Black Holes, White Dwarfs, and Neutron Stars
THE PHYSICS OF COMPACT OBJECTS
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12 Black Holes

12.1 Introduction

We have seen in previous chapters that both white dwarfs and neutron stars have a maximum possible mass. What happens to a neutron star that accretes matter and exceeds the mass limit? What is the fate of the collapsing core of a massive star, if the core mass is too large to form a neutron star? The answer, according to general relativity, is that nothing can halt the collapse. As the collapse proceeds, the gravitational field near the object becomes stronger and stronger. Eventually, nothing can escape from the object to the outside world, not even light. A black hole has been born.

A black hole is defined simply as a region of spacetime that cannot communicate with the external universe. The boundary of this region is called the surface of the black hole, or the *event horizon*.[1]

The ultimate fate of collapsing matter, once it has crossed the black hole surface, is not known. Densities for a $1 M_\odot$ object are $\sim 10^{17}$ g cm^{-3} as the black hole is formed, and are smaller for larger masses. How can we be sure that some hitherto unknown source of pressure does not become important above such extreme densities and halt the collapse? The answer is that by the time a black hole forms it is already too late to hold back the collapse: matter must move on worldlines *inside* the local light cone, and the spacetime geometry is so distorted that even an "outward" light ray does not escape. In fact, since all forms of energy gravitate in relativity, increasing the pressure energy only *accelerates* the late stages of the collapse.

If we extrapolate Einstein's equations all the way inside a black hole, they ultimately break down: a *singularity* develops. There is as yet no quantum theory of gravitation, and some people believe that the singularity would not occur in

[1] Strictly speaking, the event horizon is a three-dimensional hypersurface in spacetime (a 2-surface existing for some time interval). We shall, however, speak loosely of the event horizon or black hole surface as a 2-surface at some instant of time.

such a theory. It would be replaced by finite, though unbelievably extreme, conditions.

Exercise 12.1 Construct a density by dimensional analysis out of c, G, and $\hbar$. Evaluate numerically this "Planck density" at which quantum gravitational effects would become important.

Answer: $\rho \sim 10^{94}$ g cm^{-3}.

As long as the singularity is hidden inside the event horizon, it cannot influence the outside world. The singularity is said to be "causally disconnected" from the exterior world. We can continue to use general relativity to describe the observable universe, even though the theory breaks down inside the black hole.

One might expect that the solutions of Einstein's equations describing equilibrium black holes would be extremely complicated. After all, black holes can be formed from stars with varying mass distributions, shapes (multipole moments), magnetic field distributions, angular momentum distributions, and so on. Remarkably, the most general stationary black hole solution is known analytically. It depends on only three parameters: the mass M, angular momentum J, and charge Q of the black hole. All other information about the initial state is radiated away in the form of electromagnetic and gravitational waves during the collapse. The remaining three parameters are the only independent observable quantities that characterize a stationary black hole.[2] This situation is summarized by Wheeler's aphorism, "A black hole has no hair."

The mass of a black hole is observable, for example, by applying Kepler's Third Law for satellites in the Newtonian gravitational field far from the black hole. The charge is observable by the Coulomb force on a test charge far away. The angular momentum is observable by non-Newtonian gravitational effects. For example, a torque-free gyroscope will precess relative to an inertial frame at infinity (Lense–Thirring effect).

12.2 History of the Black Hole Idea

As early as 1795 Laplace (1795) noted that a consequence of Newtonian gravity and Newton's corpuscular theory of light was that light could not escape from an object of sufficiently large mass and small radius. In spite of this early foreshadowing of the possibility of black holes, the idea found few adherents, even after the formulation of general relativity.

In December of 1915 and within a month of the publication of Einstein's series of four papers outlining the theory of general relativity, Karl Schwarzschild (1916) derived his general relativistic solution for the gravitational field surround-

[2]See Carter (1979) for a complete discussion.

ing a spherical mass. Schwarzschild sent his paper to Einstein to transmit to the Berlin Academy. In replying to Schwarzschild, Einstein wrote, "I had not expected that the exact solution to the problem could be formulated. Your analytical treatment of the problem appears to me splendid." Although the significance of the result was apparent to both men, neither they nor anyone else knew at that time that Schwarzschild's solution contained a complete description of the external field of a spherical, electrically neutral, nonrotating black hole. Today we refer to such black holes as *Schwarzschild black holes*, in honor of Schwarzschild's great contribution.

As we described in Chapter 3, Chandrasekhar (1931b) discovered in 1930 the existence of an upper limit to the mass of a completely degenerate configuration. Remarkably, Eddington (1935) realized almost immediately that if Chandrasekhar's analysis was to be accepted, it implied that the formation of black holes would be the inevitable fate of the evolution of massive stars. He thus wrote in January 1935: "The star apparently has to go on radiating and radiating and contracting and contracting until, I suppose, it gets down to a few kilometers radius when gravity becomes strong enough to hold the radiation and the star can at last find peace." But he then went on to declare, "I felt driven to the conclusion that this was almost a reductio ad absurdum of the relativistic degeneracy formula. Various accidents may intervene to save the star, but I want more protection than that. I think that there should be a law of Nature to prevent the star from behaving in this absurd way."

As is clear from his concluding remarks, Eddington never accepted Chandrasekhar's result of the existence of an upper limit to the mass of a cold, degenerate star. This in spite of Eddington's being one of the first to understand and appreciate Einstein's theory of general relativity! (His book *The Mathematical Theory of Relativity* (1922) was the first textbook on general relativity to appear in English.) In fact, Eddington subsequently proceeded to modify the equation of state of a degenerate relativistic gas so that finite equilibrium states would exist for stars of arbitrary mass.[3]

But Eddington was not alone in his misgivings about the inevitability of collapse as the end product of the evolution of a massive star. Landau (1932), in the same paper giving his simple derivation of the mass limit (cf. Section 3.4), acknowledged that for stars exceeding the limit, "there exists in the whole quantum theory no cause preventing the system from collapsing to a point." But rather then follow the sober advice put forth at the beginning of his paper ("It seems reasonable to try to attack the problem of stellar structure by methods of theoretical physics"), Landau, in the end, retreats and declares, "As in reality such masses exist quietly as [normal] stars and do not show any such tendencies,

[3]Chandrasekhar (1980) has recently lamented Eddington's shortsightedness regarding black holes, declaring, "Eddington's supreme authority in those years effectively delayed the development of fruitful ideas along these lines for some thirty years."

we must conclude that all stars heavier than $1.5M_{\odot}$ certainly possess regions in which the laws of quantum mechanics (and therefore quantum statistics) are violated."

