2001 C1

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June 7, 2022

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Let $A = (a_1, a_2, \ldots, a_{2001})$ be a sequence of positive integers. Let m be the number of 3-element subsequences (a_i, a_j, a_k) with $1 \leq i < j < k \leq 2001$, such that $a_j = a_i + 1$ and $a_k = a_j + 1$. Considering all such sequences A, find the greatest value of m.

We claim the answer is 667^3. Indeed, we prove that replacing 2001 with $3x$, the answer is x^3.

For both solutions, we present the construction:

$$A = (\underbrace{1, 1, \ldots, 1}_{x\,1's}, \underbrace{2, 2, \ldots, 2}_{x\,2's}, \underbrace{3, 3, \ldots, 3}_{x\,3's})$$

Solution 1 (Smart). Let a, b, c denote how many elements of A are $0, 1, 2 \bmod 3$ respectively. Note that $a + b + c = 3x$. A subsequence counted by m must contain exactly 1 of each residue class, so there are at most abc such subsequences. By AM-GM, $abc \leq \left(\frac{a+b+c}{3}\right)^3 = x^3$ as desired. ■

Solution 2 (Dumb). Suppose we have A such that m is maximal. Note that A must be non-decreasing, since if $a_i > a_{i+1}$, then switching these two increases m. Furthermore, if $a_i < a_{i+1}$, then $a_{i+1} - a_i = 1$, as otherwise we may increase m by subtracting 1 to $a_{i+1}, \ldots, a_{3x}$. Hence, A is of the form

$$(\underbrace{n, \ldots, n}_{s_0 \text{ times}}, \underbrace{n+1, \ldots, n+1}_{s_1 \text{ times}}, \ldots, \underbrace{n+r-1, \ldots, n+r-1}_{s_{r-1} \text{ times}})$$

and $m = s_0s_1s_2 + \ldots + s_{r-3}s_{r-2}s_{r-1}$. (Note that $r \geq 3$, since otherwise $m = 0$). Hence, we want to prove that if $s_0 + s_1 + \ldots + s_{r-1} = 3x$, then $s_0s_1s_2 + \ldots + s_{r-3}s_{r-2}s_{r-1} \leq x^3$. we proceed by induction on r. If $r = 3$ this is immediate by AM-GM as in solution 1. If $r = 4$, then

$$s_0s_1s_2 + s_1s_2s_3 = s_1s_2(s_0 + s_3) \leq x^3$$

where the inequality comes from $r = 3$ with $s_1, s_2, s_0 + s_3$. If $r = 5$, then

$$s_0s_1s_2 + s_1s_2s_3 + s_2s_3s_4 = s_1s_2(s_0 + s_3) + s_2(s_0 + s_3)s_4 - s_2s_0s_4 < s_2s_2(s_0 + s_3) + s_2(s_0 + s_3)s_4 \leq x^3$$

where the last inequality comes from $r = 4$ with $s_1, s_2, s_0 + s_3, s_4$. As our induction hypothesis, suppose the inequality holds for $r = k \geq 5$. Then,

$$s_0s_1s_2 + \ldots + s_{k-2}s_{k-1}s_k = s_1s_2(s_0 + s_3) + s_2(s_0 + s_3)s_4 + (s_0 + s_3)s_4s_5 + \ldots + s_{k-2}s_{k-1}s_k - s_2s_0s_4 - s_0s_4s_5$$

$$< s_1s_2(s_0 + s_3) + s_2(s_0 + s_3)s_4 + (s_0 + s_3)s_4 + s_5 + \ldots + s_{k-2}s_{k-1}s_k \leq x^3,$$

where the last inequality comes from $r = k$ with $s_1, s_2, s_0 + s_3, s_4, \ldots, s_k$. This concludes the induction. ■