(x,y) = \frac{mx}{a+cx^2}$, where the parameter c is defined as $c = \frac{1}{i}$ which is also positive parameter and can be defined as the inverse measure of inhibitory effect. Because of this, to formulate our mathematical model we have considered the simplified Monod-Haldane or Holling type-IV [14,30] function which describes the phenomenon of group defense [6,8,10] where predation is decreased due to the increased ability of the prey to better defend or disguise themselves when their numbers are large enough.

Motivated by Refs. [21,22,24] in the recent study, our objective of this paper is to investigate and analyze a four-dimensional prey-predator model which consists of a prey population which is divided into two sub populations, one is juvenile prey and other is adult prey and also a predator population which is also divided into two sub-populations, one is juvenile predator and other is adult predator. Our aim is to study the dynamics of the proposed four-tier predator-prey model and also discuss the effects of important parameters related with our proposed model. Keeping these points in mind, we have studied the Hopf bifurcation of our proposed system with respect to some vital parameters which plays an important role to interact among juvenile prey, adult prey, juvenile predator and adult predator.

The organization of this paper is as follows: in section 2, the assumption and construction of model (1) are discussed. The positivity and boundedness of all solutions of the system (1) are described in section 3. Section 4 mainly analyze the equilibrium points of the system (1), their existence and the stability analysis of the equilibrium points. In section 5, we represent the fact that the existence of stability switching and Hopf bifurcation around the interior equilibrium point. Section 6 is devoted to all of our important findings which are verified numerically using MATLAB software, which is not only confirm the theoretical results, but also explore the complex dynamical behavior with respect some important parameters. A final discussion of the paper and some biological implication of our mathematical findings are given in the last section.

2. Assumptions and model formulation

Our proposed model consists of two populations such as (a) the prey population whose density is denoted by N and (b) the predator population whose density is denoted by P. Before giving our mathematical model, we make the following assumptions:

(i) Considering stage-structure on both prey and predator, the total prey population (N) is divided into two classes: one is juvenile prey (X) and other is adult prey (Y) i.e., $N = X + Y$.
Also, the predator population (P) is divided into two classes: one is juvenile predator (Z) and other is adult predator (W) i.e., $P = Z + W$.

(ii) It is assumed that only adult prey is capable of reproducing.

(iii) In the absence of predator, the birth rate $r(>0)$ of the juvenile prey is proportional to the existing adult prey population.

(iv) In real world natural ecological system, it is seen that the juvenile members of a species may not be able to direct predation. Basically, they are dependent on the mature or adult members. In that respect, we have considered the predation ability only on adult predator but not on juvenile predator. Here, the population growth of juvenile predator is considered to be depend on the consumption rate of juvenile and adult prey by their adult predator.

(v) The functional response for predation to juvenile prey is assumed to be taken as Holling type II response function. On the other hand, the functional response for predation to adult prey is assumed to be taken as Holling type IV response function instead of Holling type II response function because Holling type IV response function describes the phenomenon of group defense whereby predation is decreased, or even prevented altogether, due to the increased ability of the prey to better defend or disguise themselves when their numbers are large enough.

Based on the above assumptions, our model is formulated as follows:

$$\left.\begin{aligned}
\frac{dX}{dt} &= rY - b_1X - d_1X - \frac{m_1XW}{k_1+X}\\
\frac{dY}{dt} &= b_1X - c_1Y^2 - \frac{m_2YW}{k_2+k_3Y^2} - d_2Y\\
\frac{dZ}{dt} &= \frac{\alpha m_1XW}{k_1+X} + \frac{\beta m_2YW}{k_2+k_3Y^2} - nZ - d_3Z - \eta YW\\
\frac{dW}{dt} &= nZ - d_4W
\end{aligned}\right\} \tag{1}$$

where, r represents the birth rate of the juvenile prey is proportional to the existing adult prey, b_1 represents the transition rate from juvenile prey to adult prey, m_1 and k_1 stands for attack rate and half saturation constant respectively. Here c_1 is the coefficient of interspecific competition among adult prey. Also, d_1, d_2, d_3 and d_4 be the death rate of juvenile prey, adult prey, juvenile predator and adult predator respectively. The parameter α and β are the conversion efficiency from juvenile prey and adult prey to the growth of predator population. Here, n and η be the transition rate from juvenile predator to adult predator and the rate of anti predator behavior of adult prey to predator respectively.

3. Positivity and boundedness of solutions

3.1. Positive invariance

Consider the proposed system (1) and put them in the vector form

$$\dot{V} = \vartheta(V(t)) \tag{2}$$

where $V(t) = (v_1, v_2, v_3, v_4)^T = (X(t), Y(t), Z(t), W(t))^T$,

$V(0) = (X(0), Y(0), Z(0), W(0))^T \in R_+^4$

and

$$\vartheta(V(t)) = \begin{pmatrix} \vartheta_1(V(t)) \\ \vartheta_2(V(t)) \\ \vartheta_3(V(t)) \\ \vartheta_4(V(t)) \end{pmatrix} = \begin{pmatrix} rY - d_1X - b_1X - \dfrac{m_1XW}{k_1+X} \\ b_1X - c_1Y^2 - \dfrac{m_2YW}{k_2+k_3Y^2} - d_2Y \\ \dfrac{\alpha m_1XW}{k_1+X} + \dfrac{\beta m_2YW}{k_2+k_3Y^2} - (n+d_3)Z - \eta YW \\ nZ - d_4W \end{pmatrix}$$

It is easy to verify that $\vartheta_i(V(t))|_{v_i=0} \geq 0$, (for i = 1,2,3,4). Due to Nagumo's theorem, any solutions of (2) with initial point $V(0) = V_0 \in R^4_+$, say $V(t) = V(t; V_0)$, in such a way that $V(t) \in R^4_+$ for all $t > 0$, that is, the solutions of the system (1) remain positive throughout the region R^4_+.

