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Geodesics in Schwarzschild Metric

Supervisor
Prof. Albino Perego

Graduate Student
Federico De Paoli
227552

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Abstract

Devo fare l'abstract?

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Introduction

Bozza, forse qui va la prima subsection del capitolo uno?
Study of geodesics in Schwarzschild metric
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Chapter 1

Theory

The content of this chapter is mostly based on James B Hartle. Gravity: an introduction to Einstein's general relativity*. Cambridge University Press, 2021 and Stuart L Shapiro and Saul A Teukolsky.* Black holes, white dwarfs, and neutron stars: The physics of compact objects*. John Wiley & Sons, 2008, ch. 12.*

1.1 Introduction

1.1.1 Why the Schwarzschild Geometry

Newtonian mechanics is built upon the concept of absolute time and space. Once the concept of *inertial frame* is well-defined, physics can be done on a space described by Euclidean geometry. Free particles (particles on which no forces are acting) move in a straight line, which is the shortest distance between two points in a three-dimensional space, measured as:

$$\Delta s^2 = \Delta x^2 + \Delta y^2 + \Delta z^2 \,. \tag{1.1}$$

On the other hand, time is *just* seen as a parameter, common to every inertial frame, that can be used to determine the particle velocity and acceleration.

With the appearance of Maxwell's Equations it became clear that what they predicted (the speed of light being constant in every inertial frame) was in contrast with the description of our space given by Newtonian Mechanics, where the speed of anything changes with respect to the inertial frame chosen. Between Maxwell's Equations and Newtonian mechanics Einstein chose to modify the latter and wrote his two postulates for the theory of Special Relativity:

- The laws of physics are invariant (identical) in all inertial frames of reference;
- The speed of light in vacuum, $c = 299\,792\,458 \mathrm{m/s}$, is the same for all observers, regardless of the motion of light source or observer.

The postulates may or may not be intuitive, but simple observations based on them bring us to abandon the idea of absolute space and time and to introduce the concept of *spacetime*, together with a new way of measuring distances

$$\Delta s^2 = -c^2 \Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2 \,. \tag{1.2}$$

In special relativity distances measured this way are the same for every observer in every inertial frame possible.

The appearance of time in a formula that is supposed to give us the distance between two objects is surely destabilizing at first, but geometry teaches us that fixing the way we calculate Δs^2, more properly referred to as the *line element* $\mathrm{d}s^2$, is enough to describe the geometry of the space that we are using. Since eq. 1.2 is different from eq. 1.1, in particular there is a minus sign in front of Δt^2, we moved away from the familiar three-dimensional Euclidean geometry and are now in four-dimensional spacetime, usually referred to as *flat spacetime* or *Minkowski space*.

This new geometry allowed for a reformulation of Maxwell's Equations and brought (and explained) phenomena like time dilation, length contraction and the relativity of simultaneity. The last one in particular, the concept that the simultaneity of two events depends on the frame of reference, poses a threat to the *force* of gravity. Up until this point gravity was defined as the instantaneous force F_{12} acting on a mass m_1 at time t due to a second mass m_2:

$$F_{12} = G\frac{m_1 m_2}{|r_1(t) - r_2(t)|^2} \tag{1.3}$$

The adjective *instantaneous* in a theory where nothing can travel faster than the speed of light should already raise some concern. But looking at $r_1(t)$ and $r_2(t)$ in eq. 1.3, that are supposed to indicate the positions of the masses in the same instant of time, makes it even clearer that the force F_{12} can't be the same in all frames of reference.

Solving this issue gave birth to the theory of general relativity, where a mass is not a source of gravitational force anymore, but is responsible for bending the four-dimensional spacetime itself. This implies that when we observe a particle deviating its trajectory from a straight line in the presence of a massive object, it is not because of a force acting on it. In fact, we can consider the particle free and moving from point A to point B along the shortest path, it is just that in the curved surface bent by the mass the shortest path is not a straight line.

While this concept may not enhance our intuitive understanding, the implications and the mathematical formalism required to articulate the theory are even more challenging. If the presence of mass distorts the space we work in, changing the line element $\mathrm{d}s^2$ is therefore necessary. The details of the theories, particularly the Einstein field equations, that describe this distortion and allow us to evaluate the new $\mathrm{d}s$ from a give distribution of mass are beyond the scope of this thesis. Our focus will be on evaluating the observable effects, given the line element.

More specifically we will study one of the simplest curved spacetime that general relativity has to offer: the geometry of empty space outside a spherically symmetric source of curvature, for example, a spherical star. It is one of the simplest because of the many symmetries that presents and, luckily, is also one of the most useful.

The line element of what is more commonly know as the Schwarzschild geometry is

$$\mathrm{d}s^2 = -\left(1 - \frac{2GM}{c^2 r}\right)(c\mathrm{d}t)^2 + \left(1 - \frac{2GM}{c^2 r}\right)^{-1}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2)$$

expressed in spherical coordinates centered in the mass responsible for bending the space.

1.1.2 Notation and Formalism

In the *flat spacetime* we can introduce a coordinate basis for four-vectors

$$\mathbf{e_t} = (1, 0, 0, 0), \quad \mathbf{e_y} = (0, 1, 0, 0), \quad \mathbf{e_y} = (0, 0, 1, 0), \quad \mathbf{e_z} = (0, 0, 0, 1). \tag{1.4}$$

The set $\{\mathbf{e_t}, \mathbf{e_x}, \mathbf{e_y}, \mathbf{e_z}\}$, is often referred to as $\{\mathbf{e_0}, \mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$. Any four-vector $\mathbf{a}$ can then be written as

$$\mathbf{a} = a^t\mathbf{e_t} + a^x\mathbf{e_x} + a^y\mathbf{e_y} + a^z\mathbf{e_z} = a^0\mathbf{e_0} + a^1\mathbf{e_1} + a^2\mathbf{e_2} + a^3\mathbf{e_3} \tag{1.5}$$

where $(a_t,\ a_x,\ a_y,\ a_z)$, or equivalently $(a_0,\ a_1,\ a_2,\ a_3)$, are the components of the four-vector. Both notations will be used.

