2000 G3

Ezra Guerrero Alvarez

February 2, 2022

2000 G3

2000 G3

Let O be the circumcenter and H the orthocenter of an acute triangle ABC. Show that there exist points D, E, and F on sides BC, CA, and AB respectively such that

$$OD + DH = OE + EH = OF + FH$$

and the lines AD, BE, and CF are concurrent.

Let P be the reflection of H over $\overline{\mathrm{BC}}$ and $D = \overline{\mathrm{BC}} \cap \overline{\mathrm{OP}}$. Define E, F similarly. We claim D, E, F satisfy the required conditions. Indeed,

$$OD + DH = OD + DP = OP = R,$$

and analogously for E and F, so the length condition is met. Also, note that $\angle PHD = \angle DPH = \angle PAO$, so $\overline{\mathrm{HD}} \parallel \overline{\mathrm{AO}}$. Hence, if $D' = \overline{\mathrm{AO}} \cap \overline{\mathrm{BC}}$ we find

$$\angle OD'D = \angle HDB = \angle BDP = \angle D'DO,$$

so O lies on the perpendicular bisector of $\overline{\mathrm{DD'}}$. This implies $\overline{\mathrm{AD}}$ and $\overline{\mathrm{AO}}$ are isotomic conjugates. Therefore, $\overline{\mathrm{AD}}, \overline{\mathrm{BE}}, \overline{\mathrm{CF}}$ concur at the isotomic conjugate of O. ■

Remark: This solution is inspired by the following lemma:

Lemma 1: Isogonal Conjugates give tangent Ellipse

Let P and Q be isogonal conjugates wrt $\triangle ABC$. Then, there exists an ellipse with foci P and Q that is tangent to all three sides of $\triangle ABC$.

Considering the tangency points of the ellipse when $(P, Q) = (H, O)$ it suffices to show the three cevians concur, which follows after some angle chasing.
Indeed, using this gives a second solution:
Let D, E, F be the points of tangency of the ellipse with foci H, O that is tangent to the sides of $\triangle ABC$, which exists by the above lemma. By degenerate Brianchon on $AECDBF$, lines $\overline{\mathrm{AD}}, \overline{\mathrm{BE}}, \overline{\mathrm{CF}}$ concur. ■