2019 G4

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Let P be a point inside a triangle ABC. Let $A_1 = \overline{\text{AP}} \cap \overline{\text{BC}}$, and let A_2 denote the reflection of P over A_1. Define B_1, B_2, C_1, C_2 similarly. Prove that A_2, B_2, C_2 cannot all lie strictly inside the circumcircle of $\triangle ABC$.

We set $\triangle ABC$ as our reference triangle, with $A = (1,0,0), B = (0,1,0), C = (0,0,1)$ and $a = \overline{\text{BC}}, b = \overline{\text{CA}}, c = \overline{\text{AB}}$. Further, let $P = (p,q,r)$. Then, $A_1 = (0 : q : r) = (0, q/(q+r), r/(q+r))$. Thus,

$$A_2 = 2A_1 - P = (-p, 2q/(q+r) - q, 2r/(q+r) - r).$$

Now, if A_2 lies strictly inside the circumcircle, we must have its power to be negative. Its power is

$$-a^2 \cdot \frac{2q - q(q+r)}{q+r} \cdot \frac{2r - r(q+r)}{q+r} + b^2 p \cdot \frac{2r - r(q+r)}{q+r} + c^2 p \frac{2q - q(q+r)}{q+r},$$

which simplifies as

$$\frac{-4a^2qr + 4a^2qr(q+r) - a^2qr(q+r)^2 + 2b^2pr - b^2pr(q+r) + 2c^2pq - c^2pq(q+r)}{q+r}.$$

Recalling $p + q + r = 1$, we can rewrite this as

$$\frac{-a^2qr(p+1)^2 + b^2pr(p+1) + c^2pq(p+1)}{q+r} = \frac{p+1}{q+r} \cdot (-a^2qr(p+1) + b^2pr + c^2pq)$$

Now, if this power is negative, it implies $-a^2qr(p+1) + b^2pr + c^2pq < 0$. Adding the analogous inequalities for B_2, C_2 gives

$$a^2qr(1-p) + b^2pr(1-q) + c^2(1-r) < 0,$$

which contradicts $p, q, r < 1$. Thus, A_2, B_2, C_2 cannot all lie strictly inside (ABC). ■