Mathematical Preliminaries

Elements of Linear Algebra and Calculus

EE4375 - FEM For EE Applications

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In These Slides

- Elements of linear algebra for FDM and FEM
- Elements of calculus of functions in one variable for FDM
- Elements of calculus of functions in several variables for FDM
- Elements of calculus of functions in one variable for FEM
- Elements of calculus of functions in several variables for FEM

FDM: finite difference method - FEM: finite element method

Elements of linear algebra for FDM and FEM

In This Section

- vectors, representation of vectors, transpose of vectors, element-wise operations on vectors, equality of vectors, linear combinations of vectors, magnitude of vectors, inner product of vectors;
- matrices, representation of matrices, transpose of matrices, equality of matrices, linear combinations of matrices;
- matrix-vector multiplication (order of symbols matters)
- linear system solve
- (non-linear system solve and time-stepping)

Elements of linear algebra for FDM and FEM

Vectors (1/2)

- $\mathbf{v} \in \mathbb{R}^n$: n-dimensional column vector: $\mathbf{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = (v_i)_{1 \leq i \leq n}$
- real-valued elements v_i (unless stated otherwise)
- $\mathbf{v}^T \in \mathbb{R}^n$: transpose of $\mathbf{v}$

  n-dimensional row vector: $\mathbf{v}^T = \begin{pmatrix} v_1 & \ldots & v_n \end{pmatrix}$
- element-wise operation on $\mathbf{v}$:

  form new vector by operation on each element of $\mathbf{v}$

  element-wise-operation($\mathbf{v}$) $=$ (operation(v_i))$_{1 \leq i \leq n}$

Elements of linear algebra for FDM and FEM

Vectors (2/2)

- equal to zero-vector

  $\mathbf{v} = \mathbf{0}_n \Leftrightarrow v_i = 0$ for $1 \leq i \leq n$ (all components are zero)

- linear combination of vectors

  $\alpha, \beta \in \mathbb{R}$ numbers and $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ vectors

  $\alpha\,\mathbf{v} + \beta\,\mathbf{w} = (\alpha\, v_i + \beta\, w_i)_{1 \leq i \leq n}$ (new vector)

- Euclidean norm - magnitude of vector - distance to zero

  $\|\mathbf{v}\|_2 = \sqrt{\sum_{i=1}^{n} v_i^2} \in \mathbb{R}$ (scalar)

- inner product of vectors of same size

  $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ vectors then $\mathbf{v} \cdot \mathbf{w} = \sum_{i=1}^{n} v_i\, w_i \in \mathbb{R}$ (scalar)

Elements of linear algebra for FDM and FEM

Matrices (1/2)

- $A \in \mathbb{R}^{m \times n}$: m-rows n-columns matrix: $A = \begin{pmatrix} A_{11} & \dots & A_{1n} \\ \vdots & \vdots & \vdots \\ A_{m1} & \dots & A_{mn} \end{pmatrix}$
- real-valued elements A_{ij} (unless stated otherwise)
- $A^T \in \mathbb{R}^{n \times m}$: transpose of A defined as $(A^T)_{ij} = A_{ji}$
- $n \neq m$: A is rectangular - $n = m$: A is square

Matrices (2/2)

- equal to zero-matrix

  $A = \mathbf{0}_{m\times n} \Leftrightarrow A_{ij} = 0$ for $1 \leq i \leq m$ and $1 \leq j \leq n$ (all cmpts zero)

- linear combination of matrices of same size

  $\alpha, \beta \in \mathbb{R}$ numbers and $A, B \in \mathbb{R}^{m\times n}$ matrices

  $(\alpha\, A + \beta\, B)_{ij} = \alpha\, A_{ij} + \beta\, B_{ij}$ for $1 \leq i \leq n$ and $1 \leq j \leq m$
  (new matrix)

- norm of a matrix: see references

Elements of linear algebra for FDM and FEM

Matrix-Vector Multiplication

- $\mathbf{v} \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n \times n}$ (A is square $m = n$)
- $\mathbf{w} = A\mathbf{v} \in \mathbb{R}^n$
- $w_i = \sum_{j=1}^{n} A_{ij}\, v_j$ (i fixed, j summation index)
- exercises: repeat for A rectangular, $\mathbf{v}^T A$ and $\mathbf{v}^T A\mathbf{v}$

