The Rectifiability Theorem for Varifolds

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Abstract

The study of smooth manifolds is central to many questions in physics, topology, and geometry. However, to best approach some problems, it is necessary to deal with more general notions of surfaces than smooth manifolds. One such generalization are varifolds, which come in a much more measure theoretic flavor. There are two main notions of varifolds: rectifiable and general varifolds. In this paper, we prove the first rectifiability theorem, which states that under very mild conditions, both notions coincide.

1 Introduction

In this paper we discuss varifolds, a generalization of the smooth manifolds encountered in topology. We study rectifiable $n-$varifolds, general $n-$varifolds, and connect them by proving the first rectifiability theorem. Briefly, the theorem states that any general varifold which has tangent spaces almost everywhere is rectifiable. We assume familiarity with geometric measure theory as laid out in [1] or [2].

In Section 2 we define varifolds and some related concepts, stating results that will be useful for our discussion. In Section 3 we prove the main result of the paper using our previously developed tools. Our discussion is heavily influenced by [3].

2 Varifolds

Throughout this paper $\mathcal{H}^n$ denotes the $n-$dimensional Hausdorff measure.

2.1 Rectifiable $n-$varifold

First, we recall what it means for a set to be *countably $n-$rectifiable*:

Definition 1: Countably $n-$rectifiable Sets

A subset $m \subseteq \mathbb{R}^{n+\ell}$ is countably $n-$rectifiable if there is a set M_0 with $\mathcal{H}^n(M_0) = 0$ and Lipschitz functions $f_j : \mathbb{R}^n \to \mathbb{R}^{n+\ell}$ for $j = 1, 2, \ldots$ such that

$$M \subseteq M_0 \cup \left(\bigcup_{j=1}^{\infty} f_j(\mathbb{R}^n) \right).$$

The sets $f_j(\mathbb{R}^n)$ remind us of local charts in atlases for manifolds. Indeed, countably $n-$rectifiable sets already serve as generalizations to manifolds. Up to a measure zero set, these sets look like the union of countably many embedded $n-$manifolds. This is given formally in the following result:

Lemma 1: Unions of embedded manifolds

A set M is countably $n-$rectifiable if and only if there is a set N_0 with $\mathcal{H}^n(N_0) = 0$ and $n-$dimensional embedded C^1 submanifolds of $\mathbb{R}^{n+\ell}$ N_j for $j = 1, 2, \ldots$ such that

$$M \subseteq \cup_{j=0}^{\infty} N_j.$$

For a proof, we refer the reader to Lemma 1.2 of Chapter 3 in [3].

If we want to generalize manifolds, we should also have an appropriate description of tangent spaces. These will be the *approximate tangent spaces*. First, for $\lambda > 0$ and $x \in M$, we define $\eta_{x,\lambda} \colon \mathbb{R}^{n+\ell} \to \mathbb{R}^{n+\ell}$ to be the function

$$\eta_{x,\lambda}(y) = \lambda^{-1}(y - x).$$

Definition 2: Approximate Tangent Spaces

Suppose M is an $\mathcal{H}^n-$measurable subset of $\mathbb{R}^{n+\ell}$ and θ is a positive $\mathcal{H}^n-$measurable function on M. Additionally, suppose $\int_{M \cap K} \theta \, \mathrm{d}\mathcal{H}^n < \infty$ for all compact $K \subseteq \mathbb{R}^{n+\ell}$. For each $x \in \mathbb{R}^{n+\ell}$, we say an $n-$dimensional subspace of P_x is an approximate tangent plane of M with respect to θ if

$$\lim_{\lambda \downarrow 0} \int_{\eta_{x,\lambda}(M)} f(y)\theta(x + \lambda y) \, \mathrm{d}\mathcal{H}^n = \theta(x) \int_{P_x} f(y) \, \mathrm{d}\mathcal{H}^n$$

for all continuous functions f with compact support.
Define $TM := \{(x, P_x) \mid x \in M_*\}$, where M_* is the set of x such that M has an approximate tangent space with respect to θ.

The function θ should be thought of as a multiplicity or density function, telling us how to weigh each point. In fact, if we set $\theta \equiv 1$ on M —so every point is weighted equally— and let M be an $n-$manifold, we have that P_x and T_xM coincide.

Fortunately, countably $n-$rectifiable sets have approximate tangent spaces almost everywhere:

Theorem 1: Countably $n-$rectifiable and Tangents

Suppose M and θ satisfy the same hypotheses as Definition 2. Then, M is countably $n-$rectifiable if and only if M has an approximate tangent space P_x with respect to θ for $\mathcal{H}^n-$a.e. $x \in M$.

