2002 G4

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Circles S_1 and S_2 intersect at points P and Q. Distinct points A_1 and B_1 (not at P or Q) are selected on S_1. The lines A_1P and B_1P meet S_2 again at A_2 and B_2 respectively, and the lines A_1B_1 and A_2B_2 meet at C. Prove that, as A_1 and B_1 vary, the circumcentres of triangles A_1A_2C all lie on one fixed circle.

We use directed angles mod 180°. Note that Q is the Miquel point of quadrilateral $A_1B_1B_2A_2$. Therefore, CA_1QA_2 is cyclic. Hence, the circumcenter of $\triangle A_1A_2C$ is the circumcenter of $\triangle A_1A_2Q$. Let O be the circumcenter of this triangle and O_1, O_2 be the centers of S_1, S_2. We will show O lies on (O_1QO_2), which finishes the problem. Let M and N be the midpoints of $\overline{\mathrm{QA}_1}, \overline{\mathrm{QA}_2}$ respectively. Note that

$$\measuredangle QO_1A_1 = 2\measuredangle QPA_1 = 2\measuredangle QPA_2 = \measuredangle QO_2A_2.$$

This implies $\measuredangle A_1QO_1 = \measuredangle A_2QO_2$. Therefore,

$$\measuredangle A_2QA_1 = \measuredangle A_2QO_2 + \measuredangle O_2QO_1 + \measuredangle O_1QA_1 = \measuredangle O_2QO_1.$$

Now, since $QMON$ is cyclic we have $\measuredangle A_2QA_1 = \measuredangle NQM = \measuredangle NOM = \measuredangle O_2OO_1$. Thus, since $\measuredangle O_2OO_1 = \measuredangle O_2QO_1$, we have $O \in (O_1QO_2)$ as desired. ■