Constructing Braid Invariants via Representations

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Abstract

In this paper, we introduce braids and the braid group. Braids are convenient knotted objects to work with, as there is a group structure we can give on them. We can use linear maps on vector spaces to construct representations of the braid groups, which when suitably modified, give braid invariants. In this way, we use linear algebra to answer the topological question of when two braids are different.

1 Introduction

Braids are one of the most ubiquitous objects in mathematics. Their properties, and those of the corresponding *braid groups*, are relevant in many areas of study. The braid group arises naturally in the setting of configuration spaces and mapping class groups. Moreover, braids have interesting applications to algebraic geometry, robotics, and even public key cryptography [2]. Less surprisingly, braids are related to the topological field of knot theory. A celebrated theorem by Markov (Theorem 2.3 in [1]) exactly characterizes how to switch from knots to braids and vice-versa.

Informally, we think of a braid as a box containing some strands, fixed at the top and bottom of the box. Furthermore, we require that the strands are "pulled tight". That is, if an ant traveled from the bottom end to the top end of the strand, then its height is always increasing.

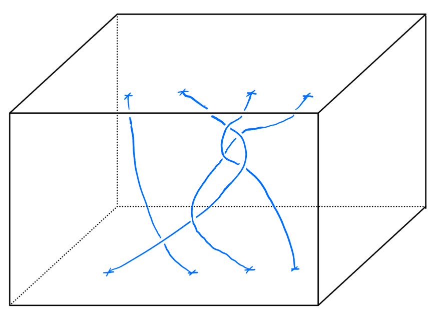


Figure 1: A $4-$stranded braid.

The central question of braid theory is determining when two braids are "the same". Intuitively, two braids are *equivalent* if, starting from one braid, we can pull its strands to obtain the other. This process should keep the ends fixed, not pass strands through each other, and maintain the strands pulled tight. Proving that two braids are equivalent is straightforward: we show how to pull the strands to get from one braid to the other. On the other hand, proving two braids are not equivalent is trickier; how are we sure we have not missed the right way to pull the strands? One way mathematicians tackle the question of distinguishing braids is via *invariants*. Simply put, a braid invariant is a way to assign some mathematical object to any given braid consistently. If we assign two braids different invariants, then they cannot be equivalent. The number of strands provides one such invariant. However, when our braids have the same number of strands, we have to come up with more sophisticated assignments.

In this paper, we show how group representations of the braid group can be used to obtain braid invariants. Since we can define a group structure on braids, group representations provide a way to assign matrices to braids. After imposing some conditions on the matrices to ensure this assignment is done consistently, we obtain interesting braid invariants.

We assume familiarity with point-set topology, basic group theory, and linear algebra. In Section 2 we bring the reader up to speed with the algebra required for the paper. In Section 3 we introduce braids and the braid group. Finally, in Section 4 we present the Yang-Baxter equation and use it to construct braid invariants. We conclude by using our techniques to explicitly show that the following two braids are not equivalent:

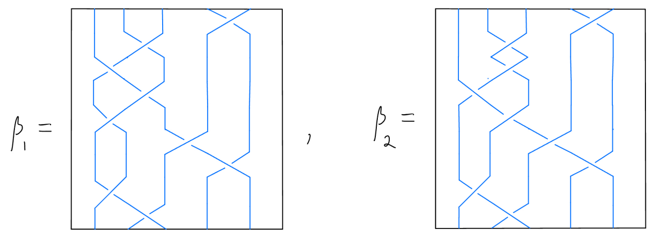


Figure 2: Two non-equivalent braids.

2 Algebra

As we will see in sections 3.2 and 4.1, there are two natural ways in which we can combine braids: putting one on top of the other, called composition, and putting one next to the other, called tensor product. When assigning matrices to our braids, composition will correspond to matrix multiplication. To take care of the other operation, we define tensor products in the context of linear algebra.

2.1 Tensor Products

Tensor products are a useful way to "mix" two vector spaces. For a more abstract and general discussion of tensor products we refer the reader to [3].

Definition 2.1

Let $V \cong \mathbb{C}^n$ and $W \cong \mathbb{C}^m$ be two finite-dimensional complex vector spaces, and let $\{e_1, \ldots, e_n\}$ and $\{f_1, \ldots, f_m\}$ be bases of V and W, respectively. Then, $V \otimes W$ is the vector space with a basis given by the formal symbols $\{e_i \otimes f_j\}$ for $i = 1, \ldots, n$ and $j = 1, \ldots, m$.
Observe that $V \otimes W \cong \mathbb{C}^{n \cdot m}$. Moreover, the tensor product satisfies the following equations:

1) $v \otimes w + v' \otimes w = (v + v') \otimes w$;
2) $v \otimes w + v \otimes w' = v \otimes (w + w')$;
3) $\lambda(v \otimes w) = (\lambda v) \otimes w = v \otimes (\lambda w)$.

