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Let $a_2, a_3, \ldots, a_n$ be positive reals with product 1, where $n \geq 3$. Show that

$$(1+a_2)^2(1+a_3)^3 \ldots (1+a_n)^n > n^n.$$

By AM-GM we have $\underbrace{\frac{1}{k-1} + \ldots + \frac{1}{k-1}}_{k-1 \text{ times}} + a_k \geq k \cdot \left(\frac{a_k}{(k-1)^{k-1}}\right)^{1/k}$, so

$$(1+a_k)^k \geq \frac{a_k \cdot k^k}{(k-1)^{k-1}}.$$

Multiplying over all values of k,

$$(1+a_2)^2(1+a_3)^3 \cdots (1+a_n)^n \geq \frac{a_2 \cdot 2^2}{(1)^1} \cdot \frac{a_3 \cdot 3^3}{(2)^2} \cdots \frac{a_n \cdot n^n}{(n-1)^{n-1}} = 1 \cdot n^n/1^1 = n^n.$$

Now we show the inequality is strict. If we had equality in the last inequality, then we must have equality in all our AM-GM inequalities, so $a_k = \frac{1}{k-1}$ for all k. However, since $n \geq 3$, their product would not be 1, proving the inequality is strict as desired. ■