Linear System Formulation

- given $\mathbf{f} \in \mathbb{R}^n$ and given $A \in \mathbb{R}^{n \times n}$ (A is again square)
- assume A is non-singular
- unknown $\mathbf{u} \in \mathbb{R}^n$
- find $\mathbf{u} \in \mathbb{R}^n$ by solving the linear system $\boxed{A\mathbf{u} = \mathbf{f}}$
- find $u_1, \dots, u_n$ such that for all $i \in \{1, \dots, n\}$ we have that

$$\sum_{j=1}^{n} A_{ij}\, u_j = f_i$$

- solution $\mathbf{u}$ exists and is unique (by construction)

Linear System Solve

- $A\mathbf{u} = \mathbf{f} \Leftrightarrow \begin{pmatrix} A_{11} & \dots & A_{1n} \\ \vdots & \vdots & \vdots \\ A_{n1} & \dots & A_{nn} \end{pmatrix} \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} = \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}$
- solve by LU-decomposition of A
- never compute A^{-1}! (add picture here)
- see assignments

Extensions

- data structures for sparse matrices (few non-zero elements only)
- iterative solution method for large sparse matrices

Recap

- given $\mathbf{f} \in \mathbb{R}^n$ and given $A \in \mathbb{R}^{n \times n}$
- find $\mathbf{u} \in \mathbb{R}^n$ such that $A\mathbf{u} = \mathbf{f}$
- using LU-decomposition of A

Elements of linear algebra for FDM and FEM

References

- bachelor courses in linear algebra

Further Reading

- https://en.wikipedia.org/wiki/Numerical_linear_algebra

Elements of calculus of functions in one var - FDM

Throughout This Section

- given $x \in \Omega = (0,1)$ interval or $0 < x < 1$
- $x = 0$ and $x = 1$ left and right boundary points
- $f(x)$: assumed known or given function with domain $x \in \Omega$

  assume given force or electrical charge distribution
- given α number
- $u(x)$: assumed unknown with domain $x \in \Omega$

  u is short for unknown - displacement or electrical potential
- $u'(x) = \dfrac{du}{dx}(x)$ and $u''(x) = \dfrac{d^2u}{dx^2}(x)$: short hand notation

Boundary Value Problem for Second Order Differential Equations

- given $x \in \Omega = (0, 1)$ with left and right boundary point
- given $f(x)$ with domain Ω
- find $u(x)$ such that $u''(x) = f(x)$ plus boundary conditions

Boundary Value Problem for Second Order Differential Equations

- given $x \in \Omega = (0,1)$ with left and right boundary point
- given: $f(x)$ given function and α given number
- find: $u(x)$ such that

$$-u''(x) = f(x) \text{ for } 0 < x < 1 \text{ (differential equation on } \Omega)$$
$$u(x=0) = 0 \text{ (Dirichlet boundary condition in } x = 0)$$
$$\frac{du}{dx}(x=1) = \alpha \text{ (Neumann boundary condition in } x = 1)$$

- differential equation not valid in $x = 0$ or $x = 1$

  boundary conditions are valid in $x = 0$ or $x = 1$

Exercise

- choose $f(x) = 1$
- find: $u(x)$ such that

$$-u''(x) = 1 \text{ for } 0 < x < 1 \text{ (differential equation on } \Omega)$$

$$u(x=0) = 0 \text{ (Dirichlet boundary condition in } x = 0)$$

$$\frac{du}{dx}(x=1) = \alpha \text{ (Neumann boundary condition in } x = 1)$$

- solve using pen and paper and plot computed solution for various values of α

Solution (1/2)

- by integration twice and using the boundary conditions

$$u(x) = -\frac{1}{2}\,x^2 + (1-\alpha)\,x$$

- solving by guessing and iteratively improving guess is allowed
- solution can be checking by checking differential equation and boundary conditions
- this examples is will guide remainder of the course

Solution (2/2)

- in example $f(x) = 1$
- for more general $f(x) = \text{whatever}(x)$ proceed as follows
- integrate using pen-and-paper: $u'(x) = \int^x f(\eta)\, d\eta + C_1$

  (here η is the integration variable)
- thus by replacing x by ξ: $u'(\xi) = \int^\xi f(\eta)\, d\eta + C_1$
- integrate once more

