# 2005 G7

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In an acute triangle $ABC$, let $D$, $E$, $F$ be the feet of the perpendiculars from the points $A$, $B$, $C$ to the lines $BC$, $CA$, $AB$, respectively, and let $P$, $Q$, $R$ be the feet of the perpendiculars from the points $A$, $B$, $C$ to the lines $EF$, $FD$, $DE$, respectively.

Prove that $p(ABC)\, p(PQR) \geq (p(DEF))^2$, where $p(T)$ denotes the perimeter of triangle $T$ .

Assume without loss of generality that the circumradius of $\triangle ABC$ is 1. We let $\angle A := \angle BAC, a := BC$ and analogously for $B$ and $C$. First, from the law of sines

$$p(ABC) = a + b + c = 2(\sin\angle A + \sin\angle B + \sin\angle C).$$

Now, using the law of sines on $\triangle BFD$ we see

$$\frac{\sin\angle B}{FD} = \frac{\sin\angle C}{BD} = \frac{\sin\angle C}{c\cos\angle B},$$

so $FD = c\sin\angle B\cos\angle B/\sin\angle C = \sin 2\angle B$. Analogously for the other two sides,

$$p(DEF) = \sin 2\angle A + \sin 2\angle B + \sin 2\angle C.$$

Now, note that $PE = AE\cos\angle B$ and $RE = EC\cos\angle B$. Thus, by the law of cosines on $\triangle PER$,

$$\begin{aligned}
PR^2 &= AE^2\cos^2\angle B + EC^2\cos^2\angle B + 2AE\cdot EC\cos^2\angle B\cos 2\angle B\\
&= \cos^2\angle B(AE^2 + EC^2 + 2AE\cdot EC - 4AE\cdot EC\sin^2\angle B)\\
&= \cos^2\angle B(b^2 - 4b\cos\angle A\sin\angle C\cdot b\sin\angle A\cos\angle C)\\
&= b^2\cos^2\angle B(1 - \sin 2\angle C\sin 2\angle A),
\end{aligned}$$

which, since $\angle B < 90°$, gives $PR = b\cos\angle B\sqrt{1-\sin 2\angle C\sin 2\angle A} = \sin 2\angle B\sqrt{1-\sin 2\angle C\sin 2\angle A}$. Therefore,

$$p(PQR) = \sum \sin 2\angle A\sqrt{1-\sin 2\angle B\sin 2\angle C}.$$

Now, since sin is positive and concave on $(0,\pi)$ we use AM-GM and Jensen to obtain

$$\sin 2\angle B\sin 2\angle C \leq \left(\frac{\sin 2\angle B + \sin 2\angle C}{2}\right)^2 \leq \sin^2\angle A.$$

This implies $\sqrt{1-\sin 2\angle B \sin 2\angle C} \geq \cos \angle A$. Finally, making use of the Cauchy-Schwarz inequality,

$$\begin{aligned}
p(ABC)p(PQR) &= \left(\sum 2\sin\angle A\right)\left(\sum \sin 2\angle A\sqrt{1-\sin 2\angle B\sin 2\angle C}\right)\\
&\geq \left(\sum 2\sin\angle A\right)\left(\sum \cos\angle A \sin 2\angle A\right)\\
&\geq \left(\sqrt{2\sin\angle A\cos\angle A\sin 2\angle A}\right)^2\\
&= \left(\sum \sin 2\angle A\right)^2 = (p(DEF))^2. \blacksquare
\end{aligned}$$