18.904 Lecture Notes (Ezra Guerrero 10/11/2023)

Dehn Twists Generate Mapping Class Groups

Goal: Prove that the mapping class group of a compact orientable surface is generated by Dehn twists.

Mapping Class Groups

Let S be a compact, orientable surface and let $\text{Homeo}^+(S, \partial S)$ denote the group of homeomorphisms of S that preserve its orientation and restrict to the identity on the boundary.

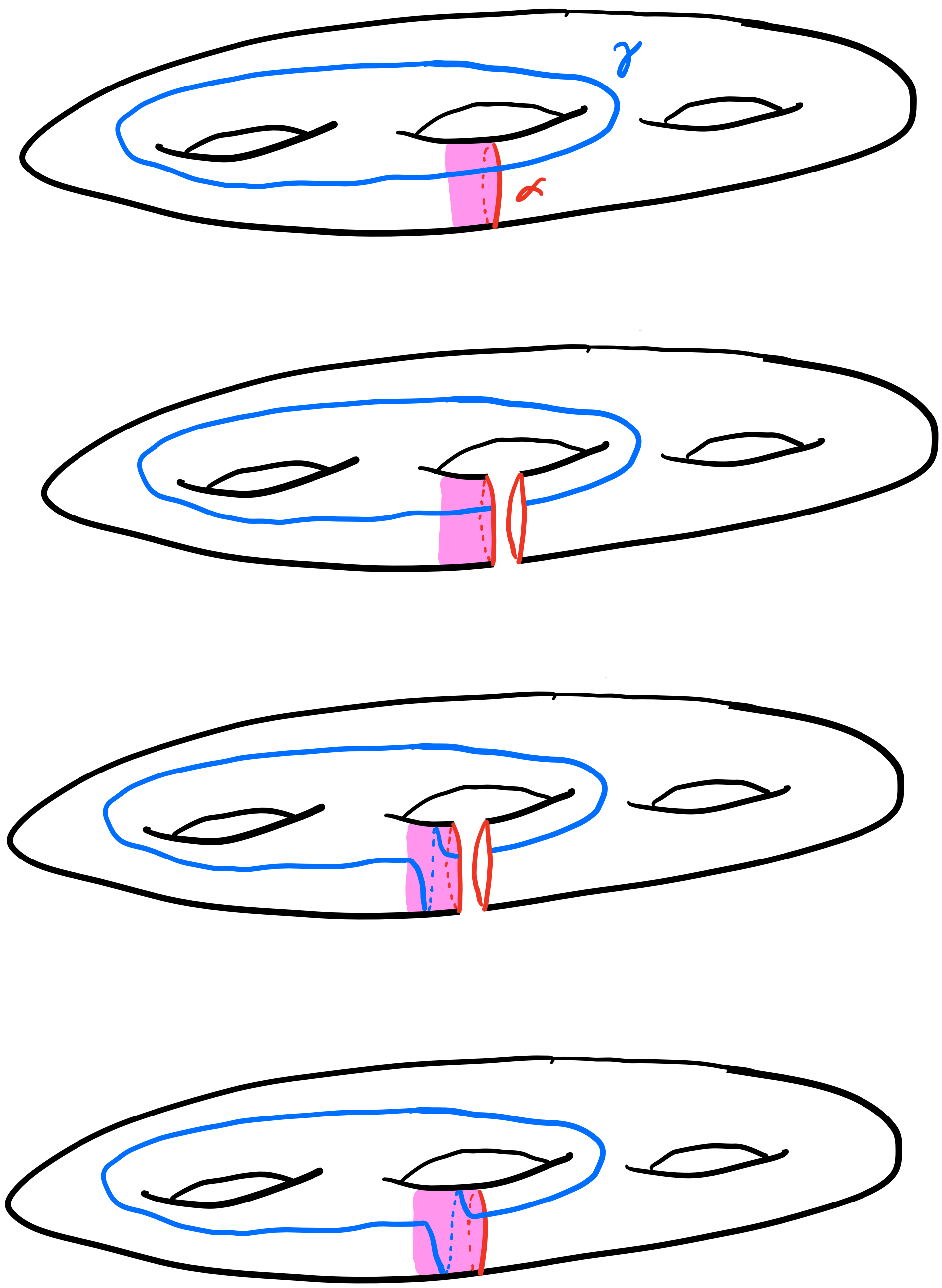
For $h \in H$, denote by $[h]$ the set of homeomorphisms from S to S homotopic to h. Call $[h]$ the mapping class of h. The set of mapping classes forms a group with operation given by

$$[f] \cdot [g] = [f \circ g].$$

We will call this group $MCG(S)$.

