# Applications of Morse Theory to Algebraic Topology

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**Abstract**

Morse theory is a powerful technique that allows us to use non-degenerate critical points of smooth functions on a manifold to give it CW-structures. This technique thus provides a more geometric way to compute algebraic invariants of the manifold. By considering the energy functional and geodesics, we can extend our tools to compute invariants of the path space of the manifold. This allows us to recover classical results like the (co)homology of $\Omega S^n$ and the Freudenthal suspension theorem. In this paper, we survey this technique and how we can apply it to obtain these algebraically flavored theorems.

# 1 Introduction

In this paper we outline how to use the techniques of Morse Theory to obtain famous results in algebraic topology. Morse theory is concerned with a specific type of smooth real-valued function on smooth manifolds, the so-called Morse functions. These satisfy that their critical points are all non-degenerate. This non-degeneracy allows us to build up our space one critical point at a time, adding a cell for each such point. This gives our space the homotopy type of a CW-complex which we can analyze to obtain results about its (co)homology and homotopy groups.

In Section 2 we define non-degeneracy and the index of a critical point. In section 3 we dive into the main results, proving the Fundamental Theorem of Morse Theory. We also provide our first application: computing the (co)homology of $\mathbb{C}P^n$ with our tools so far. In Section 4 we use Riemannian geometry to extend our results to the path space of a manifold. Finally, in Section 5, we use our tools to compute the (co)homology of $\Omega S^n$ and prove the Freudenthal suspension theorem. We strongly follow the exposition in Milnor [2].

# 2 Critical Points

Throughout this paper all manifolds are assumed to be smooth, and real-valued functions on a manifold are smooth unless explicitly stated otherwise.

Let $M$ be an $n-$manifold, $p$ a point in $M$, and $f\colon M \to \mathbb{R}$ a real-valued function. We denote the tangent space to $M$ at $p$ by $T_pM$ and the map induced by $f$ between tangent spaces by

$$\mathrm{d}f_p\colon T_pM \to T_{f(p)}\mathbb{R} \cong \mathbb{R}.$$

We call $p$ a critical point of $f$ if $\mathrm{d}f_p$ is the zero map. Otherwise, we call $p$ a regular point. Given local coordinates $(x^1, \dots, x^n)$ around $p$, we always assume without loss of generality that $p$ corresponds to the origin. In local coordinates, the map $\mathrm{d}f_p$ corresponds to the matrix

$$\begin{pmatrix} \dfrac{\partial f}{\partial x^1} & \cdots & \dfrac{\partial f}{\partial x^n}, \end{pmatrix}$$

so $p$ is a critical point if and only if all the partial derivatives are zero.

Not all critical points will be useful for our discussion. We want to restrict our attention to those that are non-degenerate:

**Definition 1: Non-degeneracy and Index of a critical point**

Given local coordinates $(x^1, \ldots, x^n)$ around $p$, define the *Hessian* of $f$ at $p$ to be the matrix

$$\mathrm{Hess}(f)_p := \left(\frac{\partial^2 f}{\partial x^i \partial x^j}(p)\right).$$

If $p$ is a critical point of $f$, we call $p$ *non-degenerate* if $\mathrm{Hess}(f)_p$ is non-singular.
Furthermore, the *index* of $p$ — often denoted $\lambda$ — is the maximal dimension of a subspace of $T_pM$ where $\mathrm{Hess}(f)_p$ is negative definite.

From this definition it is not obvious that the non-degeneracy and index of $p$ do not depend on the choice of local coordinates. In fact, we may instead define the Hessian intrinsically as a symmetric bilinear function on $T_pM$[1]. Then, our coordinate-dependent definition coincides with the matrix given above by writing this function in the basis of $T_pM$ induced by our coordinate functions. This shows that non-degeneracy and the index are intrinsic properties of $f$ and $p$. With this issue resolved, we will not concern ourselves with the intrinsic definition of the Hessian again.

The index of a non-degenerate critical point will be the most important notion in this paper. The following lemma begins to demonstrate its importance:

**Lemma 1: The Morse Lemma**

Let $p$ be a non-degenerate critical point of $f$ and $\lambda$ be its index. Then, there are local coordinates $(x^1, \ldots, x^n)$ around $p$ such that

$$f(x^1, \ldots, x^n) = f(p) - (x^1)^2 - \ldots - (x^\lambda)^2 + (x^{\lambda+1})^2 + \ldots + (x^n)^2$$

throughout the neighborhood where the local coordinates are defined.

