Constructing Braid Invariants via Representations

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Abstract

In this expository paper, we introduce the notion of braids and use tools from linear algebra to construct braid invariants. Braids are convenient knotted objects to work with, as there is a group structure we can give on them. We can use linear maps on vector spaces to construct representations of the braid groups, which when suitably modified, give braid invariants. In this way, we use linear algebra to answer the topological question of when two braids are different.

1 Introduction

In this paper we study how group representations on the so-called braid group can be used to obtain invariants of braids. Braids are a type of knotted object, which we imagine as a bunch of strings living inside a box. We are interested in knowing when two braids are different, in the sense that one cannot be deformed to the other. We use braid invariants to do this. Since we can define a group structure on braids, we use group representations to assign linear maps to braids. After imposing some conditions, we will find that these linear maps give us some interesting braid invariants.

In Section 2 we lay out the necessary algebra background for the paper. In Section 3 we introduce braids and the braid group. Finally, in Section 4 we combine these in order to obtain braid invariants via the Yang-Baxter equation. We showcase these constructions by using them to prove two braids are not equivalent. We strongly follow Jackson and Moffatt [2].

2 Background

Throughout this paper we will assume familiarity with point-set topology, vector spaces, and basic group theory. Knowledge of knot theory is helpful, but not required. Our main linear algebra tool is the tensor product, exploiting its ability to combine vector spaces and linear maps between them.

2.1 Tensor Products

We think of tensor products as a useful way to "mix" two vector spaces. For a deeper discussion of tensor products (in the context of modules over commutative rings) see [1].

Definition 1: Tensor Product of Vector Spaces

Given two vector spaces V, W over the same field K, we define their tensor product $V \otimes W$ as follows: Pick a basis $\{e_i\}_{i\in I}$ of V and a basis $\{f_j\}_{j\in J}$ of W. Then, $V \otimes W$ is the vector space with a basis given by the formal symbols $\{e_i \otimes f_j\}_{i\in I, j\in J}$. The tensor product satisfies the following equalities:

1) $v \otimes w + v' \otimes w = (v + v') \otimes w$;

2) $v \otimes w + v \otimes w' = v \otimes (w + w')$;

3) $\lambda(v \otimes w) = (\lambda v) \otimes w = v \otimes (\lambda w)$.

The above equalities give a way to define the formal symbol $v \otimes w$ [1], where $v \in V$ and $w \in W$ are arbitrary. Write $v = \sum_{i \in I} v_i e_i$ and $w = \sum_{j \in J} w_j f_j$. Then,

$$v \otimes w = \sum_{i \in I, j \in J} v_i w_j \left(e_i \otimes f_j\right).$$

From now on we will only concern ourselves with finite dimensional vector spaces. The above definition makes it clear that if V has dimension m and W has dimension n, then $V \otimes W$ is a vector space of dimension $m \cdot n$.

In this paper, we will consider $n-$fold tensor products of a vector space with itself. For brevity, we will denote $\underbrace{V \otimes \cdots \otimes V}_{n \text{ times}}$ by $V^{\otimes n}$.

2.2 Linear Maps

We are also interested in how tensor products combine linear maps. Given a vector space V and linear maps $S, T \colon V \to V$, we can similarly construct the tensor product $S \otimes T \colon V \otimes V \to V \otimes V$ by mapping

$$v \otimes w \mapsto S(v) \otimes T(w),$$

for $v, w \in V$ such that the following equalities hold:

1) $(S \otimes T)(v \otimes w + v' \otimes w') = (S \otimes T)(v \otimes w) + (S \otimes T)(v' \otimes w')$;
2) $(S \otimes T)(\lambda(v \otimes w)) = \lambda(S \otimes T)(v \otimes w)$.

These equalities guarantee that $S \otimes T$ is a linear map from $V \otimes V$ to itself, and we can check that they are consistent with the equalities satisfied by the tensor product. From the above definition, we see that if we have linear maps $S, S', T, T' \colon V \to V$, the composition of $S \otimes T$ and $S' \otimes T'$ is

$$(S \otimes T) \circ (S' \otimes T') = (S \circ S') \otimes (T \circ T').$$

A linear map $S \colon V \to V$ is invertible if it has an inverse. We denote the set of invertible linear maps $V \to V$ by $\mathrm{Aut}(V)$. If we have $S, T \in \mathrm{Aut}(V)$, then $S \otimes T \in \mathrm{Aut}(V \otimes V)$, as we can show that $S^{-1} \otimes T^{-1}$ is an inverse of $S \otimes T$.

