MAE 507: Engineering Analysis I

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Project 2

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Membrane Equation

We will begin with the membrane equations and situation that we had in class. We will start with the same assumptions. We are assuming that the membrane is elastic and homogeneous. We are also assuming that it has no resistance to bending. The tension at the edges is assumed to be large enough that the weight of the membrane can be totally neglected. We are also assuming that the deflections in the membrane will be small and so will the angle of inclination.

Now we start with the equation of motion in the transverse direction:

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial y^2}\right) = c^2\nabla^2 u$$

But we are going to want this equation in polar coordinates. To do that we are going to replace the Cartesian Laplacian operator with the Polar Laplacian operator. This will give us:

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

This will give us in turn the full membrane equation

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right)$$

For this problem however we are looking at the axisymmetric case. That means that we can look the problem as simply:

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right)$$

Separation of Variables:

We will attempt to find a solution to this problem by attempting a separation of variables using the following:

$$u(r,t) = F(r)G(t)$$

Putting this identity into the vibrating membrane equation we will get the following:

$$F(r)\ddot{G}(t) = c^2\left(F''(r)G(t) + \frac{1}{r}F'(r)G(t)\right)$$

Rearranging we get the following equality:

$$\frac{\ddot{G}(t)}{c^2 G(t)} = \frac{1}{F(r)}\left(F''(r) + \frac{1}{r}F'(r)\right) = K$$

Or:

$$\ddot{G}(t) - Kc^2G(t) = 0$$

$$F''(r) + \frac{1}{r}F'(r) - KF(r) = 0$$

Now we need to look to see what values K could be. We know that if K = 0 our solutions will look like:

$$G(t) = c_1x + c_2$$

And if K > 0 we will get an answer like:

$$G(t) = c_1e^{kx} + c_2e^{kx}$$

Which will give us either a linear or exponential response from the impact of our membrane. We know from experience that the response we are looking for does not fit either of these solutions so instead we will go with K < 0, specifically:

$$K = -k^2$$

Putting that into the equations from earlier we get instead:

$$\ddot{G}(t) + k^2c^2G(t) = 0$$

$$F''(r) + \frac{1}{r}F'(r) + k^2F(r) = 0$$

Starting with the Time equation we know that our solution is going to be the following:

$$G(t) = Acos(c_1kt) + Bsin(c_2kt)$$

Looking at the the radial equation is a bit harder. Initially we attempted creating a characteristic equation as follows:

$$R^2 + \frac{1}{r}R + k^2 = 0$$

And attempting to solve using the quadratic equation but after looking for a while there is actually a simpler solution.

If we take the the radio function and multiply through by r^2 we will get:

$$r^2F''(r) + rF'(r) + (r^2k^2 - 0^2)F(r) = 0$$

This is actually the order zero Bessel Function. This is a very well known function and the solution is:

$F(r) = c_1 J_0(kr) + c_2 Y_0(kr)$

No we need to check get the constant coefficients of this F(r)

At r=0 we will get

$F(0) = c_1 J_0(0) + c_2 Y_0(0)$

At $Y_0(0)$ the solution is -∞, which is not a solution that is realistic, therefor we will set $c_2 = 0$

Now our boundary condition states that we fixed the membrane $u(a, t) = 0$. As we don't want a trivial solution and we know that we will have to solve for coefficients later using the initial value conditions we can set $c_1 = 1$ and get the following

$F(a) = J_0(ka) = 0$

That means that ka is a root of the J_0. There are infinitely many of these roots and we can simply call them γ_n.

This means that rearranging we can get:

$k_n = \frac{\gamma_n}{a}, n = 1,2,3...$

Going back to $u(r, t) = F(r)G(t)$ we will get the following solutions:
$u(r, t) = (J_0(kr))(Acos(ckt) + Bsin(ckt))$

Which when discretized and summed becomes:

$u_n(r, t) = (J_0(k_n r))(A_n cos(ck_n t) + B_n sin(ck_n t))$

This means that our eigenvalues are:

$k_n = \frac{\gamma_n}{a}$

With the eigenfunctions being

$J_0(k_n r)$

Demonstrating Orthogonality:

We know that Bessel Functions are orthogonal and can prove it as follows:

We are looking to prove:

