2015 G4

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Let ABC be an acute triangle and let M be the midpoint of AC. A circle ω passing through B and M meets the sides AB and BC at points P and Q respectively. Let T be the point such that $BPTQ$ is a parallelogram. Suppose that T lies on the circumcircle of ABC. Determine all possible values of $\frac{BT}{BM}$.

We claim the only possibility is $\sqrt{2}$. We proceed with barycentric coordinates, with reference triangle $\triangle ABC$. Let $T = (u, v, w)$. It follows from parallel lines that $P = (u, 1-u, 0)$ and $Q = (0, 1-w, w)$. Then, we compute the equation of circle (BPQ) is

$$(BPQ) \to -a^2yz - b^2zx - c^2xy + (x+y+z)(c^2(1-u)x + a^2(1-w)z) = 0.$$

Since $M = (1:0:1)$ lies on this circle, it follows

$$c^2 + a^2 - \frac{b^2}{2} = c^2u + a^2w.$$

Since T lies on (ABC) we also have

$$-a^2vw - b^2wu - c^2uv = 0.$$

Finally, recall that $BM^2 = \frac{c^2+a^2}{2} - \frac{b^2}{4}$. Then, from the barycentric distance formula

$$BT^2 = -a^2(v-1)(w) - b^2wu - c^2(u)(v-1) = -a^2vw - b^2wu - c^2uv + a^2w + c^2u = 0 + c^2 + a^2 - \frac{b^2}{2} = 2BM^2.$$

The conclusion follows. ■