Geodesics in Schwarzschild Metric

Candidate: Federico De Paoli
Supervisor: Prof. Albino Perego

Bachelor degree in Physics,
University of Trento

September 18, 2024

Introduzione

- Perché è necessaria la Relatività Generale
- Cos'è la metrica di Schwarzschild
- Studio delle geodetiche come strumento per capire la metrica
- Soluzioni numeriche alle equazioni del moto di una particella massiva

Il moto di una particella libera

Possiamo descrivere il moto di una particella libera tramite il principio variazionale

Principio di Minima Azione

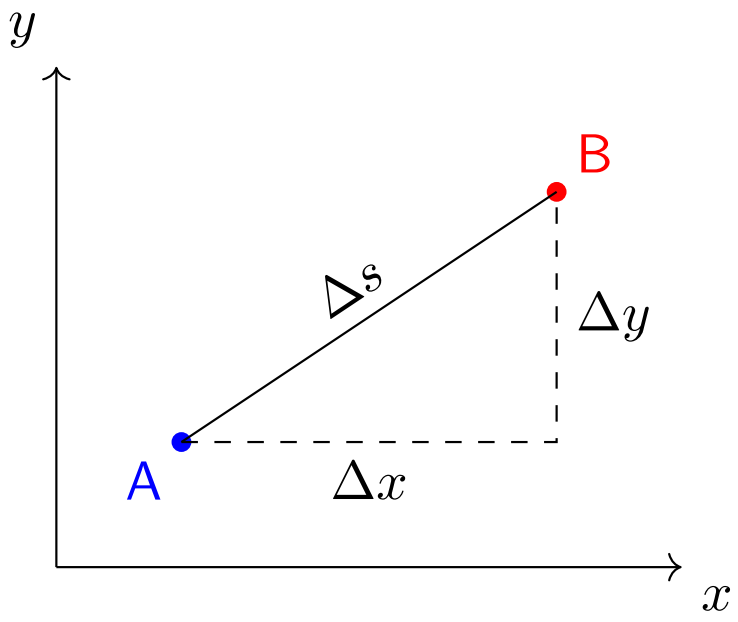
Una particella libera che si muove da A a B segue il percorso che minimizza la distanza tra i due punti.

Il problema si può quindi ricondurre a come si misura *correttamente* una **distanza**?

Misurare le Distanze

Meccanica Newtoniana

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$$



Rappresentazione 2D $(x,\, y)$ di Δs^2.

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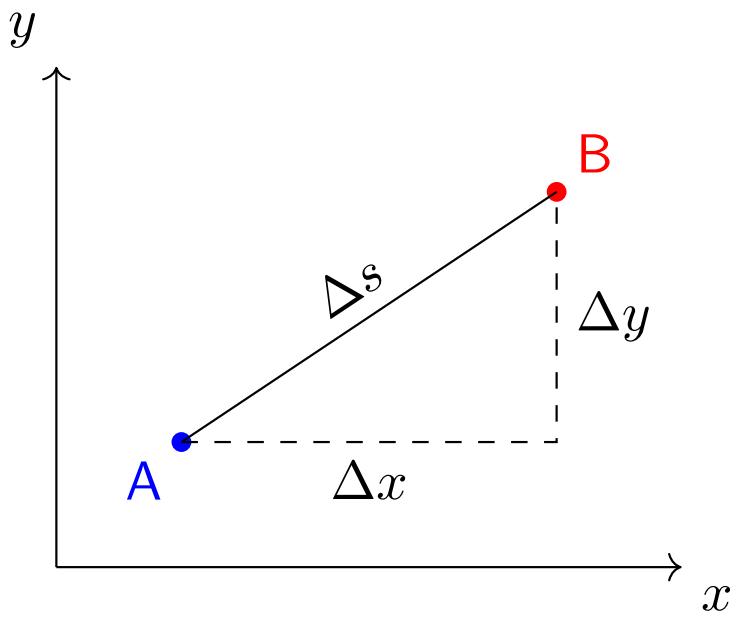
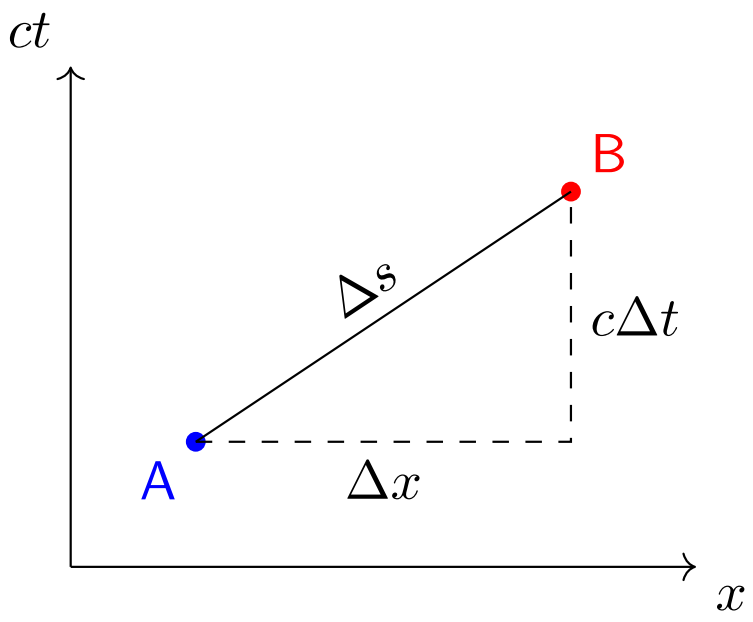


Rappresentazione 2D (x, y) di Δs^2.

Relatività Ristretta

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$$



Rappresentazione 2D (x, ct) di Δs^2.

Riformulazione della Meccanica

È utile definire la metrica

$$\eta_{\nu\mu} = \begin{array}{c} \\ t \\ x \\ y \\ z \end{array} \begin{array}{c} \begin{array}{cccc} t & x & y & z \end{array} \\ \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array} \qquad \Longrightarrow \qquad \mathrm{d}s^2 = \eta_{\nu\mu}\mathrm{d}x^\nu \mathrm{d}x^\mu$$

$\mathrm{d}s^2$ si chiama **elemento di linea**, ha le seguenti proprietà:

- è uguale per tutti i sistemi di riferimento inerziali
- determina la geometria dello spaziotempo

Il Problema della Forza di Gravità

La forza di gravità tra due masse m_1 e m_2, posizionate in $r_1(t)$ e $r_2(t)$, **non è compatibile con la relatività ristretta**. Infatti

$$F_{12} = G\frac{m_1 m_2}{|r_1(t) - r_2(t)|^2}$$

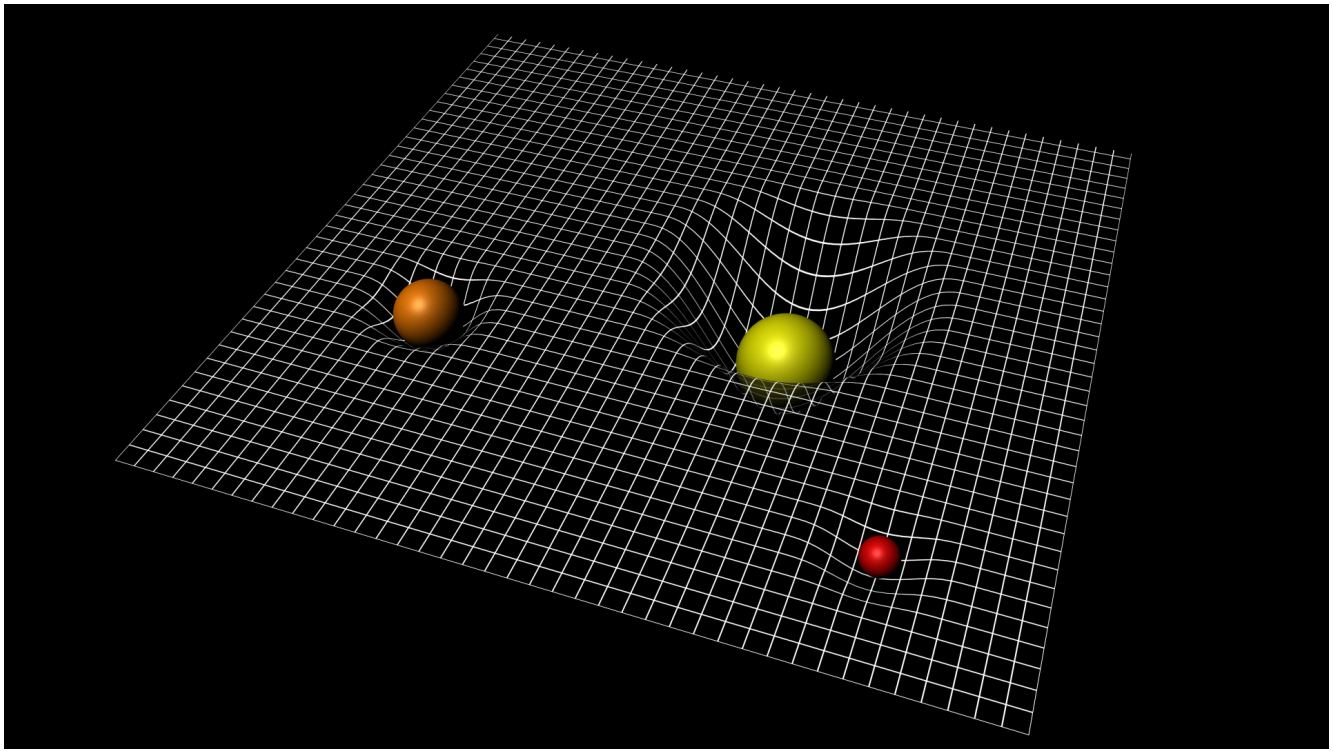
- Nella teoria di Newton dovrebbe essere istantanea
- $|r_1(t) - r_2(t)|$ dipende dal sistema di riferimento

Una Nuova Teoria

Relatività Generale

La massa di un oggetto non da origine alla forza di gravità, ma curva lo *spaziotempo* stesso.

