2001 G4

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Let M be a point in the interior of triangle ABC. Let A' lie on BC with MA' perpendicular to BC. Define B' on CA and C' on AB similarly. Define

$$p(M) = \frac{MA' \cdot MB' \cdot MC'}{MA \cdot MB \cdot MC}.$$

Determine, with proof, the location of M such that $p(M)$ is maximal. Let $\mu(ABC)$ denote this maximum value. For which triangles ABC is the value of $\mu(ABC)$ maximal?

We claim $p(M)$ is maximal when M is the incenter of $\triangle ABC$ and that $\mu(ABC)$ is maximal when $\triangle ABC$ is equilateral. Note that

$$p(M) = \sqrt{\sin\angle BAM \sin\angle MAC} \cdot \sqrt{\sin\angle ACM \sin\angle MCB} \cdot \sqrt{\sin\angle CMB \sin\angle MBA}.$$

Now, recall that on $(0, \pi)$, sin is both positive and concave. Therefore, by AM-GM and Jensen,

$$p(M) \leq \sin\left(\frac{\angle A}{2}\right) \sin\left(\frac{\angle B}{2}\right) \sin\left(\frac{\angle C}{2}\right) = \mu(ABC).$$

Equality is attained only when $\overline{\text{AM}}, \overline{\text{BM}}, \overline{\text{CM}}$ bisect the angles of the triangle, ie. when M is the incenter. Finally, using AM-GM and Jensen again,

$$\mu(ABC) \leq \left(\sin\frac{\pi}{6}\right)^3 = \frac{1}{8}.$$

Equality is attained only when $\angle A = \angle B = \angle C$, ie. when $\triangle ABC$ is equilateral. ■