These equations give a way to define the formal symbol[1] $v \otimes w$, for any $v \in V$ and $w \in W$. Write v_i for the $i-$th coordinate of the vector v. Then,

$$v \otimes w = \sum_{i=1}^{n} \sum_{j=1}^{m} v_i w_j \left(e_i \otimes f_j\right).$$

The most important tensor products for our purposes will be $n-$fold tensor products of a vector space with itself. For brevity, we denote $\underbrace{V \otimes \cdots \otimes V}_{n \text{ times}}$ by $V^{\otimes n}$.

[1] Be careful not to think that every element of $V \otimes W$ is of the form $v \otimes w$, as this is usually not the case! For example, if V is two dimensional with basis $\{e_1, e_2\}$, then $e_1 \otimes e_1 + e_2 \otimes e_2$ is an element of $V \otimes V$, but it cannot be written as $v \otimes w$ for any $v, w \in V$.

2.2 Matrices

We are also interested in how tensor products combine matrices, or linear maps between vector spaces.

Definition 2.2

Let $V \cong \mathbb{C}^n$ be a finite-dimensional complex vector space and let $S, T\colon V \to V$ be two linear maps. Then, $S \otimes T$ is the linear map $V \otimes V \to V \otimes V$ given by mapping

$$v \otimes w \mapsto S(v) \otimes T(w).$$

This map satisfies the following equations:
1) $(S \otimes T)(v \otimes w + v' \otimes w') = (S \otimes T)(v \otimes w) + (S \otimes T)(v' \otimes w')$;
2) $(S \otimes T)(\lambda(v \otimes w)) = \lambda(S \otimes T)(v \otimes w)$.

Choosing a basis of $V \cong \mathbb{C}^n$, the maps S, T become $n \times n$ complex matrices. According to Definition 2.1, the choice of basis of V gives a corresponding basis for $V \otimes V$, under which $S \otimes T$ becomes a $n^2 \times n^2$ complex matrix. This matrix corresponds to the Kronecker product, described in page 421 of [4].

For the rest of the paper, we pick a basis of V. We use the same symbol to refer to the linear map and the matrix in this basis. From the above definition, if we have $n \times n$ matrices S, S', T, T', the matrix product of $S \otimes T$ and $S' \otimes T'$ is

$$(S \otimes T) \circ (S' \otimes T') = (S \circ S') \otimes (T \circ T').$$

We denote the set of invertible linear maps $V \to V$ by $\mathrm{GL}(V)$. If we have $S, T \in \mathrm{GL}(V)$, then the above equation shows $S^{-1} \otimes T^{-1}$ is the inverse of $S \otimes T$. Thus, $S \otimes T \in \mathrm{GL}(V \otimes V)$.

Just as with $n-$fold product of vector spaces, if $S \in \mathrm{GL}(V)$, we denote $\underbrace{S \otimes \cdots \otimes S}_{n \text{ times}} \in \mathrm{GL}(V^{\otimes n})$ by $S^{\otimes n}$.

2.3 Representations

To conclude this section, we define our main tool: group representations. For a more abstract discussion, we refer the reader to [5].

Definition 2.3

Let G be a group and $V \cong \mathbb{C}^n$ be a finite-dimensional complex vector space. A representation of G is a group homomorphism $\rho\colon G \to \mathrm{GL}(V)$.

For our purposes, we will view $\mathrm{GL}(V)$ as the group of invertible $n \times n$ complex matrices. For example, two representations of the group $G = \mathbb{Z}/3\mathbb{Z}$ are given by:

$$0 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad 1 \mapsto \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}, \qquad 2 \mapsto \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix};$$

$$0 \mapsto 1, \qquad 1 \mapsto e^{2\pi i/3}, \qquad 2 \mapsto e^{4\pi i/3}.$$

3 Braids

With our algebra interlude concluded, we return to topology. First, we formalize our discussion of braids from Section 1.

3.1 Fundamentals

Definition 3.1

An $n-$stranded braid σ is a disjoint union of n copies of $[0,1]$ embedded into $\mathbb{R}^2 \times [0,1]$ such that the boundary of σ is $\{1,2,\ldots,n\} \times \{0\} \times \{0,1\}$ and such that for each $t \in [0,1]$, the intersection of $\mathbb{R}^2 \times \{t\}$ and σ consists of exactly n points. We orient every strand of the braid upward.

This definition formalizes our "box with strands" definition from the introduction. The box is $\mathbb{R}^2 \times [0,1]$ and each strand is a copy of $[0,1]$. Moreover, the strands being pulled tight corresponds to the condition $|\mathbb{R}^2 \times \{t\} \cap \sigma| = n$.