3.2. Boundedness

Theorem 1. *The solutions of the proposed system* (1) *are ultimately bounded.*

Proof. let us consider a function $\mathbf{N} = \alpha X + \beta Y + Z + W$, then the time derivative along the solutions of the system (1) is given by

$$\frac{d\mathbf{N}}{dt} = \alpha\frac{dX}{dy} + \beta\frac{dY}{dt} + \frac{dZ}{dt} + \frac{dW}{dt}$$
$$= \alpha rY - \alpha X(d_1+b_1) + \beta b_1X - \beta c_1Y^2 - \beta d_2Y - d_3Z - d_4W - \eta YZ$$

We choose a constant $\xi(>0)$, such that

$$\frac{d\mathbf{N}}{dt} + \xi\mathbf{N} = \alpha rY + \beta b_1X - \beta c_1Y^2 - \eta YZ$$
$$-\alpha X\{(d_1+b_1)-\xi\} - \beta Y(d_2-\xi) - Z(d_3-\xi) - W(d_4-\xi)$$

Define $\xi = min\{d_1, d_2, d_3, d_4\}$, then

$$\frac{d\mathbf{N}}{dt} + \xi\mathbf{N} \leq \alpha rY + \beta b_1X$$

Hence, there exist a positive number M, such that

$$\frac{d\mathbf{N}}{dt} + \xi\mathbf{N} \leq M \; so, \mathbf{N} \leq \frac{M}{\xi} + \left(\mathbf{N}(0) - \frac{M}{\xi}\right)e^{-\xi t}$$

Therefore the positive solutions of the proposed system are ultimately bounded.

4. Equilibria and stability analysis

4.1. Equilibrium points

To study the stability analysis of the system (1), the equilibrium points of the respective system (1) are necessary to calculate. Now, the equilibrium points are given below:

(i) The trivial equilibrium point $E_0 = (0,0,0,0)$.
(ii) The predator free equilibrium point $E_1 = (\overline{X}, \overline{Y}, 0, 0)$. Where $\overline{X} = \frac{r}{c_1(d_1+b_1)}\left\{\frac{b_1r}{d_1+b_1} - d_2\right\}$ and $\overline{Y} = \frac{1}{c_1}\left(\frac{b_1r}{d_1+b_1} - d_2\right)$.
(iii) The interior equilibrium point $E^* = (X^*, Y^*, Z^*, W^*)$ where

$$X^* = \frac{1}{b_1}\left\{d_2Y^* + c_1Y^{*2} + \frac{m_2Y^*W^*}{k_2+k_3Y^{*2}}\right\}$$
$$Y^* = \frac{1}{r}\left\{(d_1+b_1)X^* + \frac{m_1X^*W^*}{k_1+X^*}\right\}$$
$$Z^* = \frac{1}{n+d_3}\left\{\frac{\alpha m_1X^*W^*}{k_1+X^*} + Y^*W^*\left(\frac{\beta m_2}{k_2+k_3Y^{*2}} - \eta\right)\right\}$$
$$W^* = \frac{n}{d_4}Z^*$$

4.2. Stability analysis

The stability of the proposed system is determined by the nature of the eigenvalues of the corresponding Jacobian matrix around the point (X, Y, Z, W).

Theorem 2. *If $d_2(d_1+b_1) < k_2^2rb_1$ holds, then the system is unstable around the trivial equilibrium point E_0; otherwise the system will be locally asymptotically stable.*

Proof. The Jacobian evaluated at the trivial equilibrium point $E_0 = (0,0,0,0)$ is given by

$$[J]_{E_0} = \begin{pmatrix} -(d_1+b_1) & r & 0 & 0 \\ b_1 & -\dfrac{d_2}{k_2^2} & 0 & 0 \\ 0 & 0 & -(n+d_3) & 0 \\ 0 & 0 & n & -d_4 \end{pmatrix}$$

So, the characteristic equation is

$$\{(n+d_3)+\lambda\}\{d_4+\lambda\}\left\{\lambda^2 + \lambda\left(d_1+b_1+\frac{d_2}{k_2^2}\right) + \frac{d_2}{k_2^2}(d_1+b_1) - rb_1\right\} = 0$$

It is clear from the above characteristic equation that two eigenvalues are negative. Again, from the last expression of the above characteristic equation we must say that the other two eigenvalues should be positive if $-\left(d_1+b_1+\frac{d_2}{k_2^2}\right)>0$, which is not possible because d_1, b_1, d_2, k_2 all are positive.

