Explicit Reformulation of the Colebrook−White Equation for Turbulent Flow Friction Factor Calculation

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In this paper, we present an improvement of a mathematically equivalent representation of the Colebrook−White (CW) equation to compute friction factors for turbulent flow in rough pipes. This new form is simple and very well-suited for accurately estimating the friction factor, because no iterative calculations are necessary. Specifically, the friction factor is expressed as the sum of known simple functions and an unknown correction term. This correction term satisfies an auxiliary equation that can be accurately and easily solved with predictable error bounds over the complete range of pipe roughness and Reynolds number values encountered in practice. The simplest case, with the unknown correction term set to zero, resulted in friction factor estimates with errors of <1%. A simple linear approximation of the correction term resulted in a maximum error of 3.64×10^{-4}%, whereas friction factor estimates from a continued-fractions-based approximation had a maximum error of 1.04×10^{-10}%. These maximum errors are significantly lower than any of the explicit approximations that have been proposed to date for the CW equation. The equation presented in this study is entirely theoretical and eliminates the need for best-fit parameters or complicated initial guesses that are an integral component of the various other approximations proposed to date. The simplicity with which this new equation can be solved, coupled with its smooth and predictable error behavior, should make it the method of choice for estimating turbulent flow friction factor in rough pipes.

1. Introduction

The Colebrook−White (CW) equation[1] has been widely used to estimate the friction factor for turbulent fluid flow in rough pipes. This equation, which relates the friction factor to pipe roughness and the Reynolds number (Re), is implicit in the friction factor:

$$\frac{1}{\sqrt{f}} = -2\log_{10}\left(\frac{\epsilon/D}{3.7} + \frac{2.51}{Re\sqrt{f}}\right) \tag{1}$$

where f is the friction factor, ϵ/D the pipe roughness, and Re the Reynolds number. Although alternate formulations have been proposed,[2] the CW relation remains the most widely used approach for estimating the friction factor for turbulent flow in rough pipes.

Recognizing the implicit nature of the CW equation, f vs Re charts were constructed early in the studies on the field[3] that allowed for a graphical estimation of f when the Re and ϵ/D values were known. Since then, several approximations to the CW equation that are explicit in f have been proposed.[4−16] These approximations vary in the degree of accuracy and complexity, and comparisons of some of these approaches are available.[11,13,16,17]

All of the aforementioned approximations are empirical in nature and do not provide a truly explicit representation of the CW equation. Recently, computer algebra was used to derive an explicit relationship between the friction factor f and the pipe roughness ϵ/D and the Re, in terms of the Lambert W function.[18] Although this solution is explicit in f, the CW equation expressed in this form has limited applicability, because it can be used over only a limited portion of ϵ/D and Re values encountered commonly in practice.[19] (See sections A.1 and A.2 in the Appendix.) This important aspect of the explicit solution was not considered in the earlier work.[18]

In this study, we present an improvement of an alternate form of the CW equation.[20] This representation is not an approximation but is mathematically equivalent to the CW equation. This equation is valid over the complete range of ϵ/D and Re values and is well-suited for accurate estimation of f, because no iterative solution techniques are necessary. Information on the equation, its solution, and accuracy is presented first, followed by error estimates and their predictability. The derivation of this equation has been presented in the Appendix.

2. Theory

We will present a mathematically equivalent form of the CW equation that is well-suited for numerical solution. To facilitate the derivation and avoid carrying numerical constants, several substitutions will be made in eq 1.

Let

$$a = 2\log_{10}(e) = \frac{2}{\ln(10)} \tag{2a}$$

$$b = \frac{\epsilon/D}{3.7} \tag{2b}$$

and

$$c = \frac{2.51}{Re} \tag{2c}$$

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Equation 1 may be written, in terms of a, b, and c, as

$$\frac{1}{\sqrt{f}} = -a \ln\left(b + \frac{c}{\sqrt{f}}\right) \tag{3}$$

Let

$$d = \frac{1}{ac} = \left(\frac{\ln(10)}{5.02}\right)Re$$

and

$$u = \frac{1}{a\sqrt{f}} \tag{4}$$

Using the aforementioned substitutions, eq 3 can be expressed as

$$u = -\ln\left(b + \frac{u}{d}\right) \tag{5}$$

2.1. Alternate Formulation of the CW Equation. We propose a new form for the friction factor variable, u:

$$u = \ln\left(\frac{d}{q}\right) + \delta \tag{6}$$

where $s = bd + \ln(d)$, $q = s^{s/(s+1)}$, and δ satisfies the implicit equation

$$\delta + \ln\left(1 + \frac{\delta}{g}\right) = \ln\left(\frac{q}{g}\right) \tag{7}$$

with $g = s - \ln(q)$ or equivalently $g = bd + \ln(d/q)$. The variables q and g are functions of s and all the three variables (s, q, and g) are positive. Additional information on the derivation of eq 6 is presented in sections A.3 and A.4 of the Appendix.

