2006 G6

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Circles ω_1 and ω_2 with centers O_1 and O_2 are externally tangent at point D and internally tangent to a circle ω at points E and F respectively. Line t is the common tangent of ω_1 and ω_2 at D. Let $\overline{\mathrm{AB}}$ be the diameter of ω perpendicular to t, so that A, E, O_1 are on the same side of t. Prove that lines AO_1, BO_2, EF and t are concurrent.

Let O be the center of ω and let P be the intersection of the $O-$median of $\triangle OO_1O_2$ and $\overline{\mathrm{EF}}$. We will show P lies on the 4 lines, which implies they are concurrent. First, note that since $O_1D = O_1E, O_2D = O_1F$, and $OE = OF$, circle (DEF) is tangent to all three sides of $\triangle OO_1O_2$. Since D lies between O_1 and O_2 and E and F lie outside of $\overline{\mathrm{OO_1}}$ and $\overline{\mathrm{OO_2}}$ respectively, it follows (DEF) is the $O-$excircle of $\triangle OO_1O_2$.
We now proceed with barycentric coordinates. We set $\triangle OO_1O_2$ as our reference triangle, with $O = (1,0,0), O_1 = (0,1,0), O_2 = (0,0,1)$ and $a = \overline{\mathrm{O_1O_2}}, b = \overline{\mathrm{O_2O}}, c = \overline{\mathrm{OO_1}}$. It follows that $D = (0 : s-b : s-c), E = (-(s-c) : s : 0), F = (-(s-b) : 0 : s)$, where $s = \frac{a+b+c}{2}$. Hence, line EF has equation

$$sx + (s-c)y + (s-b)z = 0.$$

Since P lies on this line and on the $O-$median, it has coordinates $(-a : s : s)$. Now, note that since t is tangent to both ω_1 and ω_2, we have $t \perp \overline{\mathrm{O_1O_2}}$. Since $t \perp \overline{\mathrm{AB}}$ by construction, it follows $\overline{\mathrm{AB}} \parallel \overline{\mathrm{O_1O_2}}$. Also, since $O \in \overline{\mathrm{AB}}$, it follows A and B have coordinates of the form $(1, t, -t)$. Now, note that ω has radius $OE = s$. Hence, $OA^2 = s^2$. By the barycentric distance formula, since $\overrightarrow{\mathrm{OA}} = (0, t, -t)$, we see $s^2 = a^2t^2$. Hence, $t = \pm\frac{s}{a}$. Indeed, B satisfies the same equation, so one solution corresponds to A and the other to B. Since A lies on the same side of t as O_1, we have $A = \left(1, \frac{s}{a}, -\frac{s}{a}\right) = (a : s : -s)$. Similarly, $B = (a : -s : s)$. Then, points on the cevian O_1A are given by $(a : k : -s)$ and points on the cevian O_2B are given by $(a : -s : k)$. It is easy to see $P = (a : -s : -s)$ is of these forms, so P lies on both lines. Finally, we must see $P \in t$. This is equivalent to showing $\overline{\mathrm{PD}} \perp \overline{\mathrm{O_1O_2}}$. Since $\overrightarrow{\mathrm{O_2O_1}} = (0, 1, -1)$ and $\overrightarrow{\mathrm{PD}} = \left(\frac{a}{b+c}, \frac{s-b}{a} - \frac{s}{b+c}, \frac{s-c}{a} - \frac{s}{b+c}\right)$, from Evan's Favorite Forgotten Trick, it suffices to show

$$a^2\left(\frac{s-c}{a} - \frac{s}{b+c} + \frac{s}{b+c} - \frac{s-b}{a}\right) + b^2\left(-\frac{a}{b+c}\right) + c^2\left(\frac{a}{b+c}\right) = 0.$$

Indeed, simplifying the left hand side we obtain

$$a(b-c) + a \cdot \frac{c^2 - b^2}{b+c} = 0$$

as desired. Thus, P lies on AO_1, BO_2, EF, and t, so these 4 lines are concurrent. ■