2010 C2

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Let $n \geq 4$ be an integer. A *flag* is a binary string of length n. We say that a set of n flags is *diverse* if these flags can be the rows of an $n \times n$ binary matrix with the entries in its main diagonal all equal. Determine the smallest positive integer M such that among any M distinct flags, there exist n flags forming a diverse set.

We claim the answer is $M = 2^{n-2} + 1$. First, we show that if $M > 2^{n-2}$. For this, consider the 2^{n-2} flags for which the first digit is 0 and the second digit is 1. Since $n \geq 4, M \geq n$. Now, we will prove that if a set of M flags does not have a diverse subset, then $M \leq 2^{n-2}$, which will finish the proof. Indeed, by Hall's lemma, since we cannot place all 0's in the main diagonal, there must be a set of $a < n$ digits for which all but at most $a - 1$ strings have all 1's in these positions. Call these strings *big*. Similarly, there must be a set of $b < n$ digits for which all but at most $b - 1$ strings have all 0's in these positions. Call such strings *small*. If the two sets of digits had overlap, then no string can be both big and small, as that would imply having a digit that is both 0 and 1. Thus, the number of strings M satisfies

$$M \leq (a-1) + (b-1) \leq 2n - 4 \leq 2^{n-2}.$$

Otherwise, if both sets of digits don't overlap, there are at most $2^{n-(a+b)}$ strings that are both big and small (fix the $a + b$ positions we know are 0 or 1). Thus,

$$M \leq (a-1) + (b-1) + 2^{n-(a+b)} = (a+b-2) + 2^{n-2-(a+b-2)} \leq 2^{n-2}.$$

In any case, we obtain the desired inequality. ■