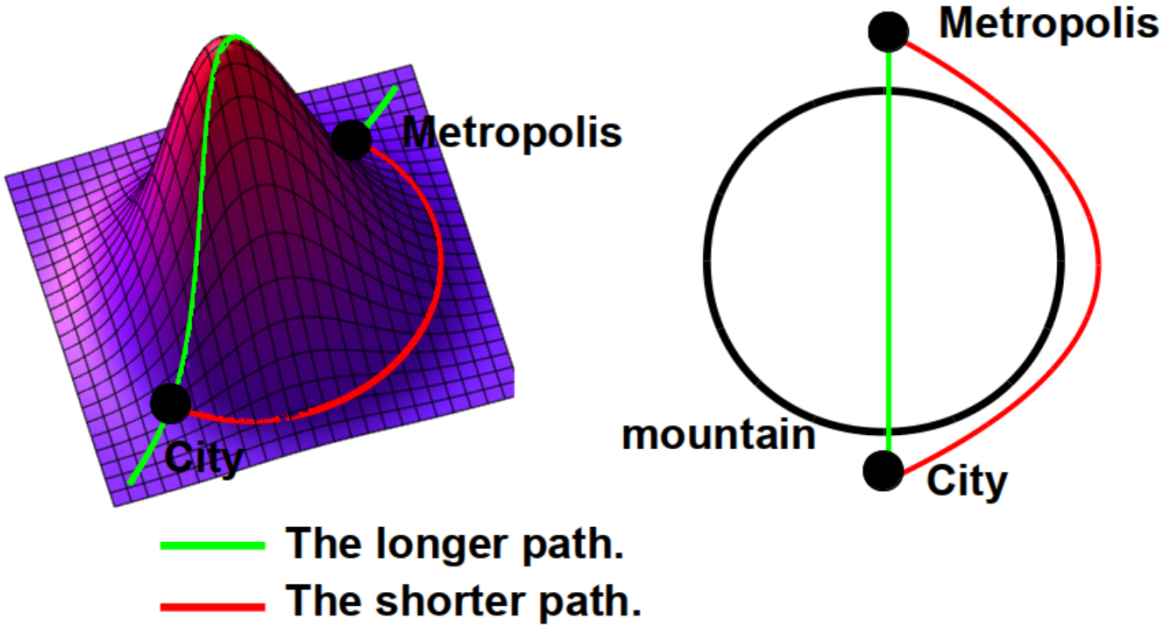
L'elemento di linea $\mathrm{d}s^2$ si dovrà calcolare in modo diverso.



https://www.esa.int

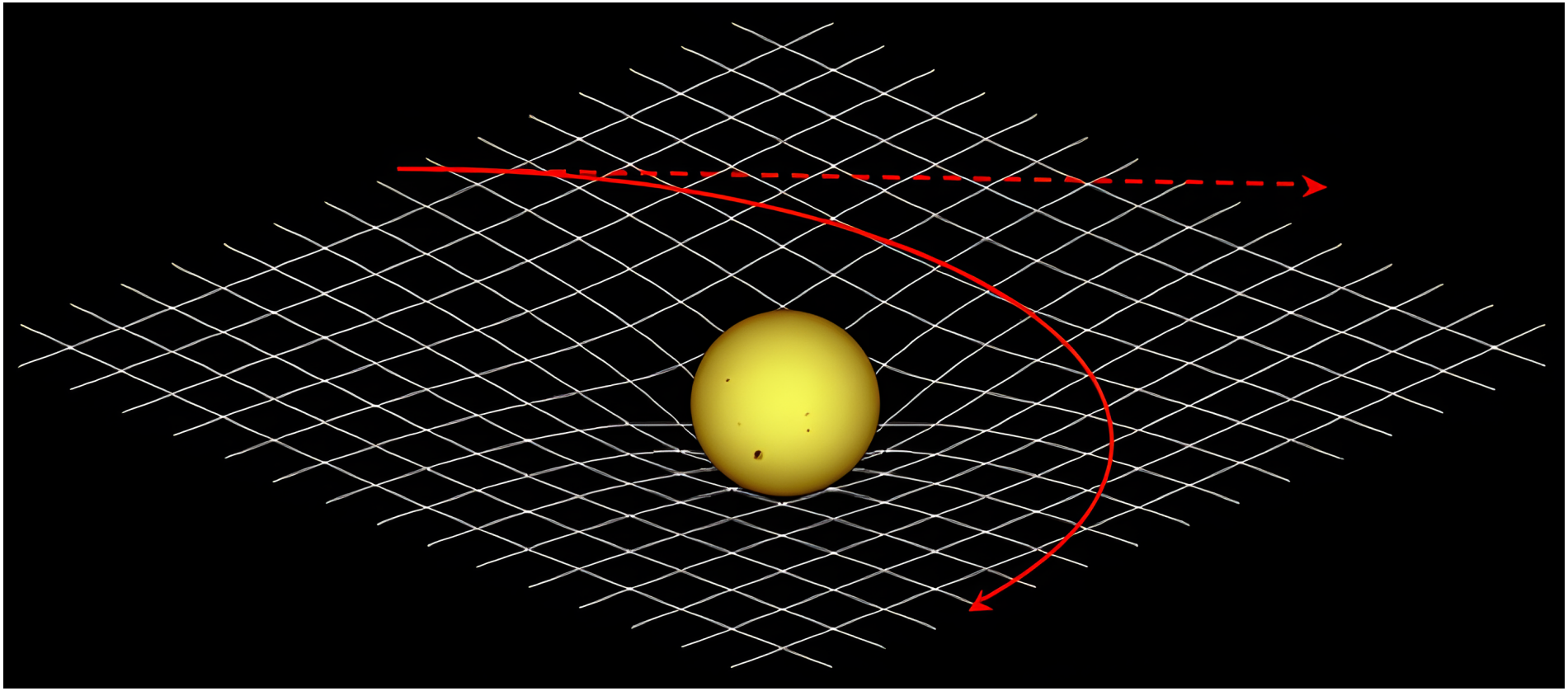
Il Moto nello Spazio Curvo

La distanza più breve tra due punti in uno spazio curvo non è più una retta



https://www.thephysicsmill.com

Geodetiche: le *Rette* dello Spazio Curvo

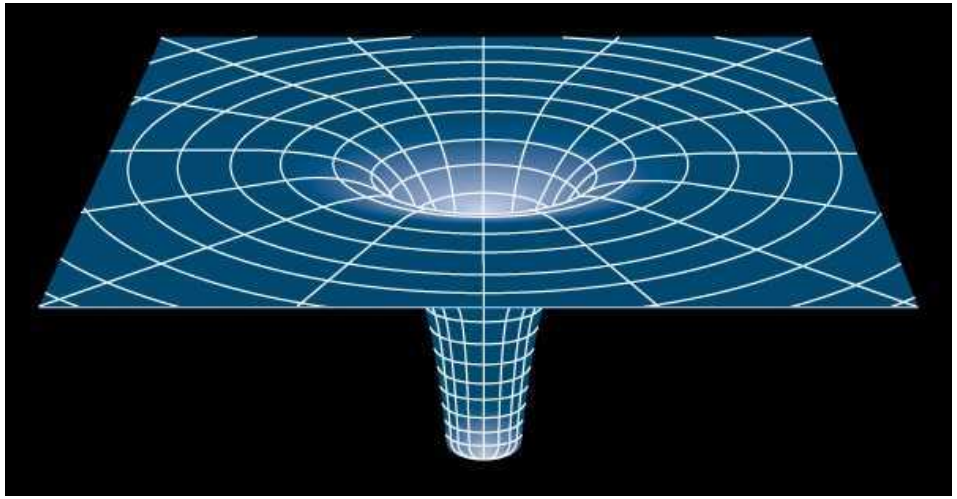


La Metrica di Schwarzschild

$$\mathrm{d}s^2 = -\left(1 - \frac{2GM}{c^2 r}\right) c^2 \mathrm{d}t^2 + \left(1 - \frac{2GM}{c^2 r}\right)^{-1} \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2)$$

