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Geodesics in Schwarzschild Metric

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Abstract

Devo davvero fare l'abstract?

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Introduction

Bozza

Study of geodesics in Schwarzschild metric

Computer simulations on the second chapter

Chapter 1

Theory

1.1 Introduction

1.1.1 Why the Schwarzschild Geometry

Newtonian mechanics is built upon the concept of absolute time and space. Once the concept of *inertial frame* is well-defined, physics can be done on a space described by Euclidean geometry. Free particles (particles on which no forces are acting) move in a straight line, which is the shortest distance between two points in a three-dimensional space, measured as:

$$\Delta s^2 = \Delta x^2 + \Delta y^2 + \Delta z^2 \,. \tag{1.1}$$

On the other hand, time is *just* seen as a parameter, common to every inertial frame, that can be used to determine the particle velocity and acceleration.

With the appearance of Maxwell's Equations it became clear that what they predicted (the speed of light being constant in every inertial frame) was in contrast with the description of our space given by Newtonian Mechanics, where the speed of anything changes with respect to the inertial frame chosen. Between Maxwell's Equations and Newtonian mechanics Einstein chose to modify the latter and wrote his two postulates for the theory of Special Relativity:

- The laws of physics are invariant (identical) in all inertial frames of reference;
- The speed of light in vacuum, $c = 299\,792\,458$m/s, is the same for all observers, regardless of the motion of light source or observer.

The postulates may or may not be intuitive, but simple observations based on them bring us to abandon the idea of absolute space and time and to introduce the concept of *spacetime*, together with a new way of measuring distances

$$\Delta s^2 = -c^2\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2 \,. \tag{1.2}$$

In special relativity distances measured this way are the same for every observer in every inertial frame possible.

The appearance of time in a formula that is supposed to give us the distance between two objects is surely destabilizing at first, but geometry teaches us that fixing the way we calculate Δs^2, more properly referred to as the *line element* $\mathrm{d}s^2$, is enough to describe the geometry of the space that we are using. Since eq. 1.2 is different from eq. 1.1, in particular there is a minus sign in front of Δt^2, we moved away from the familiar three-dimensional Euclidean geometry and are now in four-dimensional spacetime, usually referred to as *flat spacetime* or *Minkowski space*.

This new geometry allowed for a reformulation of Maxwell's Equations and brought (and explained) phenomena like time dilation, length contraction and the relativity of simultaneity. The last one in particular, the concept that the simultaneity of two events depends on the frame of reference, poses a threat to the *force* of gravity. Up until this point gravity was defined as the instantaneous force F_{12} acting on a mass m_1 at time t due to a second mass m_2:

$$F_{12} = G\frac{m_1 m_2}{|r_1(t) - r_2(t)|^2} \tag{1.3}$$

The adjective *instantaneous* in a theory where nothing can travel faster than the speed of light should already raise some concern. But looking at $r_1(t)$ and $r_2(t)$ in eq. 1.3, that are supposed to indicate the positions of the masses in the same instant of time, makes it even clearer that the force F_{12} can't be the same in all frames of reference.

Solving this issue gave birth to the theory of general relativity, where a mass is not a source of gravitational force anymore, but is responsible for bending the four-dimensional spacetime itself. This implies that when we observe a particle deviating its trajectory from a straight line in the presence of a massive object, it is not because of a force acting on it. In fact, we can consider the particle free and moving from point A to point B along the shortest path, it is just that in the curved surface bent by the mass the shortest path is not a straight line.

While this concept may not enhance our intuitive understanding, the implications and the mathematical formalism required to articulate the theory are even more challenging. If the presence of mass distorts the space we work in, changing the line element $\mathrm{d}s^2$ is therefore necessary. The details of the theories, particularly the Einstein field equations, that describe this distortion and allow us to evaluate the new $\mathrm{d}s$ from a give distribution of mass are beyond the scope of this thesis. Our focus will be on evaluating the observable effects, given the line element.

More specifically we will study one of the simplest curved spacetime that general relativity has to offer: the geometry of empty space outside a spherically symmetric source of curvature, for example, a spherical star. It is one of the simplest because of the many symmetries that presents and, luckily, is also one of the most useful.

The line element of what is more commonly know as the Schwarzschild geometry is

$$\mathrm{d}s^2 = -\left(1 - \frac{2GM}{c^2 r}\right)(c\mathrm{d}t)^2 + \left(1 - \frac{2GM}{c^2 r}\right)^{-1}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2)$$

expressed in spherical coordinates centered in the mass responsible for bending the space.