So, from the last expression of the above characteristic equation we say that one eigenvalue will be positive and other eigenvalue will be negative if $\frac{d_2}{k_2^2}(d_1+b_1) - rb_1 < 0$ i.e., $d_2(d_1+b_1) < k_2^2rb_1$.

Theorem 3. *The predator-free equilibrium point $E_1 = (\overline{X}, \overline{Y}, 0, 0)$ is locally asymptotically stable when $\overline{Y} > \frac{1}{2c_1}\left(\frac{rb_1}{d_1+b_1} - d_2\right)$* and $0 < P_4 < d_4 + \frac{d_3d_4}{n}$, *where $P_4 = \frac{\alpha m_1\overline{X}}{k_1+\overline{X}} + \frac{\beta m_2\overline{Y}}{k_2+k_3\overline{Y}^2} - \eta\overline{Y}$, otherwise unstable.*

Proof. The corresponding Jacobian matrix at E_1 is given by

$$[J]_{E_1} = \begin{pmatrix} -(d_1+b_1) & r & 0 & -P_1 \\ b_1 & -P_2 & 0 & -P_3 \\ 0 & 0 & -(n+d_3) & P_4 \\ 0 & 0 & n & -d_4 \end{pmatrix}$$

$$where, P_1 = \frac{m_1\overline{X}}{k_1+\overline{X}}$$
$$P_2 = 2c_1\overline{Y} + d_2$$
$$P_3 = \frac{m_2\overline{Y}}{k_2+k_3\overline{Y}^2}$$
$$P_4 = \frac{\alpha m_1\overline{X}}{k_1+\overline{X}} + \frac{\beta m_2\overline{Y}}{k_2+k_3\overline{Y}^2} - \eta\overline{Y}$$

So, the characteristic equation of the above Jacobian matrix is

$$\left[\lambda^2 + (n+d_3+d_4)\lambda + \{d_4(n+d_3) - nP_4\}\right]\left[\lambda^2 + P_2(d_1+b_1)\lambda + \{P_2(d_1+b_1) - rb_1\}\right] = 0$$

Now, from the first expression of the above characteristic equation we say that the expression should have two negative eigenvalues if $P_4 > 0$ and $d_4(n+d_3) - nP_4 > 0$ i.e., $P_4 < d_4 + \frac{d_3d_4}{n}$. So, combining the two restrictions we have $0 < P_4 < d_4 + \frac{d_3d_4}{n}$.

Also, from the second expression of the above characteristic equation we say that the expression should have two negative eigenvalues if

$P_2(d_1+b_1)-rb_1>0$ i.e., $P_2>\frac{rb_1}{d_1+b_1}$.. Which imply that $\overline{Y}>\frac{1}{2c_1}\left(\frac{rb_1}{d_1+b_1}-d_2\right)$.

Theorem 4. *The interior equilibrium point $E^*=(X^*,Y^*,Z^*,W^*)$ is locally asymptotically stable if $A>0, C>0, D>0$ and $ABC>C^2+A^2D$, otherwise unstable.*

Proof. The corresponding Jacobian matrix at the interior equilibrium point E^* is given by

$$[J]_{E^*}=\begin{pmatrix}-Q_1 & r & 0 & -Q_2\\ b_1 & -Q_3 & 0 & -Q_4\\ Q_5 & Q_6 & -Q_7 & Q_8\\ 0 & 0 & n & -d_4\end{pmatrix}$$

where, $Q_1=(d_1+b_1)+\dfrac{m_1k_1W^*}{(k_1+X^{*2})}$

$$Q_2=\frac{m_1X^*}{k_1+X^*}$$

$$Q_3=2c_1Y^*+d_2+\frac{m_2W^*(k_2-k_3Y^{*2})}{(k_2+k_3Y^{*2})^2}$$

$$Q_4=\frac{m_2Y^*}{(k_2+k_3Y^{*2})}$$

$$Q_5=\frac{\alpha m_1k_1W^*}{(k_1+X^{*2})}$$

$$Q_6=\frac{\beta m_2W^*(k_2-k_3Y^{*2})}{(k_2+k_3Y^{*2})^2}-\eta W^*$$

$$Q_7=(n+d_3)$$

$$Q_8=\frac{\alpha m_1X^*}{k_1+X^*}+\frac{\beta m_2Y^*}{(k_2+k_3Y^{*2})}-\eta Y^*$$

So, the corresponding characteristic equation of the above Jacobian matrix is

$$\lambda^4+A\lambda^3+B\lambda^2+C\lambda+D=0 \tag{3}$$

where, $A=Q_1+Q_3+Q_7+d_4+nB=\{(d_4+n)Q_7+(Q_1+Q_3)(d_4+n+Q_7)+Q_1Q_3+rb_1\}C=\{Q_7(Q_1+Q_3)(d_4+n)+Q_1Q_3(d_4+n+Q_7)+rb_1(d_4+Q_7)\ +nQ_4Q_6+nQ_2Q_5-rnb_1\}D=Q_1Q_3Q_7(d_4+n)+nQ_1Q_4Q_6+nQ_2Q_3Q_5+rnQ_4Q_5+b_1nQ_2Q_6+rb_1Q_7(d_4-n)$

Hence, by Routh-Hurwitz criterion, the equation (3) have all negative roots or roots have negative real part if and only if $A>0, C>0, D>0$ and $ABC>C^2+A^2D$, otherwise the system will be unstable around the interior equilibrium point.

