2008 G3

Ezra Guerrero Alvarez

February 10, 2022

2008 G3

2008 G3

Let $ABCD$ be a convex quadrilateral and let P and Q be points in $ABCD$ such that $PQDA$ and $QPBC$ are cyclic quadrilaterals. Suppose that there exists a point E on the line segment PQ such that $\angle PAE = \angle QDE$ and $\angle PBE = \angle QCE$. Show that the quadrilateral $ABCD$ is cyclic.

Let W, X, Y, Z be the points on $\overline{\mathrm{AE}}, \overline{\mathrm{BE}}, \overline{\mathrm{CE}}, \overline{\mathrm{DE}}$ respectively such that W, Z lie on $(PQDA)$ and X, Y lie on $(PQBC)$. From the angle condition it follows both $WQPZ$ and $XPQY$ are isosceles trapezoids. Now, let

$$f(X) = \pm\frac{PX}{QX},$$

where $f(X)$ is negative if X lies on or below the segment and positive otherwise. From the isosceles trapezoids, we see

$$f(W)f(Z) = f(X)f(Y) = 1.$$

Now, by the *Ratio Lemma* we have that

$$f(A)f(W) = f(B)f(X) = f(C)f(Y) = f(D)f(Z) = f(E).$$

Therefore,

$$f(A)f(D) = f(A)f(W)f(D)f(Z) = f(E)^2 = f(B)f(X)f(C)f(Y) = f(B)f(C).$$

But by the ratio lemma once more this implies $\overline{\mathrm{AD}}, \overline{\mathrm{BC}}, \overline{\mathrm{PQ}}$ concur, hence $ABCD$ is cyclic as desired. ■