2002 G3

Ezra Guerrero Alvarez

January 21, 2022

2002 G3

2002 G3

The circle S has center O, and BC is a diameter of S. Let A be a point of S such that $\angle AOB < 120°$. Let D be the midpoint of the arc AB which does not contain C. The line through O parallel to DA meets the line AC at I. The perpendicular bisector of OA meets S at E and at F. Prove that I is the incenter of the triangle CEF.

Let I' be the incenter of $\triangle CEF$. We will show that $\overline{\mathrm{OI'}} \parallel \overline{\mathrm{AD}}$ and $I' \in \overline{\mathrm{AC}}$, which implies the result. First, note that A and D are on the same arc $\widehat{EF}$ from the $\angle AOB < 120°$ condition. Then, as $\triangle EAO$ and $\triangle FAO$ are equilateral,

$$\angle EDF = \frac{1}{2}(360° - 120°) = 120°.$$

Also, $\angle EI'F = 90° + \frac{1}{2}\angle C = 90° + 30° = 120°$. Now, note that

$$\angle I'ED = \angle I'EF + \angle FED = \frac{1}{4}\angle FOC + 30° + \frac{1}{4}\angle AOB = 30° - \frac{1}{4}\angle AOB + 30° + \frac{1}{4}\angle AOB = 60°.$$

This implies that $I'EDF$ is a parallelogram. Hence, the midpoint of $\overline{\mathrm{DI'}}$ coincides with that of $\overline{\mathrm{EF}}$, which coincides with that of $\overline{\mathrm{AO}}$, implying that $DAI'O$ is a parallelogram. This gives $\overline{\mathrm{OI'}} \parallel \overline{\mathrm{AD}}$. Finally, since A is the midpoint of $\widehat{EF}$, I' lies on $\overline{\mathrm{AC}}$, so $I' = I$ as desired. ■