How to solve the Arnold conjecture

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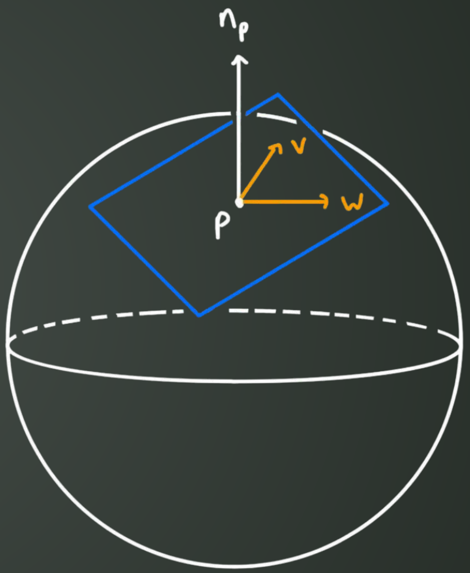
Table of contents

Symplectic Geometry

In classical mechanics, physicists solve problems by considering a *phase space* of possible states for a system.

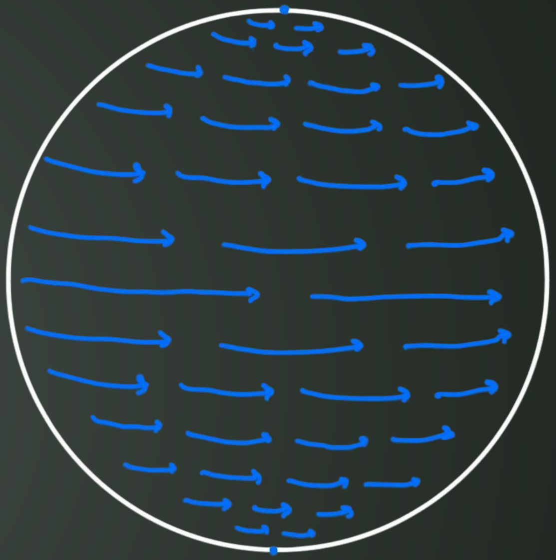
In math, phase space becomes an even-dimensional manifold endowed with a *symplectic form*, ω. Such manifolds are called *symplectic manifolds*.



$$\omega_p(v,w) = \langle v \times w, n \rangle$$

01

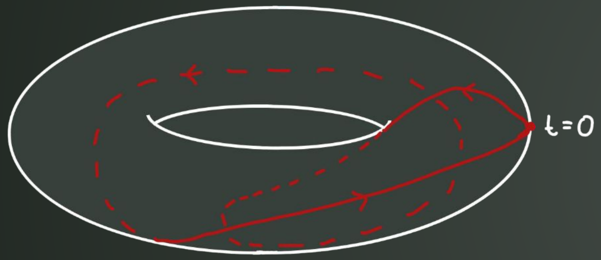
Hamiltonian Systems



$\omega(Y, X_H) = dH(Y)$

Counting periodic solutions of a differential system

Hamiltonians



$$H_t(x,y) = \frac{1}{2\pi}\left[\sin(2\pi y) + \cos(2\pi(x - t\cos(2\pi y)))\right]$$

A *time-dependent Hamiltonian* is a smooth function $H : W \times \mathbb{R} \to \mathbb{R}$. We write H_t for the Hamiltonian at time t, $H_t(p) = H(p,t)$.

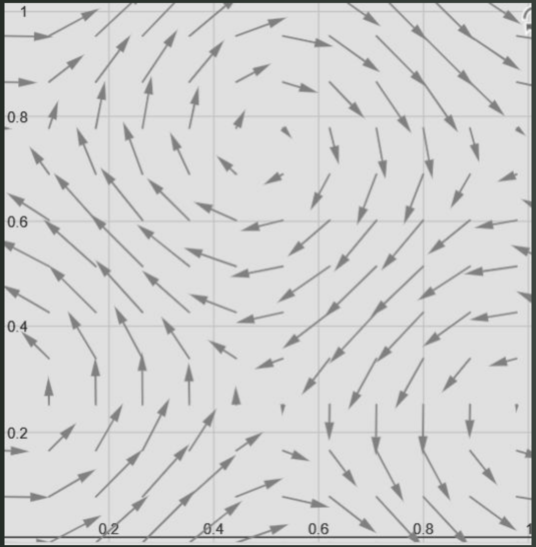
Equipped with a symplectic form, we define the *symplectic gradient* of H_t by the formula:

