2000 N2

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For a positive integer n, let $d(n)$ be the number of all positive divisors of n. Find all positive integers n such that $d(n)^3 = 4n$.

We claim the only such n are $2, 128$, and 2000, which are easily seen to work. Now, from $4n = d(n)^3$ it follows there is some integer m for which $n = 2m^3$. Then, $d(2m^3)^3 = 8m^3$, so

$$d(2m^3) = 2m.$$

Now, let $m = 2^\alpha \cdot x$, with x odd. Further, let $p_1, p_2, \dots$ be primes and $e_1, e_2, \dots$ non-negative integers such that $x = \prod p_i^{e_i}$. Then,

$$d(2^{3\alpha+1} \cdot x^3) = 2^{\alpha+1} \cdot x$$
$$(3\alpha + 2) \prod (3e_i + 1) = 2^{\alpha+1} \prod p_i^{e_i}.$$

Now, notice that the left hand side is not divisible by 3. Hence, $p_i > 3$ for all i. It follows $p_i^{e_i} \geq (3e_i + 1)$ for all i, with equality iff $e_i = 0$. Also, for $\alpha \geq 2$, we have $2^{\alpha+1} \geq (3\alpha + 2)$, with equality iff $\alpha = 2$. We separate into cases:

1. If $\alpha = 0$, then $\prod(3e_i + 1) = \prod p_i^{e_i}$. By the aforementioned inequality this gives $e_i = 0$ for all i, so $x = 1 \implies m = 1 \implies \boxed{n = 2}$.
2. If $\alpha = 1$ then $5 \prod(3e_i+1) = 4 \prod p_i^{e_i}$. Thus, $5 \in \{p_1, \dots\}$. Now, note that if the exponent of 5 is at least 2, $4 \cdot 5^e > 5 \cdot (3e+1)$ and due to the previous inequality both sides will never be equal. Hence, the exponent of 5 is 1. But then, $\prod_{p_i \neq 5}(3e_i+1) = \prod_{p_i \neq 5} p_i^{e_i}$, so $e_i = 0$. Hence, $x = 5 \implies m = 10 \implies \boxed{n = 2000}$.
3. Finally, if $\alpha \geq 2$ then $2^{\alpha+1} \geq (3\alpha + 2)$ and $p_i^{e_i} \geq (3e_i + 1)$. Hence,

$$2^{\alpha+1} \prod p_i^{e_i} \geq (3\alpha + 2) \prod (3e_i + 1) = 2^{\alpha+1} \prod p_i^{e_i}.$$

   Since equality holds it follows $\alpha = 2$ and $e_i = 0$ for all i, so $x = 1 \implies m = 4 \implies \boxed{n = 128}$. ■