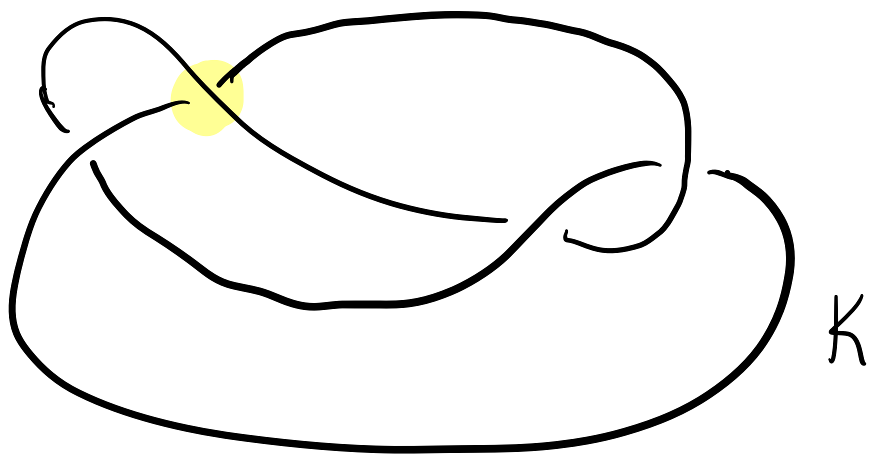
18.904 Lecture Notes (Ezra Guerrero 10/30/2023)

Calculating Homology of Cyclic Covers Using Surgery in S^3

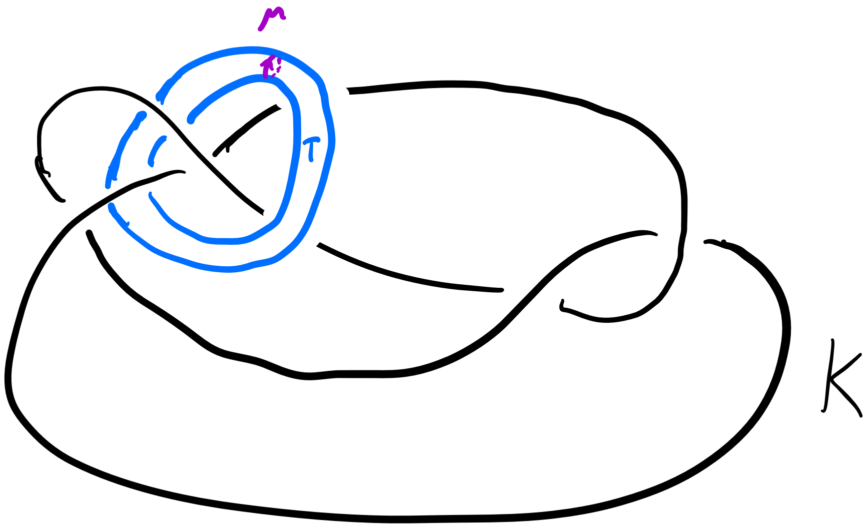
It turns out that we can use surgeries on S^3 to view knot complements in a way which gives a convenient way of visualizing its cyclic covers.

We demonstrate with the example of the figure-eight knot.



Observe that if we change the highlighted crossing in the figure-eight knot, we obtain the unknot.

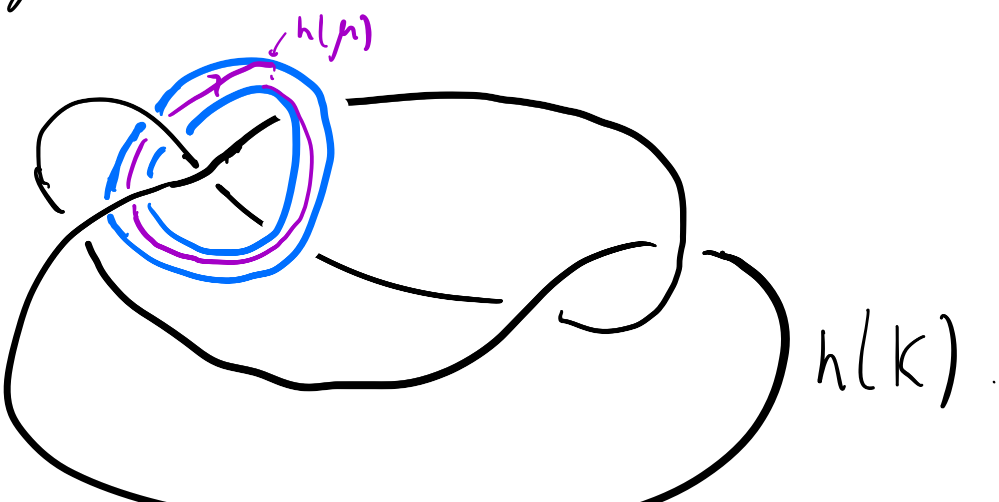
To this end, we introduce the following unknotted solid torus T:



