2007 A2

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Consider those functions $f : \mathbb{N} \mapsto \mathbb{N}$ which satisfy the condition

$$f(m+n) \geq f(m) + f(f(n)) - 1$$

for all $m, n \in \mathbb{N}$. Find all possible values of $f(2007)$.

All possible values for $f(2007)$ are $1, 2, \ldots, 2008$. First, we prove these are the only ones. Note that if $a > b$ then

$$f(a) = f(b + (a-b)) \geq f(b) + f(f(a-b)) - 1 \geq f(b),$$

so f is non-decreasing. Now, we proceed to show $f(n) \leq n+1$ for all n. Clearly, $f \equiv 1$ is a solution and satisfies this. So, assume f is not identically 1 and let α be the smallest integer such that $f(\alpha) > 1$. Suppose $f(n) > n$. Then,

$$f(f(n)) = f(f(n) - n + n) \geq f(f(n) - n) + f(f(n)) - 1,$$

so $1 \geq f(f(n) - n)$. Therefore, $f(n) - n < \alpha$. Thus, we find that $g(n) := f(n) - n$ is bounded from above. Let c be its maximum and k such that $g(k) = c$. Then, $g(2k) \leq c$ so

$$2k + c \geq f(2k) = f(k+k) \geq f(k) + f(f(k)) - 1 \geq 2f(k) - 1 = 2k + 2c - 1,$$

giving $1 \geq c$. Hence, $c = 1$ and $f(n) - n \leq 1$ as desired.
Finally, note that the following function achieves $f(2007) = r$, where $r \in \{1, 2, \ldots, 2008\}$:

$$f(1) = f(2) = \ldots = f(2006) = 1,\ f(2007 + m) = r + m \,\forall m \geq 0.$$

We verify these satisfy the condition. If $m, n < 2007$ or $m, n > 2006$ this is clear. Otherwise, let $a < 2007$ and $b = 2007 + m$. We have

$$f(a + 2007 + m) = r + m + a \geq 2r + m - 2007 = 1 + r + (r + m - 2007) - 1 = f(a) + f(f(2007 + m)) - 1.$$

This concludes the proof. ■