# Using network theory to detect concept drift

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March 22, 2024

# Overview

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- Can help us better understand data dynamics.
- Can degrade the performance of a predictive model.

# Illustrative example: Clustering in Customer Purchase Data

- $\mathbf{C}_{t_1}$ and $\mathbf{C}_{t_2}$ represent the clustering configurations at times 1 and 2.
- We say that concept drift has occurred if

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# Illustrative example: Clustering in Customer Purchase Data

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- Detection and adaptation to concept drift ensures that strategies are up-to-date.

# Unsupervised Concept Drift

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- No response variable $Y$ - lack of a clear signal.
- Difficult for the model to adapt.

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- Interpretability

# Existing methods

In the context of unsupervised learning with high-dimensional data:

- Lack of interpretable methods for visualising concept drift, especially for visualising the drift between all covariates at once.

# Framing the problem

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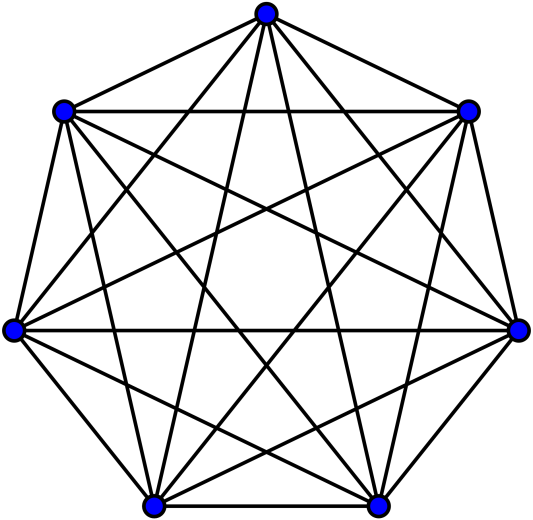

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- Adjacency matrix $\mathbf{A}^{(t)} \in \mathbb{R}^{n\times n}$,
- $\mathbf{A}_{ij}^{(t)} = Cov(\mathbf{c}_i, \mathbf{c}_j)$ at time $t$ with $\mathbf{c}_1, \ldots, \mathbf{c}_n \in \mathbb{R}^m$, representing $m$ observations.

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- $d$-dimensional representation of data.
- Embed adjacency matrices at different time points while maintaining *stability*.

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- Extension of singular value decomposition (SVD).

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- $\mathbf{U}_{\mathbf{A}} \in \mathbb{O}^{n \times d}$ and $\mathbf{V}_{\mathbf{A}} \in \mathbb{O}^{nT \times d}$ contain the left and right singular vectors respectively.

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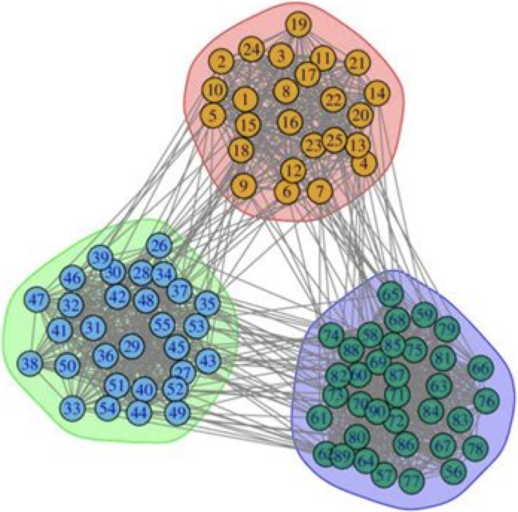
- Divide $\hat{\mathbf{Y}}$ into $T$ blocks $\hat{\mathbf{Y}}^{(t)}$,
- Rows $\hat{\mathbf{Y}}_i$ of each block are the $d$-dimensional points to be plotted to represent the drift at time $t$.

# Dynamic Stochastic Block Model



# Toy Model

Connection probabilities:

$$\mathbf{B}^{(1)} = \begin{pmatrix} 0.24 & 0.06 & 0.54 & 0.06 \\ 0.06 & 0.60 & 0.12 & 0.06 \\ 0.54 & 0.12 & 0.06 & 0.10 \\ 0.06 & 0.06 & 0.10 & 0.18 \end{pmatrix}, \ \mathbf{B}^{(2)} = \begin{pmatrix} 0.48 & 0.48 & 0.12 & 0.06 \\ 0.48 & 0.48 & 0.12 & 0.06 \\ 0.12 & 0.12 & 0.27 & 0.10 \\ 0.06 & 0.06 & 0.10 & 0.18 \end{pmatrix}.$$

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- At $t = 2$ communities 1 and 2 have merged.
- Community 4 has same connection probabilities.

# What makes a successful embedding?

- Cross-sectional stability: The embeddings for communities 1 and 2 at time 2 are close.
- Longitudinal stability: The embeddings for community 4 at times 1 and 2 are close.

# Generating our data frames

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6: $\mathbf{D}^{(t)}_{k,i} = \sum_j \mathbf{x}_j$ for j where $\mathbf{c}_i$ and $\mathbf{c}_j$ are connected
7: **end for**
8: **end for**

# Computing the embedding

- Separately compute $Cov(\mathbf{D}^{(1)})$ and $Cov(\mathbf{D}^{(2)})$.
- Let $\mathbf{A}^{(1)} = \text{Hollow}(Cov(\mathbf{D}^{(1)}))$ and $\mathbf{A}^{(2)} = \text{Hollow}(Cov(\mathbf{D}^{(2)}))$ be our adjacency matrices.