Descrive uno spaziotempo stazionario e a simmetria sferica

- stella sferica
- buco nero



https://www.physicsforums.com/

La Metrica di Schwarzschild

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2M}{r}\right)^{-1}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2)$$

$$g_{\nu\mu} = \begin{array}{c} \\ t \\ r \\ \theta \\ \phi \end{array} \begin{array}{c} \begin{array}{cccc} t & r & \theta & \phi \end{array} \\ \begin{pmatrix} -\left(1 - \frac{2M}{r}\right) & 0 & 0 & 0 \\ 0 & \frac{1}{\left(1 - \frac{2M}{r}\right)} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{pmatrix} \end{array}$$

$r = 0$ singolarità fisica

$r = 2M$ singolarità di coordinate

$r \to \infty$ metrica di Minkowski

La Metrica di Schwarzschild

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2M}{r}\right)^{-1}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2)$$

È indipendente da t e ϕ $\implies$ vettori di Killing associati

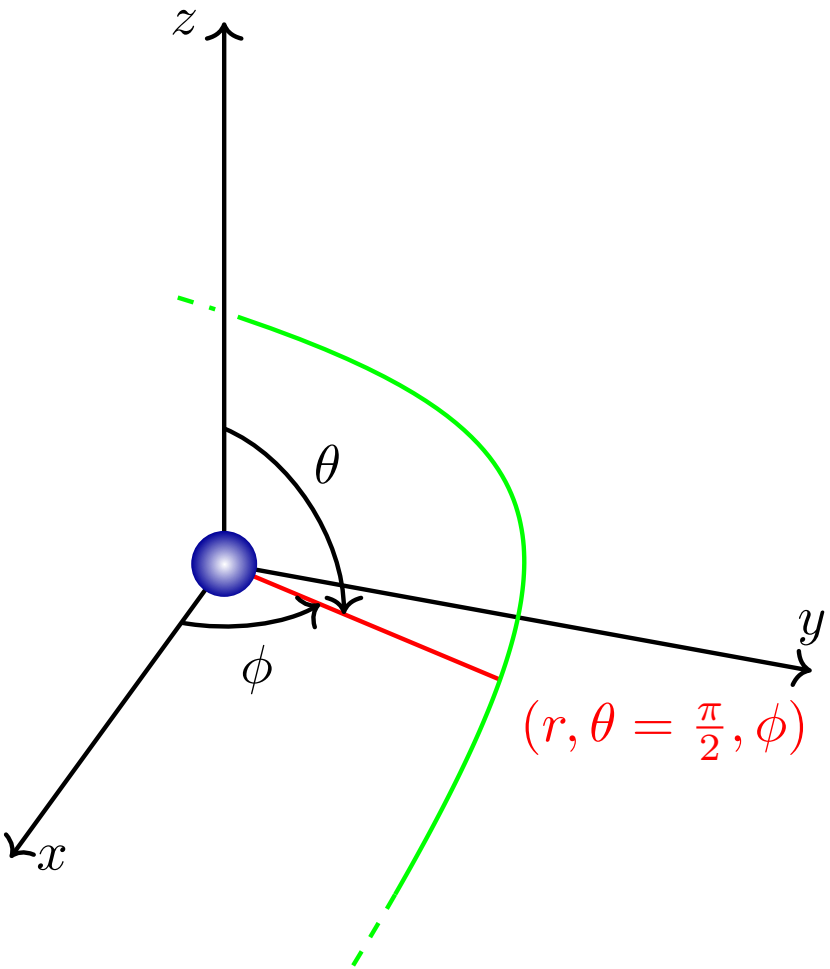
$$\xi^\mu = (1, 0, 0, 0) \quad \eta^\mu = (0, 0, 0, 1)$$

Possiamo trovare delle costanti del moto

$$e = -\xi \cdot \mathbf{u} = \left(1 - \frac{2M}{r}\right)\frac{\mathrm{d}t}{\mathrm{d}\tau}$$

$$l = \eta \cdot \mathbf{u} = r^2 \sin^2\theta \frac{\mathrm{d}\phi}{\mathrm{d}\tau}$$

L'Orbita Giace su un Piano



$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{e}{1 - \frac{2M}{r}}$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}\tau} = \frac{l}{r^2}$$

$$\mathbf{u} = \left(\frac{\mathrm{d}t}{\mathrm{d}\tau}, \frac{\mathrm{d}r}{\mathrm{d}\tau}, \frac{\mathrm{d}\theta}{\mathrm{d}\tau}, \frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)$$

$$\mathbf{u} \cdot \mathbf{u} = -1$$

Equazione radiale

Possiamo ricondurci a un'espressione famigliare

$$\mathcal{E} = \frac{1}{2}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 + \underbrace{\frac{l^2}{2r^2} - \frac{M}{r} - \frac{Ml^2}{r^3}}_{V_{\mathrm{eff}}}$$

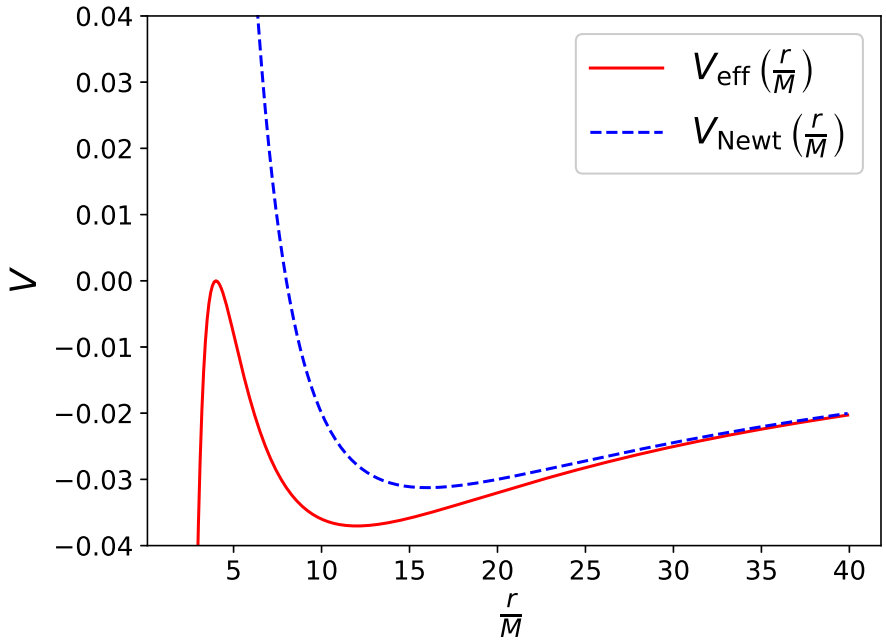
$$\mathcal{E} = \frac{e^2 - 1}{2}$$

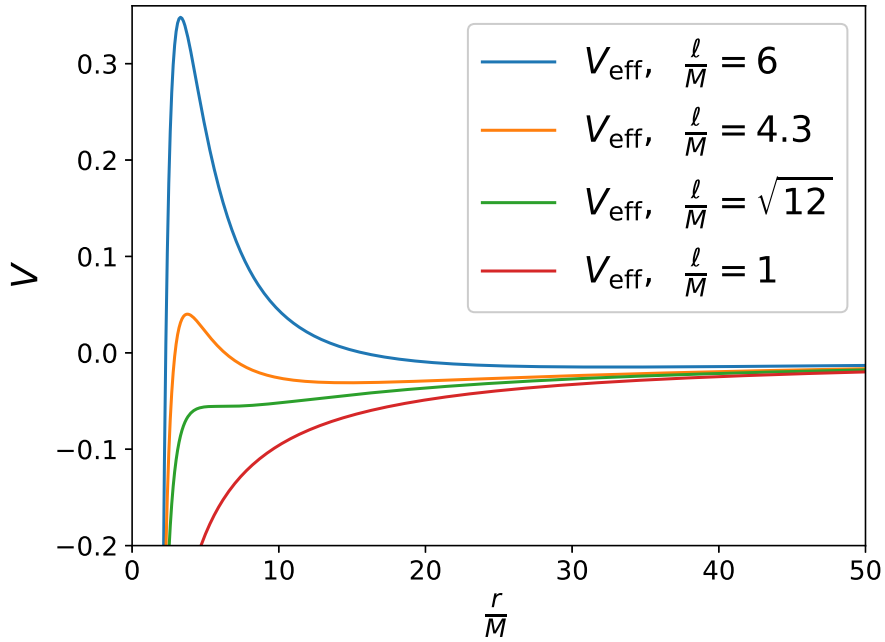
Ricordando che

$$V_{\mathrm{New}} = \frac{l^2}{2r} - \frac{M}{r}$$

Il Potenziale Efficacie V_{eff}

Per $l > \sqrt{12}M$ il potenziale efficace un minimo $(r_{\min})$ e un massimo $(r_{\max})$ locali