For a proof of this lemma, we refer the reader to Lemma 2.2 of [2] (see also exercise 11 of 1.7 in [1]). This lemma tells us that if $\lambda = 0$, then $p$ is a local minimum. Similarly, if $\lambda = n$, then $p$ is a local maximum. Thus, the index "counts the number of directions" in which $p$ is a maximum. Below, we will see that the index tells us much more, giving us a way to put a CW-structure on $M$.

# 3 Morse Theory

## 3.1 Main Theorems

Given a real-valued function $f\colon M \to \mathbb{R}$ on an $n-$manifold $M$, we will denote by $M^a := f^{-1}((-\infty, a])$ the set of all $p \in M$ such that $f(p) \le a$. Observe that if $a$ is a regular point of $f$, then $M^a$ is an $n-$manifold with boundary[2].

We want to analyze how $M^a$ changes as we increase $a$. This will give us a way to build up $M$ as a CW-complex, adding cells for each $a$ where the homotopy type of $M^a$ changes. Our first theorem tells us that this can only happen when $a$ is a critical point:

[1] Given two vectors $v, w \in T_pM$, extend $w$ to a vector field $W$. Then, $\mathrm{Hess}(f)_p(v, w) := v(W(f))(p)$, where $W(f)$ denotes the directional derivative of $f$ in the direction of $W$ and analogously with $v(\cdot)$. The discussion in page 5 of [2] gives the equivalence of both definitions.

[2] See the lemma in page 62 of [1]

**Theorem 1: Stability Under Regular Points**

Suppose $a < b$ and that $f^{-1}([a,b])$ is compact and contains no critical points of $f$. Then, $M^a$ and $M^b$ are diffeomorphic. Furthermore, $M^b$ deformation retracts onto $M^a$, so that the inclusion $M^a \hookrightarrow M^b$ is a homotopy equivalence.

The idea behind the proof is to use $f$ to push $M^b$ down onto $M^a$. Since there are no critical values in $[a,b]$, we will not get "caught" on any points in this process and we will be able to extract both the diffeomorphism and deformation retraction. For more details, see the proof of theorem 3.1 in [2].

Now we want to know how the homotopy type of $M^a$ changes as we pass through a critical value. The following theorem tells us that all we need to do is add a cell whose dimension is given by the index of the critical point:

**Theorem 2: Critical Points Add Cells**

Let $p$ be a non-degenerate critical point of $f$ with index $\lambda$. Define $c = f(p)$ and fix $\varepsilon > 0$. Suppose that $f^{-1}([c-\varepsilon, c+\varepsilon])$ is compact and contains no critical points other than $p$. Then, for sufficiently small $\varepsilon$, we have that $M^{c+\varepsilon}$ has the homotopy type of $M^{c-\varepsilon}$ with a $\lambda-$cell attached.

*Proof.* By the Morse Lemma, we can choose local coordinates $(x^1, \ldots, x^n)$ around $p$ such that

$$f(x^1, \ldots, x^n) = c - (x^1)^2 - \ldots - (x^\lambda)^2 + (x^{\lambda+1})^2 + \ldots + (x^n)^2$$

throughout the neighborhood $U$ where the local coordinates are defined. For ease of notation, define functions $\rho, \sigma \colon U \to \mathbb{R}$ by $\rho = (x^1)^2 + \ldots + (x^\lambda)^2$ and $\sigma = (x^{\lambda+1})^2 + \ldots + (x^n)^2$. Then, $f = c - \rho + \sigma$. Take $\varepsilon$ small enough so that $f^{-1}([c-\varepsilon, c+\varepsilon])$ is compact, contains no critical points other than $p$, and $U$ contains[3] the closed ball

$$\left\{(x^1, \ldots, x^n) \mid \rho + \sigma \leq 2\varepsilon\right\}.$$

Define $D^\lambda$ to be the $\lambda-$cell contained in $U$ given by $\rho \leq \varepsilon$ and $x^{\lambda+1} = \ldots = x^n = 0$. We will show that $D^\lambda$ attaches to $M^{c-\varepsilon}$ and, using an appropriately defined function, that $M^{c-\varepsilon} \cup D^\lambda$ is a deformation retraction of $M^{c+\varepsilon}$.

