# An Overview of the Arf Invariant

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**Abstract**

The Arf invariant of a knot is an interesting invariant of knots that can be characterized in various different ways. In this paper, we present three equivalent definitions of the Arf invariant, delving into quadratic forms, pass-moves, and the Alexander polynomial. We show that all three definitions are equivalent. Finally, using the full flexibility of the Arf invariant to our disposal, we show it is a concordance invariant.

## 1 Introduction

Out of all the knot invariants to be studied, there is perhaps no invariant more flexible than the Arf invariant. In simple terms, the Arf invariant assigns each knot one of two numbers: 0 or 1. However, the machinery behind how we get the number, reveals interesting relationships between many topological concepts. The first approach to the Arf invariant, via quadratic forms, reveals a satisfying machinery of Seifert surfaces and linking numbers to obtain the $\mathbb{Z}/2-$valued invariant. Moreover, this formulation of the invariant has the advantage of being easily computable. The next approach, via pass-equivalence, gives the invariant a much more elementary flavor missing from the first approach. In these terms, the invariant boils down to whether you can use pass-moves to obtian an unknot. It is a more geometric and intuitive approach, although it appears harder to compute. Our final approach, using the Alexander polynomial, illustrates how far-reaching the Arf invariant's connections are.

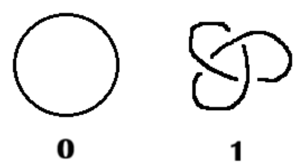


Figure 1: An unknot and a trefoil, along with their Arf invariants.

In Section 2, we introduce the Arf invariant of a knot. To do this, we discuss the classification of quadratic forms over $\mathbb{Z}/2$ and apply our results to the Seifert surface of a knot. In Section 3, we present two equivalent definitions of the Arf invariant and prove their equivalence to our original definition. Finally, in Section 4, we show that the Arf invariant is a concordance invariant of knots.

## 2 The Arf Invariant of a Knot

Let $k \subset S^3$ be a knot. Any Seifert surface, $F$, of $k$ and any choice of basis of the group $H_1(F;\mathbb{Z})$ gives rise to a quadratic form

$$q\colon H_1(F;\mathbb{Z}/2) \to \mathbb{Z}/2.$$

Motivated by this construction, which we make more explicit in subsection 2.2, we begin by classifying quadratic forms over $\mathbb{Z}/2$.

## 2.1 Quadratic Forms over $\mathbb{Z}/2$

**Definition 2.1**

Let $V$ be a finite-dimensional $\mathbb{Z}/2-$vector space. A map $q\colon V \to \mathbb{Z}/2$ is a *quadratic form* if

$$I(x,y) = q(x+y) - q(x) - q(y)$$

is a bilinear form. We call the quadratic form *non-degenerate* if $\det I \neq 0 \in \mathbb{Z}/2$.
Two forms $q, q'\colon V \to \mathbb{Z}/2$ are called *equivalent* if there exists a linear transformation $P \in \mathrm{GL}(V)$ such that $q(x) = q'(Px)$.

This definition would not change if we replaced $\mathbb{Z}/2$ by any field $K$ of characteristic different from 2. Moreover, observe that $I$ is symmetric, $I(x,y) = I(y,x)$. However, the study of quadratic forms over $\mathbb{Z}/2$ requires more care than that over $K$. For characteristic different from 2, there is a one-to-one correspondence between quadratic forms and symmetric bilinear forms given by[1]

$$q(x) \leftrightarrow \frac{1}{2} I(x,x).$$

Thus, the study of quadratic forms and the study of symmetric bilinear forms coincides. This correspondence does not exist for $\mathbb{Z}/2$, as we see already in the case of $\dim V = 1$. Observe that, for all $x \in V$, we must have

$$I(x,x) = q(2x) - 2q(x) = 0.$$

However, the symmetric bilinear form $I\colon (\mathbb{Z}/2)^2 \to \mathbb{Z}/2$ given by $I(x,y) = xy$ satisfies $I(1,1) = 1$, so it cannot correspond to any quadratic form. Indeed, all quadratic forms $\mathbb{Z}/2 \to \mathbb{Z}/2$ correspond to $I \equiv 0$, so they are all degenerate. This should not be surprising in light of the following theorem:

**Lemma 2.2**

Suppose $q\colon V \to \mathbb{Z}/2$ is a non-degenerate quadratic form. Then, $\dim V$ is even.