5. Hopf bifurcation

The aim of this section is to investigate the Hopf bifurcation analysis of our proposed system (1) around the non-negative interior equilibrium point. Here we have considered the transition rate b_1 (rate of transition from juvenile prey to adult prey), η (coefficient of anti-predator behavior), r (birth rate of juvenile prey), m_2 (attack rate of adult predator to adult prey) and n (transition rate from juvenile predator to adult predator) as a bifurcation parameter. In the following theorem we show that the transition rate b_1 exceeds the critical value $b_1=b_1^*$, the System (1) undergoes Hopf bifurcation around the positive interior equilibrium point.

Theorem 5. *In system* (1)*, the non-negative interior equilibrium point $E^*(X^*(b_1),Y^*(b_1),Z^*(b_1),W^*(b_1))$ enters into Hopf bifurcation as b_1 varies over $\mathbf{R}$.*

Suppose $\psi:(0,\infty)\to\mathbf{R}$ be the following continuously differential function of b_1

$$\psi(b_1)=A(b_1)B(b_1)C(b_1)-C^2(b_1)-D(b_1)A^2(b_1)$$

Suppose, b_1^* be a positive root of the equation $\psi(b_1)=0$.

Hence, the Hopf-bifurcation of the interior equilibrium E^* occurs at $b_1=b_1^*\in(0,\infty)$ iff

$$\psi(b_1^*)=0$$

$$A^2(AD'-B'C)-(AB-2C)(AC'-A'C)\neq 0.$$

and all other eigenvalues are of negative real parts, where $\sigma(b_1)$ is purely imaginary at $b_1=b_1^*$.

Proof. With the help of the condition $\psi(b_1^*)=0$, The characteristic equation can be written as :

$$\left(\lambda^2+\frac{C}{A}\right)\left(\lambda^2+A\lambda+\frac{AD}{C}\right)=0$$

The roots of the above equation are $\sigma_i, (i=1,2,3,4)$. Suppose, the pair of purely imaginary roots at $b_1=b_1^*$ are σ_1 and σ_2 then we have

$$\sigma_3+\sigma_4=-A \tag{4}$$

$$\omega_0^2+\sigma_3\sigma_4=B \tag{5}$$

$$\omega_0^2(\sigma_3+\sigma_4)=-C \tag{6}$$

$$\omega_0^2\sigma_3\sigma_4=D \tag{7}$$

where $\omega_0=\mathrm{Im}\sigma_1(b_1^*)$. With the help of above $\omega_0=\sqrt{\frac{C}{A}}$. Now, if σ_3 and σ_4 are complex conjugate, then from (4), it follows that $2Re\sigma_3=-A$; if they are real roots then by (5) and (6) we conclude that $\sigma_3<0$ and $\sigma_4<0$.

Now, we can verify the transversality condition

$$\left[\frac{dRe(\sigma_j(b_1))}{db_1}\right]_{b=b_1^*}\neq 0, j=3,4.$$

Due to the property of continuity of the roots of $\psi(b_1^*)$, there exists an open interval $(b_1^*-\varepsilon, b_1^*+\varepsilon)$ for some positive ε. Thus for $b_1\in(b_1^*-\varepsilon, b_1^*+\varepsilon)$, the characteristic equation (3) has no roots whose real parts are negative. Let σ_3 and σ_4 are complex conjugate for b_1. Suppose their general forms in this neighborhood are

$$\sigma_3(b_1)=\phi_1(b_1)+i\phi_2(b_1)$$

$$\sigma_4(b_1)=\phi_1(b_1)-i\phi_2(b_1)$$

To verify the transversality condition $\left[\frac{dRe(\sigma_j(b_1))}{db_1}\right]_{b=b_1^*}\neq 0, j=3,4$, substituting $\sigma_j(b_1)=\phi_1(b_1)\pm i\phi_2(b_1)$, into the characteristic equation (3) and calculating the derivative, we have

$$N_1(b_1)\phi_1'(b_1)-N_2(b_1)\phi_2'(b_1)+N_3(b_1)=0,$$

$$N_2(b_1)\phi_1'(b_1)+N_1(b_1)\phi_2'(b_1)+N_4(b_1)=0.$$

where,

$$\begin{aligned}N_1(b_1)&=4\phi_1^3(b_1)-12\phi_1(b_1)\phi_2^2(b_1)+3A(b_1)\left(\phi_1^2(b_1)-\phi_2^2(b_1)\right)\\&\quad+2B(b_1)\phi_1(b_1)+C(b_1),\\ N_2(b_1)&=12\phi_1^2(b_1)\phi_2(b_1)-4\phi_2^3(b_1)+6A(b_1)\phi_1(b_1)\phi_2(b_1)+2B(b_1)\phi_2(b_1),\\ N_3(b_1)&=\left(\phi_1^3(b_1)-3\phi_1(b_1)\phi_2^2(b_1)\right)A'(b_1)+\left(\phi_1^2(b_1)-\phi_2^2(b_1)\right)B'(b_1)\\&\quad+\phi_1(b_1)C'(b_1)+D'(b_1),\\ N_4(b_1)&=\left(3\phi_1^2(b_1)\phi_2(b_1)-\phi_2^3(b_1)\right)A'(b_1)+2\phi_1(b_1)\phi_2(b_1)B'(b_1)\\&\quad+\phi_2(b_1)C'(b_1).\end{aligned}$$

