2004 C1

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There are 10001 students at an university. Some students join together to form several clubs (a student may belong to different clubs). Some clubs join together to form several societies (a club may belong to different societies). There are a total of k societies. Suppose that the following conditions hold:

1. Each pair of students are in exactly one club.
2. For each student and each society, the student is in exactly one club of the society.
3. Each club has an odd number of students. In addition, a club with $2m+1$ students (m is a positive integer) is in exactly m societies.

Find all possible values of k.

The only possible k is 5000. Indeed, replace 10001 by $2K+1$, we prove the only possible k is K. First, note that K is achievable by taking one club with every student in it that belongs to K societies. This satisfies every condition. Now, we prove k can only be K. Indeed, suppose there are $2c_i+1$ students in club i. The first condition guarantees that $\binom{2K+1}{2} = \sum \binom{2c_i+1}{2}$, or

$$K(2K+1) = \sum c_i(2c_i+1).$$

Now, from the second condition it is clear every student is in exactly k societies. Thus, if we add the number of students in each society we obtain $k(2K+1)$. The second condition also gives that, within a society, each pair of clubs is disjoint. Hence, the number of students in a given society is the sum of the number of students in each club in the society. Summing over all societies, each club gets counted c_i times, so this is $\sum c_i(2c_i+1)$. Hence,

$$K(2K+1) = \sum c_i(2c_i+1) = k(2K+1),$$

which implies $k = K$ as desired. ■