1997 SL4

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An $n \times n$ matrix whose entries come from the set $S = \{1, 2, \ldots, 2n-1\}$ is called a *silver matrix* if, for each $i = 1, 2, \ldots, n$, the i-th row and the i-th column together contain all elements of S. Show that:

(a) there is no silver matrix for $n = 1997$;

(b) silver matrices exist for infinitely many values of n.

Suppose we have an $n \times n$ ($n > 1$) silver matrix A. Weight each element a_{ij} of the matrix with a $w_{ij} = 1$ if $i = j$ and $w_{ij} = 2$ if $i \neq j$. From the definition of a silver matrix,

$$C := \sum_{a_ij=r} w_{ij}$$

is the same for all $r \in S = \{1, 2, \ldots, 2n-1\}$. Note that $2n - 1 > n$, so that there is some element of S that does not appear in the main diagonal. From this element, it follows C is even, since w_{ij} will always be 2. Therefore, it follows that if any element of S must show up an even number of times in the main diagonal. This is clearly impossible if n is odd, which proves (a).

Now, we provide an inductive proof that silver matrices always exist for n a power of 2, which will prove (b). We have the following construction for $n = 2$:

$$\begin{bmatrix} A & C \\ B & A \end{bmatrix},$$

where $\{A, B, C\} = \{1, 2, 3\}$. The crucial bit is that the main diagonal consists of the same element. For our inductive hypothesis, assume a silver matrix exists for $n = 2^k$, with the main diagonal being the same element. Now, we construct a silver matrix for $n = 2^{k+1}$ with the main diagonal being the same element, which will conclude the induction. Let M_1 be a $2^k \times 2^k$ silver matrix, but replace S with

$$S_1 := \{1, 2, \ldots, 2^k - 1, 2^{k+2} - 2^k, \ldots, 2^{k+2} - 1\}.$$

Similarly, let M_2 and M_3 be $2^k \times 2^k$ silver matrices, but replacing S with

$$S_2 := \{2^k, \ldots, 2^{k+2} - 2^k - 2\}, S_3 := \{2^k + 1, \ldots, 2^{k+2} - 2^k - 1\}$$

respectively. M_2 and M_3 will be identical, except for their main diagonal. While M_2 has its main diagonal all equal to 2^k, M_3 has its main diagonal all equal to $2^{k+2} - 2^k - 1$. We claim

$$M := \begin{bmatrix} M_1 & M_2 \\ M_3 & M_1 \end{bmatrix}$$

is a silver matrix. Indeed, looking at the ith row and column, all elements of S_1 appear exactly once because of M_1, all elements of $S_2 \cap S_3$ appear exactly once since M_2 and M_3 are identical except for the main diagonal, and $2^k, 2^{k+2} - 2^k - 1$ appear exactly once because of the main diagonals of M_2 and M_3. Hence, M is a silver matrix as desired. ■