Noticing that $\phi_1(b_1^*) = 0$ and $\phi_2(b_1^*) = \sqrt{\frac{C(b_1^*)}{A(b_1^*)}}$, for $b_1 = b_1^*$, then we have $N_1(b_1^*) = -2C, N_2(b_1^*) = 2\sqrt{\frac{C}{A}}\left(B - \frac{2C}{A}\right), N_3(b_1^*) = D' - \frac{B'C}{A}, N_2(b_1^*) = 2\sqrt{\frac{C}{A}}\left(B - \frac{2C}{A}\right), N_3(b_1^*) = D' - \frac{B'C}{A}$, and $N_4(b_1^*) = \sqrt{\frac{C}{A}}\left(C' - \frac{A'C}{A}\right)$

Solving for $\phi_{1'}(b_1)$ at $b_1 = b_1^*$ we have,

$$Now, \left[\frac{dRe(\sigma_j(b_1))}{db_1}\right]_{b=b_1^*} = \phi_{1'}(b_1^*) = -\frac{N_2(b_1^*)N_4(b_1^*) + N_1(b_1^*)N_3(b_1^*)}{N_1^2(b_1^*) + N_2^2(b_1^*)}$$
$$= \frac{2\frac{C}{A}\left(\frac{AB-2C}{A}\right)\left(\frac{AC'-A'C}{A}\right)}{4C^2 + \frac{4C}{A}\frac{(AB-2C)^2}{A^2}}$$
$$= \frac{A^2(AD' - B'C) - (AB - 2C)(AC' - A'C)}{2A^3C + 2(AB - 2C)^2} \neq 0$$

if $A^2(AD' - B'C) - (AB - 2C)(AC' - A'C) \neq 0$

Therefore, the transversality condition hold and hence Hopf bifurcation occurs at $b_1 = b_1^*$.

6. Numerical simulation

In this section, the dynamical behavior of the proposed model (1) has been discussed numerically with the help of MATLAB software. Due to unavailability of real data of all parameters related to the model, the suppositional values of the different parameters have been considered as follows:

$$r = 1.6, d_1 = 0.1, b_1 = 0.6, c_1 = 0.6, m_1 = 0.6, k_1 = 0.01, d_2 = 0.06, \alpha = 0.6, n = 0.09, d_3 = 0.4, d_4 = 0.05, m_2 = 0.8, k_2 = 0.01, k_3 = 0.05, \beta = 1.1, \eta = 0.1.$$

Now, considering the above data, we have calculated $A = 21.4268 > 0, C = 40.2788 > 0, D = 21.6474 > 0$, and $ABC > C^2 + A^2D$. So, according to the Theorem 4, the positive interior equilibrium point $E^* = (X^*, Y^*, Z^*, W^*) = (0.0149, 0.0404, 0.1266, 0.2278)$ is locally asymptotically stable. Also, Fig. 1 shows a positive locally asymptotically stable point $E^* = (0.0149, 0.0404, 0.1266, 0.2278)$ and from that figure, it can be easily conclude that the juvenile prey, adult prey, juvenile predator and adult predator population coexist simultaneously.

Now, the dynamical behavior of our proposed model (1) is discussed by considering the transition rate from juvenile prey to adult prey (b_1) as a bifurcation parameter. Due to Hopf bifurcation, there is some periodic oscillations which is shown in Fig. 2 by considering b_1 as a bifurcating parameter. Considering the following set of parametric values $r = 2.1, d_1 = 0.1, c_1 = 0.6, m_1 = 0.6, k_1 = 0.01, d_2 = 0.06, \alpha = 0.6, n = 0.09, d_3 = 0.4, d_4 = 0.05, m_2 = 0.8, k_2 = 0.01, k_3 = 0.05, \beta = 0.4, \eta = 0.1$, we have calculated the critical value of the transition rate from juvenile prey to adult prey (b_1) as 1.28. From this figure, it is clearly observed that, if we change b_1 from 1.25 to 1.28 then, the system (1) remains stable but the system shows oscillatory behavior for $b_1 > 1.28$.. To show the oscillation in Fig. 2, the bold black circular points indicates the global minimum and global maximum point of periodic solutions to show the oscillatory behavior of our proposed model (1) when the bifurcating parameter (b_1) crosses it's threshold value. From this figure, it can be observed that if the transition rate from juvenile prey to adult prey (b_1) crosses it's critical value $(= 1.28)$, then the system (1) becomes unstable.

