2000 G4

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Let $A_1A_2\dots A_n$ be a convex polygon, $n \geq 4$. Prove that $A_1A_2\dots A_n$ is cyclic if and only if to each vertex A_j one can assign a pair (b_j, c_j) of real numbers, $j = 1, 2, \dots, n$, so that $A_iA_j = b_jc_i - b_ic_j$ for all i, j with $1 \leq i < j \leq n$.

Assignable implies Cyclic:
We have

$$\begin{aligned}
A_1A_2 \cdot A_3A_k + A_1A_k \cdot A_2A_3 &= (b_2c_1 - b_1c_2)(b_kc_3 - b_3c_k) + (b_kc_1 - b_1c_k)(b_3c_2 - b_2c_3) \\
&= c_1b_2c_3b_k - c_1b_2b_3c_k - b_1c_2c_3b_k + b_1c_2b_3c_k + c_1c_2b_3b_k - c_1b_2c_3b_k - b_1c_2b_3c_k + b_1b_2c_3c_k \\
&= c_1c_2b_3b_k + b_1b_2c_3c_k - c_1b_2b_3c_k - b_1c_2c_3b_k \\
&= b_3c_1(b_kc_2 - b_2c_k) - b_1c_3(b_kc_2 - b_2c_k) \\
&= (b_3c_1 - b_1c_3)(b_kc_2 - b_2c_k) \\
&= A_1A_3 \cdot A_2A_k.
\end{aligned}$$

Thus, by the converse of Ptolemy's theorem, A_k lies on $(A_1A_2A_3)$ for all $k \geq 4$. Hence, $A_1A_2\dots A_n$ is cyclic.
Cyclic implies Assignable:
Scale so that the circumcircle of $A_1A_2\dots A_n$ has radius 1. Then, we assign to A_j the pair

$$(\sqrt{2}\sin(\angle A_1OA_j/2), \sqrt{2}\cos(\angle A_1OA_j/2)).$$

Therefore,

$$\begin{aligned}
b_jc_i - b_ic_j &= (\sqrt{2}\sin(\angle A_1OA_j/2))(\sqrt{2}\cos(\angle A_1OA_i/2)) - (\sqrt{2}\sin(\angle A_1OA_i/2))(\sqrt{2}\cos(\angle A_1OA_j/2)) \\
&= 2\sin(\angle A_1OA_j/2 - \angle A_1OA_i/2) \\
&= 2\sin(\angle A_iOA_j/2) \\
&= 2\sin\angle A_iA_1A_j \\
&= A_iA_j,
\end{aligned}$$

where the last step follows from the law of sines on $\triangle A_iA_1A_j$. Thus, this assignment of pairs works. ■