IMO 1987/4

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Prove that there is no function $f\colon \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ satisfying $f(f(n)) = n + 1987$ for every $n \in \mathbb{Z}_{\geq 0}$.

Indeed, we prove the statement replacing 1987 by any odd positive integer M. First, note that if $f(a) = f(b)$, then $a + M = f(f(a)) = f(f(b)) = b + M$, so $a = b$. Hence, f is injective. Furthermore, evidently every integer at least M has a pre-image (take $f(a - M)$). Now, suppose $b < M$ has a pre-image. Then, there exists c such that $f(c) = b$. Now, if c had a pre-image, then there exists d such that $f(d) = c$. Hence, $d + M = f(f(d)) = f(c) = b$. But $d + M \geq M$ and $b < M$ which is impossible. Hence, c does not have a pre-image. It also follows $c < M$. Thus, at most $\frac{M-1}{2}$ integers less than M have a pre-image, implying at least $\frac{M+1}{2}$ do not have one.
Now, consider the directed graph with vertices corresponding to $\mathbb{Z}_{\geq 0}$ and edges pointing from a to $f(a)$. Suppose it has a cycle of length k. Then, going around the cycle twice, $a = f^{(2k)} = a + k \cdot M$. It follows the graph has no cycles. Since f is injective, every vertex has in-degree at most 1. It follows the graph is a union of chains. Note that every chain must "start" at some vertex of in-degree 1, since else we have some infinite sequence of non-negative integers $a_0, a_1, \ldots$ such that $a_0 = f^{(k)}(a_k)$. This implies $a_{2k} = a_0 - k \cdot M$, which directly contradicts that every a_i is non-negative. Hence, the graph is a union of chains that have a vertex of in-degree 0. Because of the previous discussion, these vertices must correspond to a number less than M. Hence, there are at least $\frac{M+1}{2}$ such chains (and at most M). Now, if the first vertex in a chain is k, it follows every non-negative integer that is $k \pmod{M}$ is in the chain (and is an even distance away from k). Now, consider $f(k)$. Since it is a distance 1 from k, it is not $k \pmod{M}$. Suppose it is $r \pmod{M}$. Then, r cannot be in any other chain, since that would imply $f(k)$ is in that other chain. Hence, $f(k) = r$. Thus, the chains give a pairing of non-negative integers less than M. However, there are M of these. Since M is odd, we cannot pair them up, so this is impossible. Hence, such a function f does not exist. ■