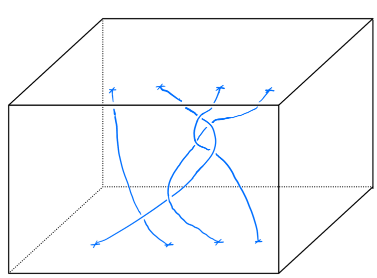
Just as with $n-$fold product of vector spaces, if $S \in \mathrm{Aut}(V)$, we denote $\underbrace{S \otimes \cdots \otimes S}_{n \text{ times}} \in \mathrm{Aut}(V^{\otimes n})$ by $S^{\otimes n}$.

Given a group G, we call a homomorphism $G \to \mathrm{Aut}(V)$ a group representation of G on V.

3 Braids

3.1 Fundamentals

Informally, we think of a braid as a box containing some strands, fixed at the top and bottom of the box, such that, if each strand is oriented from bottom to top, no strand ever goes down.



[1]Be careful not to think that every element of $V \otimes W$ is of the form $v \otimes w$, as this is usually not the case! For example, if V is two dimensional with basis $\{e_1, e_2\}$, then $e_1 \otimes e_1 + e_2 \otimes e_2$ is an element of $V \otimes V$, but a straightforward computation shows that it cannot be written as $v \otimes w$ for $v, w \in V$.

We think of two braids as equivalent if, starting from one braid, we can pull its strands continuously while keeping their ends fixed to obtain the other braid. This deformation should not pass strands through each other nor orient any strand downward. We define these concepts formally as follows:

Definition 2: Braid

An $n-$stranded braid σ is a disjoint union of n copies of $[0,1]$ embedded into $\mathbb{R}^2 \times [0,1]$ such that the boundary of σ is $\{1,2,\dots,n\} \times \{0\} \times \{0,1\}$ and such that for each $t \in [0,1]$, the intersection of $\mathbb{R}^2 \times \{t\}$ and σ consists of exactly n points. We orient every strand of the braid upward.

Definition 3: Braid Equivalence

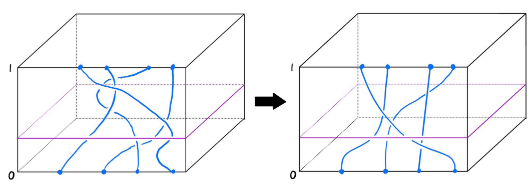
Given two braids σ and σ', we call them *equivalent* if there is a continuous family of maps

$$h_t : \mathbb{R}^2 \times [0,1] \to \mathbb{R}^2 \times [0,1]$$

such that the following hold, for all $t \in [0,1]$:

1) h_0 is the identity;
2) $h_1(\sigma) = \sigma'$;
3) $h_t \mid_{\mathbb{R}^2 \times \{0,1\}}$ is the identity;
4) $h_t \mid_{\mathbb{R}^2 \times \{s\}}$ sends $\mathbb{R}^2 \times \{s\}$ to $\mathbb{R}^2 \times \{s\}$ for all $s \in [0,1]$;
5) h_t is a homeomorphism for all $t \in [0,1]$.

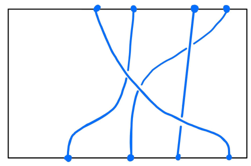
Families of maps satisfying conditions 1, 2, and 5 are called *isotopies*. We additionally require our isotopies to be *level- and boundary-preserving*. Condition 3 ensures that the endpoints of the braid are fixed throughout the isotopy ("boundary-preserving"). Condition 4 makes sure that strands are only pulled "horizontally", so that this process never orients any strand downward and the number of intersections of our braid with each plane is always exactly n ("level-preserving").



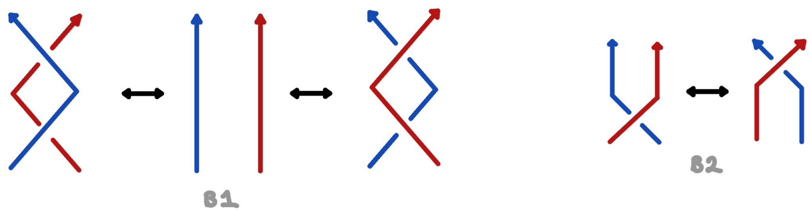
It is inconvenient to draw three-dimensional pictures every time that we wish to talk about braids. As such, we usually work with so-called *braid-diagrams*. To do this, imagine projecting our braid onto the front wall of the box. When doing so, some intersections will arise. By deforming our braid a little, we can ensure all these intersections are *double-points*, that is, points where only two distinct strings intersect. If, on top of this, the number of double-points is finite, we call the projection *regular*. Given a regular projection of a braid, the only thing remaining is to indicate at each crossing which strand is over the other.

