2015 A1

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Suppose that a sequence of $a_1, a_2, \ldots$ of positive real numbers satisfies

$$a_{k+1} \geq \frac{ka_k}{a_k^2 + (k-1)}$$

for every positive integer k. Prove that $a_1 + a_2 + \cdots + a_n \geq n$ for every $n \geq 2$.

Adding from $k = 1$ to m we get

$$a_1 + \ldots + a_m \geq \frac{m}{a_{m+1}}.$$

Now we prove the problem by induction. For the base case $n = 2$, we have

$$a_2 \geq \frac{1}{a_1},$$

so $a_1 + a_2 \geq a_1 + \frac{1}{a_1} \geq 2$ by AM-GM. Now, suppose that $a_1 + \ldots + a_m \geq m$. If $a_{m+1} \geq 1$ then clearly $a_1 + \ldots + a_{m+1} \geq m + 1$. Otherwise, $a_{m+1} < 1$. But then

$$a_1 + \ldots + a_{m+1} \geq \frac{m}{a_{m+1}} + a_{m+1} = \frac{m-1}{a_{m+1}} + \left(\frac{1}{a_{m+1}} + a_{m+1}\right) > m - 1 + 2 = m + 1,$$

finishing the induction. ■