Equilibria in First Price Auction with Different Participation Cost

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Content

Introduction

- Not all bidders can participate in an auction freely. The existence of bidders' participation costs can substantially change the outcome of an auction.
- Generally, a first price auction with participation cost is the one in which an indivisible object is allocated to one of potential buyers via a first-price auction, and in order to participate, bidders must incur a nonrefundable cost that may be the costs of traveling to an auction site, to pay for the process of learning the rules of auction, to acquire information, or more generally the opportunity cost of attending an auction, etc.
- Hence, the question of whether to participate in auctions may be more crucial than the standard question of how to bid, suggesting that such decisions should be modeled and included as part of an equilibrium.

- monotonic equilibria in which a lower participation cost results in a lower cutoff to participate in an auction
- nonmonotonic equilibria in which a lower participation cost results in a higher cutoff

The Setup

- an independent private value economic environment with one seller and $n \geq 2$ potential buyers.
- The seller is risk-neutral and values the object as 0.
- Bidder i' s valuation v_i is private information, which is independently distributed with a cumulative distribution function $F_i(v)$ that has continuously differentiable density $f_i(v)$ with full support $[0, 1]$.
- The participation costs $c_i \in (0, 1]$ for all i are common knowledge.
- The bidder with the highest bid wins the object and pays his bid price. If there is only one bidder in the auction, he wins the object and pays 0 . If the highest bids are equal for more than one bidder, the allocation is determined by a fair lottery.

The Setup

The Rational Action

- the individually rational action set for any type of bidder is $\{No\} \cup [0, 1]$, where " $\{N_o\}''$ denotes not participating in the auction. Bidder i incurs the participation cost if and only if his action is different from " $\{N_o\}''$." Let $b_i(v_i, \mathbf{c})$ denote bidder i's strategy where $c = (c_1, c_2, ..., c_n)$.
- If a bidder finds participating in this first-price auction optimal, he cannot do better than bid his max-payoff function $\lambda_i(v_i, v_i^*)$. All of our results about the uniqueness or multiplicity of the equilibria should be interpreted accordingly.

- Given the equilibrium strategies of all others, a bidder's expected payoff from participating in the auction is a nondecreasing function of his valuation. Therefore, we can focus on Bayesian-Nash equilibria in which each bidder uses a cutoff strategy denoted by $v_i^*(\mathbf{c})$, that is, he bids his valuation if it is greater than or equal to the cutoff and does not enter otherwise.
- Thus, the bidding decision function of each bidder is characterized by

$$b(\lambda_i, c) = \begin{cases} \lambda_i (v_i, v_i^*) & \text{if } v_i^*(c) \leqslant v_i \leqslant 1 \\ No\ otherwise \end{cases}$$

For notational convenience, we simply denote $v_i^*(\mathbf{c}) = v_i^*$.

DEFINITION 1

An equilibrium is a cutoff vector $(v_1^*, v_2^*, \ldots, v_n^*) \in \mathbb{R}_+^n$ and the related max payoff function such that each bidder' s action is optimal, given others' cutoff strategies.

DEFINITION 2

An equilibrium $(v_1^*, v_2^*, \ldots, v_n^*) \in \mathbb{R}_+^n$ is called a symmetric equilibrium (respectively, asymmetric equilibrium) if $v_1^* = v_2^* = \ldots = v_n^*$ (respectively, there exist two bidders i and j such that $v_i^* \neq v_j^*$).

Two Bidders with Different Participation Cost

- Two bidders who have different participation costs c_1 and c_2 with $c_1 < c_2$, and the same distribution function $F(\cdot)$.
- First assume, provisionally, that a monotonic equilibrium (v_1^*, v_2^*) exists, that is, $v_1^* < v_2^*$.
- When bidder $1'$ s valuation is $v_1 = v_1^*$, his expected payoff from participating is given by $v_1^* F(v_2^*) + 0(1 - F(v_2^*))$.
- Zero net-payoff (equilibrium) condition requires that

$$c_1 = v_1^* F(v_2^*)$$

Two Bidders with Different Participation Cost

- bidder 2's max-value function satisfies

$$\underline{b_2} = \max arg \max_b \left(F\left(b\right) - F\left(v_1^*\right)\right)\left(v_2^* - b\right)$$