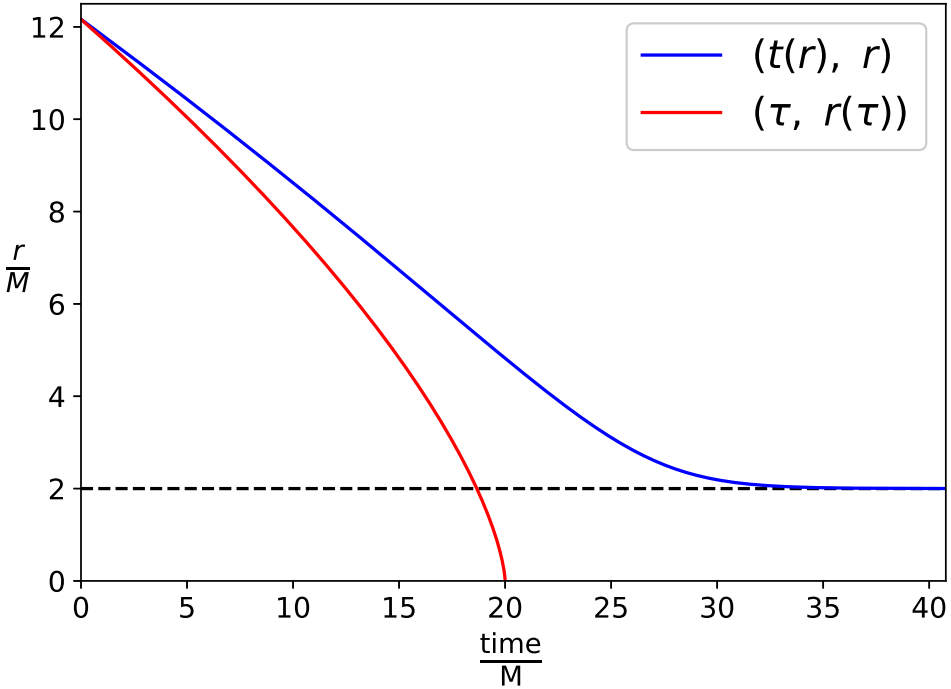




$$V_{\text{eff}}(r) = \frac{l^2}{2r^2} - \frac{M}{r} - \frac{Ml^2}{r^3}$$

Caduta Libera Radiale

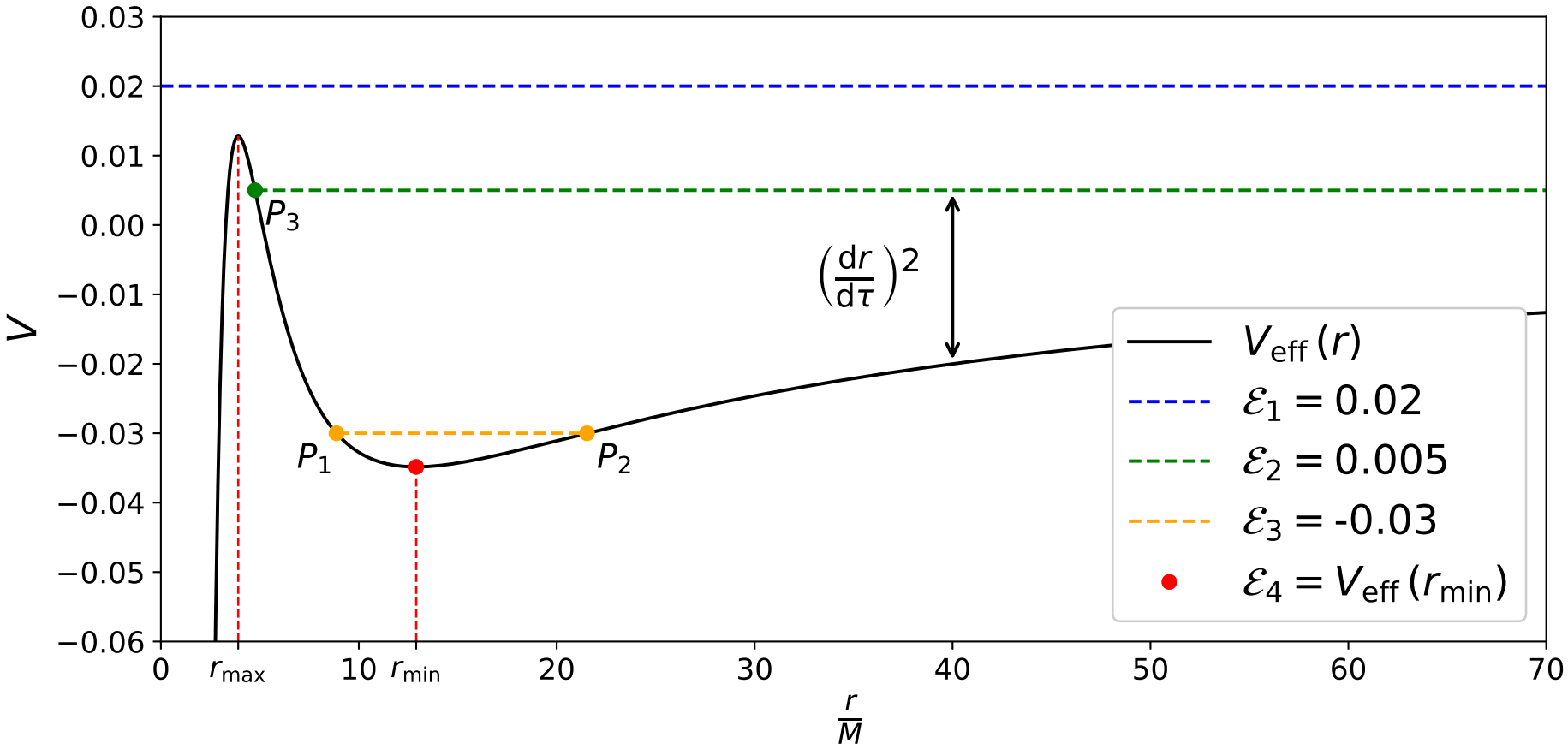
Consideriamo $l = 0$, $\mathcal{E} = 0$, c'è una soluzione analitica



$$r(\tau) = \left(\frac{3}{2}\right)^{2/3} (2M)^{1/3} (\tau_* - \tau)^{2/3}$$

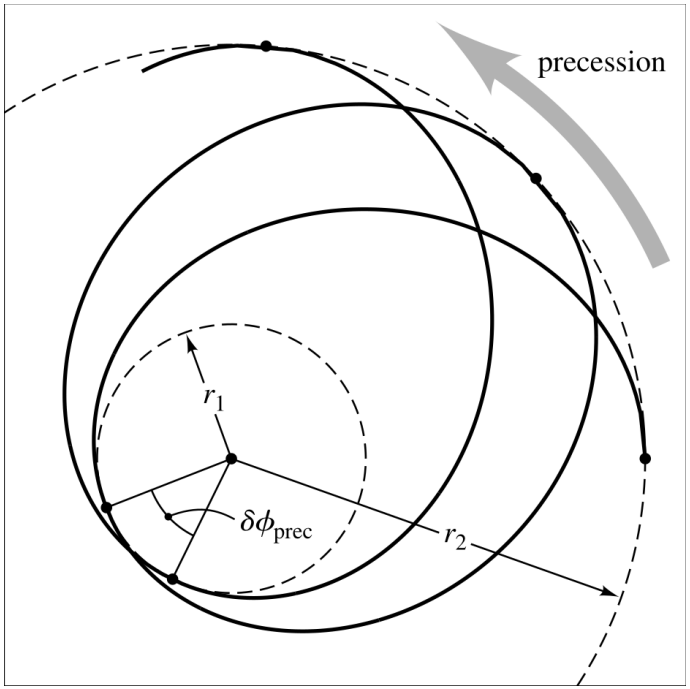
$$t(r) = t_* + 2M \left[-\frac{3}{2} \left(\frac{r}{2M}\right)^{3/2} - 2 \left(\frac{r}{2M}\right)^{1/2} + \ln \left| \frac{(r/2M)^{1/2} + 1}{(r/2M)^{1/2} - 1} \right| \right]$$

$$\mathcal{E} = \frac{1}{2}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 + V_{\mathrm{eff}}(r, l)$$



Per $l > \sqrt{12}M$

Orbite Stabili: Precessione



Da *Gravity: An Introduction to Einstein's General Relativity, Hartle*

$$\delta\phi = \Delta\phi - 2\pi$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}\tau} = \frac{\ell}{r^2}$$

$$\frac{\mathrm{d}r}{\mathrm{d}\tau} = \pm\sqrt{e^2 - \left(1 + \frac{\ell^2}{r^2}\right)\left(1 - \frac{2M}{r}\right)}$$

$$\Delta\phi = 2\int_{r_1}^{r_2} \frac{l}{r^2}\left[e^2 - \left(1 + \frac{\ell^2}{r^2}\right)\left(1 - \frac{2M}{r}\right)\right]^{-1/2} \mathrm{d}r$$

