2000 N3

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Does there exist a positive integer n such that n has exactly 2000 prime divisors and n divides 2^n+1?

We show more generally that there exists a positive integer n with exactly k prime divisors such that $n \mid 2^n+1$. We proceed by induction on k. For our base case, note that $9 \mid 513$. Now, assume for our inductive hypothesis that $n = 9\prod_{i=2}^{k} p_i \mid 2^n+1$ and $p_j \mid 2^{n/p_j \cdots p_k}+1$. By Zsigmondy, there exists a primitive prime divisor $p_{k+1} > 2$ of 2^n+1. If $p_{k+1} \in \{3, p_2, \ldots, p_k\}$, then it is not primitive, as it would divide $2^{n/p_k}+1$. Thus, $p_{k+1} \nmid n$. Now, we show $N = np_{k+1} \mid 2^N+1$. Indeed, since $n \perp p_{k+1}$ it suffices to show $n, p_{k+1} \mid 2^N+1$. We have

$$2^N+1 \equiv (2^n)^{p_{k+1}}+1 \equiv (-1)^{p_{k+1}}+1 \equiv 0 \pmod{n, p_{k+1}},$$

so $N \mid 2^N+1$, as desired. ■