- The first order condition for $\underline{b_2}$ gives

$$f\left(\underline{b_2}\right)\left(v_2^* - \underline{b_2}\right) = F\left(\underline{b_2}\right) - F\left(v_1^*\right)$$

- When bidder 2's participation cost is too large, he may never participate in the auction, the expected payoff he obtains from participating even when his value is 1 is less than his participation cost c_2.In this case, the bidder2's participation cost satisfies

$$c_2 > v_2^* F\left(v_1^*\right) + \left(F\left(\underline{b_2}\right) - F\left(v_1^*\right)\right)\left(1 - \underline{b_2}\right)$$

- In this case, we have a monotonic equilibrium with $v_1^* = c_1$ and $v_2^* > 1$.

Cao,X,Tian,G.,2010.Equilibeia in first price auctions with participation costs.Games Econ. Behav.

For bidder $j \in \{m+1, \ldots, n\}$ with $v_j = v_2^*$, he bids zero and has revenue v_2^* when none of the oth
If any other bidder in type 2 enters the auction, he will lose the bid. If only $k \leqslant m$ bidders in type 1
optimal bid $\underline{b}_k$ for bidder j is determined by

$$\underline{b}_k = \max \arg\max_b \left(F(b) - F\left(v_1^*\right)\right)^k \left(v_2^* - b\right).$$

The first order condition for $\underline{b}_k$ gives

$$\underline{b}_k + \frac{F(\underline{b}_k) - F(v_1^*)}{kf(\underline{b}_k)} = v_2^*.$$

$\underline{b}_k$ is chosen with probability $C_m^k F(v_1^*)^{m-k}(1 - F(v_1^*))^k$. $C_m^k = \frac{m!}{k!(m-k)!}$ is the combination number
dates from the n items that are available. Thus, at equilibrium, we have

$$c \geqslant v_2^* F\left(v_1^*\right)^m F\left(v_2^*\right)^{n-m-1} + F\left(v_2^*\right)^{n-m-1} \sum_{k=1}^{m} C_m^k F\left(v_1^*\right)^{m-k} \left(F(\underline{b}_k) - F\left(v_1^*\right)\right)^k \left(v_2^* - \underline{b}_k\right),$$

- To find a monotonic equilibrium, we define the following two cutoff reaction function equations:

$$xF(y) = c_1$$
$$c_2 = yF(x) + \left(F\left(\underline{b_2}\right) - F(x)\right)\left(y - \underline{b_2}\right)$$

- with $x < y$, where x corresponds to v_1^*, and y corresponds to v_2^*. It can be easily seen that we have $x \geq c_1$ and $y \geq c_2$. They can be regarded as cutoff reaction functions because the first equation shows how bidder 1 will choose a cutoff x, given bidder $2'$ s action y. Equation 2 shows how bidder 2 will choose a cutoff y, given bidder $1'$ s action x. A monotonic equilibrium $(v_1^*, v_2^*) \in [c_1, 1] \times [c_2, 1]$ is obtained when x and y satisfy these two equations simultaneously.

- We have $x = x(y) = \frac{c_1}{F(y)}$. Then, $\frac{dx}{dy} = -\frac{c_1 f(y)}{F^2(y)} < 0$. This implicitly defines y as a decreasing function of x, denoted by $y = y(x)$. Substitute $y = y(x)$ and let

$$h(x) = y(x) F(x) + \left(F\left(\underline{b_2}\right) - F(x)\right)\left(y - \underline{b_2}\right) - c_2$$

- Substitute $x = x(y)$ and let

$$\lambda(y) = yF\left(\frac{c_1}{F(y)}\right) + \left(F\left(\underline{b_2}\right) - F\left(\frac{c_1}{F(y)}\right)\right)\left(y - \underline{b_2}\right)$$

