2011 G1

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Let ABC be an acute triangle. Let ω be a circle whose centre L lies on the side BC. Suppose that ω is tangent to AB at B' and AC at C'. Suppose also that the circumcentre O of triangle ABC lies on the shorter arc $B'C'$ of ω. Prove that the circumcircle of ABC and ω meet at two points.

Note that $\angle B'OC' = 180^\circ - \frac{1}{2}(180^\circ - \angle A) = 90^\circ + \angle A/2$. Also, $\angle BOC = 2\angle A$. Since $\angle BOC < \angle B'OC'$, we have $2\angle A < 90^\circ + \angle A/2$, or $\angle A < 60^\circ$. Since $\overline{\text{AL}}$ is the angle bisector of $\angle BAC$, we see using the angle bisector theorem that

$$BL = \frac{ca}{b+c},\ LC = \frac{ab}{b+c}.$$

Let r be the radius of ω and R the radius of (ABC). We wish to show that $2r > R$. By Stewart on $\triangle OBC$, we get

$$a\left(r^2 + \frac{a^2bc}{(b+c)^2}\right) = R^2a,$$

that is

$$4r^2 = 4R^2 - \frac{4a^2bc}{(b+c)^2}$$

However, note that by the law of sines, $a = 2R\sin\angle A < R\sqrt{3}$ and by AM-GM, $4bc \le (b+c)^2$. Then,

$$4r^2 > 4R^2 - 3R^2 \cdot 1 = R^2,$$

which after taking square roots becomes $2r > R$ as desired. ■