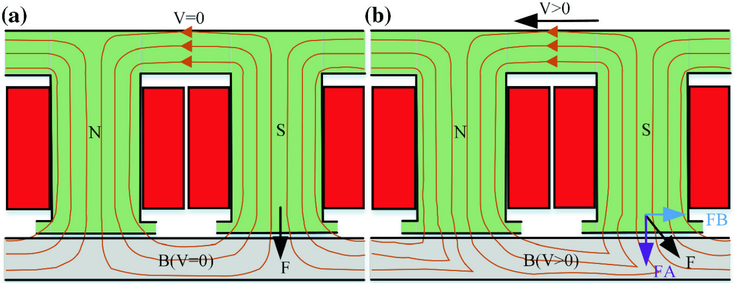
  $$u(x) = \int^x u'(\xi)\, d\xi = \int^x \int^\xi f(\eta)\, d\eta\, d\xi + C_1\, x + C_2$$

  with C_1 and C_2 from the boundary conditions

Elements of calculus of functions in one var - FDM

Example of Rotor/Stator Motion

(a) $-u''(x) = f(x)$ (b) $u''(x) + \beta\, u'(x) = f(x)$



Solve Using Available Tools

- find: $u(x)$ such that

$$-u''(x) = 1 \text{ plus boundary conditions}$$

- solve symbolically using function `dsolve()` in Sympy.jl
- solve numerically using collocation method or shooting method implemented in `BVProblem` in DifferentialEquations.jl
- important: know that methods exits and how to apply
- outside scope of course: details of the methods being methods

More Involved Models in 1D

- variable diffusion $c(x)$: $-\dfrac{d}{dx}\left[c(x)\dfrac{du(x)}{dx}\right] = f(x)$ + bc
- induced current effects: $u''(x) + \beta\, u(x) = f(x)$ + b.c.
- rotor/stator motion: $u''(x) + \beta\, u'(x) = f(x)$
- ferromagnetic saturation: $-\dfrac{d}{dx}\left[c[u'(x)]\dfrac{du(x)}{dx}\right] = f(x)$ + bc
- lots more ...
- no worries now - see later

Elements of calculus of functions in one var - FDM

From here on

Goodbye Analytics

Hello Numerics

Elements of calculus of functions in one var - FDM

First Order Derivative and Its Finite Difference Approximation

- definition: $$\frac{d\,u}{dx}(x = x_i) = \lim_{\Delta x \to 0} \frac{u(x_i + \Delta x) - u(x_i)}{\Delta x}$$
- forward difference: $$\frac{d\,u}{dx}(x = x_i) \approx \frac{u(x_i + \Delta x) - u(x_i)}{\Delta x}$$
- backward difference: $$\frac{d\,u}{dx}(x = x_i) \approx \frac{u(x_i) - u(x_i - \Delta x)}{\Delta x}$$
- central difference: $$\frac{d\,u}{dx}(x = x_i) \approx \frac{u(x_i + \Delta x) - u(x_i - \Delta x)}{2\,\Delta x}$$

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Second Order Derivative and Its Finite Difference Approximation

- set $v(x) = \dfrac{d\,u}{dx}$ then $\dfrac{d\,v}{dx} = \dfrac{d^2\,u}{dx^2}$
- use half steps: $v_{i+1/2} \approx v(x = x_{i+1/2}) = \dfrac{d\,u}{dx}(x_{i+1/2}) \approx \dfrac{u_{i+1} - u_i}{\Delta x}$
- similar to obtain $v_{i-1/2}$
- combine to obtain
  $$\frac{d^2\,u}{dx^2}(x = x_i) \approx \frac{v_{i+1/2} - v_{i-1/2}}{\Delta x} \approx \frac{u_{i-1} - 2u_i + u_{i+1}}{\Delta x^2}$$
- accuracy of second order accurate approximation using Taylor series expansions ($\mathcal{O}(\Delta x^2)$)

Accuracy of Three-Point Finite Difference Approximation

- three-point finite difference scheme

$$\frac{d^2 u}{dx^2}(x = x_i) \approx \frac{u_{i-1} - 2u_i + u_{i+1}}{\Delta x^2} = \frac{u_{i-1} - 2u_i + u_{i+1}}{h^2}$$