- To consider the existence of nonmonotonic equilibria in which $v_2^* < v_1^*$ whenever $c_1 < c_2$, we follow the above process similarly and at equilibrium get

$$c_2 = v_2^* F\left(v_1^*\right)$$

$$c_1 \geqslant v_1^* F\left(v_2^*\right) + \left(F\left(\underline{b_1}\right) - F\left(v_2^*\right)\right)\left(v_1^* - \underline{b_1}\right)$$

  where the equality holds whenever $v_1^* \leq 1$.
- To find a nonmonotonic equilibrium, we define the two cutoff reaction functions

$$y\left(x\right) = \frac{c_2}{F\left(x\right)}$$

$$\varphi\left(x\right) = xF\left(\frac{c_2}{F\left(x\right)}\right) + \left(F\left(\underline{b_1}\right) - F\left(\frac{c_2}{F\left(x\right)}\right)\right)\left(x - \underline{b_1}\right)$$

- To find a nonmonotonic equilibrium, we define the two cutoff reaction functions

$$y\left(x\right)=\frac{c_2}{F\left(x\right)}$$

$$\varphi\left(x\right)=xF\left(\frac{c_2}{F\left(x\right)}\right)+\left(F\left(\underline{b_1}\right)-F\left(\frac{c_2}{F\left(x\right)}\right)\right)\left(x-\underline{b_1}\right)$$

  Again, we use x to correspond to v_1^* and y to correspond to v_2^*. Note that we have $x \geq y \geq c_2$.

- Let c_m be the minimum of $\varphi\left(x\right)=xF\left(\frac{c_2}{F(x)}\right)+\left(F\left(\underline{b_1}\right)-F\left(\frac{c_2}{F(x)}\right)\right)\left(x-\underline{b_1}\right)$.

Proposition in Two Bidders

Proposition 1: (Existence and Uniqueness Theorem)

For the independent private values economic environment with two bidders who have different participation costs $c_2 > c_1$, we have the following conclusions:

(1) There always exists a monotonic equilibrium.

(2) Suppose $F(\cdot)$ is concave. Then the equilibrium is unique and monotonic.

(3) Suppose $F(\cdot)$ is strictly convex. Then

(3i) the monotonic equilibrium is unique when the reverse hazard rate of $F(\cdot)$, that is, when $\frac{f(\cdot)}{F(\cdot)}$ is nonincreasing,

(3ii) the nonmonotonic equilibrium is unique when $c_1 = c_m$,

(3iii) there is no nonmonotonic equilibrium when $c_1 < c_m$, and

(3iv) there are at least two nonmonotonic equilibria when $c_m < c_1 < c_2$.

Remarks

- For any power functions $F(\cdot)$ that are convex, the reverse hazard rate is a nonincreasing function. Thus, the set of such strictly convex functions is not empty. To understand why there is a unique monotonic equilibrium for this type of strictly convex distribution, see Figure 1. $\lambda(y)$ starts from v_1^s with negative slope. When $\lambda'(y)$ equals 0 at most once, $\lambda(y)$ intersects with c_2 at most once, indicating that the monotonic equilibrium is unique.
- From the proof in the Appendix, one can see $c_2 > c_m$. Then, as long as $c_2 - c_1$ is sufficiently small, we have $c_2 > c_1 > c_m$. Thus, we can conclude that when $c_2 - c_1$ is sufficiently small, there are two nonmonotonic equilibria that are given by (x_1, y_1) and (x_2, y_2) with $y_1 = y\left(x_1\right), y_2 = y\left(x_2\right)$, and $y_1 < y_2 < v_2^s < x_1 < x_m < x_2 < x_0$. Thus, when $F(\cdot)$ is strictly convex, the existence of nonmonotonic equilibrium depends on the difference of participation costs, $c_2 - c_1$.