In 1939 Oppenheimer and Snyder (1939) revived the discussion by calculating the collapse of a homogeneous sphere of pressureless gas in general relativity. They found that the sphere eventually becomes cut off from all communication with the rest of the Universe. This was the first rigorous calculation demonstrating the formation of a black hole.

Black holes and the problem of gravitational collapse were generally ignored until the 1960s, even more so than neutron stars. However, in the late 1950s, J. A. Wheeler and his collaborators began a serious investigation of the problem of collapse.[4] Wheeler (1968) coined the name "black hole" in 1968.

In 1963 R. Kerr (1963) discovered an exact family of charge-free solutions to Einstein's vacuum field equations. The charged generalization was subsequently found as a solution to the Einstein–Maxwell field equations by Newman et al. (1965). Only later was the connection of these results to black holes appreciated. We know today that the *Kerr–Newman geometry* described by these solutions provides a unique and complete description of the external gravitational and electromagnetic fields of a stationary black hole.

A number of important properties of black holes were discovered and several powerful theorems concerning black holes were proved during this period. The discovery of quasars in 1963, pulsars in 1968, and compact X-ray sources in 1962 helped motivate this intensive theoretical study of black holes. Observations of the binary X-ray source Cygnus X-1 in the early 1970s (cf. Section 13.5) provided the first plausible evidence that black holes might actually exist in space.

We turn now from history to a discussion of the physics of black holes. We shall begin our treatment with a discussion of the simplest black hole, one with $J = Q = 0$.

12.3 Schwarzschild Black Holes

We repeat here the Schwarzschild solution from Eq. (5.6.8):

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\phi^2. \tag{12.3.1}$$

We are using the geometrized units ($c = G = 1$) of Section 5.5.

[4]See Harrison et al. (1965) for an account of these investigations.

A static observer in this gravitational field is one who is at fixed r, θ, ϕ. The lapse of proper time for such an observer is given by Eq. (12.3.1) as

$$d\tau^2 = -ds^2 = \left(1 - \frac{2M}{r}\right) dt^2, \tag{12.3.2}$$

or

$$d\tau = \left(1 - \frac{2M}{r}\right)^{1/2} dt. \tag{12.3.3}$$

This simply shows the familiar gravitational time dilation (redshift) for a clock in the gravitational field compared with a clock at infinity (i.e., $d\tau < dt$). Note that Eq. (12.3.3) breaks down at $r = 2M$, which is the *event horizon* ($\equiv$ *surface of the black hole* $\equiv$ *Schwarzschild radius*). Another name for this is the *static limit*, because static observers cannot exist inside $r = 2M$; they are inexorably drawn into the central singularity, as we shall see later.

A static observer makes measurements with his or her local orthonormal tetrad (Section 5.1). Using carets to denote quantities in the local orthonormal frame, we have from Eq. (12.3.1)

$$\begin{aligned}
\vec{\mathbf{e}}_{\hat{t}} &= \left(1 - \frac{2M}{r}\right)^{-1/2} \vec{\mathbf{e}}_t, \\
\vec{\mathbf{e}}_{\hat{r}} &= \left(1 - \frac{2M}{r}\right)^{1/2} \vec{\mathbf{e}}_r, \\
\vec{\mathbf{e}}_{\hat{\theta}} &= \frac{1}{r} \vec{\mathbf{e}}_\theta, \\
\vec{\mathbf{e}}_{\hat{\phi}} &= \frac{1}{r \sin\theta} \vec{\mathbf{e}}_\phi.
\end{aligned} \tag{12.3.4}$$

This is clearly an orthonormal frame, since[5]

$$\vec{\mathbf{e}}_{\hat{t}} \cdot \vec{\mathbf{e}}_{\hat{t}} = \left(1 - \frac{2M}{r}\right)^{-1} \vec{\mathbf{e}}_t \cdot \vec{\mathbf{e}}_t = \left(1 - \frac{2M}{r}\right)^{-1} g_{tt} = -1, \text{ etc.} \tag{12.3.5}$$

12.4 Test Particle Motion

To explore the Schwarzschild geometry further, let us consider the motion of freely moving test particles. Recall from Eq. (5.2.21) that such particles move

[5]The reader may wish to review the last part of Section 5.2, which discusses the relationship between an orthonormal frame and a general coordinate system.

along geodesics of spacetime, the geodesic equations being derivable from the Lagrangian

$$2L = -\left(1 - \frac{2M}{r}\right)\dot{t}^2 + \left(1 - \frac{2M}{r}\right)^{-1}\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2, \quad (12.4.1)$$

where $\dot{t} \equiv dt/d\lambda = p^t$ is the t-component of 4-momentum, and so on. Here we have chosen the parameter λ to satisfy $\lambda = \tau/m$ for a particle of mass m.

The Euler–Lagrange equations are

$$\frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{x}^\alpha}\right) - \frac{\partial L}{\partial x^\alpha} = 0, \qquad x^\alpha = (t, r, \theta, \phi). \quad (12.4.2)$$

For θ, ϕ, and t these are, respectively,

$$\frac{d}{d\lambda}(r^2\dot{\theta}) = r^2 \sin\theta\cos\theta\,\dot{\phi}^2, \quad (12.4.3)$$

$$\frac{d}{d\lambda}(r^2\sin^2\theta\,\dot{\phi}) = 0, \quad (12.4.4)$$

$$\frac{d}{d\lambda}\left[\left(1 - \frac{2M}{r}\right)\dot{t}\right] = 0. \quad (12.4.5)$$

Instead of using the r-equation directly, it is simpler to use the fact that

$$g_{\alpha\beta}p^\alpha p^\beta = -m^2. \quad (12.4.6)$$

In other words, in Eq. (12.4.1) L has the value $-m^2/2$.

Now Eq. (12.4.3) shows that if we orient the coordinate system so that initially the particle is moving in the equatorial plane (i.e., $\theta = \pi/2$, $\dot{\theta} = 0$), then the particle remains in the equatorial plane. This result follows from the uniqueness theorem for solutions of such differential equations, since $\theta = \pi/2$ for all λ satisfies the equation. Physically, the result is obvious from spherical symmetry.