- Taylor expansion of u_{i+1} and u_{i-1} around x_i

$$u_{i+1} = u(x_{i+1}) = u_i + h\,u_i' + h^2/2\,u_i'' + h^3/6\,u_i''' + h^4/24\,u_i''''$$
$$u_{i-1} = u(x_{i-1}) = u_i - h\,u_i' + h^2/2\,u_i'' - h^3/6\,u_i''' + h^4/24\,u_i''''$$

- add Taylor expansion, subtract $2u_i$ and divide by h^2 to obtain

$$\frac{u_{i-1} - 2u_i + u_{i+1}}{h^2} = u_i'' + h^2/12\,u_i''''$$

- three-point scheme is thus second order accurate. Reducing meshwidth h by 2, reduces error in finite difference approximation by 4.

Accuracy of Three-Point Finite Difference Approx. (cont'd)

- we obtained earlier that
  $$\frac{u_{i-1} - 2u_i + u_{i+1}}{h^2} = u_i'' + h^2/12\, u_i''''$$
- error zero and thus scheme exact in case that $u''''(x) = 0$
- three-point difference scheme is thus exact for first, second and third order polynomials

Accuracy of Two-Point Finite Difference Approximation

- forward and backward difference scheme for u_i' are first order accurate
  (exercise: show this result using Taylor expansions)
- central difference scheme for u_i' is second order accurate
  (exercise: show this result using Taylor expansions)
- use ghost points and central scheme to approximate Neumann boundary conditions
  (details to be added)
  (exercise: show this result using Taylor expansions)

Finite Difference Method in Short

- start from $-u''(x) = f(x)$ + boundary conditions
- approximate differences using finite differences
- form linear system for unknown in grid points
- details coming soon

Extensions

- differential equations closer to practice
- more accurate finite difference approximation
- fast solution
- embed in digital twin for electrical machine or high voltage line

References

- bachelor courses in calculus

Further Reading

- https://en.wikipedia.org/wiki/Finite_difference

Elements of calculus of functions in one var - FDM

Recap

- boundary value problem for the Poisson equation for $u(x)$
- analytical solution methods
- finite difference approximation

$$-\frac{d^2 u}{dx^2}(x = x_i) = \frac{-u_{i-1} + 2u_i - u_{i+1}}{\Delta x^2} + \mathcal{O}(\Delta x^2)$$

Elements of calculus of functions - two vars - FDM

In This Section

- functions in two variables $u(x, y)$ and $f(x, y)$
- analytical solution method for the Laplace and Poisson equation using separation of variables
- finite difference approximation of the derivative of functions in two variables

Elements of calculus of functions - two vars - FDM

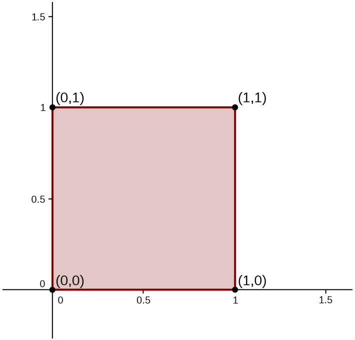
Domain of Computation

Before

Unit Interval: $\Omega = (0, 1)$

Here

Unit Square: $\Omega = (0, 1)^2$



Domain of Computation

- Before: $u(x)$ and $f(x)$ for $x \in \Omega = (0, 1)$
- Here: $u(x, y)$ and $f(x, y)$ for $(x, y) \in \Omega = (0, 1)^2$
- Goal: formulate $u''(x, y) = f(x, y)$
- However: what are $u'(x, y)$ and $u''(x, y)$?

Gradient (1/2): Definition

- unit (basic) vectors $\mathbf{i} = (1, 0) \in \mathbb{R}^2$ and $\mathbf{j} = (0, 1) \in \mathbb{R}^2$
- gradient of $u(x, y)$ using $\nabla = (\partial_x, \partial_y)$:

$$\text{grad } u(x, y) = \nabla u = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \right) = \frac{\partial u}{\partial x}\,\mathbf{i} + \frac{\partial u}{\partial y}\,\mathbf{j}$$

