2005 G5

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Let $\triangle ABC$ be an acute-angled triangle with $AB \neq AC$. Let H be the orthocenter of triangle ABC, and let M be the midpoint of the side BC. Let D be a point on the side AB and E a point on the side AC such that $AE = AD$ and the points D, H, E are on the same line. Prove that the line HM is perpendicular to the common chord of the circumscribed circles of triangle $\triangle ABC$ and triangle $\triangle ADE$.

Let $K = (ABC) \cap (ADE)$. It suffices to show K lies on (AH). Note that K is the center of a spiral similarity taking $\overline{\text{BD}}$ to $\overline{\text{CE}}$. Let X, Y be the feet of the height from B and C to $\overline{\text{CA}}$ and $\overline{\text{AB}}$, respectively. Since we want K to lie on (AXY), it suffices to show the aforementioned spiral similarity maps Y to X. Note that

$$\angle YHD = 90^\circ - \angle HDA = \frac{1}{2}\angle A = \frac{1}{2}\angle YHB.$$

Thus, $\overline{\text{YH}}$ bisects $\angle YHB$. Analogously, $\overline{\text{XH}}$ bisects $\angle CHX$. Therefore, by the angle bisector theorem,

$$\frac{YD}{DB} = \frac{HY}{HB} = \sin(90^\circ - \angle A) = \frac{HX}{HC} = \frac{XE}{EC},$$

so Y maps to X as desired. ■