With $\theta = \pi/2$, Eqs. (12.4.4) and (12.4.5) become

$$p_\phi \equiv r^2\dot{\phi} = \text{constant} \equiv l, \quad (12.4.7)$$

$$-p_t \equiv \left(1 - \frac{2M}{r}\right)\dot{t} = \text{constant} \equiv E. \quad (12.4.8)$$

These are simply the constants of the motion corresponding to the ignorable coordinates ϕ and t in Eq. (12.4.1) (cf. Section 5.2). To understand their physical significance, consider a measurement of the particle's energy made by a static observer in the equatorial plane. This locally measured energy is the time component of the 4-momentum as measured in the observer's local orthonormal frame—that is, the projection of the 4-momentum along the time basis vector:

$$E_{\text{local}} \equiv p^{\hat{t}} = -p_{\hat{t}} = -\vec{\mathbf{p}} \cdot \vec{\mathbf{e}}_{\hat{t}} = -\vec{\mathbf{p}} \cdot \left(1 - \frac{2M}{r}\right)^{-1/2} \vec{\mathbf{e}}_t$$

$$= -\left(1 - \frac{2M}{r}\right)^{-1/2} p_t,$$

that is,

$$E = \left(1 - \frac{2M}{r}\right)^{1/2} E_{\text{local}}. \tag{12.4.9}$$

For $r \to \infty$, $E_{\text{local}} \to E$, so the conserved quantity E is called the "energy-at-infinity." It is related to E_{local} by a redshift factor.

Exercise 12.2 For an alternative derivation of the redshift formula, use the fact that E is constant along the photon's path to show that

$$\frac{\nu_{\text{em}}}{\nu_{\text{rec}}} = \left(1 - \frac{2M}{r_{\text{em}}}\right)^{-1/2} \tag{12.4.10}$$

for a static emitter at $r = r_{\text{em}}$ and a receiver at $r \to \infty$. Explain why the event horizon for a Schwarzschild black hole is sometimes called the "surface of infinite redshift."

The physical interpretation of l follows from considering the locally measured value of $v^{\hat{\phi}}$, the tangential velocity component:

$$v^{\hat{\phi}} = \frac{p^{\hat{\phi}}}{p^{\hat{t}}} = \frac{p_{\hat{\phi}}}{p^{\hat{t}}} = \frac{\vec{\mathbf{p}} \cdot \vec{\mathbf{e}}_{\hat{\phi}}}{E_{\text{local}}} = \frac{\vec{\mathbf{p}} \cdot \vec{\mathbf{e}}_{\phi}/r}{E_{\text{local}}} = \frac{p_{\phi}/r}{E_{\text{local}}},$$

and so

$$l = E_{\text{local}} r v^{\hat{\phi}}. \tag{12.4.11}$$

Comparing with the Newtonian expression $mv^{\hat{\phi}}r$, we see that l is the conserved angular momentum of the particle.

We now consider separately the cases $m \neq 0$ and $m = 0$. For particles of nonzero rest mass, it is convenient to renormalize E and l to quantities expressed per unit mass. Define

$$\tilde{E} = \frac{E}{m}, \qquad \tilde{l} = \frac{l}{m}. \tag{12.4.12}$$

Then, recalling that $\lambda = \tau/m$, we find from Eqs. (12.4.6)–(12.4.8):

$$\left(\frac{dr}{d\tau}\right)^2 = \tilde{E}^2 - \left(1 - \frac{2M}{r}\right)\left(1 + \frac{\tilde{l}^2}{r^2}\right), \tag{12.4.13}$$

$$\frac{d\phi}{d\tau} = \frac{\tilde{l}}{r^2}, \tag{12.4.14}$$

$$\frac{dt}{d\tau} = \frac{\tilde{E}}{1 - 2M/r}. \tag{12.4.15}$$

Equation (12.4.13) can be solved for $r = r(\tau)$ (in general, an elliptic integral); then Eq. (12.4.14) gives $\phi(\tau)$ and Eq. (12.4.15) gives $t(\tau)$.

It is interesting to consider orbits just outside the event horizon. The locally measured value of $v^{\hat{r}}$, the radial velocity component, is given by

$$v^{\hat{r}} = \frac{p^{\hat{r}}}{p^{\hat{t}}} = \frac{p_{\hat{r}}}{p^{\hat{t}}} = \frac{\vec{\mathbf{p}} \cdot \vec{\mathbf{e}}_{\hat{r}}}{E_{\text{local}}} = \frac{p_r(1 - 2M/r)^{1/2}}{E_{\text{local}}} = \frac{p^r}{E}, \tag{12.4.16}$$

from Eqs. (12.3.4) and (12.4.9). Recalling $p^r \equiv m\, dr/d\tau$ and Eq. (12.4.13), we get

$$v^{\hat{r}} = \frac{dr}{\tilde{E}\, d\tau} = \left[1 - \frac{1}{\tilde{E}^2}\left(1 - \frac{2M}{r}\right)\left(1 + \frac{\tilde{l}^2}{r^2}\right)\right]^{1/2}. \tag{12.4.17}$$

So as $r \to 2M$, $v^{\hat{r}} \to 1$ and the particle is observed by a local static observer at r to approach the event horizon along a *radial* geodesic at the *speed of light*, independent of $\tilde{l}$.

Exercise 12.3 Show that the same observer at r finds that the tangential velocity of the particle satisfies

$$v^{\hat{\phi}} = \left(1 - \frac{2M}{r}\right)^{1/2} \frac{\tilde{l}}{r\tilde{E}}, \tag{12.4.18}$$

so that $v^{\hat{\phi}} \to 0$ as $r \to 2M$.

Exercise 12.4

(a) Show from Eq. (12.4.17) that a local observer at r finds that the velocity of a radially freely-falling particle released from rest at infinity is given by

$$v^{\hat{r}} = \left(\frac{2m}{r}\right)^{1/2}, \tag{12.4.19}$$

which has precisely the same form as the Newtonian velocity!

(b) Obtain the same result from Eq. (12.4.9), noting that $E_{\text{local}} \equiv \gamma m$.

