2006 G2

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October 15, 2021

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Let $ABCD$ be a trapezoid with parallel sides $AB > CD$. Points K and L lie on the line segments AB and CD, respectively, so that $AK/KB = DL/LC$. Suppose that there are points P and Q on the line segment KL satisfying

$$\angle APB = \angle BCD \qquad \text{and} \qquad \angle CQD = \angle ABC.$$

Prove that the points P, Q, B and C are concyclic.

We use directed angles $\mod 180^\circ$. Let $T = \overline{\mathrm{AD}} \cap \overline{\mathrm{BC}}$. By homothety, $T - L - K$. Now, from $\measuredangle APB = \measuredangle DCB = \measuredangle ABC$ we see that $\overline{\mathrm{CB}}$ is tangent to (APB). Analogously, $\overline{\mathrm{CB}}$ is tangent to (CDQ). Then, if P' is the second intersection of (CDQ) with $\overline{\mathrm{TP}}$, by tangency and homothety we have

$$\measuredangle CBP = \measuredangle BAP = \measuredangle CDP' = \measuredangle CQP' = \measuredangle CQP,$$

implying the cyclic quad. ■