We are now ready to define rectifiable $n-$varifolds. We want a varifold to be a countably rectifiable set along with a tangent space at almost every point. Since, as always, sets of measure zero are irrelevant, a varifold will be defined as an equivalence class.

Definition 3: Rectifiable $n-$varifolds

Let M be a countably $n-$rectifiable $\mathcal{H}^n-$measurable subset of $\mathbb{R}^{n+\ell}$, and let θ be a positive locally $\mathcal{H}^n-$integrable function on M. Then, the rectifiable $n-$varifold $\underline{v}(M, \theta)$ is the equivalence class of all pairs (M', θ') such that M' is countably $n-$rectifiable, $\mathcal{H}^n((M \setminus M') \cup (M' \setminus M)) = 0$, and $\theta = \theta'$ $\mathcal{H}^n-$a.e. on $M \cap M'$. We call θ the multiplicity function of $\underline{v}(M, \theta)$.

Next, we define some useful notions for a rectifiable $n-$varifold:

Definition 4: Useful Definitions

Let $V = \underline{v}(M, \theta)$ be a rectifiable $n-$varifold.

(1) (Weight Measure) There is a Radon measure μ_V, called the weight measure of V, given by

$$\mu_V(A) = \int_{A \cap M} \theta \, \mathrm{d}\mathcal{H}^n.$$

(2) (Mass) The mass $\mathbb{M}(V)$ is given by $\mu_V(\mathbb{R}^{n+\ell})$.

(3) (Tangent Space) Given a point $x \in M$, we define the tangent space $T_x M$ as the approximate tangent space of M with respect to θ.

We have seen one way to approach varifolds, starting with countably $n-$rectifiable sets and considering a multiplicity function to build tangent spaces. We now explore a slightly different approach.

2.2 General Varifolds

Let $G(n+\ell, n)$ denote the set of all $n-$dimensional subspaces of $\mathbb{R}^{n+\ell}$. For subspaces S, T, denote by ρ_S, ρ_T the orthogonal projections of $\mathbb{R}^{n+\ell}$ onto S and T, respectively. We equip $G(n+\ell, n)$ with the metric $\rho(S,T) = |\rho_S - \rho_T|$. We will use G to encode tangent spaces into our varifold directly. To do this, we define for a subset $A \subseteq \mathbb{R}^{n+\ell}$

$$G_n(A) := A \times G(n+\ell, n),$$

equipped with the product metric.

Definition 5: General Varifolds

Let U be an open subset of $\mathbb{R}^{n+\ell}$. Then, a general $n-$varifold on U is any Radon measure on $G_n(U)$.

Intuitively, the Radon measure is picking out $n-$dimensional subspaces above each point in U. We can now make similar definitions as in the rectifiable case:

Definition 6: Useful Definitions 2

Let V be a general $n-$varifold on U. Let $\pi\colon G_n(U) \to U$ be the projection mapping $(x, S) \mapsto x$.

(1) (Weight Measure) There is a Radon measure μ_v, called the weight measure of V, given by

$$\mu_V(A) = V(\pi^{-1}(A)).$$

(2) (Mass) The mass $\mathbb{M}(V)$ is given by $\mu_V(U)$.

Comparing Definitions 4 and 6, we note a striking similarity. In fact, given any rectifiable $n-$varifold we can construct a corresponding general $n-$varifold. The construction is as follows:

Suppose $\underline{v}(M, \theta)$ is a rectifiable $n-$varifold and μ is its weight measure. Let π be as in Definition 6. Then, construct the Radon measure V on $G_n(\mathbb{R}^{n+\ell})$ given by

$$V(A) = \mu(\pi(A \cap TM)).$$

It is straightforward to see that $\mu_V = \mu$. Thus, $\underline{v}(M, \theta)$ and V have the same weight measure and it is sensible to call V the associated general $n-$varifold of $\underline{v}(M, \theta)$.

This raises the question of whether we can do this operation in reverse. Can we construct a rectifiable $n-$varifold given a general $n-$varifold?

The observant reader may have noticed that we did not define Tangent Spaces for general $n-$varifolds. This is because they do not always exist. When they do, the answer to the above question is positive and general varifolds correspond to rectifiable varifolds.

