2018 G5

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Let ABC be a triangle with circumcircle Ω and incentre I. A line ℓ intersects the lines AI, BI, and CI at points D, E, and F, respectively, distinct from the points A, B, C, and I. The perpendicular bisectors x, y, and z of the segments AD, BE, and CF, respectively determine a triangle Θ. Show that the circumcircle of the triangle Θ is tangent to Ω.

We use directed angles $\mod 180°$. Let $X = y \cap z$ and define Y, Z similarly. Let T be where the reflection of ℓ over $\overline{\text{YZ}}$ meets Ω. We show the reflections of ℓ over the sides of Θ all meet at T. Let T' be the reflection of T over $\overline{\text{YZ}}$ and T'' the reflection of T over $\overline{\text{ZX}}$. Then,

$$\measuredangle EDI = \measuredangle T'DA = \measuredangle DAT = \measuredangle M_A AT.$$

Now,

$$\measuredangle BET'' = \measuredangle TBE = \measuredangle TBM_B = \measuredangle TBM_A + \measuredangle M_A BM_B = 90° - \measuredangle ACI - \measuredangle M_A AT = -\measuredangle EDI - \measuredangle DIE = \measuredangle IED = \measuredangle BED,$$

so T'' is on ℓ. Analogously with $\overline{\text{XY}}$, we conclude the reflections of ℓ over the sides of Θ concur at T.
Now, this implies that the feet of the perpendicular from T to the sides of Θ are collinear. Thus, T has a Simson line wrt. Θ so it must lie on its circumcircle. Now, this implies as well that H, the orthocenter of Θ, lies on ℓ. Now, consider the reflections of H over the sides of Θ. It is well known these lie on its circumcircle. Also, the triangle formed by these reflections is similar to $\triangle ABC$. Since H is in ℓ it follows these triangles are homothetic and by our above work lines connecting corresponding vertices concur at T. Hence, T is the center of homothety from (XYZ) to (ABC), so these are tangent. ■