AI 1 Training

Contents

1. Introduction

Linear systems of equations form a core topic in linear algebra with widespread applications in mathematics, science, engineering, and technology. Solving linear systems efficiently is crucial, and among various techniques, *Gaussian elimination* is one of the most effective and widely used.

2. Linear Systems

A linear system of equations consists of multiple linear equations involving the same set of variables:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_1 n x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_2 n x_n = b_2 \\ \vdots \\ a_{\mathrm{m1}}x_1 + a_{\mathrm{m2}}x_2 + \dots + a_{\mathrm{mn}}x_n = b_m \end{cases} \quad (2.1)$$

Where:
- a_{ij} are the coefficients,
- x_j are the unknowns,
- b_i are the constants.

3. Matrix Representation

A linear system can be written compactly in matrix form as:

$$A \cdot X = B \tag{3.1}$$

Where:
- A: $m \times n$ coefficient matrix
- X: $n \times 1$ unknown vector
- B: $m \times 1$ result vector

This representation is ideal for applying numerical methods.

3.1. Determinant of a Matrix

The determinant of a square matrix A, denoted $\det(A)$ or $|A|$, helps determine invertibility.
For a 2×2 matrix:

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \mathrm{ad} - \mathrm{bc} \tag{3.2}$$

3.2. Transpose of a Matrix

The transpose of a matrix A, denoted A^T, swaps its rows and columns.
Example:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \tag{3.3}$$

$$A^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix} \tag{3.4}$$

3.3. Inverse of a Matrix

A square matrix A has an inverse A^{-1} if:

$$A \cdot A^{-1} = I \tag{3.5}$$

, where $\det(A) \neq 0$
Methods to compute A^{-1}:
- Augmented matrix (row operations)
- Adjoint method: $A^{-1} = \frac{1}{\det(A)} \cdot \mathrm{adj}(A)$

4. Gaussian Elimination

A step-by-step method to reduce a matrix to upper triangular form:

4.1. Steps:

1. **Form the augmented matrix**: $[A \mid B]$
2. **Forward elimination**:
   - Identify pivot elements along the main diagonal
   - Use row operations to zero out entries below each pivot
3. **Back substitution**:
   - Solve for variables starting from the last row upward

4.2. Example

Given the system:

$$\begin{cases} 2x+3y-z & =5 \\ 4x+y+2z & =6 \\ -2x+5y+3z & =7 \end{cases}$$

4.2.1. Step 1: Augmented Matrix

$$\left(\begin{array}{ccc|c} 2 & 3 & -1 & 5 \\ 4 & 1 & 2 & 6 \\ -2 & 5 & 3 & 7 \end{array}\right)$$

4.2.2. Step 2: Row Reduction

- Normalize the first row
- Eliminate entries below the pivot
- Repeat for next rows

4.2.3. Step 3: Back Substitution

Find values of z, then y, and finally x.

5. Gauss–Jordan Elimination

An extension of Gaussian Elimination that produces Reduced Row Echelon Form (RREF):

5.1. Steps:

- Form the augmented matrix $[A \mid B]$
- Use row operations to form leading 1s (pivots)
- Zero out **both** above and below the pivot in each column

5.2. Resulting Form:

$$\begin{pmatrix} 1 & 0 & 0 & | & x_1 \\ 0 & 1 & 0 & | & x_2 \\ 0 & 0 & 1 & | & x_3 \end{pmatrix}$$

6. Row Echelon Form (REF)

A matrix is in *row echelon form* if:

- All nonzero rows are above zero rows
- The leading entry (pivot) in each nonzero row is 1
- Each pivot is to the right of the one in the row above
- All entries below a pivot are 0

6.1. Example

$$\begin{pmatrix} 1 & 2 & -1 & | & 4 \\ 0 & 1 & 3 & | & -2 \\ 0 & 0 & 1 & | & 5 \end{pmatrix}$$

6.2. Advantages

- Simplifies solving linear systems
- Easy to implement algorithmically
- Helps identify inconsistent or dependent systems

6.3. Visual Representation

$$\begin{pmatrix} 1 & & & \\ 0 & 1 & & * \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Interpretation:

- Zero row at the bottom
- Leading 1s at $A_{\{11\}}$ and $A_{\{22\}}$
- All entries below pivots are 0

6.4. Solving with Row Echelon Form (REF)

We are given the system:

$$\begin{cases} x+y+z & =6 \\ 2x+3y+7z & =20 \\ x+3y+4z & =13 \end{cases}$$

Step 1: Write the Augmented Matrix

$$\begin{pmatrix} 1 & 1 & 1 & | & 6 \\ 2 & 3 & 7 & | & 20 \\ 1 & 3 & 4 & | & 13 \end{pmatrix}$$

Step 2: Row Reduction to Row Echelon Form

- Keep Row 1 as is.
- Eliminate below the first pivot (Row 1, Col 1):

$R_2 := R_2 - 2 \cdot R_1$

$R_3 := R_3 - R_1$

New matrix:

$$\begin{pmatrix} 1 & 1 & 1 & | & 6 \\ 0 & 1 & 5 & | & 8 \\ 0 & 2 & 3 & | & 7 \end{pmatrix}$$

- Eliminate below the second pivot (Row 2, Col 2):

$R_3 := R_3 - 2 \cdot R_2$

Resulting Row Echelon Form:

$$\begin{pmatrix} 1 & 1 & 1 & | & 6 \\ 0 & 1 & 5 & | & 8 \\ 0 & 0 & -7 & | & -9 \end{pmatrix}$$

Step 3: Back Substitution
From the last row:

$$-7z = -9 \rightarrow z = \frac{9}{7} \tag{6.1}$$

Second row:

$$y + 5z = 8 \rightarrow y = 8 - 5 \cdot \left(\frac{9}{7}\right) = \frac{11}{7} \tag{6.2}$$

First row:

$$x + y + z = 6 \rightarrow x = 6 - \frac{11}{7} - \frac{9}{7} = \frac{22}{7} \tag{6.3}$$

6.5. Final Answer:

$$x = \frac{22}{7}, y = \frac{11}{7}, z = \frac{9}{7} \tag{6.4}$$

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