2004 G4

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In a convex quadrilateral $ABCD$, the diagonal BD bisects neither the angle ABC nor the angle CDA. The point P lies inside $ABCD$ and satisfies

$$\angle PBC = \angle DBA \quad \text{and} \quad \angle PDC = \angle BDA.$$

Prove that $ABCD$ is a cyclic quadrilateral if and only if $AP = CP$.

We set $\triangle PBD$ as our reference triangle, with $P = (1,0,0), B = (0,1,0), D = (0,0,1)$ and $a = \overline{\mathrm{BD}}, b = \overline{\mathrm{DP}}, c = \overline{\mathrm{PB}}$. Note that A and C are isogonal conjugates with respect to $\triangle PBD$. Thus, let $A = (x_0, y_0, z_0)$. Then, $C = (a^2/x_0, b^2/y_0, c^2/z_0)$. Let $T := a^2y_0z_0 + b^2z_0x_0 + c^2x_0y_0$. The circle (ABD) is given by

$$-a^2yz - b^2zx - c^2xy + T(x+y+z)x/x_0 = 0.$$

Therefore, C lies on this circle iff

$$-\frac{a^2b^2c^2}{y_0z_0} - \frac{a^2b^2c^2}{z_0x_0} - \frac{a^2b^2c^2}{x_0y_0} + T\left(\frac{a^2}{x_0} + \frac{b^2}{y_0} + \frac{c^2}{z_0}\right)a^2/x_0^2 = 0.$$

Simplifying, this is equivalent to $T^2 = (x_0bc)^2$. Now, note that $C = (a^2y_0z_0/T, b^2z_0x_0/T, c^2x_0y_0/T)$. Thus,

$$\overrightarrow{\mathrm{PA}} = (x_0 - 1, y_0, z_0) \qquad \text{and} \qquad \overrightarrow{\mathrm{PC}} = \frac{1}{T}(a^2y_0z_0 - T : b^2z_0x_0 : c^2x_0y_0).$$

Then,

$$\left|\overrightarrow{\mathrm{PA}}\right|^2 = -T + b^2z_0 + c^2y_0 = \frac{T(y_0+z_0) - a^2y_0z_0}{x_0}$$

and

$$\left|\overrightarrow{\mathrm{PC}}\right|^2 = \frac{-a^2b^2c^2x_0y_0z_0}{T^2} + \frac{b^2c^2x_0y_0 + b^2c^2 + z_0x_0}{T} = \frac{(bcx_0)^2}{T^2}\left(\frac{T(y_0+z_0) - a^2y_0z_0}{x_0}\right).$$

Thus, $AP = CP$ iff $T^2 = (bcx_0)^2$. Hence, we can conclude that $ABCD$ is cyclic if and only if $AP = CP$. ■