$$\omega(Y, X_t) = dH_t(Y)$$

We are interested in solutions of period 1 to the *Hamiltonian system*

$$\dot{x}(t) = X_t(x(t))$$

The Arnold Conjecture



<u>Conjecture 1</u>: Let W be a compact symplectic manifold and let

$$H : W \times \mathbb{R} \to \mathbb{R}$$

be a time-dependent Hamiltonian. Then, there are at least

$$\sum_k \dim H_k(W; \mathbb{Z}/2)$$

periodic solutions to the associated Hamiltonian system.

Autonomous Hamiltonians

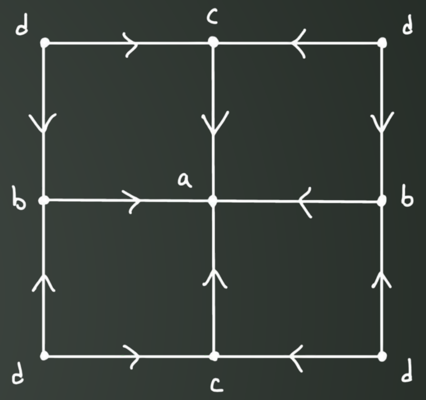
Suppose H does not depend on time. We can prove the Arnold conjecture as follows:

<u>Step 1</u>: Find the critical points of H.

<u>Step 2</u>: Compute the *index of each critical point.*

<u>Step 3</u>: Count the number of trajectories connecting critical points.

<u>Step 4</u>: Build a chain complex whose homology coincides with the singular homology of W



$$H(x,y) = \cos(2\pi x) + \cos(2\pi y)$$

$$\underset{d}{\mathbb{Z}/2} \xrightarrow{0} \underset{b}{\mathbb{Z}/2} \oplus \underset{c}{\mathbb{Z}/2} \xrightarrow{0} \underset{a}{\mathbb{Z}/2}$$

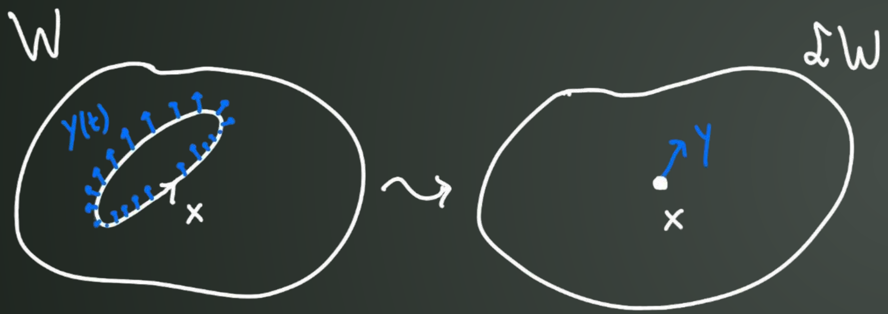
02

The Action Functional

$$\frac{\partial u}{\partial s} + J \frac{\partial u}{\partial t} + \operatorname{grad} H_t = 0$$

Periodic solutions as critical points

The Loop Space



To follow our recipe, we need to work with the space of loops. We denote the space of contractible loops by $\mathcal{L}W$.

A tangent vector at $x \in \mathcal{L}W$ is given by a vector field along x.