Another useful convention is to use Roman letters (usually i or j) to refer to indices 1, 2, 3 and Greek letters (usually μ or ν) to refer to indices 0, 1, 2, 3. Using Einstein notation the expression in eq. 1.5, can be rewritten simply as $\mathbf{a} = a^\mu \mathbf{e}_\mu$. Other useful ways to specify the components of $\mathbf{a}$ are

$$a^\mu = (a^t, a^x, a^y, a^z) \quad a^\mu = (a^t, a^i) \quad a^\mu = (a^t, \vec{a})$$

where $\vec{a} = a^i e_i$ is the tree-dimensional vector (a_x, a_y, a_z).

The length of the four-vector $\mathbf{a}$ must match the definition given with the Δs^2 in 1.2, it is useful to define the *metric* $\eta_{\nu\mu}$ so that

$$\eta_{\nu\mu} = \begin{array}{c c} & \begin{array}{cccc} t & x & y & z \end{array} \\ \begin{array}{c} t \\ x \\ y \\ z \end{array} & \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array} \qquad \Longrightarrow \qquad \mathrm{d}s^2 = \eta_{\nu\mu}\mathrm{d}x^\nu \mathrm{d}x^\mu \tag{1.6}$$

where a double sum is implied, and we rightfully notice that the minus sign has appeared again under the t component. Now we can compactly write

$$\mathbf{a} \cdot \mathbf{a} = \eta_{\mu\nu}\, a^\mu\, a^\nu = -(a^t)^2 + (a^x)^2 + (a^y)^2 + (a^z)^2 \tag{1.7}$$

Without any claim of rigorously demonstrating it, we can say that since this scalar product is built from the line element $\mathrm{d}s^2$, it is the same in every inertial frame one might choose. Quantities that have these properties are *invariant*.

When working in the Schwarzschild geometry it is useful to adopt the Schwarzschild coordinates, spherical coordinates centered at the center of the mass M, and use geometrized units, where $G = c = 1$ (Appendix A). The line element and the metric can be rewritten as

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r}\right)(\mathrm{d}t)^2 + \left(1 - \frac{2M}{r}\right)^{-1} \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2)$$

$$g_{\nu\mu} = \begin{array}{c c} & \begin{array}{cccc} t & r & \theta & \phi \end{array} \\ \begin{array}{c} t \\ r \\ \theta \\ \phi \end{array} & \begin{pmatrix} -(1 - 2M/r) & 0 & 0 & 0 \\ 0 & (1 - 2M/r)^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix} \end{array} . \tag{1.8}$$

It's worth pointing out that, given 1.11, the coordinate basis introduced in 1.4 is not normalized in this geometry, for example:

$$\mathbf{e_t} \cdot \mathbf{e_t} = g_{\nu\mu} e_t^{\mu} e_t^{\nu} = g_{00} = -(1 - 2M/r) \tag{1.9}$$

If we want an orthonormal tetrad we can define

$$\hat{\mathbf{e}}_{\mathbf{t}} = \left(1 - \frac{2M}{r}\right)^{-1/2} \mathbf{e_t} \implies \hat{\mathbf{e}}_{\mathbf{t}} \cdot \hat{\mathbf{e}}_{\mathbf{t}} = g_{\nu\mu} \hat{e}_t^{\mu} \hat{e}_t^{\nu} = g_{00} \left(1 - \frac{2M}{r}\right)^{-1} = -1 \tag{1.10a}$$

$$\hat{\mathbf{e}}_{\mathbf{r}} = \left(1 - \frac{2M}{r}\right)^{1/2} \mathbf{e_r} \implies \hat{\mathbf{e}}_{\mathbf{r}} \cdot \hat{\mathbf{e}}_{\mathbf{r}} = g_{\nu\mu} \hat{e}_r^{\mu} \hat{e}_r^{\nu} = g_{00} \left(1 - \frac{2M}{r}\right) = 1 \tag{1.10b}$$

$$\hat{\mathbf{e}}_{\theta} = \frac{1}{r} \mathbf{e_t} \implies \hat{\mathbf{e}}_{\theta} \cdot \hat{\mathbf{e}}_{\theta} = 1 \tag{1.10c}$$

$$\hat{\mathbf{e}}_{\phi} = \frac{1}{r \sin\theta} \mathbf{e_t} \implies \hat{\mathbf{e}}_{\phi} \cdot \hat{\mathbf{e}}_{\phi} = 1 \tag{1.10d}$$

1.2 Proprieties of the Metric

Let's first analyze the Schwarzschild metric in more detail:

$$\mathrm{d}s^2 = -\left(1-\frac{2M}{r}\right)(\mathrm{d}t)^2 + \left(1-\frac{2M}{r}\right)^{-1}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2) \tag{1.11}$$

There are two singularities in $r = 0$ and $r = 2M$. The first one is intrinsic to the spherical coordinate system and, since the metric is only valid in the space outside the start, doesn't concern us. The second occur at what is defined as the *Schwarzschild radius* $r_s = 2M$. Every non-black hole object has a radius larger than its Schwarzschild radius. The nature and significance of this will become clearer in the subsequent sections.

On the other hand, if we take the limit as r approaches infinity, we notice that the metric becomes asymptotically flat, approaching the metric of Minkowski space.

Finally, $\mathrm{d}s^2$ is independent of the coordinates t and ϕ. This is expected, as the mass responsible for curving the spacetime is static and spherically symmetric. The metric's independence from time and rotation implies the existence of two easy *killing vectors*:

$$\xi = (1, 0, 0, 0) \quad \text{and} \quad \eta = (0, 0, 0, 1)\,. \tag{1.12}$$

A *killing vector* is a direction in the four-dimensional spacetime along which we can freely move without changing the metric. It is a general way to describe a symmetry of the metric. Since symmetries correspond to conserved quantities they will be a key point in studying the trajectories of free particles, the *geodesics*.

