# 2002 A3

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Let $P$ be a cubic polynomial given by $P(x) = ax^3 + bx^2 + cx + d$, where $a, b, c, d$ are integers and $a \neq 0$. Suppose that $xP(x) = yP(y)$ for infinitely many pairs $x, y$ of integers with $x \neq y$. Prove that the equation $P(x) = 0$ has an integer root.

Assume for the sake of contradiction that $P(x) = 0$ has no integer roots. Now, let $(m, n)$ be such that $mP(m) = nP(n)$. Subtracting and using $m \neq n$, we see

$$a(m+n)(m^2+n^2) + b(m^2 + n(m+n)) + c(m+n) + d = 0.$$

Let $s := m + n$. Then,

$$as(2m^2 - 2ms + s^2) + b(m^2 - ms + s^2) + cs + d = 0,$$

giving

$$(2as + b)m^2 - s(2as + b)m + P(s) = 0.$$

Now, if $2as + b = 0$, then $P(s) = 0$, contradicting there are no integer roots. Hence, assume $2as + b \neq 0$. By the quadratic formula,

$$m = \frac{s \pm \sqrt{s^2 - \frac{4P(S)}{2as+b}}}{2}.$$

Since $m$ is an integer, this implies $2as + b \mid 4P(s) \mid 8a^2P(s)$. Then,

$$\frac{8a^3s^3 + 8a^2bs^2 + 8a^2cs + 8a^2d}{2as + b} = 4a^2s^2 + 2abs + 4ca - b^2 + \frac{8a^2d - 4abc + b^3}{2as + b}.$$

If $8a^2d - 4abc + b^3 \neq 0$, then $s$ can only take on finitely many values. Since given $s$ there are only two possible values of $x$, giving two values of $y$, this would contradict having infinitely many pairs which work. Thus, $8a^2d - 4abc + b^3 = 0$. This implies that

$$2s^2 + \frac{2c + bs}{a} - \frac{b^2}{2a^2} \in \mathbb{Z}.$$