$\int_0^a J_0(k_1 r) J_0(k_2 r) dr = 0$

Remember that the function J_0 is a solution to the following equations:

$$r^2F_1''(r) + rF_1'(r) + (r^2k_1^2 - 0^2)F_1(r) = 0$$

$$r^2F_2''(r) + rF_2'(r) + (r^2k_2^2 - 0^2)F_2(r) = 0$$

Now if we multiply each of these functions by the other one and subtract the results we get the equation:

$$(k_1^2 - k_2^2)\int_0^a F_1(r)F_2(r)rdr = -\int_0^a \partial r\frac{\partial}{\partial r}(r\frac{\partial F_1(r)}{\partial r})F_2(r) + \int_0^a \partial r\frac{\partial}{\partial r}(r\frac{\partial F_2(r)}{\partial r})F_1(r)$$

Now we can utilize integration by parts to get:

$$-\int_0^a \partial r\frac{\partial}{\partial r}(r\frac{\partial F_1(r)}{\partial r})F_2(r) + \int_0^a \partial r\frac{\partial}{\partial r}(r\frac{\partial F_2(r)}{\partial r})F_1(r) = r\frac{\partial F_2(r)}{\partial r}F_1(r) - r\frac{\partial F_1(r)}{\partial r}F_2(r)\,|_{r=0}^{r=a}$$

Now we can just evaluate at these values

$$a\frac{\partial F_2(a)}{\partial r}F_1(a) - a\frac{\partial F_1(r)}{\partial r}F_2(a) - (0*\frac{\partial F_2(a)}{\partial r}F_1(a) - 0*\frac{\partial F_1(r)}{\partial r}F_2(a)$$

But remember that $F_1(a) = J_0(k*a) = J_0(\frac{\gamma_1}{a}a) = 0$ with a similar logic for $F_2(a)$.

Therefor everything goes to 0 and we confirm that $\int_0^a J_0(k_1r)J_0(k_2r)dr = 0.$

Solving the Initial Value Problem:

We now have the solution to the problem as

$$u_n(r,t) = \sum_{n=1}^{\infty}(J_0(k_nr))(A_ncos(ck_nt) + B_nsin(ck_nt))$$

For our problem we are starting from an undeformed position, which means we have the following:

$$u_n(r,0) = \sum_{n=1}^{\infty}(J_0(k_nr))(A_ncos(ck_n0) + B_nsin(ck_n0)) = 0$$

$$u_n(r,0) = \sum_{n=1}^{\infty}A_nJ_0(k_nr) = 0$$

There for $A_n = 0$

Now also have an initial condition relating to the initial velocity, so we will begin by taking the derivative with respect to t and setting t to 0.

$$\dot{u}_n(r,0) = \sum_{n=1}^{\infty} ck_n J_0(k_n r) B_n cos(ck_n 0) = v(r) = \sum_{n=1}^{\infty} ck_n B_n J_0(k_n r)$$

$$B_n = \frac{2}{c\gamma_n a J_1^2(\gamma_n)} * \int_0^a (1 - \frac{r^2}{a^2}) J_0(k_n r) r dr$$