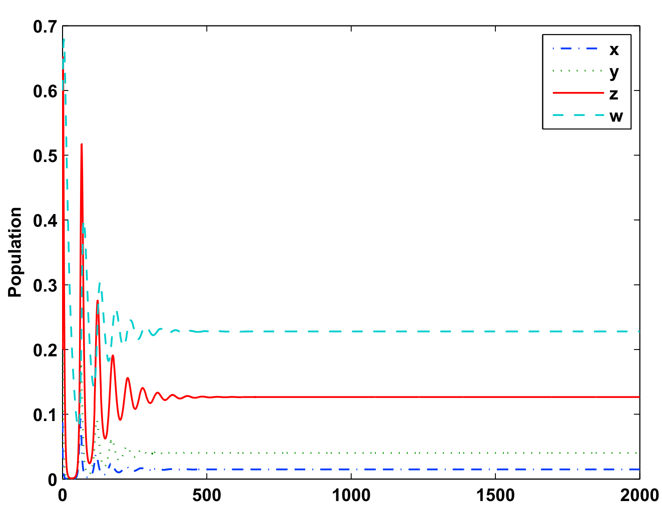

Fig. 1. Stability of the interior equilibrium of the system (1).

Now, we discuss the dynamical behavior of our proposed model (1) by considering b_1 as a bifurcation parameter, the following parametric values have been taken as:

$$r = 2.1, d_1 = 0.1, c_1 = 0.6, m_1 = 0.6, k_1 = 0.01, d_2 = 0.06, \alpha = 0.6, n = 0.09, d_3 = 0.4, d_4 = 0.05, m_2 = 0.8, k_2 = 0.01, k_3 = 0.05, \beta = 0.4, \eta = 0.1.$$

For this set of parametric values we have calculated the critical value of the transition rate from juvenile prey to adult prey (b_1) as 1.28. From the Fig. 3, it is clearly observed that if we change b_1 from 1.25 to 1.28 then, the system (1) remains stable in $1.25 \leq b_1 \leq 1.28$ and the system shows oscillatory behavior for $b_1 > 1.28$. From this figure, it can be concluded that the transition rate from juvenile prey to adult prey (b_1) has a significant role to stabilize the system.

Again, we plot the bifurcation diagram of the proposed system (1) with respect to the coefficient of anti-predator behavior (η) on the evolution of the juvenile predator and adult predator which is shown in the Fig. 4. Using the following set of parameter values:

$$r = 3.8, d_1 = 0.1, b_1 = 0.6, c_1 = 0.6, m_1 = 1.2, k_1 = 0.01, d_2 = 0.06, \alpha = 0.6, n = 0.09, d_3 = 0.4, d_4 = 0.05, m_2 = 0.8, k_2 = 0.01, k_3 = 0.05, \beta = 0.4,$$

the bifurcation diagram of the system (1) is plotted as a function of the control parameter $\eta \in (0.0, 0.2)$. From this figure, it is seen that the system (1) provides stable solution when $\eta = \eta^*(> 0.07)$ and the system shows unstable solution within the interval $0 < \eta < 0.07$. From this figure, it can be concluded that the anti-predator behavior (η) plays a vital role on the stability of our proposed system.

Now, to discuss the dynamical behavior of our proposed model (1) by considering birth rate r of juvenile prey as a bifurcation parameter, the following parametric values have been considered as:

$$d_1 = 0.1, b_1 = 0.6, c_1 = 0.6, m_1 = 0.6, k_1 = 0.01, d_2 = 0.06, \alpha = 0.6, n = 0.09, d_3 = 0.4, d_4 = 0.05, m_2 = 0.8, k_2 = 0.01, k_3 = 0.05, \beta = 1.1, \eta = 0.1$$

Using this set of values, the bifurcation diagram of the system (1) is plotted as a function of the control parameter $r \in (3.0, 4.5)$ which is shown in Fig. 5. From this figure, it is observed that the system (1) provides stable solution when $3.0 < r < 3.45$ and the system exhibits unstable solution when $r > r^*(= 3.45)$ due to Hopf bifurcation of the proposed system. From this figure, it can be concluded that the birth rate of juvenile prey (r) has an important effect on the stability of the system (1).

Considering the following set of parametric values:

$$r = 3.8, d_1 = 0.1, b_1 = 0.6, c_1 = 0.6, m_1 = 0.8, k_1 = 0.01, d_2 = 0.06, \alpha = 0.6, n = 0.02, d_3 = 0.4, d_4 = 0.05, k_2 = 0.01, k_3 = 0.05, \beta = 0.4, \eta = 0.3,$$

