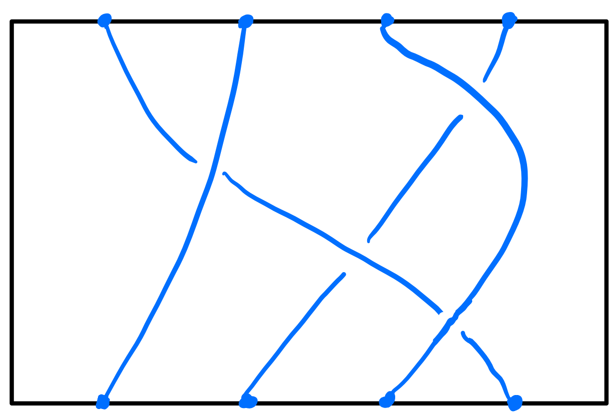
18.904 Lecture Notes (Ezra Guerrero 12/11/2023)

Representations to get Braid Invariants

We can use modules to construct representations of the Braid Group. Such representations lead to polynomial invariants of braids!

What is a braid?

We think of a braid as a box containing some strings, fixed at the top and bottom of the box, such that if you oriented each string from the bottom to the top, no string ever goes down.

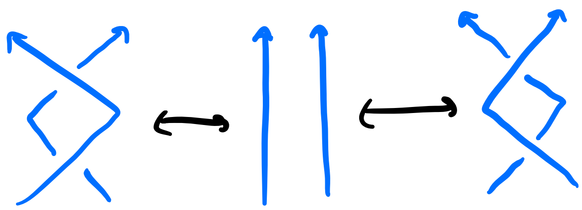


We think of two braids as equivalent if we can move the strings around continuously without making any string go down and keeping the endpoints fixed.

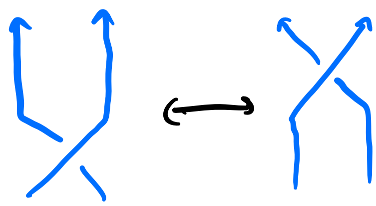
Just as with knots, we draw braids by projecting them onto braid diagrams, where one string goes over the other at each crossing.

Braid Moves

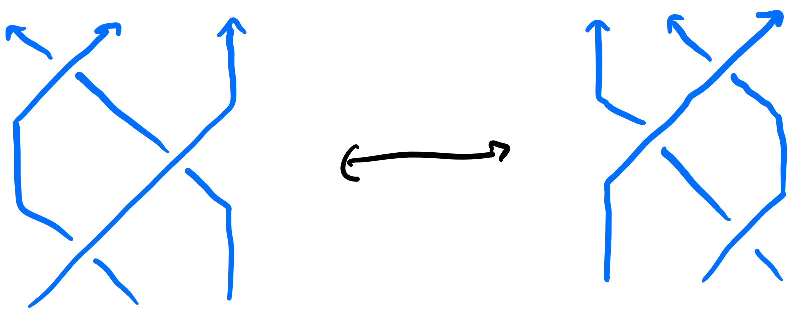
Of course, we need a way to tell whether two braid diagrams represent equivalent braids. We have an analogue of the Reidemeister moves:



B1



B2



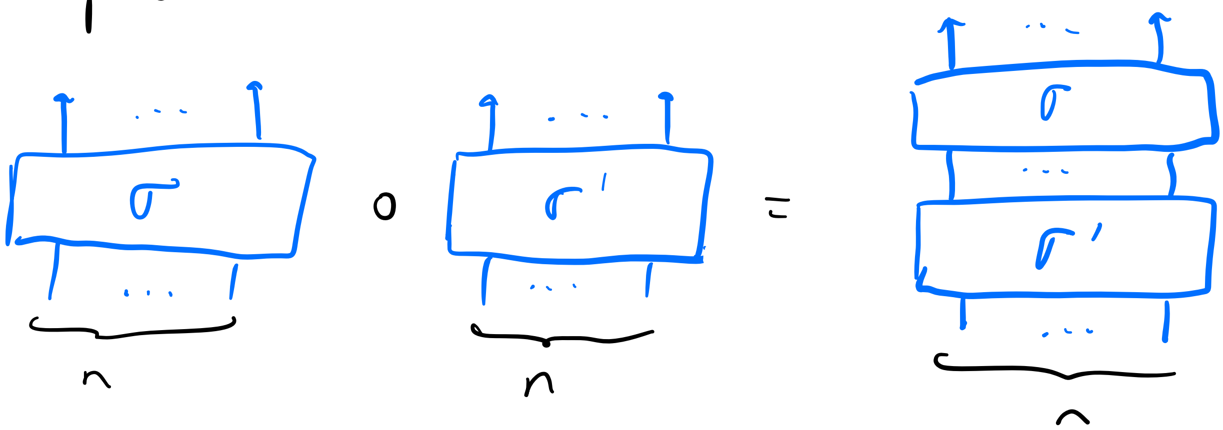
B3

We can prove two braid diagrams represent equivalent braids if and only if they are related by a (finite) sequence of braid moves.