We start by considering the four-momentum $\mathbf{p}$ of a particle of mass m, defined as

$$p^\mu := mu^\mu = m\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \tag{1.13}$$

where u is the four-velocity of the particle, x its position and τ the proper time. Therefore, the quantities

$$E = -\xi\cdot\mathbf{p} = -g_{00}\,p^0 = m\left(1-\frac{2M}{r}\right)\frac{\mathrm{d}x^t}{\mathrm{d}\tau}$$

$$L = \eta\cdot\mathbf{p} = g_{33}\,p^\phi = mr^2\sin^2\theta\frac{\mathrm{d}x^\phi}{\mathrm{d}\tau}$$

will be conserved along the geodesic. We already named them E and L as they are respectively the energy and the angular momentum at large r and low velocities. To simplify the expressions used in the discussion will use renormalized quantities

$$e = \frac{E}{m} = \left(1-\frac{2M}{r}\right)\frac{\mathrm{d}x^t}{\mathrm{d}\tau} \tag{1.14a}$$

$$\ell = \frac{L}{m} = r^2\sin^2\theta\frac{\mathrm{d}x^\phi}{\mathrm{d}\tau}\,. \tag{1.14b}$$

e and ℓ are respectively the conserved energy and angular momentum per unit rest mass.

In the next sections the normalization of the four-velocity $\mathbf{u}$ will be really useful too

$$\mathbf{u}\cdot\mathbf{u} = g_{\nu\mu}u^\nu u^\mu = -1 \qquad \text{for } m \neq 0 \tag{1.15a}$$

$$\mathbf{u}\cdot\mathbf{u} = g_{\nu\mu}u^\nu u^\mu = 0 \qquad \text{for } m = 0\,. \tag{1.15b}$$

It's not a property of the metric, but it's valid for every $g_{\nu\mu}$. Equations 1.15 can be derived like this

$$\begin{aligned}\mathrm{d}s^2 &= g_{\nu\mu}\mathrm{d}x^\nu \mathrm{d}x^\mu \\ \frac{\mathrm{d}s^2}{\mathrm{d}\tau^2} &= g_{\nu\mu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\,.\end{aligned}$$

From here we use $\mathrm{d}s^2 = 0$ for a light ray, or the definition of proper time $\mathrm{d}\tau^2 = -\mathrm{d}s^2$ for a massive particle.

1.3 Gravitational Redshift

Let's consider a static observer in r. When the observer measures the energy of a photon, that corresponds to the t component of $\mathbf{p}$, they do that using their local orthonormal tetrad that we described in 1.10.

Referring to $p^{\hat{t}}$ as the value measured in the orthonormal tetrad $\{\hat{e}_t, \hat{e}_r, \hat{e}_\theta, \hat{e}_\phi\}$ and p^t as the value measured with the coordinate basis $\{e_t, e_r, e_\theta, e_\phi\}$, the energy measured in r will be

$$E(r) = p^{\hat{t}} = \mathbf{p}\cdot\mathbf{\hat{e}_t} = \mathbf{p}\cdot\mathbf{e_t}\left(1-\frac{2M}{r}\right)^{-1/2} = \mathbf{p}\cdot\xi\left(1-\frac{2M}{r}\right)^{-1/2}$$

$$\left(1-\frac{2M}{r}\right)^{1/2} E(r) = \mathbf{p}\cdot\xi = \text{const}\,. \tag{1.16}$$

Where we used the expression for $\hat{e}_t$ from eq. 1.10a and notice that $\mathbf{e_t} = \xi$ from eq. 1.4 and eq. 1.12. Solving for the constant $\mathbf{p}\cdot\xi$ we find the expression in 1.16. The relationship between the energy of a photon measured at r' and the one measured at r from two static observers using their own tetrad is

$$\left(1-\frac{2M}{r'}\right)^{1/2} E(r') = \left(1-\frac{2M}{r}\right)^{1/2} E(r)$$

Taking the limit as r' approaches infinity and using $E = \hbar\omega$ for the energy of the photon

$$\omega_\infty = \omega_*\left(1-\frac{2M}{r}\right)^{1/2}\,. \tag{1.17}$$

Here, ω_∞ is the frequency measured by a distant observer at $r \gg r_s$, while w_* denotes the frequency measured at a specific distance r. Photons observed at a certain distance from a star exhibit a lower frequency compared to the one they have at the point of emission.

1.4 Particle Orbits

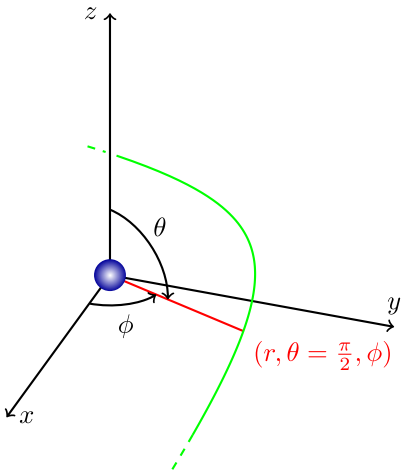

Figure 1.1: Visual representation of the spherical coordinates used. The green line represents a possible trajectory on the xy plane.

To further analyze the Schwarzschild geometry, we will now explore the behavior of a test particle within it. A *test particle* refers to an object with mass so small that its influence on the surrounding spacetime is negligible, allowing us to analyze its motion without altering the geometry of the spacetime itself.
As already established from eq. 1.14b the angular momentum is conserved during the motion of a particle in this geometry. This implies that the particle's orbit must lie within a plane. Without loosing generality we can imagine the particle path to stay in the xy plane, fixing $\theta = \pi/2$ and, consequently,

$$u^\theta = \frac{\mathrm{d}\theta}{\mathrm{d}\tau} = 0$$

$$\ell = r^2 \sin^2\theta \frac{\mathrm{d}\phi}{\mathrm{d}\tau} = r^2 \frac{\mathrm{d}\phi}{\mathrm{d}\tau}$$

Refer to Figure 1.1 for a visual representation.