*Proof.* We proceed by induction on $\dim V$. We have already established that if $\dim V = 1$ all quadratic forms are degenerate. If $\dim V = 2$, then it is even. This establishes our base case.

Suppose that we know the result for $\dim V < n$. Suppose that $\dim V = n > 2$ and choose a basis in $V$. Then, the form $I(\cdot,\cdot)$ is given by a matrix $I$ that satisfies

$$I(x,y) = \langle x, Iy \rangle$$

where $\langle \cdot, \cdot \rangle$ is the standard inner product. Furthermore, since $q$ is non-degenerate, $\det I = 1$, so $I$ is invertible. Let $a_1 \in V$ be an element of our basis. There is some $b_1'$ such that[2] $\langle a_1, b_1' \rangle = 1$. Let $b_1 = I^{-1} b_1'$. Then, $I(a_1, b_1) = 1$, so $a_1$ and $b_1$ are linearly independent. Choose a basis with $a_1$ and $b_1$ as the first two basis vectors. After some elementary operations in this basis, the matrix for $I$ becomes

$$\left(\begin{array}{cc|c} 0 & 1 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 0 & I' \end{array}\right).$$

The matrix $I'$ corresponds to the symmetric bilinear form obtained by restricting $q$ to the complement of $\mathrm{span}\{a_1, b_1\}$, $V'$. Hence, we can apply our inductive hypothesis to $V'$ and conclude that its dimension is even. Thus, $\dim V = \dim V' + 2$ is also even, concluding the induction. □

[1] See equation (1.3) in [2].
[2] For example, find some $i$ such that $a_1(i) = 1$ and set $b_1'(j) = \delta_{ij}$.

Observe that following the procedure outlined in this proof, we obtain a basis $\{a_1, b_1, \ldots, a_g, b_g\}$ of $V$ such that $I(a_i, a_j) = I(b_i, b_j) = 0$ and $I(a_i, b_j) = \delta_{ij}$. A basis satisfying this property is called a *symplectic basis.*

So, if we want to understand non-degenerate quadratic forms over $\mathbb{Z}/2$, it suffices to consider even dimensions. We now shift to understanding the smallest non-trivial example: $\dim V = 2$.

**Example 2.3**

Let $a, b$ be a basis of $V = (\mathbb{Z}/2)^2$. Define $q_i \colon V \to \mathbb{Z}/2$ for $i = 0, 1$ by $q_i(a) = q_i(b) = i$ and $q_i(a + b) = 1$. Then, $q_0$ and $q_1$ are the only non-degenerate quadratic forms on $V$ up to equivalence.

*Proof.* The only non-degenerate symmetric bilinear form on $V$ is given by $I(a, a) = I(b, b) = 0$ and $I(a, b) = 1$. In particular, $\{a, b\}$ is a symplectic basis.

First, we show that $q_0$ and $q_1$ are not equivalent. Suppose they were equivalent. Then,

$$N_0 = |\{x \in V \mid q_0(x) = 0\}| = |\{x \in V \mid q_1(x) = 0\}| = N_1.$$

However, we have $N_0 = 3$ and $N_1 = 1$ from the definitions, so we have obtained a contradiction. Hence, $q_0$ and $q_1$ are not equivalent.

Now, we show any other non-degenerate quadratic form, $q$, is equivalent to either $q_0$ or $q_1$. Since $q$ can only correspond to one bilinear form, we have already exhausted most options for $q$. By symmetry, the only possibility left to consider is $q(a) = 0$ and $q(b) = 1$. Then,

$$q(a + b) = I(a, b) + q(a) + q(b) = 1 + 0 + 1 = 0,$$

so changing basis to $\{a, a + b\}$ shows that $q$ and $q_0$ are equivalent. □

Having completely characterized non-degenerate quadratic forms in dimension 2, we wish to turn to the general case. To distinguish $q_0$ from $q_1$, we counted the number of elements that got mapped to 0. Thus, given any quadratic form on $(\mathbb{Z}/2)^2$, we can determine if it is equivalent to $q_0$ or $q_1$ by counting the number of elements that get mapped to 0. This "democratic process" will be captured by the following quantity:

