2000 N4

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Find all triplets of positive integers (a, m, n) such that $a^m + 1 \mid (a+1)^n$.

We claim all such triples are $(1, m, n)$ for $m, n \in \mathbb{Z}^+$, $(a, 1, n)$ for $a, n \in \mathbb{Z}^+$, and $(2, 3, n)$ for $n \in \mathbb{Z}_{\geq 2}$, which are all easily seen to work. Now, we see they are the only ones.
If $a = 1$ then it is clear all pairs (m, n) work. Similarly, if $m = 1$ it is clear all pairs (a, n) work. Thus, assume $a, m > 1$. Then, by Zsigmondy's theorem, we have either $a = 2, m = 3$ or $a^m + 1$ has a primitive prime divisor p. But if $a^m + 1 \mid (a+1)^n$, then there cannot be such a primitive prime divisor giving a contradiction. Hence, $a = 2, m = 3$ from where $9 \mid 3^n$, so $n \geq 2$. ■