Solution of eq 6 is mathematically equivalent to the solution of the CW equation. The equivalence may be observed by substituting for u in terms of δ in eq 5 or by substituting for δ in terms of u in eq 7, as shown below. Using the properties of logarithms, eq 7 may be written as

$$\delta + \ln\left(\frac{d}{q}\right) + \ln\left(\frac{g + \delta}{d}\right) = 0 \tag{8}$$

Substituting for g, in terms of b, d, and q, and simplifying gives

$$\delta + \ln\left(\frac{d}{q}\right) + \ln\left(\frac{bd + \delta + \ln(d/q)}{d}\right) = 0 \tag{9}$$

The common sum, $\delta + \ln(d/q)$, may be replaced by u (eq 6), and, after some simplification, eq 9 reduces to the CW equation:

$$u + \ln\left(b + \frac{u}{d}\right) = 0 \tag{10}$$

Equation 6 separates the solution for u into a known explicit component, $\ln(d/q)$, and an unknown component (δ) that satisfies eq 7, an implicit expression. The explicit component, which can be readily computed, is the dominant portion of the solution and may be considered as an initial guess for u, whereas δ must be determined. Hence, solution of the CW equation has been reduced to solving eq 7 for δ. The usefulness of this approach can be questioned, because we are essentially shifting from the original implicit equation (eq 1) to another implicit equation (eq 7). However, the solution for δ is reduced to approximating $\ln(1 + (\delta/g))$, where the absolute value of δ/g is $\ll 1$. A first-order Taylor series approximation provides highly accurate solutions to the convergent series expansion of the logarithmic term. In addition, the contribution of δ to the value of u is <1%, as will be shown later. Thus, besides being an exact representation of the CW equation, eq 6 also provides accurate estimates of f, as will be shown in subsequent sections.

2.2. Solution of the New Equation for $\boldsymbol{\delta}$. Equation 7 may be written in a form suited for numerical solution as

$$\delta + \ln\left(1 + \frac{\delta}{g}\right) = z \tag{11}$$

where $z = \ln(q/g)$. The only unknown quantity is δ and the other quantities, q and g, are functions of a single variable s. A solution to eq 11 requires an approximation for the logarithmic term. As shown in Appendix A.4, g is positive and the choice of g has minimized δ, resulting in $|\delta|/g \ll 1$. Consequently, many approximations that provide accurate solutions to eq 11 are available, and we will consider two of them.

A simple and obvious approach consists of approximating $\ln[1 + (\delta/g)]$ by the linear term of the convergent Taylor series expansion. Equation 11 may be written as

$$\delta + \frac{\delta}{g} \simeq z \tag{12}$$

Thus, an expression for δ can be written as

$$\delta_{\mathrm{LA}} = \left(\frac{g}{g + 1}\right)z \tag{13}$$

This approximation is called the linear approximation (LA) and, as expected, results in an error on the order of $(\delta/g)^2$. A much more accurate solution may be obtained through continued-fractions approximations (CFA) of the logarithmic term,[21] where the resulting error is proportional to $(\delta/g)^4$. The solution for δ, using the CFA, is

$$\delta_{\mathrm{CFA}} = \delta_{\mathrm{LA}}\left[1 + \frac{z/2}{(g + 1)^2 + (z/3)(2g - 1)}\right] \tag{14}$$

The equations involved in estimating the friction factor f, using the above approach, are presented in Table 1, where a complete listing of the variables involved is also presented for completeness. If necessary, iterations may be performed to further improve the accuracy of the solution. However, as observed in the Results section, the accuracy obtained without iterations are adequate for all practical applications. Details of the iterative process and related convergence issues will be discussed in a separate publication.

3. Results

3.1. Discretization of the ϵ/D and Re Space. The accuracy of eq 6 for estimating f from the CW equation was tested over a rectangular space of ϵ/D and Re values. A set of 20 ϵ/D values, corresponding to those used by Moody,[3] were selected, and these spanned a range from 10^{-6} to 5×10^{-2}. For each ϵ/D value, 500 Re values, distributed uniformly in the logarithmic space covering a range from 4×10^3 to 10^8 were chosen. Accuracy of the solution at these 10 000 points (20 × 500 ϵ/D and Re values) was used to establish the magnitude of the errors in f.

3.2. Verification of the Accuracy of the Solution. Friction factor values were computed over the 20 × 500 grid of ϵ/D and Re values, using three different approaches for the computa-

Table 1. Listing of Equations Used to Compute the Friction Factor *f*, Using the Approach Presented in This Study

1. The Original Colebrook−White (CW) Equation

$$u = -\ln\left(b + \frac{u}{d}\right)$$

$$u = \frac{1}{a\sqrt{f}}$$

$$a = \frac{2}{\ln(10)}$$

$$b = \frac{\epsilon/D}{3.7}$$

$$d = \left(\frac{\ln(10)}{5.02}\right)Re$$

2. Alternate Mathematically Exact Representation of the CW Equation

$$u = \ln\left(\frac{d}{q}\right) + \delta \quad \text{or} \quad \frac{1}{\sqrt{f}} = a\left[\ln\left(\frac{d}{q}\right) + \delta\right], \text{ where } s = bd + \ln(d) \text{ and } q = s^{[s/(s+1)]}$$