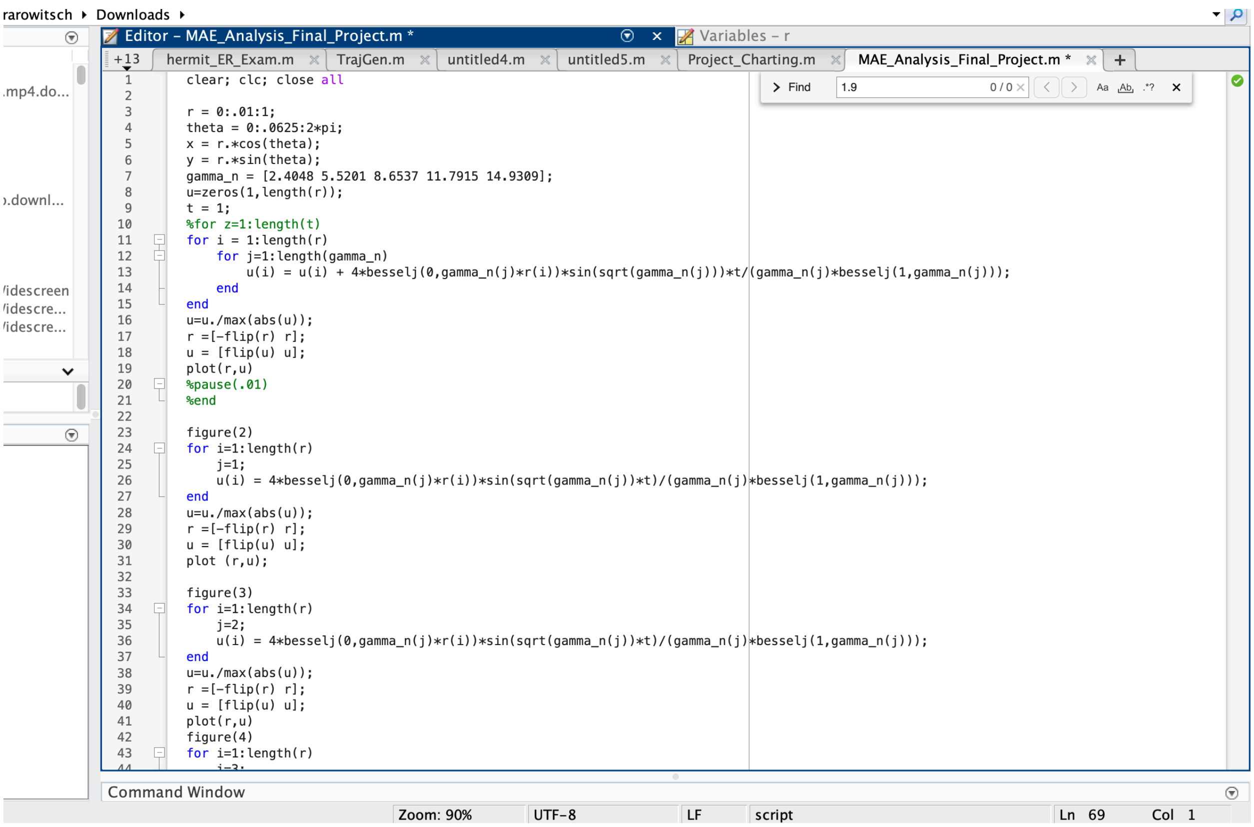
$$B_n = \frac{4J_1(\gamma_n)}{c\gamma_n a J_1^2(\gamma_n)} = \frac{4}{c\gamma_n a J_1(\gamma_n)}$$

Putting this all back into the original $u_n(r, t)$ equation we get:

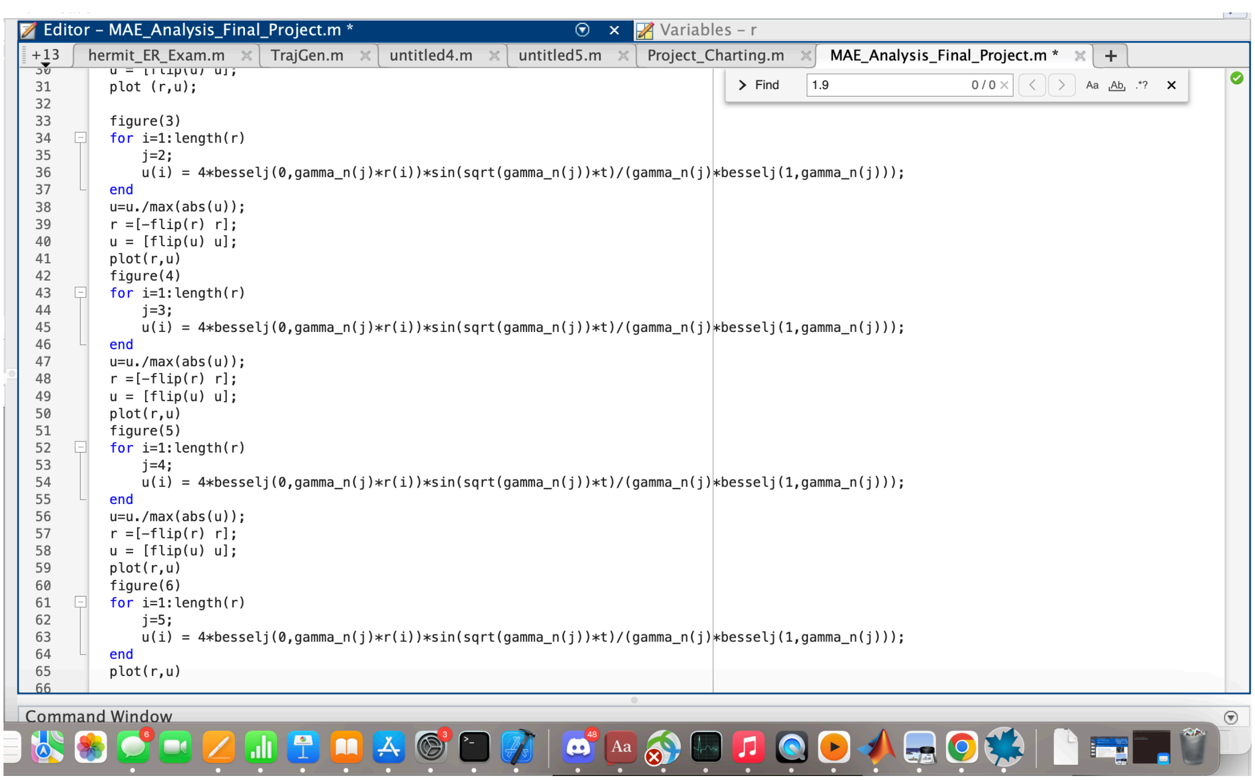
$$u_n(r, t) = \sum_{n=1}^{\infty} \frac{4J_0(k_n r) sin(ck_n t)}{c\gamma_n a J_1(\gamma_n)}$$

