2015 A2

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Determine all functions $f : \mathbb{Z} \to \mathbb{Z}$ with the property that

$$f(x - f(y)) = f(f(x)) - f(y) - 1$$

holds for all $x, y \in \mathbb{Z}$.

We claim the only solutions are $f \equiv -1$ and $f \equiv x + 1$ which are easily seen to work. Now, plugging in $x = 0, y = f(0)$ we obtain

$$f(-f(f(0))) = -1.$$

Let $u = -f(f(0))$, then setting $y = u$,

$$f(x + 1) = f(f(x)).$$

Hence, the given equation becomes $f(x - f(y)) = f(x + 1) - f(y) - 1$. Substituting $x = f(n) - 1, y = n$ we get

$$\begin{aligned} f(f(n) - 1 - f(n)) &= f(f(n) - 1 + 1) - f(n) - 1 \\ f(-1) &= f(n + 1) - f(n) - 1 \\ f(-1) + 1 &= f(n + 1) - f(n). \end{aligned}$$

Now, since the LHS is a constant, we get $f(n + 1) - f(n)$ is constant for all n. Since the domain of f is $\mathbb{Z}$, it follows $f(x) = kx + c$ for some $k, c \in \mathbb{Z}$. Substituting in the equation we find

$$c + 1 = (k^2 - k)(x + y) + 2kc.$$

Setting $x + y = 0$, we find $c \mid c + 1$, so $c \in \{-1, 1\}$. Each of these cases give the aforementioned solutions. ■