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- Column concatenate: $\mathbf{A} = [\mathbf{A}^{(1)} | \mathbf{A}^{(2)}] \in \mathbb{R}^{n \times 2n}$.
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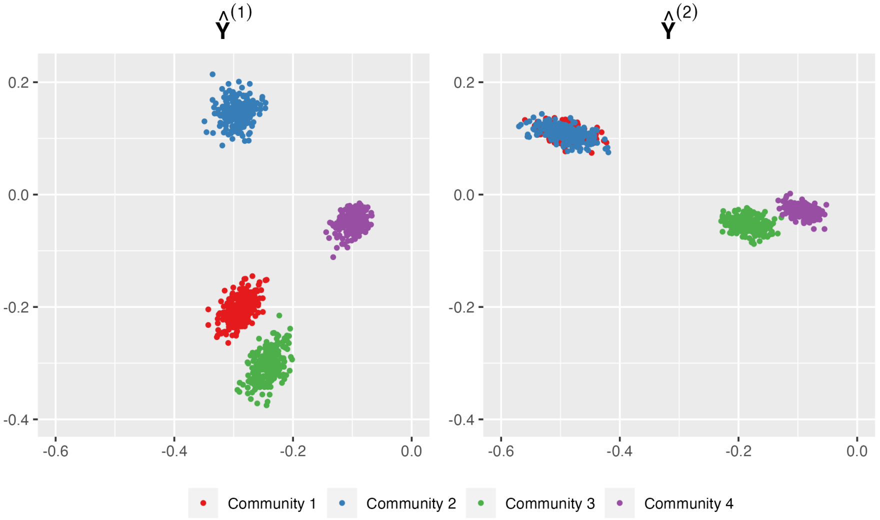
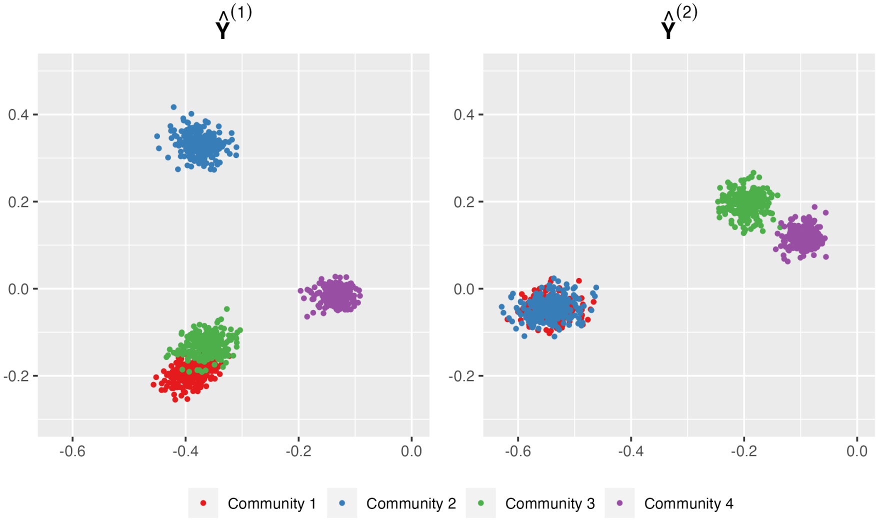
- Plot first 2 entries for each row of $\hat{\mathbf{Y}}^{(1)}$ and $\hat{\mathbf{Y}}^{(2)}$.

# Stability of UASE



# Comparison: Independent Spectral Embedding



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- The cross sectional, and longitudinal stability of UASE.
- That each embedding $\hat{\mathbf{Y}}^{(t)}$ converges to some underlying 'true' or 'noise-free' $\tilde{\mathbf{Y}}^{(t)}$ as the size of n increases, up to rotation.
- That the points  point clouds converge to a multivariate Gaussian distribution.

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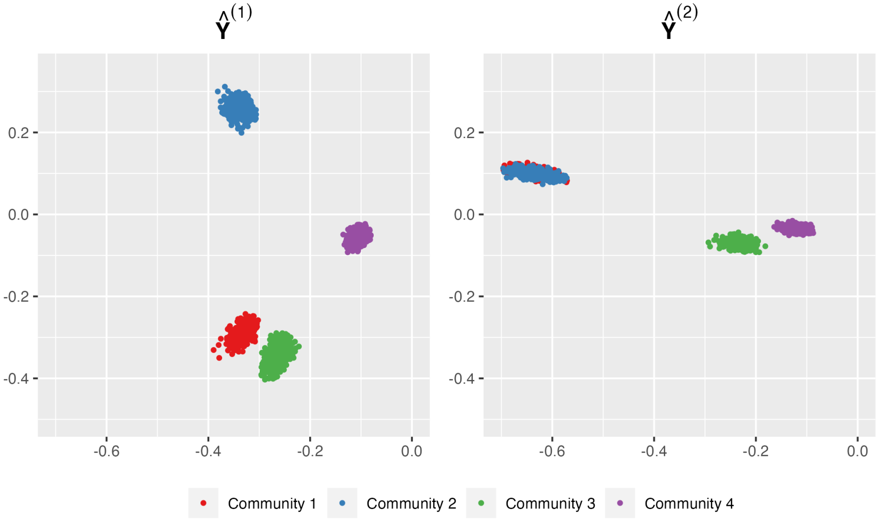

$\hat{Y}^{(1)}$
$\hat{Y}^{(2)}$
0.2
0.0
-0.2
-0.4
-0.6
-0.4
-0.2
0.0
Community 1
Community 2
Community 3
Community 4