Definition 3.2

Given two braids σ and σ', we call them *equivalent* if there is a continuous family of maps

$$h_t : \mathbb{R}^2 \times [0,1] \to \mathbb{R}^2 \times [0,1]$$

such that the following hold, for all $t \in [0,1]$:

1) h_0 is the identity;
2) $h_1(\sigma) = \sigma'$;
3) $h_t |_{\mathbb{R}^2 \times \{0,1\}}$ is the identity;
4) $h_t |_{\mathbb{R}^2 \times \{s\}}$ sends $\mathbb{R}^2 \times \{s\}$ to $\mathbb{R}^2 \times \{s\}$ for all $s \in [0,1]$;
5) h_t is a homeomorphism for all $t \in [0,1]$.

This definition formalizes the notion of pulling the strands. Conditions 1 and 2 tell us that we go from braid σ to braid σ'. Condition 3 ensures we keep the ends fixed and condition 4 keeps the strands pulled tight. Finally, condition 5 makes sure strands do not pass through each other.

Families of maps satisfying conditions 1, 2, and 5 are called *isotopies*. The additional conditions 3 and 4 make our isotopies *level-* and *boundary-preserving*.

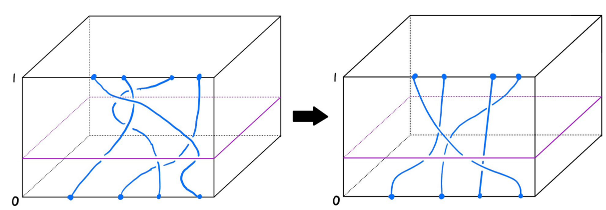


Figure 3: A level- and boundary-preserving isotopy of a 4$-$stranded braid.

It is inconvenient to draw three-dimensional pictures every time that we wish to talk about braids. So, we usually work with *braid-diagrams*. Imagine projecting our braid onto the front wall of the box. When doing so, some intersections will arise. By deforming our braid a little, we can ensure all these intersections are *double-points*, that is, points where only two distinct strands intersect. If, on top of this, the number of double-points is finite, we call the projection *regular*. Given a regular projection of a braid, the only thing needed to recover the original braid is to indicate at each crossing which strand is over the other.

Definition 3.3

An $n-$stranded braid diagram is the image of a regular projection of an $n-$stranded braid, where at each double-point we indicate which strand is over the other with a line break.

For example, Figure 2 pictured the braid diagrams of two braids.

Given two braid diagrams, we are interested in knowing when they represent equivalent braids. To answer this question, we introduce the three braid moves. They are the following local changes to a braid diagram:

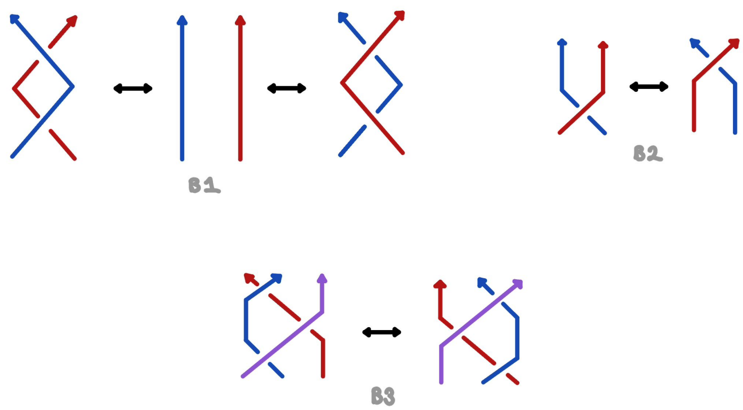


Figure 4: B1: pulling a strand over another, B2: pulling a crossing up, and B3: switching top and bottom.

The following theorem tells us that braid moves are the only interesting changes we can make to a braid diagram:

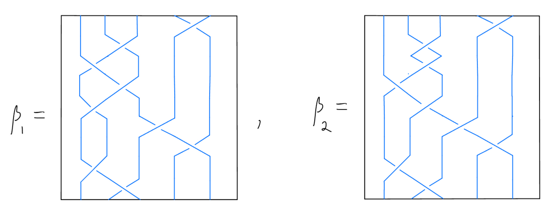
Theorem 3.4
Two braid diagrams represent equivalent braids if and only if there is a finite sequence of braid moves and level- and boundary-preserving isotopies that take one diagram to the other.

For an outline of the proof, see Theorem 4.5 in [6].

In light of this theorem, we can replace thinking about braids up to braid equivalence by thinking about braid diagrams up to the braid moves.

