2000 G2

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March 1, 2022

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Two circles G_1 and G_2 intersect at two points M and N. Let AB be the line tangent to these circles at A and B, respectively, so that M lies closer to AB than N. Let CD be the line parallel to AB and passing through the point M, with C on G_1 and D on G_2. Lines AC and BD meet at E; lines AN and CD meet at P; lines BN and CD meet at Q. Show that $EP = EQ$.

We use directed angles mod 180°. Also, let $m(X, Y)$ denote the midpoint of $\overline{\text{XY}}$. We have

$$\measuredangle BAE = \measuredangle MCA = \measuredangle MAB = -\measuredangle BAM,\ \measuredangle EBA = \measuredangle BDM = \measuredangle ABM = -\measuredangle MBA,$$

and $AB = AB$. Thus, by SAS we have

$$\triangle AMB \overset{-}{\cong} \triangle AEB.$$

It follows that $AM = AE$ and $MB = EB$, so $\overline{\text{AB}}$ is the perpendicular bisector of $\overline{\text{EM}}$. Therefore, $\overline{\text{EM}} \perp \overline{\text{AB}} \parallel \overline{\text{PQ}}$, so it suffices to show M is the midpoint of $\overline{\text{PQ}}$. To this end, note that $ABQP$ is a trapezoid whose non-parallel sides meet at N. Therefore, by homothety $m(A, B) - m(P, Q) - N$. However, by Power of a point we know $m(A, B)$ lies on the radical axis of both circles, id est $\overline{\text{MN}}$. Therefore, $m(P, Q)$ lies on $\overline{\text{MN}}$ and must therefore be M. Since $MP = MQ$ and $\overline{\text{EM}} \perp \overline{\text{PQ}}$, the conclusion follows. ■