2018 N5

Ezra Guerrero Alvarez

November 18, 2021

2018 N5

2018 N5

Positive integers x, y, z, t satisfy $xy - zt = x + y = z + t$. Can xy and zt both be perfect squares?

We claim the answer is no. First, note that if x and y have different parity, then z and t have different parity as well. But then $zy - zt$ is even and $x + y$ is odd. Thus, we must have $x \equiv y \pmod 2$ and $z \equiv t \pmod 2$. Thus, let

$$p := \frac{x+y}{2}, q := \frac{x-y}{2}, r := \frac{z+t}{2}, s := \frac{z-t}{2}.$$

Then, the equation rewrites as $p = r$ and $(s-q)(s+q) = 2p$. Thus, s and q must have the same parity as well. Letting

$$a := \frac{s+q}{2}, b := \frac{s-q}{2},$$

we see that $p = 2ab$. Thus, we have

$$\begin{aligned} x &= 2ab + a - b \\ y &= 2ab - a + b \\ z &= 2ab + a + b \\ t &= 2ab - a - b. \end{aligned}$$

This characterizes all solutions to the given equation, so long as $(a, b) \neq (1, 1)$ since this gives $t = 0$. Now, suppose $xy = c^2$. Also, note that

$$xy = (2ab)^2 - (a-b)^2 \qquad \text{and} \qquad zt = (2ab)^2 - (a+b)^2.$$

Claim 1

$(c-2)^2 < zt < c^2$.

Proof. The RHS is clear from the above equations. For the LHS we have that if $(a, b) \neq (1, 1)$ then

$$\begin{aligned} a^2 + b^2 + 1 &< 3(ab)^2 \\ (ab)^2 + (a-b)^2 + 2ab + 1 &< (2ab)^2 \\ (ab+1) &< \sqrt{xy} \\ 4ab + 4 &< 4c \\ xy - 4c + 4 &< zt \\ (c-2)^2 &< zt. \end{aligned}$$

□

Now, recall that xy and zt have the same parity since their difference is even. Since there are no perfect squares between $(c-2)^2$ and c^2 that have the same parity as c, this shows they cannot both be perfect squares. $\blacksquare$