Dehn Twists

Given a surface S and a simple closed curve α on it, imagine cutting along α, twisting one of the resulting boundary components $360°$ to the right, and carefully re-gluing. This is a homeomorphism, called a "Dehn Twist about α" and denoted T_α.



T_α and how it affects a curve γ.

Geometric Intersection Number

Given two homotopy classes of simple closed curves a and b, we define the geometric intersection number $i(a,b)$ to be the minimum of $|\alpha \cap \beta|$ over all representatives α, β of a, b respectively.

Lemma (Single Intersection Twisting)

Let a, b be the homotopy classes of two simple closed curves such that $i(a,b) = 1$. Then, $T_a T_b(a) = b$.

Proof: We can prove that any pair of simple closed curves that intersect once differ by a homeomorphism of the surface. Thus, it suffices to check for a single pair of curves. The desired equality is equivalent to $T_b(a) = T_a^{-1}(b)$. This is straightforward to check from the following picture:

$T_b(a)$ $T_a(b)$

$\square$

Lemma (Main)

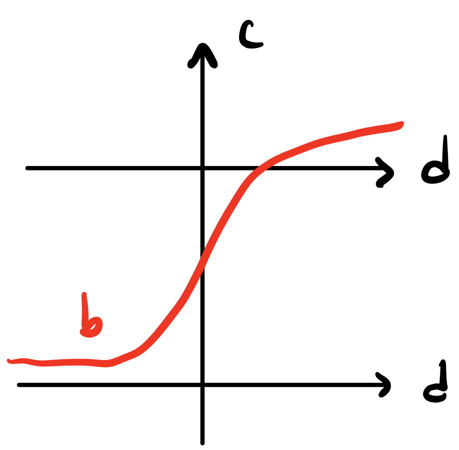
If c and d are simple closed curves in a compact, orientable surface S, then there is a product h of Dehn twists so that

$$i(c, h(d)) \leq 2.$$

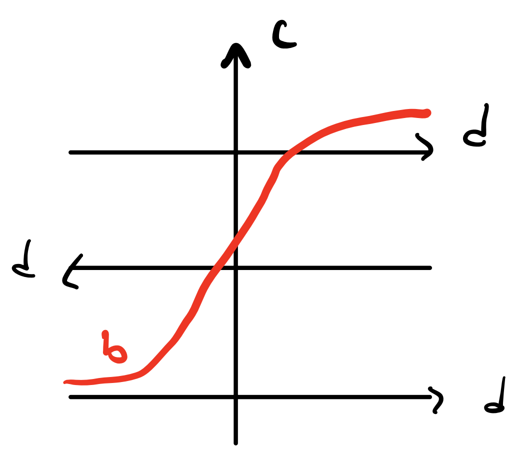
Proof: We will show that if $i(c,d) \geq 3$, then there is a simple closed curve b so that

$$i(c, T_b(d)) < i(c,d).$$

Orient c and d arbitrarily. If $i(c,d) \geq 3$, then there are either two consecutive intersections of the same sign or three consecutive intersections with alternating signs (consecutive from c's point of view).

