2002 G2

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Let ABC be a triangle for which there exists an interior point F such that $\angle AFB = \angle BFC = \angle CFA$. Let the lines BF and CF meet the sides AC and AB at D and E respectively. Prove that

$$AB + AC \geq 4DE.$$

Let X and Y be points outside $\triangle ABC$ such that $\triangle ACY$ and $\triangle ABX$ are equilateral. It is easy to see that $B-D-F-Y$, $C-E-F-X$ and $(AFCY)$ and $(AFDX)$ cyclic. Then,

$$\frac{FY}{FD} = 1 + \frac{DY}{FD} = 1 + \frac{[AYD]}{[AFD]} = 1 + \frac{[AYC]}{[AFC]} \geq 1 + 3 = 4.$$

Analogously $FX \geq 4FE$. Now, by the law of cosines,

$$\begin{aligned} 4DE &= 4\sqrt{FD^2 + FE^2 + FD \cdot FE} \\ &\leq \sqrt{FX^2 + FY^2 + FX \cdot FY} \\ &= XY \\ &\leq AX + AY \\ &= AB + AC. \blacksquare \end{aligned}$$