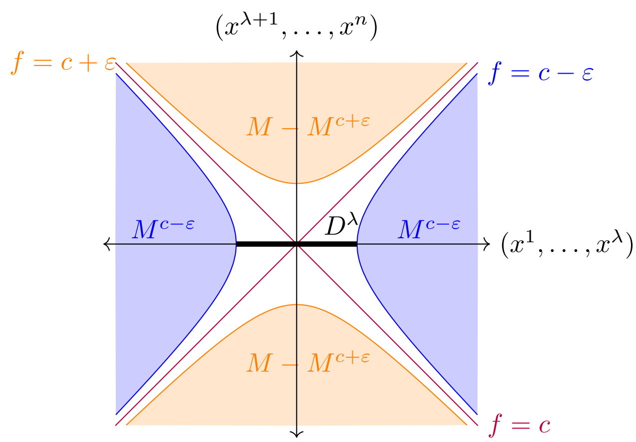


Figure 1: What we have so far, collapsing $x^1, \ldots, x^\lambda$ to one axis and $x^{\lambda+1}, \ldots, x^n$ to the other.

Indeed, observe that

$$D^\lambda \cap M^{c-\varepsilon} = \{(x^1, \ldots, x^\lambda, 0, \ldots, 0) \mid \rho = \varepsilon\} = \partial D^\lambda,$$

[3]This is an abuse of notation. Technically, we should say the image of $U$ under the local chart contains this ball.

so $D^\lambda$ is attached to $M^{c-\varepsilon}$. Now, let $\mu\colon \mathbb{R} \to \mathbb{R}$ be a function satisfying that

$$\mu(0) > \varepsilon, \quad \mu(r) = 0 \text{ for } r \geq 2\varepsilon, \text{ and } -1 < \mu'(r) \leq 0 \text{ for all } r \in \mathbb{R}.$$

We will use $\mu$ to bump $f$ slightly around $p$ conveniently. Observe that since $\mu$ is a non-increasing function, $\mu(r) \geq 0$ for $r < 2\varepsilon$. Define $F\colon M \to \mathbb{R}$ to be the function that coincides with $f$ outside of $U$, and

$$F = f - \mu(\rho + 2\sigma) \text{ in } U.$$

Now, outside the region where $\rho + 2\sigma \leq 2\varepsilon$ we have $\mu(\rho + 2\sigma) = 0$, so $F = f$. Inside this region, since $\rho \geq 0$, we have

$$F \leq f = c - \rho + \sigma \leq c + \frac{1}{2}\rho + \sigma \leq c + \varepsilon.$$

Hence, inside this region the values of $F$ and $f$ are always at most $c + \varepsilon$. Therefore, $F(q) \leq c + \varepsilon$ precisely when $f(q) \leq c + \varepsilon$, so $F^{-1}((-\infty, c+\varepsilon]) = M^{c+\varepsilon}$.

Now, $\mathrm{d}\rho = 2(x^1 \mathrm{d}x^1 + \ldots + x^\lambda \mathrm{d}x^\lambda)$ and $\mathrm{d}\sigma = 2(x^{\lambda+1}\mathrm{d}x^{\lambda+1} + \ldots + x^n \mathrm{d}x^n)$ are linearly independent unless $x^1 = \ldots = x^n = 0$. Therefore, since

$$\begin{aligned} \frac{\partial F}{\partial \rho} &= -1 - \mu'(\rho + 2\sigma) < 0 \\ \frac{\partial F}{\partial \sigma} &= 1 + 2\mu'(\rho + 2\sigma) \geq 1, \end{aligned}$$

it follows that $\mathrm{d}F = \dfrac{\partial F}{\partial \rho}\mathrm{d}\rho + \dfrac{\partial F}{\partial \sigma}\mathrm{d}\sigma$ can only be 0 at the origin, so $F$ has the same critical points as $f$.

Next, since $F \leq f$ and $F^{-1}((-\infty, c+\varepsilon]) = M^{c+\varepsilon}$, we get that $F^{-1}([c-\varepsilon, c+\varepsilon]) \subseteq f^{-1}([c-\varepsilon, c+\varepsilon])$. As a closed subset of a compact set, it follows $F^{-1}([c-\varepsilon, c+\varepsilon])$ is compact. Furthermore, it could only contain $p$ as a critical point, but by construction

$$F(p) = c - \mu(0) < c - \varepsilon,$$

so it does not contain any critical points. Hence, we may apply Theorem 1 to $F$ to conclude that the set $F^{-1}((-\infty, c-\varepsilon])$ is a deformation retraction of $F^{-1}((-\infty, c+\varepsilon]) = M^{c+\varepsilon}$.

