2017 G3

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Let O be the circumcenter of an acute triangle ABC. Line OA intersects the altitudes of ABC through B and C at P and Q, respectively. The altitudes meet at H. Prove that the circumcenter of triangle PQH lies on a median of triangle ABC.

Let T be the circumcenter of $\triangle PQH$. Also, denote by E, F, M the feet of the altitudes from B and C and the midpoint of $\overline{\text{BC}}$, respectively. First,

$$\angle PQH = \angle QAC + \angle ACQ = 90^\circ - \angle B + 90^\circ - \angle A = \angle C.$$

Therefore, $\angle THE = \angle THP = 90^\circ - \angle C = \angle HAE$. Therefore, $\overline{\text{TH}}$ is tangent to (AEF). Similarly,

$$\angle BPT = \angle HPT = 90^\circ - \angle C = \angle BAP,$$

so $\overline{\text{TP}}$ is tangent to (ABP). Finally, note that

$$\angle MBP = \angle CBE = 90^\circ - \angle C = \angle BAP,$$

so $\overline{\text{MB}}$ is tangent to (ABP). Recall that $\overline{\text{ME}}$ is tangent to (AEF). Then, since $TH^2 = TP^2$ and $MB^2 = ME^2$, both T and M lie on the radical axis of both circles. But A lies on both circles, so this radical axis is the $A-$median, as desired. ■