The four-velocity of our test particle can then be written as

$$u^\mu = \left(\frac{\mathrm{d}x^t}{\mathrm{d}\tau}, \frac{\mathrm{d}x^r}{\mathrm{d}\tau}, \frac{\mathrm{d}x^\theta}{\mathrm{d}\tau}, \frac{\mathrm{d}x^\phi}{\mathrm{d}\tau}\right) = \left(\frac{\mathrm{d}t}{\mathrm{d}\tau}, \frac{\mathrm{d}r}{\mathrm{d}\tau}, 0, \frac{\ell}{r^2}\right) .$$

Where we simplified the notation using $t = x^t$ and $r = x^r$. Thanks to the normalization of $\mathbf{u}$ (eq. 1.15a) and using $\theta = \pi/2$ again, we can write

$$-1 = g_{\nu\mu} u^\nu u^\mu = -\left(1 - \frac{2M}{r}\right)\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 + \left(1 - \frac{2M}{r}\right)^{-1}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 + \frac{\ell^2}{r^2}$$

By using the conserved energy per unit rest mass found in 1.14a to eliminate the dependence from t and rearranging the expression, we have

$$e^2 = \left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 + \left(1 + \frac{\ell^2}{r^2}\right)\left(1 - \frac{2M}{r}\right) . \tag{1.18}$$

To compare eq. 1.18 with the Newtonian case we can expand the multiplication, subtract 1 from both sides and divide by a factor of 2:

$$\mathcal{E} = \frac{e^2 - 1}{2} = \frac{1}{2}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 + \frac{\ell^2}{2r^2} - \frac{M}{r} - \frac{M\ell^2}{r^3} . \tag{1.19}$$

By defining this dimensionless constant $\mathcal{E}$, we can now see the derivative of r as the kinetic energy (per unit rest mass) and the other terms as an effective potential acting on the particle, defined as

$$V_{\text{eff}}(r) = \frac{\ell^2}{2r^2} - \frac{M}{r} - \frac{M\ell^2}{r^3} . \tag{1.20}$$

To better understand eq. 1.20, we can express it in $\mathcal{LMT}$ units, substituting $\ell \to \ell/c$ and $M \to GM/c^2$

$$V_{\text{eff}}(r) = \frac{1}{c^2}\left(\frac{\ell^2}{2r^2} - \frac{GM}{r} - \frac{GM\ell^2}{c^2r^3}\right)$$

The first two terms are identical to the Newtonian potential for a particle with angular momentum (per mass) ℓ, orbiting around an object of mass M. The third one is new, ignorable for $GM\ell^2 \ll c^2r^3$, and it is proportional to r^{-3}.

Figure 1.2 shows the effect of the r^{-3} term: the infinite centrifugal barrier of the Newtonian potential disappears in V_{eff} and a particle with enough energy can fall to the center of the massive object.

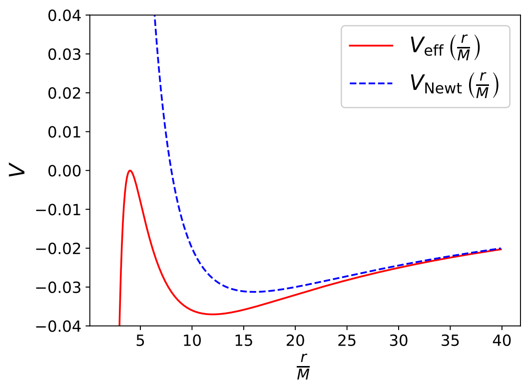

Figure 1.2: Effective potential defined in eq. 1.20 against the Newtonian potential, $\frac{\ell}{M} = 4$. The r^{-3} term dominates for $r \sim r_s$ and the particle can fall into the massive object. On the other hand the Newtonian potential presents its characteristic infinite centrifugal barrier.

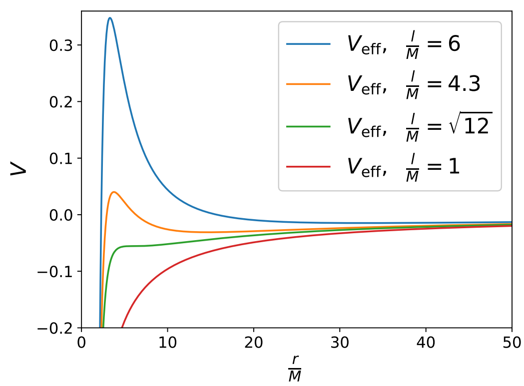

Figure 1.3: Plot of V_{eff} for $\frac{\ell}{M} = [1, \sqrt{12}, 4.3, 6]$. For $\frac{\ell}{M} = \sqrt{12}$ (green) there is only one stationary point and it is not stable. For $\frac{\ell}{M} = 1$ (red) there is no stationary point. The only stable points can be found for $\frac{\ell}{M} > \sqrt{12}$, in $r_{\min}$, defined in eq. 1.21a.

Taking the derivative of V_{eff} with respect to r gives us the stationary points for $\frac{\ell}{M} \geq \sqrt{12}$.

$$r_{+/-} = \frac{\ell^2}{2M}\left[1 \pm \sqrt{1 - 12\left(\frac{M}{\ell}\right)^2}\right] \qquad \frac{\ell}{M} > \sqrt{12} \tag{1.21a}$$

$$r_{\text{ISCO}} = 6M \qquad \frac{\ell}{M} = \sqrt{12} \tag{1.21b}$$

In eq. 1.21a r_- and r_+ correspond to an unstable point and to a stable one respectively.