Exercise 12.5 A particle moves along a geodesic from r and ϕ to $r + dr$ and $\phi + d\phi$ in time dt. A local static observer at (r, ϕ) measures the proper length of the particle's path to have increased by $ds(t, \theta, \phi = \text{const}) = g_{rr}^{1/2}\, dr (= d\hat{r})$ and $ds(t, r, \theta = \text{const}) = g_{\phi\phi}^{1/2}\, d\phi (= d\hat{\phi})$ in the r and ϕ directions, respectively, during this time; the proper time for this motion as measured on the observer's clock lasts $[-ds^2(r, \theta, \phi = \text{const})]^{1/2} = (-g_{00})^{1/2}\, dt (= d\hat{t})$. [Note that $d\hat{t}$ for the *observer* is *not* equal to $d\tau$ appearing, e.g., in Eqs. (12.4.13)–(12.4.15) for the particle!] Use the expressions for these measurements together with Eqs. (12.4.13)–(12.4.15) to rederive Eqs. (12.4.17) and (12.4.18).

The simplest geodesics are those for radial infall, ϕ = constant. This occurs if $\tilde{l} = 0$, and Eq. (12.4.13) becomes

$$\frac{dr}{d\tau} = -\left(\tilde{E}^2 - 1 + \frac{2M}{r}\right)^{1/2}. \tag{12.4.20}$$

By considering the limit $r \to \infty$ of Eq. (12.4.20), we see that there are three cases: (i) $\tilde{E} < 1$, particle falls from rest at $r = R$, say; (ii) $\tilde{E} = 1$, particle falls from rest at infinity; (iii) $\tilde{E} > 1$, particle falls with finite inward velocity from infinity, $v \equiv v_\infty$.

Exercise 12.6

(a) Integrate Eq. (12.4.20) for the case $\tilde{E} < 1$, so that $1 - \tilde{E}^2 = 2M/R$, to get ($\tau = 0$ at $r = R$):

$$\tau = \left(\frac{R^3}{8M}\right)^{1/2}\left[2\left(\frac{r}{R} - \frac{r^2}{R^2}\right)^{1/2} + \cos^{-1}\left(\frac{2r}{R} - 1\right)\right]. \tag{12.4.21}$$

(b) Introduce the "cycloid parameter" η by

$$r = \frac{R}{2}(1 + \cos\eta), \tag{12.4.22}$$

and show that

$$\tau = \left(\frac{R^3}{8M}\right)^{1/2}(\eta + \sin\eta). \tag{12.4.23}$$

(c) Integrate Eq. (12.4.15) for t in terms of η to get ($t = 0$ at $r = R$):

$$\frac{t}{2M} = \ln\left|\frac{(R/2M - 1)^{1/2} + \tan(\eta/2)}{(R/2M - 1)^{1/2} - \tan(\eta/2)}\right| + \left(\frac{R}{2M} - 1\right)^{1/2}\left[\eta + \frac{R}{4M}(\eta + \sin\eta)\right]. \tag{12.4.24}$$

Note the following important results for radial infall: from Eq. (12.4.21), the *proper time* to fall from rest at $r = R > 2M$ to $r = 2M$ is *finite*. In fact, the proper time to fall to $r = 0$ is $\pi(R^3/8M)^{1/2}$, also finite. However, from Eqs. (12.4.23) and (12.4.24), the *coordinate time* (proper time for an observer at infinity) to fall to $r = 2M$ is *infinite* [at $r = 2M$, $\tan(\eta/2) = (R/2M - 1)^{1/2}$]. These results are displayed in Figure 12.1.

Exercise 12.7

(a) Find $\tau(r)$ and $t(r)$ for radial infall when $\tilde{E} = 1$.

(b) Find $\tau(r)$, $r(\eta)$, $\tau(\eta)$, and $t(\eta)$ when $\tilde{E} > 1$. You can get these from Eqs. (12.4.21)–(12.4.24) by defining R such that $2M/R = \tilde{E}^2 - 1$ and changing the sign of R in these equations. Show that $2M/R = v_\infty^2/(1 - v_\infty^2)$.

Answer: See Lightman et al. (1975), p. 407.

Turn now to nonradial motion. The elliptic integrals resulting from Eqs. (12.4.13)–(12.4.15) are not particularly informative, but we can get a general picture of the orbits by considering an "effective potential,"

$$V(r) \equiv \left(1 - \frac{2M}{r}\right)\left(1 + \frac{\tilde{l}^2}{r^2}\right). \tag{12.4.25}$$

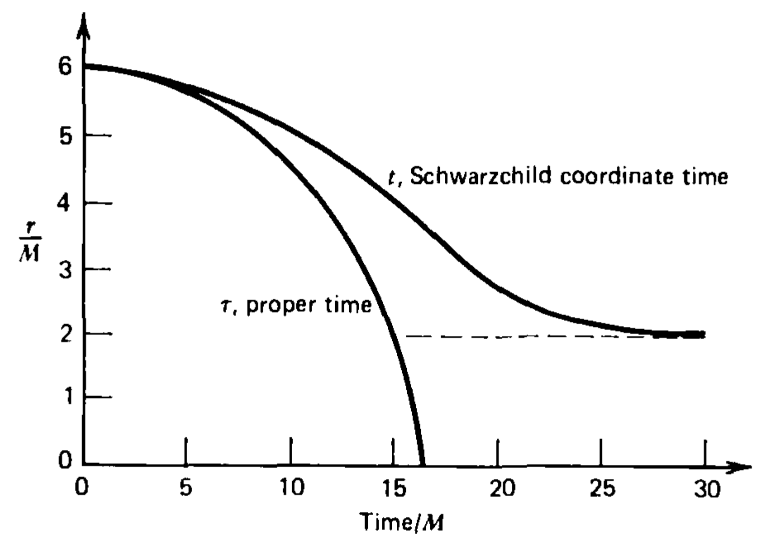


Figure 12.1 Fall from rest toward a Schwarzschild black hole as described (a) by a comoving observer (proper time τ) and (b) by a distant observer (Schwarzschild coordinate time t). In the one description, the point $r = 0$ is attained, and quickly [see Eq. (12.4.23)]. In the other description, $r = 0$ is never reached and even $r = 2M$ is attained only asymptotically [Eq. (12.4.24)]. [From *Gravitation* by Charles W. Misner, Kip S. Thorne, and John Archibald Wheeler, W. H. Freeman and Company. Copyright © 1973.]

