18.904 Lecture Notes (Ezra Guerrero 09/27/2023)

Connect Sum and Additivity of Genus

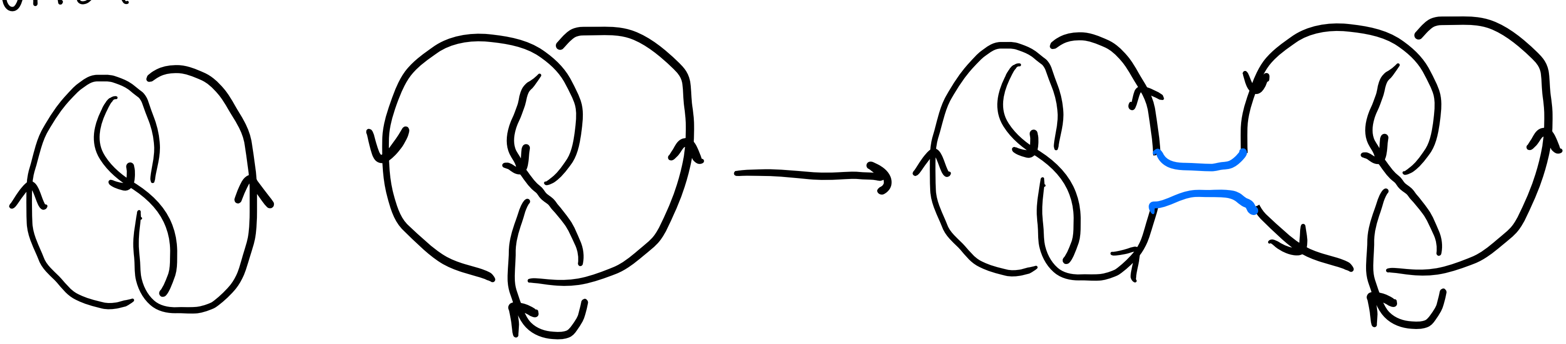
Goal: Prove that, for knots in $\mathbb{R}^3$, genus is additive.

$$g(K_1 \# K_2) = g(K_1) + g(K_2).$$

First, let's remember what this all means...

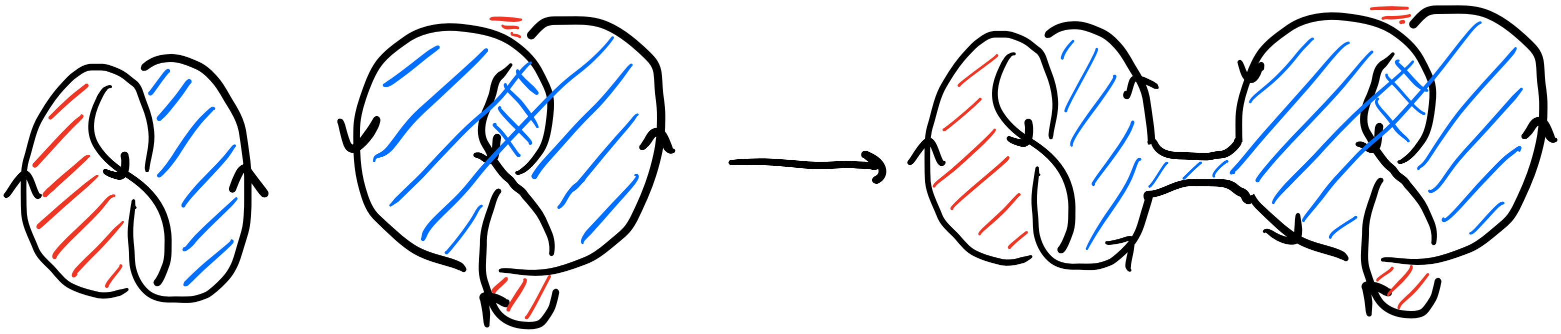
Connect sum

Given two knots K_1, K_2, we define their connect sum $K_1 \# K_2$ by removing little arcs from K_1 and K_2 and then connecting them with a tube, preserving orientation.



K_1 K_2 $K_1 \# K_2$

In the same vein, given two Seifert surfaces, one of K_1 and one of K_2, we can construct a Seifert surface for the connect sum:

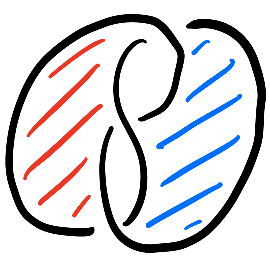


K_1 $\qquad$ K_2 $\qquad$ $K_1 \# K_2$

Genus

Given a knot K, its genus $g(K)$ is the minimal genus over all its Seifert surfaces. We call a Seifert surface of K minimal if its genus is precisely that minimum.

