2011 G4

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October 14, 2021

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Let ABC be an acute triangle with circumcircle Ω. Let B_0 be the midpoint of AC and let C_0 be the midpoint of AB. Let D be the foot of the altitude from A and let G be the centroid of the triangle ABC. Let ω be a circle through B_0 and C_0 that is tangent to the circle Ω at a point $X \neq A$. Prove that the points D, G and X are collinear.

We use directed angles $\mod 180^\circ$. Let Y be the point on ω such that $\overline{\mathrm{AY}} \parallel \overline{\mathrm{BC}}$, A_0 be the midpoint of $\overline{\mathrm{BC}}$, T be the A_0-altitude of $\triangle A_0B_0C_0$, and X' be the second intersection of $\overline{\mathrm{YD}}$ with ω. Considering a homothety with center G and ratio $-\frac{1}{2}$, (ABC) is taken to the nine-point circle of $\triangle ABC$, so the image of Y must be the intersection of $\overline{\mathrm{BC}}$ and the nine-point circle that is not A_0. Thus, $Y-G-D$. Thus, it suffices to show $(X'B_0C_0)$ is tangent to (ABC).
Again by the homothety with center G, note that D maps to T, so $D-G-T$. Now, we have

$$\measuredangle C_0TX' = \measuredangle AYX' = \measuredangle ABX' = \measuredangle C_0BX',$$

so $C_OTX'B$ is cyclic. Analogously, $B_0TX'C$ is cyclic. Thus,

$$\measuredangle C_0X'B = \measuredangle C_0TB = 90^\circ - \measuredangle BTA_0 = 90^\circ - \measuredangle A_0TC = \measuredangle CTB_0 = \measuredangle CX'B_0.$$

Let P and Q be the second intersections of $\overline{\mathrm{X'C_0}}$ and $\overline{\mathrm{X'B_0}}$ with ω. The last angle equality implies that $PQCB$ is an isosceles trapezoid, giving $\overline{\mathrm{B_0C_0}} \parallel \overline{\mathrm{QP}}$. Thus, there is a homothety centered at X' taking $(X'B_0C_0)$ to (ABC), implying these circles are tangent at X', as desired. ■