Equation (12.4.13) then becomes

$$\left(\frac{dr}{d\tau}\right)^2 = \tilde{E}^2 - V(r). \tag{12.4.26}$$

For a fixed value of $\tilde{l}$, V is depicted schematically in Figure 12.2. Shown on the diagram are three horizontal lines corresponding to different values of $\tilde{E}^2$. From Eq. (12.4.26) we see that the distance from the horizontal line to V gives $(dr/d\tau)^2$. Consider orbit 1, the horizontal line labeled 1 corresponding to a particle coming in from infinity with energy $\tilde{E}^2$. When the particle reaches the value of r corresponding to point A, $dr/d\tau$ passes through zero and changes sign—the particle returns to infinity. Such an orbit is *unbound*, and A is called a *turning point*. Orbit 2 is a *capture* orbit; the particle plunges into the black hole. Orbit 3 is a *bound* orbit, with two turning points A_1 and A_2. The point B corresponds to a *stable circular* orbit. If the particle is slightly perturbed away from B, the orbit remains close to B. The point C is an *unstable circular* orbit; a particle placed in such an orbit will, upon experiencing the slightest inward radial perturbation, fall toward the black hole and be captured. If it is perturbed outward, it flies off to infinity. Orbits like 1 and 3 exist in the Newtonian case for motion in a central gravitational field; capture orbits are unique to general relativity.

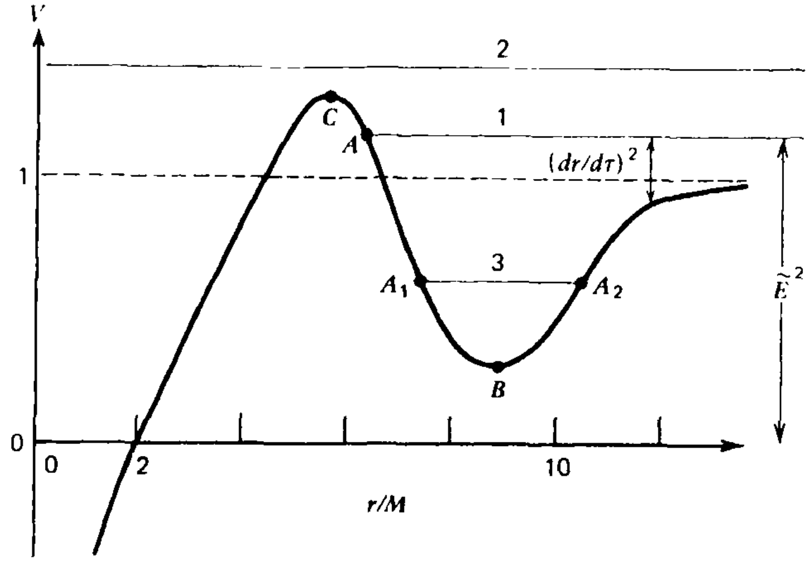


Figure 12.2 Sketch of the effective potential profile for a particle with *nonzero* rest mass orbiting a Schwarzschild black hole of mass M. The three horizontal lines labeled by different values of $\tilde{E}^2$ correspond to an (1) unbound, (2) capture, and (3) bound orbit, respectively. See text for details.

Exercise 12.8 Show that Eq. (12.4.26) reduces to the familiar Newtonian expression for particle motion in a central gravitational field when $2M/r \ll 1$.

Exercise 12.9

(a) Show that $\partial V/\partial r = 0$ when

$$Mr^2 - \tilde{l}^2 r + 3M\tilde{l}^2 = 0, \tag{12.4.27}$$

and hence that there are no maxima or minima of V for $\tilde{l} < 2\sqrt{3}\,M$.

(b) Show that $V_{\max} = 1$ for $\tilde{l} = 4M$.

The variation of V with $\tilde{l}$ is shown in Figure 12.3.

Circular orbits occur when $\partial V/\partial r = 0$ and $dr/d\tau = 0$. Equations (12.4.26) and (12.4.27) give

$$\tilde{l}^2 = \frac{Mr^2}{r - 3M}, \tag{12.4.28}$$

$$\tilde{E}^2 = \frac{(r - 2M)^2}{r(r - 3M)}. \tag{12.4.29}$$

Thus circular orbits exist down to $r = 3M$, the limiting case corresponding to a photon orbit ($\tilde{E} = E/m \to \infty$). The circular orbits are stable if V is concave up; that is, $\partial^2 V/\partial r^2 > 0$ and unstable if $\partial^2 V/\partial r^2 < 0$ (Why?).

Exercise 12.10 Show the circular Schwarzschild orbits are stable if $r > 6M$, unstable if $r < 6M$.

Exercise 12.11

(a) Show that in Newtonian theory, a distant nonrelativistic test particle can only be captured by a star of mass M and radius R if

$$\tilde{l} < \tilde{l}_{\text{crit}} \simeq (2MR)^{1/2}.$$

(b) Taking into account general relativity, can particles with much larger values of angular momentum be captured by neutron stars? by white dwarfs?

The binding energy per unit mass of a particle in the last stable circular orbit at $r = 6M$ is, from Eq. (12.4.29),

$$\tilde{E}_{\text{binding}} = \frac{m - E}{m} = 1 - \left(\frac{8}{9}\right)^{1/2} = 5.72\%. \tag{12.4.30}$$

