2013 N1

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Let $\mathbb{Z}_{>0}$ be the set of positive integers. Find all functions $f : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ such that

$$m^2 + f(n) \mid mf(m) + n$$

for all positive integers m and n.

We claim the only solution is $f(n) = n$ for all n, which is easily seen to work. Now, plugging in $m = n = 2$ we obtain

$$4 + f(2) \mid 2f(2) + 2.$$

Since $4 + f(2) \mid 8 + 2f(2)$, it follows $4 + f(2) \mid 6$. But $4 + f(2) \geq 5$, so $4 + f(2) = 6$, implying $f(2) = 2$. Now, substituting $m = 2$ gives

$$\begin{aligned} 4 + f(n) &\mid 4 + n \\ \implies 4 + f(n) &\leq 4 + n \\ f(n) &\leq n \end{aligned}$$

for all n. It is clear $f(1) \geq 1$. For all $m > 1$, we see

$$\begin{aligned} m^2 + f(m) &\mid mf(m) + m \\ \implies m^2 + f(m) &\leq mf(m) + m \\ m(m-1) &\leq f(m)(m-1) \\ m &\leq f(m). \end{aligned}$$

Thus, for all $n \in \mathbb{Z}_{>0}$, $n \leq f(n) \leq n$, so $f(n) = n$ for all n as desired. ■