2016 G1

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In convex pentagon $ABCDE$ with $\angle B > 90°$, let F be a point on $\overline{AC}$ such that $\angle FBC = 90°$. It is given that $FA = FB$, $DA = DC$, $EA = ED$, and rays $\overline{AC}$ and $\overline{AD}$ trisect $\angle BAE$. Let M be the midpoint of $\overline{CF}$. Let X be the point such that $AMXE$ is a parallelogram. Show that $\overline{FX}$, $\overline{EM}$, $\overline{BD}$ are concurrent.

Let D' be the second intersection of $\overleftrightarrow{AB}$ with the circumcircle of $\triangle BFC$. Now, since $\angle BAC = \angle CAD = \angle ACD$ and $\angle EDA = \angle EAD = \angle DAC$ we have $\overline{AB} \parallel \overline{CD}$ and $\overline{AC} \parallel \overline{ED}$. Now, note that

$$\angle CAD = \angle CAB = \angle FBA = \angle FCD',$$

so $\overline{CD'} \parallel \overline{AD}$. Hence, since $DA = DC$, $ADCD'$ is a rhombus. Thus, $\overline{AC}$ is the perpendicular bisector of $\overline{DD'}$. However, $\overline{AC}$ is also a diameter of (FBC), so reflecting over it tells us D lies on this circle as well. Now, note that E, D, X are collinear because of $\overline{AC} \parallel \overline{ED}$. Thus, $\angle CDX = 180° - \angle EDC = \angle EDA = \frac{1}{2}\angle CAE = \frac{1}{2}\angle CMX$. Thus, since M is the center of (FBC), X lies on the circle as well.
Furthermore, we have

$$\angle FMX = 180° - \angle MAE = 2(90° - \angle CAD) = 2\angle BCD,$$

so $FDXB$ must be an isosceles trapezoid. Finally, note that $\angle FXD = \angle FCD = \angle FAD$, so parallelogram $AMXE$ implies $AFXD$ and $FMDE$ are parallelograms. Since $MF = MD$, we have $EF = ED$, so $\overleftrightarrow{ME}$ is the perpendicular bisector of $\overline{FD}$. Since $FDXB$ is an isosceles trapezoid, it follows by symmetry that $\overline{FX}, \overline{EM}, \overline{BD}$ concur. $\blacksquare$