The Action Functional

For each $x \in \mathcal{L}W$, pick an extension $u\colon D^2 \to W$. We define

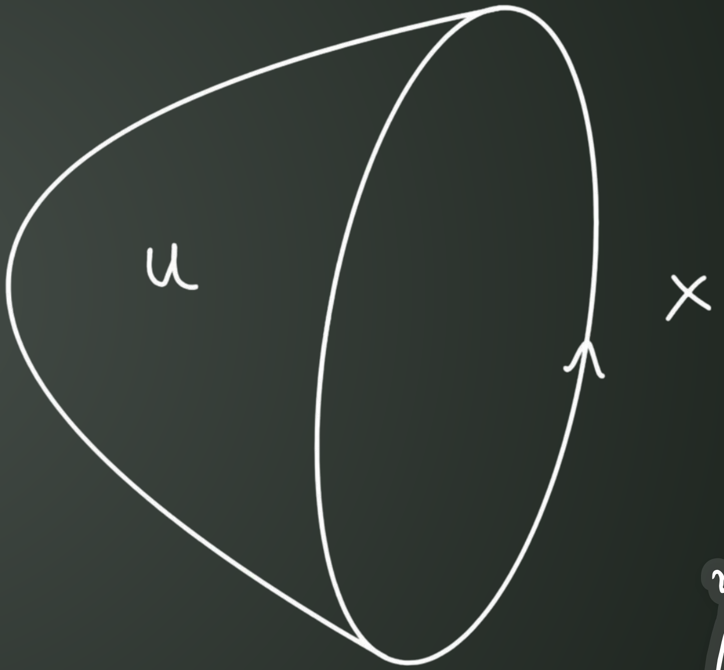
$$\mathcal{A}_H(x) = -\int_{D^2} u^\star\omega + \int_0^1 H_t(x(t))\mathrm{d}t$$

This is well-defined under suitable assumptions. Thus, we have a function

$$\mathcal{A}_H\colon \mathcal{L}W \to \mathbb{R}$$

We can compute its derivative:

$$(d\mathcal{A}_H)_x(Y) = \int_0^1 \omega_{x(t)}(\dot{x}(t) - X_t(x), Y)\mathrm{d}t$$



The Floer Equation

To define trajectories, we need a notion of gradient on our loop space.

We use an *almost complex structure* J, and compute

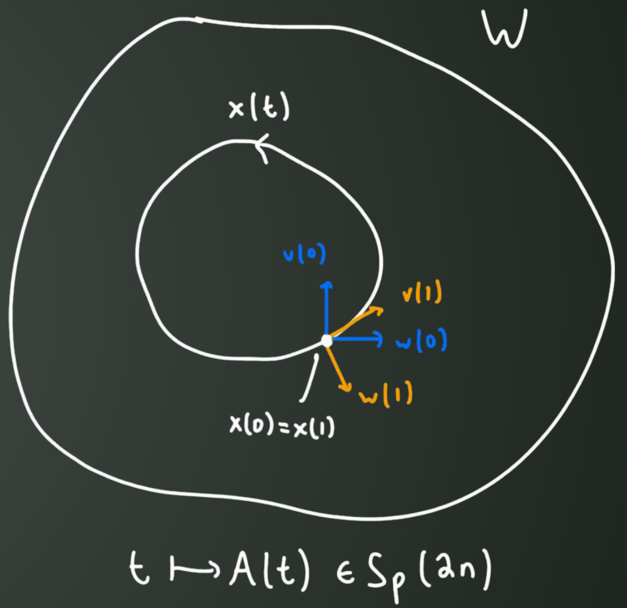
$$\operatorname{grad} \mathcal{A}_H = J\dot{x} + \operatorname{grad} H_t$$

Thus, the trajectories of the (negative) gradient are solutions $u\colon \mathbb{R}_s \times S^1_t \to W$ to the differential equation

$$\frac{\partial u}{\partial s} + J\frac{\partial u}{\partial t} + \operatorname{grad} H_t = 0$$