Mode shapes and Convergence Characteristics:

We are now going to look at some of the Convergence characteristics and mode shapes that this system produces. This is generated by code written in Matlab which I will attach as pictures below.



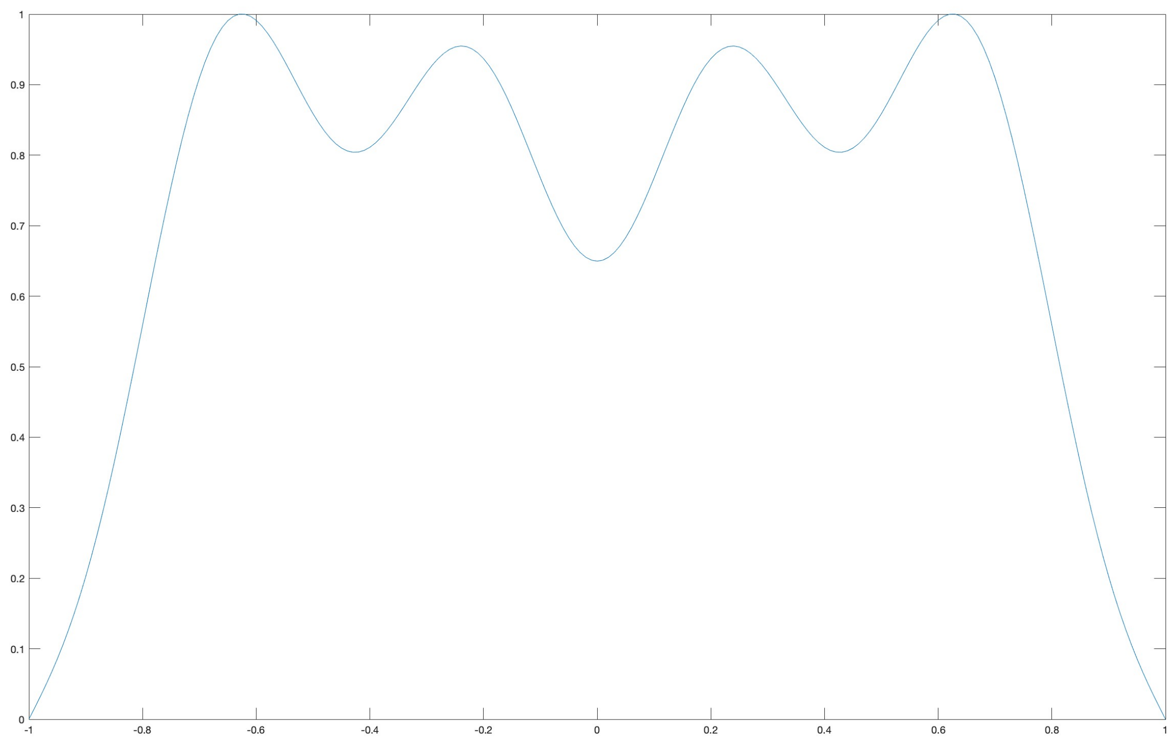
```
clear; clc; close all

r = 0:.01:1;
theta = 0:.0625:2*pi;
x = r.*cos(theta);
y = r.*sin(theta);
gamma_n = [2.4048 5.5201 8.6537 11.7915 14.9309];
u=zeros(1,length(r));
t = 1;
%for z=1:length(t)
for i = 1:length(r)
    for j=1:length(gamma_n)
        u(i) = u(i) + 4*besselj(0,gamma_n(j)*r(i))*sin(sqrt(gamma_n(j)))*t/(gamma_n(j)*besselj(1,gamma_n(j)));
    end
end
u=u./max(abs(u));
r =[-flip(r) r];
u = [flip(u) u];
plot(r,u)
%pause(.01)
%end

figure(2)
for i=1:length(r)
    j=1;
    u(i) = 4*besselj(0,gamma_n(j)*r(i))*sin(sqrt(gamma_n(j))*t)/(gamma_n(j)*besselj(1,gamma_n(j)));
end
u=u./max(abs(u));
r =[-flip(r) r];
u = [flip(u) u];
plot (r,u);

figure(3)
for i=1:length(r)
    j=2;
    u(i) = 4*besselj(0,gamma_n(j)*r(i))*sin(sqrt(gamma_n(j))*t)/(gamma_n(j)*besselj(1,gamma_n(j)));
end
u=u./max(abs(u));
r =[-flip(r) r];
u = [flip(u) u];
plot(r,u)
figure(4)
for i=1:length(r)
```