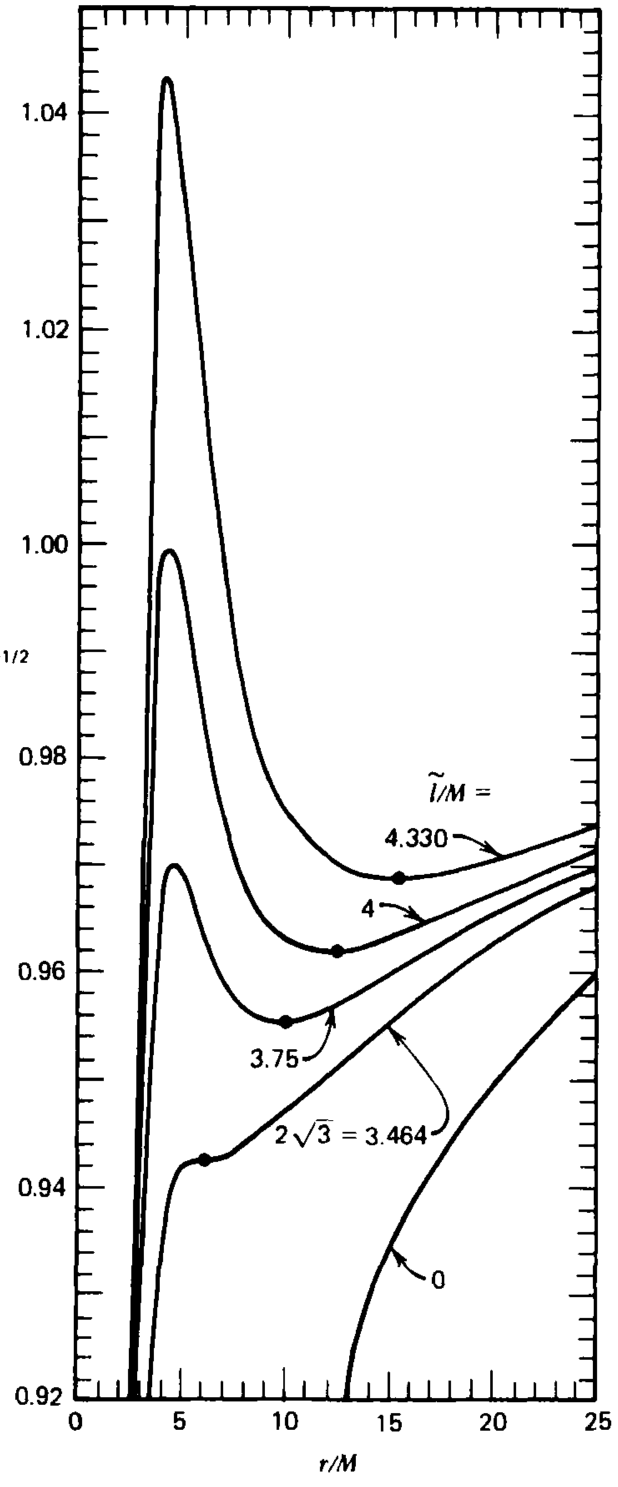


Figure 12.3 The effective potential profile for *nonzero* rest-mass particles of various angular momenta $\tilde{l}$ orbiting a Schwarzschild black hole of mass M. The dots at local minima locate radii of stable circular orbits. Such orbits exist only for $\tilde{l} > 2\sqrt{3}\, M$. [From *Gravitation* by Charles W. Misner, Kip S. Thorne, and John Archibald Wheeler, W. H. Freeman and Company. Copyright © 1973.]

This is the fraction of rest-mass energy released when, say, a particle originally at rest at infinity spirals slowly toward a black hole to the innermost stable circular orbit, and then plunges into the black hole. Thus, the conversion of rest mass to other forms of energy is potentially much more efficient for accretion onto a black hole than for nuclear burning, which releases a maximum of only 0.9% of the rest mass (H → Fe). This high efficiency will be important in our discussion of accretion disks around black holes (Section 14.5). It is the basis for invoking black holes as the energy source in numerous models seeking to explain astronomical observations of huge energy output from compact regions (e.g., Cygnus X-1; quasars; double radio galaxies, etc.).

Exercise 12.12

(a) Use Eq. (12.4.18) to show that the velocity of a particle in the innermost stable circular orbit as measured by a local static observer is $v^{\hat{\phi}} = \frac{1}{2}$ ($c = 1$).

(b) Suppose the particle in part (a) is emitting monochromatic light at frequency ν_{em} in its rest frame. Show that the frequency received at infinity varies periodically between

$$\frac{\sqrt{2}}{3}\nu_{\text{em}} < \nu_\infty < \sqrt{2}\,\nu_{\text{em}}.$$

Hint: Write $\nu_\infty/\nu_{\text{em}} = (\nu_\infty/\nu_{\text{stat}})(\nu_{\text{stat}}/\nu_{\text{em}})$, where ν_{stat} is the frequency measured by the local static observer and is related to ν_{em} by the special relativistic Doppler formula.

(c) Compute the orbital period for the particle as measured by the local static observer and by the observer at infinity.

Hint: Since $d\hat{\phi} = r\,d\phi$, the proper circumference of the orbit is simply $2\pi r$.

Answer: $T_{\text{stat}} = 24\pi M$, $T_\infty = T_{\text{stat}}/(2/3)^{1/2} = 4.5 \times 10^{-4}\,\text{s}\,(M/M_\odot)$

Exercise 12.13

(a) Show that the angular velocity as measured from infinity, $\Omega \equiv d\phi/dt$, has the same form in the Schwarzschild geometry as for circular orbits in Newtonian gravity—namely,

$$\Omega = \left(\frac{M}{r^3}\right)^{1/2}. \tag{12.4.31}$$

(b) Use this result to confirm the value of T_∞ found in Exercise 12.12.

In our later discussion of accretion onto black holes, we will need to know the capture cross section for particles falling in from infinity. This is simply

$$\sigma_{\text{capt}} = \pi b_{\max}^2, \tag{12.4.32}$$

where $b_{\max}$ is the maximum impact parameter of a particle that is captured. To express b in terms of $\tilde{E}$ and $\tilde{l}$, consider the definition of the impact parameter (cf. Fig. 12.4)

$$b = \lim_{r \to \infty} r \sin\phi. \tag{12.4.33}$$

Now for $r \to \infty$, Eqs. (12.4.13) and (12.4.14) give

$$\frac{1}{r^4}\left(\frac{dr}{d\phi}\right)^2 \simeq \frac{\tilde{E}^2 - 1}{\tilde{l}^2}. \tag{12.4.34}$$

Substituting $r \simeq b/\phi$, we identify

$$\frac{1}{b^2} = \frac{\tilde{E} - 1}{\tilde{l}^2}, \tag{12.4.35}$$

or in terms of the velocity at infinity, $\tilde{E} = (1 - v_\infty^2)^{-1/2}$,

$$\begin{aligned} \tilde{l} &= b v_\infty \left(1 - v_\infty^2\right)^{-1/2} \\ &\to b v_\infty \quad \text{for } v_\infty \ll 1. \end{aligned} \tag{12.4.36}$$

