2003 G4

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Let Γ_1, Γ_2, Γ_3, Γ_4 be distinct circles such that Γ_1, Γ_3 are externally tangent at P, and Γ_2, Γ_4 are externally tangent at the same point P. Suppose that Γ_1 and Γ_2; Γ_2 and Γ_3; Γ_3 and Γ_4; Γ_4 and Γ_1 meet at A, B, C, D, respectively, and that all these points are different from P. Prove that

$$\frac{AB \cdot BC}{AD \cdot DC} = \frac{PB^2}{PD^2}.$$

Let $\gamma_1, \gamma_2, \gamma_3, \gamma_4, A', B', C', D'$ be the images of $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, A, B, C, D$ under an inversion with center P and radius 1. From the given conditions, we see the quadrilateral formed by lines $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ is a parallelogram. Furthermore, this quadrilateral is $A'B'C'D'$. Therefore, $A'B' = D'C'$. From the inversion distance formula, this gives

$$\frac{AB}{PA \cdot PB} = \frac{DC}{PD \cdot PC},$$

or $\frac{AB}{DC} = \frac{PA}{PC} \cdot \frac{PB}{PD}$. Analogously, we obtain

$$\frac{BC}{AD} = \frac{PC}{PA} \cdot \frac{PB}{PD}.$$

Multiplying these two equations,

$$\frac{AB \cdot BC}{AD \cdot DC} = \frac{PA}{PC} \cdot \frac{PB}{PD} \cdot \frac{PC}{PA} \cdot \frac{PB}{PD} = \frac{PB^2}{PD^2},$$

as desired. ■