2016 G2

Ezra Guerrero Alvarez

January 14, 2022

2016 G2

2016 G2

Let ABC be a triangle with circumcircle Γ and incenter I and let M be the midpoint of $\overline{BC}$. The points D, E, F are selected on sides $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ such that $\overline{ID} \perp \overline{BC}$, $\overline{IE} \perp \overline{AI}$, and $\overline{IF} \perp \overline{AI}$. Suppose that the circumcircle of $\triangle AEF$ intersects Γ at a point X other than A. Prove that lines XD and AM meet on Γ.

We use directed angles mod 180°. Let D' be the reflection of D over M. It is well known that $\overline{\mathrm{IM}} \parallel \overline{\mathrm{AD'}}$. Also, since X is the spiral center of the spiral similarity sending $\overline{\mathrm{BC}}$ to $\overline{\mathrm{FE}}$, we see $\triangle XIM \overset{+}{\sim} \triangle XFB$. Let $T = \overline{\mathrm{XM}} \cap \overline{\mathrm{AD'}}$. Then,

$$\measuredangle ATX = \measuredangle IMX = \measuredangle FBX = \measuredangle ABX,$$

so T lies on Γ. Therefore, by the *Butterfly Theorem* it follows that $\overline{\mathrm{AM}} \cap \overline{\mathrm{XD}}$ lies on Γ as desired. ■