Consider now a nonrelativistic particle moving towards the black hole ($\tilde{E} \simeq 1$, $v_\infty \ll 1$). From Exercise 12.9, we know that it is captured if $\tilde{l} < 4M$. Thus

$$b_{\max} = \frac{4M}{v_\infty}, \tag{12.4.37}$$

which gives a capture cross section

$$\sigma_{\text{capt}} = \frac{4\pi(2M)^2}{v_\infty^2}. \tag{12.4.38}$$

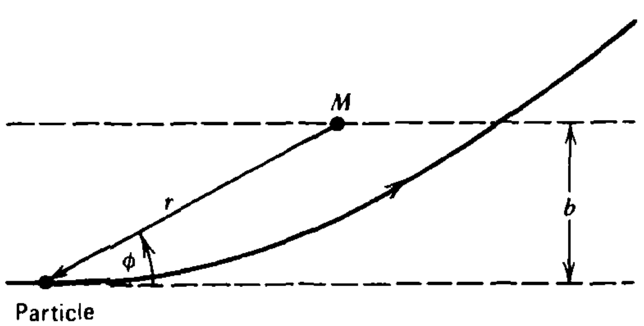


Figure 12.4 Impact parameter b for a particle with trajectory $r = r(\phi)$ about mass M.

This value should be compared with the geometrical capture cross section of a particle by a sphere of radius R in Newtonian theory:

$$\sigma_{\text{Newt}} = \pi R^2 \left(1 + \frac{2M}{v_\infty^2 R}\right). \tag{12.4.39}$$

A black hole thus captures nonrelativistic particles like a Newtonian sphere of radius $R = 8M$.

12.5 Massless Particle Orbits in the Schwarzschild Geometry

For $m = 0$ (e.g., a photon), Eqs. (12.4.6)–(12.4.8) become

$$\frac{dt}{d\lambda} = \frac{E}{1 - 2M/r}, \tag{12.5.1}$$

$$\frac{d\phi}{d\lambda} = \frac{l}{r^2}, \tag{12.5.2}$$

$$\left(\frac{dr}{d\lambda}\right)^2 = E^2 - \frac{l^2}{r^2}\left(1 - \frac{2M}{r}\right). \tag{12.5.3}$$

Now by the Equivalence Principle, we know that the particle's worldline should be independent of its energy. We can see this by introducing a new parameter

$$\lambda_{\text{new}} = l\lambda. \tag{12.5.4}$$

Writing

$$b \equiv \frac{l}{E} \tag{12.5.5}$$

and dropping the subscript "new," we find

$$\frac{dt}{d\lambda} = \frac{1}{b(1 - 2M/r)}, \tag{12.5.6}$$

$$\frac{d\phi}{d\lambda} = \frac{1}{r^2}, \tag{12.5.7}$$

$$\left(\frac{dr}{d\lambda}\right)^2 = \frac{1}{b^2} - \frac{1}{r^2}\left(1 - \frac{2M}{r}\right). \tag{12.5.8}$$

The worldline depends only on the parameter b, which is the particle's *impact parameter*, and not on l or E separately. Taking the limit $m \to 0$ of Eq. (12.4.35), we see that b of Eq. (12.5.5) is the same quantity defined in the previous section for massive particles.

We can understand photon orbits by means of an effective potential

$$V_{\text{phot}} = \frac{1}{r^2}\left(1 - \frac{2M}{r}\right), \tag{12.5.9}$$

so that Eq. (12.5.8) becomes

$$\left(\frac{dr}{d\lambda}\right)^2 = \frac{1}{b^2} - V_{\text{phot}}(r). \tag{12.5.10}$$

Clearly the distance from a horizontal line of height $1/b^2$ to V_{phot} gives $(dr/d\lambda)^2$. The quantity V_{phot} has a maximum of $1/(27M^2)$ at $r = 3M$; it is displayed in Figure 12.5. We see that the critical impact parameter separating capture from scattering orbits is given by $1/b^2 = 1/(27M^2)$, or

$$b_c = 3\sqrt{3}\,M. \tag{12.5.11}$$

The capture cross section for photons from infinity is thus

$$\sigma_{\text{phot}} = \pi b_c^2 = 27\pi M^2. \tag{12.5.12}$$

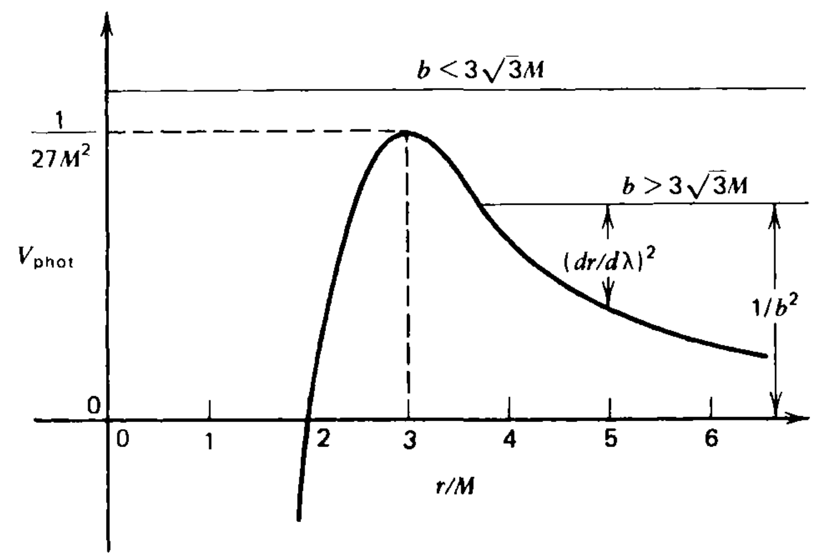


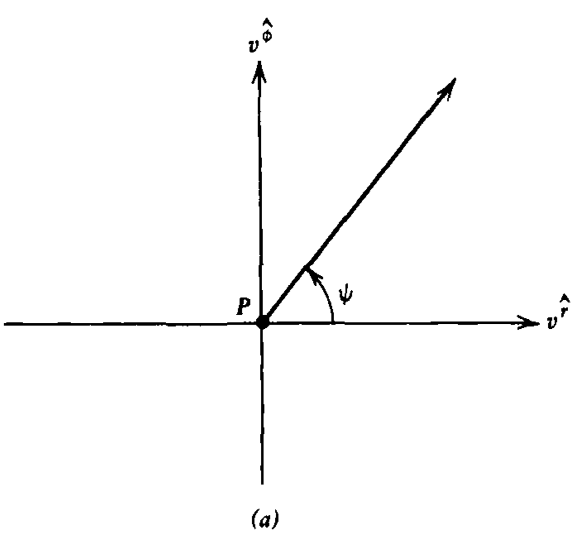
Figure 12.5 Sketch of the effective potential profile for a particle with *zero* rest mass orbiting a Schwarzschild black hole of mass M. If the particle falls from $r = \infty$ with impact parameter $b > 3\sqrt{3}\,M$ it is scattered back out to $r = \infty$. If, however, $b < 3\sqrt{3}\,M$ the particle is captured by the black hole.