The case described by eq. 1.21b represents the *innermost stable circular orbit* (ISCO). As the name suggests, it is the smallest stable orbit that a particle can theoretically follow. It is shown in green in Figure 1.3.

For $\frac{\ell}{M} < \sqrt{12}$ there are no stationary points and the particle is destined to fall towards the mass (red line in Figure 1.3).

As in the Newtonian case a bound orbit exists only when V_{eff} has a stable stationary point and the total energy given to the particle is not greater than $V_{\text{eff}}(r_-)$ (a more detailed explanation in Section 1.6).

1.5 The Simplest Geodesics: Radial Infalls

The simplest case we can consider is a radial infall where ϕ stays constant. From eq. 1.14b this implies $\ell = 0$.

Eq. 1.18 becomes

$$e^2 = \left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 + 1 - \frac{2M}{r}$$

$$\frac{\mathrm{d}r}{\mathrm{d}\tau} = -\sqrt{e^2 - 1 + \frac{2M}{r}}\,. \tag{1.22}$$

Where we chose the negative root as the radius is decreasing. Looking at the parameter e in eq. 1.22 we can distinguish 3 cases:

- $e^2 < 1$: the particle has to start from a finite radius $r = R$ for the argument of the square root to be positive;
- $e = 1$: the particle starts at rest from $r = \infty$ (that implies $\frac{\mathrm{d}t}{\mathrm{d}\tau} = 1$ at infinity so $e = 1$ from eq. 1.14a);
- $e^2 > 1$: the particle starts from $r = \infty$, but with some inward velocity.

We choose to analyze the case where $e = 1$ so that eq. 1.22 can be solved analytically. We get

$$\frac{\mathrm{d}r}{\mathrm{d}\tau} = -\sqrt{\frac{2M}{r}}\,. \tag{1.23}$$

$$r^{1/2}\mathrm{d}r = -(2M)^{1/2}\mathrm{d}\tau$$

$$r(\tau) = \left(\frac{3}{2}\right)^{2/3}(2M)^{1/3}(\tau_* - \tau)^{2/3}\,. \tag{1.24}$$

Where τ_* is an integration constant that fixes the proper time τ when the particle arrives at $r = 0$. An observer that falls together with the particle will measure a finite time when he reaches $r = 2M$ and $\tau = \tau_*$ at $r = 0$.

On the contrary the experience from an observer far away from the source of curvature, that measures the Schwarzschild time t it's much different. To see this we can solve eq. 1.23 in respect to the Schwarzschild time t. Using the chain rule to derive r in respect of t and eliminating the dependence from τ using eq. 1.14a with $e = 1$, we get

$$-\sqrt{\frac{2M}{r}} = \frac{\mathrm{d}r}{\mathrm{d}\tau} = \frac{\mathrm{d}t}{\mathrm{d}\tau}\frac{\mathrm{d}r}{\mathrm{d}t} = \left(1 - \frac{2M}{r}\right)^{-1}\frac{\mathrm{d}r}{\mathrm{d}t}$$

$$\frac{\mathrm{d}t}{\mathrm{d}r} = -\left(\frac{2M}{r}\right)^{-1/2}\left(1 - \frac{2M}{r}\right)^{-1}$$

The integration is not as immediate as the previous one, but can be resolved with simple techniques. The result, fixing a similar time constant t_* as before, is

$$t = t_* + 2M\left[-\frac{3}{2}\left(\frac{r}{2M}\right)^{3/2} - 2\left(\frac{r}{2M}\right)^{1/2} + \ln\left|\frac{(r/2M)^{1/2} + 1}{(r/2M)^{1/2} - 1}\right|\right]\,. \tag{1.25}$$

We can't find $r(t)$ explicitly as was done previously, but the expression in 1.25 already tells us that for $r \to 2M$ the time measured by a far observer goes to infinity. This implies that the distant observer will never see the particle reach $r = 2M$ and enter the Schwarzschild radius. They will instead measure a signal infinitely redshifted as described by eq. 1.17, asymptotically going to $\omega_\infty = 0$. That is why the Schwarzschild radius is also referred to as a *source of infinite redshift*. Figure 1.4 shows eq. 1.24 and eq 1.25, the integration constants τ_* and t_* where chosen to make both equations start from the same point $r \simeq 12M$ at $\tau = t = 0$.

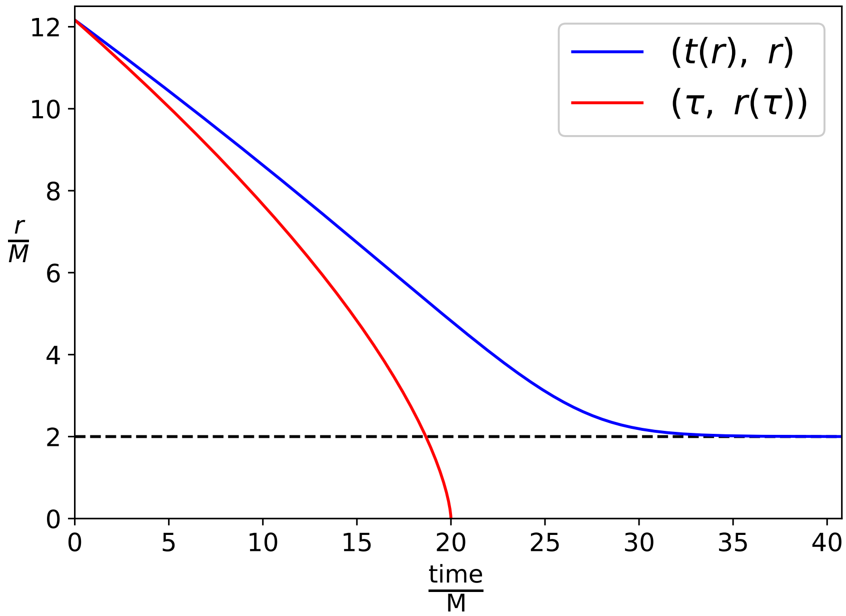

Figure 1.4: Eq. 1.24 and eq. 1.25 describing the fall from $r \simeq 12M$ on. The integration constants τ_* and t_* where fixed so that both equations started from the same $r(0)$. $r(\tau)$ gets to 0 quickly (at $\tau = \tau_*$), while $t(r)$ goes to infinity for $r \to r_s = 2M$. The Schwarzschild radius r_s is represented with the dashed black line.

