2000 C2

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A staircase-brick with 3 steps of width 2 is made of 12 unit cubes. Determine all integers n for which it is possible to build a cube of side n using such bricks.

We claim the answer is all positive multiples of 12. Evidently, if n is an integer that works, we have $12 \mid n^3$, so $6 \mid n$. Hence, $n = 6k$. Now, note that we may place two staircase-bricks together to form a $2 \times 3 \times 4$ prism. Since $2, 3, 4 \mid 12$, it readily follows that the cube can be built when k is even. Now we prove the cube cannot be built if k is odd. Suppose it can and consider a tiling of a $6k \times 6k \times 6k$ cube with staircase-bricks. We separate the brick into six 6×6 layers. In each layer, the staircase-brick fills either a $2 \times 1, 2 \times 2$, or 2×3. If the staircase-brick fills a 2×2, then it must fill a 2×1 and 2×3 in the above and below layers. Hence, the bottom-most layer has no 2×2 filled by staircase-bricks. It follows the layer right on top is filled only by 2×2. It is straightforward to see that the only way to tile a $6k \times 6k$ layer using 2×2 pieces is to lay out the pieces in a $3k \times 3k$ grid. Therefore, we use $9k^2$ pieces. Now, look at the bottom layer. It is filled by x 2×3 and y 2×1. Since it is the bottom layer, these must be part of one of the $9k^2$ pieces above. Hence, $x + y = 9k^2$. On the other hand, summing areas we see $6x + 2y = 36k^2$, or $3x + y = 18k^2$. Thus, $2x = 9k^2$. However, k is odd, so this is impossible. Hence, the cube cannot be built when k is odd as desired. ■