**Definition 2.4**

Let $q \colon V \to \mathbb{Z}/2$ be a non-degenerate quadratic form and let $\{a_i, b_i\}_{i=1}^g$ be a sympletic basis of $V$. The *Arf Invariant* of $q$ is

$$\operatorname{Arf}(q) = \sum_{i=1}^{g} q(a_i)q(b_i) \in \mathbb{Z}/2.$$

We can compute the Arf invariants of the forms from Example 2.3:

$$\operatorname{Arf}(q_0) = q_0(a)q_0(b) = 0 \cdot 0 = 0$$
$$\operatorname{Arf}(q_1) = q_1(a)q_q(b) = 1 \cdot 1 = 1.$$

Note that they have different Arf invariants. Moreover, if $q$ is the quadratic form given by $q(a) = 0, q(b) = 1$, we find $\operatorname{Arf}(q) = 0$, and as we found before, $q$ and $q_0$ are equivalent. Thus, the Arf invariant gives a complete classification of non-degenerate quadratic forms in dimension 2. In fact, it classifies quadratic forms in every dimension:

**Theorem 2.5**

Two non-degenerate quadratic forms on a finite-dimensional $\mathbb{Z}/2-$vector space are equivalent if and only if they have the same Arf invariant.

For a discussion on why Definition 2.4 does not depend on the choice of symplectic basis and the proof of this theorem, we refer the reader to Lecture 9 in [5].

Let $\dim V = 2g$ and define $\varphi_0 = gq_0$ and $\varphi_1 = q_1 + (g-1)q_0$. The important takeaway is that

$$\mathrm{Arf}(\varphi_0) = 0 \quad \text{and} \quad \mathrm{Arf}(\varphi_1) = 1,$$

so every non-degenerate quadratic form on $V$ is equivalent to either $\varphi_0$ or $\varphi_1$.

## 2.2 Seifert Surfaces

We are now ready to return to topology. Recall that a Seifert Surface, $F$, of an oriented knot $k$ is a connected compact oriented surface whose oriented boundary is $k$. Consider any curve $x \subset F$. Since $F$ is oriented, we can define a normal field on it compatible with this orientation. The *positive push-off* $x^+$ of $x$ is the curve parallel to $x$ that lies just above $F$ in the direction of this normal field. We can use these push-offs to define the following useful object:

**Definition 2.6**

Let $F$ be a genus $g$ Seifert surface of a knot $k$. Let $x_1, \ldots, x_{2g} \subset F$ be simple curves that generate a basis of $H_1(F;\mathbb{Z})$. The associated *Seifert matrix*, $S$, is the matrix given by

$$S_{ij} = \mathrm{lk}(x_i, x_j^+).$$

The Seifert Matrix is related to the intersection form $I$ on $H_1(F;\mathbb{Z}/2)$. Each intersection point will generate linking between the push-offs of the two intersecting curves. Indeed, chasing definitions we obtain that

$$I = S^T - S.$$

We can use Seifert matrices to obtain many knot invariants. With the work from Section 2.1, we obtain:

**Definition 2.7**

Let $k$ be a knot and let $S$ be a Seifert matrix of $k$. Let $Q = S + S^T$ and define

$$q(x) = \frac{1}{2}Q(x,x) \pmod 2.$$

Then, we call $\mathrm{Arf}(q)$ the Arf invariant of $k$, and write $\mathrm{Arf}(q) = \mathrm{Arf}(k)$.

First, observe that this is well-defined. We have

$$Q(x,x) = S(x,x) + S^T(x,x) = 2\,\mathrm{lk}(x, x+),$$

so $\frac{1}{2}Q(x,x)$ is always an integer. Over $\mathbb{Z}/2$, we have $Q = I$. Moreover,

$$q(x+y) - q(x) - q(y) = S(x+y, x+y) - S(x,x) - S(y,y) = S(x,y) + S(y,x),$$

so $I$ is the associated bilinear form. In particular, $q$ is a non-degenerate quadratic form and this definition makes sense.

**Theorem 2.8**

The invariant $\mathrm{Arf}(k)$ does not depend on the choice of Seifert matrix. Thus, $\mathrm{Arf}(k)$ is a knot invariant.

