2002 N3

Ezra Guerrero Alvarez

October 27, 2021

2002 N3

2002 N3

Let $p_1, p_2, \ldots, p_n$ be distinct primes greater than 3. Show that $2^{p_1 p_2 \cdots p_n} + 1$ has at least 4^n divisors.

We will prove the stronger claim that $N := 2^{p_1 p_2 \cdots p_n} + 1$ has at least 2^{2^n} divisors.
For a subset $\mathcal{A}$ of $\{p_1, \ldots, p_n\}$ let $\pi(\mathcal{A})$ denote the product of its elements, with $\pi(\emptyset) = 1$ by convention. Then, since $\pi(\mathcal{A}) \mid p_1 \cdots p_n$ and this product is odd, it is well known that

$$2^{\pi(\mathcal{A})} + 1 \mid N.$$

Now, call a prime factor q of $2^{\pi(\mathcal{A})} + 1$ *qualified* if it does not divide $2^{\pi(\mathcal{C})} + 1$ for any $\mathcal{C} \subseteq \{p_1, \ldots, p_n\}$ with $\pi(\mathcal{C}) < \pi(\mathcal{A})$. By Zsigmondy's, we know that there is a qualified prime for every subset of $\{p_1, \ldots, p_n\}$ (here we use $p_i > 3$ to avoid the exception). Thus, since each of these qualified primes, which by definition are distinct, divide N, we see that N has at least 2^n prime divisors. Therefore, N has at least 2^{2^n} divisors. ■