MA581 HW1

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1.18

a) To determine $\bigcap_{n=1}^{4} A_n$, we can start from what we know so far. We know that $A_n = [0, \frac{1}{n}]$. Then, we can get that:

- For $n = 1$, $A_1 = [0, 1]$
- For $n = 2$, $A_2 = [0, \frac{1}{2}]$
- For $n = 3$, $A_3 = [0, \frac{1}{3}]$
- For $n = 4$, $A_4 = [0, \frac{1}{4}]$

So, $\bigcap_{n=1}^{4} A_n = [0, \frac{1}{4}]$

To determine $\bigcup_{n=1}^{4} A_n$, We also can start from $A_n = [0, \frac{1}{n}]$.

- For $n = 1$, $A_1 = [0, 1]$
- For $n = 2$, $A_2 = [0, \frac{1}{2}]$
- For $n = 3$, $A_3 = [0, \frac{1}{3}]$
- For $n = 4$, $A_4 = [0, \frac{1}{4}]$

So, $\bigcup_{n=1}^{4} A_n = [0, 1]$

b) Given that $A_n = [0, \frac{1}{n}]$, we need to find the intersection of all such intervals as n ranges over the natural numbers $\mathbb{N}$.

- For $n = 1$, $A_1 = [0, 1]$.
- For $n = 2$, $A_2 = [0, \frac{1}{2}]$.
- And so on, as n increases, the interval $[0, \frac{1}{n}]$ gets smaller, approaching 0.

The intersection of all these sets A_n is the set of points that are common to all intervals $[0, \frac{1}{n}]$. The only number that is common to all these intervals, as n goes to infinity, is 0. Therefore, the intersection of all A_n is the set containing just 0, i.e., $\bigcap_{n=1}^{\infty} A_n = \{0\}$.

To find $\bigcup_{n=1}^{\infty} A_n$ as n ranges over the natural numbers $\mathbb{N}$, consider the following:

- Each interval starts from 0 and extends up to $\frac{1}{n}$, with n being any natural number.
- As n increases, the upper limit of these intervals, $\frac{1}{n}$, decreases, but the smallest interval is when $n = 1$, which is $[0, 1]$.

Since the intervals include all numbers from 0 to $\frac{1}{n}$ for any n in the natural numbers, and since these intervals overlap, the union of all these intervals A_n will cover every number from 0 up to 1. Therefore, the union of all A_n is $[0, 1]$, i.e., $\bigcup_{n=1}^{\infty} A_n = [0, 1]$.

1.31

a) To verify the statement that $A = (A \cap B) \cup (A \cap B^c)$, consider the following:

1. The intersection $A \cap B$ includes all elements common to both A and B.
2. The complement B^c includes all elements not in B.
3. Therefore, $A \cap B^c$ includes elements in A but not in B.
4. By union, $(A \cap B) \cup (A \cap B^c)$ includes all elements in A, either in B or not in B.

Hence, we conclude that $A = (A \cap B) \cup (A \cap B^c)$, covering all elements in A.

b) Given that $A \cap B = \varnothing$, we understand this to mean that there are no elements common to both sets A and B. This is the definition of disjoint sets. From this premise, we conclude the following:

1. Since A and B share no elements, every element of A must not be present in B. This is directly implied by the definition of disjoint sets.
2. Considering the universal set U, which contains all elements under consideration, the complement of B (B^c) consists of all elements not in B.
3. Therefore, all elements of A must belong to B^c, as they cannot be in B. This leads us to conclude $A \subset B^c$.

In formal notation, we express this as:

$$A \cap B = \varnothing \implies A \subset B^c$$

This logical deduction confirms that if A and B are disjoint, then A must be a subset of the complement of B, denoted as B^c.

c) Given the statement $A \subset B \implies B^c \subset A^c$, we analyze it as follows:

1. The assumption $A \subset B$ implies that all elements of A are also elements of B, though B may contain additional elements.
2. The complement of B (B^c) is defined as $U - B$, which includes all elements not in B.
3. Similarly, the complement of A (A^c) is $U - A$, encompassing all elements not in A. Given that A is a subset of B, it follows that A^c includes everything in B^c and possibly more, as A^c would also include elements exclusive to B.
4. Thus, we conclude that $B^c \subset A^c$, as every element not in B is also not in A.

