2020 G2

Ezra Guerrero Alvarez

November 18, 2021

2020 G2

2020 G2

Consider the convex quadrilateral $ABCD$. The point P is in the interior of $ABCD$. The following ratio equalities hold:

$$\angle PAD : \angle PBA : \angle DPA = 1 : 2 : 3 = \angle CBP : \angle BAP : \angle BPC.$$

Prove that the following three lines meet in a point: the internal bisectors of angles $\angle ADP$ and $\angle PCB$ and the perpendicular bisector of segment AB.

Let R and S be points on $\overline{\text{DA}}$ and $\overline{\text{BC}}$ such that $\overleftrightarrow{\text{BR}}$ and $\overleftrightarrow{\text{AS}}$ are the interior angle bisectors of $\angle PBA$ and $\angle BAP$ respectively. Then, the angle ratios give

$$\angle PAR = \angle PBR,$$

so $RABP$ is cyclic. Since $\angle RBA = \angle PBR$, we have $\angle RPA = \angle PAR$. Thus, $\angle PRD = 2\angle PAR$ from the exterior angle theorem. Also, from the angle ratios we have $\angle RPD = 2\angle PAR$. Hence, $\triangle DRP$ is isosceles with D as its vertex, so the internal angle bisector of $\angle ADP$ is the perpendicular bisector of $\overline{\text{RP}}$. Analogously, $SBAP$ is cyclic and the interior angle bisector of $\angle PCB$ is the perpendicular bisector of $\overline{\text{PS}}$. Since $ARPSB$ is cyclic, the perpendicular bisectors of $\overline{\text{RP}}$, $\overline{\text{PS}}$, and $\overline{\text{AB}}$ concur at its circumcenter, giving the desired result. ■

Remark. I solved this in contest!!!