2007 A4

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Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying $f\left(x + f\left(y\right)\right) = f\left(x+y\right) + f\left(y\right)$ for all pairs of positive reals x and y. Here, $\mathbb{R}^+$ denotes the set of all positive reals.

We claim the only solution is $f \equiv 2x$, which we easily check works. Now, we show it is the only one. Note that if $y > f(y)$ for some y, then we may substitute $x = y - f(y)$ to obtain

$$f(y) = f(2y - f(y)) + f(y),$$

but since $f(2y - f(y)) > 0$, this is a contradiction. Hence, $y \leq f(y)$. Furthermore, if $y = f(y)$ then $f(x+y) = f(x+y) + f(y)$, contradicting again so $y < f(y)$. Thus, define $g \colon \mathbb{R}^+ \to \mathbb{R}^+$ as

$$g(x) := f(x) - x$$

The given equation becomes

$$g(x + y + g(y)) = g(x+y) + y$$

Now, for $a > b$ positive reals, plugging in $x = \frac{a-b}{2}$ and $y = \frac{a+b}{2} + g(b)$ we get

$$\begin{aligned}
g\left(a + g(b) + g\left(\frac{a+b}{2} + g(b)\right)\right) &= g(a + g(b)) + \frac{a+b}{2} + g(b) \\
g\left(a + b + g(b) + g\left(\frac{a+b}{2}\right)\right) &= g(a) + b + \frac{a+b}{2} + g(b) \\
g\left(a + b + g\left(\frac{a+b}{2}\right)\right) + b &= g(a) + b + \frac{a+b}{2} + g(b) \\
g(a+b) + \frac{a+b}{2} + b &= g(a) + b + \frac{a+b}{2} + g(b) \\
g(a+b) &= g(a) + g(b).
\end{aligned}$$

Thus, g is Cauchy. Since g is also bounded from below, it follows $g(x) = kx$ for some $k > 0$, which upon substituting in the functional equation yields $k = 1$. This implies f must be $2x$ as desired. ■