1.6 Stable Orbits

In Section 1.4 we analyzed the role that $\frac{\ell}{M}$ has on the effective potential V_{eff} defined in eq. 1.20 and found out that for $\frac{\ell}{M} > \sqrt{12}$ it has two stationary points $r_{+/-}$ defined in eq. 1.21a.

For clarity, we rewrite eq. 1.19 below, accompanied by a visual representation in Figure 1.5.

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 = \mathcal{E} - V_{\text{eff}}(r) \tag{1.26}$$

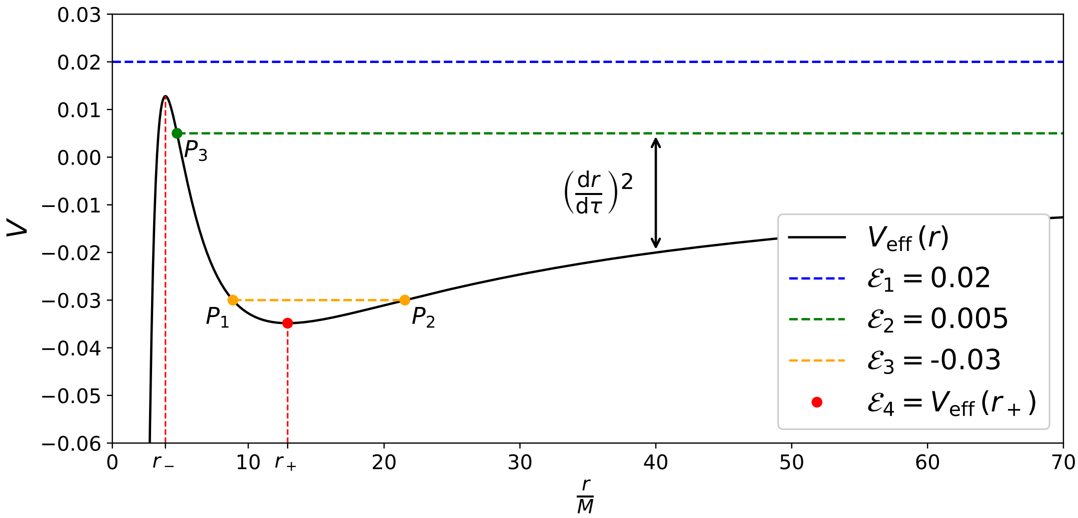

Figure 1.5: In black the effective potential with $\ell/M = 4.1$. The dashed lines, along with the red dot, represent possible values of $\mathcal{E}$ that give 4 different scenarios. Refer to the text for a detailed explanation. *This figure inspired from* ***shapiro2008black****, page 245, Figure 12.2.*

From eq. 1.26 we know that the particle can change its distance r from the massive object only if there is the energy $\mathcal{E} - V_{\text{eff}}$ to do so. In Figure 1.5, V_{eff} with $\ell/M = 4.1$ is represented along with 4 different values of $\mathcal{E}$.

The red full dot represents the case where the particle has exactly the energy required to stay in the local minimum of the potential. Here there is no *spare* energy that the particle can use to move along r. This is the case of a circular orbit, and it will be explored in Section 1.6.1

In the yellow case, $\mathcal{E} = -0.03$, the particle has some *spare* energy (not enough to escape) to change its radius between the two turning points P_1 and P_2 This results in an ellipse-like shape around the massive object. [1]

Finally, the green case, $\mathcal{E} = 0.005$, and the blue one, $\mathcal{E} = 0.02$, are not stable orbits. In the first case, the particle has one turning point P_3 and has enough energy to escape the potential well and go back towards infinity. In the latter, the particle as a bigger energy than $V_{\text{eff}}(r_-)$ and can fall into the massive object.

[1]As we will see in Section 1.6.2 it's not exactly an ellipse as in the Newtonian case.

1.6.1 Circular Orbits

As we have seen from Figure 1.5 a particle can draw a circular orbits if it has an energy $\mathcal{E} = V_{\text{eff}}(r_+)$. It is possible to find the relationship between the energy e and angular momentum ℓ that makes this possible.

First of all, the four-velocity of the particle will be

$$u^\mu = \left(\frac{\mathrm{d}t}{\mathrm{d}\tau}, 0, 0, \frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right) = \left(\frac{\mathrm{d}t}{\mathrm{d}\tau}, 0, 0, \frac{\mathrm{d}t}{\mathrm{d}\tau}\Omega\right) = u^t(1, 0, 0, \Omega) \tag{1.27}$$

Where we defined $\Omega := \frac{\mathrm{d}\phi}{\mathrm{d}t}$: the rate of which ϕ changes with respect to the Schwarzschild time t. Using eq. 1.14a and eq. 1.14b, Ω can be rewritten as

$$\Omega = \frac{\mathrm{d}\phi}{\mathrm{d}t} = \frac{\mathrm{d}\tau}{\mathrm{d}t}\frac{\mathrm{d}\phi}{\mathrm{d}\tau} = \frac{1}{r^2}\left(1 - \frac{2M}{r}\right)\frac{\ell}{e}\,. \tag{1.28}$$

A second equation for $\frac{\ell}{e}$ comes with the restriction that the particle must orbit at the minimum of V_{eff}, from eq. 1.21a

$$r = \frac{\ell^2}{2M}\left[1 + \sqrt{1 - 12\left(\frac{M}{\ell}\right)^2}\,\right]\,. \tag{1.29}$$

