2006 G3

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Let $ABCDE$ be a convex pentagon such that

$$\angle BAC = \angle CAD = \angle DAE \qquad \text{and} \qquad \angle ABC = \angle ACD = \angle ADE.$$

The diagonals BD and CE meet at P. Prove that the line AP bisects the side CD.

Note that there is a spiral similarity with center A taking $B \to C \to D \to E$. Thus, the spiral similarity takes $\overline{\text{BD}}$ to $\overline{\text{CE}}$. Therefore, since $P = \overline{\text{BD}} \cap \overline{\text{CE}}$, we have $AEDP$ and $ABCP$ cyclic. Thus, since $\angle CBP = \angle DCP$ from the spiral similarity, we have

$$\angle DCP = \angle CBP = \angle CAP,$$

giving $(ABCP)$ tangent to $\overline{\text{CD}}$. Analogously, $(AEDP)$ is tangent to $\overline{\text{CD}}$. Thus, since $\overline{\text{AP}}$ is the radical axis of these two circles, it bisects their common external tangent $\overline{\text{CD}}$. ■