3. Equation for *δ*

$$\delta + \ln\left(1 + \frac{\delta}{g}\right) = z, \quad \text{where} \quad z = \ln\left(\frac{q}{g}\right) \quad \text{and} \quad g = bd + \ln\left(\frac{d}{q}\right)$$

4. Solutions for *δ*

(1) linear approximation (LA)	$\delta_{LA} = \left(\frac{g}{g+1}\right)z$
(2) continued-fractions approximation (CFA)	$\delta_{CFA} = \delta_{LA}\left[1 + \frac{z/2}{(g+1)^2 + (z/3)(2g-1)}\right]$

tion of δ. These included setting $\delta = 0$; computing δ from the linear approximation (δ_{LA}; see eq 13); and estimating δ using the CFA (δ_{CFA}; see eq 14). To estimate the errors associated with these approaches, error-free values *f* values were necessary that would serve as the reference. To obtain these error-free *f* values, δ values from the CFA were improved using a single iteration, and arbitrary precision calculations revealed that the resulting *f* values were characterized by a maximum absolute error of 3.7×10^{-52}. Therefore, these values were chosen to be the true values of *f* against which *f* estimates for $\delta = 0$, δ_{LA}, and δ_{CFA} were compared. All computations were performed using arbitrary arithmetic precision and the error in *f* was estimated as

$$\text{Absolute Error} = f_{\text{true value}} - f_{\text{estimated}} \qquad (15a)$$

$$\text{Percentage Error} = \left(\frac{f_{\text{true value}} - f_{\text{estimated}}}{f_{\text{true value}}}\right) \times 100 \qquad (15b)$$

It is important to note that the accuracy of *f* estimates from the iterative improvement of δ_{CFA} extends well beyond the range of commonly used double precision calculations, and this approach is not necessary for routine estimation of *f*. It has only been used here to provide error-free values of *f* that serve as a reference to evaluate the error associated with estimating *f* using the methods that we propose in this study.

A plot of the maximum error in *f* for each of the 20 ϵ/D values is shown in Figure 1 for the three different approaches through which δ was calculated. It is important to note that 500 values of *f* were computed (corresponding to 500 *Re* values logarithmically spaced in the range of 4×10^3 to 10^8) for each value of ϵ/D, and only the maximum of these 500 values is shown in Figure 1. Errors in *f* decrease with increasing ϵ/D values, with the rate of decrease being slow in the range of $10^{-6} < \epsilon/D < 10^{-3}$, followed by a rapid decline for further increases in ϵ/D values. The error profiles were smooth in all cases and the maximum absolute error values for *f* estimates across the 20×500 grid of ϵ/D and *Re* values were 2.91×10^{-2}, 1.07×10^{-5}, and 3.04×10^{-12} for the $\delta = 0$, δ_{LA}, and δ_{CFA} cases, respectively. Thus, for these maximum error values, a two-orders-of-magnitude improvement in accuracy was observed for δ_{LA} over $\delta = 0$, whereas a much larger seven-orders-of-magnitude improvement was observed for δ_{CFA} over δ_{LA}.

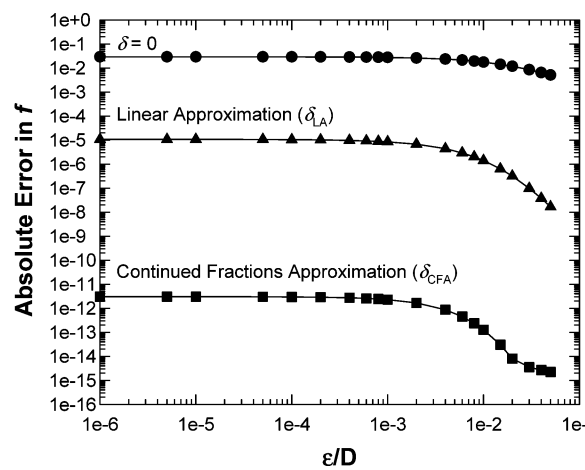

Figure 1. Maximum absolute error estimates during estimation of the friction factor (*f*), using the methods presented in this study. The three cases correspond to $\delta = 0$, δ estimated from eq 13 using the linear approximation (δ_{LA}), and δ estimated from eq 14 using the continued fractions approximation (δ_{CFA}). The maximum absolute errors in *f* across the 20×500 grid of ϵ/D and *Re* values were 2.91×10^{-2}, 1.07×10^{-5}, and 3.04×10^{-12} for the $\delta = 0$, δ_{LA}, and δ_{CFA} cases, respectively.