Using T we commence our surgery. Start by removing the interior of T. Now, we can twist what remains obtaining a homeomorphism

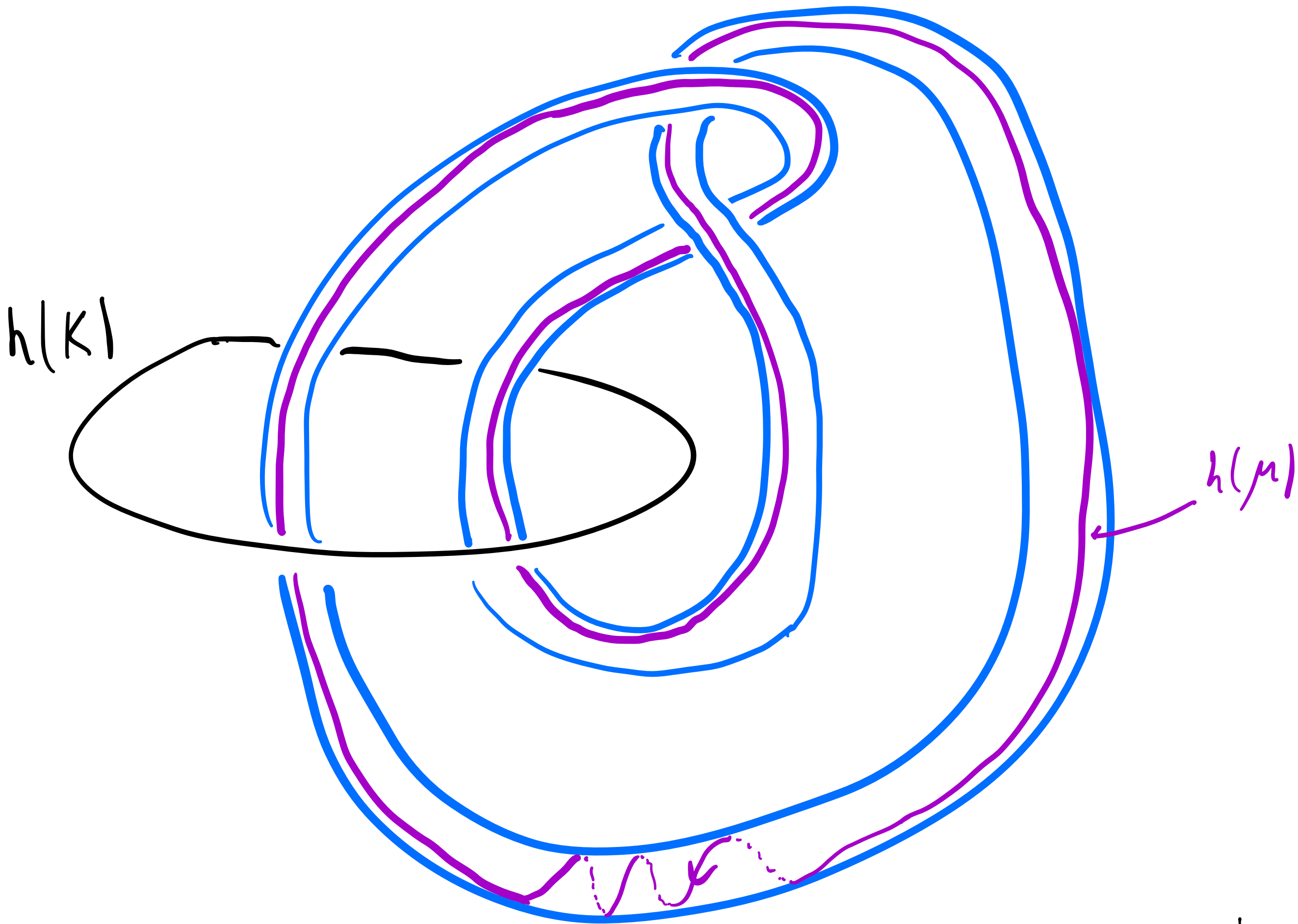
$$h : S^3 - \mathring{T} \to S^3 - \mathring{T}$$