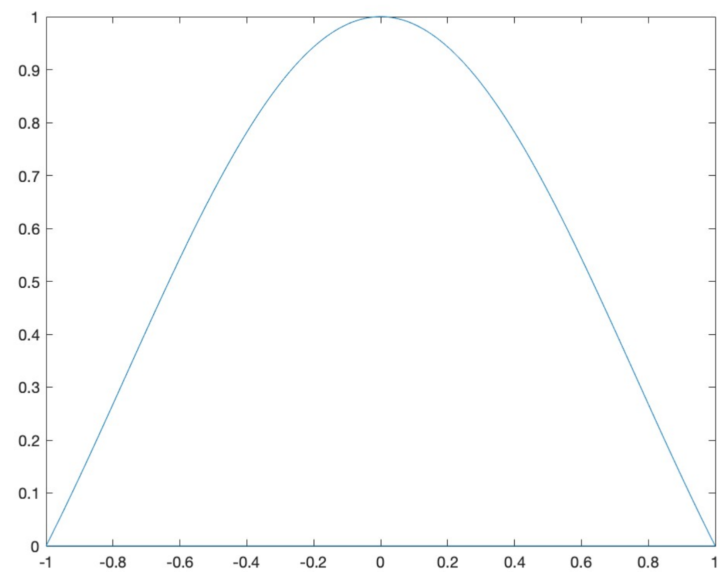
```
plot (r,u);

figure(3)
for i=1:length(r)
    j=2;
    u(i) = 4*besselj(0,gamma_n(j)*r(i))*sin(sqrt(gamma_n(j))*t)/(gamma_n(j)*besselj(1,gamma_n(j)));
end
u=u./max(abs(u));
r =[-flip(r) r];
u = [flip(u) u];
plot(r,u)
figure(4)
for i=1:length(r)
    j=3;
    u(i) = 4*besselj(0,gamma_n(j)*r(i))*sin(sqrt(gamma_n(j))*t)/(gamma_n(j)*besselj(1,gamma_n(j)));
end
u=u./max(abs(u));
r =[-flip(r) r];
u = [flip(u) u];
plot(r,u)
figure(5)
for i=1:length(r)
    j=4;
    u(i) = 4*besselj(0,gamma_n(j)*r(i))*sin(sqrt(gamma_n(j))*t)/(gamma_n(j)*besselj(1,gamma_n(j)));
end
u=u./max(abs(u));
r =[-flip(r) r];
u = [flip(u) u];
plot(r,u)
figure(6)
for i=1:length(r)
    j=5;
    u(i) = 4*besselj(0,gamma_n(j)*r(i))*sin(sqrt(gamma_n(j))*t)/(gamma_n(j)*besselj(1,gamma_n(j)));
end
plot(r,u)
```

