2002 N2

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Let $n \geq 2$ be a positive integer, with divisors $1 = d_1 < d_2 < \ldots < d_k = n$. Prove that $d_1d_2 + d_2d_3 + \ldots + d_{k-1}d_k$ is always less than n^2, and determine when it is a divisor of n^2.

Note that

$$\sum d_i d_{i+1} = n^2 \sum \frac{1}{d_i d_{i+1}} \leq n^2 \sum \frac{d_{i+1} - d_i}{d_i d_{i+1}} = n^2 \sum \left(\frac{1}{d_i} - \frac{1}{d_{i+1}} \right) = n^2 \left(1 - \frac{1}{n} \right) < n^2.$$

Now, we show this sum only divides n^2 when n is prime. Indeed, when n is prime the sum equals n, so it divides n^2 trivially. Otherwise, if n is composite, the sum is strictly greater than $d_{k-1}d_k = n^2/d_2$. However, since d_2 is the smallest prime factor of n, and consequently of n^2, there cannot be a divisor of n^2 between n^2/d_2 and n^2 exclusive. Hence, the sum does not divide n^2 when n is composite. ∎