2006 A2

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The sequence of real numbers $a_0, a_1, a_2, \ldots$ is defined recursively by

$$a_0 = -1, \qquad \sum_{k=0}^{n} \frac{a_{n-k}}{k+1} = 0 \quad \text{for } n \geq 1.$$

Show that $a_n > 0$ for all $n \geq 1$.

The problem statement gives

$$\frac{a_n}{1} + \ldots + \frac{a_1}{n} = \frac{1}{n+1}$$
$$\frac{a_{n+1}}{1} + \ldots + \frac{a_1}{n+1} = \frac{1}{n+2}.$$

Now, these two equations give

$$a_{n+1} = \sum_{k=1}^{n} a_k \left(\frac{n+1}{n+2} \cdot \frac{1}{(n+1-k)(n+2-k)} \right).$$

Now we finish the problem by induction. Since $a_1 = \frac{1}{2}$, using the above equation as our inductive hypothesis and nothing that the expression inside parentheses is always positive, we can conclude all a_n are positive for $n \geq 1$. $\blacksquare$