In formal notation, we express this as:

$$A \subset B \implies B^c \subset A^c$$

1.32

a) *Proof.* Let x be an arbitrary element. We will show that x belongs to the left-hand side if and only if x belongs to the right-hand side.

Left to right: If $x \in (\bigcap_n A_n)^c$, this means that $x \notin \bigcap_n A_n$, which means there exists at least one n such that $x \notin A_n$. Therefore, $x \in A_n^c$ for some n, which means that $x \in \bigcup_n A_n^c$

Right to left: Assume that $x \in \bigcup_n A_n^c$, this means that there exists at least one n such that $x \notin A_n$. Therefore, x cannot be in every A_n, which implies that $x \notin \bigcap_n A_n$. Thus, $x \in (\bigcap_n A_n)^c$.

□

b) *Proof.* Again, let x be an arbitrary element. We will prove that x belongs to the left-hand side if and only if x belongs to the right-hand side.

Left to right: Assume $x \in (\bigcup_n A_n)^c$. This means that $\forall x \notin \bigcup_n A_n$, which implies that for all n, $x \notin A_n$. Therefore, $x \in A_n^c$ for all n, which means $x \in \bigcap_n A_n^c$

Right to left: Assume $x \in \bigcap_n A_n^c$. This means for all n, $x \in A_n^c$, or equivalently, $x \notin A_n$. Therefore, x cannot be in any A_n, implying $x \notin \bigcup_n A_n$. Thus, $x \in (\bigcup_n A_n^c)$.

□

2.11

a) In a random experiment where 12 jurors are chosen from a pool of 18 individuals (10 men and 8 women), we are interested in determining the size of the sample space. The sample space consists of all possible ways to select 12 jurors out of the 18 individuals available.

To calculate the size of the sample space, we use the combination formula given by:

$$C(n,k) = \frac{n!}{k!(n-k)!}$$

where n is the total number of individuals available for selection, k is the number of individuals to be selected, and $n!$ denotes the factorial of n.

For our specific case:

$$C(18,12) = \frac{18!}{12!(18-12)!}$$

Plugging the values into the formula, we get:

$$C(18,12) = \frac{18!}{12! \cdot 6!} = 18,564$$

Therefore, the sample space for this random experiment contains 18,564 possible outcomes, which represents all the ways in which 12 jurors can be selected from a pool of 10 men and 8 women.

b) Here, we use M for men and W for women.
Event A stands for at least half of the 12 jurors are men, then

$$A = \{(6M, 6W), (7M, 5W), (8M, 4W), (9M, 3W), (10M, 2W)\}$$

We can get the same thing for event B which is at least half of the 8 women are on the jury

$$B = \{(4W, 8M), (5W, 7M), (6W, 6M), (7W, 5M), (8W, 4M)\}$$

Then, for $A \cap B$ it means that at least half of the jurors are men and at least half of the women are on the jury. Then

$$A \cap B = \{(6M, 6W), (7M, 5M), (8M, 4W)\}$$

$A \cup B$ stands for the event that at least half of the jurors are men or at least half of the women are on the jury. Then,

$$A \cup B = \{(6M, 6W), (7M, 5W), (8M, 4W), (9M, 3W), (10M, 2W), (4W, 8M)\}$$

$A \cap B^c$ stands for at least half of the jurors are men and fewer than half of the women are on the jury. Then,we can get that

$$A \cap B^c = \{(9M, 3W), (10M, 2W)\}$$

c) For the first two parts, A and B are not mutually exclusive since their interception are non-empty. A and B^c is also not mutually exclusive since their set is also non-empty. A^c and B^c will be mutually exclusive because if A^c occur, it implies B^c must occur. In other words, if A^c is ture, then B^c is necessarily true, and vice versa.