2019 G3

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In triangle ABC point A_1 lies on side BC and point B_1 lies on side AC. Let P and Q be points on segments AA_1 and BB_1, respectively, such that $\overline{\mathrm{PQ}} \parallel \overline{\mathrm{AB}}$. Point P_1 is chosen on ray PB_1 beyond B_1 such that $\angle PP_1C = \angle BAC$. Point Q_1 is chosen on ray QA_1 beyond A_1 such that $\angle CQ_1Q = \angle CBA$. Prove that points P_1, Q_1, P, Q are cyclic.

We set $\triangle ABC$ as our reference triangle, with $A = (1,0,0), B = (0,1,0), C = (0,0,1)$ and $a = \overline{\mathrm{BC}}, b = \overline{\mathrm{CA}}, c = \overline{\mathrm{AB}}$. Since $\overline{\mathrm{PQ}} \parallel \overline{\mathrm{AB}}$, we have $P = (u_1 : v_1 : 1)Q = (u_2 : v_2 : 1)$ where $u_1 + v_1 = u_2 + v_2$. Now, $A_1 = (0 : v_1 : 1)$ and $B_1 = (u_2 : 0 : 1)$. Let X and Y be the intersection points of $\overline{\mathrm{PB_1}}$ and $\overline{\mathrm{QA_1}}$ with side $\overline{\mathrm{AB}}$ respectively. By the angle condition, $AXCP_1$ and $BYCQ_1$ are cyclic. Also, by Reims' it suffices to show P_1Q_1XY is cyclic. By radical center, it then suffices to show $\overline{\mathrm{A_1Y}}, \overline{\mathrm{B_1X}}$ and the radical axis of $(ACX), (BCY)$ concur. Now, using that P, B_1, X are collinear, if $X = (x_1 : y_1 : 0)$,

$$\begin{vmatrix} u_1 & v_1 & 1 \\ u_2 & 0 & 1 \\ x_1 & y_1 & 0 \end{vmatrix} = 0,$$

which implies $x_1 : y_1 = u_1 - u_2 : v_1$. Thus, $X = (u_1 - u_2 : v_1 : 0)$. Analogously, $Y = (u_2 : v_2 - v_1 : 0)$. From here, via determinants we obtain

$$\begin{aligned} \overline{\mathrm{A_1Y}}&\colon\ +x(v_2 - v_1) - y(u_2) + z(v_1u_2) = 0 \\ \overline{\mathrm{B_1X}}&\colon\ -x(v_1) + y(u_1 - u_2) + z(v_1u_2) = 0 \end{aligned}$$

Now we compute the equations of (ACX) and (BCY). Substituting the points A, C, X into the formula, we see that

$$-c^2(u_1 - u_2)v_1 + (v_2)\beta v_1 = 0,$$

so $\beta = \frac{c^2(u_1-u_2)}{v_2}$. Thus,

$$(ACX)\colon\ -a^2yz - b^2zx - c^2yz + (x+y+z)\frac{c^2(u_1-u_2)}{v_2}y = 0.$$

Analogously,

$$(ACX)\colon\ -a^2yz - b^2zx - c^2yz + (x+y+z)\frac{c^2(v_2-v_1)}{u_1}x = 0.$$

Subtracting these two and using $u_1 - u_2 = v_2 - v_1$, we see the radical axis of both circles is given by $v_2x - u_1y = 0$. Now,

$$\begin{vmatrix} v_2 & -u_1 & 0 \\ v_2 - v_1 & -u_2 & v_1u_2 \\ -v_1 & u_1 - u_2 & v_1u_2 \end{vmatrix} = \begin{vmatrix} v_2 & -u_1 & 0 \\ v_2 & -u_1 & 0 \\ -v_1 & u_1 - u_2 & v_1u_2 \end{vmatrix} = 0,$$

so the three lines concur as desired. ■