4. Discussion

The discussion is divided into three subsections. We start with a comparison of the accuracy of the new method with existing explicit approximations of the CW equation. Subsequently, a detailed error analysis is presented for computation of δ using the linear approximation, and it will be shown that the error bounds in *f* are predictable for the approach presented in this study. Finally, some recommendations will be made for practical

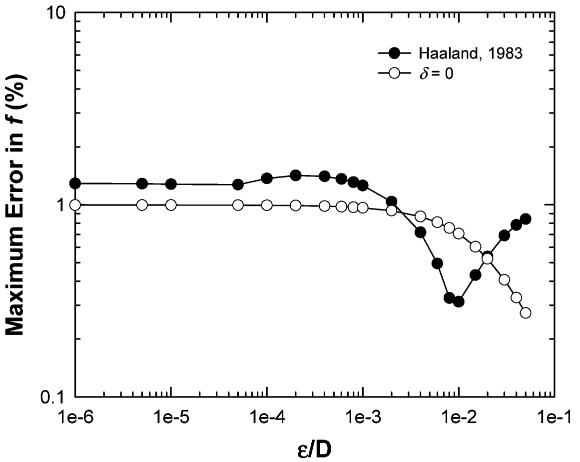

Figure 2. Comparison of maximum percent errors in f for the $\delta = 0$ case and the best currently available non-iterative explicit approximation of the CW equation (Haaland;[7] see eq 16). The maximum percent error in f across the 20 × 500 grid of ϵ/D and Re values was 1% for the $\delta = 0$ case and 1.42% for the Haaland equation.

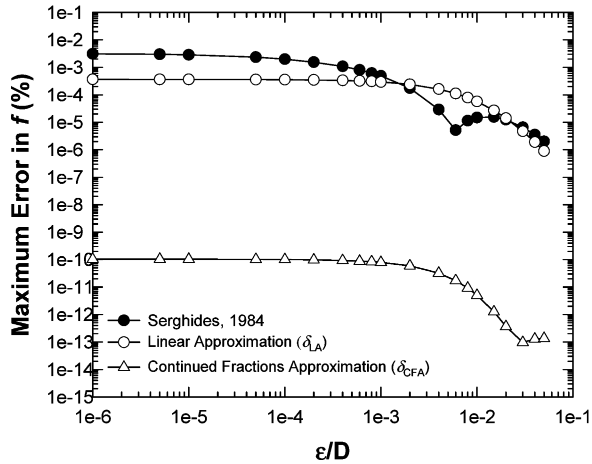

Figure 3. Comparison of the maximum percent error in f for the δ_{LA} and δ_{CFA} cases with the best currently available iterative explicit approximation of the CW equation (Serghides;[13] see eq 17). The maximum percent errors in f across the 20 × 500 grid of ϵ/D and Re values were 3.64×10^{-4}% and 1.04×10^{-10}% for the δ_{LA} and δ_{CFA} cases, respectively, and 3.1×10^{-3}% for the Serghides equation.

utilization of the solution techniques that have been presented in this work.

4.1. Comparison of the New Formulation with Existing Explicit Approximations of the CW Equation. Numerous explicit approximations of the CW equation have been presented, and several reviews of these approximations are available.[11,13,16,17] Although we have made an extensive comparison of all currently available approximations of the CW equation, with the approach presented in this study, only a brief summary is presented here. The currently available approximations can be broadly classified into iterative and non-iterative methods, depending on the way they were developed and their application. The non-iterative methods[6,7,9,10,12,14,15] typically have 1−3 parameters that were obtained though curve-fitting or other empirical approaches. The iterative approximations[4,5,11,13,16] involved 1−3 iterations during the solution process and provide more-accurate estimates of f than those obtained from the non-iterative approximations.

The accuracy of all the above-mentioned approximations was investigated over the 20 × 500 grid of ϵ/D and Re values. Of all the non-iterative approximations, the method proposed by Haaland[7] was the best, with a maximum error in f of 1.42%.

$$\frac{1}{\sqrt{f}} = -1.8 \log_{10}\left[\left(\frac{\epsilon/D}{3.7}\right)^{1.11} + \frac{6.9}{Re}\right] \tag{16}$$

Among the iterative approximations, the method of Serghides,[13] which involves three iterations, was the most accurate, with a maximum error in f of 3.1×10^{-3}%.

$$f = \left\{A - \frac{(B-A)^2}{(C - 2B + A)}\right\}^{-2} \tag{17a}$$

$$A = -2 \log_{10}\left(\frac{\epsilon/D}{3.7} + \frac{12}{Re}\right) \tag{17b}$$

$$B = -2 \log_{10}\left(\frac{\epsilon/D}{3.7} + \frac{2.51A}{Re}\right) \tag{17c}$$

$$C = -2 \log_{10}\left(\frac{\epsilon/D}{3.7} + \frac{2.51B}{Re}\right) \tag{17d}$$

Figure 2 shows the maximum percentage errors in f, as a function of ϵ/D, for the Haaland equation and the $\delta = 0$ case. The simplest approach presented in this study ($\delta = 0$), with a maximum error of 1%, is more accurate than the best currently available non-iterative approximation of the CW equation (eq 16 with a maximum f error of 1.42%). When δ was computed using the simple linear approximation (δ_{LA}), the maximum error in f was 3.64×10^{-4}% (see Figure 3), which clearly shows its superiority over the best currently available iterative explicit approximation of the CW equation (eq 17, with a maximum f error of 3.10×10^{-3}%). When δ was computed using the CFA, the resulting estimates of f were ~7 orders of magnitude more accurate than those obtained from eq 17 (see Table 2). In addition to the improved accuracy of the methods proposed in this study over the currently available approximations, it is also important to note that their error properties are not arbitrary, and this will be examined in detail in the subsequent section.