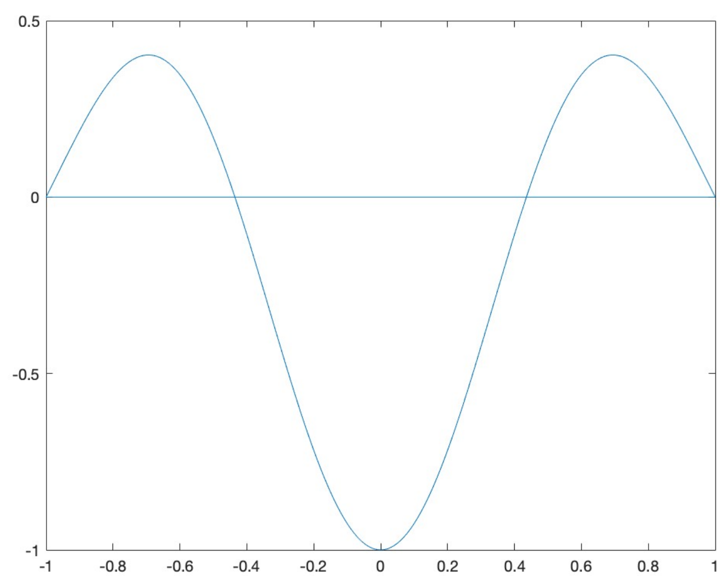
For this system we have a natural frequency once every 0 on the zeroth order Bessel function with our natural frequency $k_n = \dfrac{\gamma_n}{a}$.

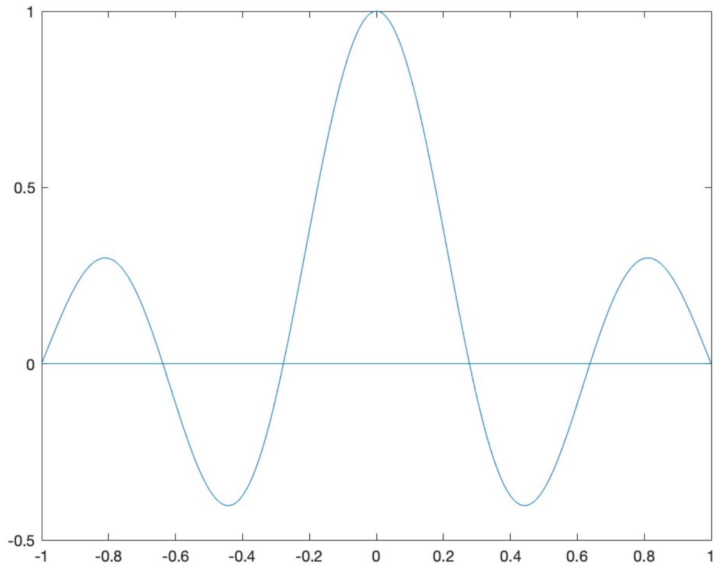
First we have the entire sum which looks like the following:

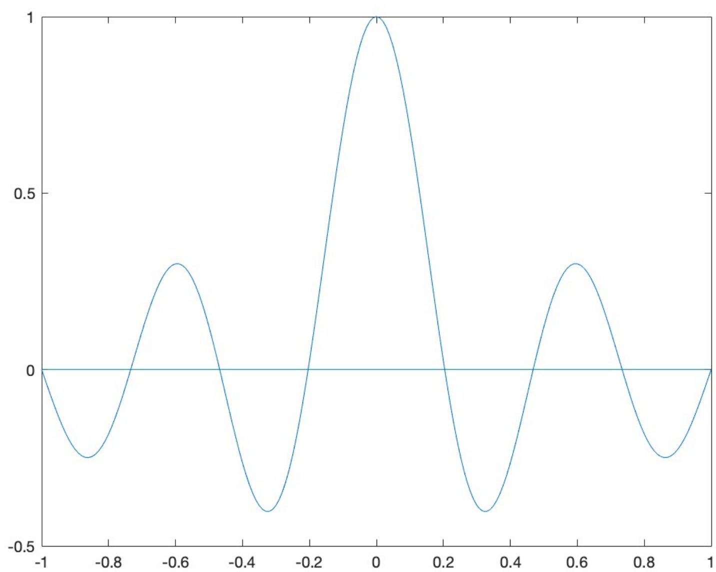


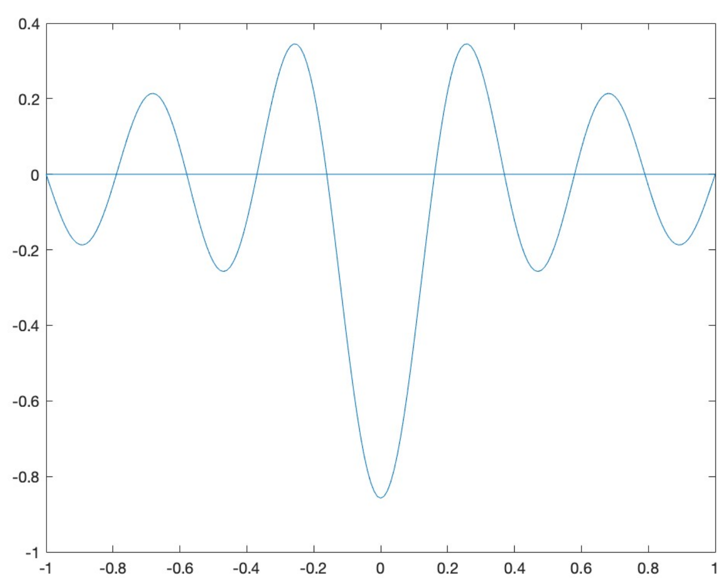
The following images are the first 5 natural frequencies of the system which I have plotted. They are a cross section across the center of the membrane over the domain [-a a] with both the radius and the max u(r,t) being normalized to a maximum of 1. Taken at t=1 with all other parameters being 1 as well.






1
0.5
0
-0.5
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1


1
0.5
0
-0.5
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1


0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1

Starting with the top figure we see the first frequency mode, with the entire membrane moving up and down in one motion. From there we see going next figure down that at the first natural frequency the membrane crosses the 0 plane twice, once on each radius. Each of the frequency modes above this follow the same pattern, crossing the u=0 line n-1 times for the associated k_n.

Non-Axisymmetric Case

If we do not have an axisymmetric case that means that we will have to go back to our original circular membrane equation. This time we cannot just assume that $\theta = 0$ so we will will have the full equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2})$$

From there we will again attempt separation of variables, this time with the following definition:

$$u(r, t) = F(r)G(t)\Theta(\theta)$$

Which will turn our original equation into

$$F(r)\ddot{G}(t)\Theta(\theta) = c^2(F''(r)G(t)\Theta(\theta) + \frac{1}{r}F'(r)G(t)\Theta(\theta) + \frac{1}{r^2}\Theta''(\theta)G(t)F(r))$$

And by dividing both sides through by $c^2\Theta(\theta)F(r)G(t)$ we will get

$$\frac{\ddot{G}(t)}{c^2G(t)} = \frac{F''(r)}{F(r)} + \frac{F'(r)}{rF(r)} + \frac{\Theta''(\theta)}{r^2\Theta(\theta)} = -k^2$$

From here our G(t) equation is already separated. This means that we will have to go on to separate the thetas and radial portion. From here if we take that “middle” portion and multiply through by r^2 we can again separate the variables to form another set of equations, this time getting:

$$-\frac{\Theta(\theta)}{\Theta(\theta)} = C$$

$$\frac{r^2F(r)}{F(r)} + \frac{rF(r)}{F(r)} + k^2r^2 = C$$

From here we could follow the above logic, finding the periodic solution to this system as well. We would then end up needing to use the boundary conditions to again find the eigenvalues and eigenvectors. From there we could form another Fourier styles summation and the initial values to again find the coefficients and get a final solution in a similar style to what we had for the axisymmetric case. To complete this we would need another boundary condition describing the θ position, as well as another initial value describing how the θ is connected to the velocity.

References:

Weisstein, Eric W. "*Bessel Function Zeros.*" From *MathWorld--A Wolfram Web Resource*. https://mathworld.wolfram.com/BesselFunctionZeros.html

Dobrushkin, Vladimir. *Orthogonality of Bessel Functions*, Brown University, https://www.cfm.brown.edu/people/dobrush/am34/Mathematica/ch7/besselo.html.

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