3.2 The Braid Group

Given two braid diagrams, we have a way to prove that they represent the same braid. It suffices to provide a finite sequence of braid moves that takes one diagram to the other. However, how can we show that two braid diagrams represent legitimately different braids? For example, recall the braids β_1 and β_2:



Playing around with them, we can convince ourselves that these braids are not equivalent. To prove this formally, we begin by defining a group structure on braids.

Denote by $\mathcal{B}$ the set of all braids (up to equivalence) and by $\mathcal{B}_n$ the set of all braids with n strands.

Definition 3.5
Let $\sigma, \sigma' \in \mathcal{B}_n$ be two $n-$stranded braids. Their composition $\sigma \circ \sigma'$ is the $n-$stranded braid obtained by stacking σ on top of σ'.

This operation turns $\mathcal{B}_n$ into a group:

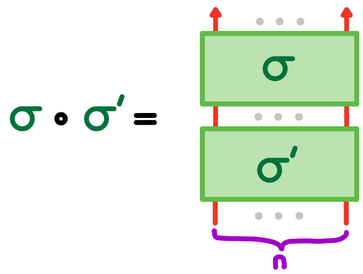


Figure 5: The composition $\sigma \circ \sigma'$ of two $n-$stranded braids.

Lemma 3.6
The set $\mathcal{B}_n$ with braid composition $\circ$ is a group.

Proof. We check $(\mathcal{B}_n, \circ)$ satisfies all the axioms of a group:

(i) (Closed) Since the composition of two $n-$stranded braids is an $n-$stranded braid, we have that $\mathcal{B}_n$ is closed under $\circ$.

(ii) (Associativity) Checking associativity is straightforward.

(iii) (Identity) Consider the $n-$stranded braid with no crossings. If we stack it on top or below any braid, σ, we obtain σ back, so $\mathcal{B}_n$ has an identity element.

Thus, we only have to find inverses to establish our group structure. For this, we first find a set of generators of $\mathcal{B}_n$, which will allow us to explicitly express inverses. For each $i = 1, \ldots, n-1$, we define σ_i as the $n-$stranded braid with exactly one crossing: the $i-$th strand going over the $i+1$th strand. That is,

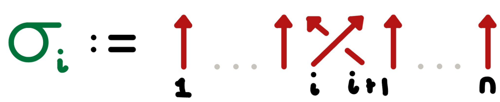


Figure 6: Generators of $\mathcal{B}_n$

Similarly, σ_i^{-1} is the $n-$stranded braid with exactly one crossing: the $i-$th strand going under the $i+1$th strand. We can split a braid horizontally into "levels" where each level has exactly one crossing. This shows that the set $\{\sigma_i, \sigma_i^{-1} \mid i = 1, \ldots, n-1\}$ generates $\mathcal{B}_n$. An example in $\mathcal{B}_4$ is given below:

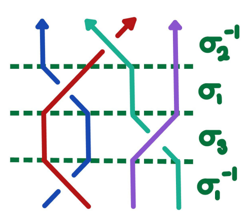


Figure 7: Writing a braid in terms of the generators of $\mathcal{B}_4$.

Therefore, we can write any braid as $\sigma = \sigma_{j_1}^{\delta_1} \circ \sigma_{j_2}^{\delta_2} \circ \ldots \circ \sigma_{j_m}^{\delta_m}$, for some $j_k \in \{1, \ldots, n-1\}$ and $\delta_k \in \{-1, 1\}$.

Proof of Lemma 3.6, continued.

(iv) (Inverses) Observe that $\sigma_i \circ \sigma_i^{-1}$ and $\sigma_i^{-1} \circ \sigma_i$ look like the left and right pictures of braid move B1. Therefore, we can change them to look like the middle picture, which shows σ_i and σ_i^{-1} are inverses. Therefore, the inverse of σ will be given by

$$\sigma^{-1} := \sigma_{j_m}^{-\delta_m} \circ \ldots \circ \sigma_{j_2}^{-\delta_2} \circ \sigma_{j_1}^{-\delta_1}.$$

□

Using our generators, we can give a group presentation of $\mathcal{B}_n$ as $\langle \sigma_i, \sigma_i^{-1} \mid \mathcal{R} \rangle$, where the relations in $\mathcal{R}$ encode the remaining braid moves B2 and B3. These are

$$\mathcal{R} = \begin{cases} \sigma_i \circ \sigma_j = \sigma_j \circ \sigma_i & \text{if } |i-j| \geq 2 \\ \sigma_i \circ \sigma_{i+1} \circ \sigma_i = \sigma_{i+1} \circ \sigma_i \circ \sigma_{i+1} & i = 1, \ldots, n-2 \end{cases}$$

The first set of relations is called *long commutativity* and comes from move B2. The second set of relations come from move B3. Because of this presentation, we can replace thinking about braid diagrams up to the braid moves by thinking about the braid group.