4.2. Error Analysis of the Solution Presented in this Study. The error in the friction factor variable, u (and, hence, in the friction factor f) results from the error in the solution of the equation for δ (eq 7). This error in the solution of δ is determined by the accuracy in approximating the logarithmic term in eq 7, which, in turn, is determined by the magnitude of δ/g. We will first estimate the magnitudes of δ/g and relate them to the resulting errors in the approximation of the logarithmic term. This will be followed by error estimates in δ that result from the solution of eq 7.

4.2.1. Estimation of the Magnitude of δ/g. The linear approximation will be used to obtain an estimate of the magnitude of δ/g. The linear approximation, presented as eq 13, can be rearranged as

$$\frac{\delta}{g} = \frac{z}{g+1} \tag{18}$$

where $s = bd + \ln(d)$ and b and d are constants that are proportional to ϵ/D and Re, respectively (see eqs 2 and 4). The magnitude of δ/g is a function of s and q. Let us examine the behavior of s over the range of ϵ/D (from 10^{-6} to 5×10^{-2}) and Re (from 4×10^3 to 10^8) values encountered in practice. Over this space of parameter values, s reaches its minimum value on the boundary where ϵ/D and Re are the smallest ($\epsilon/D = 10^{-6}$; $Re = 4 \times 10^3$). The smallest s value is ~7.515. The value of s increases as ϵ/D or Re increases. It reaches a maximum value for the maximum pipe roughness and Reynolds number ($\epsilon/D = 5 \times 10^{-2}$; $Re = 10^8$) and is ~6.2×10^5. Hence,

Table 2. Maximum Errors in Friction Factor Estimates from 10 000 *f* Values Spanning a 20 × 500 Grid of ϵ/D and Reynolds Number (*Re*) Values Using the Methods Presented in This Study, along with Those from the Best Currently Available Non-iterative and Iterative Explicit Approximations of the CW Equation

Friction Factor Estimation Method	Maximum Absolute Error in Friction Factor Estimates	Maximum Percentage Error in Friction Factor Estimates (%)
Haaland approximation (eq 16)[a]	5.82×10^{-2}	1.42
$\delta = 0$ (this study)	2.91×10^{-2}	1.00
Serghides approximation (eq 17)[b]	1.41×10^{-4}	3.10×10^{-3}
δ_{LA} (this study)	1.07×10^{-5}	3.64×10^{-4}
δ_{CFA} (this study)	3.04×10^{-12}	1.04×10^{-10}

[a] Best currently available non-iterative approximation. [b] Best currently available iterative approximation.

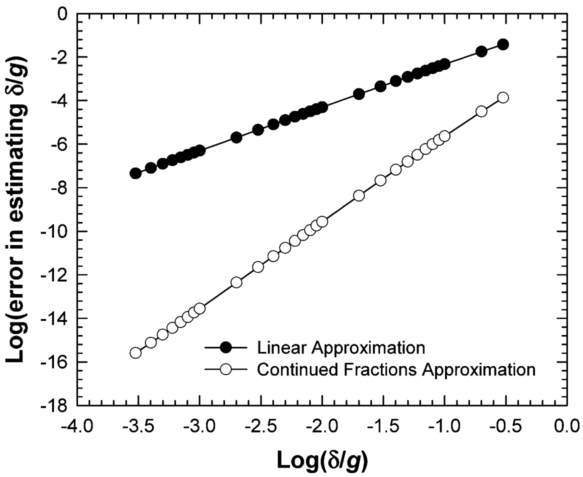

Figure 4. Error profiles for the LA and CFA approximations of $\ln[1 + (\delta/g)]$ for varying values of δ/g.

a suitable range for s may be considered to extend from 7.5 to 6.2×10^5. The functional forms of g, z, and δ/g are dependent on q. For $q = s^{s/(s+1)}$ (see subsections A.4 and A.5 in the Appendix), the largest δ/g value occurs when s is a minimum. All the δ/g values are positive and the maximum value is $<5.0 \times 10^{-3}$.

4.2.2. Error in Approximating $\ln(1 + \delta/g)$. To facilitate the discussion, let $x = \delta/g$. The linear and continued-fractions approximations for $\ln(1 + x)$, respectively, are

$$\ln(1 + x) = x \qquad \text{(linear approximation)} \tag{19}$$

$$\ln(1 + x) = \frac{x(6 + x)}{6 + 4x} \qquad \text{(continued-fractions approximation)} \tag{20}$$

The error is defined as

$$\text{Error} = \ln(1 + x)_{\text{Approximate Value}} - \ln(1 + x)_{\text{Actual Value}} \tag{21}$$

The errors in the approximation may be simply represented by a constant multiplied by a power of x. For positive values of x, the convergent alternating Taylor series guarantees the error in LA to be $<0.5x^2$. Figure 4 shows the variation in the error defined by eq 21, as a function of x, in logarithmic coordinates. The error profiles appear as straight lines in Figure 4, with slopes of ∼2 and ∼4 for the LA and CFA approximations, respectively. The CFA shows a remarkable quadratic convergence, and the error is $<0.05x^4$. Hence, these two relations may be used to estimate the error in approximating $\ln(1 + x)$ over the useful range of s values.

