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Let p be a prime number and let A be a set of positive integers that satisfies the following conditions:

(i) the set of prime divisors of the elements in A consists of $p-1$ elements;

(ii) for any nonempty subset of A, the product of its elements is not a perfect p-th power.

What is the largest possible number of elements in A ?

We claim the answer is $(p-1)^2$. This is achievable by taking the set

$$A = \{q_i^{pj+1} | (i,j) \in \{1,\ldots,p-1\}^2\}$$

where $\{q_i\}_{i=1}^{\infty}$ is the sequence of primes. Now we show $|A| > (p-1)^2$ is impossible. Let $n = |A|$. Note that by condition (i) we can represent the elements of A as vectors in $\mathbb{F}_p^{p-1}$. Let these be

$$v_i = (e_{i1}, \ldots, e_{i(p-1)}) \text{ for } 1 \leq i \leq n.$$

Now, we define $f_j \colon \mathbb{F}_p^n \to \mathbb{F}_p^n$ as

$$f_j(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i^{p-1} e_{ji}.$$

Consider the system of equations $f_j(x_1, \ldots, x_n) = 0$ for $1 \leq j \leq p-1$. Note that $(0, \ldots, 0)$ is a trivial solution. Suppose for the sake of contradiction that $n > (p-1)^2$. Then, we have

$$n > (p-1)^2 = (p-1) \cdot (p-1) = \sum_{j=1}^{p-1} \deg(f_j).$$

Thus, quoting *Chevalley's Theorem* there must exist a nontrivial solution to the system. However, since $x_i^{p-1} \in 0, 1$, this nontrivial solution specifies a nonempty subset of A for which the product is a perfect $p-$th power, yielding the desired contradiction. ■