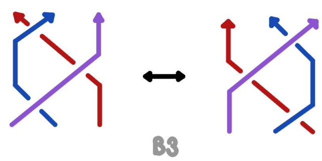
Definition 4: Braid Diagram

An $n-$stranded braid diagram is the image of a regular projection of an $n-$stranded braid, where at each double-point we indicate with a line break which strand is over the other.



Given two braid diagrams, we are interested in knowing when they represent equivalent braids. To answer this question for knots, we have the Reidemester moves. Analogously, for braids we have the three braid moves. They are the following local changes to a braid diagram:





We have a theorem analogous to the Reidemester theorem, which we state without proof[2].

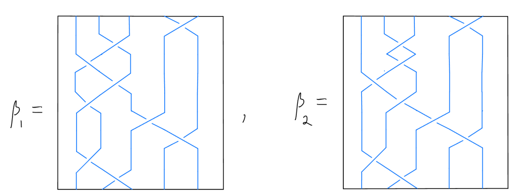
Theorem 1: Braid Moves give Equivalent Braids

Two braid diagrams represent equivalent braids if and only if there is a finite sequence of braid moves and level- and boundary-preserving isotopies that take one diagram to the other.

In light of this theorem, we think of braids up to braid equivalence and braid diagrams up to the braid moves as equivalent.

3.2 The Braid Group

Given two braid diagrams, we have a way to prove that they represent the same braid. It suffices to provide a finite sequence of braid moves that takes one diagram to the other. However, how can we show that two braid diagrams represent legitimately different braids? For example, consider the following two braids:

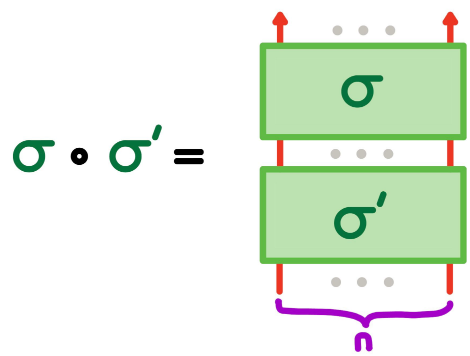


Playing around with them, we can convince ourselves that these braids are not equivalent. In order to prove this formally, we will define a group structure on braids, which will allow us to use our algebra techniques to assign braid invariants. A braid invariant will be some object assigned to each braid such that equivalent

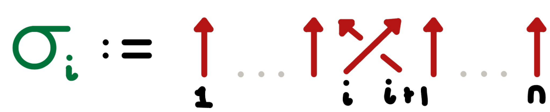
[2]For an outline of the proof, see Theorem 4.5 in [2].

braids are assigned the same object. Then, we will show that the above two braids have different objects assigned to them, so they are not equivalent.

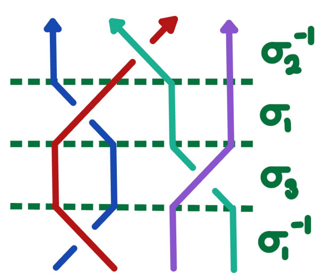
We denote by $\mathcal{B}$ the set of all braids (up to equivalence) and by $\mathcal{B}_n$ the set of all braids with n strands. Given two braids $\sigma, \sigma' \in \mathcal{B}_n$, we define their composition $\sigma \circ \sigma'$ as the braid obtained by stacking σ on top of σ'. With pictures, this looks like the following:



We want to check that $(\mathcal{B}_n, \circ)$ is a group. It is clear that $\mathcal{B}_n$ is closed under $\circ$, and it is not hard to convince oneself that the operation is associative. Furthermore, the $n-$stranded braid with no crossings acts as an identity element. Thus, we only have to find inverses to establish our group structure. For this, we first find a set of generators of $\mathcal{B}_n$, which will allow us to explicitly express inverses. For each $i = 1, \ldots, n-1$, we define σ_i as the $n-$stranded braid with exactly one crossing: the $i-$th strand going over the $i+1$th strand. That is,