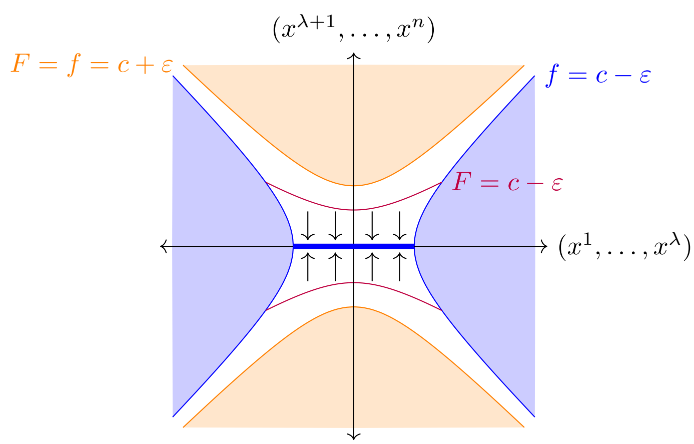


Figure 2: Arrows showing a deformation retraction of $F^{-1}((-\infty, c-\varepsilon])$ onto $M^{c-\varepsilon} \cup D^\lambda$.

It only remains to show that $F^{-1}((-\infty, c-\varepsilon])$ is homotopy equivalent to $M^{c-\varepsilon} \cup D^\lambda$. In fact, we can write an explicit deformation retraction, shown above pictorially. Writing out the explicit expression involves considering several cases, so we do not worry ourselves with the details. These are written out in detail in the proof of Theorem 3.2 in [2]. $\square$

We may modify the proof of this theorem to show that if we have non-degenerate critical points $p_1, \ldots, p_k$ with indices $\lambda_1, \ldots, \lambda_k$, all with $f(p_j) = c$, then $M^{c+\varepsilon}$ has the homotopy type of $M^{c-\varepsilon}$ with one $\lambda_j-$cell attached for each $j = 1, \ldots, k$.

Our tools so far only work for non-degenerate critical points. A function whose critical points are all non-degenerate is called a *Morse function*. Our results then give us the most important result in Morse Theory:

**Theorem 3: The Fundamental Theorem of Morse Theory**

Let $f$ be a Morse function on $M$ and suppose each $M^a$ is compact. Then, $M$ has the homotopy type of a countable CW-complex with one cell of dimension $\lambda$ for each critical point of index $\lambda$.

*Proof.* Since critical points are isolated[4] and each $M^a$ is compact, it follows there are countably many critical values $c_1 < c_2 < \ldots$ of $f$. Furthermore, the sequence $\{c_i\}$ has no cluster points.

Suppose we have a regular value $a$, and assume $M^a$ has the homotopy type of a CW-complex $K$. Let $c$ be the smallest critical value greater than $a$. Then, from Theorems 1 and 2, we know that for sufficiently small $\varepsilon$, $M^a$ is a deformation retraction of $M^{c-\varepsilon}$ and $M^{c+\varepsilon}$ has the homotopy type of $M^{c-\varepsilon}$ with cells of appropriate dimensions attached. We can now use cellular approximation to pass to our CW-complex. We conclude that $M^{c+\varepsilon}$ has the same homotopy type as $K$ with one cell of dimension $\lambda$ attached to its $\lambda - 1$ skeleton for each critical point of index $\lambda$ in the preimage of $c$.

With this observation in hand, we now proceed by induction. Observe that $M^a = \emptyset$ for $a < c_1$. Since $\emptyset$ is a CW-complex, this is our base case. Then, induction and Theorem 1 tell us that every $M^a$ has the homotopy type of a CW-complex. Take regular values $a_1, a_2, \ldots$ such that $a_1 < c_1$ and there is exactly one critical value in each $(a_i, a_{i+1})$. Then, we have a sequence of homotopy equivalences

$$\begin{tikzcd}
M^{a_1} \arrow[r, hook] \arrow[d] & M^{a_2} \arrow[r, hook] \arrow[d] & M^{a_3} \arrow[r, hook] \arrow[d] & \ldots \\
K^1 \arrow[r, hook] & K^2 \arrow[r, hook] & K^3 \arrow[r, hook] & \ldots
\end{tikzcd}$$

where $K^j$ is the CW complex obtained from $K^{j-1}$ by adding the appropriate cells. Observe that $M$ is the direct limit of the $M^{a_1} \subseteq M^{a_2} \subseteq \ldots$. Thus, by Whitehead's theorem, as $M$ is a manifold, and hence is dominated by CW-complexes, the above sequence induces a homotopy equivalence in the direct limit, establishing the result. $\square$

