# 2016 G4

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Let $ABC$ be a triangle with $AB = AC \neq BC$ and let $I$ be its incenter. The line $BI$ meets $AC$ at $D$, and the line through $D$ perpendicular to $AC$ meets $AI$ at $E$. Prove that the reflection of $I$ in $AC$ lies on the circumcircle of triangle $BDE$.

We use barycentric coordinates with reference triangle $ABC$. Furthermore, let $BC = a, CA = AB = b$. Let $I'$ be the reflection of $I$ in $\overline{\mathrm{AC}}$ and $N$ the midpoint of $\overline{\mathrm{II'}}$. Since $D = \left(\frac{a}{a+b}, 0, \frac{b}{a+b}\right)$ and $\overline{\mathrm{DE}} \perp \overline{\mathrm{CA}}$, setting $E = (1-2t, t, t)$ we find

$$t = \frac{2b^3}{(a+b)(4b^2-a^2)},$$

so

$$E = (4ab^2 - a^3 - a^2b : 2b^3 : 2b^3).$$

It is clear that $N = \left(\frac{a}{2b}, 0, 1-\frac{a}{2b}\right)$. Then,

$$I' = 2N - I = (a(a+b) : -b^2 : 3b^2 - a^2).$$

Now, let the equation of $(BDI')$ be

$$-a^2yz - b^2zx - c^2xy + (x+y+z)(x\alpha + y\beta + z\gamma) = 0.$$

Since $B$ lies on the circle, $\beta = 0$. From $D$ on the circle, we find

$$a(a+b)\alpha + b(a+b)\gamma = ab^2(b).$$

From $I'$ on the circle and a bunch of simplification, we have

$$a(a+b)\alpha + (3b^2 - a^2)\gamma = ab^2(b-a).$$

Thus, we find

$$\alpha = \frac{2b^5}{(a+b)(a+2b)(b-a)}, \gamma = -\frac{a^2b^2}{(a+2b)(b-a)}.$$

Plugging in $E$ we see

$$\begin{aligned}
&-4a^2b^6 - 4b^5(4ab^2 - a^3 - a^2b) + \frac{2b^5(4ab^3 - a^3 - a^2b)(2b-a)}{b-a} - \frac{2a^2b^5(2b-a)(a+b)}{b-a} \\
&= 2b^5\left(2a^3 - 8ab^2 + \frac{8ab^3 - 2a^3b - 8a^2b^2 + 2a^4}{b-a}\right) \\
&= 2b^5(2a^3 - 8ab^2 - 2a^3 + 8ab^2) = 0,
\end{aligned}$$

so $E$ lies on the circle as desired. ■