2005 A2

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We denote by $\mathbb{R}^+$ the set of all positive real numbers.

Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ which have the property:

$$f(x)f(y) = 2f(x + yf(x))$$

for all positive real numbers x and y.

We claim the only solution is $f \equiv 2$, which clearly works. Now, we will prove by induction that $f(x) \geq 2^{(2^k-1)/2^k}$ for all non-negative integers k. Our base case is $k = 0$, or $f(x) \geq 0$. Indeed, suppose there exists $t \in \mathbb{R}^+$ such that $f(t) < 1$. Then, substituting $(x, y) = \left(t, \frac{t}{1-f(t)}\right)$ gives

$$f(t)f\left(\frac{t}{1-f(t)}\right) = 2f\left(\frac{t}{1-f(t)}\right),$$

so $1 > f(t) = 2$. This is a contradiction, so $f(x) \geq 1$ for all x. For our inductive step, suppose there exists t for which $f(t) < 2^{(2^{k+1}-1)/2^{k+1}}$. Then, setting $(x, y) = (t, t)$ we get

$$2^{(2^{k+1}-1)/2^k} > f(t)^2 = 2f(t + tf(t)),$$

so $f(t + tf(t)) < 2^{(2^k-1)/2^k}$, contradicting the inductive hypothesis. This concludes the induction. Taking $k \to \infty$, we see $f(x) \geq 2$ for all x. Now,

$$2f(x) \leq f(x)f\left(\frac{r}{f(x))}\right) = 2f(x + r),$$

so it follows f is non-decreasing. But then,

$$f(x)f(y) \geq 2f(2y).$$

Thus, if there exists a u such that $f(u) = 2$, the above inequality gives

$$2f(y) \geq 2f(2y) \geq 2f(y),$$

so $f(y) = f(2y)$ for all y. This along the non-decreasing condition suffices to show f is constant, so $f \equiv 2$. Otherwise, $f(x) > 2$ for all x. Hence, we can improve to f being strictly increasing, and thus injective. Now, substituting $(x, y) = \left(a, \frac{a}{f(a)}\right)$ and $(x, y) = \left(\frac{a}{f(a)}, a\right)$ and using injectivity we find

$$2a = \frac{a}{f(a)} + af\left(\frac{a}{f(a)}\right) > 2a,$$

which is a contradiction. Hence, the only solution is $f \equiv 2$. ■