To define tangent spaces for general varifolds we first need some definitions. Given an $n-$varifold V on U, a point $x \in U$ and $\lambda > 0$, we define $V_{x,\lambda}$ as the $n-$varifold on U given by

$$V_{x,\lambda}(A) = \lambda^{-n} V(\{(\lambda y + x, S) \mid (y, S) \in A\} \cap G_n(U)).$$

Given $T \in G(n+\ell, n)$, define $\underline{v}(T)$ to be the Radon measure on $G_n(U)$ given by

$$\underline{v}(T)(A) = \int_{T \cap A} 1 \, \mathrm{d}\mathcal{H}^n.$$

Definition 7: Tangent Space of a General Varifold

Given a subspace $T \in G(n+\ell, n)$, a point $x \in U$, and a number $\theta \in (0, \infty)$, we say that an $n-$varifold V has tangent space T with multiplicity θ if

$$\lim_{\lambda \downarrow 0} V_{x,\lambda} = \theta \underline{v}(T),$$

where we take the limit as Radon measures on the left hand side.

We will now show that if V has such a tangent space μ_V-a.e., then V is rectifiable.

3 The Rectifiability Theorem

In order to prove this theorem, we will appeal to a technical lemma:

Lemma 2: Separating Integration

Let V be a general $n-$varifold on U. Then, for any Borel set $A \subseteq U$ and for μ_V-a.e. $x \in U$ there is a Radon measure ν_V^x on $G(n+\ell, n)$ such that for any nonnegative continuous function $\beta \colon G(n+\ell, n) \to \mathbb{R}$, we have

$$\int_{G(n+\ell,n)} \beta(S) \, \mathrm{d}\eta_V^x(S) = \lim_{\rho \downarrow 0} \frac{\int_{G_n(B_\rho(x))} \beta(S) \, \mathrm{d}V(y, S)}{\mu_V(B_\rho(x))}$$

and

$$\int_{G_n(A)} \beta(S) \, \mathrm{d}V(x, S) = \int_A \int_{G(n+\ell,n)} \beta(S) \, \mathrm{d}\eta_V^x(S) \, \mathrm{d}\mu_V(x),$$

where $B_\rho(x)$ denotes the ball of radius ρ centered at x.

For a proof, we refer the reader to [3].

We are now ready to prove the main result of this paper:

Theorem 2: The First Rectifiability Theorem

Suppose V is an $n-$varifold on U with a tangent space T_x with multiplicity $\theta(x)$ for μ_V-a.e. $x \in U$. Then, V is a rectifiable $n-$varifold.

Proof. We will show that the set $M := \{x \in \operatorname{spt}(V) \mid T_x, \theta(x) \text{ exist } \}$ is countably $n-$rectifiable. With this, we will conclude $V = \underline{v}(M, \theta)$.
Indeed, note the similarities between Definition 7 and Definition 2. It follows from Definition 7 that if $x \in M$, then

$$\lim_{\lambda \downarrow 0} \int_{\eta_{x,\lambda}(M)} f(y)\theta(x+\lambda y)\,\mathrm{d}\mathcal{H}^n = \theta(x) \int_{T_x} f(y)\,\mathrm{d}\mathcal{H}^n.$$

Therefore, by Definition 2, we know T_x is an approximate tangent plane of M with respect to θ. Since T_x exists for all $x \in M$, Theorem 1 gives us that M is countably $n-$rectifiable. Furthermore, it is straightforward to check that μ_V is given by

$$\mu_V(A) = \int_{A \cap M} \theta\,\mathrm{d}\mathcal{H}^n.$$

Using this in the first equality of Lemma 2, we find that

$$\int_{G(n+\ell,n)} \beta(S)\,\mathrm{d}\eta_V^x(S) = \theta(x)\beta(T_x).$$

Therefore, for any Borel set $A \subseteq U$, we get from the second equality of Lemma 2 that

$$\int_{G_n(A)} \beta(S)\,\mathrm{d}V(x,S) = \int_{A \cap M} \beta(T_x)\,\mathrm{d}\mu_V(x).$$

Since A and β are arbitrary, we conclude that for any non-negative f on $G_n(U)$,

$$\int_{G_n(U)} f(x,S)\,\mathrm{d}V(x,S) = \int_M f(x,T_x)\,\mathrm{d}\mu_V(x),$$

so V coincides with the rectifiable $n-$manifold $\underline{v}(M,\theta)$, as desired. $\square$

References

[1] Lawrence C. Evans and Ronald F. Gariepy. *Measure Theory and Fine Properties of Functions*. 2015.

[2] Pertti Mattila. *Geometry of Sets and Measures in Euclidean Spaces*. 1995.

[3] Leon Simon. *Introduction to Geometric Measure Theory*. 2014.