whose image looks like

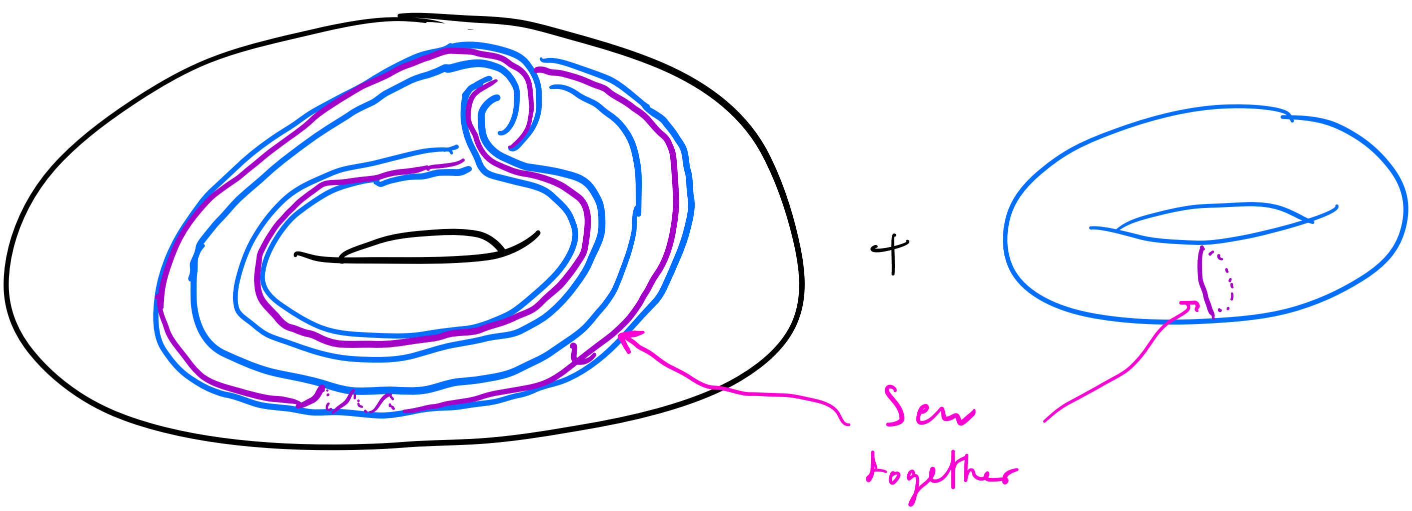


Note: h does not extend to a homeomorphism of S^3, as otherwise, the figure-eight knot is trivial!

Now, since $h(K)$ is unknotted, we can change the picture by a homeomorphism of S^3 to



With this picture we can finish our surgery. We can view the complement of the figure-eight knot as an open solid torus, with an open solid torus removed, and replaced with a torus with meridian running along $h(\mu)$.