## 3.2 The CW-structure of $\mathbb{C}P^n$

As a quick application of the fundamental theorem, we can now compute the CW-structure of $\mathbb{C}P^n$. We will view $\mathbb{C}P^n$ as equivalence classes of complex numbers $(z_0, \ldots, z_n)$. Furthermore, we always assume we pick a representative such that $(z_0, \ldots, z_n) \in S^{2n+1} \subseteq \mathbb{C}^{n+1}$. We denote an equivalence class by $(z_0 \colon \ldots \colon z_n)$. Consider the Morse function

$$f(z_0 \colon \ldots \colon z_n) = \sum_{j=0}^{n} j|z_j|^2.$$

Also, recall the standard open $U_k := \{(z_0 \colon \ldots \colon z_n) \mid z_k \neq 0\}$. For $j \neq k$, define $x_j + iy_j = \frac{|z_k|}{z_k} z_j$. Then, $(x_0, y_0, \ldots, x_n, y_n) \colon U_k \to \mathbb{R}^{2n}$ gives a local coordinate system mapping $U_k$ to the unit ball. We find that

$$f(x_0, y_0, \ldots, x_n, y_n) = k + \sum_{j \neq k} (j - k)(x_j^2 + y_j^2)$$

holds throughout $U_k$. Thus, we see the only critical point of $f$ in $U_k$ is the point $(0 \colon \ldots \colon z_k = 0 \colon \ldots \colon z_n)$. Furthermore, by the Morse Lemma, its index is $2k$. Since the $U_0, \ldots, U_n$ cover $\mathbb{C}P^n$, we have found all critical points of $f$ along with their respective indices. Hence, the Fundamental Theorem tells us that $\mathbb{C}P^n$

[4] See, for example, the paragraph at the end of page 41 of [1]

has the homotopy type of a CW-complex with one cell of each dimension $0, 2, \dots, 2n$. In particular, we immediately know its (co)homology:

$$H_k(\mathbb{C}P^n) = H^k(\mathbb{C}P^n) = \begin{cases} \mathbb{Z}, & \text{if } 0 \le k \le 2n \text{ is even} \\ 0, & \text{else.} \end{cases}$$

# 4 The Path Space

## 4.1 Riemannian Geometry

We now take a brief detour into Riemannian geometry to set up some useful notions that will allow us to extend our tools to the path space of any manifold.

**Definition 2: Metric and Connection**

Let $M$ be a manifold. A Riemannian *metric* on $M$ is a smoothly-varying family of inner products $\langle -, - \rangle_p$ on $T_pM$. We usually suppress the $p$ subscript from our notation, as which inner product we are using will be clear from context.
A connection $\nabla$ on $M$ is a bilinear map that takes two vector fields $X, Y$ and returns a vector field $\nabla_X Y$. This assignment must satisfy the following properties for any real-valued function $f$

- We have $\nabla_{fX} Y = f \nabla_X Y$;
- (Chain Rule) $\nabla_X(fY) = X(f)Y + f\nabla_X Y$, where $X(f)$ denotes the directional derivative of $f$ in the direction of $X$.

We want to use these notions to study curves on $M$. Because of this, sometimes we will want to study vector fields defined only along a curve and not on the entire manifold. For a curve $c$, we call a smooth mapping $t \mapsto V_t \in T_{c(t)}M$ a *vector field along* $c$. Our first example is the *velocity* of $c$, given by $\dot{c} := \mathrm{d}c(\partial_t)$, where $\partial_t$ is the standard vector field on $\mathbb{R}$. We can now define what it means to differentiate along a curve.

**Definition 3: Covariant Derivative**

Let $c$ be a curve on $M$. Given any vector field $V$ along $C$, the *covariant derivative* of $V$, denoted $\frac{\mathrm{D}}{\mathrm{d}t}V$ is the unique vector field along $c$ satisfying the following properties: For all vector fields $V, W$ along $c$ and real-valued $f$,

- (Linearity) $\dfrac{\mathrm{D}}{\mathrm{d}t}(V + W) = \dfrac{\mathrm{D}}{\mathrm{d}t}V + \dfrac{\mathrm{D}}{\mathrm{d}t}W$;
- (Chain Rule) $\dfrac{\mathrm{D}}{\mathrm{d}t}(fV) = \dfrac{\mathrm{d}f}{\mathrm{d}t}V + f\dfrac{\mathrm{D}}{\mathrm{d}t}V$;
- (Extension) If $Y$ is a vector field defined on $M$ such that $V_t = Y_{c(t)}$, then $\dfrac{\mathrm{D}}{\mathrm{d}t}V = \nabla_{\dot{c}}Y$.