The two example braids β_1 and β_2 are elements of $\mathcal{B}_5$. Written in terms of our generators, we see

$$\beta_1 = \sigma_4 \circ \sigma_2 \circ \sigma_1^{-1} \circ \sigma_2 \circ \sigma_1 \circ \sigma_3 \circ \sigma_4^{-1} \circ \sigma_1 \circ \sigma_2^{-1}, \tag{1}$$
$$\beta_2 = \sigma_4 \circ \sigma_2 \circ \sigma_2 \circ \sigma_1^{-1} \circ \sigma_2 \circ \sigma_3 \circ \sigma_4^{-1} \circ \sigma_1 \circ \sigma_2^{-1}. \tag{2}$$

4 Representations of Braids

Now that we have endowed braids with a group structure, we can use our algebraic techniques to construct braid invariants. We begin by breaking down braids into simple *elmentary pieces*.

4.1 Assigning a Map

Definition 4.1
Let $\sigma, \sigma' \in \mathcal{B}$ be two braids (not necessarily with the same number of strands). Their tensor product $\sigma \otimes \sigma'$ is the braid obtained by putting σ to the left of σ'.

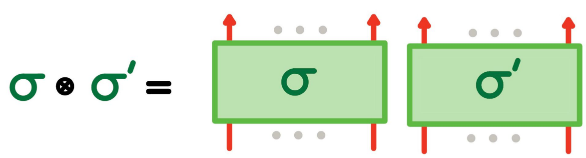


Figure 8: The tensor product $\sigma \otimes \sigma'$ of two braids.

Using these, we can break down any generator of the braid group $\mathcal{B}_n$ into the following three elementary pieces:

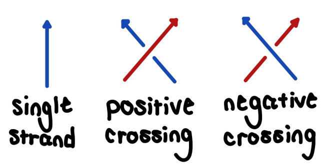


Figure 9:

Let V be some finite-dimensional complex vector space and pick some $R \in \mathrm{GL}(V \otimes V)$. For each k we define a map $\rho\colon \mathcal{B}_k \to \mathrm{GL}(V^{\otimes n})$ as follows:

To each elementary piece we assign the matrix

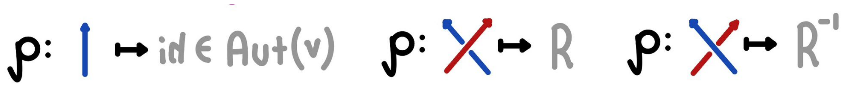


Figure 10: Maps assigned to each elementary piece.

where $\mathrm{id} \in \mathrm{GL}(V)$ denotes the identity on V. Then, we extend this map via the tensor product, so that

$$\rho(\sigma \otimes \sigma') = \rho(\sigma) \otimes \rho(\sigma').$$

This assigns an invertible matrix to each generator of $\mathcal{B}_n$. Namely, ρ maps the generator σ_i to the linear map $\mathrm{id}^{\otimes(i-1)} \otimes R \otimes \mathrm{id}^{\otimes(n-i-1)} \in \mathrm{GL}(V^{\otimes n})$. Finally, we extend the map via composition, so that for $\sigma, \sigma' \in \mathcal{B}_n$,

$$\rho(\sigma \circ \sigma') = \rho(\sigma) \circ \rho(\sigma').$$

In this way, we have assigned a matrix to every braid.

4.2 The Yang-Baxter Equation

Our goal assigning these linear maps is to obtain invariants of braids. In other words, since we identify equivalent braids with elements of a braid group, we wish to show that ρ gives a group representation of $\mathcal{B}_n$. It turns out that this is not always the case. A general map R will not always make the assignment ρ a homomorphism. However, there is a special type of linear map R that will work.

Definition 4.2: R-Matrix

Let V be a finite vector space. The Yang-Baxter equation in $\mathrm{GL}(V \otimes V \otimes V)$ is

$$(\mathrm{id} \otimes R) \circ (R \otimes \mathrm{id}) \circ (\mathrm{id} \otimes R) = (R \otimes \mathrm{id}) \circ (\mathrm{id} \otimes R) \circ (R \otimes \mathrm{id}).$$

A solution $R \in \mathrm{GL}(V \otimes V)$ is called an *R-matrix*.

We claim that when R is an R-matrix, the matrix assigned to each braid by ρ is indeed a braid invariant. This is equivalent to proving the following theorem:

Theorem 4.3

Let V be a finite-dimensional complex vector space and $R \in \mathrm{GL}(V \otimes V)$ be an R-matrix. Then, the map

$$\rho\colon \mathcal{B}_n \to \mathrm{GL}(V^{\otimes n})\colon \sigma_i \mapsto \mathrm{id}^{\otimes(i-1)} \otimes R \otimes \mathrm{id}^{\otimes(n-i-1)},$$

for $i = 1, \ldots, n-1$ and extended homomorphically gives a representation of the braid group $\mathcal{B}_n$.

