2013 G2

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Let ω be the circumcircle of a triangle ABC. Denote by M and N the midpoints of the sides AB and AC, respectively, and denote by T the midpoint of the arc BC of ω not containing A. The circumcircles of the triangles AMT and ANT intersect the perpendicular bisectors of AC and AB at points X and Y, respectively; assume that X and Y lie inside the triangle ABC. The lines MN and XY intersect at K. Prove that $KA = KT$.

We use directed angles $\mod 180^\circ$. Let O be the center of ω, P be a point on ω such that $\overline{\mathrm{PT}} \parallel \overline{\mathrm{AC}}$, and $X' = \overline{\mathrm{PT}} \cap \overline{\mathrm{ON}}$. We have

$$\measuredangle ABP = \measuredangle ATP = \measuredangle TAC = \measuredangle TAB,$$

so $\overline{\mathrm{PB}} \parallel \overline{\mathrm{AT}}$. Thus, $APBT$ is an isosceles trapezoid, giving $PT = AB$. Since X' is the midpoint of $\overline{\mathrm{PT}}$ (lies on its perpendicular bisector), this gives $PX' = BM$. Thus, since $BP = PB$ and

$$\measuredangle PBM = \measuredangle PBA = \measuredangle PTA = \measuredangle TPB = \measuredangle X'PB$$

we have $PMX'B$ is an isosceles trapezoid. Thus, $\overline{\mathrm{MX'}} \parallel \overline{\mathrm{PB}} \parallel \overline{\mathrm{AT}}$. Hence, $\measuredangle TAM = \measuredangle ATX'$ implies X' lies on (AMT). This gives $X' = X$, so $\overline{\mathrm{MX}} \parallel \overline{\mathrm{AT}}$. Analogously, $\overline{\mathrm{NY}} \parallel \overline{\mathrm{AT}}$.
Now, since $\overline{\mathrm{MX}} \parallel \overline{\mathrm{AT}} \parallel \overline{\mathrm{NY}}$, these three lines have the same perpendicular bisector. This implies that $MNYX$ is an isosceles trapezoid, so $K = \overline{\mathrm{MN}} \cap \overline{\mathrm{XY}}$ must lie on this common perpendicular bisector. Hence, $KA = KT$ as desired. ■