The Braid Group

We denote by $\mathcal{B}$ the set of all braids and let $\mathcal{B}_n$ denote the set of n-stranded braids.

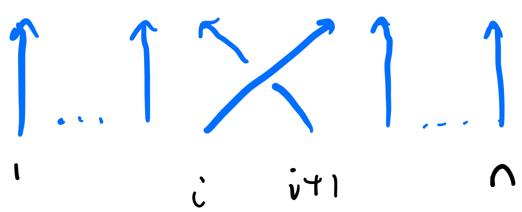
Consider two braids $\sigma, \sigma' \in \mathcal{B}_n$. We can compose the two braids into the braid $\sigma \circ \sigma'$ by "stacking" σ on top of σ'.

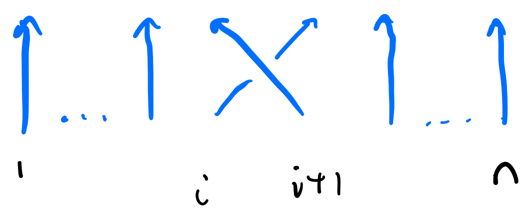


Now, $\mathcal{B}_n$ is clearly closed under composition and we can show that composition is associative. Furthermore, the braid with no crossings is an identity element, so $\mathcal{B}_n$ is at least a monoid.

We can describe a set of generators as follows:

For $i = 1, \ldots, n-1$ let σ_i be the braid with one <u>positive</u> crossing between the i-th and $(i+1)$-th strands and σ_i^{-1} be the braid with one <u>negative</u> crossing between these same two strands.

$\sigma_i :=$ 

$\sigma_i^{-1} :=$ 

Using the braid move B2 we observe $\sigma_i \circ \sigma_i^{-1} = \sigma_i^{-1} \circ \sigma_i = 1$, so σ_i^{-1} is the inverse of σ_i. Then,

$$\{\sigma_i, \sigma_i^{-1} : i = 1, \dots, n-1\}$$

is a set of generators for $\mathcal{B}_n$ and we can represent each braid σ as

$$\sigma = \sigma_{j_1}^{\delta_1} \circ \sigma_{j_2}^{\delta_2} \circ \dots \circ \sigma_{j_K}^{\delta_K},$$

with $\delta_m = \pm 1$ for each m. It is clear

$$\sigma^{-1} := \sigma_{j_K}^{-\delta_K} \circ \dots \circ \sigma_{j_2}^{-\delta_2} \circ \sigma_{j_1}^{-\delta_1}$$

is the inverse of σ, so $\mathcal{B}_n$ is a group.

Using $\{\sigma_i : i = 1, \dots, n-1\}$ as our set of generators, we can give a group presentation of $\mathcal{B}_n$

$$\mathcal{B}_n = \langle \sigma_1, \dots, \sigma_{n-1} \mid \mathcal{R} \rangle$$

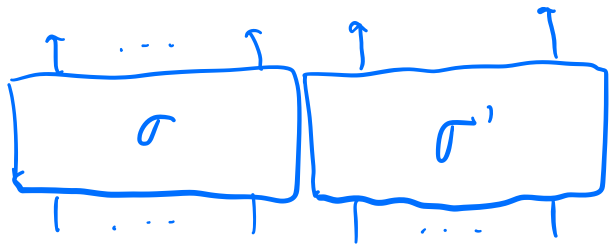
where $\mathcal{R}$ is the following set of relations:

$$\begin{cases} \sigma_i \circ \sigma_j = \sigma_j \circ \sigma_i & \text{if } |i-j| \geq 2 \\ \sigma_i \circ \sigma_{i+1} \circ \sigma_i = \sigma_{i+1} \circ \sigma_i \circ \sigma_{i+1} & \text{for } i = 1, \dots, n-2. \end{cases}$$

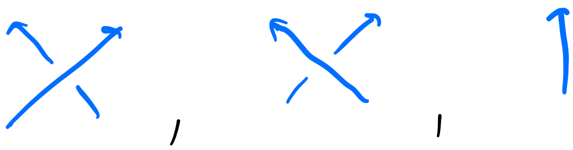
These arise from writing B2 and B3 in terms of braid generators.