Using eq. 1.18 the derivative of r vanishes and we can write everything as

$$e^2 = \left(1 + \frac{\ell^2}{r^2}\right)\left(1 - \frac{2M}{r}\right)\,. \tag{1.30}$$

Instead of substituting r from eq. 1.29 into eq. 1.30, it is easier to solve eq. 1.29 for $1/\ell^2$ obtaining

$$\frac{1}{\ell^2} = \frac{1}{Mr} - \frac{3}{r^2}\,, \tag{1.31}$$

and then use it to rewrite eq. 1.30 as

$$\frac{e^2}{\ell^2} = \left(\frac{1}{\ell^2} + \frac{1}{r^2}\right)\left(1 - \frac{2M}{r}\right) = \left(\frac{1}{Mr} - \frac{3}{r^2} + \frac{1}{r^2}\right)\left(1 - \frac{2M}{r}\right) = \frac{1}{Mr}\left(1 - \frac{2M}{r}\right)^2$$

$$\frac{\ell}{e} = \sqrt{Mr}\left(1 - \frac{2M}{r}\right)^{-1} \tag{1.32}$$

We can finally substitute eq. 1.32 into eq. 1.28 to get

$$\Omega^2 = \frac{M}{r^3} \tag{1.33}$$

that gives the angular velocity observed from infinity of a particle in a circular orbit. With the value of Ω found in eq. 1.33 we can find the normalization of the four-velocity defined in 1.27.

$$-1 = \mathbf{u} \cdot \mathbf{u} = g_{\nu\mu} u^\nu u^\mu = (u^t)^2 \left[-\left(1 - \frac{2M}{r}\right) + r^2 \Omega^2 \right]$$
$$(u^t)^2 = \left[1 - \frac{2M}{r} - \frac{M}{r}\right]^{-1}$$
$$u^t = \left(1 - \frac{3M}{r}\right)^{-1/2}$$

Therefore, we have

$$u^\mu = \left(1 - \frac{3M}{r}\right)^{-1/2} \left(1, 0, 0, \sqrt{\frac{M}{r^3}}\right) \tag{1.34}$$

1.6.2 General Shapes and Precession

To complete the discussion on bound orbit we can have a look at a more general case, where $V_{\text{eff}} < \mathcal{E} < 0$. If we want to characterize the shape of the orbit, as always restricted to the xy plane, we need to express ϕ as a function of r. To do so we can use eq. 1.14b and eq. 1.18 and rewrite them respectively as

$$\frac{\mathrm{d}\phi}{\mathrm{d}\tau} = \frac{\ell}{r^2} \tag{1.35a}$$
$$\frac{\mathrm{d}r}{\mathrm{d}\tau} = \pm\sqrt{e^2 - \left(1 + \frac{\ell^2}{r^2}\right)\left(1 - \frac{2M}{r}\right)}. \tag{1.35b}$$

This time we had no reason to keep one sign or the other in eq. 1.35b. Dividing 1.35b into 1.35a gives

$$\frac{\mathrm{d}\phi}{\mathrm{d}r} = \pm\frac{\ell}{r^2} \left[e^2 - \left(1 + \frac{\ell^2}{r^2}\right)\left(1 - \frac{2M}{r}\right)\right]^{-1/2} \tag{1.36}$$

The function $\phi(r)$ can be found simply by integrating the right-hand side. The result can be expressed in terms of elliptic functions but not in a very enlightening way for those not familiar with them[2].

An interesting parameter to study is the *precession*, defined as

$$\delta\phi_{\text{prec}} = \Delta\phi - 2\pi \tag{1.37}$$

To define $\Delta\phi$ consider the turning points P_1 and P_2 in Figure 1.5. *One orbit* is complete when the particle starts from the inner turning point, P_1, and gets back to it. We can equivalently consider P_2 as a starting and ending point. $\Delta\phi$ is the angle swept during one orbit. It's also useful to notice that the angle swept in one orbit is twice the angle swept between P_1 and P_2, thus we can rewrite $\Delta\phi$ as

$$\Delta\phi = \int_{P_1}^{P_2} \frac{\mathrm{d}\phi}{\mathrm{d}r}\mathrm{d}r = 2\ell \int_{r_1}^{r_2} \frac{1}{r^2} \left[e^2 - \left(1 + \frac{\ell^2}{r^2}\right)\left(1 - \frac{2M}{r}\right)\right]^{-1/2} \mathrm{d}r \tag{1.38}$$

[2] **hartle2021gravity**, page 202.

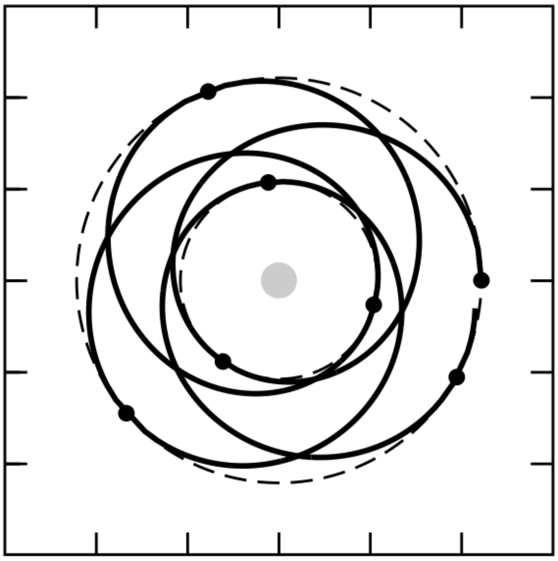

Figure 1.6: placeholder

By the definition of the turning points, r_1 and r_2, the integral we need to compute is between the two zeros of the denominator contained in square brackets. To check the correspondence with the Newtonian case we can solve the integral in 1.38 neglecting the r^{-3} term and remembering the definition of $\mathcal{E}$ from eq. 1.19:

$$\Delta\phi = 2\ell \int_{r_1}^{r_2} \frac{1}{r^2} \left[\mathcal{E} - \frac{\ell^2}{r^2} + \frac{2M}{r}\right]^{-1/2} \mathrm{d}r$$

Making the substitution $u = 1/r$ and using the root of the denominator as the bounds of integration we get

$$\Delta\phi = 2 \int_{u_2}^{u_1} \frac{\mathrm{d}u}{\sqrt{(u_1 - u)(u - u_2)}} = 2\pi \qquad \forall u_1 \neq u_2 \tag{1.39}$$

where $u_{1/2} = \frac{1}{r_{1/2}}$, but the integral is π for every u_1 and u_2.