- $-\nabla u$: direction of largest increase in u - direction of diffusion

Gradient (2/2): Example

- electrical potential: $\phi(x, y)$

MIND THE GAP

- electrical field vector: $\mathbf{E}(x, y) = -\nabla\phi(x, y)$
- electrical field components: $E_x(x, y) = -\dfrac{\partial\phi}{\partial x}$ and $E_y(x, y) = -\dfrac{\partial\phi}{\partial y}$
- or $\mathbf{E}(x, y) = E_x(x, y)\,\mathbf{i} + E_y(x, y)\,\mathbf{j} = -\dfrac{\partial\phi}{\partial x}\,\mathbf{i} - \dfrac{\partial\phi}{\partial y}\,\mathbf{j}$
- how about magnetic flux $\mathbf{B}(x, y)$?

Divergence (1/3): Recap on Inner Product

- inner product denoted by $\cdot$
- performs a contraction

  suppose that $\mathbf{a} = (a_x, a_y)$ and $\mathbf{b} = (b_x, b_y)$, then

  $$\mathbf{a} \cdot \mathbf{b} = (a_x, a_y) \cdot (b_x, b_y) = a_x\, b_x + a_y\, b_y \in \mathbb{R} \text{ (single leg only)}$$

- example: electrical field $\mathbf{E}(x, y)$ - electrical displacement $\mathbf{D}(x, y)$

  $$\mathbf{E}(x, y) \cdot \mathbf{D}(x, y) \text{ (electrostatic energy density)}$$

Divergence (2/3): Definition

- vector field

$$\mathbf{F}(x,y) = P(x,y)\,\mathbf{i} + Q(x,y)\,\mathbf{j} \text{ (vector)}$$

- examples: $\mathbf{F}(x,y) = \nabla u(x,y)$ or $\mathbf{F}(x,y) = \mathbf{E}(x,y)$
- divergence of the vector field $\mathbf{F}(x,y)$

$$\text{div }\mathbf{F}(x,y) = \nabla \cdot \mathbf{F}(x,y) = \frac{\partial P(x,y)}{\partial x} + \frac{\partial Q(x,y)}{\partial y} \text{ (scalar)}$$

- interpretation: $\nabla \cdot \mathbf{F}(x,y)$: change in F over small volume

Divergence (3/3): Example

- electrical charge density $\rho(x, y)$ (scalar)
- electrical displacement

  $\mathbf{D}(x, y) = (D_x(x, y),\ D_y(x, y)) = D_x(x, y)\,\mathbf{i} + D_y(x, y)\,\mathbf{j}$ (vector)
- Gauss's Law: $\nabla \cdot \mathbf{D}(x, y) = \rho(x, y)$ or

$$\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} = \rho \text{ (scalar)}$$

- how about Ampère's Law?

Stay Tuned!

Laplacian (1/2): Definition

- if $u(x,y)$ scalar function, then $\nabla u(x,y)$ is a vector function
- if $\mathbf{F}$ vector function, then $\nabla \cdot \mathbf{F}$ is a scalar function
- put $\mathbf{F} = \nabla u(x,y)$, then

$$\nabla \cdot \mathbf{F} = \nabla \cdot \nabla u(x,y) = \nabla \cdot \left(\frac{\partial u}{\partial x} \mathbf{i} + \frac{\partial u}{\partial y} \mathbf{j} \right) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

- Laplacian - double derivative for $u(x,y)$

$$u'' \Rightarrow \triangle u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

Laplacian (2/2): Example in Electrostatics

- $\mathbf{x} = (x, y) \in \Omega$ - $f(\mathbf{x})$: known/given electrical charge density (possibly denoted by $\rho(\mathbf{x})$ elsewhere)
- unknown/desired electrical potential $u(\mathbf{x})$ (possibly denoted by $\phi(\mathbf{x})$ elsewhere)
- $\mathbf{E} = -\nabla u(\mathbf{x})$ and $\mathbf{D} = \epsilon\,\mathbf{E}$
- Gauss's Law:
  $\operatorname{div}\mathbf{D} = f(\mathbf{x}) \Leftrightarrow -\operatorname{div}\,(\epsilon\,\mathbf{E}) = f(\mathbf{x}) \Leftrightarrow -\operatorname{div}\,(\operatorname{grad} u(\mathbf{x})) = f(\mathbf{x})/\epsilon$
- Poisson equation for the electric potential:
  (assume $\epsilon = 1$ - beware of the minus sign!) $\boxed{-\triangle\, u(\mathbf{x}) = f(\mathbf{x})}$
- similar for magnetostatics: stay tuned