Linear Maps from Braid Diagrams

Given any two braids $\sigma, \sigma' \in \mathcal{B}$, we define their tensor product $\sigma \otimes \sigma'$ as putting σ' to the right of σ. That is,

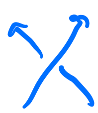
$\sigma \otimes \sigma' :=$ 

We can use the tensor product to express each generator of $\mathcal{B}_n$ in terms of the "elementary pieces"



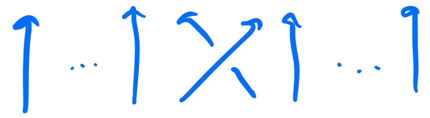
Thus, via composition and tensor products we can express any braid diagram in terms of these pieces.

Now, take some $R \in \mathrm{Aut}(V \otimes V)$. For now, take V a vector space, later we will take free, finite rank modules. We associate to each elementary piece a map as follows:

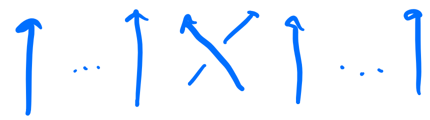
$\rho_e :$  $\mapsto id \in \mathrm{Aut}(V)$ $\quad \rho_R :$  $\mapsto R$ $\quad \rho_R :$  $\mapsto R^{-1}$

Denote by ρ_R this linear map.

We can extend ρ_R to generators via the tensor product:

$\sigma_i =$  $\longmapsto \underbrace{id \otimes \cdots \otimes id}_{i-1} \otimes R \otimes \underbrace{id \otimes \cdots \otimes id}_{n-i-1}$

1 $\quad$ i $\quad$ $i+1$ $\quad$ n

$\sigma_i^{-1} =$  $\longmapsto \underbrace{id \otimes \cdots \otimes id}_{i-1} \otimes R^{-1} \otimes \underbrace{id \otimes \cdots \otimes id}_{n-i-1}$

1 $\quad$ i $\quad$ $i+1$ $\quad$ n

Finally, we extend ρ to the braid σ by setting

$$\rho(\sigma' \circ \sigma'') = \rho(\sigma') \circ \rho(\sigma'')$$

Let's work out ρ_R in detail when V has rank 1. Let $\{e_0\}$ be a basis of V. Then, $V \otimes V$ has rank 1 with basis $\{e_0 \times e_0\}$. Hence, R will be represented by a 1×1 matrix $[a]$. How does ρ_R look on a generator?

$$\rho_R(\sigma_i) = id^{\otimes(i-1)} \otimes R \otimes id^{\otimes(n-i-1)} \in End(V^{\otimes n}).$$

Note that $V^{\otimes n}$ has basis $\{e_0^{\otimes n}\}$, so we compute

$$\begin{aligned} \rho_R(\sigma_i) : e_0^{\otimes n} \longmapsto\ & e_0^{\otimes(i-1)} \otimes R(e_0 \otimes e_0) \otimes e_0^{\otimes(n-i-1)} \\ =\ & e_0^{\otimes i-1} \otimes a(e_0 \otimes e_0) \otimes e_0^{\otimes(n-i-1)} \\ =\ & a\, e_0^{\otimes n}. \end{aligned}$$

Hence, $\rho_R(\sigma_i)$ is represented by $[a]$. We can similarly compute $\rho_R(\sigma_i^{-1})$ is represented by $[a^{-1}]$.

Thus,

$$\rho_R(\sigma) = \rho_R(\sigma_{j_1}^{\delta_1}) \circ \rho_R(\sigma_{j_2}^{\delta_2}) \circ \cdots \circ \rho_R(\sigma_{j_k}^{\delta_k})$$

is represented by $[a^{\delta_1}][a^{\delta_2}]\cdots[a^{\delta_k}] = [a^{\delta_1+\cdots+\delta_k}]$.

Now, observe that $\delta_i = 1$ if there is a positive crossing while $\delta_i = -1$ if there is a negative crossing, so the exponent of a is precisely

$$\#\text{positive crossings} - \#\text{negative crossings},$$

which is a <u>braid</u> invariant known as the writhe.

Thus, we have used our linear map to construct a braid invariant. More interesting choices of V yield more complicated and exciting braid invariants.