2017 N2

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Let $p \geq 2$ be a prime number. Eduardo and Fernando play the following game making moves alternately: in each move, the current player chooses an index i in the set $\{0, 1, 2, \ldots, p-1\}$ that was not chosen before by either of the two players and then chooses an element a_i from the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Eduardo has the first move. The game ends after all the indices have been chosen .Then the following number is computed:

$$M = a_0 + a_1 10 + a_2 10^2 + \cdots + a_{p-1} 10^{p-1} = \sum_{i=0}^{p-1} a_i . 10^i.$$

Eduardo wins by setting $a_0 = 0$ always. If $p = 2, 5$ then this is clear, Eduardo may relax the rest of the game. Otherwise, assume $p \perp 10$ and let $d = \text{ord}_p(10)$. Then, $p - 1 = dk$. We have two cases:

1. If d is even, split $[1, p-1]$ into intervals of the form $[dx + 1, d(x + 1)]$. If Fernando picks a_i, Eduardo chooses $a_j = a_i$, where $j = i \pm \frac{d}{2}$, whichever is still in the same interval as i. Since $10^{\frac{d}{2}} \equiv -1 \pmod{p}$, when the game ends
$$M \equiv 0 + \ldots + 0 \equiv 0 \pmod{p},$$
as desired.

2. If d is odd, then k must be even. Then, pair the elements of $[1, p-1]$ into $\frac{k}{2}$ pairs of numbers d apart. If Fernando picks a_i, Eduardo chooses $a_j = 9 - a_i$, where j is i's pair. Then, $10^i a_i + 10^j a_j \equiv 9 \cdot 10^i$, so when the game ends
$$M = \ell \cdot \underbrace{99\ldots9}_{d \text{ times}} \equiv \ell \cdot (10^d - 1) \equiv 0 \pmod{p},$$
as desired. ■