2010 A1

Ezra Guerrero Alvarez

November 27, 2021

2010 A1

2010 A1

Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$,

$$f(\lfloor x \rfloor y) = f(x) \lfloor f(y) \rfloor .$$

We claim the only solutions are $f(x) \equiv 0$ and $f(x) = c$ for any $c \in [1, 2)$, which clearly work. Setting $x = y = 0$ gives $f(0) = f(0) \lfloor f(0) \rfloor$, so $f(0) = 0$ or $\lfloor f(0) \rfloor = 1$. In the second case, setting $y = 0$ we find $f(0) = f(x)$ for all x, so f is constant. This constant is in $[1, 2)$ since $\lfloor f(0) \rfloor = 1$. In the first case, suppose there exists u such that $\lfloor f(u) \rfloor \neq 0$. Then, $(x, y) = (0.5, u)$ gives

$$0 = f(0.5) \lfloor f(u) \rfloor ,$$

so $f(0.5) = 0$. Then, $(x, y) = (2, 0.5)$ gives $f(1) = 0$. Finally, $x = 1$ gives $f(y) = 0$ for all y, contradicting the existence of u. Thus, $\lfloor f(x) \rfloor = 0$ for all x. But then, setting $x = 1$ in the original gives $f(y) = 0$ for all y, concluding the proof. ■