Raggi di luce

Le uniche differenze sono la normalizzazione della velocità

$$\mathbf{u} \cdot \mathbf{u} = 0$$

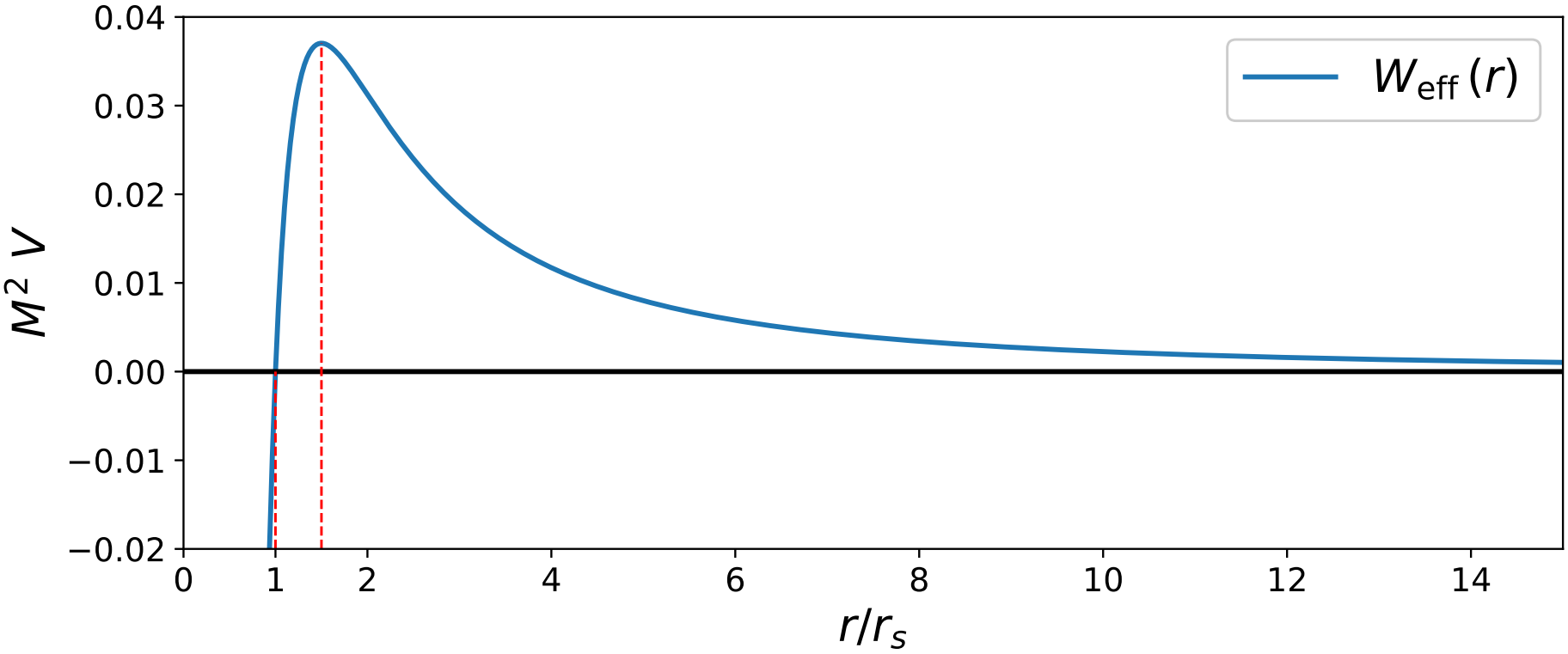
e l'utilizzo di un parametro λ al posto di τ

$$\frac{\mathrm{d}t}{\mathrm{d}\lambda} = \frac{e}{1 - \frac{2M}{r}} \qquad \frac{\mathrm{d}\phi}{\mathrm{d}\lambda} = \frac{l}{r^2}$$

$$\implies e^2 = \left(\frac{\mathrm{d}r}{\mathrm{d}\lambda}\right)^2 + \ell^2 \underbrace{\frac{1}{r^2}\left(1 - \frac{2M}{r}\right)}_{W_{\mathrm{eff}}}$$

Il Potenziale Efficacie W_{eff}

Ha solo un massimo $W_{\text{eff}}(r = 3M) = \frac{1}{27M^2}$



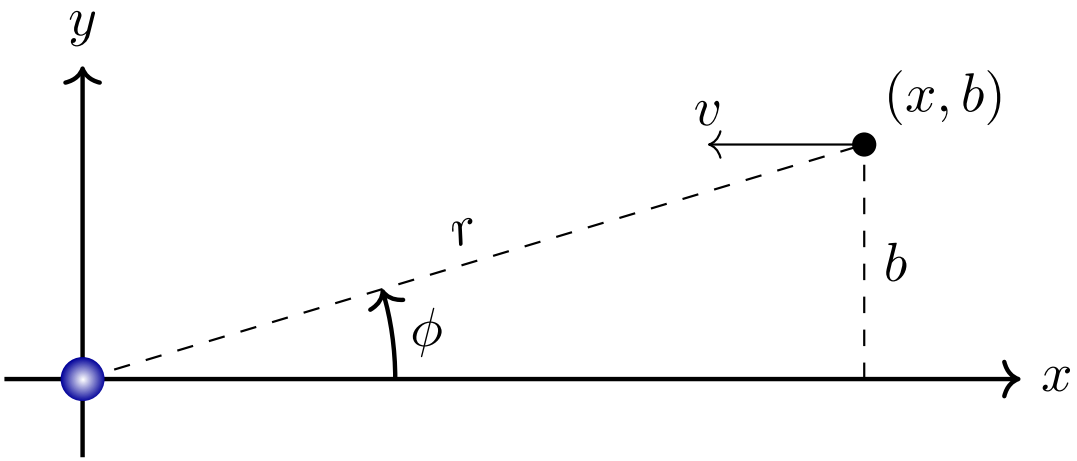
$$W_{\text{eff}}(r) = \frac{1}{r^2}\left(1 - \frac{2M}{r}\right)$$

Il Parametro d'Impatto

$$\frac{e^2}{l^2} = \frac{1}{l^2}\left(\frac{\mathrm{d}r}{\mathrm{d}\lambda}\right)^2 + W_{\mathrm{efff}}$$

Grazie alla possibilità di ridefinire $\lambda \to l^2\lambda$ possiamo scrivere

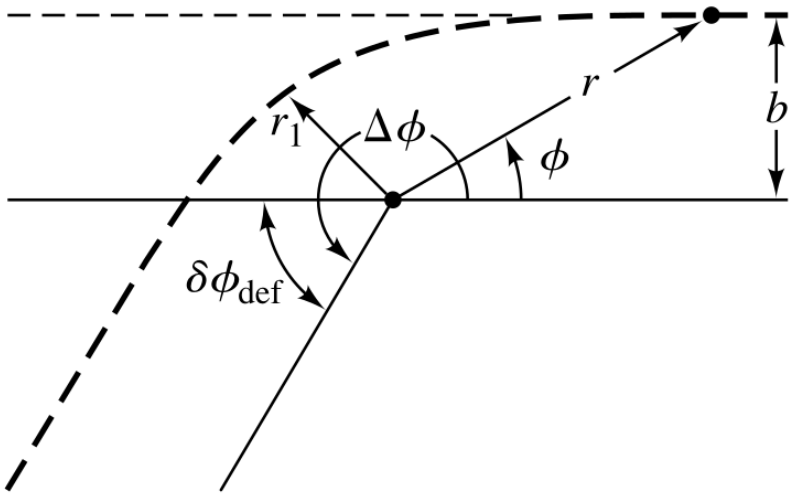
$$\frac{1}{b^2} = \left(\frac{\mathrm{d}r}{\mathrm{d}\lambda}\right)^2 + W_{\mathrm{eff}}(r) \qquad \text{dove} \quad b = \left|\frac{\ell}{e}\right|$$



Se $x \gg b$ allora b è il parametro d'impatto

Deflessione della Luce

$$\delta\phi = 2\int_{r_1}^{\infty} \frac{1}{r^2}\left[\frac{1}{b^2} - \frac{1}{r^2}\left(1 - \frac{2M}{r}\right)\right]^{-1/2} \mathrm{d}r$$



