2001 G2

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Consider an acute-angled triangle ABC. Let P be the foot of the altitude of triangle ABC issuing from the vertex A, and let O be the circumcenter of triangle ABC. Assume that $\angle C \geq \angle B + 30^\circ$. Prove that $\angle A + \angle COP < 90^\circ$.

We begin by showing the following claim:

Claim 1

We have

$$\sin\angle A \sin\angle B \cos\angle C < \frac{1}{4}.$$

Proof.

$$\begin{aligned}
\sin\angle A \sin\angle B \cos\angle C &= \frac{1}{2}\sin\angle A(\sin(\angle B + \angle C) + \sin(\angle B - \angle C)) \\
&= \frac{1}{2}\sin\angle A(\sin\angle A - \sin(\angle B - \angle C)) \\
&\leq \frac{1}{2}\sin\angle A\left(\sin\angle A - \frac{1}{2}\right) \\
&< \frac{1}{4},
\end{aligned}$$

where the last inequality comes from the quadratic on $\sin\angle A$ being increasing on $[1/2, 1)$ and negative when $\sin\angle A < \frac{1}{2}$. □

Now, this claim implies

$$r^2 = \frac{ab}{4\sin\angle A \sin\angle B} > ab\cos\angle C = a \cdot PC.$$

Now, from power of a point

$$OP^2 = r^2 - PC \cdot PB = r^2 - a \cdot PC + PC^2 > PC^2,$$

so $OP > PC$. Thus, looking at triangle POC, we get $90^\circ - \angle A = \angle PCO > \angle COP$ as desired. ■