We say a vector field $V$ along $c$ is *parallel* if $\frac{\mathrm{D}}{\mathrm{d}t}V = 0$ at all $t$.

We say that a connection is *compatible with the metric* if for any curve $c$ and any parallel vector fields $P, P'$ along $c$, we have that $\langle P, P' \rangle$ is a constant function. We say that a connection is *symmetric* if the identity $\nabla_X Y - \nabla_Y X = XY - YX$ holds throughout $M$ for any vector fields $X, Y$.

There are many possible connections we can give a Riemannian manifold. However, a theorem of Riemannian geometry tells us that there exists a unique symmetric connection that is compatible with the metric. We will always refer to this unique connection.

**Definition 4: Length and Minimal Geodesics**

A curve $\gamma$ is called a geodesic if $\dot{\gamma}$ is parallel along $\gamma$.
The *length* of a curve $\gamma\colon [a,b] \to M$ is the integral

$$L(\gamma) := \int_a^b \sqrt{\langle \dot{\gamma}, \dot{\gamma} \rangle}\, \mathrm{d}t.$$

A geodesic is called *minimal* if its length is less than or equal to the length of any other curve joining its endpoints. Any curve whose length is minimal in this sense must be a geodesic.

Geodesics are the equivalent of straight lines in our manifold. In fact, geodesics in $\mathbb{R}^n$ are precisely the straight lines. Observe that in $\mathbb{R}^n$, every geodesic is a minimal geodesic. In fact, it is true that any sufficiently small segment of a geodesic is minimal. However, unlike in $\mathbb{R}^n$, geodesics need not be unique. For example, geodesics in $S^n$ are given by great circle arcs. Thus, antipodal points in $S^n$ are connected by infinitely many minimal geodesics.

## 4.2 The Path Space

We are now ready to talk about the path space of a manifold $M$. We want to view it as an infinite dimensional manifold. This leads us to the following definitions:

**Definition 5: The Path Space**

Denote the set of paths from $p$ to $q$ by $\Omega(M;p,q)$. When clear, we will simply call this space $\Omega$.
The tangent space of $\Omega$ at a path $\omega$ is the vector space of vector fields $W$ along $\omega$ that are 0 at the endpoints. We denote this tangent space by $T_\omega\Omega$.
A *variation* of a path $\omega$ is a function $\alpha\colon (-\varepsilon, \varepsilon) \to \Omega$ such that $\alpha(0) = \omega$. We say that a path $\omega$ is a critical path for a functional $F\colon \Omega \to \mathbb{R}$ if and only if

$$\left.\frac{\mathrm{d}F(\alpha(u))}{\mathrm{d}u}\right|_{u=0} = 0$$

for every variation $\alpha$.
We can define an $n-$*parameter variation* similarly. Let $U$ be a neighborhood of the origin in $\mathbb{R}^n$. Then, $\alpha\colon U \to \Omega$ is an $n-$parameter variation if $\alpha(0,\ldots,0) = \omega$.

In order to apply our Morse Theory techniques to the path space, we will need an appropriate Morse function. This will be the energy functional. For a path $\omega\colon [a,b] \to M$, we define its *energy* as

$$E(\omega) = \int_a^b \langle \dot{\omega}, \dot{\omega} \rangle\, \mathrm{d}t.$$

It turns out that the critical paths of $E$ are precisely the geodesics. Its minima are the minimal geodesics. We can also define the Hessian of $E$. For $W_1, W_2 \in T_\omega\Omega$, choose a $2-$parameter variation $\alpha\colon U \to \Omega$ such that $\frac{\partial \alpha}{\partial x_j} = W_j$ for $j = 1,2$. Then, we define $\mathrm{Hess}(E)_\omega$ to be

$$\left.\frac{\partial^2 E(\alpha(x_1,x_2))}{\partial x_1 \partial x_2}\right|_{(x_1,x_2)=(0,0)}$$

Just as before, the Hessian is a well-defined symmetric bilinear function of $W_1, W_2$. In particular, it does not depend on the choice of variation. By picking an appropriate $2-$parameter variation[5], we can show that if $\gamma$ is a minimal geodesic, then $\mathrm{Hess}(E)_\gamma$ is positive semi-definite, so the index of $\gamma$ is zero.