Boundary conditions (1/4)

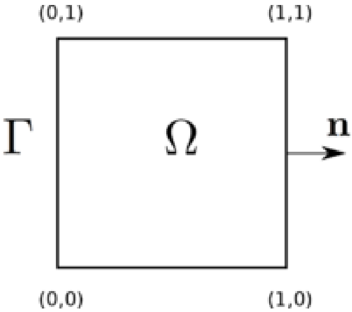
- Before: $u''(x) = f(x)$
- Now: $u''(x, y) = f(x, y)$ actually means $\triangle u = f(x, y)$
- what about the boundary conditions?
- Before: $\Omega = (0, 1)$ has boundary $x = 0$ and $x = 1$
- Now: $\Omega = (0, 1)^2$ has boundary Γ

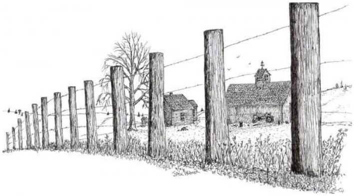
Elements of calculus of functions - two vars - FDM

Boundary conditions (2/4)

$\Omega = (0,1)^2$: Unit Square

Γ boundary of Ω - **n**: outward normal on Γ





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Boundary conditions (3/4)

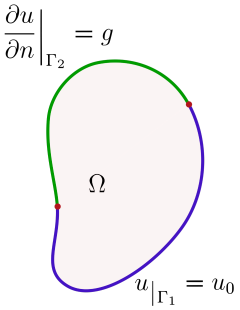
- direction or vector: $\mathbf{s} = (s_x, s_y) = s_x\,\mathbf{i} + s_y\,\mathbf{j}$
- directional derivative of $u(x, y)$ in the direction of $\mathbf{s}$

$$\frac{\partial u}{\partial s} = \underbrace{\nabla u \cdot \mathbf{s}}_{innerproduct} = s_x \frac{\partial u}{\partial x} + s_y \frac{\partial u}{\partial x} \text{ (scalar)}$$

- choose $\mathbf{s} = \mathbf{n}$ or $\mathbf{s} = \mathbf{t}$: normal or tangential derivative
- example: electromagnetic force/torque computation using the Maxwell stress tensor

Boundary conditions (4/4)

Two Types of Boundary Conditions: $\Gamma = \Gamma_D \cup \Gamma_N$



- Dirichlet condition on Γ_D

  fix u

  (equivalent of $x = 0$ in 1D)

- Neumann condition on Γ_N

  fix $\frac{\partial u}{\partial n}$

  (equivalent of $x = 1$ in 1D)

Elements of calculus of functions - two vars - FDM

Boundary Value Problem for
Second Order Differential Equations

- given $(x, y) \in \Omega = (0, 1)^2$ with $\Gamma = \Gamma_D \cup \Gamma_N$ the boundary of Ω
- given: $f(x, y)$ given function and α given number
- find: $u(x, y)$ such that

$$
\begin{aligned}
-\triangle u(x, y) &= f(x, y) \text{ for } 0 < x, y < 1 \text{ (differential equation on } \Omega) \\
u &= 0 \text{ (Dirichlet boundary condition on } \Gamma_D) \\
\frac{\partial u}{\partial n} &= \alpha \text{ (Neumann boundary condition on } \Gamma_N)
\end{aligned}
$$

- differential equation invalid on Γ - boundary conditions valid on Γ

Elements of calculus of functions - two vars - FDM

Exercise: Verify Solution

- suppose given: $u(x,y) = x\,(x-1)\,y\,(y-1)$
- then $\frac{\partial u}{\partial x} = (2\,x-1)\,y\,(y-1)$ and $\frac{\partial^2 u}{\partial x^2} = 2\,y\,(y-1)$
- thus $-\triangle\, u(x,y) = -2\,x\,(x-1) - 2\,y\,(y-1)$
- furthermore $u(x,y) = 0$ on Γ
- thus $u(x,y)$ solves

$$\begin{aligned} -\triangle\, u(x,y) &= -2\,x\,(x-1) - 2\,y\,(y-1) \text{ on } \Omega \\ u &= 0 \text{ on } \Gamma \end{aligned}$$

