2003 G3

Ezra Guerrero Alvarez

January 21, 2022

2003 G3

2003 G3

Let ABC be a triangle and let P be a point in its interior. Denote by D, E, F the feet of the perpendiculars from P to the lines BC, CA, AB, respectively. Suppose that

$$AP^2 + PD^2 = BP^2 + PE^2 = CP^2 + PF^2.$$

Denote by I_A, I_B, I_C the excenters of the triangle ABC. Prove that P is the circumcenter of the triangle $I_AI_BI_C$.

We have

$$\begin{aligned} AP^2 + PD^2 &= BP^2 + PE^2 \\ AP^2 - PE^2 &= BP^2 - PD^2 \\ AE^2 &= BD^2 \\ AE &= BD. \end{aligned}$$

Analogously, $BF = CE$ and $CD = AF$. It follows that D, E, F are the extouch points. Hence, P is the concurrency point of the perpendiculars to the sides through the extouch points. Now, consider a homothety with center I and ratio 2. This sends O to V, the *Bevan Point*. Since it also sends the midpoints of arcs $\widehat{BC}, \widehat{CA}, \widehat{AB}$ to the excenters, V is the circumcenter of $\triangle I_AI_BI_C$. By the homothety, $\overline{\mathrm{I_AV}} \perp \overline{\mathrm{BC}}$ and analogously with the other sides, so V is the concurrency point of the perpendiculars to the sides through the extouch points. Hence, $P = V$ as desired. $\blacksquare$