2002 A1

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Find all functions f from the reals to the reals such that

$$f\left(f(x)+y\right)=2x+f\left(f(y)-x\right)$$

for all real x, y.

We claim the only solutions are $f(x)=x+c$ which are easily seen to work. Now, plugging in $y=-f(x)$ we see

$$f(0)-2x=f(f(-f(x))-x),$$

implying f is surjective. Now, suppose we have $f(a)=f(b)$ and let u be such that $f(u)=a+b$. Then, setting $x=a, y=u$ and $x=b, y=u$ we get

$$\begin{aligned} f(f(a)+u) &= 2a+f(a+b-a)=2a+f(b) \\ f(f(b)+u) &= 2b+f(a+b-b)=2b+f(a). \end{aligned}$$

Thus, $2a=2b$ so $a=b$ and f is injective. Finally, setting $x=0$,

$$f(f(0)+y)=f(f(y)) \implies f(y)=y+f(0),$$

as desired. ■