AI 1 Training

Contents

1. Linear Algebra

1.1. Introduction

Linear systems of equations form a core topic in linear algebra with widespread applications in mathematics, science, engineering, and technology. Solving linear systems efficiently is crucial, and among various techniques, *Gaussian elimination* is one of the most effective and widely used.

1.2. Linear Systems

Definition 1.2.1 (Linear System)

A system of linear equations consists of multiple linear equations involving the same set of variables:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_1nx_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_2nx_n = b_2 \\ \vdots \\ a_{\mathrm{m}1}x_1 + a_{\mathrm{m}2}x_2 + \ldots + a_{\mathrm{mn}}x_n = b_m \end{cases} \tag{1.1}$$

Where:
- a_{ij} are the coefficients,
- x_j are the unknowns,
- b_i are the constants.

1.3. Matrix Representation

Theorem 1.3.1 (Matrix Representation)

A linear system can be written compactly in matrix form as:

$$A \cdot x = b \tag{1.2}$$

Where:
- A: $m \times n$ coefficient matrix
- x: $n \times 1$ unknown vector
- b: $m \times 1$ result vector

This representation is ideal for applying numerical methods.

1.3.1. Determinant of a Matrix

Definition 1.3.2 (Determinant of a Matrix)

The determinant of a square matrix A, denoted $\det(A)$ or $|A|$, helps determine invertibility.

Definition 1.3.3 (Determinant of 2×2 matrix)

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \mathrm{ad} - \mathrm{bc} \tag{1.3}$$

1.3.2. Transpose of a Matrix

Definition 1.3.4 (Transpose of a Matrix)

The transpose of a matrix A, denoted A^T, swaps its rows and columns.

Example.

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \tag{1.4}$$

$$A^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix} \tag{1.5}$$

1.3.3. Inverse of a Matrix

Definition 1.3.5 (Inverse of a Matrix)

A square matrix A has an inverse A^{-1} if:

$$A \cdot A^{-1} = A^{-1} \cdot A = I \tag{1.6}$$

Methods to compute A^{-1}:
- Augmented matrix (row operations)
- Adjoint method:

$$A^{-1} = \frac{1}{\det(A)} \cdot \operatorname{adj}(A) \tag{1.7}$$

1.4. Gaussian Elimination

Definition 1.4.1 (Gaussian Elimination)

A step-by-step method to reduce a matrix to upper triangular form, consisting of:

1. **Form the augmented matrix**: $[A \mid B]$
2. **Forward elimination**:
   - Identify pivot elements along the main diagonal
   - Use row operations to zero out entries below each pivot
3. **Back substitution**:
   - Solve for variables starting from the last row upward

Example. Given the system:

$$\begin{cases} 2x + 3y - z & = 5 \\ 4x + y + 2z & = 6 \\ -2x + 5y + 3z & = 7 \end{cases} \tag{1.8}$$

1. Augmented Matrix

$$\begin{pmatrix} 2 & 3 & -1 & | & 5 \\ 4 & 1 & 2 & | & 6 \\ -2 & 5 & 3 & | & 7 \end{pmatrix} \tag{1.9}$$

2. Row Reduction
   - Normalize the first row
   - Eliminate entries below the pivot
   - Repeat for next rows

3. Back Substitution
   Find values of z, then y, and finally x.

1.5. Gauss–Jordan Elimination

Definition 1.5.1 (Row Echelon Form of Matrix)

A matrix is in *row echelon form* if:

- All nonzero rows are above zero rows
- The leading entry (pivot) in each nonzero row is 1
- Each pivot is to the right of the one in the row above
- All entries below a pivot are 0

Example.

$$\begin{pmatrix} 1 & 2 & -1 & | & 4 \\ 0 & 1 & 3 & | & -2 \\ 0 & 0 & 1 & | & 5 \end{pmatrix} \tag{1.10}$$

Definition 1.5.2 (Gauss–Jordan Elimination)

An extension of Gaussian Elimination that produces Reduced Row Echelon Form (RREF), consisting of:
1. Form the augmented matrix $[A \mid B]$
2. Use row operations to form leading 1s (pivots)
3. Zero out **both** above and below the pivot in each column

1.5.1. Advantages

- Simplifies solving linear systems
- Easy to implement algorithmically
- Helps identify inconsistent or dependent systems

1.5.2. Visual Representation

$$\begin{pmatrix} 1 & & & \\ 0 & 1 & & * \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{1.11}$$

Interpretation:
- Zero row at the bottom
- Leading 1s at A_{11} and A_{22}
- All entries below pivots are 0

Example. Solving with Row Echelon Form (REF)
We are given the system:

$$\begin{cases} x + y + z & = 6 \\ 2x + 3y + 7z & = 20 \\ x + 3y + 4z & = 13 \end{cases} \tag{1.12}$$

1. **Write the Augmented Matrix**

$$\begin{pmatrix} 1 & 1 & 1 & | & 6 \\ 2 & 3 & 7 & | & 20 \\ 1 & 3 & 4 & | & 13 \end{pmatrix} \tag{1.13}$$

2. **Row Reduction to Row Echelon Form**
   - Keep Row 1 as is.

- Eliminate below the first pivot (Row 1, Col 1):

$$R_2 := R_2 - 2 \cdot R_1 \tag{1.14}$$

$$R_3 := R_3 - R_1 \tag{1.15}$$

New matrix:

$$\begin{pmatrix} 1 & 1 & 1 & | & 6 \\ 0 & 1 & 5 & | & 8 \\ 0 & 2 & 3 & | & 7 \end{pmatrix} \tag{1.16}$$

- Eliminate below the second pivot (Row 2, Col 2):

$$R_3 := R_3 - 2 \cdot R_2 \tag{1.17}$$

Resulting Row Echelon Form:

$$\begin{pmatrix} 1 & 1 & 1 & | & 6 \\ 0 & 1 & 5 & | & 8 \\ 0 & 0 & -7 & | & -9 \end{pmatrix} \tag{1.18}$$

3. **Back Substitution**
From the last row:

$$-7z = -9 \to z = \frac{9}{7} \tag{1.19}$$

Second row:

$$y + 5z = 8 \to y = 8 - 5 \cdot \left(\frac{9}{7}\right) = \frac{11}{7} \tag{1.20}$$

First row:

$$x + y + z = 6 \to x = 6 - \frac{11}{7} - \frac{9}{7} = \frac{22}{7} \tag{1.21}$$

Final Answer:

$$x = \frac{22}{7}, y = \frac{11}{7}, z = \frac{9}{7} \tag{1.22}$$

1.6 Vector

1.7 Vector Space

1.8 Linear Transformation

1.9 Introduction to SVMs

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