2006 G1

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October 22, 2021

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Let ABC be triangle with incenter I. A point P in the interior of the triangle satisfies

$$\angle PBA + \angle PCA = \angle PBC + \angle PCB.$$

Show that $AP \geq AI$, and that equality holds if and only if $P = I$.

We have

$$2(\angle PBC + \angle PCB) = \angle PBA + \angle PCA + \angle PBC + \angle PCB = 180^\circ - \angle BAC.$$

Thus, $\angle BPC = 180^\circ - (90^\circ - \frac{1}{2}\angle BAC) = 90^\circ + \frac{1}{2}\angle BAC = \angle BIC$. Hence, P is on (BIC). From the incenter-excenter lemma, if L is the midpoint of $\widehat{BC}$, we have $A - I - L$ and $LP = LI$. Now, by the triangle inequality,

$$AP + PL \geq AL = AI + IL,$$

implying $AP \geq AI$, with equality iff $A - P - L$, that is, when $P = I$. ■