Proof. Using our presentation of the braid group $\mathcal{B}_n$, it suffices to prove that the following hold:

(i) For all $i, j = 1, \ldots, n-1$, we have $\rho(\sigma_i \circ \sigma_j) = \rho(\sigma_i) \circ \rho(\sigma_j)$;

(ii) For all $i = 1, \ldots, n-1$, we have $\rho(\sigma_i^{-1}) = \rho(\sigma_i)^{-1}$;

(iii) For all $i, j = 1, \ldots, n-1$ such that $|i - j| \geq 2$, we have $\rho(\sigma_i \circ \sigma_j) = \rho(\sigma_j \circ \sigma_i)$;

(iv) For all $i = 1, \ldots, n-2$ we have $\rho(\sigma_i \circ \sigma_{i+1} \circ \sigma_i) = \rho(\sigma_{i+1} \circ \sigma_i \circ \sigma_{i+1})$.

Now, (i) and (ii) hold from our construction of ρ, so it only remains to check (iii) and (iv). For (iii), assume without loss of generality that $i < j$ (so $i \leq j-2$). Observe that

$$\begin{aligned}
\rho(\sigma_i \circ \sigma_j) &= \rho(\sigma_i) \circ \rho(\sigma_j) \\
&= (\mathrm{id}^{\otimes(i-1)} \otimes R \otimes \mathrm{id}^{\otimes(n-i-1)}) \circ (\mathrm{id}^{\otimes(j-1)} \otimes R \otimes \mathrm{id}^{\otimes(n-j-1)}) \\
&= \mathrm{id}^{\otimes(i-1)} \otimes R \otimes \mathrm{id}^{\otimes(j-i-2)} \otimes R \otimes \mathrm{id}^{\otimes(n-j-1)} \\
&= (\mathrm{id}^{\otimes(j-1)} \otimes R \otimes \mathrm{id}^{\otimes(n-j-1)}) \circ (\mathrm{id}^{\otimes(i-1)} \otimes R \otimes \mathrm{id}^{\otimes(n-i-1)}) \\
&= \rho(\sigma_j) \circ \rho(\sigma_i) \\
&= \rho(\sigma_j \circ \sigma_i).
\end{aligned}$$

Finally, it remains to check (iv). For $n \leq 2$, this holds vacuously. So, assume $n \geq 3$. Recall that R is an R-matrix. Hence,

$$(\mathrm{id} \otimes R) \circ (R \otimes \mathrm{id}) \circ (\mathrm{id} \otimes R) = (R \otimes \mathrm{id}) \circ (\mathrm{id} \otimes R) \circ (R \otimes \mathrm{id}).$$

Taking the left tensor product with $\mathrm{id}^{\otimes(i-1)}$ and the right tensor product with $\mathrm{id}^{\otimes(n-i-2)}$ gives

$$\begin{aligned}
&(\mathrm{id}^{\otimes i} \otimes R \otimes \mathrm{id}^{\otimes(n-i-2)}) \circ (\mathrm{id}^{\otimes i-1} \otimes R \otimes \mathrm{id}^{\otimes(n-i-1)}) \circ (\mathrm{id}^{\otimes i} \otimes R \otimes \mathrm{id}^{\otimes(n-i-2)}) \\
&= (\mathrm{id}^{\otimes(i-1)} \otimes R \otimes \mathrm{id}^{\otimes(n-i-1)}) \circ (\mathrm{id}^{\otimes i} \otimes R \otimes \mathrm{id}^{\otimes(n-i-2)}) \circ (\mathrm{id}^{\otimes(i-1)} \otimes R \otimes \mathrm{id}^{\otimes(n-i-1)})
\end{aligned}$$

so $\rho(\sigma_{i+1} \circ \sigma_i \circ \sigma_{i+1}) = \rho(\sigma_i \circ \sigma_{i+1} \circ \sigma_i)$, as desired. □

Observe that property (iv) implies that if ρ is a representation, then R has to be a solution of the Yang-Baxter equation.

4.3 Constructing Invariants

Equipped with the technology needed to generate braid invariants from representations, we compute a particular example. We will let V be a two-dimensional vector space with basis $\{e_0, e_1\}$. Then, the tensor product $V \otimes V$ has dimension 4 and a canonical ordered basis $\{e_0 \otimes e_0, e_0 \otimes e_1, e_1 \otimes e_0, e_1 \otimes e_1\}$.

