2004 G2

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Let Γ be a circle and let d be a line such that Γ and d have no common points. Further, let AB be a diameter of the circle Γ; assume that this diameter AB is perpendicular to the line d, and the point B is nearer to the line d than the point A. Let C be an arbitrary point on the circle Γ, different from the points A and B. Let D be the point of intersection of the lines AC and d. One of the two tangents from the point D to the circle Γ touches this circle Γ at a point E; hereby, we assume that the points B and E lie in the same halfplane with respect to the line AC. Denote by F the point of intersection of the lines BE and d. Let the line AF intersect the circle Γ at a point G, different from A.

Prove that the reflection of the point G in the line AB lies on the line CF.

We use directed angles mod 180°. Let M and E' be the intersections of $\overline{\text{AM}}$ and $\overline{\text{AE}}$ with d, respectively. Note that $AEMF$ is cyclic with diameter $\overline{\text{AF}}$. We have

$$\measuredangle DEF = \measuredangle DEB = \measuredangle EAB = \measuredangle EAM = \measuredangle EFM = \measuredangle EFD,$$

so $DE = DF$. Since $\triangle FEE'$ is right, it follows D is the circumcenter of (FEE'). Now, note that the inversion with center A and radius $\sqrt{AC \cdot AD}$ sends Γ to d, implying $AG \cdot AF = AE \cdot AE'$ so $G \in (FEE')$. Therefore, $DG = DE$ and $\overline{\text{DG}}$ is tangent to Γ. By tangents,

$$-1 = (A, C; E, G) \stackrel{F}{=} (G, \overline{\text{FC}} \cap \Gamma; B, A).$$

Since $\overline{\text{AB}}$ is a diameter, it follows $\overline{\text{FC}}$ passes through the reflection of G in line $\overline{\text{AB}}$, as desired. $\blacksquare$