4.2.3. Error in Estimating δ. We will examine the error in estimating δ that results from the error in approximating $\ln(1 + x)$. Let $\Phi(\delta/g)$ be an approximation for $\ln[1 + (\delta/g)]$ and let θ be the error associated with it:

$$\theta = \Phi\left(\frac{\delta}{g}\right) - \ln\left(1 + \frac{\delta}{g}\right) \tag{22}$$

The auxiliary equation for δ is

$$\delta + \ln\left(1 + \frac{\delta}{g}\right) = z$$

and may be written as

$$\delta + \Phi\left(\frac{\delta}{g}\right) = z + \theta \tag{23}$$

For the LA, $\Phi(\delta/g) = \delta/g$ and eq 23 becomes

$$\delta\left(\frac{1 + g}{g}\right) = z + \theta \tag{24}$$

which can be rewritten as

$$\delta = \delta_a + \left(\frac{g}{g + 1}\right)\theta \tag{25}$$

where δ_a is the approximate solution. Thus, the error in δ can be written as

$$\delta_a - \delta = -\left(\frac{g}{g + 1}\right)\theta \tag{26}$$

Because g is positive, the error in δ clearly is the negative of the error in the LA of $\ln[1 + (\delta/g)]$ and also smaller by a factor of $g/(g + 1)$. Consequently, the error in the $\ln[1 + (\delta/g)]$ approximation serves as an upper bound for the error in δ and may be used as a conservative estimate of the error in δ. The same conclusions also hold for the CFA.

4.2.4. Error Prediction in the Alternate CW Formulation. Because the error in u and, hence, in the friction factor f results from the error in δ, the error in δ may be used to predict the error in u. The largest value of $|\delta|/g$ is <0.005. We can obtain a conservative estimate of the error in δ for both the LA and CFA approximations, using the estimates of the error in $\ln[1 + (\delta/g)]$ as upper bounds. For the LA at $|\delta|/g = 4.97 \times 10^{-3}$, the theoretically predicted value of the maximum absolute error in the friction factor was 1.48×10^{-5}, whereas the actual value was 1.05×10^{-5}. For the CFA, the theoretically predicted maximum absolute error was 3.04×10^{-11}, in comparison with the actual value of 2.99×10^{-12}. In both cases, the theoretical error estimate clearly is higher than the computed value, which is a reflection of the conservative nature of the theoretical predictions.

4.3. Practical Application of the Approach Presented in this Study. We have presented an alternate representation of the CW equation as

$$\frac{1}{\sqrt{f}} = a\left[\ln\left(\frac{d}{q}\right) + \delta\right] \tag{27}$$

where a is a constant, d is a function of Re only, and q and δ are functions of both ϵ/D and Re (see Table 1). Evaluation of the $\ln(d/q)$ term is trivial, as observed from the definitions of d and q from Table 1. The correction term δ, however, satisfies an implicit relationship (eq 7) and we have presented three methods in this study by which δ can be estimated from eq 7 without requiring iterative numerical solutions. In the simplest case, δ can be set to zero, which provides f estimates with maximum errors of <1% (see Table 2). The other two cases include the LA and CFA approximations that result in maximum errors of 3.64×10^{-4}% and 1.04×10^{-10}%, respectively (see Table 2).

Given the empirical nature of the CW equation, f estimates with 1% accuracy should be adequate for most practical applications. Thus, the $\delta = 0$ case presented in this study should be adequate for most routine f calculations:

$$\frac{1}{\sqrt{f}} = a \ln\left(\frac{d}{q}\right) \tag{28}$$

which can be rewritten, in terms of ϵ/D and Re, as

$$\frac{1}{\sqrt{f}} = 0.8686 \ln\left(\frac{0.4587Re}{s^{s/(s+1)}}\right) \tag{29}$$

where $s = 0.1240(\epsilon/D)Re + \ln(0.4587Re)$. If additional accuracy in f is desired, then δ can be computed using either the LA or CFA approximations.