Similarly, σ_i^{-1} is the $n-$stranded braid with exactly one crossing: the $i-$th strand going under the $i+1$th strand. Then, the set $\{\sigma_i, \sigma_i^{-1} \mid i = 1, \ldots, n-1\}$ generates $\mathcal{B}_n$. We can see this by splitting a braid horizontally into "levels" where only one crossing occurs per level. An example in $\mathcal{B}_4$ is given below:



Therefore, for some $j_k \in \{1, \ldots, n-1\}$ and $\delta_k \in \{-1, 1\}$, we can write any braid as $\sigma = \sigma_{j_1}^{\delta_1} \circ \sigma_{j_2}^{\delta_2} \circ \ldots \circ \sigma_{j_m}^{\delta_m}$. Now, observe that braid move B1 implies that σ_i and σ_i^{-1} are inverses. Therefore, the inverse of σ will be given by

$$\sigma^{-1} := \sigma_{j_m}^{-\delta_m} \circ \ldots \circ \sigma_{j_2}^{-\delta_2} \circ \sigma_{j_1}^{-\delta_1}.$$

Hence, $\mathcal{B}_n$ is a group as we wanted to show. Using our generators, we can give a group presentation of $\mathcal{B}_n$ as $\langle \sigma_i, \sigma_i^{-1} \mid \mathcal{R} \rangle$, where the relations in $\mathcal{R}$ encode the remaining braid moves B2 and B3. These are

$$\begin{cases} \sigma_i \circ \sigma_j = \sigma_j \circ \sigma_i & \text{if } |i-j| \geq 2 \\ \sigma_i \circ \sigma_{i+1} \circ \sigma_i = \sigma_{i+1} \circ \sigma_i \circ \sigma_{i+1} & i = 1, \ldots, n-2 \end{cases}$$

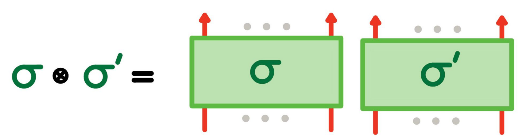
The first set of relations is called *long commutativity* and comes from move B2. The second set of relations come from move B3. In this way, we can think of braid diagrams up to the braid moves and the braid group as equivalent.
The two example braids above are elements of $\mathcal{B}_5$. Written in terms of our generators, we see

$$\beta_1 = \sigma_4 \circ \sigma_2 \circ \sigma_1^{-1} \circ \sigma_2 \circ \sigma_1 \circ \sigma_3 \circ \sigma_4^{-1} \circ \sigma_1 \circ \sigma_2^{-1},$$
$$\beta_2 = \sigma_4 \circ \sigma_2 \circ \sigma_2 \circ \sigma_1^{-1} \circ \sigma_2 \circ \sigma_3 \circ \sigma_4^{-1} \circ \sigma_1 \circ \sigma_2^{-1}.$$

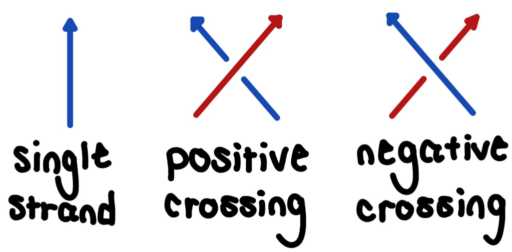
4 Representations of Braids

4.1 Assigning a Map

We have already discussed how to decompose braids via composition. Now, we show another way we can break down braids into simpler pieces. Given any two braids $\sigma, \sigma' \in \mathcal{B}$ (not necessarily with the same number of strands), we define their tensor product $\sigma \otimes \sigma'$ as putting σ to the left of σ'. With pictures, this looks like the following:

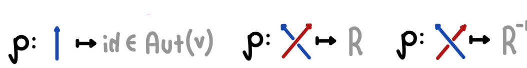


Using these, we can break down any generator of the braid group $\mathcal{B}_n$ (and their inverses) into the following three elementary pieces:



We will do this in order to assign a linear map to each braid. We hope to assign maps in such a way that we obtain braid invariants.

