2017 G4

Ezra Guerrero Alvarez

October 5, 2021

2017 G4

2017 G4

In triangle ABC, let ω be the A-excircle. Let D, E, F be the points where ω is tangent to BC, CA, AB respectively. The circle AEF intersects BC at P and Q. Let M be the midpoint of AD. Prove that (MPQ) is tangent to ω.

Let I_A be the A-excenter. Then, since $\angle AFI_A = \angle AEI_A = 90°$, $(APFI_AEQ)$ has diameter AI_A. Let N be the midpoint of $\overline{\mathrm{AI_A}}$. Then, from midpoints $\overline{\mathrm{MN}} \parallel \overline{\mathrm{DI_A}} \perp \overline{\mathrm{PQ}}$. Since $NP = NQ$, M lies on the perpendicular bisector of $\overline{\mathrm{PQ}}$ so M is the midpoint of arc $\widehat{PMQ}$. Now, let T be the second intersection of $\overrightarrow{\mathrm{AD}}$ with ω and K be the midpoint of $\overline{\mathrm{DT}}$. Since $I_AK \perp DT$, K lies on $(APFI_AEQ)$. Hence,

$$DM \cdot DT = \frac{1}{2}DA \cdot 2DK = DA \cdot DK = DP \cdot DQ,$$

giving T on (MPQ). Now, let O be the intersection of lines $\overline{\mathrm{TI_A}}$ and $\overline{\mathrm{MN}}$. We have

$$\angle OMT = \angle I_ADT = \angle I_ATD = \angle OTM,$$

so O lies on the perpendicular bisector of $\overline{\mathrm{MT}}$. Since it also lies on the perpendicular bisector of $\overline{\mathrm{PQ}}$, O is the center of $(MPTQ)$. Since O, I_A, T are collinear, (MPQ) and ω are tangent at T. ■