5. Conclusions

We have presented an alternate representation (eq 6) of the Colebrook–White (CW) equation that is mathematically equivalent to the original CW expression (eq 1). This relationship for the friction factor (f) separates the solution into a known dominant component (>99%) and an unknown component (the correction term). The implicit equation for the unknown component may be solved directly using either a linear approximation (LA) or a more-accurate continued-fractions approximation (CFA) for a convergent logarithmic term. No iterative numerical schemes are necessary for the solution of this equation. The approximations that we present for the estimation of the correction term are simple, stable, and highly accurate. Moreover, errors in the resulting friction factor estimates are smooth and theoretically predictable. When the correction term was set to zero, the maximum error in f was 1%, and this was more accurate than any currently available non-iterative approximation of the CW equation. When the correction term was estimated using the LA, the errors in f were $<3.64 \times 10^{-4}$%, which is better than any iterative approximation of the CW equation proposed to date. The CFA solution results in maximum errors that were $<1.04 \times 10^{-10}$%. Our approach obviates the use of empirical methods for approximating the CW equation, which is an approach that has been widely used to date. The exact mathematical equivalence, coupled with a simple and accurate solution valid over the full range of observed pipe roughness and Reynolds number values, suggests that the new relation should be considered the preferred equation for computation of the turbulent flow friction factor in rough pipes.

Appendix

Here, we derive the analytical solution for the Colebrook–White (CW) equation, in terms of the Lambert W function. Familiarity with the W function is not assumed. It is shown that the equation in this form has limited numerical applicability. We follow this derivation with an extension of the analytical equation and develop a mathematically equivalent alternate form of the CW equation.

We start with the alternate representation of the CW equation in eq 5:

$$u = -\ln\left(b + \frac{u}{d}\right) \tag{A.1}$$

which may be rewritten as

$$u = \ln(d) - \ln(p + u) \tag{A.2}$$

where $p = bd$. Adding p to both sides and rearranging the terms, we get

$$(p + u) + \ln(p + u) = p + \ln(d) \tag{A.3}$$

The left-hand side of eq A.3 can be expressed as the logarithm of a product:

$$\ln[(p + u)e^{(p+u)}] = p + \ln(d) \tag{A.4}$$

or, in terms of the inverse function, an exponential (eq A.4) may be written as

$$(p + u)e^{p+u} = e^{p+\ln(d)} = e^{s} \tag{A.5}$$

where $s = p + \ln(d)$.

A.1. Analytical Solution. Consider a function, $\psi(x)$, which is defined for real values of x by

$$\psi(x) = xe^{x} \tag{A.6}$$

For $x \geq -1$, the function $\psi(x)$ is a one-to-one relationship; i.e., distinct x values are mapped to distinct ψ values. The principal branch of the Lambert W function is defined as the inverse of $\psi(x)$:[22]

$$W\{\psi(x)\} = x \qquad (-1 \leq x < \infty) \tag{A.7}$$

and

$$\psi\{W(y)\} = y \qquad \left(-\frac{1}{e} \leq y < \infty\right) \tag{A.8}$$

For the CW equation, only large positive arguments of the W function will be needed. Equation A.5 may be written as

$$\psi(p + u) = e^{s} \tag{A.9}$$

where both p and u are positive. From the previously described inverse property, we have

$$W\{\psi(p + u)\} = p + u \tag{A.10}$$

Consequently, eq A.9 reduces to

$$p + u = W(e^{s}) \tag{A.11}$$

or

$$u = W(e^{s}) - p \tag{A.12}$$

In terms of the original variables, eq A.12 becomes

$$\frac{1}{\sqrt{f}} = \frac{2}{\ln(10)} W\left[\frac{\ln(10)}{5.02} Re \times e^{(\ln(10)/18.574)(\epsilon/D)Re}\right] - \frac{1}{9.287}\left(\frac{\epsilon}{D}\right) Re \quad (A.13)$$

An equivalent expression was presented by Keady,[18] which was obtained through symbolic algebra in Maple software (Waterloo Maple, Inc.). Equation A.13 is an explicit expression that relates the friction factor f with the relative pipe roughness ϵ/D and Reynolds number Re.

A.2. Limitations of the Analytical Solution. Equation A.12 can be rewritten in terms of the constants a, b, c, and d as

$$\frac{1}{\sqrt{f}} = aW(de^{bd}) - \frac{b}{c} \quad (A.14)$$

It follows from eq A.14 that, for high values of the product $\epsilon/D \times Re$, the term de^{bd} can become larger than the largest floating point number used in routine computing. Under these circumstances, eq A.14 can no longer be used to compute the friction factor f. Using the definitions of b and d from Table 1, de^{bd} can be written in terms of ϵ/D and Re as

$$de^{bd} = \left(\frac{\ln(10) \times Re}{5.02}\right) e^{\{[(\epsilon/D)/3.7](\ln(10)Re/5.02)\}} \quad (A.15)$$

which can be approximated as

$$de^{bd} \approx \frac{Re}{2.1802}\, e^{(\epsilon/D)Re/8.0666} \quad (A.16)$$

For typical IEEE-compliant machines, the largest usable floating values for double precision is 1.79×10^{308}.[23] This implies that the term de^{bd} as defined in eq A.16 must be $<1.79 \times 10^{308}$ to be able to use eq A.14 for f computations. A graphical description of the constraint defined by eq A.16 is shown in Figure A-1. For all combinations of ϵ/D and Re that lie below the line $\overline{AB}$, eq A.14 can be used for the estimation of f. However, for points that lie above $\overline{AB}$, the value of de^{bd} in eq A.16 is $>1.79 \times 10^{308}$, which precludes the use of eq A.14 for estimating f. Because it is common in practice to encounter ϵ/D and Re pairs that lie above line $\overline{AB}$, eq A.14 has limited practical utility for f calculations.