1.1.2 Notation and Formalism

In the *flat spacetime* we can introduce a coordinate basis for four-vectors

$$\mathbf{e_t} = (1,0,0,0), \quad \mathbf{e_x} = (0,1,0,0), \quad \mathbf{e_y} = (0,0,1,0), \quad \mathbf{e_z} = (0,0,0,1). \tag{1.4}$$

The set $\{\mathbf{e_t}, \mathbf{e_x}, \mathbf{e_y}, \mathbf{e_z}\}$, is often referred to as $\{\mathbf{e_0}, \mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$. Any four-vector $\mathbf{a}$ can then be written as

$$\mathbf{a} = a^t\mathbf{e_t} + a^x\mathbf{e_x} + a^y\mathbf{e_y} + a^z\mathbf{e_z} = a^0\mathbf{e_0} + a^1\mathbf{e_1} + a^2\mathbf{e_2} + a^3\mathbf{e_3} \tag{1.5}$$

where $(a_t,\ a_x,\ a_y,\ a_z)$, or equivalently $(a_0,\ a_1,\ a_2,\ a_3)$, are the components of the four-vector. Both notations will be used.

Another useful convention is to use Roman letters (usually i or j) to refer to indices 1, 2, 3 and Greek letters (usually μ or ν) to refer to indices 0, 1, 2, 3. Using Einstein notation the expression in eq. 1.5, can be rewritten simply as $\mathbf{a} = a^\mu\mathbf{e}_\mu$. Other useful ways to specify the components of $\mathbf{a}$ are

$$a^\mu = (a^t, a^x, a^y, a^z) \quad a^\mu = (a^t, a^i) \quad a^\mu = (a^t, \vec{a})$$

where $\vec{a} = a^i e_i$ is the tree-dimensional vector (a_x, a_y, a_z).

The length of the four-vector $\mathbf{a}$ must match the definition given with the Δs^2 in 1.2, it is useful to define the *metric* $\eta_{\nu\mu}$ so that

$$\eta_{\nu\mu} = \begin{array}{c c} & \begin{array}{cccc} t & x & y & z \end{array} \\ \begin{array}{c} t \\ x \\ y \\ z \end{array} & \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array} \implies \mathrm{d}s^2 = \eta_{\nu\mu}\mathrm{d}x^\nu\mathrm{d}x^\mu \tag{1.6}$$

where a double sum is implied, and we rightfully notice that the minus sign has appeared again under the t component. Now we can compactly write

$$\mathbf{a}\cdot\mathbf{a} = \eta_{\mu\nu}\, a^\mu\, a^\nu = -(a^t)^2 + (a^x)^2 + (a^y)^2 + (a^z)^2 \tag{1.7}$$

Without any claim of rigorously demonstrating it, we can say that since this scalar product is built from the line element $\mathrm{d}s^2$, it is the same in every inertial frame one might choose. Quantities that have these properties are *invariant*.

When working in the Schwarzschild geometry it is useful to adopt the Schwarzschild coordinates, spherical coordinates centered at the center of the mass M, and use geometrized units, where $G = c = 1$ (Appendix A). The line element and the metric can be rewritten as

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r}\right)(\mathrm{d}t)^2 + \left(1 - \frac{2M}{r}\right)^{-1}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2)$$

$$g_{\nu\mu} = \begin{array}{c c} & \begin{array}{cccc} t & r & \theta & \phi \end{array} \\ \begin{array}{c} t \\ r \\ \theta \\ \phi \end{array} & \begin{pmatrix} -(1-2M/r) & 0 & 0 & 0 \\ 0 & (1-2M/r)^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix} \end{array}. \tag{1.8}$$

It's worth pointing out that, given 1.15, the coordinate basis introduced in 1.4 is not normalized in this geometry, for example:

$$\mathbf{e_t} \cdot \mathbf{e_t} = g_{\nu\mu} e_t^{\mu} e_t^{\nu} = g_{00} = -(1 - 2M/r) \tag{1.9}$$

If we want an orthonormal tetrad we can define

$$\hat{\mathbf{e}}_{\mathbf{t}} = \left(1 - \frac{2M}{r}\right)^{-1/2} \mathbf{e_t} \qquad \Longrightarrow \qquad \hat{\mathbf{e}}_{\mathbf{t}} \cdot \hat{\mathbf{e}}_{\mathbf{t}} = g_{\nu\mu} \hat{e}_t^{\mu} \hat{e}_t^{\nu} = g_{00} \left(1 - \frac{2M}{r}\right)^{-1} = -1 \tag{1.10}$$

$$\hat{\mathbf{e}}_{\mathbf{r}} = \left(1 - \frac{2M}{r}\right)^{1/2} \mathbf{e_r} \qquad \Longrightarrow \qquad \hat{\mathbf{e}}_{\mathbf{r}} \cdot \hat{\mathbf{e}}_{\mathbf{r}} = g_{\nu\mu} \hat{e}_r^{\mu} \hat{e}_r^{\nu} = g_{00} \left(1 - \frac{2M}{r}\right) = 1 \tag{1.11}$$