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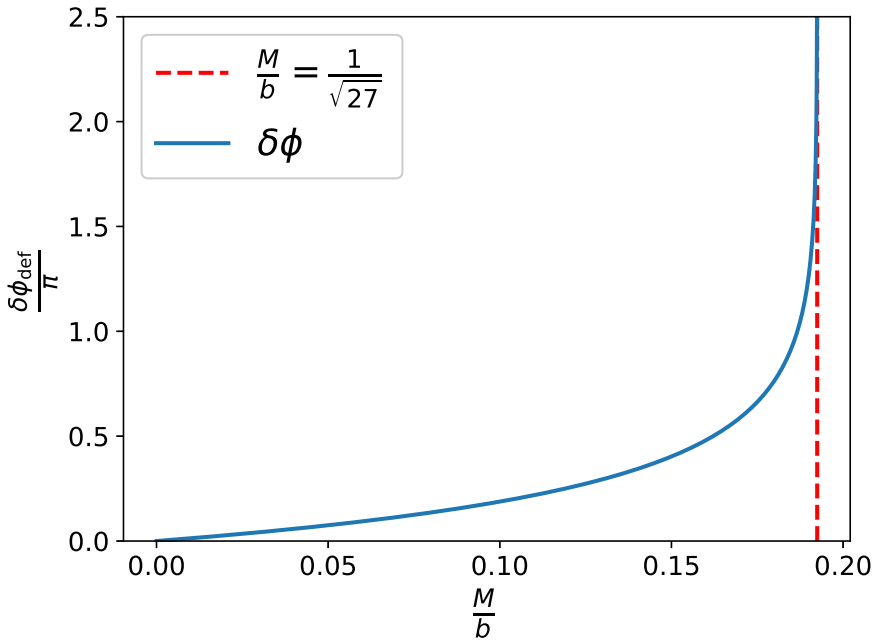
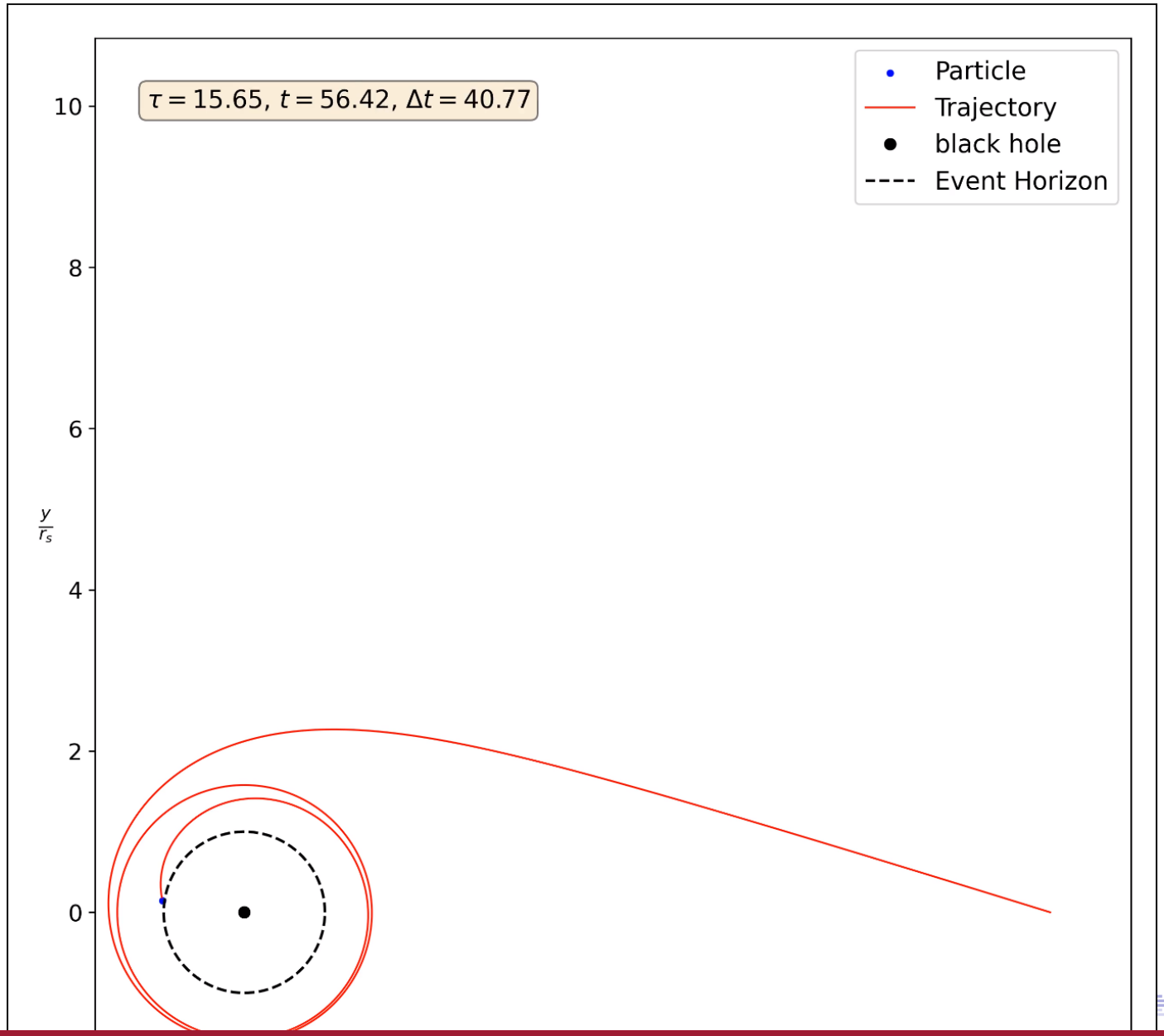


Simulazione

Equazioni del moto per una particella massiva

$$\begin{cases} \dfrac{1}{2}\left(\dfrac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 = \mathcal{E} - V_{\text{eff}}(r) \\ \dfrac{\mathrm{d}\phi}{\mathrm{d}\tau} = \dfrac{l}{r^2} \\ \dfrac{\mathrm{d}t}{\mathrm{d}\tau} = \dfrac{e}{1-2M/r} \end{cases}$$

Video on the computer



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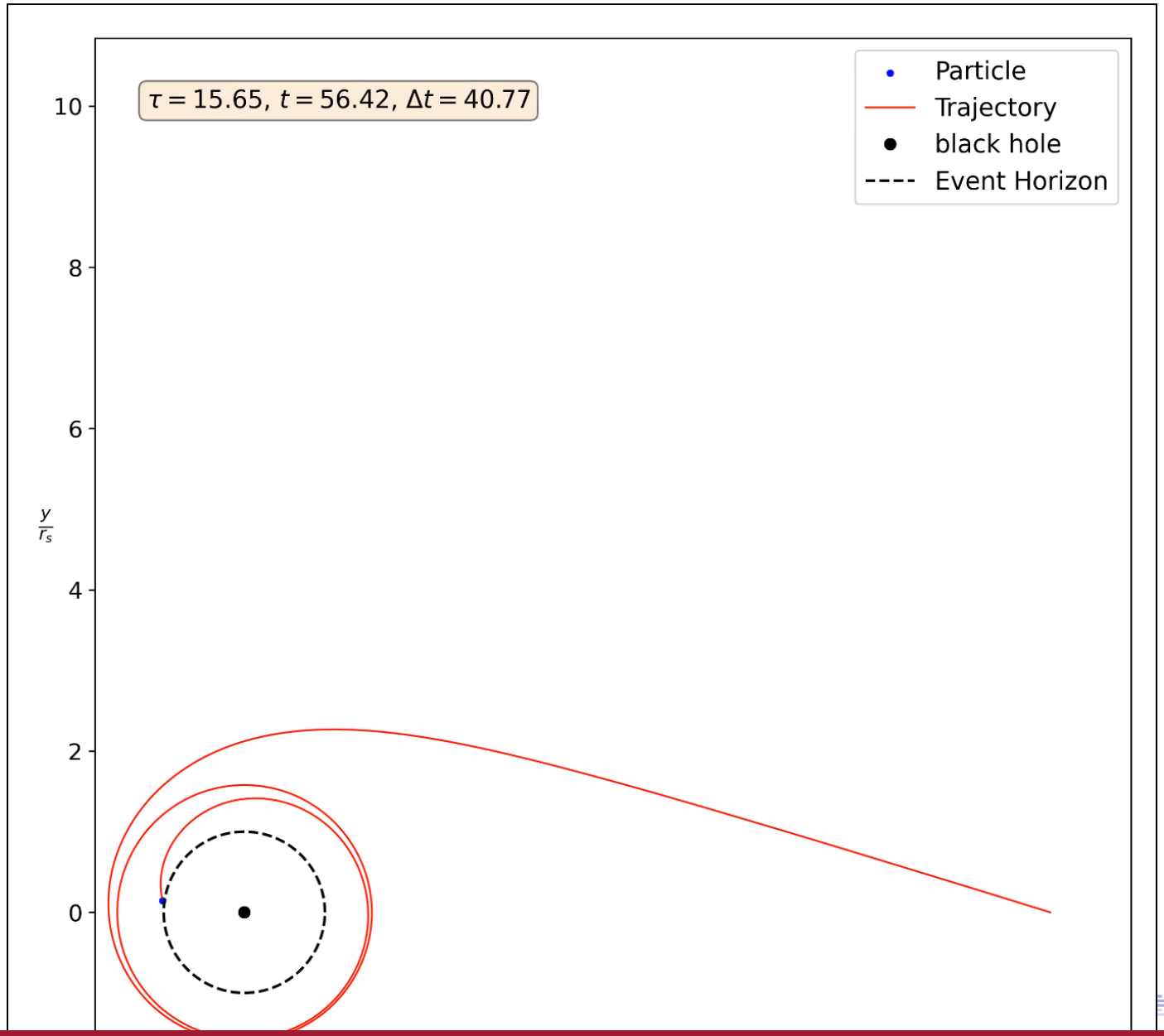