Remarks

- Figure 2 can help us to understand the proof in the Appendix and the points mentioned above. $\varphi(x)$ starts from $y = v_2^s$ with negative slope. When $c_2 - c_1$ is small enough, it intersects with c_1; that is, a nonmonotonic equilibrium exists. When $c_2 - c_1$ is big enough so that $c_1 < c_m, \varphi(x)$ and c_1 cannot intersect; that is, no nonmonotonic equilibrium exists. From the figure, when c_1 is close to c_2, there are at least two intersection points for $y = \varphi(x)$ and $y = c_1$, which means there are at least two nonmonotonic equilibria, say, (x_1, y_1) and (x_2, y_2)

Remarks

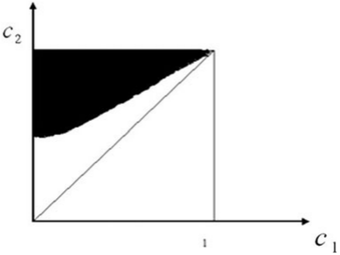

FIGURE 1. UNIQUENESS FOR CONVEX CASE

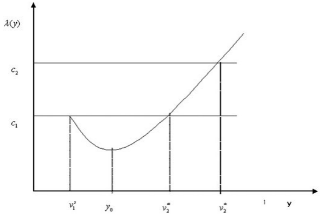

FIGURE 2. EXISTENCE OF NONMONOTONIC EQUILIBRIA FOR CONVEX CASE

Remarks

- Our result shows that the strict convexity of the distribution function alone is not a sufficient condition for the existence of a nonmonotonic equilibrium, unless the difference $c_2 - c_1$ is small enough, which implies that one can refine equilibria and always eliminate nonequilibria by making participation costs for bidders sufficiently different when necessary.
- In the proof of Proposition 1 , the condition that $F(\cdot)$ is concave can be weakened to $\frac{F(v)}{v}$ nondecreases for all $v \in [c_1, 1]$, and the condition that $F(\cdot)$ is strictly convex can be weakened to $\frac{F(v)}{v}$ decreases with v for all $v \in [c_2, 1]$.

Remarks

- When multiple equilibria exist and $F(\cdot)$ is nomonotonic, there cannot exist mixed strategies in which a bidder uses different cutoffs with positive probability. Indeed, if one bidder behaves in this way, the expected payoff from participating of his opponent can be uniquely determined, which is still a nondecreasing function of his valuation and thus there is only one cutoff.
- A non-truth-telling equilibrium may exist when bidders do not use dominant strategies. Suppose bidder 1 bids zero and bidder 2 bids one when they participate. Bidder 1 wins only when bidder 2 does not enter, hence at equilibrium $v_1^* F\left(v_2^*\right) = c_1$. Bidder 2 always wins once he enters and pays nothing. At equilibrium we have $v_2^* = c_2$. Thus $v_1^* = \frac{c_1}{F(c_2)}$. Therefore, if bidders do not use dominant bidding strategy, we have other cutoff equilibria.

Remarks

- Existence of a reserve price r does not affect parts (1) and (2) of Proposition 1. It can be easily shown that the condition of strict convexity of $F(\cdot)$ in part (3) needs to be replaced by $F\left(v_2'^s\right) - \left(v_2'^s - r\right) f\left(v_2'^s\right) < 0$, where $v_2'^s$ is the revised symmetric cutoff equilibrium when both bidders have participation cost c_2, which is defined by $\left(v_2'^s - r\right) F\left(v_2'^s\right) = c_2$.

Wonder

- Consider the difference between c_1 and c_2. If the difference is too large, which means, $c_1 < c_m < c_2$, there is no nonmontonic equilibrium. Only the difference is sufficiently small can the nonmontonic equilibrium exists.
- One may wonder what would happen at the limits of monotonic and nonmonotonic equilibria as $c_2 - c_1 \to 0$. Should a monotonic equilibrium converge to a symmetric equilibrium ?

Proposition in Two Bidders

PROPOSITION 2 (Limit Theorem)

For the independent private values economic environment with two bidders having participation costs $c_2 > c_1$, we have the following conclusions:
(1) Suppose $F(\cdot)$ is concave. The unique monotonic equilibrium (no nonmonotonic equilibrium) converges to the unique symmetric equilibrium as $c_2 - c_1 \to 0$.
(2) Suppose $F(\cdot)$ is strictly convex with nonincreasing reverse hazard rate. The unique monotonic equilibrium converges to an asymmetric equilibrium as $c_2 - c_1 \to 0$.
(3) Suppose $F(\cdot)$ is strictly convex. When $c_2 - c_1 \to 0$, there are two nonmonotonic equilibria, of which one converges to the unique symmetric equilibrium and the other converges to an asymmetric equilibrium.