Pick some $R \in \mathrm{Aut}(V \otimes V)$, where V is some finite dimensional vector space. For each n we define a map $\rho\colon \mathcal{B}_n \to \mathrm{Aut}(V^{\otimes n})$ as follows:
To each elementary piece we assign the linear map:



where id $\in \mathrm{Aut}(V)$ denotes the identity on V.
Then, we extend this map via the tensor product, so that

$$\rho(\sigma \otimes \sigma') = \rho(\sigma) \otimes \rho(\sigma').$$

Observe that this assigns an invertible linear map to each generator of $\mathcal{B}_n$. Namely, ρ maps the generator σ_i to the linear map $\mathrm{id}^{\otimes(i-1)} \otimes R \otimes \mathrm{id}^{\otimes(n-i-1)} \in \mathrm{Aut}(V^{\otimes n})$. Finally, we extend the map homomorphically via composition, so that for $\sigma, \sigma' \in \mathcal{B}_n$,

$$\rho(\sigma \circ \sigma') = \rho(\sigma) \circ \rho(\sigma').$$

In this way, we have assigned a linear map to every braid.

4.2 The Yang-Baxter Equation

Our goal assigning these linear maps is to obtain invariants of braids. Therefore, we hope that $\rho(\sigma) = \rho(\sigma')$ whenever σ and σ' are separated by a sequence of braid moves. In other words, since we identify equivalent braids with elements of a braid group, we wish to show that ρ gives a group representation of $\mathcal{B}_n$. It turns out that this is not always the case. A general map R will not always make the assignment ρ a homomorphism. However, there is a special type of linear map R that will work.

Definition 5: R-Matrix

Let V be a finite vector space. The Yang-Baxter equation in $\mathrm{Aut}(V \otimes V \otimes V)$ is

$$(\mathrm{id} \otimes R) \circ (R \otimes \mathrm{id}) \circ (\mathrm{id} \otimes R) = (R \otimes \mathrm{id}) \circ (\mathrm{id} \otimes R) \circ (R \otimes \mathrm{id}).$$

A solution $R \in \mathrm{Aut}(V \otimes V)$ is called an *R-matrix*.

We claim that when R is an R-matrix, the linear map assigned to each braid by ρ is indeed a braid invariant. This is equivalent to proving the following theorem:

Theorem 2: Braid Group Representation

Let V be a finite dimensional vector space and $R \in \mathrm{Aut}(V \otimes V)$ be an R-matrix. Then, the map

$$\rho \colon \mathcal{B}_n \to \mathrm{Aut}(V^{\otimes n}) \colon \sigma_i \mapsto \mathrm{id}^{\otimes (i-1)} \otimes R \otimes \mathrm{id}^{\otimes (n-i-1)},$$

for $i = 1, \ldots, n-1$ and extended homomorphically gives a representation of the braid group $\mathcal{B}_n$.

Proof. Using our presentation of the braid group $\mathcal{B}_n$, it suffices to prove that the following hold:

(i) For all $i, j = 1, \ldots, n-1$, we have $\rho(\sigma_i \circ \sigma_j) = \rho(\sigma_i) \circ \rho(\sigma_j)$;

(ii) For all $i = 1, \ldots, n-1$, we have $\rho(\sigma_i^{-1}) = \rho(\sigma_i)^{-1}$;

(iii) For all $i, j = 1, \ldots, n-1$ such that $|i-j| \geq 2$, we have $\rho(\sigma_i \circ \sigma_j) = \rho(\sigma_j \circ \sigma_i)$;

(iv) For all $i = 1, \ldots, n-2$ we have $\rho(\sigma_i \circ \sigma_{i+1} \circ \sigma_i) = \rho(\sigma_{i+1} \circ \sigma_i \circ \sigma_{i+1})$.

Now, (i) and (ii) hold from our construction of ρ, so it only remains to check (iii) and (iv). For (iii), assume without loss of generality that $i < j$ (so $i \leq j-2$). Observe that

$$\begin{aligned}
\rho(\sigma_i \circ \sigma_j) &= \rho(\sigma_i) \circ \rho(\sigma_j) \\
&= (\mathrm{id}^{\otimes (i-1)} \otimes R \otimes \mathrm{id}^{\otimes (n-i-1)}) \circ (\mathrm{id}^{\otimes (j-1)} \otimes R \otimes \mathrm{id}^{\otimes (n-j-1)}) \\
&= \mathrm{id}^{\otimes (i-1)} \otimes R \otimes \mathrm{id}^{\otimes (j-i-2)} \otimes R \otimes \mathrm{id}^{\otimes (n-j-1)} \\
&= (\mathrm{id}^{\otimes (j-1)} \otimes R \otimes \mathrm{id}^{\otimes (n-j-1)}) \circ (\mathrm{id}^{\otimes (i-1)} \otimes R \otimes \mathrm{id}^{\otimes (n-i-1)}) \\
&= \rho(\sigma_j) \circ \rho(\sigma_i) \\
&= \rho(\sigma_j \circ \sigma_i).
\end{aligned}$$

