2010 G3

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Let $A_1A_2\dots A_n$ be a convex polygon. Point P inside this polygon is chosen so that its projections $P_1,\dots,P_n$ onto lines $A_1A_2,\dots,A_nA_1$ respectively lie on the sides of the polygon. Prove that for arbitrary points $X_1,\dots,X_n$ on sides $A_1A_2,\dots,A_nA_1$ respectively,

$$\max\left\{\frac{X_1X_2}{P_1P_2},\dots,\frac{X_nX_1}{P_nP_1}\right\}\geq 1.$$

Let θ_j be the measure of the (counter-clockwise) angle $\angle X_{j-1}PX_j$, where we take $X_0 = X_n$. If we had $\theta_j + \angle A_j < 180^\circ$ for all j, then adding over all of them would give

$$360^\circ + (n-2)180^\circ < n180^\circ,$$

which is a clear contradiction. Hence, there exists some j for which $\theta_j \geq 180^\circ - \angle A_j$. This implies that P lies inside of the circumcircle of $\triangle X_{j-1}A_jX_j$. Therefore, if its diameter is D, we have $A_jP \leq D$. Then, by the law of sines, we have

$$\frac{X_{j-1}X_j}{\sin\angle A_j} = D, \quad \frac{P_{j-1}P_j}{\sin\angle A_j} = A_jP,$$

noting that A_jP is the diameter of $(P_{j-1}A_jP_j)$. Hence,

$$\frac{X_{j-1}X_j}{P_{j-1}P_j} = \frac{D}{A_jP} \geq 1,$$

implying the desired result. ■