To calculate the observed emission from gas near a black hole we must know those propagation directions, as measured by a static observer, for which a photon emitted at radius r can escape to infinity. Referring to Figure 12.5, we see that a photon at $r \geqslant 3M$ escapes only if (i) $v^{\hat{r}} > 0$, or (ii) $v^{\hat{r}} < 0$ and $b > 3\sqrt{3}\,M$. In terms of the angle ψ between the propagation direction and the radial direction (see Figure 12.6), we have since $|\mathbf{v}| = 1$,

$$v^{\hat{\phi}} = \sin\psi, \qquad v^{\hat{r}} = \cos\psi. \tag{12.5.13}$$

But Eqs. (12.4.12) and (12.4.18) give, with $b = l/E$,

$$v^{\hat{\phi}} = \frac{b}{r}\left(1 - \frac{2M}{r}\right)^{1/2}. \tag{12.5.14}$$



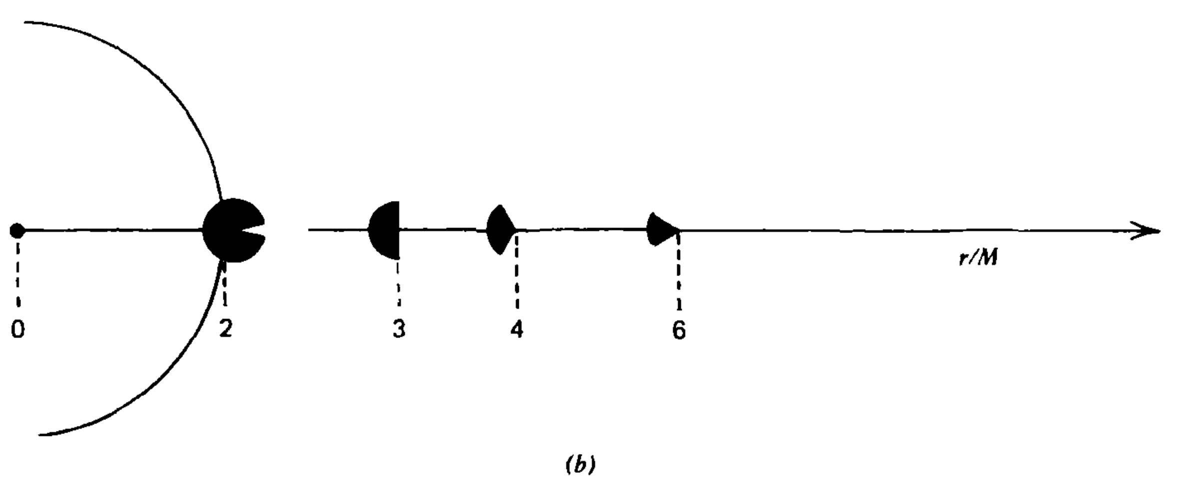


Figure 12.6 (*a*) The angle ψ between the propagation direction of a photon and the radial direction at a given point P. (*b*) Gravitational capture of radiation by a Schwarzschild black hole. Rays emitted from each point into the interior of the *shuded* conical cavity are captured. The indicated capture cavities are those measured in the orthonormal frame of a local static observer.

Thus an inward-moving photon escapes the black hole if

$$\sin\psi > \frac{3\sqrt{3}\,M}{r}\left(1 - \frac{2M}{r}\right)^{1/2}. \tag{12.5.15}$$

At $r = 6M$, escape requires $\psi < 135°$; at $r = 3M$, $\psi < 90°$ so that all inward-moving photons are captured (i.e., 50% of the radiation from a stationary, isotropic emitter at $r = 3M$ is captured).

Exercise 12.14 Show that an outward-directed photon emitted between $r = 2M$ and $r = 3M$ escapes if

$$\sin\psi < \frac{3\sqrt{3}\,M}{r}\left(1 - \frac{2M}{r}\right)^{1/2}.$$

Only the outward-directed radial photons escape as the source approaches $r = 2M$. See Figure 12.6 for a diagram of these effects.

12.6 Nonsingularity of the Schwarzschild Radius

The metric (12.3.1) appears singular at $r = 2M$; the coefficient of dt^2 goes to zero, while the coefficient of dr^2 becomes infinite. However, we cannot immediately conclude that this behavior represents a true physical singularity. Indeed, the coefficient of $d\phi^2$ becomes zero at $\theta = 0$, but we know this is simply because the polar coordinate system itself is singular there. The coordinate singularity at $\theta = 0$ can be removed by choosing another coordinate system (e.g., stereographic coordinates on the 2-sphere).

We already have a clue that the Schwarzschild radius $r = 2M$ is only a *coordinate singularity*. Recall that a radially infalling particle does not notice anything strange about $r = 2M$; there is nothing special about $r(\tau)$ at this point. However, the coordinate time t becomes infinite as $r \to 2M$. This strongly suggests the presence of a coordinate singularity rather than a physical singularity.

There are many different coordinate transformations that can be used to show explicitly that $r = 2M$ is not a physical singularity. We shall exhibit one—the *Kruskal*[6] *coordinate system*. It is defined by the transformation

$$u = \left(\frac{r}{2M} - 1\right)^{1/2} e^{r/4M} \cosh\frac{t}{4M}, \tag{12.6.1}$$

$$v = \left(\frac{r}{2M} - 1\right)^{1/2} e^{r/4M} \sinh\frac{t}{4M}. \tag{12.6.2}$$

[6] Kruskal (1960); Szekeres (1960).