Since we want to construct invariants, our 4×4 R-matrix has to satisfy the Yang-Baxter equation. Additionally, for simplicity we will assume R is of the following form, for scalars a, b, c, d, e, f:

$$R = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & c & 0 \\ 0 & d & e & 0 \\ 0 & 0 & 0 & f \end{pmatrix}$$

Matrices of this form are said to have the *charge conservation* condition, and satisfy that $R_{i,j}^{k,\ell} = 0$ if $i + j \neq k + \ell$. Here, $R_{i,j}^{k,\ell}$ denotes the entry of R in the $(i,j)-$th column and $(k,\ell)-$th row.

Observe that $V \otimes V \otimes V$ has dimension 8. Furthermore, we can split $V^{\otimes 3}$ into a direct sum of four spaces: $V^{\otimes 3} = \oplus_{i=0}^{3} V_i$, where V_i are the vector spaces spanned by the following bases:

$$\begin{aligned}
V_0 &= \mathrm{span}(\{e_0 \otimes e_0 \otimes e_0\}) \\
V_1 &= \mathrm{span}(\{e_0 \otimes e_0 \otimes e_1, e_0 \otimes e_1 \otimes e_0, e_1 \otimes e_0 \otimes e_0\}) \\
V_2 &= \mathrm{span}(\{e_0 \otimes e_1 \otimes e_1, e_1 \otimes e_0 \otimes e_1, e_1 \otimes e_1 \otimes e_0\}) \\
V_3 &= \mathrm{span}(\{e_1 \otimes e_1 \otimes e_1\}).
\end{aligned}$$

Observe that V_i is spanned by the $e_a \otimes e_b \otimes e_c$ satisfying $a + b + c = i$. Thus, R is block diagonal on these spaces. Therefore, we restrict our attention to calculating $R \otimes \mathrm{id}$ and $\mathrm{id} \otimes R$ on each V_i. After some computation, we arrive at the following results:

$$(R \otimes \mathrm{id})\mid_{V_0} = (\mathrm{id} \otimes R)\mid_{V_0} = \begin{pmatrix} a \end{pmatrix}$$

$$(R \otimes \mathrm{id})\mid_{V_1} = \begin{pmatrix} a & 0 & 0 \\ 0 & b & c \\ 0 & d & e \end{pmatrix}, (\mathrm{id} \otimes R)\mid_{V_1} = \begin{pmatrix} b & c & 0 \\ d & e & 0 \\ 0 & 0 & a \end{pmatrix}$$

$$(R \otimes \mathrm{id})\mid_{V_2} = \begin{pmatrix} b & c & 0 \\ d & e & 0 \\ 0 & 0 & f \end{pmatrix}, (\mathrm{id} \otimes R)\mid_{V_2} = \begin{pmatrix} f & 0 & 0 \\ 0 & b & c \\ 0 & d & e \end{pmatrix}$$

$$(R \otimes \mathrm{id})\mid_{V_3} = (\mathrm{id} \otimes R)\mid_{V_3} = \begin{pmatrix} f \end{pmatrix}.$$

Now, the Yang-Baxter equation tells us that

$$(\mathrm{id} \otimes R) \circ (R \otimes \mathrm{id}) \circ (\mathrm{id} \otimes R) = (R \otimes \mathrm{id}) \circ (\mathrm{id} \otimes R) \circ (R \otimes \mathrm{id}).$$

We want this equation to be satisfied upon restricting to each V_i. It clearly holds when restricting to V_0 and V_3. The other two restrictions imply a, b, c, d, e, f solve the following system of equations:

$$\begin{aligned} b(ab + cd - a^2) &= 0, \\ b(bf + cd - f^2) &= 0, \\ e(cd + ae - a^2) &= 0, \\ e(cd + ef - f^2) &= 0, \\ bce = bde = be(e - b) &= 0. \end{aligned}$$

Observe that this system is symmetric in b and e. Hence, we assume without loss of generality that $e = 0$ and $b \neq 0$. Under this assumptions, the above system simplifies to

$$\begin{aligned} ab + cd - a^2 &= 0, \\ bf + cd - f^2 &= 0. \end{aligned}$$

We will leave everything in terms of a, c, d. The first equation gives $b = a - cd/a$. Solving the quadratic equation for f in the second equation, we find $f = a$ or $f = -cd/a$. Hence, we get two $R-$matrices:

$$R_1 = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a - cd/a & c & 0 \\ 0 & d & 0 & 0 \\ 0 & 0 & 0 & a \end{pmatrix}, \qquad R_2 = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a - cd/a & c & 0 \\ 0 & d & 0 & 0 \\ 0 & 0 & 0 & -cd/a \end{pmatrix}.$$

4.4 Telling Two Braids Apart

Recall the two braids β_1 and β_2. We now have the tools to show that they are not equivalent.

