2016 N1

Ezra Guerrero Alvarez

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For any positive integer k, denote the sum of digits of k in its decimal representation by $S(k)$. Find all polynomials $P(x)$ with integer coefficients such that for any positive integer $n \geq 2016$, the integer $P(n)$ is positive and
$$S(P(n)) = P(S(n)).$$

We claim the only solutions are $P(x) = 1, 2, \ldots, 9$ and $P(x) = x$, which clearly work. We show they are the only ones.
Let $P(x) = \sum a_i x^i$. Setting $x = 10^k$ for $k \geq 4$,

$$\sum a_i = P(1) = S(P(10^k)) \leq \sum S(a_i).$$

This is clearly impossible unless the a_i are digits. Now, setting $x = 9 \cdot 10^k$ for $k \geq 3$,

$$\sum 9^i a_i = P(9) = S(P(9 \cdot 10^k)) \leq \sum S(9^i a_i).$$

This is clearly impossible unless $a_i = 0$ for all $i \geq 2$ and $a_1 \in \{0, 1\}$. If $a_1 = 0$ it follows $a_0 \in \{1, 2, \ldots, 9\}$. If $a_1 = 1$, then
$$S(n + a_0) = S(n) + a_0.$$

Taking $n = 2019$, it's easy to see $a_0 = 0$. This concludes the proof. ■