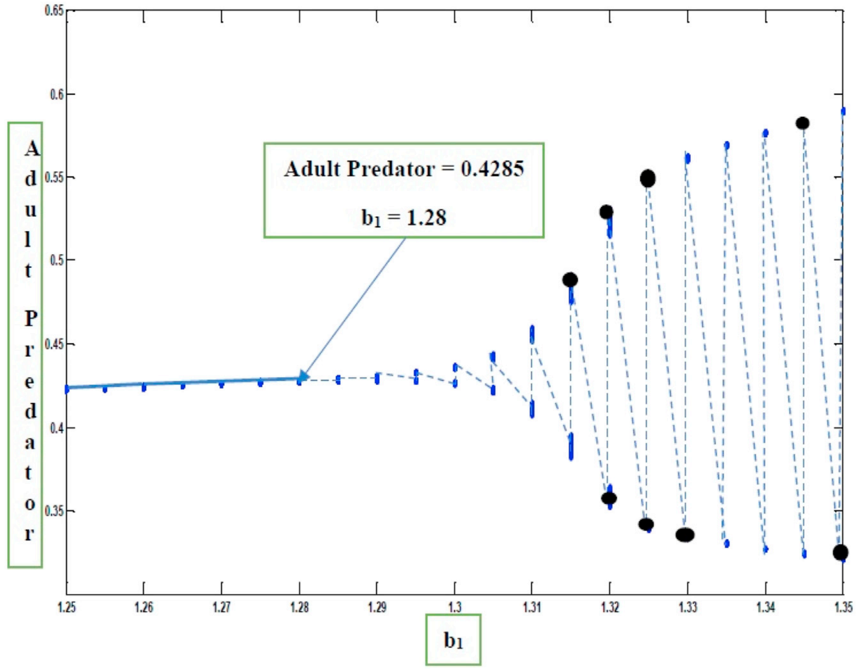

Fig. 2. Periodic oscillation of Adult predator population with respect to b_1.

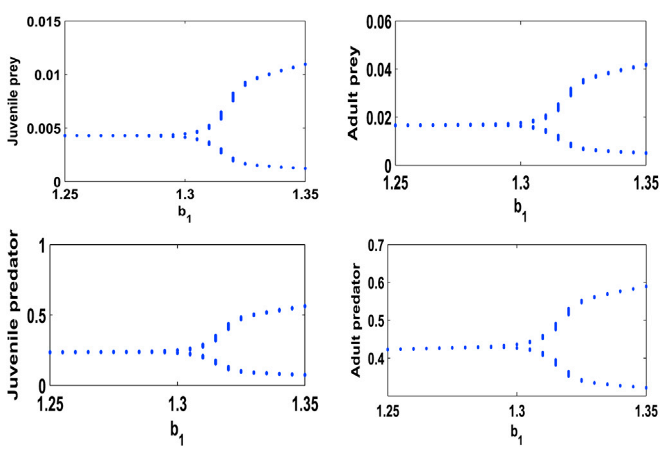

Fig. 3. Bifurcation diagram of system (1) for b_1.

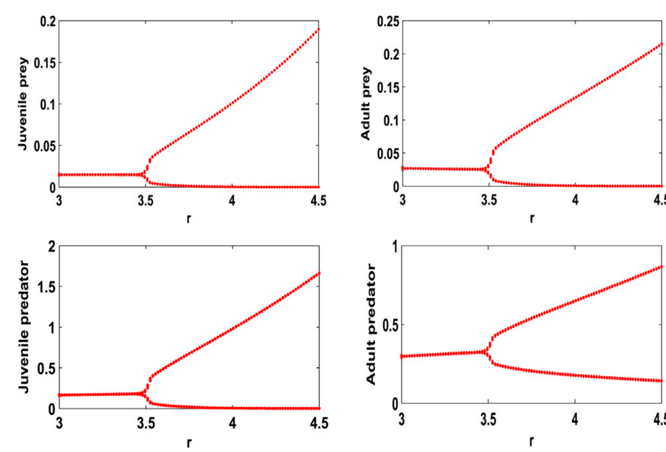

Fig. 5. Bifurcation diagram of system (1) for r.

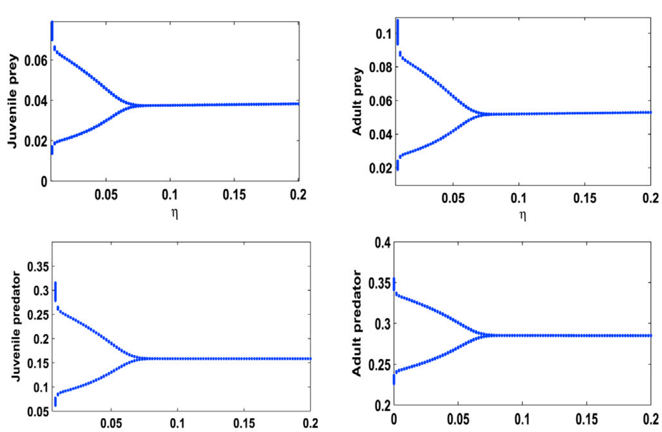

Fig. 4. Bifurcation diagram of system (1) for η.

the bifurcation diagram of our proposed system has been plotted as a function of the control parameter $m_2 \in (1.1, 1.3)$ in Fig. 6. From this figure, it is seen that the system (1) shows the stable solution within $1.1 < m_2 < 1.165$ and unstable solution when $m_2 > 1.165 (= m_2^*)$. Also, from this figure it can be concluded that the attack rate (m_2) of adult predator to adult prey plays a vital role on the stability of our proposed system (1).