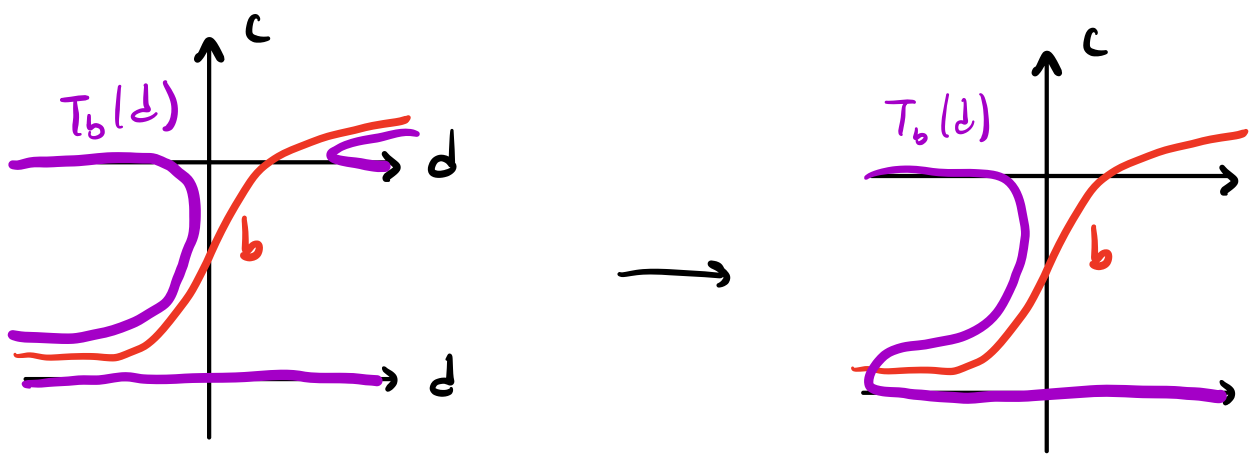


Consecutive intersections of same sign



Consecutive Intersections of alternating sign

In each case, the curve b drawn above satisfies what we need. For example, below we draw $T_b(d)$ in purple (for the first case) which after some pushing around it is evident it intersects less times.



We can do a similar computation for the second case, proving the lemma. $\square$

We are now ready to prove the goal:

__Theorem__

The mapping class group of a compact, orientable surface is generated by Dehn twists.

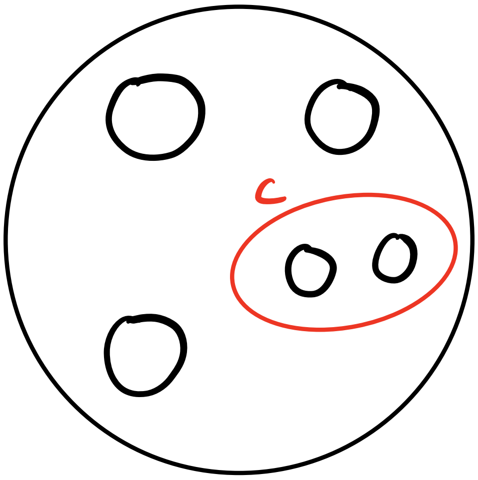
__Proof__: We proceed by induction on the genus g of our surface S.

__Base case__: $g=0$. We proceed by induction on the number n of boundary components.

If $n=0,1,2,3$ we have a sphere, disk, annulus, or pair of pants, which we verify separately.

Suppose $n \geq 4$ and let $f \in MCG(S)$. Let c be a curve that cuts off a pair of pants in S.

Observe that on the other side of c we have a surface of genus g and $n-1$ boundary components.



By the main lemma there is a product of Dehn twists such that $i(c, h \circ f(c)) \leq 2$.

Now, since c is a separating curve, the intersection number must be 0 or 2.

We can check that if d is a curve in S, $c \neq d$ and $i(c,d) = 0, 2$ then c and d surround different sets of boundary components. Since h, f act as the identity on the boundary of S, c and $h \circ f(c)$ surround the same boundary components, so $h \circ f(c) = c$.

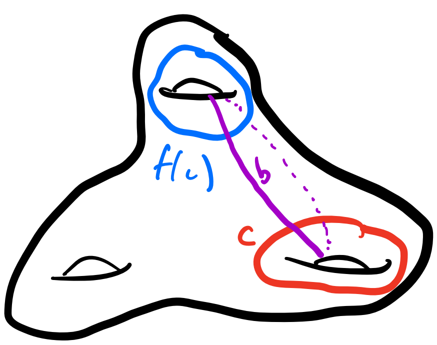
This means without loss of generality we may assume f fixes c.

In other words, the homotopy class of homeomorphisms f fixes the homotopy class of curves c.

From the isotopy extension theorem of differential topology, we can choose a representative homeomorphism of f that fixes pointwise a representative curve of c. Cutting along this curve we obtain two surfaces and our representative of f induces a homeomorphism on each of these. By induction, the corresponding mapping classes are products of Dehn twists, so the original mapping is a product of these same twists!

<u>Inductive Step</u>: Let $g \geq 1$ and assume by induction every surface of genus $g-1$ satisfies the theorem.

Observe that for any nonseparating curve c, cutting our surface along it produces a surface of genus $g-1$ and two additional boundary components.

