2012 G6

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January 16, 2022

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Let ABC be a triangle with circumcenter O and incenter I. The points D, E and F on the sides BC, CA and AB respectively are such that $BD + BF = CA$ and $CD + CE = AB$. The circumcircles of the triangles BFD and CDE intersect at $P \neq D$. Prove that $OP = OI$.

We use directed angles mod 180°. Let V be the reflection of I over O and T_A, T_B, T_C be the extouch points. Furthermore, let $X = (BT_CT_A) \cap (BFD)$ and $Y = (CT_BT_A) \cap (CED)$. First, note that by the incenter-excenter lemma and homothety, V is the circumcenter of the ex-triangle, also known as the *Bevan point*. It follows by this homothety that $\overline{\mathrm{VT_A}} \perp \overline{\mathrm{BC}}$ and analogously for the others. Thus, V is the Miquel point of $\triangle T_AT_BT_C$ with respect to $\triangle ABC$. Now, note that

$$\measuredangle VYP = \measuredangle VYC + \measuredangle CYP = 90^\circ + \measuredangle CDP = 90^\circ + \measuredangle BDP = \measuredangle VXB + \measuredangle BXP = \measuredangle VXP,$$

so $XVYP$ is cyclic. By spiral similarity construction, we see X is the center of a spiral similarity taking $\overline{\mathrm{DT_A}}$ to $\overline{\mathrm{FT_C}}$. However, since $BT_A + BT_C = AC = BD + DF$, it follows

$$DT_A = FT_C.$$

Hence, $\triangle XDT_A \cong \triangle XFT_C$. Thus, $XD = XF$, and X is the midpoint of arc $\widehat{DF}$. Hence, $B - I - X$. Since $\overline{\mathrm{BX}} \perp \overline{\mathrm{XV}}$, it follows X lies on the circle with diameter $\overline{\mathrm{IV}}$. Analogously, Y lies on this circle. Therefore, P lies on this circle and because O is its center, $OP = OI$. ■