2018 A2

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Find all integers $n \geq 3$ for which there exist real numbers $a_1, a_2, \dots, a_n$ satisfying

$$a_i a_{i+1} + 1 = a_{i+2}$$

for $i = 1, 2, \dots, n$, where indices are taken modulo n.

We claim the answer is all n that are multiples of 3. For these, the sequence $(-1, -1, 2, \dots, -1, -1, 2)$ satisfies the given equations. Now, note that

$$a_{i+2}^2 - a_{i+2} = a_i a_{i+1} a_{i+2} = a_i a_{i+3} - a_i.$$

Thus, we have

$$0 = 2 \cdot \sum (a_{i+2}^2 - a_{i+2} - a_i a_{i+3} + a_i) = \sum (a_i^2 - 2a_i a_i + 3 + a_{i+3}^2),$$

so we have $\sum (a_i - a_{i+3})^2 = 0$. Thus, from the trivial inequality it is clear that $a_i = a_{i+3}$ for all i. If $3 \nmid n$, this would imply that all a_i are equal. However, the equation $a^2 + 1 = a$ has no real solutions, so we conclude n must be a multiple of 3. ■