Construction of Analytical Solution

- separation of variables $u(x, y) = X(x)\,Y(y)$
- $X(x)$ and $Y(y)$ solved separately as 1D problems
- see References section for details
- in this course: not applied analytically
- in this course: applied as cheat-sheet to solve numerically

More Involved Models in 2D

- as before, replace $u(x)$ by $u(x, y)$

From here on

Goodbye Analytics

Hello Numerics

Elements of calculus of functions - two vars - FDM

Mesh Generation

- equidistant mesh in both x and y-direction (insert figure here)
- mesh spacing: $\Delta x = \Delta y = h = 1/N$
- $x_i = (i-1)\,h$ and $y_j = (j-1)\,h$ for $1 \leq i, j \leq N+1$
- interior nodes: (x_i, y_j) for $2 \leq i, j \leq N$
- east boundary where $x = 0$: fix $i = 1$

  similar for other 3 boundaries

Finite Difference Discretization (1/2)

- apply finite difference discretization in both x and y direction
- as before by varying x and keeping y fixed

$$\frac{\partial^2 u}{\partial x^2}(x = x_i, y = y_j) \approx \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2} \quad (\leftrightarrow)$$

- likewise in y-direction and keeping x fixed

$$\frac{\partial^2 u}{\partial y^2}(x = x_i, y = y_j) \approx \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{h^2} \quad (\updownarrow)$$

Finite Difference Discretization (2/2)

- add contributions in x and y-direction and multiply with -1

$$-\triangle u(x = x_i, y = y_j) \approx \frac{-u_{i-1,j} - u_{i,j-1} + 4u_{i,j} - u_{i+1,j} - u_{i,j+1}}{h^2}$$

- treat boundary conditions
- formulate system of linear equations for $u_{i,j}$
- solve for $u_{i,j}$

Elements of calculus of functions - two vars - FDM

References

- bachelor courses in calculus

Further Reading

- https://en.wikipedia.org/wiki/Separation_of_variables
- https://en.wikipedia.org/wiki/Finite_difference
- https://en.wikipedia.org/wiki/Kronecker_sum_of_discrete_Laplacians

Recap

- boundary value problem for the Poisson equation for $u(x, y)$
- analytical solution methods
- finite difference approximation

$$-\triangle u(x = x_i, y = y_j) \approx \frac{-u_{i-1,j} - u_{i,j-1} + 4u_{i,j} - u_{i+1,j} - u_{i,j+1}}{h^2}$$

Strong and Weak Equality of Functions

- strong equality of functions on Ω
- weak equality of functions on Ω

Elements of calculus of functions in one var: FEM

Derivative: Integration by Parts in one variable (1/2)

- fundamental theorem of calculus:

  assume $0 < x < 1$ or $x \in \Omega = (0,1)$ and $F'(x) = \dfrac{d\,F(x)}{dx}$

$$\int_\Omega F'(x)\,dx = \int_0^1 F'(x)\,dx = [F(x)]_0^1 = F(1) - F(0)$$

- choose $F(x) = \frac{d\,u}{dx}\,v(x)$ (motivation later)

  observe $u(x)$ has prime and $v(x)$ has no prime

- then $F'(x) = \dfrac{d}{dx}\left(\dfrac{d\,u}{dx}v(x)\right) = u''(x)\,v(x) + u'(x)\,v'(x)$

Derivative: Integration by Parts in one variable (2/2)

- subsequently from $\int_0^1 F'(x)\,dx = [F(x)]_0^1$

$$\int_0^1 \left[u''(x)\,v(x) + u'(x)\,v'(x)\right]\,dx = [u'(x)\,v(x)]_0^1$$

- after rearranging terms

$$\boxed{-\int_0^1 u''(x)\,v(x)\,dx = \int_0^1 u'(x)\,v'(x)\,dx - [u'(x)\,v(x)]_0^1}$$

LHS: $u(x)$ has double prime and $v(x)$ has no prime

RHS: both $u(x)$ and $v(x)$ have one prime - additional fudge term

(used later)