Finally, it remains to check (iv). For $n \leq 2$, this holds vacuously. So, assume $n \geq 3$. Recall that R is an R-matrix. Hence,

$$(\mathrm{id} \otimes R) \circ (R \otimes \mathrm{id}) \circ (\mathrm{id} \otimes R) = (R \otimes \mathrm{id}) \circ (\mathrm{id} \otimes R) \circ (R \otimes \mathrm{id}).$$

Taking the left tensor product with $\mathrm{id}^{\otimes (i-1)}$ and the right tensor product with $\mathrm{id}^{\otimes (n-i-2)}$ gives

$$\begin{aligned}
&(\mathrm{id}^{\otimes i} \otimes R \otimes \mathrm{id}^{\otimes (n-i-2)}) \circ (\mathrm{id}^{\otimes i-1} \otimes R \otimes \mathrm{id}^{\otimes (n-i-1)}) \circ (\mathrm{id}^{\otimes i} \otimes R \otimes \mathrm{id}^{\otimes (n-i-2)}) \\
= &(\mathrm{id}^{\otimes (i-1)} \otimes R \otimes \mathrm{id}^{\otimes (n-i-1)}) \circ (\mathrm{id}^{\otimes i} \otimes R \otimes \mathrm{id}^{\otimes (n-i-2)}) \circ (\mathrm{id}^{\otimes (i-1)} \otimes R \otimes \mathrm{id}^{\otimes (n-i-1)})
\end{aligned}$$

so $\rho(\sigma_{i+1} \circ \sigma_i \circ \sigma_{i+1}) = \rho(\sigma_i \circ \sigma_{i+1} \circ \sigma_i)$, as desired. $\square$

Observe that property (iv) in fact implies that R has to be a solution of the Yang-Baxter equation if ρ is to be a representation.
Now that we have shown that $R-$matrices give rise to braid invariants, we construct some in order to tell apart braids.

4.3 Constructing Invariants

Equipped with the technology needed to generate braid invariants from representations, we compute a particular example. We will let V be a two-dimensional vector space with basis $\{e_0, e_1\}$. Then, the tensor product $V \otimes V$ has dimension 4 and a canonical ordered basis $\{e_0 \otimes e_0, e_0 \otimes e_1, e_1 \otimes e_0, e_1 \otimes e_1\}$. We can describe any linear map $R \in \mathrm{Aut}(V \otimes V)$ by its action on these basis elements. Then, we can represent R as a matrix, each column representing the image of a basis element when written in terms of this same basis.
Since we want to construct invariants, our 4×4 R-matrix has to satisfy the Yang-Baxter equation. Additionally, for simplicity we will assume R is of the following form, for scalars a, b, c, d, e, f:

$$R = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & c & 0 \\ 0 & d & e & 0 \\ 0 & 0 & 0 & f \end{pmatrix}$$

Matrices of this form are said to have the *charge conservation* condition, and satisfy that $R_{i,j}^{k,\ell} = 0$ if $i + j \neq k + \ell$. Here, $R_{i,j}^{k,\ell}$ denotes the entry of R in the $(i,j)-$th column and $(k,\ell)-$th row.
Now, we want R to be a solution to the Yang-Baxter equation. To this end, observe that $V \otimes V \otimes V$ has dimension 8. Furthermore, we can split $V^{\otimes 3}$ into a direct sum of four spaces: $V^{\otimes 3} = \oplus_{i=0}^{3} V_i$, where V_i are the vector spaces spanned by the following bases:

$$\begin{aligned}
V_0 &= \mathrm{span}(\{e_0 \otimes e_0 \otimes e_0\}) \\
V_1 &= \mathrm{span}(\{e_0 \otimes e_0 \otimes e_1, e_0 \otimes e_1 \otimes e_0, e_1 \otimes e_0 \otimes e_0\}) \\
V_2 &= \mathrm{span}(\{e_0 \otimes e_1 \otimes e_1, e_1 \otimes e_0 \otimes e_1, e_1 \otimes e_1 \otimes e_0\}) \\
V_3 &= \mathrm{span}(\{e_1 \otimes e_1 \otimes e_1\}).
\end{aligned}$$

