2004 G3

Ezra Guerrero Alvarez

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Let O be the circumcenter of an acute-angled triangle ABC with $\angle B < \angle C$. The line AO meets the side BC at D. The circumcenters of the triangles ABD and ACD are E and F, respectively. Extend the sides BA and CA beyond A, and choose on the respective extensions points G and H such that $AG = AC$ and $AH = AB$. Prove that the quadrilateral $EFGH$ is a rectangle if and only if $\angle ACB - \angle ABC = 60°$.

Let ρ denote the reflection over the exterior angle bisector of $\angle A$. Then, $\rho(C) = G$ and $\rho(B) = H$. Now, since $\overline{\mathrm{AD}}$ passes through the circumcenter of $\triangle ABC$ and it is well known that the circumcenter and orthocenter are isogonal conjugates, we find $\overline{\mathrm{AD}} \perp \overline{\mathrm{GH}}$. Now, note that $\angle C - \angle B = 60°$ if and only if $\angle ADC = 30°$. This occurs if and only if $\triangle AFC$ and $\triangle AEB$ are equilateral. From centers and isosceles triangles, these triangles are equilateral if and only if $AG = AF$ and $AH = AE$.

Now, this implies $\angle AFG = \angle FAD$, ie $\overline{\mathrm{FG}} \parallel \overline{\mathrm{AD}}$. Analogously, we would have $\overline{\mathrm{EH}} \perp \overline{\mathrm{GH}} \perp \overline{\mathrm{FG}}$. Since $FE = GH$ from SAS on $\triangle AGH$ and $\triangle AFE$, this implies $EFGH$ is an isosceles trapezoid with right angles, ie, a rectangle. On the other hand, if $EFGH$ is a rectangle, then $\angle FGA = 90° - \angle C$ and $\overline{\mathrm{FG}} \parallel \overline{\mathrm{AD}}$. But then $\angle AFG = \angle FAD = 90° - \angle C$, so $AG = AF$. $\blacksquare$