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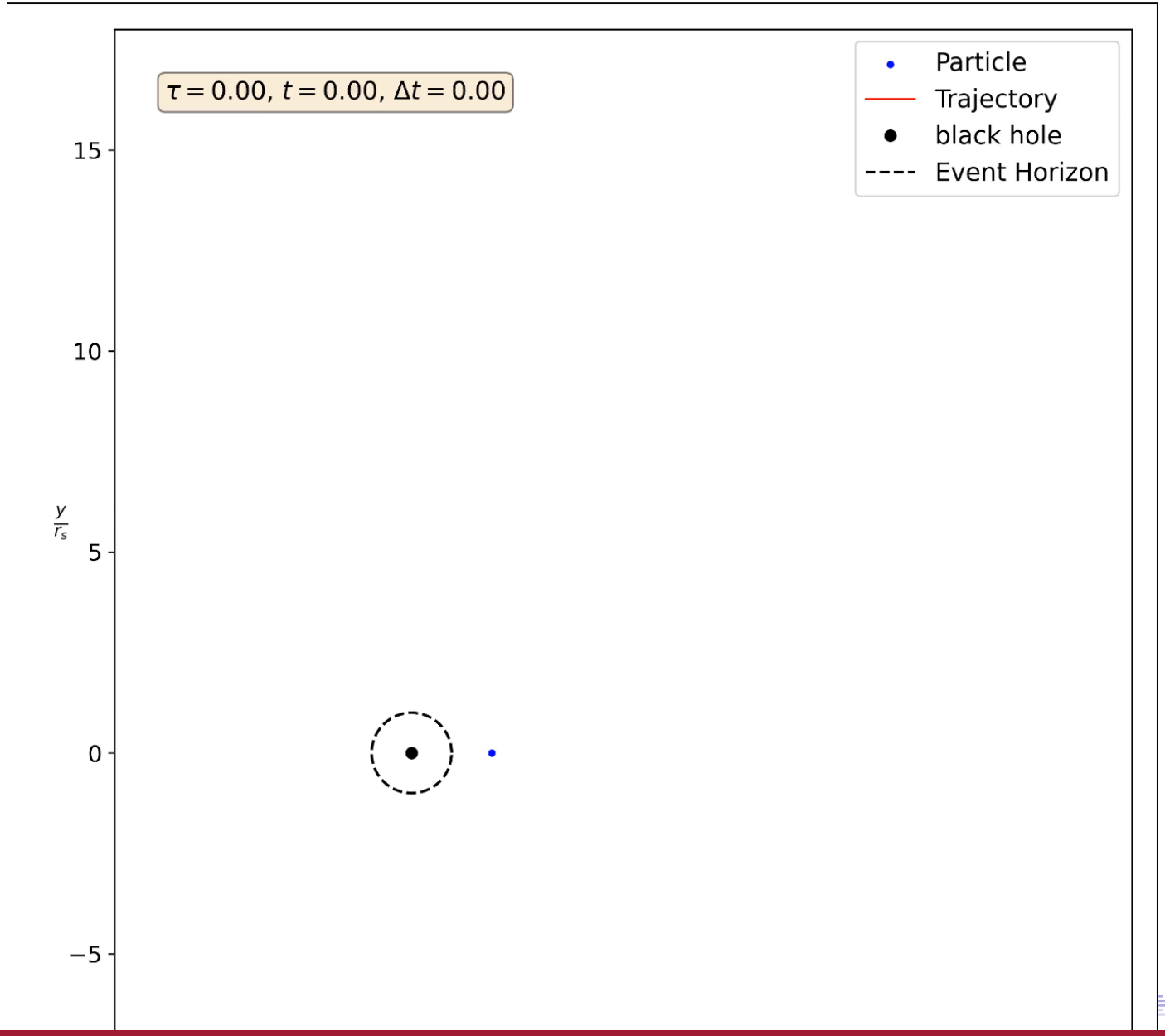

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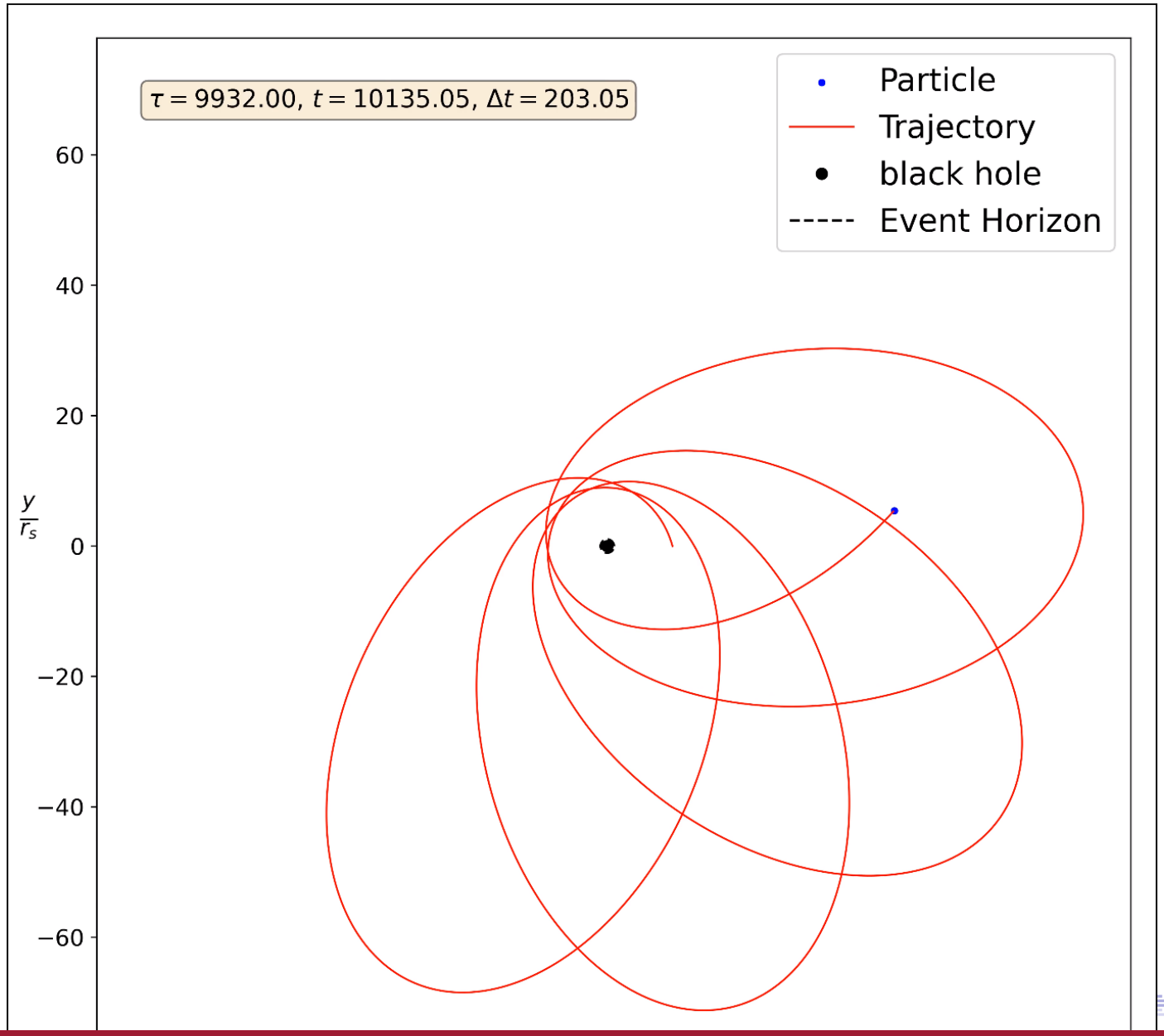

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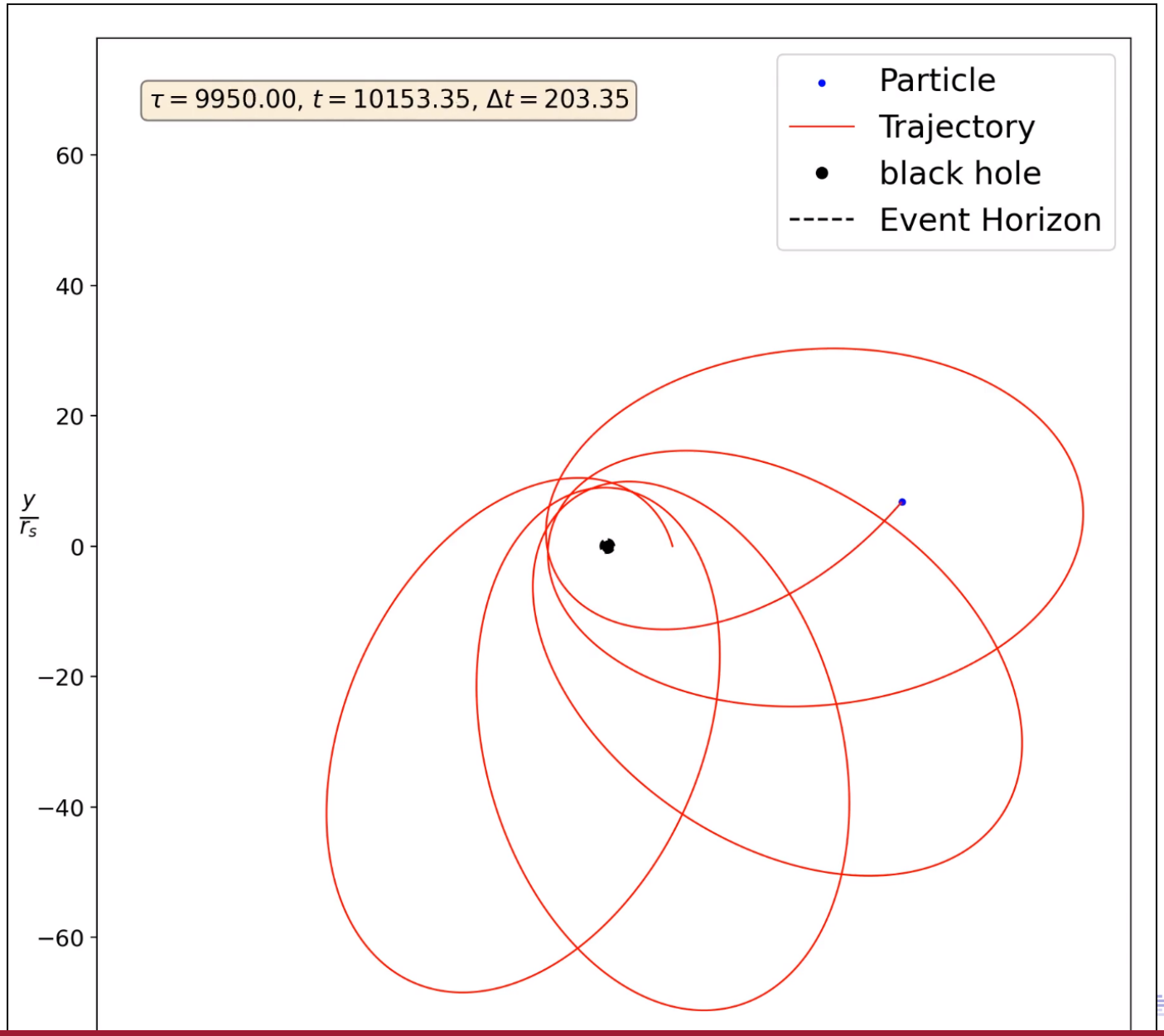

