2001 A3

Ezra Guerrero Alvarez

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Let $x_1, x_2, \ldots, x_n$ be arbitrary real numbers. Prove the inequality

$$\frac{x_1}{1+x_1^2} + \frac{x_2}{1+x_1^2+x_2^2} + \cdots + \frac{x_n}{1+x_1^2+\cdots+x_n^2} < \sqrt{n}.$$

We proceed by induction on n. Our base case $n = 1$ is dealt with AM-GM: ($x_1 = 0$ is clear)

$$\frac{x_1}{1+x_1^2} \le \frac{x_1}{2\,|x_1|} \le \frac{1}{2} < \sqrt{1}.$$

Now, for our inductive step note that we may assume the x_i are positive, since changing them to positive increases the LHS and if one of them is 0 we are done by the inductive hypothesis. Now, if

$$\frac{x_n}{1+x_1^2+\ldots+x_n^2} \le \sqrt{n} - \sqrt{n-1},$$

we finish by the inductive hypothesis. Thus, assume the opposite inequality holds. Then,

$$(\sqrt{n} - \sqrt{n-1})x_n^2 - x_n + (\sqrt{n} - \sqrt{n-1})(1 + \ldots + x_{n-1}^2) < 0.$$

This can happen only if the discriminant of the quadratic in x_n is positive, so this implies

$$4(2n - 1 - 2\sqrt{n^2 - n})(1 + \ldots + x_{n-1}^2) < 1.$$

But then, using the inductive hypothesis

$$\sqrt{n-1} > \sum \frac{x_i}{1 + \ldots + x_i^2} > \sum \frac{x_i}{1 + \ldots + x_{n-1}^2} > 4(2n - 1 - 2\sqrt{n^2 - n}) \sum x_i.$$

[FAKESOLVE ALERT :( ]