Theorem 4.4

Braids β_1 and β_2 are not equivalent.

Proof. From section 3.2 we have that, as elements of $\mathcal{B}_5$,

$$\begin{aligned} \beta_1 &= \sigma_4 \circ \sigma_2 \circ \sigma_1^{-1} \circ \sigma_2 \circ \sigma_1 \circ \sigma_3 \circ \sigma_4^{-1} \circ \sigma_1 \circ \sigma_2^{-1}, \\ \beta_2 &= \sigma_4 \circ \sigma_2 \circ \sigma_2 \circ \sigma_1^{-1} \circ \sigma_2 \circ \sigma_3 \circ \sigma_4^{-1} \circ \sigma_1 \circ \sigma_2^{-1}. \end{aligned}$$

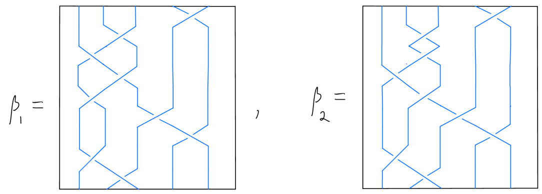


Figure 11: Non-equivalent braids.

We wish to show $\beta_1 \neq \beta_2$. We will suppose for the sake of contradiction that they are equal. Since $\mathcal{B}_5$ is a group, we can compose the equality $\beta_1 = \beta_2$ with $(\sigma_4 \circ \sigma_2)^{-1}$ on the left and with $(\sigma_3 \circ \sigma_4^{-1} \circ \sigma_1 \circ \sigma_2^{-1})^{-1}$ on the right to obtain

$$\sigma_1^{-1} \circ \sigma_2 \circ \sigma_1 = \sigma_2 \circ \sigma_1^{-1} \circ \sigma_2.$$

If β_1 and β_2 are equivalent, this equality has to hold. Now, observe that this equality does not involve the fourth and fifth strands of the braid, so it suffices to show that it does not hold in $\mathcal{B}_3$.

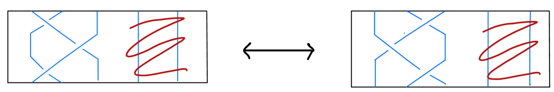


Figure 12: Reducing the problem to 3−stranded braids.

To do this, consider $a = c = 1$ and $d = 2$ for the matrix R_1 from section 4.3. This gives the R-matrix

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

which induces a representation $\rho \colon \mathcal{B}_3 \to \mathrm{GL}(V^{\otimes 3})$. We have

$$\rho(\sigma_1^{-1} \circ \sigma_2 \circ \sigma_1) = (R^{-1} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes R) \circ (R \otimes \mathrm{id}),$$
$$\rho(\sigma_2 \circ \sigma_1^{-1} \circ \sigma_2) = (\mathrm{id} \otimes R) \circ (R^{-1} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes R).$$

We will show that these two linear maps are different by showing that they map $e_0 \otimes e_0 \otimes e_1 \in V^{\otimes 3}$ to different elements. We compute

$$R^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 1 & 1/2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

so we have

$$\begin{aligned}(R^{-1} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes R) \circ (R \otimes \mathrm{id})(e_0 \otimes e_0 \otimes e_1) &= (R^{-1} \otimes \mathrm{id})(\mathrm{id} \otimes R)(e_0 \otimes e_0 \otimes e_1) \\ &= (R^{-1} \otimes \mathrm{id})(-e_0 \otimes e_0 \otimes e_1 + 2e_0 \otimes e_1 \otimes e_0) \\ &= -e_0 \otimes e_0 \otimes e_1 + 2e_1 \otimes e_0 \otimes e_0\end{aligned}$$

and

$$\begin{aligned}(\mathrm{id} \otimes R) \circ (R^{-1} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes R)(e_0 \otimes e_0 \otimes e_1) &= (\mathrm{id} \otimes R) \circ (R^{-1} \otimes \mathrm{id})(-e_0 \otimes e_o \otimes e_1 + 2e_0 \otimes e_1 \otimes e_0) \\ &= (\mathrm{id} \otimes R)(-e_0 \otimes e_0 \otimes e_1 + 2e_1 \otimes e_0 \otimes e_0) \\ &= e_0 \otimes e_0 \otimes e_1 - 2e_0 \otimes e_1 \otimes e_0 + 2e_1 \otimes e_0 \otimes e_0.\end{aligned}$$

These are manifestly not equal, so $\rho(\sigma_1^{-1} \circ \sigma_2 \circ \sigma_1) \neq \rho(\sigma_2 \circ \sigma_1^{-1} \sigma_2)$. Therefore, braids β_1 and β_2 are not equivalent, just as we wanted to prove. $\square$

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