2006 G4

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A point D is chosen on the side AC of a triangle ABC with $\angle C < \angle A < 90^\circ$ in such a way that $BD = BA$. The incircle of ABC is tangent to AB and AC at points K and L, respectively. Let J be the incenter of triangle BCD. Prove that the line KL intersects the line segment AJ at its midpoint.

We set $\triangle ABC$ as our reference triangle, with $A = (1,0,0), B = (0,1,0), C = (0,0,1)$ and $a = \overline{\text{BC}}, b = \overline{\text{CA}}, c = \overline{\text{AB}}$. Let E be the foot of the height from B to $\overline{\text{AC}}$, P be the point where the interior angle bisector of $\angle BDC$ meets $\overline{\text{BC}}$ and M be the midpoint of $\overline{\text{AJ}}$. First, since $\triangle ABD$ is isosceles, we know E is the midpoint of $\overline{\text{AD}}$. Then, since $E = (S_bSc : 0 : S_aS_b) = (S_c/b^2, 0, S_a/b^2)$ we have

$$D = 2M - A = \left(\frac{a^2-c^2}{b^2}, 0, \frac{2S_a}{b^2}\right) = \left(\frac{a^2-c^2}{b}, 0, \frac{2S_a}{b}\right).$$

Note that $\frac{a^2-c^2}{b} + 0 + \frac{2S_a}{b} = b$. Therefore, $AD = \frac{2S_a}{b}$ and $DC = \frac{a^2-c^2}{b}$. By the angle bisector theorem, this implies

$$CP \colon BP = DC \colon BD = \frac{a^2-c^2}{b} : c = (a^2-c^2) \colon bc.$$

Hence, $P = (0 : a^2-c^2 : bc)$. Now, since J lies on line $\overline{\text{CI}}$, we have $J = (a : b : t)$ for some t. Since J lies on $\overline{\text{DP}}$ we then have

$$\begin{vmatrix} 0 & a^2-c^2 & bc \\ a^2-c^2 & 0 & 2S_a \\ a & b & t \end{vmatrix} = 0.$$

Solving for t gives

$$t = \frac{2aS_a + b^2c}{a^2-c^2},$$

so $J = \left(a : b : \frac{2aS_a+b^2c}{a^2-c^2}\right)$. Normalizing,

$$J = \left(\frac{a(a^2-c^2)}{b(a+c)(a+b-c)}, \frac{b(a^2-c^2)}{b(a+c)(a+b-c)}, \frac{2aS_a+b^2c}{b(a+c)(a+b-c)}\right).$$

Now, $M = (A+J)/2$ so

$$M = \left(\frac{a(a^2-c^2)+b(a+c)(a+b-c)}{2b(a+c)(a+b-c)}, \frac{b(a^2-c^2)}{2b(a+c)(a+b-c)}, \frac{2aS_a+b^2c}{2b(a+c)(a+b-c)}\right).$$

Thus, $M = (a(a^2-c^2)+b(a+c)(a+b-c) : b(a^2-c^2) : 2aS_a+b^2c)$. We wish to show $K = (s-b : s-a : 0), L = (s-c : 0 : s-a)$, and M are collinear. This happens iff

$$\begin{vmatrix} a(a^2-c^2)+b(a+c)(a+b-c) & b(a^2-c^2) & 2aS_a+b^2c \\ s-c & 0 & s-a \\ s-b & s-a & 0 \end{vmatrix} = 0.$$

This happens iff $(s-c)(s-a)(2aS_a+b^2c) = (s-a)((s-a)(a(a^2-c^2)+b(a+c)(a+b-c))-(s-b)b(a^2-c^2))$.

$$\begin{aligned}
(s-c)(s-a)(2aS_a+b^2c) &= (s-a)((s-a)(a(a^2-c^2)+b(a+c)(a+b-c))-(s-b)b(a^2-c^2)) \\
(s-c)(2aS_a+b^2c) &= (s-a)(a(a^2-c^2)+2b(a+c)(s-c))-(s-b)b(a^2-c^2) \\
(s-c)(2aS_a+b^2c) &= -(a^2-c^2)(a-b)(s-c)+2b(a+c)(s-c)(s-a) \\
2aS_a+b^2c &= (a+c)(b^2+ca-a^2) \\
a(-a^2+b^2+c^2)+b^2c &= ab^2-a^3+cb^2+c^2a.
\end{aligned}$$

Since these two quantities are equal, this proves the result. ■