2005 G4

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Let $ABCD$ be a fixed convex quadrilateral with $BC = DA$ and BC not parallel with DA. Let two variable points E and F lie of the sides BC and DA, respectively and satisfy $BE = DF$. The lines AC and BD meet at P, the lines BD and EF meet at Q, the lines EF and AC meet at R.
Prove that the circumcircles of the triangles PQR, as E and F vary, have a common point other than P.

We use directed angles mod 180°. Let T be the center of the rotation mapping $\overline{\mathrm{BC}}$ to $\overline{\mathrm{DA}}$, which exists since they are not parallel. Note that this rotation maps E to F. It follows the following quadrilaterals are cyclic:

$$BCTP, ADTP, BETQ, DFQT, CERT, AFTR.$$

We will show that $T \in (PQR)$. Since T does not depend on E and F, this will finish the problem. We have

$$\measuredangle PTQ = \measuredangle PTD + \measuredangle DTQ = \measuredangle PAD + \measuredangle DFQ = \measuredangle RAF + \measuredangle AFR = -\measuredangle FRA = \measuredangle ARF = \measuredangle PRQ,$$

so $PQTR$ is cyclic as desired. ■