See Lecture 9 in [5].

We can now compute the Arf invariant of the unknot. Figure 2 shows a basis of the homology of a genus 1 Seifert surface, $F$, of the unknot. In this figure, the choice of normal points to the "outside" of the torus.

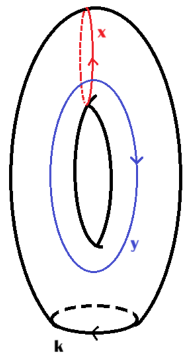


Figure 2: A basis $x$ (red), $y$ (blue) for the homology of $F$.

We see that $y^+$ is not linked to $x$, so $\mathrm{lk}(x, y^+) = 0$. Furthermore, $\mathrm{lk}(y, x^+) = -1$. Hence, the Seifert matrix we obtain is

$$S = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}.$$

Then, the quadratic form $q$ satisfies $q(x, x) = q(y, y) = 0$, so $\mathrm{Arf}(q) = 0$. Hence, the Arf invariant of the unknot is 0.

Via similar calculations we can show that the Arf invariant of the trefoil knot is 1.

# 3 Equivalent Definitions

We now present two equivalent definitions of the Arf Invariant for a knot.

## 3.1 Pass-Equivalence

**Definition 3.1**

A *pass-move* is a change in the knot diagram in which we pass two oppositely oriented strands through two oppositely oriented strands; see Figure 3. Two knots are said to be *pass-equivalent* if there is a series of pass-moves that takes us from one to the other.

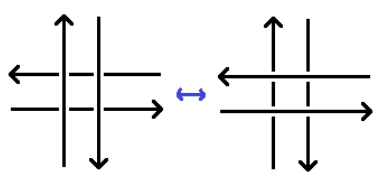

Figure 3: A pass move

At first, the pass-move may seem a rather unintuitive move to consider[3]. However, the move becomes much more sensible when considering Seifert surfaces. Isotope the Seifert surface of a knot so that it is embedded as a disk with bands attached. Then, the edges of each band will be oppositely oriented. Thus, a pass-move on the knot will correspond to passing two bands of the Seifert surface through each other; see Figure 4.

**Lemma 3.2**

Every knot $k$ is pass-equivalent to either the unknot or the trefoil knot.

[3]see page 93 in [4]

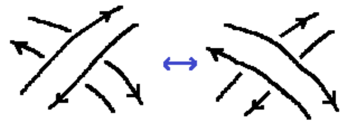

Figure 4: A pass-move on the Seifert surface

The idea is to use pass-moves to simplify the Seifert surface of a knot as much as possible. Doing so, we bring each knot to be either the unknot or a connected sum of trefoil knots. In the latter case, we can show the trefoil is pass-equivalent to its mirror image, which will allow us to cancel trefoils in pairs. Thus, our knot will be pass-equivalent to either the unknot or a single trefoil knot. For a more detailed proof; see section 8.2 in [1].

**Definition 3.3**
Let $k$ be a knot. We define $\mathrm{Arf}(k) = 0$ if $k$ is pass-equivalent to the unknot and $\mathrm{Arf}(k) = 1$ if $k$ is pass-equivalent to the trefoil knot.

This is a very geometric definition, completely disjoint from the abstract algebra we used to define the Arf invariant in section 2. That these two definitions coincide should be no surprise given the relevance of Seifert surfaces in both contexts.

**Theorem 3.4**
Definition 3.3 is equivalent to Definition 2.7.

*Proof.* Let $k$ be a knot. We will use $\mathrm{Arf}(k)$ to denote the Arf invariant according to Definition 2.7. Since we have already established that $\mathrm{Arf}(\text{unknot}) \neq \mathrm{Arf}(\text{trefoil})$, it suffices to show that if $k$ and $k'$ are pass-equivalent, then $\mathrm{Arf}(k) = \mathrm{Arf}(k')$.