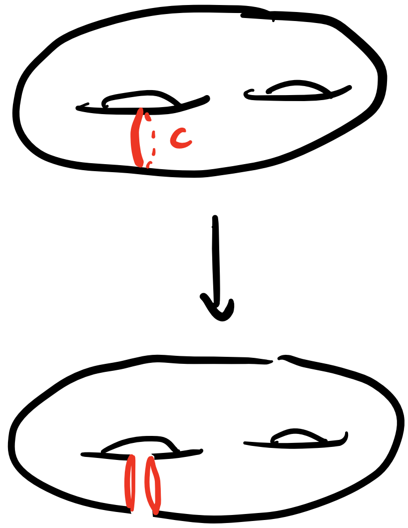


As before, given $f \in MCG(S)$, we can use the main lemma to argue that without loss of generality, $i(c, f(c)) = 0, 1, 2$.

In any of these cases, we can find a curve b with $i(c,b) = i(b, f(c)) = 1$.

Then, the single intersection twisting lemma gives us a way to modify f via Dehn twists so that $f(c) = c$.

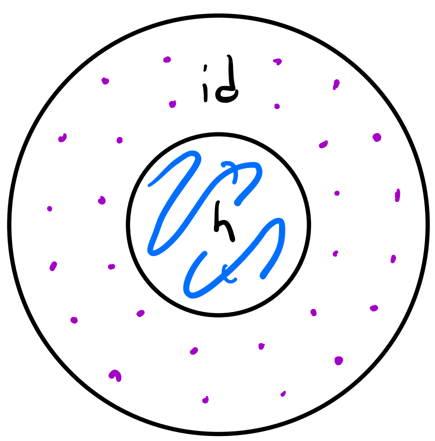
As before, this gives a mapping class of the surface of genus $g-1$ obtained by cutting along c. By induction, that mapping class is equal to a product of Dehn twists, so f itself is a product of Dehn twists as desired.



This concludes the induction, proving the theorem. ■

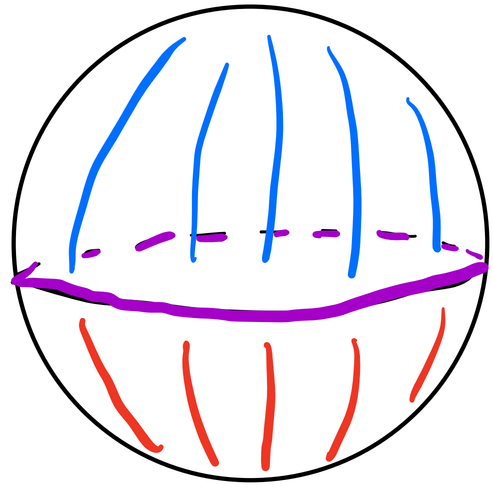
Disk, Sphere, Annulus, Pair of Pants

Given a homeomorphism h of D, we can homotope it to the identity as follows: at time t, apply h on the subdisk of radius $1-t$ and the identity elsewhere. Since h fixes the boundary of the disk, this is continuous and gives a homotopy from h to the identity.



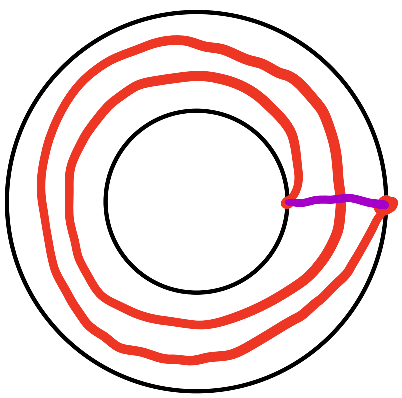
For the sphere, we can use isotopy extension to modify any homeomorphism into one that fixes the equator. This homeomorphism must then send each hemisphere to itself. However, each hemisphere is a disk, so we conclude we can homotope our homeomorphism into the identity.

Thus, the mapping class group of both the disk and the sphere is trivial, and so generated by Dehn twists vacuously.



For the annulus it turns out its mapping class group is $\mathbb{Z}$, generated by the Dehn twist about its core curve.

To prove this it suffices to show an arc connecting two given points on different boundary components is completely determined up to homotopy by how many times it winds around the annulus.



With this, we can construct an isomorphism to $\mathbb{Z}$ using an arc that does not wind around and mapping f to how many times $f(\alpha)$ winds around.

We can do a very similar thing to prove the mapping class group of the pair of pants is $\mathbb{Z}^3$, generated by the Dehn twists around the boundaries.