Proposition in Two Bidders

PROPOSITION 3 (Comparative Static Theorem)

For the independent private values economic environment with two bidders, suppose the values of bidders are drawn from a distribution function $F(\cdot)$ and the participation costs c_1 and c_2 are common knowledge. Then for the monotonic equilibrium, an increase in participation cost c_i increases i' s cutoff v_i^* but decreases the opponent's cutoff v_j^* for $j \neq i$. Specially, when $F(\cdot)$ is concave, which gives us a unique and monotonic equilibrium, an increase in participation cost c_i increases i 's cutoff v_i^* but decreases the opponent's cutoff v_j^* for $j \neq i$.

Extension

- TYPE ASYMMETRIC EQUILIBRIA
- BIDDERS WITH DIFFERENT VALUATION DISTRIBUTIONS
- POSITIVE LOWER BOUND OF SUPPORTS

Type Asymmetric Equilibria

- I give a brief discussion on allowing asymmetric cutoffs within a group. To allow such a possibility, suppose there are only two different cutoff points used by bidders. Bidders $i = 1, \ldots, m$ with participation cost c_1 use v_1^* and bidders $j = m+1, \ldots, n$ with participation cost c_2 use v_2^* as the cutoff point, with $c_1 < c_2$, which are defined as type1 and 2. Initially, consider the monontonic equilibrium, we assume $v_1^* < v_2^*$. Then at equilibrium, we have

$$c_1 = v_1^* F\left(v_1^*\right)^{m-1} F\left(v_2^*\right)^{n-m}$$

Type Asymmetric Equilibria

- For bidder $j \in \{m+1, \ldots, n\}$ with $v_j = v_2^*$, he bids zero and has revenue v_2^* when none of the others enters the auction. If any other bidder in type 2 enters the auction, he will lose the bid. If only $k \leqslant m$ bidders in type 1 enter the auction, the optimal bid $\underline{b}_k$ for bidder j is determined by

$$\underline{b}_k = \max \arg\max_b \left(F(b) - F\left(v_1^*\right)\right)^k \left(v_2^* - b\right).$$

- The first order condition for $\underline{b}_k$ gives

$$\underline{b}_k + \frac{F\left(\underline{b}_k\right) - F\left(v_1^*\right)}{kf\left(\underline{b}_k\right)} = v_2^*.$$

Type Asymmetric Equilibria

- $\underline{b}_k$ is chosen with probability $C_m^k F\left(v_1^*\right)^{m-k}\left(1-F\left(v_1^*\right)\right)^k . C_m^k=\frac{m!}{k!(m-k)!}$ is the combination number for choosing k candidates from the m items that are available.
- Thus, at equilibrium, we have

$$c_2 \geqslant v_2^* F\left(v_1^*\right)^m F\left(v_2^*\right)^{n-m-1}+F\left(v_2^*\right)^{n-m-1} \sum_{k=1}^m C_m^k F\left(v_1^*\right)^{m-k}\left(F\left(\underline{b}_k\right)-F\left(v_1^*\right)\right)^k\left(v_2^*-\underline{b}_k\right)$$

where the first part is the expected revenue when none of the others enters the auction, which happens with probability $F\left(v_1^*\right)^m F\left(v_2^*\right)^{n-m-1}$; the second part is the expected revenue when no other bidders in type 2 participate in the auction and there are exactly $k \leqslant m$ bidders in type 1 in the auction, which happens with probability $F\left(v_2^*\right)^{n-m-1} F\left(v_1^*\right)^{m-k}$. The inequality holds whenever bidders in type 2 do not participate in the auction, i.e., $v_2^*>1$.