[5] See Lemma 13.6 in [2]

**Definition 6: Jacobi Fields and Conjugate Points**

Suppose $\gamma$ is a geodesic from $p$ to $q$. A vector field in the nullspace of $\text{Hess}(E)_\gamma$ is called a Jacobi field along $\gamma$. If the dimension of this nullspace is positive, we say that $p$ and $q$ are *conjugate* along $\gamma$. In this case, we call the dimension of the nullspace the *multiplicity* of $p$ and $q$ as conjugate points. Thus, $\gamma$ is non-degenerate if and only if $p$ and $q$ are not conjugate along $\gamma$.

Jacobi fields satisfy a certain linear, second order differential equation, called the Jacobi differential equation. If $M$ has dimension $n$, it has $2n$ linearly independent solutions. However, since vector fields must vanish at the endpoints, we can impose the initial condition that our Jacobi field vanishes at $t = 0$. This immediately tells us that the dimension of the nullspace is at most $n$. It is not hard[6] to improve this bound to $n - 1$.

Returning to our example with $S^n$, two points are conjugate along some geodesic if and only if they are the same point or antipodes. In either case, they are conjugate along any great circle arc with multiplicity $n-1$. Conjugate points are important, as they allow us to compute index of a geodesic:

**Theorem 4: The Index Theorem**

The index $\lambda$ of $\text{Hess}(E)_\gamma$ is equal to the number of points $\gamma(t)$, with $0 < t < 1$ such that $\gamma(t)$ is conjugate to $\gamma(0)$ along $\gamma$, counted with multiplicity. This index is always finite.

A proof is given in Chapter 15 of [2]. We have not yet given $\Omega$ a topology. The topology we need to give $\Omega$ to make $E$ continuous is not the usual compact-open topology. However, these two different topologies yield homotopy equivalent spaces, so we will not distinguish between them. Furthermore, recall that $\Omega$ with the usual compact-open topology has the homotopy type of a CW-complex. The techniques we have developed will allow us to compute the CW-structure explicitly.

So far, we have been building an analogy between the path space and our previous work with manifolds. However, we can only apply our previous results if we are working with actual manifolds. Luckily, we can approximate the path space with a finite dimensional manifold, allowing us to reap the benefits of our analogy.

**Lemma 2: Approximating $\Omega^c$**

As before, let $\Omega(M;p,q)^c := E^{-1}((-\infty, c])$. Furthermore, consider a subdivision of the unit interval $0 = t_0 < t_1 < \ldots < t_k = 1$. Let $\Omega(t_0, \ldots, t_k)$ be the subset of $\Omega$ consisting of paths $\omega$ such that $\omega(0) = p, \omega(1) = q$, and $\omega\mid_{[t_i, t_{i+1}]}$ is a geodesic for each $i = 0, \ldots, k-1$. Let

$$B = \text{Int}(\Omega^c) \cap \Omega(t_0, \ldots, t_k).$$

Then, for all sufficiently fine subdivisions $(t_0, \ldots, t_k)$, we can give $B$ the structure of a smooth manifold in a natural way. If $E'$ is the restriction of $E$ to $B$, we have that $E'$ is smooth. Furthermore, for each $a < c$, the set $B^a$ is compact and a deformation retract of $\Omega^a$. The critical points of $E'$ and $E$ coincide, and the index and nullity of $E'$ and $E$ at each critical point also match.

This lemma will allow us to extend the Fundamental Theorem to the path space, as we now have a concrete manifold to apply the theorem to. Putting everything together, we obtain

[6] In fact, $t\dot{\gamma}$ satisfies the differential equation, but does not vanish at $q$. See Remark 14.2 in [2]

**Theorem 5: The Fundamental Theorem for Path Spaces**

Let $M$ be a complete Riemannian manifold and let $p, q$ be two points that are not conjugate along any geodesic. Then, $\Omega(M; p, q)$ has the homotopy-type of a countable CW-complex with one cell of dimension $\lambda$ for each geodesic from $p$ to $q$ of index $\lambda$

*Proof.* Picking a sequence of regular values $a_1 < a_2 < \dots$ such that there is exactly one critical point of $E$ in each $(a_i, a_{i+1})$, we obtain from Lemma 2 and Theorem 2 a sequence of homotopy equivalences

$$\begin{tikzcd} \Omega^{a_1} \arrow[r, hook] \arrow[d] & \Omega^{a_2} \arrow[r, hook] \arrow[d] & \Omega^{a_3} \arrow[r, hook] \arrow[d] & \dots \\ K^1 \arrow[r, hook] & K^2 \arrow[r, hook] & K^3 \arrow[r, hook] & \dots \end{tikzcd}$$