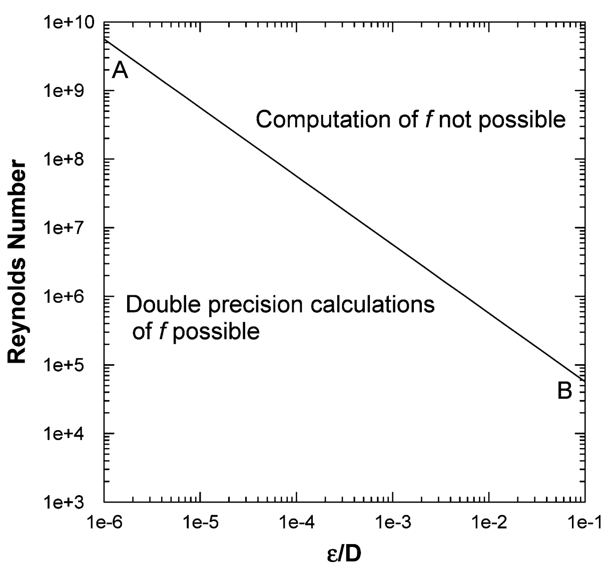

Figure A-1. Graphical representation of the applicability of eq A.14 for computation of the friction factor f in the Re vs ϵ/D space. The region below line $\overline{AB}$ can be used for double-precision calculations, whereas no f calculations are possible for Re and ϵ/D pairs that lie above line $\overline{AB}$.

A.3. Alternate Form for the CW Equation. In eq A.5, p and u are positive quantities and the function $(p + u)e^{p+u}$ is a one-to-one relationship. Consequently, if a set of parameters $g + \delta > 0$ satisfy the relation

$$(g + \delta)e^{g+\delta} = e^{s} \quad (A.17)$$

the one-to-one condition dictates that

$$p + u = g + \delta \quad (A.18)$$

or

$$u = g + \delta - p \quad (A.19)$$

where g is a known function of the parameters b and d, whereas δ is unknown and must be determined. Note that finding u ($u = 1/(a\sqrt{f})$) has been replaced by solving the associated equation (eq A.17) for δ. The solution of δ, followed by its substitution in eq A.19, is mathematically equivalent to the solution of the CW equation. The benefits, as illustrated next, come from the fact that eq A.17 can be solved accurately and stably for any realistic set of ϵ/D and Re values.

A.4. Derivation of the Alternate Equation for the Friction Factor *f*. The function g in eq A.17 may be considered as an initial guess and δ as a correction factor for u. For commonly observed pipe roughness and Reynolds numbers, e^{s} can exceed the largest number used in routine computation, as illustrated in section A.2. To eliminate e^{s}, we will define g explicitly as a sum of s and h:

$$g = s + h \quad (A.20)$$

where h is a function of the parameters b and d and the choices for h will be discussed in the next section.

For a given choice of h, eq A.19 may be simplified to

$$\begin{aligned} u &= s + h + \delta - p \\ &= \ln(d) + h + \delta \end{aligned} \quad (A.21)$$

Equation A.21 may be written as

$$(g + \delta)e^{s+h+\delta} = e^{s} \quad (A.22)$$

This can be simplified by canceling the e^{s} term from both sides:

$$g\left(1 + \frac{\delta}{g}\right)e^{\delta} = q \quad (A.23)$$

where $q = e^{-h}$ or, equivalently, $h = -\ln(q)$.

If one takes the logarithm of both sides,

$$\delta + \ln\left(1 + \frac{\delta}{g}\right) = \ln\left(\frac{q}{g}\right) \quad (A.24)$$

the relation for u, given in eq A.21, becomes

$$u = \ln\left(\frac{d}{q}\right) + \delta \quad (A.25)$$

The next section shows that a good choice for h is $h = -[s/(s + 1)] \ln(s)$, which results in $q = s^{s/(s+1)}$. The relation for u (eq A.25) and the equation for δ (eq A.24) comprise the new relation for f proposed in the body of the paper.

A.5. Functional Form for *h*(*b*,*d*). Our formulation allows for a fairly general choice for $h(b,d)$. The choices for h are limited by requiring the functional form to be simple while making the correction term (δ) small. A small value of $|\delta|/g$ leads to rapid convergence of eq A.24. A good choice for h can be determined by setting $\delta = 0$ in eq A.25. Substituting for q and g in terms of h and solving for h yields

$$h + \ln(s + h) = 0 \tag{A.26}$$

This can be rewritten as

$$h + \ln\left(1 + \frac{h}{s}\right) = -\ln(s) \tag{A.27}$$

Because s is positive and $s \gg |h|$, we will use a linear approximation (LA) of the Taylor series expansion of the logarithmic term to define h:

$$h = -\left(\frac{s}{s+1}\right)\ln(s) \tag{A.28}$$

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