2020 G1

Ezra Guerrero Alvarez

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Let ABC be an isosceles triangle with $BC = CA$, and let D be a point inside side AB such that $AD < DB$. Let P and Q be two points inside sides BC and CA, respectively, such that $\angle DPB = \angle DQA = 90°$. Let the perpendicular bisector of PQ meet line segment CQ at E, and let the circumcircles of triangles ABC and CPQ meet again at point F, different from C.
Suppose that P, E, F are collinear. Prove that $\angle ACB = 90°$.

We use directed angles $\mod 180°$. Let M be the midpoint of $\overline{\mathrm{AB}}$, D' be the second intersection of $\overline{\mathrm{CD}}$ with (ABC) and T be the second intersection of $\overline{\mathrm{FB}}$ with (CPQ). First, since $\measuredangle CPD = \measuredangle CQD = 90°$, we have $D \in (PCFQ)$. Then,

$$\measuredangle FDQ = \measuredangle FPQ = \measuredangle EPQ = \measuredangle PQE = \measuredangle PQC = \measuredangle PDC,$$

so $\overline{\mathrm{DC}}$ and $\overline{\mathrm{DF}}$ are isogonal with respect to $\triangle PDQ$. Since $\overline{\mathrm{CD}}$ is a diameter, it follows $\overline{\mathrm{DF}}$ is perpendicular to $\overline{\mathrm{PQ}}$. Now, note that F is the Miquel point of $ABPQ$ and $D'BPD$, so by spiral similarity it follows $\overline{\mathrm{FD'}}$ is perpendicular to $\overline{\mathrm{AB}}$. Now,

$$\measuredangle FBA = \measuredangle FCA = \measuredangle FCQ = \measuredangle FTQ,$$

so $\overline{\mathrm{BA}} \parallel \overline{\mathrm{TQ}}$. Also, since $\triangle ABC$ is isosceles, $\overline{\mathrm{CM}} \perp \overline{\mathrm{AB}}$, so $M \in (PTDQFC)$. Thus, $MDQT$ is an isosceles trapezoid. This means

$$\measuredangle AFD' = \measuredangle ACD' = \measuredangle QCD = \measuredangle MFT = \measuredangle MFB,$$

so $\overline{\mathrm{FD'}}$ and $\overline{\mathrm{FM}}$ are isogonal with respect to $\triangle BFA$. Since $\overline{\mathrm{FD'}} \perp \overline{\mathrm{AB}}$, it follows $\overline{\mathrm{FM}}$ passes through the center of (ABC), so it must be M. Thus, $\overline{\mathrm{AB}}$ is a diameter and $\measuredangle ACB = 90°$ as desired. ■