$$\hat{\mathbf{e}}_{\theta} = \frac{1}{r} \mathbf{e_t} \qquad \Longrightarrow \qquad \hat{\mathbf{e}}_{\theta} \cdot \hat{\mathbf{e}}_{\theta} = 1 \tag{1.12}$$

$$\hat{\mathbf{e}}_{\phi} = \frac{1}{r \sin\theta} \mathbf{e_t} \qquad \Longrightarrow \qquad \hat{\mathbf{e}}_{\phi} \cdot \hat{\mathbf{e}}_{\phi} = 1 \tag{1.13}$$

$$\tag{1.14}$$

1.2 Proprieties of the Metric

Let's first analyze the Schwarzschild metric in more detail:

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r}\right)(\mathrm{d}t)^2 + \left(1 - \frac{2M}{r}\right)^{-1} \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2) \tag{1.15}$$

There are two singularities in $r = 0$ and $r = 2M$. The first one is intrinsic to the spherical coordinate system and, since the metric is only valid in the space outside the start, doesn't concern us. The second occur at what is defined as the *Schwarzschild radius* $r_s = 2M$. Every non-black hole object has a radius larger than its Schwarzschild radius. The nature and significance of this will become clearer in the subsequent sections.

On the other hand, if we take the limit as r approaches infinity, we notice that the metric becomes asymptotically flat, approaching the metric of Minkowski space.

Finally, $\mathrm{d}s^2$ is independent of the coordinates t and ϕ. This is expected, as the mass responsible for curving the spacetime is static and spherically symmetric. The metric's independence from time and rotation implies the existence of two easy *killing vectors*:

$$\xi = (1, 0, 0, 0) \quad \text{and} \quad \eta = (0, 0, 0, 1)\,. \tag{1.16}$$

A *killing vector* is a direction in the four-dimension spacetime along which we can freely move without changing the metric. This means that the spacetime metric $g_{\nu\mu}$ remains unchanged, and with it distances ($\Delta s^2 = g_{\mu\nu} x^\nu x^\mu$) and other quantities evaluated with it.

Chapter 2

Simulations

Computed in `C` and animated in `python` (hopefully, if I have enough time)

Conclusions

Appendix A

Geometrized Units

In general relativity the constants $G \simeq 6.674\,30 \times 10^{-11}\mathrm{N\,m^2\,kg^{-2}}$ and $c = 299\,792\,458\mathrm{m/s}$ appears quite often, so it's useful to redefine our units of measurements to cancel them out. Let's take the Schwarzschild line element as an example

$$\mathrm{d}s^2 = -\left(1 - \frac{2GM}{c^2 r}\right)(c\mathrm{d}t)^2 + \left(1 - \frac{2GM}{c^2 r}\right)^{-1}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2)\,. \tag{A.1}$$

Time always appears next to the speed of light, ct. This is effectively as if we were measuring time with the distance that light can cover in t seconds.

The mass M is measured in kg in S.I. units. If we multiply it by G and divide by c^2 we get

$$\left[\frac{GM}{c^2}\right] = \frac{\mathrm{N\,m^2\,kg^{-2}\,kg}}{\mathrm{m^2\,s^{-2}}} = \frac{\mathrm{N}}{\mathrm{kg}}\mathrm{s}^2 = \mathrm{m}\,.$$

So, in a less intuitive way, we can measure the mass as a distance too.

Substituting $\hat{t} = ct$ and $\hat{M} = \frac{GM}{c^2}$ in eq. A.1 gives

$$\mathrm{d}s^2 = -\left(1 - \frac{2\hat{M}}{r}\right)\mathrm{d}\hat{t}^2 + \left(1 - \frac{2\hat{M}}{r}\right)^{-1}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2)\,.$$

In this way we went from a $\mathcal{LMT}$(Length Mass Time) units system, to an $\mathcal{L}$ one. Since it will be clearly said when this convention, loosely referred to as $G = c = 1$, is in use we will omit the *hat* in the new defined variables.

To go back to $\mathcal{LMT}$ units we just need to substitute back $t \to ct$ and $M \to \frac{GM}{c^2}$, being extra careful on cases like the speed, that is derived from time and therefor inherits a $1/c$ factor.

Table A shows some typical value for geometrized units.

	S.I. units	Geometrized units
Mass of the Earth	5.97×10^{24} kg	4.43 mm
Mass of the Sun	1.99×10^{30} kg	1.48 km
M87 black hole	$6.5 \times 10^9 M_\odot$	64.2 au

Table A.1: Some common masses of the universe expressed in unit of length.

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