1.7 Light Ray orbits

When working with light, more properly photons, we define the four-velocity using the *affine parameter* λ.

$$u^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\,.$$

The conserved quantities e and ℓ defined in 1.14a and 1.14b are the same, with the only difference of using λ instead of the proper time τ.

$$e = \left(1 - \frac{2M}{r}\right)\frac{\mathrm{d}t}{\mathrm{d}\lambda}$$
$$\ell = r^2 \sin^2\theta \frac{\mathrm{d}\phi}{\mathrm{d}\lambda}\,.$$

We can therefore restrict the motion of the light ray motion to the xy plane as we did with massive particle and write four-velocity as

$$u^\mu = \left(\frac{\mathrm{d}t}{\mathrm{d}\lambda}, \frac{\mathrm{d}r}{\mathrm{d}\lambda}, 0, \frac{\ell}{r^2}\right)\,.$$

The only difference lies, as described in eq. 1.15b, in the normalization of $\mathbf{u}$, that is $\mathbf{u}\cdot\mathbf{u} = 0$. So, with the same intent we had when we found eq. 1.18 we can write

$$0 = g_{\nu\mu}u^\nu u^\mu = -\left(1 - \frac{2M}{r}\right)\left(\frac{\mathrm{d}t}{\mathrm{d}\lambda}\right)^2 + \left(1 - \frac{2M}{r}\right)^{-1}\left(\frac{\mathrm{d}r}{\mathrm{d}\lambda}\right)^2 + \frac{\ell^2}{r^2}$$

and then rearrange

$$\frac{1}{b^2} = \frac{1}{\ell^2}\left(\frac{\mathrm{d}r}{\mathrm{d}\lambda}\right)^2 + W_{\mathrm{eff}}(r)\,. \tag{1.41}$$

Where we defined another effective potential, one that acts on light rays only, as

$$W_{\mathrm{eff}}(r) = \frac{1}{r^2}\left(1 - \frac{2M}{r}\right) \tag{1.42}$$

and a different constant for the energy

$$b^2 = \frac{\ell^2}{e^2}\,. \tag{1.43}$$

Chapter 2

Simulations

Computed in `C` and animated in `python`

$$
\begin{cases}
\left(\dfrac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 = e^2 - \left(1 - \dfrac{2M}{r}\right)\left(1 + \dfrac{l^2}{r^2}\right) & \text{(2.1a)} \\
\dfrac{\mathrm{d}\phi}{\mathrm{d}\tau} = \dfrac{l}{r^2} & \text{(2.1b)} \\
\dfrac{\mathrm{d}t}{\mathrm{d}\tau} = \dfrac{e}{1 - 2M/r} & \text{(2.1c)}
\end{cases}
$$

Conclusions

Appendix A

Geometrized Units

In general relativity the constants $G \simeq 6.674\,30 \times 10^{-11}\mathrm{N\,m^2\,kg^{-2}}$ and $c = 299\,792\,458\mathrm{m/s}$ appears quite often, so it's useful to redefine our units of measurements to cancel them out. Let's take the Schwarzschild line element as an example

$$\mathrm{d}s^2 = -\left(1 - \frac{2GM}{c^2 r}\right)(c\mathrm{d}t)^2 + \left(1 - \frac{2GM}{c^2 r}\right)^{-1}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2)\,. \tag{A.1}$$

Time always appears next to the speed of light, ct. This is effectively as if we were measuring time with the distance that light can cover in t seconds.

The mass M is measured in kg in S.I. units. If we multiply it by G and divide by c^2 we get

$$\left[\frac{GM}{c^2}\right] = \frac{\mathrm{N\,m^2\,kg^{-2}\,kg}}{\mathrm{m^2\,s^{-2}}} = \frac{\mathrm{N}}{\mathrm{kg}}\mathrm{s}^2 = \mathrm{m}\,.$$

So, in a less intuitive way, we can measure the mass as a distance too.

Substituting $\hat{t} = ct$ and $\hat{M} = \frac{GM}{c^2}$ in eq. A.1 gives

$$\mathrm{d}s^2 = -\left(1 - \frac{2\hat{M}}{r}\right)\mathrm{d}\hat{t}^2 + \left(1 - \frac{2\hat{M}}{r}\right)^{-1}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2)\,.$$

In this way we went from a $\mathcal{LMT}$(Length Mass Time) units system, to an $\mathcal{L}$ one. Since it will be clearly said when this convention, loosely referred to as $G = c = 1$, is in use we will omit the *hat* in the new defined variables.

To go back to $\mathcal{LMT}$ units we just need to substitute back $t \to ct$ and $M \to \frac{GM}{c^2}$, being extra careful on cases like the speed, that is derived from time and therefor inherits a $1/c$ factor.

Table A.1 shows some typical masses values expressed in geometrical units.

	S.I. units	Geometrized units
Mass of the Earth	5.97×10^{24} kg	4.43 mm
Mass of the Sun	1.99×10^{30} kg	1.48 km
M87 black hole	$6.5 \times 10^{9} M_\odot$	64.2 au

Table A.1: Some common masses of the universe expressed in unit of length.

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