Observe that V_i is spanned by the $e_a \otimes e_b \otimes e_c$ satisfying $a + b + c = i$. This fact alongside the charge conservation condition on R implies that $V_0, \ldots, V_3$ are invariant under $R \otimes \mathrm{id}$ and $\mathrm{id} \otimes R$. Therefore, we restrict our attention to calculating $R \otimes \mathrm{id}$ and $\mathrm{id} \otimes R$ on each V_i. After some computation, we arrive at the following results:

$$(R \otimes \mathrm{id}) \mid_{V_0} = (\mathrm{id} \otimes R) \mid_{V_0} = \begin{pmatrix} a \end{pmatrix}$$

$$(R \otimes \mathrm{id}) \mid_{V_1} = \begin{pmatrix} a & 0 & 0 \\ 0 & b & c \\ 0 & d & e \end{pmatrix}, (\mathrm{id} \otimes R) \mid_{V_1} = \begin{pmatrix} b & c & 0 \\ d & e & 0 \\ 0 & 0 & a \end{pmatrix}$$

$$(R \otimes \mathrm{id}) \mid_{V_2} = \begin{pmatrix} b & c & 0 \\ d & e & 0 \\ 0 & 0 & f \end{pmatrix}, (\mathrm{id} \otimes R) \mid_{V_2} = \begin{pmatrix} f & 0 & 0 \\ 0 & b & c \\ 0 & d & e \end{pmatrix}$$

$$(R \otimes \mathrm{id}) \mid_{V_3} = (\mathrm{id} \otimes R) \mid_{V_3} = \begin{pmatrix} f \end{pmatrix}.$$

Now, the Yang-Baxter equation tells us that

$$(\mathrm{id} \otimes R) \circ (R \otimes \mathrm{id}) \circ (\mathrm{id} \otimes R) = (R \otimes \mathrm{id}) \circ (\mathrm{id} \otimes R) \circ (R \otimes \mathrm{id}).$$

We want this equation to be satisfied upon restricting to each V_i. It clearly holds when restricting to V_0 and V_3. The other two restrictions imply a, b, c, d, e, f solve the following system of equations:

$$\begin{aligned} b(ab + cd - a^2) &= 0, \\ b(bf + cd - f^2) &= 0, \\ e(cd + ae - a^2) &= 0, \\ e(cd + ef - f^2) &= 0, \\ bce = bde = be(e - b) &= 0. \end{aligned}$$

Observe that this system is symmetric in b and e. Hence, we assume without loss of generality that $e = 0$ and $b \neq 0$. Under this assumptions, the above system simplifies to

$$\begin{aligned} ab + cd - a^2 &= 0, \\ bf + cd - f^2 &= 0. \end{aligned}$$

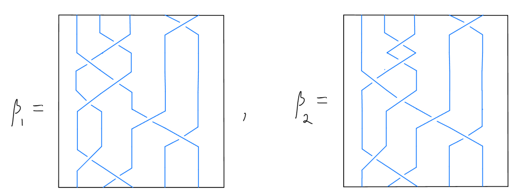
We will leave everything in terms of a, c, d. The first equation gives $b = a - cd/a$. Solving the quadratic equation for f in the second equation, we find $f = a$ or $f = -cd/a$. Hence, we get two $R-$matrices:

$$R_1 = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a - cd/a & c & 0 \\ 0 & d & 0 & 0 \\ 0 & 0 & 0 & a \end{pmatrix}, \qquad R_2 = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a - cd/a & c & 0 \\ 0 & d & 0 & 0 \\ 0 & 0 & 0 & -cd/a \end{pmatrix}.$$

Since these matrices are R-matrices they will give rise to invariants of braids. It turns out that these invariants are related to familiar polynomials from knot theory: R_1 is related to the Jones polynomial and R_2 to the Alexander polynomial.

4.4 Telling Two Braids Apart

Recall the two braids we were considering in Section 3.2:



We now have the tools to show that they are not equivalent.

