2008 G7

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Let $ABCD$ be a convex quadrilateral with $BA \neq BC$. Denote the incircles of triangles ABC and ADC by ω_1 and ω_2 respectively. Suppose that there exists a circle ω tangent to ray BA beyond A and to the ray BC beyond C, which is also tangent to the lines AD and CD. Prove that the common external tangents to ω_1 and ω_2 intersect on ω.

Let J be the center of ω, W, X, Y, Z be the tangency points of ω with lines $\overline{\mathrm{AB}}, \overline{\mathrm{CD}}, \overline{\mathrm{DA}}, \overline{\mathrm{BC}}$, $\triangle PQR$ be the intouch triangle of $\triangle ABC$ (with P, Q opposite A, B), S be the tangency point of ω_2 and $\overline{\mathrm{AC}}$, and T be the point on ω such that $\overline{\mathrm{JT}} \perp \overline{\mathrm{AC}}$. Using the given tangencies we have

$$\begin{aligned}
2AQ &= AQ + AR \\
&= AQ + BW - BR - AW \\
&= AQ + BZ - BP - AY \\
&= AQ + PZ - AD - DY \\
&= AQ + PC + CZ - AD - DY \\
&= AQ + QC + CX - AD - DX \\
&= AC + CD - AD.
\end{aligned}$$

However, it is well known that $2CS = AC + CD - AD$, so $AQ = CS$. Now, since Q and S are isotomic, this implies that $\overline{\mathrm{BS}}$ and $\overline{\mathrm{DQ}}$ pass through the antipodes of Q and S in ω_1 and ω_2 respectively. However, considering homotheties with centers B and D taking ω_1 and ω_2 to ω respectively, it is clear that this implies that $\overline{\mathrm{BT}}$ and $\overline{\mathrm{DT}}$ pass through the antipodes of Q and S in ω_1 and ω_2 respectively. Thus, it follows there is a positive homothety centered at T taking ω_1 to ω_2, so we must have that the common external tangents of ω_1 and ω_2 meet at $T \in \omega$. ■