Integration: Quadrature by Trapezoidal Rule

- trapezoidal rule: $0 \leq a < b \leq 1$

$$\int_a^b g(x)\, dx \approx \frac{b-a}{2} \left[g(a) + g(b) \right]$$

- Simpson rule: $0 \leq a < b \leq 1$

$$\int_a^b g(x)\, dx \approx \frac{b-a}{6} \left[g(a) + 4\, g(\frac{a+b}{2}) + g(b) \right]$$

Function Spaces

- $\Omega = (0, 1)$
- $V(\Omega)$
- $V_0(\Omega)$

Elements of calculus of functions in one var: FEM

Fourier Analysis

- Fourier analysis
- basis of function space

Elements of calculus of functions in one var: FEM

Finite Dimensional Approximation of a Function Space

- set of basis functions

Extensions

- to be decided

Elements of calculus of functions in one var - FEM

References

- bachelor courses in calculus

Further Reading

- https://en.wikipedia.org/wiki/Integration_by_parts
- https://en.wikipedia.org/wiki/Trapezoidal_rule

Recap

- integration by parts in 1D

Two Small Exercises

- suppose that $\mathbf{F} = v \nabla u$
- Exercise 1: compute $\nabla \cdot \mathbf{F} = \nabla \cdot (v \nabla u)$
- Exercise 2: compute $\mathbf{F} \cdot \mathbf{n} = (v \nabla u) \cdot \mathbf{n}$

Two Small Exercises

- suppose that $\mathbf{F} = v\,\nabla u$
- Ex 1: solution $\nabla \cdot \mathbf{F} = \nabla u \cdot \nabla v + v \,\triangle\, u$
- Ex 2: solution $\mathbf{F} \cdot \mathbf{n} = (v\,\nabla u) \cdot \mathbf{n} = v\,(\nabla u \cdot \mathbf{n}) = v\,\frac{\partial u}{\partial n}$ (by definition of $\frac{\partial u}{\partial n}$)

Derivative: Integration by Parts in two variables (1/2)

- Gauss Integration Theorem: $(x, y) \in \Omega$ with boundary Γ

$$\int_\Omega \nabla \cdot \mathbf{F} \, d\Omega = \int_\Gamma \mathbf{F} \cdot \mathbf{n} \, ds$$

- choose: $\mathbf{F} = v \nabla u = (v \frac{\partial u}{\partial x}, v \frac{\partial u}{\partial y})$

Derivative: Integration by Parts in two variables (1/2)

- then

$$\int_\Omega \nabla \cdot \mathbf{F}\, d\Omega = \int_\Omega \nabla \cdot (v \nabla u)\ d\Omega = \int_\Omega [\nabla u \cdot \nabla v + v\ \triangle\ u]\ d\Omega$$
$$= \int_\Gamma \mathbf{F} \cdot \mathbf{n}\, ds = \int_\Gamma \frac{\partial u}{\partial n}\, v\, ds$$

- after rearranging terms:

$$\int_\Omega (-\triangle\ u)\, v\, d\Omega = \int_\Omega \nabla u \cdot \nabla v - \int_\Gamma \frac{\partial u}{\partial n}\, v\, ds$$

Derivative: Integration by Parts in two variables (2/2)

- integration by parts formula becomes

$$\int_{\Omega} (-\bigtriangleup u)\, v \, d\Omega = \int_{\Omega} \nabla u \cdot \nabla v - \int_{\Gamma} \frac{\partial u}{\partial n} \, v \, ds$$

- observe that as before:

  LHS: double derivatives in u - no derivatives on v

  RHS: first order derivatives on both u and v - additional term on the boundary

Elements of calculus of functions in n vars: FEM

Integration: Quadrature by Trapezoidal Rule

- trapezoidal rule: **e** triangle with vertices $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$

$$\int_{\mathbf{e}} g(x,y)\, d\mathbf{e} \approx \frac{\text{area}(\mathbf{e})}{3}\left[g(\mathbf{x}_1) + g(\mathbf{x}_2) + g(\mathbf{x}_3)\right]$$

- more accurate rules exist (Gauss quadrature)

Elements of calculus of functions in n vars: FEM

Example in Magnetostatics

- curl
-
-

Elements of calculus of functions in n vars - FEM

Extensions

- to be decided

Elements of calculus of functions in n vars - FEM

References

- bachelor courses in calculus

Further Reading

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Recap

- integration by parts in 2D