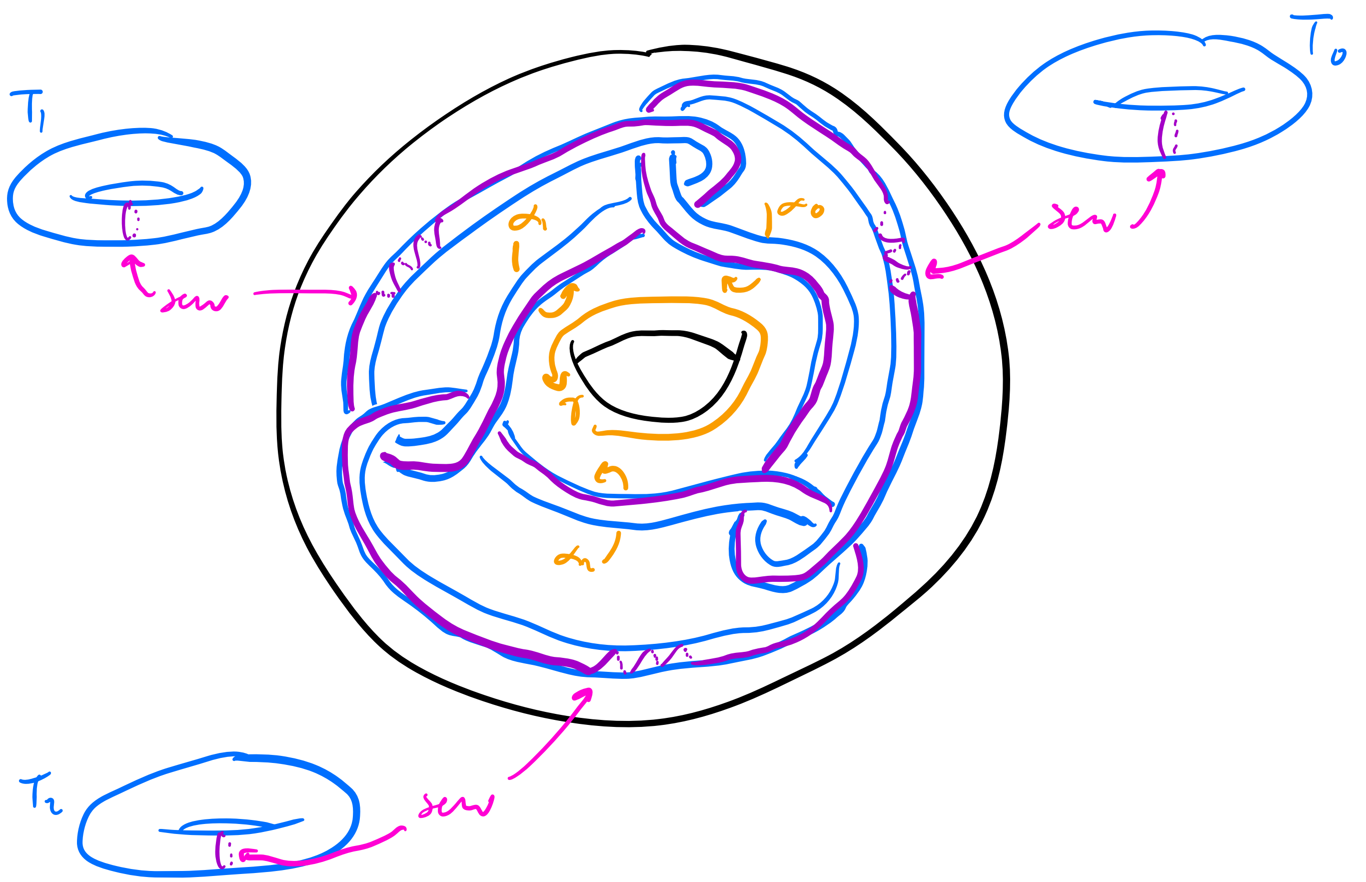
What happens if we try to do this with the unknot?

No weird surgery is needed! We can just view its complement as $S^1 \times \mathring{D}^2$. Then, its K-fold cyclic cover is

$$S^1 \times \mathring{D}^2 \xrightarrow{\ p\ } S^1 \times \mathring{D}^2$$

where p is multiplication by K in the first factor and identity on the second.

Let's return to the figure-eight knot. We can visualize its 3-fold cyclic cover $\tilde{X}_3$ as follows:



This allows us to compute the homology of $\tilde{X}_3$.

First, we have four unrelated homology generators $\alpha_0, \alpha_1, \alpha_2, \gamma$, pictured above. Now, we add relations as we sew in our tori. Imagine sewing in T_0 in two stages: first add a thickened meridinal disk bounded by the curve shown, then add the rest of T_0, which is an open ball. Adding the open ball does nothing to homology. Adding the disk imposes a relation by zeroing out the curve. In this case, we impose the relations

$$\alpha_1 + \alpha_2 - 3\alpha_0 = 0 \qquad (R_0)$$

$$\alpha_2 + \alpha_0 - 3\alpha_1 = 0 \qquad (R_1)$$

$$\alpha_0 + \alpha_1 - 3\alpha_2 = 0 \qquad (R_2)$$

so

$$H_1(\tilde{X}_3) = \langle \alpha_0, \alpha_1, \alpha_2, \gamma \mid R_0, R_1, R_2 \rangle.$$

It is straightforward to check that

$$H_1(\tilde{X}_3) \cong \mathbb{Z} \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/4.$$