Again, we consider a Seifert surface of $k$, isotoped so that it is embedded as a disk with bands attached. In such an embedding, there is a clear choice of generators for the homology; see Figure 5. Let $q$ be the quadratic form obtained from the associated Seifert matrix. Then, $q(x)$ counts the number of full twists in a neighborhood of $x$, modulo 2. The pass-moves will not affect this number. If we pass two bands corresponding to different generators, the neighborhoods of each basis curve are unaffected, leaving $q$ unchanged. If instead we pass a band through itself, the effect will be adding or removing an even number of full twists. For example, if we pass the band corresponding to the blue basis curve through itself, we change the orientation of the full twist, which is not perceived mod 2. Hence, if $k$ is pass-equivalent to the unknot it will satisfy $\mathrm{Arf}(k) = 0$ and if it is pass-equivalent to the trefoil it satisfies $\mathrm{Arf}(k) = 1$. $\square$

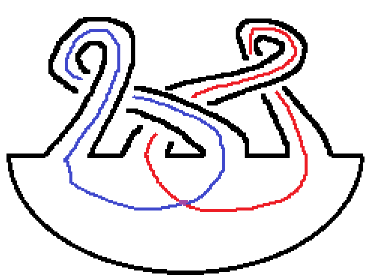

Figure 5: Natural choice of generators for disk with bands embedding of Seifert surface.

## 3.2 The Alexander Polynomial

**Definition 3.5**

Let $k$ be a knot and let $S$ be a Seifert matrix of $k$. The *Alexander Polynomial* of $k$ is given by

$$\Delta_k(t) = \det(t^{1/2}S - t^{-1/2}S^T).$$

It is straightforward to check that $\Delta_k(t)$ is a knot invariant. Furthermore, it satisfies $\Delta_k(t) = \Delta_k(t^{-1})$ in general. The Alexander polynomial is often alternatively defined by looking at the action of Laurent polynomials on homology. We can also relate it to the Conway polynomial, which is defined using skein relations. Our definition has the advantage of mentioning Seifert matrices explicitly, so the connection with the Arf invariant will not be lost. Of course, all three definitions are equivalent.

The Alexander polynomial is a very powerful invariant. Using the Seifert matrix derived from Figure 2, we see that

$$\Delta_{\text{unknot}}(t) = \det\begin{pmatrix} 0 & -t^{-1/2} \\ t^{-1/2} & 0 \end{pmatrix} = 1.$$

Giving the generators from Figure 5 a suitable orientation, we obtain the Seifert matrix

$$S = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix},$$

so that

$$\Delta_{\text{trefoil}}(t) = \det\begin{pmatrix} t^{-1/2} - t^{1/2} & -t^{-1/2} \\ t^{1/2} & t^{-1/2} - t^{1/2} \end{pmatrix} = t^{-1} - 1 + t^1.$$

Unsurprisingly, we can relate these expressions to the Arf Invariant.

**Definition 3.6**

Let $\Delta_k(t)$ be the Alexander polynomial of $k$. We define

$$\text{Arf}(k) = \frac{1}{2}\Delta_k''(1) \mod 2.$$

We double check it coincides with our previous notions: $\Delta_{\text{unknot}}''(t) = 0$, so $\text{Arf}(\text{unknot}) = 0$ and $\Delta_{\text{trefoil}}''(t) = 2t^{-3}$, so $\text{Arf}(\text{trefoil}) = 1$.

Although mysterious, the equivalence of this definition follows from a computation from our definitions:

**Theorem 3.7**

Definition 3.6 is equivalent to Definition 2.7.

*Proof.* Suppose $S$ is a Seifert surface for $k$ of size $2g \times 2g$ and let $Q = S + S^T$. There exist[4] an odd integer $a$, an integer matrix $P$ with odd determinant, odd integers $c_1, \ldots, c_g$ and integers $p_1, q_1, \ldots, p_g, q_g$ such that $a^2 \cdot Q = P^T DP$, where

$$D = \begin{pmatrix} 2p_1 & c_1 \\ c_1 & 2q_1 \end{pmatrix} \oplus \ldots \oplus \begin{pmatrix} 2p_g & c_g \\ c_g & 2q_g \end{pmatrix}.$$

Thus, there is some symplectic basis $\{a_1, b_1, \ldots, a_g, b_g\}$ such that

$$a^2 Q(a_i, a_i) = 2p_i \quad \text{and} \quad a^2 Q(b_i, b_i) = 2q_i.$$

[4]see the proof of Theorem 9.8 in [5]