Type Asymmetric Equilibria

- Additionally, consider the nonmonontonic equilibrium, we assume $v_1^* > v_2^*$.
  Similarly, at equilibrium we have

$$c_2 = v_2^* F\left(v_1^*\right)^m F\left(v_2^*\right)^{n-m-1}$$

and

$$c_1 \geq v_1^* F\left(v_1^*\right)^{m-1} F\left(v_2^*\right)^{n-m} +$$

$$F\left(v_1^*\right)^{m-1} \sum_{k=1}^{n-m} C_{n-m}^k F\left(v_2^*\right)^{n-m-k} \left(F\left(\underline{b_k}\right) - F\left(v_2^*\right)\right)^k \left(v_1^* - \underline{b_k}\right)$$

Type Asymmetric Equilibria

Proposition 4

In an economic environment with bidders of two types,
(1)There exists monontonic asymmetric equilibrium at which $m \leqslant n-1$ bidders use the cutoff point v_1^* and the others use the cutoff point v_2^* that satisfy

$$c_1 = v_1^* F\left(v_1^*\right)^{m-1} F\left(v_2^*\right)^{n-m}$$

$$c_2 \geq v_2^* F\left(v_1^*\right)^{m} F\left(v_2^*\right)^{n-m-1} +$$

$$F\left(v_2^*\right)^{n-m-1} \sum_k = 1^m C_m^k F\left(v_1^*\right)^{m-k} \left(F\left(b_k\right) - F\left(v_1^*\right)\right)^k \left(v_2^* - b_k\right)$$

with equality whenever $v_2^* \leqslant 1$ and $v_1^* < v_2^*$, where

$$\underline{b}_k = \max \arg\max_b \left(F(b) - F\left(v_1^*\right)\right)^k \left(v_2^* - b\right), C_m^k = \frac{m!}{k!(m-k)!}$$

Type Asymmetric Equilibria

Proposition 4

(2)If $F(\cdot)$ is strictly convex.
(2i) the nonmonotonic equilibrium is unique when $c_n = 0$,
(2ii) there is no nonmonotonic equilibrium when $c_n > 0$, and
(2iii) there are at least two nonmonotonic equilibria when $c_n < 0$.

Bidders With Different Valuation Distributions

- Now consider the case where we have a strong bidder 1 with value distribution $F_1(\cdot)$ and a weak bidder 2 with value distribution $F_2(\cdot)$ so that $F_1(v) < F_2(v)$ for all $v \in (0, 1)$. There participation cost are c_1 and c_2, with $c_1 < c_2$. We investigate the existence of equilibria and equilibrium behaviour.
- First assume, provisionally, that the cutoff points v_1^* and v_2^* satisfy $v_1^* < v_2^*$. Then for a strong bidder i with $v_i = v_1^*$, he can win the object only when all the other strong and weak bidders do not participate in the auction. (If any strong bidder i' enters the auction, he must have a value greater than v_1^* and thus bids higher than bidder i; or if any weak bidder j enters, then it must be the case that $v_j \geqslant v_2^* > v_1^*$. As shown in the previous section, bidder i will lose the item for sure.) Thus, at equilibrium we have

$$c_1 = v_1^* F_2\left(v_2^*\right)$$

Bidders With Different Valuation Distributions

- For a weak bidder j with $v_j = v_2^*$,at equilibrium we have

$$c_2 \geqslant v_2^* F_1(v_1^*) + \left(F_1(\underline{b_2}) - F_1(v_1^*)\right)\left(v_2^* - \underline{b_2}\right)$$

define

$$\lambda(y) = yF_1(x) + \left(F_1(\underline{b_2}) - F_1(x)\right)\left(y - \underline{b_2}\right)$$

- Similarly, for the existence of a nonmontonic equilibrium, considering the following two equations:

$$c_2 = v_2^* F_1(v_1^*)$$

$$c_1 \geqslant v_1^* F_2(v_2^*) + \left(F_2(\underline{b_1}) - F_2(v_2^*)\right)\left(v_1^* - \underline{b_1}\right)$$

define

$$\phi(x) = xF_2(y) + \left(F_2(\underline{b_1}) - F_2(y)\right)\left(x - \underline{b_1}\right)$$