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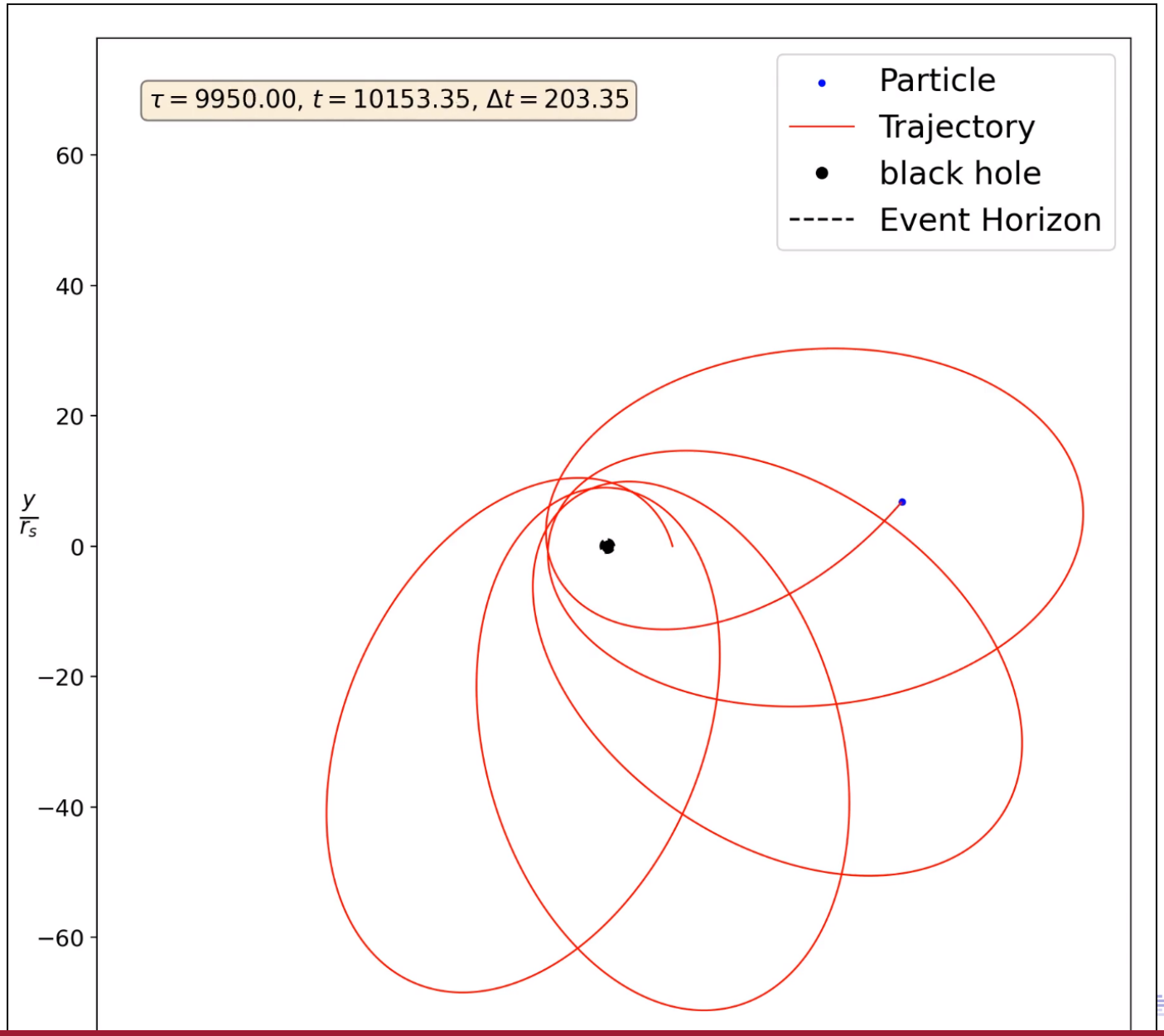

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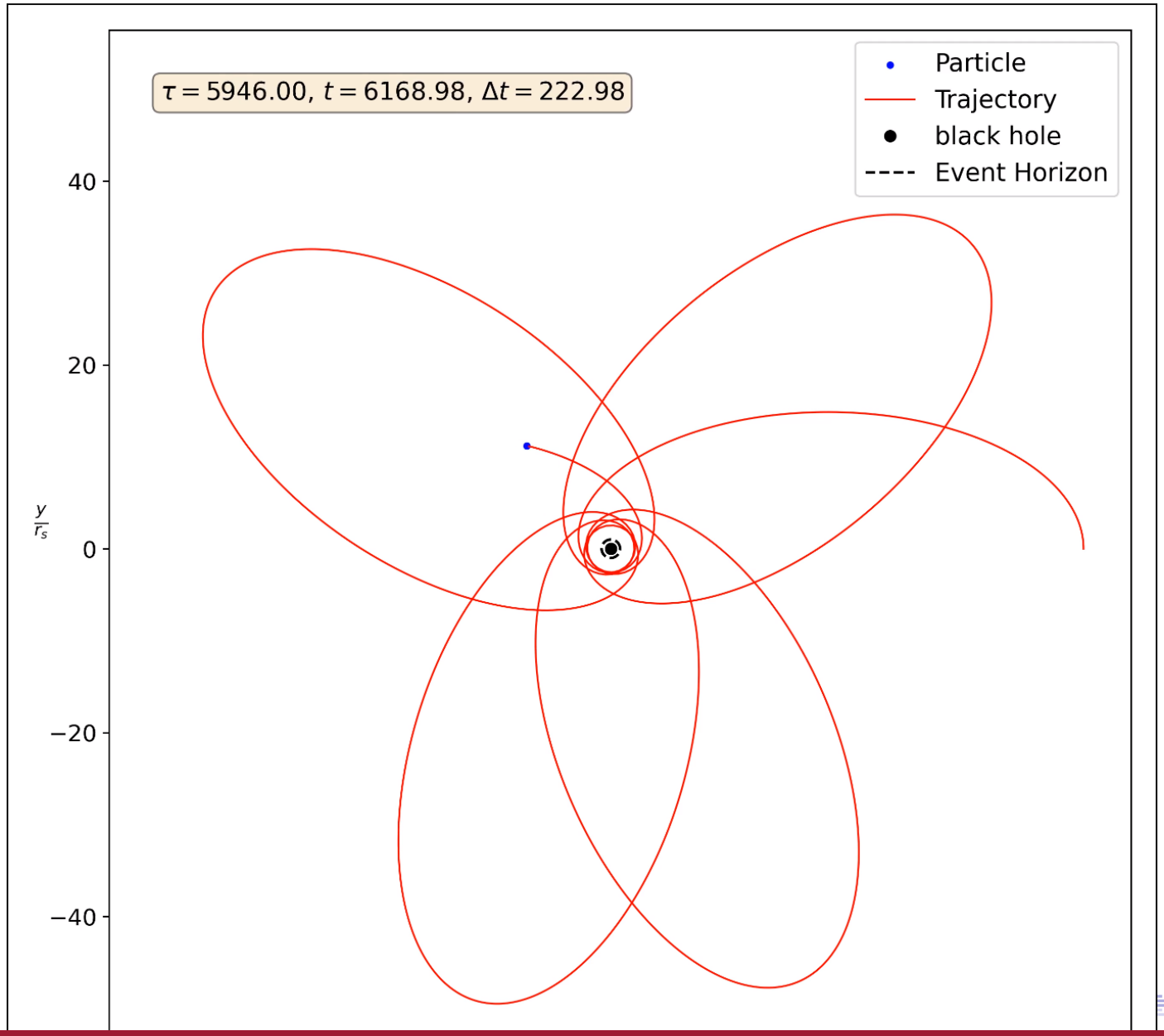

Conclusions

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