2001 N5

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Let $a > b > c > d$ be positive integers and suppose that

$$ac + bd = (b + d + a - c)(b + d - a + c).$$

Prove that $ab + cd$ is not prime.

The given equation rearranges to $a^2 + c^2 - ac = b^2 + d^2 + bd$. From the Law of Cosines it follows there is a quadrilateral $PQRS$ with $PQ = a, QR = d, RS = b, SP = c$ and $\angle P = 60°, \angle R = 120°$. Therefore, this quadrilateral is cyclic. From the extended Ptolemy's theorem (Ptolomeo en Esteroides) we have

$$a^2 + c^2 - ac = QS^2 = \frac{(ab + cd)(ad + bc)}{ac + bd}.$$

Now, since $a > b > c > d$ the rearrangement inequality gives $ab + cd > ac + bd > ad + bc$. Now, if $ab + cd$ is prime, it follows $ac + bd \perp ac + bd$, so $ac + bd \mid ad + bc$. But $ac + bd > ad + bc$ so this is impossible. Hence, $ab + cd$ is not prime. ■