2001 G3

Ezra Guerrero Alvarez

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Let ABC be a triangle with centroid G. Determine, with proof, the position of the point P in the plane of ABC such that $AP{\cdot}AG + BP{\cdot}BG + CP{\cdot}CG$ is a minimum, and express this minimum value in terms of the side lengths of ABC.

We claim the position of the point P such that the expression is minimal si when $P = G$. At this value, it is easy to see that

$$AP{\cdot}AG + BP{\cdot}BG + CP{\cdot}CG = AG^2 + BG^2 + CG^2 = \frac{1}{3}(a^2 + b^2 + c^2).$$

Note that we have

$$\begin{aligned} AP \cdot AG &\geq AP \cdot AG \cdot \cos\angle PAG \\ &= \vec{AP} \bullet \vec{AG} \\ &= (\vec{P} - \vec{A}) \bullet (\vec{G} - \vec{A}) \\ &= \vec{P} \bullet (\vec{G} - \vec{A}) + R^2 - \vec{A} \bullet \vec{G}, \end{aligned}$$

where R is the circumradius of $\triangle ABC$ and equality when P is on ray $\overrightarrow{\mathrm{AG}}$. Observe that

$$\vec{A} \bullet \vec{G} = \frac{1}{3}(R^2 + \vec{A} \bullet \vec{B} + \vec{A} \bullet \vec{C}) = R^2 - \frac{b^2 + c^2}{6}.$$

Hence,

$$AP \cdot AG \geq \vec{P} \bullet (\vec{G} - \vec{A}) + \frac{b^2 + c^2}{6}.$$

Adding the similar inequalities for B and C,

$$AP \cdot AG + BP \cdot BG + CP \cdot CG \geq \frac{1}{3}(a^2 + b^2 + c^2),$$

with equality when P lies on $\overrightarrow{\mathrm{AG}}, \overrightarrow{\mathrm{BG}}, \overrightarrow{\mathrm{CG}}$, that is, when $P = G$. $\blacksquare$