To discuss the dynamical behavior of our proposed model (1) by considering transition rate (n), from juvenile predator to adult predator as a bifurcation parameter. Using the following set of parametric values:

$$r = 3.8, d_1 = 0.1, b_1 = 0.6, c_1 = 0.6, m_1 = 1.2, k_1 = 0.01, d_2 = 0.06, \alpha = 0.6, d_3 = 0.4, d_4 = 0.05, m_2 = 0.8, k_2 = 0.01, k_3 = 0.05, \beta = 0.4, \eta = 0.3$$

the bifurcation diagram of the system (1) is plotted as a function of the control parameter $n \in (0.1, 0.4)$ in Fig. 7. From this figure, it is observed

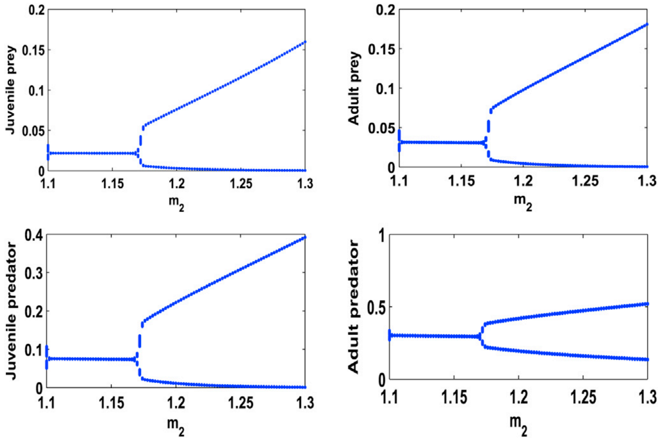

Fig. 6. Bifurcation diagram of system (1) for m_2.

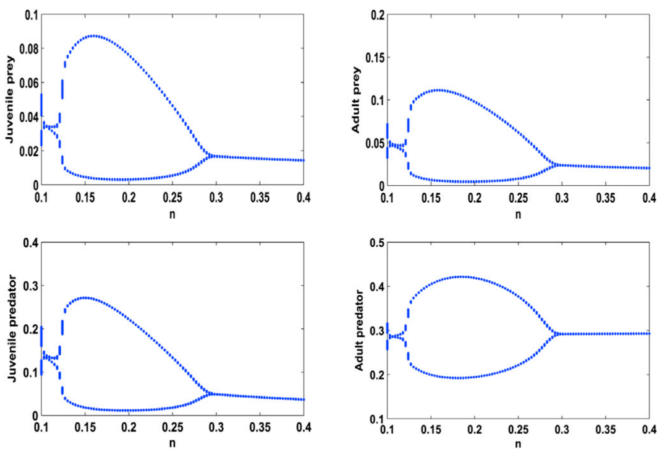

Fig. 7. Bifurcation diagram of system (1) for n.

that the system undergoes a unstable solution within the interval $0.1 < n < 0.29$ and stable solution when $n > 0.29(= n^*)$. Also, it can be concluded that the transition rate (n) has a major role to the stability of our proposed system (1).

7. Conclusion

In order to understand the dynamics of stage-structured prey-predator interaction, we have formulated and analyzed a predator-prey model in which the stage-structure have been considered on both prey as well as predator population. For the reason of stage-structure, the prey population has been classified into two sub populations such as juvenile prey and adult prey, on the other hand the predator population is also classified into two sub populations such as juvenile predator and adult predator. In this model, it is assumed that only the adult predator has the ability of predation and the adult predator consumes both juvenile prey and adult prey. As the adult prey have the capability of counterattack by grouping to their predator, so the predation term of adult predator to adult prey follows Holling type-IV response function, which include group defense. Also, it is assumed that the predation term of adult predator to juvenile prey follows Holling type-II functional response. Then, the positivity and boundedness of all solutions of the proposed system (1) has been discussed. After that, the different equilibrium points have been calculated and local stability of the system (1) around these equilibrium points have been analyzed. From local stability, it is perceived that the stability of trivial equilibrium point depends on $r, b_1, d_1,$ and d_2 (See Theorem 2). The predator free equilibrium points mainly depends on the transition rate (b_1) from juvenile prey to adult prey, death rate of juvenile prey (d_1), death rate of adult prey (d_2) and the transition rate (n) from juvenile predator to adult predator (See Theorem 3).

From the numerical simulation, it is observed that the system will be stable if $b_1 < b_1^*(= 1.28)$ and unstable if $b_1 > b_1^*(= 1.28)$. Similarly, if the coefficient of anti-predator behavior is $\eta > \eta^*(= 0.07)$ then the system (1) is stable and the system (1) undergoes a Hopf bifurcation at $0 < \eta < 0.07$. Again, we clearly observe that the system (1) becomes stable if $r < r^*(= 3.45)$ and the system (1) goes through a Hopf bifurcation if $r > r^*(= 3.45)$. Also, if the attack rate (m_2) of adult predator to adult prey is $m_2 < m_2^*(= 1.165)$, then the system (1) becomes stable and the system shows periodic solution through Hopf bifurcation if $m_2 > m_2^*(= 1.165)$. Our proposed system (1) undergoes a Hopf bifurcation if the transition rate (n) from juvenile predator to adult predator is $0.1 < n < 0.29$ and the system (1) provides stable solution if $n = n^*(> 0.29)$. From the above presented numerical result, it can be concluded that the parameters $b_1, \eta, r, m_2,$ and n has significant role to the system stability.

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