03

Making a Chain Complex



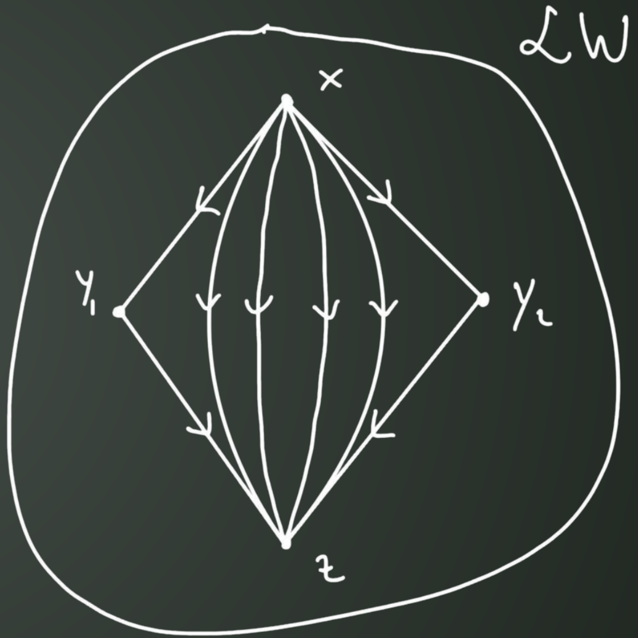
The space of trajectories and the differential

The Space of Trajectories

Denote the space of trajectories from x to y by $\mathcal{L}(x, y)$. We can show that $\mathcal{L}(x, y)$ is a manifold of dimension $\mu(x) - \mu(y) - 1$

Thus, if $\mu(x) - \mu(y) = 1$, $\mathcal{L}(x, y)$ is a 0-dimensional manifold, and if $\mu(x) - \mu(z) = 2$, $\mathcal{L}(x, z)$ is a 1-dimensional manifold.

If we include *broken trajectories*, then $\overline{\mathcal{L}}(x, z)$ is compact, with boundary consisting of broken trajectories.



The Floer Complex

$$\cdots \longrightarrow CF_k(W) \xrightarrow{\partial} CF_{k-1}(W) \xrightarrow{\partial} \cdots \xrightarrow{\partial} CF_0(W)$$

$$CF_k(W) = (\mathbb{Z}/2)^{\#\text{ crit. pts. of } \mu = k} \qquad \partial(x) = \sum_{\mu(y)=\mu(x)-1} n(x,y)\, y$$

04

Floer Homology

$$\#\,\text{periodic solutions} \geq \sum_{k=0} \dim HF_k(W;\mathbb{Z}/2)$$

A concrete computation

Invariance of Floer Homology

We can prove that the homology of the Floer complex does not depend on the choice of H or J.

Picking a suitable pair (H, J), we can show that this homology coincides with the singular homology of W.

$$\#\begin{matrix}\text{periodic}\\ \text{solutions}\end{matrix} \geq \sum_{k\geq 0} \dim HF_k(W;\mathbb{Z}/2) = \sum_{k\geq 0} \dim H_k(W;\mathbb{Z}/2)$$