For example, the trefoil has genus 1. Recall that we showed $g(K) = 0$ if and only if K is the unknot, and the following Seifert surface of the trefoil has genus 1:



Additivity of Genus

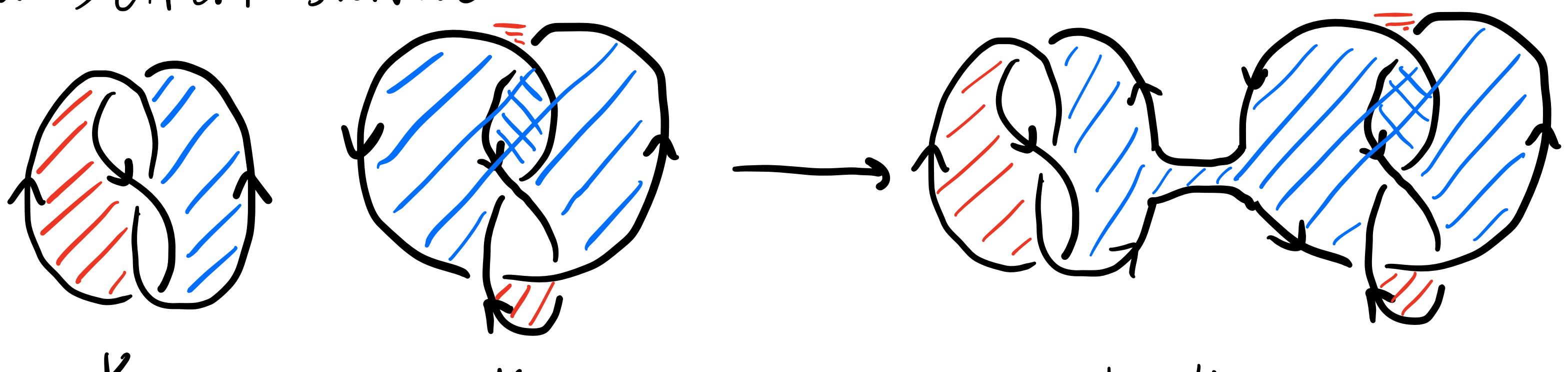
Let K_1 and K_2 be knots. Then,

$$g(K_1 \# K_2) = g(K_1) + g(K_2)$$

We prove that $g(K_1 \# K_2) \leq g(K_1) + g(K_2)$ and $g(K_1 \# K_2) \geq g(K_1) + g(K_2)$, establishing the result.

- $g(K_1 \# K_2) \leq g(K_1) + g(K_2)$.

Given K_1, K_2 consider minimal Seifert surfaces M_1, M_2 of K_1, K_2 respectively. As described before, we construct a Seifert surface for their connect sum:



K_1 $\qquad$ K_2 $\qquad$ $K_1 \# K_2$

The genus of this new "boundary connect sum" Seifert surface has to be $g(K_1) + g(K_2)$, so by minimality we establish

$$g(K_1 \# K_2) \leq g(K_1) + g(K_2).$$

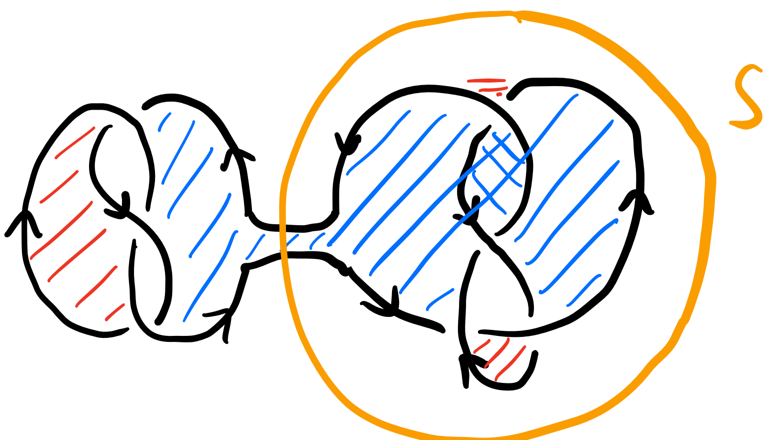
• $g(K_1 \# K_2) \geq g(K_1) + g(K_2)$.

Let M be a minimal Seifert surface of $K_1 \# K_2$. We will construct from M another Seifert surface M' which can be split into Seifert surfaces M_1, M_2 of K_1, K_2 respectively. It will follow that

$$g(K_1) + g(K_2) \leq g(M_1) + g(M_2) = g(M') = g(K_1 \# K_2)$$

which is what we want to prove.