Bidders With Different Valuation Distributions

Proposition 5 (Existence and Uniqueness Theorem)

For a two-bidder economy with different continuously differentiable distribution functions $F_1(v)$ and $F_2(v)$ and different costs $c_1 < c_2$, we have the following results:
(1) There always exists an equilibrium (v_1^*, v_2^*).
(2) Suppose $F_1(\cdot)$ and $F_2(\cdot)$ are both concave and $F_1(v) < F_2(v)$ for all $v \in (0,1)$. Then there exists a unique equilibrium and it is monotonic.
(3) Suppose $F_1(\cdot)$ and $F_2(\cdot)$ are both concave and $F_1(v) > F_2(v)$ for all $v \in (0,1)$. Let v_1^s and v_2^S satisfy $c_1 = v_1^S F_2\left(v_1^S\right)$ and $c_2 = v_2^S F_1\left(v_2^S\right)$, respectively. Then, we have
(i) If $v_1^s < v_2^s$, there is a unique equilibrium and it is monotonic;
(ii) If $v_1^s > v_2^s$, there is a unique equilibrium and it is nonmonotonic, satisfying $v_1^* > v_2^*$;
(iii) If $v_1^s = v_2^s = v^s$, there is a unique equilibrium and it is a special nonmonotonic equilibrium, satisfying $v_1^* = v_2^* = v^s$.

Positive Lower Bound of Support

- The support of valuations affects the existence of equilibria. When the lower bound of the support of the valuation is not zero, there may be an equilibrium in which one bidder always participates in the auction and the other never participates in the auction.

Positive Lower Bound of Support

Suppose the support of the distribution function $F(\cdot)$ is $[v_l, v_h]$. There are six cases for consideration.

- $v_l < v_h < c_1 < c_2$. It is clear that both bidders never participate in the auction.
- $v_l < c_1 < v_h < c_2$. Bidder 2 never participates in the auction. Bidder 1 participates in the auction if $v_1 \geq c_1$ and does not participate otherwise.
- $c_1 < v_l < v_h < c_2$. Bidder 2 never participates, and bidder 1 always participates.
- $v_l < c_1 < c_2 < v_h$. The analysis and results are the same as those in Section 3 that deal with the special case where $v_l = 0$ and $v_h = 1$

Positive Lower Bound of Support

- $c_1 < v_l < c_2 < v_h$. We may have an equilibrium in which bidder 1 always participates, and bidder 2 never participates. For this to be true, we need $v_h - v_l < c_2$; that is, the maximum revenue bidder 2 gets from participating in the auction must be smaller than his participation cost. When $c_2 \leq v_h - v_l$, bidder 2 will choose a cutoff $v_2^* \in [c_2, v_h]$. If there is an equilibrium in which bidder 1 never participates, then bidder 2 uses $v_2^* = c_2$. To have such an equilibrium, we need
  $c_1 > v_h F(c_2) + \left(F\left(\underline{b_1}\right) - F(c_2)\right)\left(v_h - \underline{b_1}\right)$
  A sufficient condition for this is

$$v_h + c_2 F(c_2) < c_1 + c_2$$

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Positive Lower Bound of Support

- $c_1 < c_2 < v_l < v_h$. It is possible to have an equilibrium in which bidder 1 always participates in the auction, and bidder 2 never participates. For this to be an equilibrium, we need $v_h - v_l < c_2$. Another possible equilibrium is bidder 2 always participates in the auction, and bidder 1 never participates. For this to be an equilibrium, we need $v_h - v_l < c_1$. When both bidders choose a cutoff inside the support of valuations, we can use the same analysis as in Section 3 to investigate the equilibrium and the corresponding properties.

Conclusion

This paper investigates equilibria of first-price auctions when bidders have private valuations and different participation costs that are common knowledge. We identify two types of equilibria: monotonic and nonmonotonic equilibria. We show that there always exists an equilibrium that is monotonic, and further that, it is unique when $F(\cdot)$ is concave or strictly convex with nonincreasing reverse hazard rate.

The End