Therefore, the quadratic form $q$ satisfies $q(a_j) = p_j$ and $q(b_j) = q_j$, so

$$\operatorname{Arf}(k) = \sum_{j=1}^{g} p_j q_j.$$

On the other hand, note that $\Delta_k(-1) = \det(iQ)$. Since

$$(a^2)^{2g} \det(iQ) = \det(iD) = (\det P)^2 \cdot \prod_{j=1}^{g} (c_j^2 - 4p_j q_j),$$

we obtain

$$\Delta_k(-1) = 1 + 4 \sum_{j=1}^{g} p_j q_j \mod 8.$$

That is, $\Delta_k(-1) = 1 + 4\operatorname{Arf}(k)$ modulo 8. It is not hard to check from the properties of the Alexander polynomial that this gives the desired result. $\square$

# 4 Concordance Invariance

We conclude by proving that the Arf invariant is a concordance invariant.

**Definition 4.1**

Two knots $k_0$ and $k_1$ are said to be *concordant* if there is an embedding $f\colon k_0 \times [0,1] \hookrightarrow S^3 \times [0,1]$ such that $f(k_0 \times \{0\}) = k_0 \times \{0\}$ and $f(k_0 \times \{1\}) = k_1 \times \{1\}$. A knot is called *slice* if it is concordant to the unknot.

Note that knot concordance is an equivalence relation. Denote by $\overline{k_1}$ the mirror image of $k_1$. Suppose that $k_0$ is concordant to $k_1$, and $f$ is an embedding exhibiting the concordance. We also have a natural embedding $g$ exhibiting the concordance of $\overline{k_1}$ with itself. Taking connected sums, $k_0 \# \overline{k_1}$ is concordant to $k_1 \# \overline{k_1}$, which is easily seen to be slice. Thus, $k_0$ is concordant to $k_1$ if and only if $k_0 \# \overline{k_1}$ is slice.

Hence, in order to prove the Arf invariant is a concordance invariant, it suffices to show (a) the Arf invariant is additive with respect to connected sums and (b) the Arf invariant of a slice knot is 0.

**Lemma 4.2**

Let $k_0$ and $k_1$ be two knots. Then,

$$\operatorname{Arf}(k_0 \# k_1) = \operatorname{Arf}(k_0) + \operatorname{Arf}(k_1).$$

*Proof.* Suppose that $\operatorname{Arf}(k_0) = \operatorname{Arf}(k_1)$. Then, according to section 3.1, $k_0$ and $k_1$ are both pass-equivalent to either the unknot or the trefoil knot. In either case, $k_0 \# k_1$ is pass-equivalent to the unknot, so $\operatorname{Arf}(k_0 \# k_1) = 0 = \operatorname{Arf}(k_0) + \operatorname{Arf}(k_1)$.

Suppose that $\operatorname{Arf}(k_0) \neq \operatorname{Arf}(k_1)$. Then, without loss of generality $k_0$ is pass-equivalent to the unknot and $k_1$ is pass-equivalent to the trefoil. Then, $k_0 \# k_1$ is pass-equivalent to the trefoil, so $\operatorname{Arf}(k_0 \# k_1) = 1 = 0 + 1 = \operatorname{Arf}(k_0) + \operatorname{Arf}(k_1)$. Thus, the Arf invariant is additive. $\square$

**Theorem 4.3**

Let $k_0$ and $k_1$ be two knots. If they are concordant, then

$$\operatorname{Arf}(k_0) = \operatorname{Arf}(k_1).$$

*Proof.* Since $k_0$ and $k_1$ are concordant, we have that $k = k_0 \# \overline{k_1}$ is slice. From Theorem 2 in [3], we have that $\Delta_k(t) = f(t)f(t^{-1})$ for some Laurent polynomial $f$. Then, from section 3.2 we know that

$$1 + 4\,\mathrm{Arf}(k) = \Delta_k(-1) = f(-1)^2 \mod 8.$$

Since $f(-1)^2$ must be 1 modulo 8, it follows that $\mathrm{Arf}(k) = 0$. Hence,

$$0 = \mathrm{Arf}(k) = \mathrm{Arf}(k_0) - \mathrm{Arf}(k_1)$$

and the result follows. □

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