2013 G6

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November 18, 2021

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Let the excircle of triangle ABC opposite the vertex A be tangent to the side BC at the point A_1. Define the points B_1 on CA and C_1 on AB analogously, using the excircles opposite B and C, respectively. Suppose that the circumcenter of triangle $A_1B_1C_1$ lies on the circumcircle of triangle ABC. Prove that triangle ABC is right-angled.

Let M_A, M_B, M_C be the midpoints of arcs $\widehat{\mathrm{CAB}}, \widehat{\mathrm{ABC}}, \widehat{\mathrm{BCA}}$ and I_A, I_B, I_C be the $A, B, C-$excenters of $\triangle ABC$ respectively. It is well known that $\triangle ABC$ and $\triangle M_AM_BM_C$ are the orthic and medial triangles of $\triangle I_AI_BI_C$ respectively. Now, since $\overline{\mathrm{BC}}$ and $\overline{\mathrm{I_BI_C}}$ are anti-parallel and $\overline{\mathrm{I_AA_1}} \perp \overline{\mathrm{BC}}$, we know the circumcenter O of $\triangle I_AI_BI_C$ lies on I_AA_1. Analogously, O lies on I_BB_1. Thus, A_1B_1C lie on the circle with diameter $\overline{\mathrm{OC}}$. Since $\angle OM_CC = 90°$, M_C also lies on this circle, so we have $A_1B_1CM_C$ is cyclic. Analogously, $B_1C_1AM_A$ and $C_1A_1BM_B$ are cyclic.
These cyclic quads imply there is a spiral similarity with center M_A taking $\overline{\mathrm{C_1B_1}}$ to $\overline{\mathrm{BC}}$ (and analogous). Since M_A is on the perpendicular bisector of $\overline{\mathrm{BC}}$, it must be on the perpendicular bisector of $\overline{\mathrm{C_1B_1}}$ as well (and analogous). Now, if the circumcenter of $\triangle A_1B_1C_1$ lies on the circumcircle of $\triangle ABC$, $\triangle A_1B_1C_1$ must be obtuse. WLOG assume $\angle B_1A_1C_1 > 90°$, then it is clear that the circumcenter must be M_A since it lies on the circle, on the perpendicular bisector and is on the right side of $\overline{\mathrm{B_1C_1}}$. Now,

$$\angle BAC = \angle B_1M_AC_1 = \angle B_1MA_1 + \angle A_1MC_1 = 2\angle M_CM_AA_1 + 2\angle A_1M_AM_B = 2\angle M_CM_AM_B = 180° - \angle BAC,$$

giving $\angle BAC = 90°$. ■