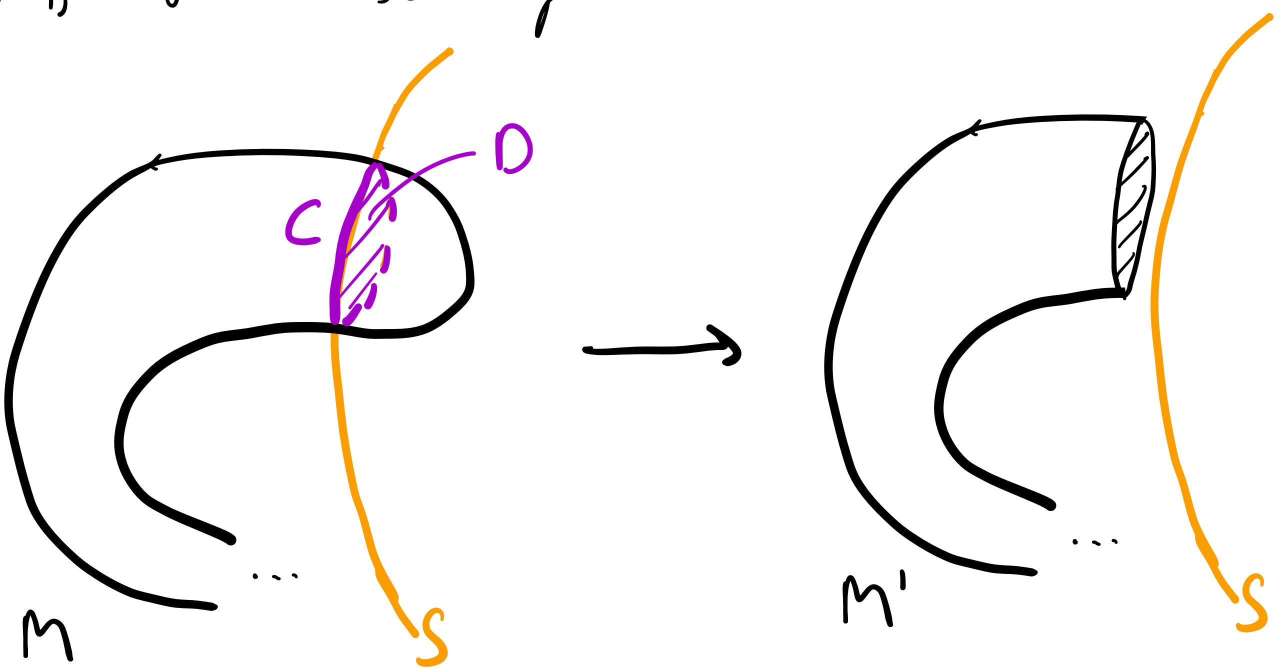
To do this, consider a sphere S splitting $K_1 \# K_2$ into K_1 and K_2. We can do this in such a way that $M \cap S$ consists of one dimensional pieces. Indeed, we can make it so that $M \cap S$ has exactly two boundary points: where $K_1 \# K_2$ goes through S.



$K_1 \# K_2$

First, suppose $M \cap S$ consists of only one arc (like above). Then, this arc splits M into two Seifert Surfaces, one of a knot isotopic to K_1, and the other of a knot isotopic to K_2. This is exactly what we needed.

The only thing that we have to consider now is if $M \cap S$ consists of more than one arc. All other components of the intersection must be simple closed curves, as S is not intersecting ∂M.



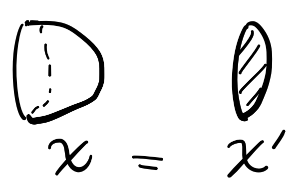
We show how to remove these one by one. It follows, by induction, that we can reduce to the case when $M \cap S$ is a single arc, which we know how to conclude.

Let $C \subset M \cap S$ be one of these simple closed curves and D be the disk it bounds. Note that $\mathrm{int}(D) \cap M = \emptyset$.

C separates M into two pieces. One of them includes ∂M, while the other does not. We construct M' by replacing the piece of $M \setminus C$ that does not contain ∂M by D and push the resulting manifold away from S near D.

It's not hard to see M' can be bicollared, $\partial M' = \partial M$ and M' intersects S in strictly less pieces. Thus, if we check the procedure we just did does not increase genus, we will be done.

However, it is not hard to see (via triangulations, e.g.) that the procedure does not change Euler characteristic. Hence, the genus cannot change, so M' is a minimal surface of $K_1 \# K_2$. We continue until the Seifert Surface intersects only once, where we apply the argument above.



Thus, $g(K_1 \# K_2) \geq g(K_1) + g(K_2)$.

Combining both inequalities,

$$g(K_1 \# K_2) = g(K_1) + g(K_2). \blacksquare$$