2016 G5

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Let D be the foot of perpendicular from A to the Euler line (the line passing through the circumcentre and the orthocentre) of an acute scalene triangle ABC. A circle ω with centre S passes through A and D, and it intersects sides AB and AC at X and Y respectively. Let P be the foot of altitude from A to BC, and let M be the midpoint of BC. Prove that the circumcenter of triangle XSY is equidistant from P and M.

Let L, N, Q, R be the midpoint of $\overline{\mathrm{AC}}, \overline{\mathrm{AB}}$ and the feet of the heights from B and C respectively. Let K be the center of $(ADRHO)$ and T be the center of $(ADNOL)$. Note that

$$\triangle RKQ \overset{+}{\sim} \triangle NTL \overset{+}{\sim} \triangle XYS,$$

as they are all isosceles and have vertex angle $2\angle A$. Furthermore, we note $R - N - X$, $Q - L - Y$ and since they all lie on the perpendicular bisector of $\overline{\mathrm{AD}}$, $K - T - S$. Thus, if we let O_1, O_2, O_3 be the centers of $\triangle RKQ$, $\triangle NTL$ and $\triangle XSY$, by the gliding principle, since

$$RKQO_1 \overset{+}{\sim} NTLO_2 \overset{+}{\sim} XYSO_3$$

it follows that $O_1 - O_2 - O_3$. However, note that O_1 is the nine-point center and O_2 lies on the perpendicular bisector of $\overline{\mathrm{NL}}$. Since $NLMP$ is an isosceles trapezoid, it follows both O_1 and O_2 lie on the perpendicular bisector of $\overline{\mathrm{PM}}$, so O_3 does as well as required. ■