Again, we pass to the direct limit. Since $\Omega$ is known to have the homotopy type of a CW-complex, Whitehead's theorem again tells us that this sequence induces a homotopy equivalence in the limit, establishing the result. □

# 5 Applications to Algebraic Topology

## 5.1 The (co)homology of $\Omega S^n$

We wish to compute the CW-structure of $\Omega S^n = \Omega(S^n; p, p)$, for any $p \in S^n$. Since $p$ is conjugate to itself, we cannot apply the Fundamental Theorem right away. However, we can find a path from $p$ to any point $q \neq p$ that is not its antipode. This path induces a homotopy equivalence $\Omega S^n \simeq \Omega(S^n; p, q)$ in the natural way. Since $p$ and $q$ are not conjugate along any geodesic, we can now apply the Fundamental Theorem.

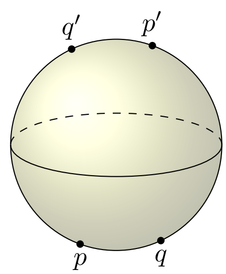


Figure 3: The sphere $S^n$, with $p, q$ and their antipodes $p', q'$.

Denote by $p'$ and $q'$ the antipodes of $p$ and $q$ respectively. The shortest geodesics from $p$ to $q$ are given by the great circle arcs $pq$ and $pq'p'q$. Then, all other geodesics are given by going around the great circle $pq'p'qp$ or $pqp'q'p$ some number of times and then following along $pq$ or $pq'p'q$. In other words, we get geodesics $\gamma_0, \gamma_1, \gamma_2, \gamma_3 \dots$ given by the great circle arcs $pq, pq'pq'q, pqp'q'pq, pq'p'qpq'p'q$, and so on. Observe that $\gamma_k$ has index $k(n-1)$, since it contains $k$ points conjugate to $p$ in its interior. Therefore, the fundamental theorem asserts that $\Omega S^n$ has the homotopy type of a CW-complex with one cell in each dimension $0, n-1, 2(n-1)$, and so on. Thus, for $n > 2$ we can immediately compute its (co)homology:

$$H_k(\Omega S^n) = H^k(\Omega S^n) = \begin{cases} \mathbb{Z}, & \text{if } (n-1) \mid k \\ 0, & \text{else.} \end{cases}$$

## 5.2 The Freudenthal Suspension Theorem

In order to prove Freudenthal suspension, we will need one last theorem. Let $d$ be the distance from $p$ to $q$. Then, $\Omega^d$ is the space of minimal geodesics from $p$ to $q$.

**Theorem 6: Space of Minimal Geodesics**

If the space of minimal geodesics is a manifold and every non-minimal geodesic from $p$ to $q$ has index at least $\lambda_0$, then the homotopy group $\pi_k(\Omega, \Omega^d)$ is trivial for $0 \leq k < \lambda_0$.

The result of this theorem is very intuitive. The fundamental theorem tells us that we get $\Omega$ by attaching cells of dimension at least $\lambda_0$ to $\Omega^d$. Therefore, by skeletal approximation, if $k < \lambda_0$, we can homotope any map $D^k \to \Omega$ so that the image of $D^k$ lies entirely in $\Omega^d$. In other words, any map $(D^k, S^{k-1}) \to (\Omega, \Omega^d)$ is homotopic to a map to $(\Omega^d, \Omega^d)$, so the relative homotopy group $\pi_k(\Omega, \Omega^d)$ must be trivial.

From the long exact sequence of a pair, we obtain that $\pi_k(\Omega^d) \cong \pi_k(\Omega)$ for $0 \leq k \leq \lambda_0 - 2$. However, recall that $\pi_k(\Omega) \cong \pi_{k+1}(M)$. Hence, $\pi_k(\Omega^d) \cong \pi_{k+1}(M)$.

We now apply this to $S^{n+1}$. Taking two antipodal points, the space of minimal geodesics is given by the set of all great circle arcs between the points. We can identify each great circle arc with a point in the equator, so we obtain that $\Omega^d \cong S^n$. Since every non-minimal geodesic between antipodal points has index at least $2n$, our result above gives us that

$$\pi_k(S^n) \cong \pi_{k+1}(S^{n+1}),$$

for $k \leq 2n - 2$. This is the Freudenthal suspension theorem.

# References


[1] Victor Guillemin and Alan Pollack. *Differential Topology.* AMS Chelsea Publishing, 1974.

[2] John Milnor. *Morse Theory.* Princeton University Press, 1973.