2011 G2

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Let $A_1A_2A_3A_4$ be a non-cyclic quadrilateral. Let O_1 and r_1 be the circumcentre and the circumradius of the triangle $A_2A_3A_4$. Define O_2, O_3, O_4 and r_2, r_3, r_4 in a similar way. Prove that

$$\frac{1}{O_1A_1^2 - r_1^2} + \frac{1}{O_2A_2^2 - r_2^2} + \frac{1}{O_3A_3^2 - r_3^2} + \frac{1}{O_4A_4^2 - r_4^2} = 0.$$

We set $\triangle A_1A_2A_3$ as our reference triangle, with $A_1 = (1, 0, 0)$, $A_2 = (0, 1, 0)$, $A_3 = (0, 0, 1)$ and $a = \overline{\mathrm{A_2A_3}}$, $b = \overline{\mathrm{A_3A_1}}$, $c = \overline{\mathrm{A_1A_2}}$. Let $A_4 = (p, q, r)$. Let $T = a^2qr + b^2rp + c^2pq$. Then, we find circle $A_1A_2A_4$ has equation

$$-a^2yz - b^2zx - c^2xy + (x + y + z)\left(\frac{T}{r} \cdot z\right) = 0.$$

Therefore, $O_3A_3^2 - r_3^2$, which is the power of A_3 with respect to this circle, equals

$$\frac{T}{r}.$$

Analogously,

$$O_2A_2^2 - r_2^2 = \frac{T}{q} \text{ and } O_1A_1^2 - r_1^2 = \frac{T}{p}.$$

Since $O_4A_4^2 - r_4^2 = -T$, we find the desired sum equals

$$\frac{p}{T} + \frac{q}{T} + \frac{r}{T} - \frac{1}{T} = \frac{1}{T} - \frac{1}{T} = 0,$$

as desired. ■