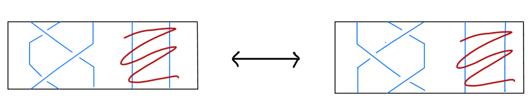
Proof. From section 3.2 we have that, as elements of $\mathcal{B}_5$,

$$\begin{aligned} \beta_1 &= \sigma_4 \circ \sigma_2 \circ \sigma_1^{-1} \circ \sigma_2 \circ \sigma_1 \circ \sigma_3 \circ \sigma_4^{-1} \circ \sigma_1 \circ \sigma_2^{-1}, \\ \beta_2 &= \sigma_4 \circ \sigma_2 \circ \sigma_2 \circ \sigma_1^{-1} \circ \sigma_2 \circ \sigma_3 \circ \sigma_4^{-1} \circ \sigma_1 \circ \sigma_2^{-1}. \end{aligned}$$

We wish to show $\beta_1 \neq \beta_2$. We will suppose for the sake of contradiction that they are equal. Since $\mathcal{B}_5$ is a group, we can compose the equality $\beta_1 = \beta_2$ with $(\sigma_4 \circ \sigma_2)^{-1}$ on the left and with $(\sigma_3 \circ \sigma_4^{-1} \circ \sigma_1 \circ \sigma_2^{-1})^{-1}$ on the right to obtain

$$\sigma_1^{-1} \circ \sigma_2 \circ \sigma_1 = \sigma_2 \circ \sigma_1^{-1} \circ \sigma_2.$$

If β_1 and β_2 are equivalent, this equality has to hold. Now, observe that this equality does not involve the fourth and fifth strands of the braid, so it suffices to show that it does not hold in $\mathcal{B}_3$.



To do this, consider $a = c = 1$ and $d = 2$ for the matrix R_1 from section 4.3. This gives the R-matrix

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

which induces a representation $\rho\colon \mathcal{B}_3 \to \mathrm{Aut}(V^{\otimes 3})$. We have

$$\begin{aligned} \rho(\sigma_1^{-1} \circ \sigma_2 \circ \sigma_1) &= (R^{-1} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes R) \circ (R \otimes \mathrm{id}), \\ \rho(\sigma_2 \circ \sigma_1^{-1} \circ \sigma_2) &= (\mathrm{id} \otimes R) \circ (R^{-1} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes R). \end{aligned}$$

We will show that these two linear maps are different by showing that they map $e_0 \otimes e_0 \otimes e_1 \in V^{\otimes 3}$ to different elements. We compute

$$R^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 1 & 1/2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

so we have

$$\begin{aligned} (R^{-1} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes R) \circ (R \otimes \mathrm{id})(e_0 \otimes e_0 \otimes e_1) &= (R^{-1} \otimes \mathrm{id})(\mathrm{id} \otimes R)(e_0 \otimes e_0 \otimes e_1) \\ &= (R^{-1} \otimes \mathrm{id})(-e_0 \otimes e_0 \otimes e_1 + 2e_0 \otimes e_1 \otimes e_0) \\ &= -e_0 \otimes e_0 \otimes e_1 + 2e_1 \otimes e_0 \otimes e_0 \end{aligned}$$

and

$$\begin{aligned} (\mathrm{id} \otimes R) \circ (R^{-1} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes R)(e_0 \otimes e_0 \otimes e_1) &= (\mathrm{id} \otimes R) \circ (R^{-1} \otimes \mathrm{id})(-e_0 \otimes e_o \otimes e_1 + 2e_0 \otimes e_1 \otimes e_0) \\ &= (\mathrm{id} \otimes R)(-e_0 \otimes e_0 \otimes e_1 + 2e_1 \otimes e_0 \otimes e_0) \\ &= e_0 \otimes e_0 \otimes e_1 - 2e_0 \otimes e_1 \otimes e_0 + 2e_1 \otimes e_0 \otimes e_0. \end{aligned}$$

These are manifestly not equal, so $\rho(\sigma_1^{-1} \circ \sigma_2 \circ \sigma_1) \neq \rho(\sigma_2 \circ \sigma_1^{-1} \sigma_2)$. Therefore, braids β_1 and β_2 are not equivalent, just as we wanted to prove. □

5 Conclusions

We have seen how we can use group representations of $\mathcal{B}_n$ to obtain braid invariants. This will give an invariant exactly when the underlying linear map is an R-matrix, so finding solutions to the Yang-Baxter equation allows us to construct such invariants. However, as we saw from the example above, solving this equation for even the smallest of cases is a laborious task. Thus, we usually assume several conditions in order to simplify our computations. When we are successful, this leads to the construction of useful braid invariants, which we used to tell apart two braids. Furthermore, these invariants turn out to be related to well-known invariants from knot theory, such as the Jones polynomial. It is truly remarkable how we can use linear algebra techniques to produce useful constructions that answer questions in the topological theory of braids.

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