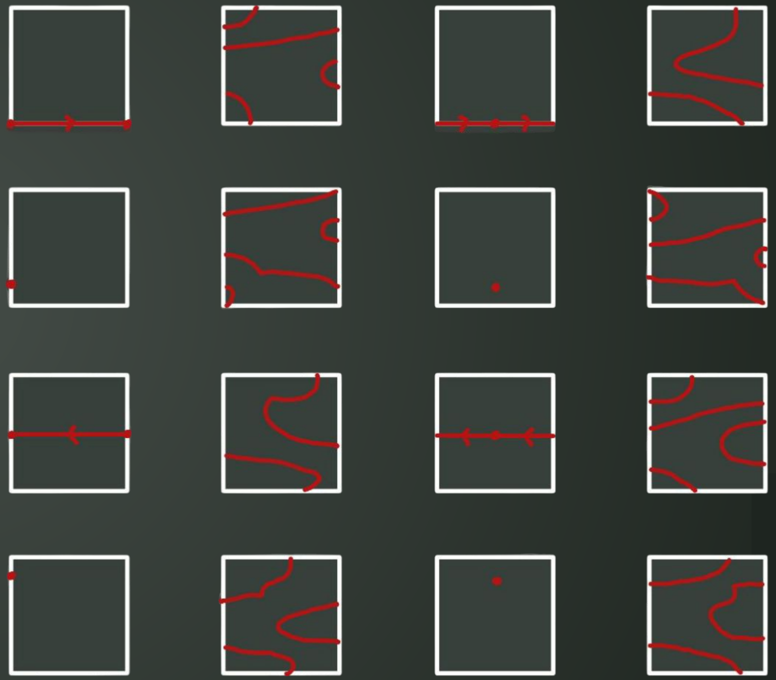
An Example

Consider the Hamiltonian

$$H_t(x,y) = \frac{1}{2\pi}\left[\sin(2\pi y) + \cos(2\pi(x - t\cos(2\pi y)))\right]$$

defined on the torus $T^2 = \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$.

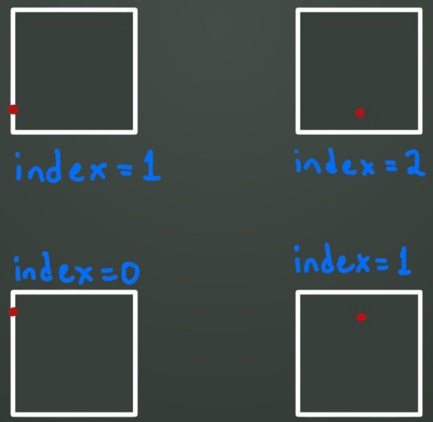
It has four contractible periodic solutions.



Example (Continued)

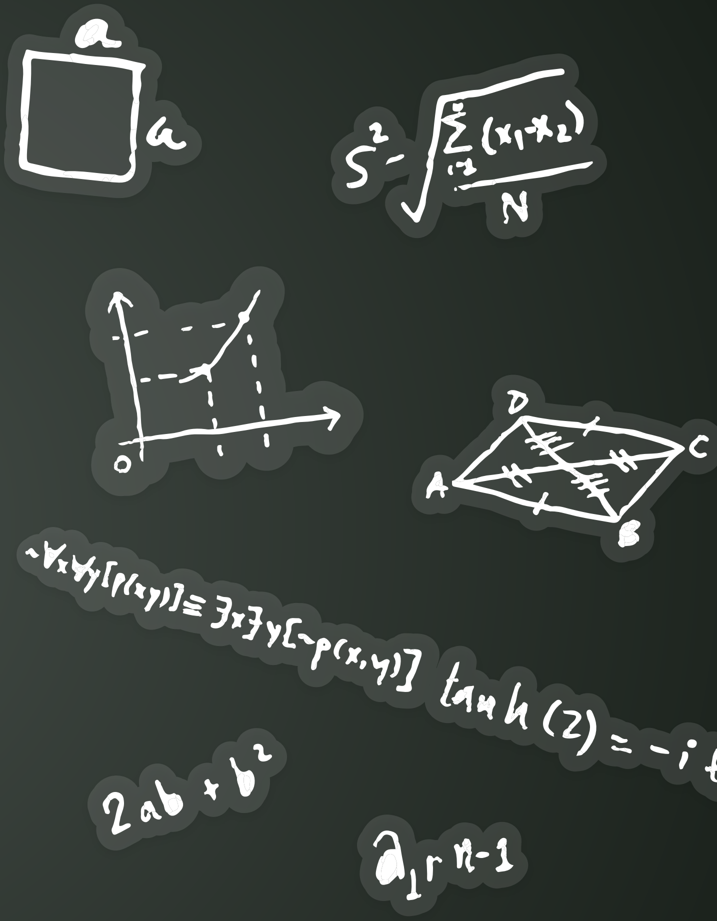
According to the indices, the Floer complex must be

$$\mathbb{Z}/2 \longrightarrow \mathbb{Z}/2 \oplus \mathbb{Z}/2 \longrightarrow \mathbb{Z}/2$$



Thank you!

Questions?



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Followed Audin and Damian's *Morse Theory and Floer Homology*